REPUBLIQUE DU CAMEROUN
Paix – Travail – Patrie
****
UNIVERSITE DE YAOUNDE I
FACULTE DES SCIENCES
DEPARTEMENT DE MATHEMATIQUES
****
CENTRE DE RECHERCHE ET DE FORMATION
DOCTORALE EN SCIENCES, TECHNOLOGIE ET GEOSCIENCES
LABORATOIRE D'ALGEBRE, GEOMETRIE ET APPLICATIONS

REPUBLIC OF CAMEROUN
Peace – Work – Fatherland
****
UNIVERSITY OF YAOUNDE I
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS
****
POSTGRADUATE SCHOOL OF SCIENCE,
TECHNOLOGY AND GEOSCIENCES
LABORATORY OF ALGEBRA, GEOMETRY AND APPLICATIONS

RESIDUAL TRANSFER IN FUZZY ALGEBRAIC STRUCTURES

THESIS
Submitted in partial fulfilment of the requirements for the award of the degree of Doctorat/ Ph.D in Mathematics

Par : **TCHOFFO FOKA Samuel Vedric**
Master in Mathematics

Sous la direction de
TONGA Marcel
Associate Professor
University of Yaoundé I

Année Académique : 2020

REPUBLIQUE DU CAMEROUN
Paix-Travail-Patrie
....
UNIVERSITE DE YAOUNDE I

FACULTE DES SCIENCES
B.P 812 Yaoundé
Tel / Fax: (237) 22 23 44 96

REPUBLIC OF CAMEROON
Peace-Work-Fatherland
....
THE UNIVERSITY OF YAOUNDE I

FACULTY OF SCIENCE
P.O box 812 Yaounde
Tel / Fax: (237) 22 23 44 96

DEPARTEMENT DE MATHEMATIQUES
DEPARTMENT OF MATHEMATICS

ATTESTATION DE CORRECTION DE LA THESE DE DOCTORAT/PH.D

Nous soussignés, membres du jury de soutenance de la thèse de Doctorat/Ph.D de Monsieur **TCHOFFO FOKA Samuel Vedric**, Matricule **05V271**, intitulée : « **Residual Transfer in Fuzzy Algebraic Structures**» soutenue le 27 août 2020, attestons que toutes les corrections demandées par le jury de soutenance ont été effectuées.

En foi de quoi, la présente attestation lui est délivrée pour servir et valoir ce que de droit.

Yaoundé, le 03/09/2020

Président :	**ANDJIGA Nicolas Gabriel, Professeur**	**Université de Yaoundé I ;**
Membres :	**NKUIMI JUGNIA Célestin, Maître de Conférences**	**Université de Yaoundé I ;**
	TEMGOUA ALOMO Etienne Romuald, Maître de Conférences	**Université de Yaoundé I ;**

DEDICATION

*Dedicated to my grandfather, **FOKA Samuel Clovis**.*
May his soul rest in peace.

Acknowledgements

This work wouldn't have been possible without the help and support of some people. I could not possibly itemize my gratitude in detail, but I would like to mention some of the most important names.

First, I would like to thank my advisor, **Pr. TONGA Marcel**, for introducing me in the area of universal algebra, and his many useful suggestions in the preparation of this thesis.

Next, for their various remarks during research seminars, I thank all the members of the research team of the Laboratory of Algebra, Geometry and Applications of the faculty of science of the university of Yaoundé I; especially to Dr. KADJI Albert and FOTSO TATUENE Armand.

Finally, many thanks to all members of my family for their constant supports during my studies.

Contents

Abstract

This thesis attempts a combination of three important areas of mathematics, namely universal algebra, residuation theory and fuzzy set theory. A fuzzy subalgebra of a universal algebra $\mathcal{A} := (A;\ F^A)$ of type $\mathcal{F}$ under a residuated lattice $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$, called an L-fuzzy subalgebra of $\mathcal{A}$, is a map from A to L which is $\wedge$-compatible with the fundamental operations of $\mathcal{A}$. This notion was introduced by V. Murali [29] in 1991, under the unit interval $[0,\ 1]$ of real numbers, and generalized by B. Šešelja [35] in 1996, under partially ordered sets.

Given a residuated lattice $\mathcal{L}$ and a universal algebra $\mathcal{A}$ of type $\mathcal{F}$ with a residuated lattice $\mathcal{S}ub(\mathcal{A}) := \big(Sub(\mathcal{A});\ \cap,\ \sqcup,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ Sg(\emptyset),\ A\big)$ on the set of its subuniverses, the set $Fu(A, L)$ of L-fuzzy subsets of A forms a residuated lattice $\mathcal{F}u(A, L) := \big(Fu(A, L);\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ \underline{0},\ \underline{1}\big)$ that extends both $\mathcal{L}$ and the Boolean algebra $\mathcal{P}(A)$ of subsets of A. The set $Fs(\mathcal{A}, L)$ of L-fuzzy subalgebras of $\mathcal{A}$ forms a bounded lattice $\mathbb{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \sqcup;\ \chi_{Sg(\emptyset)},\ \underline{1}\big)$, but not necessarily a residuated lattice, which extends both the bounded lattices of $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$. When $\mathcal{L}$ is a finite linearly ordered Brouwerian algebra, $Fs(\mathcal{A}, L)$ forms an algebraic residuated lattice $\mathcal{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \sqcup,\ \otimes,\ \hookrightarrow,\ \looparrowright;\ \chi_{Sg(\emptyset)},\ \underline{1}\big)$ that extends both $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$.

The condition on the residuated lattice $\mathcal{L}$ of the preceding result being rather restrictive, it is natural to look for some classes of algebras for which the latter is more general. In this thesis, two solutions to this problem are proposed, in the classes of mono-unary algebras and rings, and some of their properties are investigated.

Key Words: Universal algebra, Lattice, Residuated lattice, Brouwerian algebra, MV-algebra, Boolean algebra, Mono-unary algebra, Ring, Łukasiewicz ring, Subuniverse, Ideal, L-fuzzy subalgebra, L-fuzzy ideal, Category, Functor.

Résumé

Cette thèse tente une combinaison de trois domaines importants des mathématiques, à savoir l'algèbre universelle, la théorie des résidus et la théorie des ensembles flous. Une sous-algèbre floue d'une algèbre universelle $\mathcal{A} := (A;\ F^A)$ de type $\mathcal{F}$ sous un treillis résidué $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$, appelée une L-sous-algèbre floue de $\mathcal{A}$, est une application de A vers L qui est $\wedge$-compatible avec les opérations fondamentales de $\mathcal{A}$. Cette notion a été introduite par V. Murali [29] en 1991, sous l'intervalle unité $[0,\ 1]$ des nombres réels, et généralisée par B. $\check{S}e\check{s}$elja [35] en 1996, sous les ensembles partiellement ordonnés.

Étant donnés un treillis résidué $\mathcal{L}$ et une algèbre universelle $\mathcal{A}$ de type $\mathcal{F}$ avec un treillis résidué $\mathcal{S}ub(\mathcal{A}) := \big(Sub(\mathcal{A});\ \cap,\ \sqcup,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ Sg(\emptyset),\ A\big)$ sur l'ensemble de ses sous-univers, l'ensemble $Fu(A, L)$ des L-sous-ensembles flous de A forme un treillis résidué $\mathcal{F}u(A, L) := \big(Fu(A, L);\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ \underline{0},\ \underline{1}\big)$ qui prolonge à la fois $\mathcal{L}$ et l'algèbre de Boole $\mathcal{P}(A)$ des sous-ensembles de A. L'ensemble $Fs(\mathcal{A}, L)$ des L-sous-algèbres floues de $\mathcal{A}$ forme un treillis borné $\mathbb{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \sqcup;\ \chi_{Sg(\emptyset)},\ \underline{1}\big)$, mais pas nécessairement un treillis résidué, qui prolonge à la fois les treillis bornés de $\mathcal{L}$ et $\mathcal{S}ub(\mathcal{A})$. Lorsque $\mathcal{L}$ est une algèbre de Brouwer finie et linéairement ordonnée, $Fs(\mathcal{A}, L)$ forme un treillis résidué algébrique $\mathcal{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \sqcup,\ \otimes,\ \hookrightarrow,\ \looparrowright;\ \chi_{Sg(\emptyset)},\ \underline{1}\big)$ qui prolonge à la fois $\mathcal{L}$ et $\mathcal{S}ub(\mathcal{A})$.

La condition sur le treillis résidué $\mathcal{L}$ du résultat précédent étant plutôt restrictive, il est naturel de chercher des classes d'algèbres pour lesquelles cette dernière est plus générale. Dans cette thèse, deux solutions à ce problème sont proposées, dans les classes des algèbres mono-unaires et des anneaux, et certaines de leurs propriétés sont étudiées.

<u>Mots clés:</u> Algèbre universelle, Treillis, Treillis résidué, Algèbre de Brouwer, MV-algèbre, Algèbre de Boole, Algèbre mono-unaire, Anneau, Anneau de Łukasiewicz, Sous-univers, Idéal, L-sous-algèbre floue, L-idéal flou, Catégorie, Foncteur.

INTRODUCTION

It is well known that life is uncertain, knowledge is limited, measures are imprecise, and future events can only be predicted with some confidence. Because of this, traditional mathematics, supported by Boolean logic, is unable to model complex systems. In 1965, L.A. Zadeh [41] introduced fuzzy set theory, which led to a revision of mathematics, to formalize the concept of set membership under uncertainty. In order to satisfy the needs of fuzzy reasoning, several kinds of algebraic structures were then considered.

Since the introduction of the idea of residuation by R. Dedekind [11] in 1894, several researchers have approached it in a general way. In 1939, M. Ward and R.P. Dilworth [40] introduced the notion of residuated lattice, as the lattices on which a multiplication or residuation operation is defined. During the same year, R.P. Dilworth [12] introduced the notion of non-commutative residuated lattice and investigated some of its properties among which decompositions into primary and semi-primary elements. In 1990, V. Nov$\acute{a}$k [31, 32] introduced first-order fuzzy logic and proved that the algebra of this logic is a residuated lattice. Since then, there has been substantial research regarding some specific classes of residuated lattices as RL-monoids, MTL-algebras, BL-algebras, MV-algebras,... (See, [10, 16, 20, 34]).

In 1967, J.A. Goguen [18] generalized the Zadeh's concept of fuzzy subset to L-fuzzy subset, replacing the unit interval $[0,\ 1]$ of real numbers by the underlying set L of an appropriate structure of truth values. He described one of his motivating examples as follows:

«A housewife faces a fairly typical optimization problem in her grocery shopping. She must select among all possible grocery bundles one that meets as well as several criteria of optimality such as coast, nutritional value, quality and variety. The partial ordering of the bundles is an intrinsic quality of this problem. It seems to be unnatural to describe the criteria of optimality by a linear ordering as the unit interval. Why should the nutritional value of a given product be described by 0.6 (instead of 0.65, or any other value from $[0,\ 1]$), and why should a product with a high nutritional value be better than a product with a high quality since those criteria are usually incomparable?».

In 1988, U.M. Swamy and K.L.N. Swamy [37] used the Goguen's concept to introduce the concept of L-fuzzy ideals of a ring, where L is the underlying set of a complete meet-distributive lattice. In 1996, B. Šešelja [35] generalized the Murali's concept, of fuzzy subalgebra of a universal algebra [29], to L-fuzzy subalgebra, where L is the underlying set of a partially ordered set $\mathcal{L}$, by considering compatibility rather on levels sets. He also characterized classes of algebras for which the partially ordered set of L-fuzzy subalgebras is a lattice, and pointed out the fact that its definition coincides with that of V. Murali when $\mathcal{L}$ is a bounded lattice.

Given a residuated lattice $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ and a universal algebra $\mathcal{A} := (A;\ F^A)$ of type $\mathcal{F}$ with a residuated lattice structure $\mathcal{S}ub(\mathcal{A}) := \big(Sub(\mathcal{A});\ \cap,\ \sqcup,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ Sg(\emptyset),\ A\big)$ on the set of its subuniverses, this thesis investigates possibilities to define a residuated lattice structure on the set $Fs(\mathcal{A}, L)$ of L-fuzzy subalgebras of $\mathcal{A}$ which extends both $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$. The dissertation contains four chapters and a conclusion in which the main results of the research are summarized, indications for future work are given and open problems are suggested.

In **Chapter 1**, we give the mathematical background on universal algebra, residuation theory and fuzzy sets theory, and collect some results that will be used later.

In **Chapter 2**, given a complete meet-distributive residuated lattice $\mathcal{L}$ and a universal algebra $\mathcal{A}$, we set up a mimetic construction of the L-fuzzy subalgebra of $\mathcal{A}$ generated by an L-fuzzy subset of A, and characterize atoms and co-atoms of the lattice $\mathbb{F}s(\mathcal{A}, L)$. When $\mathcal{L}$ is algebraic, we characterize compact elements of $\mathbb{F}s(\mathcal{A}, L)$ and show that the latter is algebraic. Furthermore, when $\mathcal{L}$ is a finite linearly ordered Brouwerian algebra and $\mathbb{S}ub(\mathcal{A})$ supports a quantale structure $\mathcal{S}ub(\mathcal{A})$, we show that $\mathbb{F}s(\mathcal{A}, L)$ supports an algebraic quantale which is both an extension of $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$. Finally, given a complete residuated lattice $\mathcal{L}$ and a mono-unary algebra $\mathcal{A}$, we define a residuated lattice structure $\mathcal{F}s(\mathcal{A}, L)$ on the set of L-fuzzy subalgebras of $\mathcal{A}$ which is both an extension of $\mathcal{L}$ and the Heyting algebra $\mathcal{S}ub(\mathcal{A})$ on the set of subuniverses of $\mathcal{A}$. Also, we show that $\mathcal{F}s(\mathcal{A}, L)$ is an MV-algebra (resp., a Boolean algebra) if and only if $\mathcal{L}$ is an MV-algebra (resp., a Boolean algebra) and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra.

In **Chapter 3**, given a complete meet-distributive residuated lattice $\mathcal{L}$ and a unital ring $\mathcal{A}$, we define a residuated lattice structure $\mathcal{F}id(\mathcal{A}, L)$ on the set of L-fuzzy ideals of $\mathcal{A}$ which is both an extension of $\mathcal{L}$ and the residuated lattice $\mathcal{I}d(\mathcal{A})$ on the set of ideals of $\mathcal{A}$. Furthermore, we show that $\mathcal{F}id(\mathcal{A}, L)$ is commutative (a Brouwerian algebra, a Boolean algebra) if and only if so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$. Also, we characterize prime elements of $\mathcal{F}id(\mathcal{A}, L)$ and investigate some embedding properties of the lattice of its filters. Finally, we introduce the concept of Łukasiewicz rings under $\mathcal{L}$ and establish its connection with rings whose L-fuzzy ideals form an MV-algebra.

In **Chapter 4**, given a complete meet-distributive residuated lattice $\mathcal{L}$, we characterize L-fuzzy ideals of a quotient ring, and investigate some of their properties. Finally, we define some functors from the category of unital rings to the category of po-monoids, and study some of their properties.

Chapter One

Mathematical background

In this thesis, we assume familiarity with the most basic concepts from mathematical logic, set theory, order theory, lattice theory (as found in [6]), ring theory (as found in [25]) and category theory (as found in [36]).

1.1 Universal algebra

We recall here some basic concepts from universal algebra (See, [6] for a detailed exposition). Recall that the notion of universal algebra, sometimes called general algebra or algebra for short, was introduced to extract, whenever possible, the common elements of several seemingly different types of algebraic structures.

Definition 1.1.1. *A type (or language) of algebras is a pair $\mathcal{F} := \langle F;\ \sigma\rangle$, where F is a set of function symbols and σ a map from F to the set $\mathbb{N}$ of nonnegative integers.*

For any f in F, $\sigma(f)$ is called the arity (or rank) of f, and f is said to be an $\sigma(f)$-ary function symbol. Furthermore, $F = \bigcup\limits_{n\in\mathbb{N}} F_n$, where each F_n is the set of n-ary function symbols in F.

Definition 1.1.2. *An algebra of type $\mathcal{F}$ is a pair $\mathcal{A} := (A;\ F^A)$; where, A is a nonempty set (called the universe of $\mathcal{A}$), $F^A := \{f^A :\ f \in F\}$ and each $f^A : A^{\sigma(f)} \to A$ is an $\sigma(f)$-ary operation on A, called a fundamental operation of $\mathcal{A}$.*

If $F = \{f_1, f_2, \ldots, f_n\}$ with $\sigma(f_1) \geq \sigma(f_2) \geq \ldots \geq \sigma(f_n)$, then we also write $\big(A;\ f_1^A,\ f_2^A, \ldots, f_n^A\big)$ and $\langle\sigma(f_1), \sigma(f_2), \ldots, \sigma(f_n)\rangle$ for $\mathcal{A}$ and $\mathcal{F}$, respectively.

Example 1.1.3. ***(a)*** *A Heyting algebra is an algebra $(A;\ \wedge,\ \vee,\ \twoheadrightarrow;\ 0,\ 1)$ of type $\langle 2, 2, 2, 0, 0\rangle$ such that $(A;\ \wedge,\ \vee;\ 0,\ 1)$ is a distributive bounded lattice and which satisfies for any $x, y, z \in A$: $(x \twoheadrightarrow y) \wedge x = x \wedge y$, $(x \twoheadrightarrow y) \wedge y = y$, $x \twoheadrightarrow (y \wedge z) = (x \twoheadrightarrow y) \wedge (x \twoheadrightarrow z)$, $(x \vee y) \twoheadrightarrow z = (x \twoheadrightarrow z) \wedge (y \twoheadrightarrow z)$ and $x \twoheadrightarrow x = 1$.*

(b) *A Boolean algebra is an algebra* $(A;\ \wedge,\ \vee;\ ';\ 0,\ 1)$ *of type* $\langle 2,2,1,0,0\rangle$ *such that* $(A;\ \wedge,\ \vee,\ \twoheadrightarrow;\ 0,\ 1)$*, where* $x \twoheadrightarrow y = x' \vee y$ *for all* $x, y \in A$*, is a Heyting algebra;* x' *is then called the complement of* x*.*
In particular, the set $P(E)$ *of subsets of a set* E*, called the power set of* E*, forms a Boolean algebra* $\mathcal{P}(E) := \big(P(E);\ \cap,\ \cup;\ \overline{(\ )};\ \emptyset,\ E\big)$*; where,* $\overline{B} = E \setminus B$ *for all* $B \in P(E)$*.*

The class of Boolean lattices is precisely the class of reducts of Boolean algebras to $\{\wedge,\ \vee;\ 0,\ 1\}$.

Definition 1.1.4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two algebras of the same type* $\mathcal{F}$*.*
(i) $\mathcal{B}$ *is called a subalgebra of* $\mathcal{A}$ *if* $B \subseteq A$ *and for any* n*-ary* f *in* F*,* f^B *is the restriction of* f^A *to* B^n*.*
(ii) *A mapping* $h : A \to B$ *is called a homomorphism from* $\mathcal{A}$ *to* $\mathcal{B}$ *if for any* n*-ary* f *in* F*, we have* $h\big(f^A(a_1, \ldots, a_n)\big) = f^B\big(h(a_1), \ldots, h(a_n)\big)$ *for all* $a_1, \ldots, a_n \in A$*. If in addition:*

- h *is one-to-one, then it is called an embedding of* $\mathcal{A}$ *into* $\mathcal{B}$*, and* $\mathcal{A}$ *is said to be embedded into* $\mathcal{B}$*;*
- h *is onto, then it is called an epimorphism from* $\mathcal{A}$ *to* $\mathcal{B}$*, and* $\mathcal{B}$ *is said to be a homomorphic image of* $\mathcal{A}$*;*
- h *is bijective, then it is called an isomorphism from* $\mathcal{A}$ *to* $\mathcal{B}$*, and* $\mathcal{A}$ *is said to be isomorphic to* $\mathcal{B}$*.*

Remark 1.1.5. *Let* $\mathcal{A} = (A;\ \wedge_A,\ \vee_A)$ *and* $\mathcal{B} = (B;\ \wedge_B,\ \vee_B)$ *be two complete lattices. A mapping* $h : A \to B$ *is a:*

- *complete lattice morphism if and only if*

$$h(\bigwedge L) = \bigwedge_{a \in L} h(a) \text{ and } h(\bigvee L) = \bigvee_{a \in L} h(a) \text{ for all } L \subseteq A;$$

- *complete lattice embedding if and only if it is a one-to-one complete lattice morphism.*

Definition 1.1.6. *Let* $\mathcal{A}$ *be an algebra of type* $\mathcal{F}$*. A subset* B *of* A *is called a subuniverse of* $\mathcal{A}$ *if for any* n*-ary* f *in* F*, we have* $f^A(a_1, \ldots, a_n) \in B$ *for all* $a_1, \ldots, a_n \in B$*.*

Remark 1.1.7. ***(a)*** *The ideals of a lattice* $\mathcal{L} = (L;\ \sqcap,\ \sqcup)$ *are just subuniverses of the algebra* $\overline{\mathcal{L}} = (L;\ \sqcup;\ (m_a)_{a \in L})$*, where* $m_a(x) = a \sqcap x$ *for all* $a, x \in L$*.*
(b) *The normal subgroups of a group* $\mathcal{G} = (G;\ \cdot,\ ^{-1},\ e)$ *are just subuniverses of the algebra* $\overline{\mathcal{G}} = (G;\ \cdot;\ ^{-1},\ (m_a)_{a \in G};\ e)$*, where* $m_a(x) = axa^{-1}$ *for all* $a, x \in G$*.*
(c) *The ideals of a ring* $\mathcal{R} = (R;\ +,\ \cdot;\ -;\ 0)$ *are just subuniverses of the algebra* $\overline{\mathcal{R}} = (R;\ +;\ -,\ (l_a)_{a \in R},\ (r_a)_{a \in R};\ 0)$*, where* $l_a(x) = ax$ *and* $r_a(x) = xa$ *for all* $a, x \in R$*.*

Definition 1.1.8. *Let $Sub(\mathcal{A})$ be the set of subuniverses of $\mathcal{A}$.*
(i) *The subuniverse of $\mathcal{A}$ generated by a subset X of A, denoted by $Sg_{\mathcal{A}}(X)$ or simply $Sg(X)$, is $\bigcap\{B \in Sub(\mathcal{A}) : X \subseteq B\}$; i.e., the smallest subuniverse of $\mathcal{A}$ containing X.*
(ii) *$\mathcal{A}$ is called $\mathcal{F}$-trivial if $\mathcal{F}_0 \neq \emptyset$ and $A = Sg(\emptyset)$.*

Example 1.1.9. *If $\mathcal{A}$ is a semigroup (resp., a group), then $Sg_{\mathcal{A}}(\emptyset) = \emptyset$ (resp., $Sg_{\mathcal{A}}(\emptyset) = \{e\}$).*

Note that our definition of a $\mathcal{F}$-trivial algebra does not always coincide with the definition of trivial algebra; that is, an algebra with a single element (See, [6]).

Proposition 1.1.10. *(See, [6], Corollary 3.3.) The set of subuniverses of $\mathcal{A}$ forms an algebraic lattice $\mathbb{S}ub(\mathcal{A}) := \big(Sub(\mathcal{A});\ \cap,\ \sqcup;\ Sg(\emptyset),\ A\big)$; where, $\cap$ is the intersection of sets and $\sqcup$ is defined by: $B \sqcup C := Sg(B \cup C)$ for all $B, C \in Sub(\mathcal{A})$. Furthermore, compact elements of $\mathbb{S}ub(\mathcal{A})$ are exactly of the form $Sg(X)$; where, X is a finite subset of A.*

Theorem 1.1.11. *(See, [6], Theorem 3.5.) Every algebraic lattice is isomorphic to the lattice of subuniverses of an algebra.*

Definition 1.1.12. *Let X be a set of variables and $\mathcal{F}$ a type of algebras. The set $T(X, \mathcal{F})$ of terms of type $\mathcal{F}$ over X is the smallest set satisfying the following conditions:*
- *$X \cup F_0 \subseteq T(X, \mathcal{F})$.*
- *If $t_1, \ldots, t_n \in T(X, \mathcal{F})$ and $f \in F_n$, then $f(t_1, \ldots, t_n) \in T(X, \mathcal{F})$.*

Usually, the set X of variables is omitted if it is understood or of no particular importance.

Definition 1.1.13. *Given an algebra $\mathcal{A}$ of type $\mathcal{F}$ and a term $t(x_1, \ldots, x_n)$ of type $\mathcal{F}$, the evaluation (or term function) t^A of $t(x_1, \ldots, x_n)$ on $\mathcal{A}$ is the n-ary operation on A defined as follows:*
- *if $t(x_1, \ldots, x_n)$ is a variable x_i, then $t^A(a_1, \ldots, a_n) = a_i$ for all $a_1, \ldots, a_n \in A$ (t^A is the i-th projection on A^n);*
- *if $t(x_1, \ldots, x_n)$ is of the form $f\big(t_1(x_1, \ldots, x_n), \ldots, t_k(x_1, \ldots, x_n)\big)$, where $f \in F_k$, then*

$$t^A(a_1, \ldots, a_n) = f^A\big(t_1^A(a_1, \ldots, a_n), \ldots, t_k^A(a_1, \ldots, a_n)\big) \text{ for all } a_1, \ldots, a_n \in A.$$

Definition 1.1.14. ***(i)*** *An identity of type $\mathcal{F}$ is an expression of the form $t \approx s$, where t and s are terms of type $\mathcal{F}$.*
(ii) *A class $\mathcal{K}$ of algebras of type $\mathcal{F}$ is called equational if there is a set of identities Σ such that $\mathcal{K} = Mod_{\mathcal{F}}(\Sigma)$, that is the set of algebras of type $\mathcal{F}$ satisfying Σ; in this case we say that $\mathcal{K}$ is axiomatized by Σ.*

Theorem 1.1.15. *(See, [6], Theorem 11.9. (Birkhoff)) A class of algebras of the same type is equational if and only if it is a variety (closed under the operator* $\mathbf{H}$*,* $\mathbf{S}$ *and* $\mathbf{P}$*).*

1.2 Residuation theory

1.2.1 Residuated lattices

We gather here some definitions and results on residuated lattices, most of them being well known (See, [10, 12, 16, 20, 22]).

Definition 1.2.1. *[20] An algebra* $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ e)$ *of type* $\langle 2,2,2,2,2,0\rangle$ *is called a residuated lattice-ordered monoid (or residuated lattice for short) if it satisfies the following conditions:*
(RL1) $(L;\ \wedge,\ \vee)$ *is a lattice;*
(RL2) $(L;\ \ominus,\ e)$ *is a monoid;*
(RL3) *for any* $x, y, z \in L$*,*
(a) $x \ominus y \leq z$ *if and only if* $x \leq y \twoheadrightarrow z$*,*
(b) $x \ominus y \leq z$ *if and only if* $y \leq x \multimap z$*;*
where, $\leq$ *is the partial order of the lattice.*

Definition 1.2.2. *A residuated lattice* $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ e)$ *is said to be complete if its lattice* $(L;\ \wedge,\ \vee)$ *is complete.*

Let us now adopt the notion of quantale, which is not usual, but which is equivalent to that of complete residuated lattices.

Definition 1.2.3. *A quantale is an algebra* $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ e)$ *of type* $\langle 2,2,2,2,2,0\rangle$ *satisfying the following conditions:*
(Q1) $(L;\ \wedge,\ \vee)$ *is a complete lattice;*
(Q2) $(L;\ \ominus,\ e)$ *is a monoid;*
(Q3) $a \ominus (\bigvee X) = \bigvee\limits_{x\in X} a \ominus x$ *and* $(\bigvee X) \ominus a = \bigvee\limits_{x\in X} x \ominus a$ *for all* $a \in L$ *and* $X \subseteq L$*;*
(Q4) $x \twoheadrightarrow y = \bigvee\{z \in L :\ z \ominus x \leq y\}$ *and* $x \multimap y = \bigvee\{z \in L :\ x \ominus z \leq y\}$ *for all* $x, y \in L$*.*

The usual definition of quantale [33] is simply the $\{\twoheadrightarrow,\ \multimap\}$-reduct (reduct to $\{\wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ e\}$) of the above definition. In practice, we will very often confuse the signatures of the two definitions.

Remark 1.2.4. *[20] An algebra* $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ e)$ *of type* $\langle 2,2,2,2,2,0\rangle$ *is a residuated lattice if and only if* $(L;\ \wedge,\ \vee)$ *is a lattice,* $(L;\ \ominus,\ e)$ *is a monoid,* $\ominus$ *is order-preserving in each argument and the inequality* $x \ominus y \leq z$ *has a* *largest solution for x (namely $y \twoheadrightarrow z$) and for y (namely $x \multimap z$). Intuitively, the residual operations $\twoheadrightarrow$ and $\multimap$ serve as generalized division operations, and are called left residue and right residue of $\ominus$, respectively.*

Example 1.2.5. *[2] A lattice-ordered group (called l-group for short) is an algebra $(G;\ \wedge,\ \vee,\ \cdot;\ ^{-1};\ e)$ of type $\langle 2,2,2,1,0\rangle$ such that $(G;\ \wedge,\ \vee)$ is a lattice and $(G;\ \cdot,\ ^{-1},\ e)$ is a group compatible with the lattice order. It induces a residuated lattice $(G;\ \wedge,\ \vee,\ \cdot,\ \twoheadrightarrow,\ \multimap;\ e)$; where, $x \twoheadrightarrow y = y \cdot x^{-1}$ and $x \multimap y = x^{-1} \cdot y$ for all $x, y \in G$.*

Proposition 1.2.6. *[20] In a residuated lattice, the following hold (whenever $\bigwedge$ and $\bigvee$ exist) for any $a \in L$, $B, C \subseteq L$ and $\dashrightarrow \in \{\twoheadrightarrow,\ \multimap\}$:*
(1) $(\bigvee B) \ominus (\bigvee C) = \bigvee_{b\in B,\ c\in C} b \ominus c.$
(2) $a \dashrightarrow (\bigwedge B) = \bigwedge_{b\in B} (a \dashrightarrow b)$ *and* $(\bigvee B) \dashrightarrow a = \bigwedge_{b\in B} (b \dashrightarrow a).$
Furthermore, the following identities or quasi-identities and their mirror images (obtained by replacing $x\ominus y$ by $y\ominus x$ and interchanging $x \twoheadrightarrow y$ with $x \multimap y$) also hold:
(3) $\big((x \twoheadrightarrow y) \ominus x\big) \vee y = y.$
(4) *If $x \wedge y = x$, then $x \ominus z = (x \ominus z) \wedge (y \ominus z)$, $y \twoheadrightarrow z = (y \twoheadrightarrow z) \wedge (x \twoheadrightarrow z)$ and $z \twoheadrightarrow x = (z \twoheadrightarrow x) \wedge (z \twoheadrightarrow y)$.*
(5) $e \twoheadrightarrow x = x.$
(6) $e = e \wedge (x \twoheadrightarrow x).$

Proposition 1.2.7. *[20] The class of residuated lattices is a finitely based equational class $\mathcal{RL} := Mod(\Sigma)$, where Σ consists of the defining equations for lattices and monoids together with the identities $x = x \wedge \big[y \twoheadrightarrow \big((x \ominus y) \vee z\big)\big]$, $x \ominus (y \vee z) = (x \ominus y) \vee (x \ominus z)$, $\big[(y \twoheadrightarrow x) \ominus y\big] \vee x = x$ and their mirror images.*

Definition 1.2.8. *[20] A residuated lattice $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ e)$ is called:*
- *commutative if $x \ominus y = y \ominus x$ for all $x, y \in L$, in this case $\twoheadrightarrow = \multimap$ and it is simply written $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow;\ e)$;*
- *a Brouwerian algebra if $x \ominus y = x \wedge y$ for all $x, y \in L$;*
- *integral if $x \leq e$ for all $x \in L$ (e is the top element of L).*

Definition 1.2.9. *In the rest of this work, by a residuated lattice we will designate an algebra $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ of type $\langle 2,2,2,2,2,0,0\rangle$ such that $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 1)$ is an integral residuated lattice and 0 is the bottom element of L.*

Residuated lattices are sometimes called non-commutative residuated lattices, pseudo-residuated lattices or bounded integral residuated lattices (See, [10, 22, 34]).

Example 1.2.10. ***(a)*** *The Gödel structure is the residuated lattice* $\mathcal{L} = (L;\ \wedge,\ \vee,\ \wedge,\ \twoheadrightarrow;\ 0,\ 1)$ *given by* $L = [0,\ 1]$*,* $x \wedge y = \min(x, y)$*,* $x \vee y = \max(x, y)$ *and*

$$x \twoheadrightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases} \quad \text{for all } x, y \in L.$$

(b) *The product (or Gaines) structure is the residuated lattice* $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow;\ 0,\ 1)$ *given by* $L = [0,\ 1]$*,* $x \wedge y = \min(x, y)$*,* $x \vee y = \max(x, y)$*,* $x \ominus y = xy$ *(the usual multiplication of real numbers) and*

$$x \twoheadrightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y/x & \text{otherwise.} \end{cases} \quad \text{for all } x, y \in L.$$

(c) *The Łukasiewicz structure of order* $p \in \mathbb{N}^\star$ *is the residuated lattice* $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow;\ 0,\ 1)$ *given by* $L = [0,\ 1]$*,* $x \wedge y = \min(x, y)$*,* $x \vee y = \max(x, y)$*,*

$$x \ominus y = \sqrt[p]{\max(0, x^p + y^p - 1)} \text{ and } x \twoheadrightarrow y = \min\left(1, \sqrt[p]{1 - x^p + y^p}\right) \text{ for all } x, y \in L.$$

If $p = 1$*, we obtain the Łukasiewicz structure.*

Proposition 1.2.11. *(See, [22]) In a residuated lattice* $\mathcal{L}$*, for any* $x \in L$*,*

$$\overline{x} := x \twoheadrightarrow 0 \text{ and } \widetilde{x} := x \multimap 0 \text{ (mirror image of } \overline{x})$$

are called left annihilator and right annihilator of x*, respectively. Furthermore, the following identities and quasi-identities and their mirror images hold for any* x, y *in* L*:*

(7) $x \ominus 0 = 0$ *and* $\overline{0} = 1$*.*

(8) $x = x \wedge y$ *if and only if* $x \twoheadrightarrow y = 1$*.*

(9) $x = x \wedge y$ *implies* $\overline{y} = \overline{y} \wedge \overline{x}$*.*

(10) $\overline{x} \ominus x = 0$*,* $x = x \wedge \widetilde{\overline{x}}$ *and* $\overline{\widetilde{\overline{x}}} = \overline{x}$*.*

(11) $x \ominus y = (x \ominus y) \wedge (x \wedge y)$ *and* $\big((x \twoheadrightarrow y) \ominus x\big) \vee (x \wedge y) = x \wedge y$*.*

For any $x \in L$ and a non negative integer n, x^n is defined inductively by $x^0 = e$ and $x^{n+1} = x^n \ominus x$.

Definition 1.2.12. *A residuated lattice* $\mathcal{L}$ *is called:*

- *an RL-monoid if* $(x \twoheadrightarrow y) \ominus x = x \wedge y = x \ominus (x \multimap y)$ *for all* $x, y \in L$*;*
- *a* MTL*-algebra if* $(x \twoheadrightarrow y) \vee (y \twoheadrightarrow x) = 1 = (x \multimap y) \vee (y \multimap x)$ *for all* $x, y \in L$*;*
- *a* BL*-algebra if it is both an* RL*-monoid and a* MTL*-algebra;*
- *a Gödel algebra if it is both a* MTL*-algebra and a Brouwerian algebra;*
- *an* MV*-algebra if it is a* BL*-algebra satisfying* $\widetilde{\overline{x}} = x = \overline{\widetilde{x}}$ *for all* $x \in L$*;*
- *an* n*-fold Boolean algebra if* $x \vee \overline{x^n} = 1 = x \vee \widetilde{x^n}$ *for all* $x \in L$*;*
- *trivial if* $L = \{0,\ 1\}$*.*

Definition 1.2.13. *A residuated lattice $\mathcal{L}$ is called:*
• *meet-distributive if so is its lattice, that is* $r \wedge \left(\bigvee B\right) = \bigvee_{b\in B} (r \wedge b)$ *for all* $r \in L$ *and* $B \subseteq L$*, whenever both* $\bigvee$ *exist;*
• *join-distributive if so is its lattice, that is* $r \vee \left(\bigwedge B\right) = \bigwedge_{b\in B} (r \vee b)$ *for all* $r \in L$ *and* $B \subseteq L$*, whenever both* $\bigwedge$ *exist.*

Remark 1.2.14.
• *Gödel and Gaines structures are BL-algebras, and Łukasiewicz structures are MV-algebras (See, [7]).*
• *A Heyting algebra* $(A;\ \wedge,\ \vee,\ \twoheadrightarrow;\ 0,\ 1)$ *may be viewed as a Brouwerian algebra* $(A;\ \wedge,\ \vee,\ \wedge,\ \twoheadrightarrow,\ \twoheadrightarrow;\ 0,\ 1)$*, and conversely.*
• *A residuated lattice is a Boolean algebra if and only if it is both a Heyting algebra and an MV-algebra, if and only if it is both a BL-algebra and an 1-fold Boolean algebra (See, [4]).*
• *The identities* $\overline{x \wedge y} = \overline{x} \vee \overline{y}$ *and* $\widetilde{x \wedge y} = \widetilde{x} \vee \widetilde{y}$ *hold in every MTL-algebra (See, [10], Proposition 4.1).*
• *The lattice of an RL-monoid (resp., a complete RL-monoid) is distributive (resp., meet-distributive) (See, [10], Proposition 4.7).*

Definition 1.2.15. *A residuated lattice $\mathcal{L}$ is said to be:*
(i) *$\ominus$-distributive (or product-distributive) if* $x \ominus (y \wedge z) = (x \ominus y) \wedge (x \ominus z)$ *and* $(y \wedge z) \ominus x = (y \ominus x) \wedge (z \ominus x)$ *for all* $x, y, z \in L$*;*
(ii) *completely $\ominus$-distributive (or product-distributive) if* $r \ominus (\bigwedge B) = \bigwedge_{b\in B} (r \ominus b)$ *and* $(\bigwedge B) \ominus r = \bigwedge_{b\in B} (b \ominus r)$ *for all* $r \in L$ *and* $B \subseteq L$ *whenever both* $\bigwedge$ *exist.*

Definition 1.2.16. *A residuated lattice $\mathcal{L}$ is said to be:*
(i) *join-implicative if for any* $x, y, z \in L$, $x \twoheadrightarrow (y \vee z) = (x \twoheadrightarrow y) \vee (x \twoheadrightarrow z)$ *and* $x \multimap (y \vee z) = (x \multimap y) \vee (x \multimap z)$*;*
(ii) *completely join-implicative if for any* $x \in L$ *and* $B \subseteq L$, $x \twoheadrightarrow \left(\bigvee B\right) = \bigvee_{b\in B} x \twoheadrightarrow b$ *and* $x \multimap \left(\bigvee B\right) = \bigvee_{b\in B} x \multimap b$ *whenever both* $\bigvee$ *exist.*

Note that Brouwerian algebras and linearly ordered residuated lattices are product-distributive, and Boolean algebras are join-implicative.

Example 1.2.17. *Let* $L = \{0,\ n,\ a,\ b,\ c,\ d,\ e,\ f,\ m,\ 1\}$ *be a lattice such that* $0 < n < a < c < e < m < 1$, $0 < n < b < d < f < m < 1$, $b < c$ *and* $d < e$*; where,* a, b, c, d *and* e, f *are incomparable, respectively. Define the binary operations* $\ominus$ *and* $\twoheadrightarrow$ *by the two tables below:*

$\ominus$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

$\twoheadrightarrow$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

.

As Example 3.7. in [27] shows, $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow;\ 0,\ 1)$ *is a distributive residuated lattice which is not:*

- *product-distributive, since* $m \ominus (a \wedge b) = m \ominus n = 0 \neq n = a \wedge b = (m \ominus a) \wedge (m \ominus b)$;
- *join-implicative, since* $c \twoheadrightarrow (a \vee b) = 1 \neq m = e \vee f = (c \twoheadrightarrow a) \vee (c \twoheadrightarrow b)$.

Some researchers have found some logics that have some subclasses of the variety of residuated lattices as models (See, [23, 31, 32]). For example, MV-algebras (resp., BL-algebras, MTL-algebras) are the algebraic counterpart of Łukasiewicz logic (resp., Basic Logic, Monoidal T-norm Logic).

1.2.2 Prime elements and filters of a residuated lattice

In this subsection, $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a residuated lattice.

Prime elements

Definition 1.2.18. *A* $\ominus$*-prime (or prime) element of* $\mathcal{L}$ *is a proper element* p *of* L $(p \neq 1)$ *such that: for any* $x, y \in L$, $x \ominus y \leq p$ *implies* $x \leq p$ *or* $y \leq p$.

A $\ominus$-prime element of $\mathcal{L}$ is $\wedge$-prime (that is a prime element of the lattice of $\mathcal{L}$), but the converse is not necessarily true as the following example shows.

Example 1.2.19. *Let $\mathcal{L}$ be the Gaines structure (See, Example 1.2.10). Since $\mathcal{L}$ is linearly ordered, each of its proper elements are $\wedge$-prime. For any $x \in]0,\ 1[$, we have $\sqrt{x} \ominus \sqrt{x} = x$ and $\sqrt{x} \nleq x$. Thus, 0 is the only $\ominus$-prime element of $\mathcal{L}$.*

Proposition 1.2.20. *A maximal element (co-atom) of $\mathcal{L}$ is a prime element of $\mathcal{L}$.*

Proof. Let p be a maximal element of $\mathcal{L}$. For any $x, y \in L$ such that $x \ominus y \leq p$, $x \nleq p$ and $y \nleq p$, we have $1 = 1 \ominus 1 = (x \vee p) \ominus (y \vee p) = (x \ominus y) \vee (x \ominus p) \vee (p \ominus y) \vee (p \ominus p) \leq p$ and, $p = 1$; which is a contradiction. Hence, p is a prime element of $\mathcal{L}$. $\square$

Definition 1.2.21. *Let $Spec(\mathcal{L})$ be the set of prime elements of $\mathcal{L}$. The radical of an element x of L, denoted by $\sqrt{x}$, is defined by:*

$$\sqrt{x} = \bigwedge \{p \in Spec(\mathcal{L}) :\ x \leq p\}, \text{ whenever } \bigwedge \text{ exists.}$$

Definition 1.2.22. *A $\ominus$-primary (or primary) element of $\mathcal{L}$ is a proper element p of $\mathcal{L}$ such that: for any $x, y \in L$, $x \ominus y \leq p$ implies $x \leq p$ or $y \leq \sqrt{p}$.*

A primary element of $\mathcal{L}$ is also called a right primary element of $\mathcal{L}$. If $\mathcal{L}$ is commutative, then right primary and left primary elements of $\mathcal{L}$ are confused; furthermore, any prime element of $\mathcal{L}$ is a primary element of $\mathcal{L}$.

Definition 1.2.23. ***(i)*** *An element x of $\mathcal{L}$ is said to have a primary decomposition (or to be decomposable into primary elements, or primary decomposable) if there exist primary elements $p_1, \ldots, p_n$ of $\mathcal{L}$ such that $x = \bigwedge_{1 \leq i \leq n} p_i$.*
(ii) *If any proper element of $\mathcal{L}$ has a primary decomposition, then $\mathcal{L}$ is said to be primary decomposable.*

Example 1.2.24. ***(a)*** *Let $\mathcal{L}$ be the Gödel structure. Since $Spec(\mathcal{L}) = [0,\ 1[$ is also the set of all primary elements of $\mathcal{L}$, $\mathcal{L}$ is primary decomposable.*
(b) *Let $\mathcal{L}$ be the Gaines structure. Since $Spec(\mathcal{L}) = \{0\}$, we have $\sqrt{0} = 0$ and $\sqrt{x} = \bigwedge \emptyset = 1$ for all $x \in]0,\ 1]$; thus, $[0,\ 1[$ is the set of all primary elements of $\mathcal{L}$. So, $\mathcal{L}$ is primary decomposable.*
(c) *Let $\mathcal{L}$ be the Łukasiewicz structure. We have $Spec(\mathcal{L}) = \emptyset$; indeed,*
• *0 is not a prime element of $\mathcal{L}$, since $\frac{1}{2} \ominus \frac{1}{2} = max\big(0,\ \frac{1}{2} + \frac{1}{2} - 1\big) = max(0, 0) = 0$ and $\frac{1}{2} \nleq 0$;*
• *any $p \in]0,\ 1[$ is not a prime element of $\mathcal{L}$, since $\sqrt{p} \ominus \sqrt{p} = max\big(0,\ \sqrt{p} + \sqrt{p} - 1\big) = max\big(0,\ 2\sqrt{p} - 1\big) \leq p$ and $\sqrt{p} \nleq p$.*
It follows that $\bigwedge \emptyset = 1$ is the only radical of $\mathcal{L}$. Consequently, $[0,\ 1[$ is the set

of all primary elements of $\mathcal{L}$. Hence, $\mathcal{L}$ is primary decomposable.
(d) *Let $\mathcal{L}$ be the Łukasiewicz structure of order 2. We have $Spec(\mathcal{L}) = \emptyset$; indeed,*
• 0 *is not a prime element of $\mathcal{L}$, since*

$$\tfrac{\sqrt{2}}{2} \ominus \tfrac{\sqrt{2}}{2} = \sqrt{max\big(0,\ (\tfrac{\sqrt{2}}{2})^2 + (\tfrac{\sqrt{2}}{2})^2 - 1\big)} = 0 \text{ and } \tfrac{\sqrt{2}}{2} \nleq 0;$$

• *any $p \in]0,\ 1[$ is not a prime element of $\mathcal{L}$, since*

$$\sqrt{p} \ominus \sqrt{p} = \sqrt{max\big(0,\ 2p - 1\big)} \leq p \text{ and } \sqrt{p} \nleq p.$$

It follows that $\bigwedge \emptyset = 1$ is the only radical of $\mathcal{L}$. Consequently, $[0,\ 1[$ is the set of all primary elements of $\mathcal{L}$. Hence, $\mathcal{L}$ is primary decomposable.

Proposition 1.2.25. *(See, [30]) Let p and q be two primary elements of $\mathcal{L}$ such that $\sqrt{p} = \sqrt{q}$. Then the following hold:*
(1) $\sqrt{p} = \sqrt{p \wedge q} = \sqrt{q}$.
(2) *$p \wedge q$ is a primary element of $\mathcal{L}$.*

Definition 1.2.26. *A primary decomposition of an element is called normal (or short [30]) when superfluous are removed and the primary components with the same radical are combined.*

Filters

Definition 1.2.27. *(See, [10, 22]) A nonempty subset F of L is called a $\ominus$-filter (or filter) of $\mathcal{L}$ if it satisfies the following conditions for any $x, y \in L$:*
(F1) *$x \in F$ and $y \in F$ imply $x \ominus y \in F$.*
(F2) *$x \leq y$ and $x \in F$ imply $y \in F$.*

A $\ominus$-filter of $\mathcal{L}$ is a $\wedge$-filter of $\mathcal{L}$ (that is a filter of the lattice of $\mathcal{L}$), but the converse is not necessarily true (See, [10], Remark 3.5). For any nonempty subset F of L, the following are equivalent (See, [10, 22]):
(1) F is a filter of $\mathcal{L}$.
(2) $1 \in F$ and for any $x, y \in L$, ($x \in F$ and $x \twoheadrightarrow y \in F$) imply $y \in F$.
(3) $1 \in F$ and for any $x, y \in L$, ($x \in F$ and $x \multimap y \in F$) imply $y \in F$.
The filter of $\mathcal{L}$ generated by a subset X of L is given by

$$[X) := \{y \in L : \ x_1 \ominus x_2 \ominus \ldots \ominus x_n \leq y \text{ for some } x_1, \ldots, x_n \in X,\ n \in \mathbb{N}^\star\};$$

in particular, the principal filter generated by an element x of L is given by $[x) = \{a \in L : \ x^n \leq a \text{ for some } n \geq 1\}$. The set $Fil(\mathcal{L})$ of filters of $\mathcal{L}$ forms a Heyting algebra $\mathcal{F}il(\mathcal{L}) := \big(Fil(\mathcal{L});\ \cap,\ \sqcup,\ \Rightarrow;\ \{1\},\ L\big)$; where, $\cap$ is the intersection of sets, $F_1 \sqcup F_2 := [F_1 \cup F_2)$ and $F_1 \Rightarrow F_2 := \{x \in L : \ [x) \cap F_1 \subseteq F_2\}$. The lattice of $\mathcal{F}il(\mathcal{L})$ is algebraic and its compact elements are exactly the principal filters of $\mathcal{L}$ (See, [10, 22]).

1.2.3 Łukasiewicz semi-rings and MV-algebras

The following approach to MV-algebras was initiated by G. Georgescu and A. Iorgulescu in [17] to generalize the commutative one introduced by C.C. Chang in [9].

Definition 1.2.28. *An MV-algebra is an algebra $\mathcal{M} := (M;\ \oplus,\ \odot;\ ^-,\ ^\sim;\ 0,\ 1)$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ satisfying the following conditions:*
(MV1) *$(M;\ \oplus,\ 0)$ is a monoid and $1^- = 0 = 1^\sim$.*
(MV2) *For any $x \in M$, $x \oplus 1 = 1 = 1 \oplus x$ and $(x^-)^\sim = x$.*
(MV3) *For any $x, y \in M$, $(y^\sim \oplus x^\sim)^- = x \odot y = (y^- \oplus x^-)^\sim$, $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$ and $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$.*

Remark 1.2.29. *(See, Definition 1.2.12) The two definitions of MV-algebras are equivalent through the following transfer:*
- *If $(M;\ \oplus,\ \odot;\ ^-,\ ^\sim;\ 0,\ 1)$ is an MV-algebra, then $(M;\ \wedge,\ \vee,\ \odot,\ \multimap,\ \twoheadrightarrow;\ 0,\ 1)$ is a distributive MV-algebra; where, the operations $\wedge$, $\vee$, $\multimap$ and $\twoheadrightarrow$ on M are given by $x \wedge y = x \odot (x^- \oplus y) = y \odot (y^- \oplus x) = (x \oplus y^\sim) \odot y = (y \oplus x^\sim) \odot x$, $x \vee y = (x \odot y^-) \oplus y = x \oplus (x^\sim \odot y)$, $x \multimap y = y \oplus x^\sim$ and $x \twoheadrightarrow y = x^- \oplus y$ and the order $\leq$ on M is given by $x \leq y$ iff $x^- \oplus y = 1$ iff $y \oplus x^\sim = 1$.*
- *If $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is an MV-algebra, then $(L;\ \oplus,\ \ominus;\ ^\sim,\ ^-;\ 0,\ 1)$ is an MV-algebra; where the binary operation $\oplus$ on L is given by $x \oplus y = \widetilde{\overline{y} \ominus \overline{x}} = \overline{\widetilde{y} \ominus \widetilde{x}} = \widetilde{y} \twoheadrightarrow x = \overline{x} \multimap y$.*

Example 1.2.30. ***(a)*** *The MV-algebra $\big([0,\ 1];\ \oplus,\ \odot;\ ^-,\ ^-;\ 0,\ 1\big)$, where $x \oplus y = \min\{1,\ x + y\}$, $x \odot y = \max\{0,\ x + y - 1\}$ and $x^- = 1 - x$, is the Łukasiewicz structure. It is also called the Łukasiewicz chain.*
(b) *Let $\mathcal{G} = (G;\ \wedge,\ \vee,\ +;\ -;\ 0)$ be an arbitrary l-group, u a strong unit of $\mathcal{G}$ (u is a positive element of G and for any $g \in G$ there exists an integer $n \geq 1$ such that $-nu \leq g \leq nu$) and $\Gamma(G, u)$ the lattice interval $[0,\ u]$ of $\mathcal{G}$. Then $\big(\Gamma(G, u);\ \oplus,\ \odot;\ ^-,\ ^\sim;\ 0,\ u\big)$ is an MV-algebra, where $x \oplus y = (x + y) \wedge u$, $x \odot y = (x - u + y) \vee 0$, $x^- = u - x$ and $x^\sim = -x + u$.*

Every MV-algebra is meet-distributive and join-distributive (See, [34], $psmv - c_{22}$ and $psmv - c_{23}$ of Theorem 4.6.), completely product-distributive (See, [34], $psmv - c_{26}$ and $psmv - c_{27}$ of Theorem 4.6.) and isomorphic to an MV-algebra of the form

$$\big(\Gamma(G, u);\ \oplus,\ \odot;\ ^-,\ ^\sim;\ 0,\ u\big) \text{ (See, [13], Theorem 3.9.).}$$

Every complete MV-algebra is commutative (that is $\oplus$ is commutative) (See, [13], Theorem 4.2. and [14], Proposition 6.4.14).

Proposition 1.2.31. *(See, [21], Proposition 2.1.) From an MV-algebra $\mathcal{M} = (M;\ \oplus,\ \odot;\ ^-,\ ^\sim;\ 0,\ 1)$, one can extract the algebra $(M;\ \vee,\ \odot;\ ^-,\ ^\sim;\ 0,\ 1)$*

of type $\langle 2,2,1,1,0,0\rangle$, which is called a Łukasiewicz semi-ring since it satisfies the following conditions:
(LS1) $(M;\ \vee,\ \odot)$ *is an additively idempotent semi-ring with an additive identity* 0 *and a multiplicative identity* 1.
(LS2) $^-$ *and* $^\sim$ *satisfy the following conditions for any* $x, y \in M$:
(i) $x \odot y = 0$ *iff* $x \leq y^\sim$ *iff* $y \leq x^-$; *where,* $x \leq y$ *iff* $x \vee y = y$.
(ii) $((x^\sim \odot y)^\sim \odot x^\sim)^- = x \vee y = (x^\sim \odot (y \odot x^-)^\sim)^-$.
(iii) $(x^\sim \odot y^\sim)^- = (x^- \odot y^-)^\sim$.

Remark 1.2.32. *Let* $(M;\ \vee,\ \odot;\ ^-,\ ^\sim;\ 0,\ 1)$ *be a Łukasiewicz semi-ring with the relation* $\leq$ *defined for any* $x, y \in M$ *by:* $x \leq y$ *if and only if* $x \vee y = y$. *Then the following are satisfied (See, [21], Lemma 2.2.):*
- *The relation* $\leq$ *is an order on* M *that is compatible with* $\vee$ *and* $\odot$.
- $0^- = 1 = 0^\sim$ *and* $1^- = 0 = 1^\sim$.
- *For any* $x \in M$, $x^\sim \odot x = 0 = x \odot x^-$ *and* $(x^-)^\sim = (x^\sim)^-$.
- *For any* $x, y \in M$, $x \leq y$ *implies* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$.
- $(M;\ \wedge,\ \vee,\ \odot;\ 0,\ 1)$ *is a bounded lattice-ordered semi-ring; where, for any* $x, y \in M$, $(x^- \vee y^-)^\sim = x \wedge y = (x^\sim \vee y^\sim)^-$.

Proposition 1.2.33. *(See, [21], Proposition 2.3. and Proposition 2.5.)*
(1) *A Łukasiewicz semi-ring* $(M;\ \vee,\ \odot;\ ^-,\ ^\sim;\ 0,\ 1)$ *induces an* MV*-algebra* $(M;\ \oplus,\ \odot;\ ^-,\ ^\sim;\ 0,\ 1)$; *where,* $x \oplus y = (y^\sim \odot x^\sim)^-$ *for all* $x, y \in M$.
(2) *There is a duality between* MV*-algebras and Łukasiewicz semi-rings.*

1.3 L-fuzzy subsets of a set

In this section, $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a complete residuated lattice, unless otherwise specified.

1.3.1 Residuated lattice of L-fuzzy subsets

Definition 1.3.1. *Let* A *be a nonempty set. A fuzzy subset of* A *under* $\mathcal{L}$, *or an* L*-fuzzy subset of* A, *is a map from* A *to* L.

For any $B \subseteq A$, $a \in A$ and $r, s \in L$, the following functions from A to L are L-fuzzy subsets of A:

$$B_r^s(x) := \begin{cases} s & \text{if } x \in B, \\ r & \text{if not.} \end{cases} \quad \text{for all } x \in A,$$

$B_r := B_0^r$, $B^s := B_s^1$, $a_r^s := \{a\}_r^s$, $a_r := a_0^r$ (L-fuzzy point of A), $B_1 =: \chi_B := B^0$ (characteristic function of B), $\chi_a := \chi_{\{a\}}$ and $A_r =: \underline{r} := \emptyset^r$ (constant L-fuzzy subset of A with value r). For any L-fuzzy subset μ of A and $r \in L$, the sets

$$\begin{aligned} Supp(\mu) &:= \{x \in A : \ \mu(x) \neq 0\} \\ Im(\mu) &:= \{\mu(x) : \ x \in A\} \\ U(\mu, r) &:= \{x \in A : \ \mu(x) \geq r\} \end{aligned}$$

are called the support, the image and the r-level set (or r-cut) of μ, respectively. The partial order relation $\leq$ on the set $Fu(A, L)$ of L-fuzzy subsets of A is defined as follows: for any $\mu, \nu \in Fu(A, L)$,

$$\mu \leq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

The relation $<$ on $Fu(A, L)$ is defined as follows: for any $\mu, \nu \in Fu(A, L)$,

$$\mu < \nu \text{ if and only if } \mu \leq \nu \text{ and there is } x \in A \text{ such that } \mu(x) < \nu(x).$$

The set $Fu(A, L)$ forms a complete lattice $\mathbb{F}u(A, L) := \big(Fu(A, L);\ \wedge,\ \vee;\ \underline{0},\ \underline{1}\big)$ and a residuated lattice $\mathcal{F}u(A, L) := \big(Fu(A, L);\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ \underline{0},\ \underline{1}\big)$; where, the binary operations $\wedge$, $\vee$, $\ominus$, $\twoheadrightarrow$, $\multimap$ are defined componentwise. Since the class of residuated lattices is a variety, $\mathcal{L}$ and $\mathcal{F}u(A, L)$ satisfy the same residuated lattice identities.

Remark 1.3.2. • *The map $\phi : L \to Fu(A, L)$, given by $\phi(r) = \underline{r}$ for all $r \in L$, is a complete residuated lattice embedding of $\mathcal{L}$ into $\mathcal{F}u(A, L)$.*
• *The map $\psi : P(A) \to Fu(A, L)$, given by $\psi(B) = B_1$ for all $B \in P(A)$, is a complete residuated lattice embedding of the Boolean algebra $\mathcal{P}(A)$ into $\mathcal{F}u(A, L)$.*

1.3.2 L-fuzzy subalgebras of an algebra

In the rest of this section, unless otherwise specified, $\mathcal{A} := (A;\ F^A)$ is an algebra of type $\mathcal{F}$.

Let f be an n-ary operation on A. The n-ary operation f^+ on $P(A)$ is defined by: for any $B_1, \ldots, B_n \in P(A)$,

$$f^+(B_1, \ldots, B_n) := \{f(x_1, \ldots, x_n) : \ x_1 \in B_1, \ldots, x_n \in B_n\}.$$

By the Zadeh's extension principle [41], f induces on $Fu(A, L)$ an n-ary operation $\widehat{f}$ defined by: for any $\mu_1, \ldots, \mu_n \in Fu(A, L)$,

$$\widehat{f}\big(\mu_1, \ldots, \mu_n\big)(y) := \bigvee\{\bigwedge_{1 \leq i \leq n} \mu_i(x_i) : \ (x_1, \ldots, x_n) \in f^{-1}(y)\} \text{ for all } y \in A.$$

For any $f \in F_0$, we have $(f^A)^+ = \{f^A\}$ and $\widehat{f^A} = \chi_{f^A}$.

Lemma 1.3.3. *Let $\{B_i\}_{1 \leq i \leq n} \subseteq P(A)$, $\{r_i\}_{1 \leq i \leq n} \subseteq L$ and f be an n-ary operation on A. Then $\widehat{f}\big((B_1)_{r_1}, \ldots, (B_n)_{r_n}\big) = \big(f^+(B_1, \ldots, B_n)\big)_{\bigwedge_{1 \leq i \leq n} r_i}$.*

Proof. If there is $1 \leq i_0 \leq n$ such that $B_{i_0} = \emptyset$, then $(B_{i_0})_{r_{i_0}} = \underline{0}$ and $f^+(B_1, \dots, B_n) = \emptyset$; thus, $\widehat{f}\big((B_1)_{r_1}, \dots, (B_n)_{r_n}\big) = \underline{0} = \big(f^+(B_1, \dots, B_n)\big)_{\bigwedge\limits_{1 \leq i \leq n} r_i}$. Now, suppose that $B_i \neq \emptyset$ for all $1 \leq i \leq n$. For any $y \in f^+(B_1, \dots, B_n)$, there are $a_1 \in B_1, \dots, a_n \in B_n$ such that $(a_1, \dots, a_n) \in f^{-1}(y)$; thus,

$$\bigwedge_{1 \leq i \leq n} r_i \geq \widehat{f}\big((B_1)_{r_1}, \dots, (B_n)_{r_n}\big)(y) \geq \bigwedge_{1 \leq i \leq n} (B_i)_{r_i}(a_i) = \bigwedge_{1 \leq i \leq n} r_i$$

and, $\widehat{f}\big((B_1)_{r_1}, \dots, (B_n)_{r_n}\big)(y) = \bigwedge\limits_{1 \leq i \leq n} r_i$. Now, let $y \notin f^+(B_1, \dots, B_n)$. For any $(x_1, \dots, x_n) \in f^{-1}(y)$, there is $1 \leq i_0 \leq n$ such that $x_{i_0} \notin B_{i_0}$; thus, $(B_{i_0})_{r_{i_0}}(x_{i_0}) = 0$ and, $\bigwedge\limits_{1 \leq i \leq n} (B_i)_{r_i}(x_i) = 0$. So,

$$\widehat{f}\big((B_1)_{r_1}, \dots, (B_n)_{r_n}\big)(y) = \bigvee\{0\} = 0.$$

Hence, $\widehat{f}\big((B_1)_{r_1}, \dots, (B_n)_{r_n}\big) = \big(f^+(B_1, \dots, B_n)\big)_{\bigwedge\limits_{1 \leq i \leq n} r_i}$. □

Definition 1.3.4. *The universal algebras* $\mathcal{A}^+ := \big(P(A);\ \{(f^A)^+ :\ f \in F\}\big)$ *and* $\widehat{\mathcal{A}} := \big(Fu(A, L);\ \{\widehat{f^A} :\ f \in F\}\big)$ *are respectively called the power (complex) algebra and the L-fuzzy algebra induced by* $\mathcal{A}$*.*

Proposition 1.3.5. *The function* $\phi : P(A) \to Fu(A, L)$*, given by* $\phi(B) = B_1$ *for all* $B \in P(A)$*, is an embedding of* $\mathcal{A}^+$ *into* $\widehat{\mathcal{A}}$*.*

Proof. For any $f \in F_0$, we have $\phi\big((f^A)^+\big) = \phi(\{f^A\}) = \{f^A\}_1 = \chi_{f^A} = \widehat{f^A}$. For any $f \in F_n$ and $X_1, \dots, X_n \in P(A)$, from Lemma 1.3.3, we have

$$\phi\big((f^A)^+(X_1, \dots, X_n)\big) = \big((f^A)^+(X_1, \dots, X_n)\big)_{\bigwedge\limits_{1 \leq i \leq n} 1} = \widehat{f^A}\big(\phi(X_1), \dots, \phi(X_n)\big).$$

Since ϕ is clearly one-to-one, it is an embedding of $\mathcal{A}^+$ into $\widehat{\mathcal{A}}$. □

Definition 1.3.6. *An L-fuzzy subset* μ *of* A *is called an L-fuzzy subalgebra of* $\mathcal{A}$ *if it satisfies the following conditions:*
(FS1) *For any* $f \in F_0$*,* $\mu(f^A) = 1$*.*
(FS2) *For any* $f \in F_n$ *and* $a_1, \dots, a_n \in A$*,* $\mu\big(f^A(a_1, \dots, a_n)\big) \geq \bigwedge\limits_{1 \leq i \leq n} \mu(a_i)$*.*

Proposition 1.3.7. *Let* μ *be an L-fuzzy subset of* A*. Then* μ *is an L-fuzzy subalgebra of* $\mathcal{A}$ *if and only if the following conditions are satisfied:*
(1) *For any* $f \in F_0$*,* $\widehat{f^A} \leq \mu$*.*
(2) *For any* $f \in F_n$*,* $\widehat{f^A}(\mu_1, \dots, \mu_n) \leq \mu$*; where,* $\mu_1 = \dots = \mu_n = \mu$*.*

Proof. ($\Rightarrow$) Assume that μ is an L-fuzzy subalgebra of $\mathcal{A}$. For any $f \in F_0$, we have $\widehat{f^A}(f^A) = 1 = \mu(f^A)$ and $\widehat{f^A}(y) = 0 \leq \mu(y)$ for all $y \neq f^A$ in A; thus, $\widehat{f^A} \leq \mu$. Now, let $f \in F_n$. For any $y \in A$ such that $(f^A)^{-1}(y) \neq \emptyset$, we have $\bigwedge\limits_{1 \leq i \leq n} \mu(a_i) \leq \mu\big(f^A(a_1, \dots, a_n)\big) = \mu(y)$ for all $(a_1, \dots, a_n) \in (f^A)^{-1}(y)$; thus,

$\widehat{f^A}(\mu,\ldots,\mu)(y) \leq \mu(y)$. For any $y \in A$ such that $(f^A)^{-1}(y) = \emptyset$, we have $\widehat{f^A}(\mu,\ldots,\mu)(y) = \bigvee \emptyset = 0 \leq \mu(y)$. Hence, $\widehat{f^A}(\mu,\ldots,\mu) \leq \mu$.
($\Leftarrow$) Assume conditions **(1)** and **(2)** are satisfied. For any $f \in F_0$, we have $\mu(f^A) \geq \widehat{f^A}(f^A) = 1$ and, $\mu(f^A) = 1$. For any $f \in F_n$ and $a_1,\ldots,a_n \in A$, we have $(a_1,\ldots,a_n) \in (f^A)^{-1}\big(f^A(a_1,\ldots,a_n)\big)$; thus,

$$\mu\big(f^A(a_1,\ldots,a_n)\big) \geq \widehat{f^A}(\mu,\ldots,\mu)\big(f^A(a_1,\ldots,a_n)\big) \geq \bigwedge_{1\leq i\leq n} \mu(a_i).$$

Hence, μ is an L-fuzzy subalgebra of $\mathcal{A}$. □

Proposition 1.3.8. *Let μ be an L-fuzzy subset of A.*
(1) If μ is an L-fuzzy subalgebra of $\mathcal{A}$, then all its cuts are empty or subuniverses of $\mathcal{A}$.
(2) If $U(\mu,1) \neq \emptyset$, then μ is an L-fuzzy subalgebra of $\mathcal{A}$ if and only if all its cuts are subuniverses.

Proof. **(1)** Assume that μ is an L-fuzzy subalgebra of $\mathcal{A}$. Let $r \in L$ such that $U(\mu,r) \neq \emptyset$. For any $f \in F_0$, we have $\mu(f^A) = 1 \geq r$ and, $f^A \in U(\mu,r)$. For any $f \in F_n$ and $a_1,\ldots,a_n \in U(\mu,r)$, we have $\mu\big(f^A(a_1,\ldots,a_n)\big) \geq \bigwedge_{1\leq i\leq n} \mu(a_i) \geq \bigwedge_{1\leq i\leq n} r = r$ and, $f^A(a_1,\ldots,a_n) \in U(\mu,r)$. Hence, $U(\mu,r)$ is a subuniverse of $\mathcal{A}$.
(2) Assume that $U(\mu,1) \neq \emptyset$. By **(1)**, it suffices to show the second implication. So, assume that cuts of μ are subuniverses of $\mathcal{A}$. For any $f \in F_0$, we have $f^A \in U(\mu,1)$ and, $\mu(f^A) = 1$. For any $f \in F_n$ and $a_1,\ldots,a_n \in A$, we have $a_1,\ldots,a_n \in U\big(\mu, \bigwedge_{1\leq i\leq n} \mu(a_i)\big)$; thus, $f^A(a_1,\ldots,a_n) \in U\big(\mu, \bigwedge_{1\leq i\leq n} \mu(a_i)\big)$ and, $\mu\big(f^A(a_1,\ldots,a_n)\big) \geq \bigwedge_{1\leq i\leq n} \mu(a_i)$. Hence, μ is an L-fuzzy subalgebra of $\mathcal{A}$. □

Lemma 1.3.9. *Let μ be an L-fuzzy subalgebra of $\mathcal{A}$. For any n-ary term $t(x_1,\ldots,x_n)$ of type $\mathcal{F}$, $\mu\big(t^A(a_1,\ldots,a_n)\big) \geq \bigwedge_{1\leq i\leq n} \mu(a_i)$ for all $a_1,\ldots,a_n \in A$.*

Proof. We use induction on the n-ary term $t(x_1,\ldots,x_n)$ of type $\mathcal{F}$.
If $t(x_1,\ldots,x_n)$ is a nullary function symbol, then for any $a_1,\ldots,a_n \in A$, $t^A(a_1,\ldots,a_n)$ is a nullary fundamental operation; thus, $\mu\big(t^A(a_1,\ldots,a_n)\big) = 1 \geq \bigwedge_{1\leq i\leq n} \mu(a_i)$.
If $t(x_1,\ldots,x_n)$ is a variable x_j $(1 \leq j \leq n)$, then $\mu\big(t^A(a_1,\ldots,a_n)\big) = \mu(a_j) \geq \bigwedge_{1\leq i\leq n} \mu(a_i)$ for all $a_1,\ldots,a_n \in A$.
Now, suppose that $t(x_1,\ldots,x_n) = f\big(t_1(x_1,\ldots,x_n),\ldots,t_m(x_1,\ldots,x_n)\big)$, where $f \in F_m$ and for any $1 \leq j \leq m$, $\mu\big(t_j^A(b_1,\ldots,b_n)\big) \geq \bigwedge_{1\leq i\leq n} \mu(b_i)$ for all $b_1,\ldots,b_n \in A$. For any $a_1,\ldots,a_n \in A$, we have

$$\begin{aligned}\mu\big(t^A(a_1,\dots,a_n)\big) &= \mu\big[f^A\big(t_1^A(a_1,\dots,a_n),\dots,t_m^A(a_1,\dots,a_n)\big)\big]\\ &\geq \bigwedge_{1\leq j\leq m}\mu\big(t_j^A(a_1,\dots,a_n)\big)\\ &\geq \bigwedge_{1\leq j\leq m}\bigwedge_{1\leq i\leq n}\mu(a_i)\\ &= \bigwedge_{1\leq i\leq n}\mu(a_i).\end{aligned}$$

Hence, the desired result follows. □

Proposition 1.3.10. *Let μ be an L-fuzzy subalgebra of $\mathcal{A}$.*
(1) For any $a \in Sg(\emptyset)$, we have $\mu(a) = 1$.
(2) For any $a, b \in A$ such that $Sg(a) = Sg(b)$, we have $\mu(a) = \mu(b)$.

Proof. **(1)** For any $a \in Sg(\emptyset) \setminus F_0^A$, we have $a = t^A(f^A,\dots,f^A)$ for some term $t(x_1,\dots,x_n)$ and $f \in F_0$; thus, $\mu(a) \geq \bigwedge_{1\leq i\leq n}\mu(f^A) = \mu(f^A) = 1$ and, $\mu(a) = 1$.
(2) Let $a, b \in A$ such that $Sg(a) = Sg(b)$. If $a \in Sg(\emptyset)$, then $b \in Sg(\emptyset)$ and, $\mu(a) = 1 = \mu(b)$. Now, suppose that $a \notin Sg(\emptyset)$. Since $a \in Sg(b)$, we have $a = t^A(b,\dots,b)$ for some term $t(x_1,\dots,x_n)$; thus, $\mu(a) \geq \bigwedge_{1\leq i\leq n}\mu(b) = \mu(b)$. A similar reasoning shows that $\mu(b) \geq \mu(a)$. Hence, $\mu(a) = \mu(b)$. □

For any $\mu \in Fu(A, L)$, μ_* is the L-fuzzy subset of A given by

$$\mu_* := \mu \vee \chi_{Sg(\emptyset)};$$

furthermore, $\mu_* = \mu$ if and only if $Sg(\emptyset) \subseteq U(\mu, 1)$.

Proposition 1.3.11. *Let B be a subuniverse of $\mathcal{A}$ and $r, s \in L$ such that $r \leq s$. Then $(B_r^s)_*$ is an L-fuzzy subalgebra of $\mathcal{A}$.*

Proof. For any $f \in F_0$, we have

$$(B_r^s)_*(f^A) \geq \chi_{Sg(\emptyset)}(f^A) = 1 \text{ and, } (B_r^s)_*(f^A) = 1.$$

Now, let $f \in F_n$ and $a_1,\dots,a_n \in A$.

- If $f^A(a_1,\dots,a_n) \in Sg(\emptyset)$, then $(B_r^s)_*\big(f^A(a_1,\dots,a_n)\big) = 1$, since

$$(B_r^s)_*\big(f^A(a_1,\dots,a_n)\big) \geq \chi_{Sg(\emptyset)}\big(f^A(a_1,\dots,a_n)\big) = 1.$$

- If $f^A(a_1,\dots,a_n) \in B \setminus Sg(\emptyset)$, then $a_{i_0} \notin Sg(\emptyset)$ for some $1 \leq i_0 \leq n$; thus,

$$(B_r^s)_*\big(f^A(a_1,\dots,a_n)\big) = s \geq (B_r^s)_*(a_{i_0}) \geq \bigwedge_{1\leq i\leq n}(B_r^s)_*(a_i).$$

- If $f^A(a_1,\dots,a_n) \notin B$, then $a_{i_0} \notin B$ for some $1 \leq i_0 \leq n$; thus,

$$(B_r^s)_*\big(f^A(a_1,\dots,a_n)\big) = r = (B_r^s)_*(a_{i_0}) \geq \bigwedge_{1\leq i\leq n}(B_r^s)_*(a_i).$$

Hence, $(B_r^s)_*$ is an L-fuzzy subalgebra of $\mathcal{A}$. □

Chapter Two

Residuated lattice of L-fuzzy subalgebras of an algebra

In this chapter, unless otherwise specified, $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a complete residuated lattice and $\mathcal{A} := (A;\ F^A)$ is an algebra of type $\mathcal{F}$. In Section 2.1, given a complete meet-distributive residuated lattice $\mathcal{L}$ and an algebra $\mathcal{A}$, we set up a mimetic construction of the L-fuzzy subalgebra of $\mathcal{A}$ generated by an L-fuzzy subset of A. We also characterize atoms, co-atoms (when $\mathcal{L}$ is distributive) and compact elements of the lattice $\mathbb{F}s(\mathcal{A}, L)$, and show that the latter is algebraic (when $\mathcal{L}$ is algebraic). When $\mathcal{L}$ is a finite linearly ordered Brouwerian algebra and $\mathbb{S}ub(\mathcal{A})$ supports a quantale structure $\mathcal{S}ub(\mathcal{A})$, we show that $\mathbb{F}s(\mathcal{A}, L)$ supports an algebraic quantale structure which is both an extension of $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$. In Section 2.3, given a mono-unary algebra $\mathcal{A}$, we define a residuated lattice structure $\mathcal{F}s(\mathcal{A}, L)$ on the set of L-fuzzy subalgebras of $\mathcal{A}$ which is both an extension of $\mathcal{L}$ and the Heyting algebra $\mathcal{S}ub(\mathcal{A})$ on the set of subuniverses of $\mathcal{A}$. Furthermore, we show that $\mathcal{F}s(\mathcal{A}, L)$ is an MV-algebra (resp., a Boolean algebra) if and only if $\mathcal{L}$ is an MV-algebra (resp., a Boolean algebra) and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra.

2.1 Lattice of L-fuzzy subalgebras of an algebra

2.1.1 Lattice of L-fuzzy subalgebras

Proposition 2.1.1. *The set $Fs(\mathcal{A}, L)$ of L-fuzzy subalgebras of $\mathcal{A}$ is closed under the infimum of $\mathcal{F}u(A, L)$.*

Proof. Let $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fs(\mathcal{A}, L)$. For any $f \in F_0$, we have

$$\big(\bigwedge_{\lambda\in\Lambda} \mu_\lambda\big)(f^A) = \bigwedge_{\lambda\in\Lambda} \mu_\lambda(f^A) = \bigwedge_{\lambda\in\Lambda} 1 = 1.$$

For any $f \in F_n$ and $a_1, \ldots, a_n \in A$, we have

$$\begin{aligned}\big(\bigwedge_{\lambda\in\Lambda}\mu_\lambda\big)\big(f^A(a_1,\dots,a_n)\big) &= \bigwedge_{\lambda\in\Lambda}\mu_\lambda\big(f^A(a_1,\dots,a_n)\big)\\ &\geq \bigwedge_{\lambda\in\Lambda}\ \bigwedge_{1\leq i\leq n}\mu_\lambda(a_i)\\ &\geq \bigwedge_{\lambda\in\Lambda}\ \bigwedge_{1\leq i\leq n}\big(\bigwedge_{\lambda\in\Lambda}\mu_\lambda(a_i)\big)\\ &= \bigwedge_{1\leq i\leq n}\big(\bigwedge_{\lambda\in\Lambda}\mu_\lambda(a_i)\big)\\ &= \bigwedge_{1\leq i\leq n}\big(\bigwedge_{\lambda\in\Lambda}\mu_\lambda\big)(a_i).\end{aligned}$$

Hence, $\bigwedge_{\lambda\in\Lambda}\mu_\lambda$ is an L-fuzzy subalgebra of $\mathcal{A}$. □

For any L-fuzzy subset μ of A, the L-fuzzy subset of A given by $\bigwedge\{\nu\in Fs(\mathcal{A},L):\ \mu\leq\nu\}$, and denoted by $Fsg(\mu)$, is according to Proposition 2.1.1 the smallest L-fuzzy subalgebra of $\mathcal{A}$ containing μ. $Fs(\mathcal{A},L)$ forms the complete lattice $\mathbb{F}s(\mathcal{A},L):=\big(Fs(\mathcal{A},L);\ \wedge,\ \sqcup;\ \chi_{Sg(\emptyset)},\ \underline{1}\big)$, where the binary operation $\sqcup$ is defined by: $\mu\sqcup\nu=Fsg(\mu\vee\nu)$ for all $\mu,\nu\in Fs(\mathcal{A},L)$. Furthermore, for any L-fuzzy subsets μ and ν of A, we have $\mu\leq Fsg(\mu)$, $Fsg\big(Fsg(\mu)\big)=Fsg(\mu)$, and $Fsg(\mu)\leq Fsg(\nu)$ whenever $\mu\leq\nu$.

Proposition 2.1.2. *Let B be a subset of A and $r,s\in L$ such that $r\leq s$. Then $Fsg(B_r^s)=\big(Sg(B)_r^s\big)_*$.*

Proof. By Proposition 1.3.11, $\big(Sg(B)_r^s\big)_*$ is an L-fuzzy subalgebra of $\mathcal{A}$ containing B_r^s. Finally, let μ be an L-fuzzy subalgebra of $\mathcal{A}$ containing B_r^s.
- For any $a\in Sg(\emptyset)$, we have $\big(Sg(B)_r^s\big)_*(a)=s\vee 1=1=\mu(a)$.
- For any $a\in Sg(B)\setminus Sg(\emptyset)$, there are a term $t(x_1,\dots,x_n)$ and $a_1,\dots,a_n\in B$ such that $a=t^A(a_1,\dots,a_n)$; thus,

$$\big(Sg(B)_r^s\big)_*(a)=s=\bigwedge_{1\leq i\leq n}B_r^s(a_i)\leq\bigwedge_{1\leq i\leq n}\mu(a_i)\leq\mu\big(t^A(a_1,\dots,a_n)\big)=\mu(a).$$

- For any $a\notin Sg(B)$, we have $a\notin B$; thus,

$$\big(Sg(B)_r^s\big)_*(a)=r=B_r^s(a)\leq\mu(a).$$

So, $\big(Sg(B)_r^s\big)_*\leq\mu$. Hence, $Fsg(B_r^s)=\big(Sg(B)_r^s\big)_*$. □

Proposition 2.1.3. *The map $\phi: Sub(\mathcal{A})\to Fs(\mathcal{A},L)$, given by $\phi(B)=B_1$ for all $B\in Sub(\mathcal{A})$, is a complete lattice embedding (See, Remark 1.1.5) of $\mathbb{S}ub(\mathcal{A})$ into $\mathbb{F}s(\mathcal{A},L)$.*

Proof. Let $\{B_\lambda\}_{\lambda\in\Lambda}\subseteq Sub(\mathcal{A})$.
We have $\bigvee_{\lambda\in\Lambda}(B_\lambda)_1\leq\bigvee_{\lambda\in\Lambda}(\bigsqcup_{\lambda\in\Lambda}B_\lambda)_1=(\bigsqcup_{\lambda\in\Lambda}B_\lambda)_1$. Now, let μ be an L-fuzzy subalgebra of $\mathcal{A}$ containing $\bigvee_{\lambda\in\Lambda}(B_\lambda)_1$. For any $a\in\bigsqcup_{\lambda\in\Lambda}B_\lambda$, there are a term $t(x_1,\dots,x_n)$ of type $\mathcal{F}$, $\lambda_1,\dots,\lambda_n\in\Lambda$ and $a_{\lambda_1}\in B_{\lambda_1},\dots,a_{\lambda_n}\in B_{\lambda_n}$ such that $a=t^A(a_{\lambda_1},\dots,a_{\lambda_n})$; thus,

$$\mu(a) \geq \bigwedge_{1 \leq i \leq n} \mu(a_{\lambda_i}) \geq \bigwedge_{1 \leq i \leq n} (B_{\lambda_i})_1(a_{\lambda_i}) = \bigwedge_{1 \leq i \leq n} 1 = 1$$

and, $\mu(a) = 1$. So, $(\bigsqcup_{\lambda \in \Lambda} B_\lambda)_1 \leq \mu$. Hence,

$$\phi\big(\bigsqcup_{\lambda \in \Lambda} B_\lambda\big) = \bigsqcup_{\lambda \in \Lambda} (B_\lambda)_1 = (\bigsqcup_{\lambda \in \Lambda} B_\lambda)_1 = \bigsqcup_{\lambda \in \Lambda} \phi(B_\lambda).$$

Furthermore,

$$\phi\big(\bigcap_{\lambda \in \Lambda} B_\lambda\big) = (\bigcap_{\lambda \in \Lambda} B_\lambda)_1 = \bigwedge_{\lambda \in \Lambda} (B_\lambda)_1 = \bigwedge_{\lambda \in \Lambda} \phi(B_\lambda).$$

Since ϕ is clearly one-to-one, the result follows from the above. □

Proposition 2.1.4. *Suppose that $\mathcal{A}$ is not $\mathcal{F}$-trivial. The map $\psi : L \to Fs(\mathcal{A}, L)$, given by $\psi(r) = (\underline{r})_*$ for all $r \in L$, is a complete lattice embedding (See, Remark 1.1.5) of the lattice of $\mathcal{L}$ into $\mathbb{F}s(\mathcal{A}, L)$.*

Proof. Let $\{r_\lambda\}_{\lambda \in \Lambda} \subseteq L$.

- For any $a \in Sg(\emptyset)$, we have

$$\big(\bigwedge_{\lambda \in \Lambda} (\underline{r_\lambda})_*\big)(a) = \bigwedge_{\lambda \in \Lambda} (\underline{r_\lambda})_*(a) = \bigwedge_{\lambda \in \Lambda} 1 = 1 = \big(\underline{\bigwedge_{\lambda \in \Lambda} r_\lambda}\big)_*(a).$$

- For any $a \notin Sg(\emptyset)$, we have

$$\big(\bigwedge_{\lambda \in \Lambda} (\underline{r_\lambda})_*\big)(a) = \bigwedge_{\lambda \in \Lambda} (\underline{r_\lambda})_*(a) = \bigwedge_{\lambda \in \Lambda} r_\lambda = \big(\underline{\bigwedge_{\lambda \in \Lambda} r_\lambda}\big)_*(a).$$

Thus, $\psi\big(\bigwedge_{\lambda \in \Lambda} r_\lambda\big) = \big(\underline{\bigwedge_{\lambda \in \Lambda} r_\lambda}\big)_* = \bigwedge_{\lambda \in \Lambda} (\underline{r_\lambda})_* = \bigwedge_{\lambda \in \Lambda} \psi(r_\lambda)$; and, $\psi\big(\bigvee_{\lambda \in \Lambda} r_\lambda\big) = \bigvee_{\lambda \in \Lambda} \psi(r_\lambda)$ by similar arguments. Since $\mathcal{A}$ is not $\mathcal{F}$-trivial, ψ is one-to-one. Hence, ψ is a complete lattice embedding. □

Note: If $\mathcal{A}$ is $\mathcal{F}$-trivial, then $Fs(\mathcal{A}, L) = \{\underline{1}\}$ and ψ is a constant map with value $\underline{1}$.

Theorem 2.1.5. *Suppose that $F_0 = \emptyset$. Then $\mathbb{F}s(\mathcal{A}, L)$ is linearly ordered if and only if one of the following conditions is satisfied:*
(1) *$\mathcal{L}$ is linearly ordered and $\mathbb{S}ub(\mathcal{A})$ is trivial.*
(2) *$\mathbb{S}ub(\mathcal{A})$ is linearly ordered and $\mathcal{L}$ is trivial.*

Proof. ($\Rightarrow$) Assume that $\mathbb{F}s(\mathcal{A}, L)$ is linearly ordered. Suppose condition **(1)** does not hold. $\mathbb{S}ub(\mathcal{A})$ is linearly ordered by Proposition 2.1.3. Since $\mathcal{L}$ is linearly ordered by the hypothesis, $\mathbb{S}ub(\mathcal{A})$ is nontrivial by the fact that condition (**1**) does not hold; thus, there is $B \in Sub(\mathcal{A})$ such that $B \neq \emptyset$ and $B \neq A$. For any $r \in L$, we have $\underline{r} \leq B_1$ or $B_1 \leq \underline{r}$; thus, $r = \underline{r}(a) \leq B_1(a) = 0$ for some $a \notin B$ or $1 = B_1(b) \leq \underline{r}(b) = r$ for some $b \in B$; so, $r = 0$ or $r = 1$; that is $r \in \{0,\ 1\}$. Hence, $L = \{0,\ 1\}$ and, $\mathcal{L}$ is trivial. Therefore, one of conditions **(1)** and **(2)** is satisfied.

($\Leftarrow$) Suppose condition **(1)** holds. Let $\mu \in Fs(\mathcal{A}, L)$. For any $x, y \in A$ such

that $\mu(x) \leq \mu(y)$, we have $x \in A = U(\mu, \mu(y))$ and, $\mu(x) = \mu(y)$. Thus, $\mu = \underline{r}$ for some $r \in L$. Hence, $Fs(\mathcal{A}, L) = \{\underline{r} : \ r \in L\}$. Therefore, $\mathbb{F}s(\mathcal{A}, L)$ is linearly ordered, since $\mathcal{L}$ is linearly ordered.
Now, suppose condition **(2)** holds. Let $\mu \in Fs(\mathcal{A}, L)$. For any $x \notin U(\mu, 1)$, we have $\mu(x) \neq 1$ and, $\mu(x) = 0$. Thus, $\mu = B_1$ for some $B \in Sub(\mathcal{A})$. So, $Fs(\mathcal{A}, L) = \{B_1 : \ B \in Sub(\mathcal{A})\}$. Hence, $\mathbb{F}s(\mathcal{A}, L)$ is linearly ordered, since $\mathbb{S}ub(\mathcal{A})$ is linearly ordered. □

2.1.2 Atoms, co-atoms and compact elements

Theorem 2.1.6. *Atoms of $\mathbb{F}s(\mathcal{A}, L)$ are exactly of the form $\big(Sg(a)_r\big)_*$, where r and $Sg(a)$ are atoms of $\mathcal{L}$ and $\mathbb{S}ub(\mathcal{A})$, respectively.*

Proof. ($\Rightarrow$) Let μ be an atom of $\mathbb{F}s(\mathcal{A}, L)$. Since there is $a \in A \setminus Sg(\emptyset)$ such that $\mu(a) \neq 0$, we have $\chi_{Sg(\emptyset)} < \big(Sg(a)_{\mu(a)}\big)_* \leq \mu$ and, $\mu = \big(Sg(a)_{\mu(a)}\big)_*$. Since $\mu \neq \chi_{Sg(\emptyset)}$, we have $\mu(a) \neq 0$ and $Sg(a) \neq Sg(\emptyset)$.
• For any $r \in L$ such that $0 < r \leq \mu(a)$, we have $\chi_{Sg(\emptyset)} < \big(Sg(a)_r\big)_* \leq \big(Sg(a)_{\mu(a)}\big)_*$; thus, $\big(Sg(a)_r\big)_* = \big(Sg(a)_{\mu(a)}\big)_*$ and, $r = \mu(a)$. Hence, $\mu(a)$ is an atom of $\mathcal{L}$.
• For any $B \in Sub(\mathcal{A})$ such that $Sg(\emptyset) \subset B \subseteq Sg(a)$, we have $\chi_{Sg(\emptyset)} < \big(B_{\mu(a)}\big)_* \leq \big(Sg(a)_{\mu(a)}\big)_*$; thus, $\big(B_{\mu(a)}\big)_* = \big(Sg(a)_{\mu(a)}\big)_*$ and, $B = Sg(a)$. Hence, $Sg(a)$ is an atom of $\mathbb{S}ub(\mathcal{A})$.
($\Leftarrow$) Let s be an atom of $\mathcal{L}$ and $a \in A$ such that $Sg(a)$ is an atom of $\mathbb{S}ub(\mathcal{A})$. We have $\big(Sg(a)_s\big)_* \neq \chi_{Sg(\emptyset)}$, since $s \neq 0$ and $Sg(a) \neq Sg(\emptyset)$. Now, let $\mu \in Fs(\mathcal{A}, L)$ such that $\chi_{Sg(\emptyset)} < \mu \leq \big(Sg(a)_s\big)_*$. Since there is $b \in Sg(a) \setminus Sg(\emptyset)$ such that $0 < \mu(b) \leq s$, we have $Sg(a) = Sg(b)$ and, $\mu(a) = \mu(b) = s$.
• For any $x \in Sg(a) \setminus Sg(\emptyset)$, we have $\mu(x) \geq \mu(a) = s = \big(Sg(a)_s\big)_*(x) \geq \mu(x)$ and, $\mu(x) = s$.
• For any $x \notin Sg(a)$, we have $0 \leq \mu(x) \leq \big(Sg(a)_s\big)_*(x) = 0$ and, $\mu(x) = 0$.
It follows that $\mu = \big(Sg(a)_s\big)_*$. Hence, $\big(Sg(a)_s\big)_*$ is an atom of $\mathbb{F}s(\mathcal{A}, L)$. □

Lemma 2.1.7. *Suppose that $\mathcal{L}$ is distributive and let $\mu \in Fs(\mathcal{A}, L)$ and $r \in L$. Then $\underline{r} \vee \mu$ is an L-fuzzy subalgebra of $\mathcal{A}$.*

Proof. For any $f \in F_0$, we have

$$\big(\underline{r} \vee \mu\big)(f^A) = r \vee \mu(f^A) = r \vee 1 = 1.$$

For any $f \in F_n$ and $a_1, \ldots, a_n \in A$, we have

$$\begin{aligned}(\underline{r}\vee\mu)\big(f^A(a_1,\ldots,a_n)\big) &= r\vee\mu\big(f^A(a_1,\ldots,a_n)\big)\\ &\geq r\vee\big(\bigwedge_{1\leq i\leq n}\mu(a_i)\big)\\ &= \bigwedge_{1\leq i\leq n}\big(r\vee\mu(a_i)\big)\\ &= \bigwedge_{1\leq i\leq n}\big(\underline{r}\vee\mu\big)(a_i).\end{aligned}$$

Hence, $\underline{r}\vee\mu$ is an L-fuzzy subalgebra of $\mathcal{A}$. □

Theorem 2.1.8. *Suppose that $\mathcal{L}$ is distributive and $\mathcal{A}$ is not $\mathcal{F}$-trivial. Then co-atoms of $\mathbb{F}s(\mathcal{A},L)$ are exactly of the form B^s, where s and B are co-atoms of $\mathcal{L}$ and $\mathbb{S}ub(\mathcal{A})$, respectively.*

Proof. ($\Rightarrow$) Let μ be a co-atom of $\mathbb{F}s(\mathcal{A},L)$. For any $a,b\notin U(\mu,1)$, we have $\mu\leq\underline{\mu(a)}\vee\mu<\underline{1}$ and $\mu\leq\underline{\mu(b)}\vee\mu<\underline{1}$; thus, $\underline{\mu(a)}\vee\mu=\mu=\underline{\mu(b)}\vee\mu$ and, $\mu(a)=\mu(b)$. It follows that $\mu=\big(U(\mu,1)\big)^s$ for some $s\in L$.
Since $\mu\neq\underline{1}$, we have $s\neq 1$ and $U(\mu,1)\neq A$.
• For any $r\in L$ such that $s<r\leq 1$, we have $\mu<\underline{r}\vee\mu\leq\underline{1}$ and, $\underline{r}\vee\mu=\underline{1}$; thus, $r=r\vee s=1$. Hence, s is a co-atom of $\mathcal{L}$.
• For any $D\in Sub(\mathcal{A})$ such that $U(\mu,1)\subset D\subseteq A$, we have $\mu<D^s\leq\underline{1}$ and, $D^s=\underline{1}$; thus, $D=A$. Hence, $U(\mu,1)$ is a co-atom of $\mathbb{S}ub(\mathcal{A})$.
($\Leftarrow$) Let s and B be co-atoms of $\mathcal{L}$ and $\mathbb{S}ub(\mathcal{A})$, respectively. We have $B^s\neq\underline{1}$, since $s\neq 1$ and $B\neq A$. For any $\mu\in Fs(\mathcal{A},L)$ such that $B^s<\mu\leq\underline{1}$, we have $B=U(B^s,1)\subseteq U(\mu,1)\subseteq A$ and $a\notin B$ such that $s<\mu(a)\leq 1$; thus, $B\subseteq U(\mu,1)\subseteq A$ and $a\in U(\mu,1)\setminus B$; so, $B\subset U(\mu,1)\subseteq A$ and, $U(\mu,1)=A$; *i.e.*, $\mu=\underline{1}$. Hence, B^s is a co-atom of $\mathbb{F}s(\mathcal{A},L)$. □

Theorem 2.1.9. *Suppose that $\mathcal{L}$ is meet-distributive, and let $\mu\in Fu(A,L)$ and $\mu_\star\in Fu(A,L)$ defined by: $\mu_\star(x)=\bigvee\{r\in L:\ x\in Sg\big(U(\mu,r)\big)\}$ for all $x\in A$. Then $\mu_\star$ is the smallest L-fuzzy subalgebra of $\mathcal{A}$ containing μ.*

Proof. For any $a\in A$, we have $a\in U\big(\mu,\mu(a)\big)\subseteq Sg\big(U\big(\mu,\mu(a)\big)\big)$ and, $\mu(a)\leq\mu_\star(a)$. Thus, $\mu\leq\mu_\star$. We next show that $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.
For any $f\in F_0$, we have $\mu_\star(f^A)=\bigvee L=1$. Now, let $f\in F_n$ and $a_1,\ldots,a_n\in A$. For any $r_1,\ldots,r_n\in L$ such that $a_1\in Sg\big(U(\mu,r_1)\big),\ldots,a_n\in Sg\big(U(\mu,r_n)\big)$, we have $a_1,\ldots,a_n\in Sg\big(U(\mu,\bigwedge_{1\leq i\leq n}r_i)\big)$ and, $f^A(a_1,\ldots,a_n)\in Sg\big(U(\mu,\bigwedge_{1\leq i\leq n}r_i)\big)$; thus, $\mu_\star\big(f^A(a_1,\ldots,a_n)\big)\geq\bigwedge_{1\leq i\leq n}r_i$. So, $\mu_\star\big(f^A(a_1,\ldots,a_n)\big)\geq\bigwedge_{1\leq i\leq n}\mu_\star(a_i)$.
Hence, $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.
Finally, let ν be an L-fuzzy subalgebra of $\mathcal{A}$ containing μ. Let $u\in A\setminus Sg(\emptyset)$. For any $r\in L$ such that $u\in Sg\big(U(\nu,r)\big)$, there are a term $t(x_1,\ldots,x_n)$ of type $\mathcal{F}$ and $u_1,\ldots,u_n\in U(\nu,r)$ such that $u=t^A(u_1,\ldots,u_n)$; thus,

$$r \leq \bigwedge_{1\leq i\leq n} \nu(u_i) \leq \nu\big(t^A(u_1,\ldots,u_n)\big) = \nu(u).$$

So,

$$\mu_\star(u) \leq \bigvee\{r \in L: \ u \in Sg\big(U(\nu,r)\big)\} \leq \nu(u).$$

Hence, $\mu_\star \leq \nu$. Therefore, $\mu_\star$ is the smallest L-fuzzy subalgebra of $\mathcal{A}$ containing μ. □

The above result generalizes that obtained by M. Tonga in [38].

Proposition 2.1.10. *Suppose that $\mathcal{L}$ is meet-distibutive, and let $a \in A\setminus Sg(\emptyset)$ and $c \in L$. Then $Fsg\big(a_c\big)$ is a compact element of $\mathbb{F}s(\mathcal{A},L)$ if and only if c is a compact element of $\mathcal{L}$.*

Proof. ($\Rightarrow$) Assume that $Fsg\big(a_c\big)$ is a compact element of $\mathbb{F}s(\mathcal{A},L)$. Let $\{r_i\}_{i\in I} \subseteq L$ such that $c \leq \bigvee_{i\in I} r_i$. Since

$$Fsg(a_c) \leq Fsg(a_{\bigvee_{i\in I} r_i}) = Fsg(\bigvee_{i\in I} a_{r_i}) = \bigsqcup_{i\in I} Fsg(a_{r_i}),$$

there is $\{i_1,\ldots,i_p\} \subseteq I$ such that $Fsg(a_c) \leq \bigsqcup_{1\leq j\leq p} Fsg(a_{r_{i_j}}) = Fsg(a_{\bigvee_{1\leq j\leq p} r_{i_j}})$; thus, $c = Fsg(a_c)(a) \leq Fsg(a_{\bigvee_{1\leq j\leq p} r_{i_j}})(a) = \bigvee_{1\leq j\leq p} r_{i_j}$. Hence, c is a compact element of $\mathcal{L}$.
($\Leftarrow$) Assume that c is a compact element of $\mathcal{L}$. Let $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fs(\mathcal{A},L)$ such that $Fsg(a_c) \leq \bigsqcup_{\lambda\in\Lambda} \mu_\lambda$. Since $c \leq \big(\bigvee_{\lambda\in\Lambda} \mu_\lambda\big)_\star(a)$ and c is a compact element of $\mathcal{L}$, there are $r_1,\ldots,r_n \in L$ such that $a \in \bigcap_{1\leq i\leq n} Sg\big(U(\bigvee_{\lambda\in\Lambda} \mu_\lambda, r_i)\big)$ and $c \leq \bigvee_{1\leq i\leq n} r_i$. For any $1 \leq i \leq n$, there are a term $t_i(x_{i1},\ldots,x_{ik_i})$ of type $\mathcal{F}$ and $u_{i1},\ldots,u_{ik_i} \in A$ such that $a = t_i^A(u_{i1},\ldots,u_{ik_i})$ and $r_i \leq \bigvee_{\lambda\in\Lambda} \mu_\lambda(u_{ij})$ for all $1 \leq j \leq k_i$; thus, $r_i \leq \bigwedge_{1\leq j\leq k_i} \big(\bigvee_{\lambda\in\Lambda} \mu_\lambda(u_{ij})\big) = \bigvee_{(\lambda_{i1},\ldots,\lambda_{ik_i})\in\Lambda^{k_i}} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij})$. So,

$$c \leq \bigvee_{1\leq i\leq n} \bigvee_{(\lambda_{i1},\ldots,\lambda_{ik_i})\in\Lambda^{k_i}} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij});$$

i.e., $c \leq \bigvee_{\big((\lambda_{i1},\ldots,\lambda_{ik_i})\big)_{1\leq i\leq n}\in \prod_{1\leq i\leq n}\Lambda^{k_i}} \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij})$. Since c is a compact element of $\mathcal{L}$, there is a finite subset Ω of Λ such that

$$c \leq \bigvee_{\big((\lambda_{i1},\ldots,\lambda_{ik_i})\big)_{1\leq i\leq n}\in \prod_{1\leq i\leq n}\Omega^{k_i}} \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij});$$

thus,

$$\begin{aligned}
c &\leq \bigvee_{1\leq i\leq n} \bigvee_{(\lambda_{i1},\ldots,\lambda_{ik_i})\in\Omega^{k_i}} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij}) \\
&= \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)(u_{ij}) \\
&\leq \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star(u_{ij}) \\
&\leq \bigvee_{1\leq i\leq n} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star\big(t^A(u_{i1},\ldots,u_{ik_i})\big) \\
&= \bigvee_{1\leq i\leq n} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star(a) \\
&= \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star(a) \\
&= \big(\bigsqcup_{\lambda\in\Omega} \mu_\lambda\big)(a).
\end{aligned}$$

For any $u \in Sg(a) \setminus Sg(\emptyset)$, we have

$$Fsg(a_c)(u) = c \leq \big(\bigsqcup_{\lambda\in\Omega} \mu_\lambda\big)(a) \leq \big(\bigsqcup_{\lambda\in\Omega} \mu_\lambda\big)(u).$$

For any $u \notin Sg(a)$, we have

$$Fsg(a_c)(u) = 0 \leq \big(\bigsqcup_{\lambda\in\Omega} \mu_\lambda\big)(u).$$

Thus, $Fsg(a_c)(u) \leq \big(\bigsqcup_{\lambda\in\Omega} \mu_\lambda\big)(u)$ for all $u \in A$; *i.e.*, $Fsg(a_c) \leq \bigsqcup_{\lambda\in\Omega} \mu_\lambda$.
Hence, $Fsg(a_c)$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$. □

Note: For any $a \in Sg(\emptyset)$ and $c \in L$, $Fsg(a_c) = \chi_{Sg(\emptyset)}$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$.

Definition 2.1.11. *For any compact element c of $\mathcal{L}$ and $a \in A$, $Fsg(a_c)$ will be called a compact principal L-fuzzy subalgebra of $\mathcal{A}$.*

Theorem 2.1.12. *Suppose that $\mathcal{L}$ is distributive and algebraic.*
(1) *Compact elements of $\mathbb{F}s(\mathcal{A}, L)$ are exactly finite suprema of compact principal L-fuzzy subalgebras of $\mathcal{A}$.*
(2) *$\mathbb{F}s(\mathcal{A}, L)$ is an algebraic lattice.*

Proof. **(1)** A finite supremum of compact principal L-fuzzy subalgebras of $\mathcal{A}$ is a finite supremum of compact elements of $\mathbb{F}s(\mathcal{A}, L)$ by Proposition 2.1.10; thus, it is a compact element of $\mathbb{F}s(\mathcal{A}, L)$.
Conversely, let μ be a compact element of $\mathbb{F}s(\mathcal{A}, L)$. Since $\mu = \bigsqcup_{a\in A} Fsg\big(a_{\mu(a)}\big)$, there are $a_1, \ldots, a_n \in A$ such that $\mu = \bigsqcup_{1\leq i\leq n} Fsg\big((a_i)_{\mu(a_i)}\big)$. Since $\mathcal{L}$ is algebraic, for any $1 \leq i \leq n$, there is a family $\{c_j\}_{j\in I_i}$ of compact elements of $\mathcal{L}$ such that $\mu(a_i) = \bigvee_{j\in I_i} c_j$. It follows that

$$\begin{aligned}\mu &= \bigsqcup_{1\leq i\leq n} Fsg\big((a_i)_{\bigvee\limits_{j\in I_i} c_j}\big)\\ &= \bigsqcup_{1\leq i\leq n} Fsg\big(\bigvee_{j\in I_i}(a_i)_{c_j}\big)\\ &= \bigsqcup_{1\leq i\leq n}\bigsqcup_{j\in I_i} Fsg\big((a_i)_{c_j}\big)\\ &= \bigsqcup_{(j_1,\dots,j_n)\in\prod\limits_{1\leq i\leq n} I_i}\ \bigsqcup_{1\leq i\leq n} Fsg\big((a_i)_{c_{j_i}}\big).\end{aligned}$$

Since μ is compact, there is a family $\{K_i\}_{1\leq i\leq n}$ of finite sets such that $K_i\subseteq I_i$ for all $1\leq i\leq n$ and $\mu = \bigsqcup\limits_{(j_1,\dots,j_n)\in\prod\limits_{1\leq i\leq n} K_i}\ \bigsqcup\limits_{1\leq i\leq n} Fsg\big((a_i)_{c_{j_i}}\big)$. Hence, by Proposition 2.1.10, μ is a finite supremum of compact principal L-fuzzy subalgebras of $\mathcal{A}$.

(2) Since $\mathbb{F}s(\mathcal{A},L)$ is complete, it suffices to show that it is compactly generated. So, let $\mu\in Fs(\mathcal{A},L)$. Since $\mathcal{L}$ is algebraic, for any $a\in A$, there is a family $\{c_{i,a}\}_{i\in I_a}$ of compact elements of $\mathcal{L}$ such that $\mu(a)=\bigvee\limits_{i\in I_a} c_{i,a}$. Hence, $\mu = \bigsqcup\limits_{a\in A} Fsg\big(a_{\bigvee\limits_{i\in I_a} c_{i,a}}\big) = \bigsqcup\limits_{a\in A}\bigsqcup\limits_{i\in I_a} Fsg\big(a_{c_{i,a}}\big)$, and for each $a\in A$ and $i\in I_a$, $Fsg\big(a_{c_{i,a}}\big)$ is compact by Proposition 2.1.10. Therefore, $\mathbb{F}s(\mathcal{A},L)$ is algebraic. □

2.2 Quantale of L-fuzzy subalgebras of an algebra

In this section, $\mathcal{L}$ is a finite linearly ordered Brouwerian algebra and the lattice $\mathbb{S}ub(\mathcal{A})$, of subuniverses of $\mathcal{A}$, supports a quantale structure whose product is denoted by $\odot$.

2.2.1 Quantale structure

Proposition 2.2.1. *Let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be a family of L-fuzzy subalgebras of $\mathcal{A}$ and $r\in L$. Then $U\big(\bigsqcup\limits_{\lambda\in\Lambda}\mu_\lambda, r\big)=\bigsqcup\limits_{\lambda\in\Lambda} U(\mu_\lambda,r)$.*

Proof. Let $a\in U\big(\bigsqcup\limits_{\lambda\in\Lambda}\mu_\lambda, r\big)$. Since $r\leq Fsg\big(\bigvee\limits_{\lambda\in\Lambda}\mu_\lambda\big)(a)$, there is $s\in L$ such that $a\in Sg\big(U(\bigvee\limits_{\lambda\in\Lambda}\mu_\lambda, s)\big)$ and $r\leq s$; thus, there are a term $t(x_1,\dots,x_n)$ of type $\mathcal{F}$ and $a_1,\dots,a_n\in U\big(\bigvee\limits_{\lambda\in\Lambda}\mu_\lambda, s\big)$ such that $a=t^A(a_1,\dots,a_n)$. For any $1\leq i\leq n$, there is $\lambda_i\in\Lambda$ such that $s\leq\mu_{\lambda_i}(a_i)$ and, $r\leq\mu_{\lambda_i}(a_i)$; thus, $a_i\in U(\mu_{\lambda_i},r)\subseteq\bigsqcup\limits_{\lambda\in\Lambda}U(\mu_\lambda,r)$. So, $a=t^A(a_1,\dots,a_n)\in\bigsqcup\limits_{\lambda\in\Lambda}U(\mu_\lambda,r)$. Consequently,

$$U\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda, r\big) \subseteq \bigsqcup_{\lambda\in\Lambda} U(\mu_\lambda, r); \text{ and, } U\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda, r\big) = \bigsqcup_{\lambda\in\Lambda} U(\mu_\lambda, r);$$

since, $\bigsqcup_{\lambda\in\Lambda} U(\mu_\lambda, r) \subseteq \bigsqcup_{\lambda\in\Lambda} U\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda, r\big) = U\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda, r\big)$. □

Proposition 2.2.2. *Let $\mu, \nu \in Fs(\mathcal{A}, L)$. The L-fuzzy subset $\mu\otimes\nu$ of A, given by $\big(\mu\otimes\nu\big)(x) = \bigvee\{r \in L: \ x \in U(\mu, r)\odot U(\nu, r)\}$ for all $x \in A$, is an L-fuzzy subalgebra of $\mathcal{A}$.*

Proof. For any $f \in F_0$, we have $\big(\mu\otimes\nu\big)(f^A) = \bigvee L = 1$. Now, let $f \in F_n$ and $a_1, \dots, a_n \in A$. For any $r_1, \dots, r_n \in L$ and $a_1 \in U(\mu, r_1)\odot U(\nu, r_1), \dots, a_n \in U(\mu, r_n)\odot U(\nu, r_n)$, we have $a_1, \dots, a_n \in U\big(\mu, \bigwedge_{1\le i\le n} r_i\big)\odot U\big(\nu, \bigwedge_{1\le i\le n} r_i\big)$ and, $f^A(a_1, \dots, a_n) \in U\big(\mu, \bigwedge_{1\le i\le n} r_i\big)\odot U\big(\nu, \bigwedge_{1\le i\le n} r_i\big)$; thus, $\big(\mu\otimes\nu\big)\big(f^A(a_1, \dots, a_n)\big) \ge \bigwedge_{1\le i\le n} r_i$. So, $\big(\mu\otimes\nu\big)\big(f^A(a_1, \dots, a_n)\big) \ge \bigwedge_{1\le i\le n}\big(\mu\otimes\nu\big)(a_i)$. Hence, $\mu\otimes\nu$ is an L-fuzzy subalgebra of $\mathcal{A}$. □

Lemma 2.2.3. *Let μ and ν be two L-fuzzy subalgebras of $\mathcal{A}$ and $r \in L$. Then $U(\mu\otimes\nu, r) = U(\mu, r)\odot U(\nu, r)$.*

Proof. For any $a \in U(\mu, r)\odot U(\nu, r)$, we have

$$r \le \big(\mu\otimes\nu\big)(a) \text{ and, } a \in U(\mu\otimes\nu, r).$$

Thus, $U(\mu, r)\odot U(\nu, r) \subseteq U(\mu\otimes\nu, r)$.
Now, let $a \in U(\mu\otimes\nu, r)$. Since $r \le (\mu\otimes\nu)(a)$, there is $s \in L$ such that

$$r \le s \text{ and } a \in U(\mu, s)\odot U(\nu, s);$$

thus, $a \in U(\mu, r)\odot U(\nu, r)$. So, $U(\mu\otimes\nu, r) \subseteq U(\mu, r)\odot U(\nu, r)$. Hence, $U(\mu\otimes\nu, r) = U(\mu, r)\odot U(\nu, r)$. □

Theorem 2.2.4. $\mathcal{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \sqcup,\ \otimes;\ \underline{1}\big)$ *is a quantale.*

Proof. We have already proved that $\mathbb{F}s(\mathcal{A}, L)$ is a complete lattice. Now, let $\mu, \nu, \delta \in Fs(\mathcal{A}, L)$. For any $a \in A$, we have

$$\begin{aligned}
\big((\mu\otimes\nu)\otimes\delta\big)(a) &= \bigvee\{r \in L: \ a \in U(\mu\otimes\nu, r)\odot U(\delta, r)\}\\
&= \bigvee\{r \in L: \ a \in \big(U(\mu, r)\odot U(\nu, r)\big)\odot U(\delta, r)\}\\
&= \bigvee\{r \in L: \ a \in U(\mu, r)\odot\big(U(\nu, r)\odot U(\delta, r)\big)\}\\
&= \bigvee\{r \in L: \ a \in U(\mu, r)\odot U(\nu\otimes\delta, r)\}\\
&= \big(\mu\otimes(\nu\otimes\delta)\big)(a).
\end{aligned}$$

Thus, $(\mu\otimes\nu)\otimes\delta = \mu\otimes(\nu\otimes\delta)$. For any $\mu \in Fs(\mathcal{A}, L)$, we have

$$\big(\mu\otimes\underline{1}\big)(a) = \bigvee\{r \in L: \ a \in U(\mu, r)\odot A\} = \bigvee\{r \in L: \ a \in U(\mu, r)\} = \mu(a)$$

for all $a \in A$; thus, $\mu \otimes \underline{1} = \mu$; and, $\underline{1} \otimes \mu = \mu$ by similar arguments. Hence, $\big(Fs(\mathcal{A}, L);\ \otimes,\ \underline{1}\big)$ is a monoid.
For any $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fs(\mathcal{A}, L)$ and $\mu \in Fs(\mathcal{A}, L)$, we have

$$\begin{aligned}
\big(\mu \otimes \bigsqcup_{\lambda\in\Lambda} \mu_\lambda\big)(a) &= \bigvee\{r \in L:\ a \in U(\mu, r) \odot U\big(\bigsqcup_{\lambda\in\Lambda} \mu_\lambda, r\big)\} \\
&= \bigvee\{r \in L:\ a \in U(\mu, r) \odot \bigsqcup_{\lambda\in\Lambda} U(\mu_\lambda, r)\} \\
&= \bigvee\{r \in L:\ a \in \bigsqcup_{\lambda\in\Lambda} U(\mu, r) \odot U(\mu_\lambda, r)\} \\
&= \bigvee\{r \in L:\ a \in \bigsqcup_{\lambda\in\Lambda} U(\mu \otimes \mu_\lambda, r)\} \\
&= \bigvee\{r \in L:\ a \in U\big(\bigsqcup_{\lambda\in\Lambda} \mu \otimes \mu_\lambda, r\big)\} \\
&= \big(\bigsqcup_{\lambda\in\Lambda} \mu \otimes \mu_\lambda\big)(a) \text{ for all } a \in A;
\end{aligned}$$

thus, $\mu \otimes \big(\bigsqcup_{\lambda\in\Lambda} \mu_\lambda\big) = \bigsqcup_{\lambda\in\Lambda} \mu \otimes \mu_\lambda$; and, $\big(\bigsqcup_{\lambda\in\Lambda} \mu_\lambda\big) \otimes \mu = \bigsqcup_{\lambda\in\Lambda} \mu_\lambda \otimes \mu$ by similar arguments. Hence, $\mathcal{F}s(\mathcal{A}, L)$ is a quantale. □

Remark 2.2.5.
(1) *The map $\phi : Sub(\mathcal{A}) \to Fs(\mathcal{A}, L)$, given by $\phi(B) = B_1$ for all $B \in Sub(\mathcal{A})$, is a quantale embedding of $\mathcal{S}ub(\mathcal{A})$ into $\mathcal{F}s(\mathcal{A}, L)$.*
(2) *If $\mathcal{A}$ is not $\mathcal{F}$-trivial, then the map $\psi : L \to Fs(\mathcal{A}, L)$, given by $\psi(r) = (\underline{r})_*$ for all $r \in L$, is a quantale embedding of $\mathcal{L}$ into $\mathcal{F}s(\mathcal{A}, L)$.*

In the rest of this work the results of the preceding remark will be generalized in mono-unary algebras and rings. Furthermore, in each case the residual operations will be explixed.

2.3 Residual transfer in fuzzy mono-unary algebras

Note that by «Residual transfer in a fuzzy algebra» we mean the embedding of a residuated lattice (through all its operations, residual operations included) on the subuniverses of an algebra into a residuated lattice on the set of fuzzy subalgebras of the same algebra.

2.3.1 Lattice of L-fuzzy subalgebras

Definition 2.3.1. *(See, [5])*
(i) *A mono-unary algebra, also called a unar, is an algebra of type $\langle 1\rangle$; that is an algebra with one unary operation and no other operation.*
(ii) *An element x of a mono-unary algebra $(A;\ f)$ is said to be cyclic if there*

is some integer $p \geq 1$ such that $f^p(x) = x$; where, for any nonnegative integer n, f^n is defined inductively by: $f^0(a) = a$ and $f^{n+1}(a) = f\big(f^n(a)\big)$ for all $a \in A$.

Let $\mathcal{A} := (A;\ f)$ be a mono-unary algebra. The Heyting algebra of subuniverses of $\mathcal{A}$ is given by $\mathcal{S}ub(\mathcal{A}) := \big(Sub(\mathcal{A});\ \cap,\ \cup,\ \Rightarrow;\ \emptyset,\ A\big)$; where, the binary operation $\Rightarrow$ is defined for any $B, C \in Sub(\mathcal{A})$ by:

$$B \Rightarrow C := \bigcup\{D \in Sub(\mathcal{A}) :\ D \cap B \subseteq C\}.$$

Remark 2.3.2. *An L-fuzzy subset μ of A is an L-fuzzy subalgebra of $\mathcal{A}$ if and only if $\mu\big(f(x)\big) \geq \mu(x)$ for all $x \in A$, if and only if all its levels sets are subuniverses of $\mathcal{A}$.*

The set of L-fuzzy subalgebras of $\mathcal{A}$ forms a complete lattice $\mathbb{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \vee;\ \underline{0},\ \underline{1}\big)$. The subuniverse of $\mathcal{A}$ generated by an element x of A is given by $Sg(x) = \{f^k(x) :\ k \in \mathbb{N}\}$. Now, define $C_x^k := \{a \in A :\ f^k(a) = x\}$ for all $x \in A$ and $k \in \mathbb{N}$.

Theorem 2.3.3. *Let μ be an L-fuzzy subset of A. The L-fuzzy subalgebra of $\mathcal{A}$ generated by μ is defined by: $\mu_\star(x) = \bigvee\limits_{k \in \mathbb{N}} \bigvee\limits_{a \in C_x^k} \mu(a)$ for all $x \in A$.*

Proof. Since $\mu_\star(x) \geq \bigvee\limits_{a \in C_x^0} \mu(a) = \bigvee\{\mu(x)\} = \mu(x)$ for all $x \in A$, we have $\mu \leq \mu_\star$. We next show that $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.
For any $x \in A$, we have

$$\mu_\star\big(f(x)\big) = \bigvee_{k \in \mathbb{N}} \bigvee_{a \in C_{f(x)}^k} \mu(a) = \Big[\bigvee_{a \in C_{f(x)}^0} \mu(a)\Big] \vee \Big[\bigvee_{k \in \mathbb{N}} \bigvee_{a \in C_{f(x)}^{k+1}} \mu(a)\Big];$$

since, $C_x^k \subseteq C_{f(x)}^{k+1}$, we have $\mu_\star\big(f(x)\big) \geq \bigvee\limits_{k \in \mathbb{N}} \bigvee\limits_{a \in C_{f(x)}^{k+1}} \mu(a) \geq \bigvee\limits_{k \in \mathbb{N}} \bigvee\limits_{a \in C_x^k} \mu(a) = \mu_\star(x)$.
Hence, $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.
Finally, let ν be an L-fuzzy subalgebra of $\mathcal{A}$ which contains μ. Let $x \in A$. For any $k \in \mathbb{N}$ and $a \in C_x^k$, we have $\nu(x) = \nu\big(f^k(a)\big) \geq \nu(a) \geq \mu(a)$. Thus, $\nu(x) \geq \bigvee\limits_{a \in C_x^k} \mu(a)$ for all $k \in \mathbb{N}$; *i.e.*, $\nu(x) \geq \bigvee\limits_{k \in \mathbb{N}} \bigvee\limits_{a \in C_x^k} \mu(a)$; *i.e.*, $\nu(x) \geq \mu_\star(x)$. So, $\mu \leq \nu$. Hence, $\mu_\star = Fsg(\mu)$. □

Lemma 2.3.4. *Let $x \in A$. Then $Sg(x)$ is an atom of $\mathbb{S}ub(\mathcal{A})$ if and only if x is cyclic.*

Proof. Assume that $Sg(x)$ is an atom of $\mathbb{S}ub(\mathcal{A})$. Since $\emptyset \subset Sg\big(f(x)\big) \subseteq Sg(x)$, we have $Sg\big(f(x)\big) = Sg(x)$; thus, there is $n \in \mathbb{N}$ such that $f^n\big(f(x)\big) = x$; so, $f^{n+1}(x) = x$. Hence, x is cyclic.
Conversely, assume that x is cyclic of order n. Let B be a subuniverse of $\mathcal{A}$ such that $\emptyset \subset B \subseteq Sg(x)$. Since there is $m \leq n$ such that $f^m(x) \in B$, we have $x = f^n(x) = f^{n-m}\big(f^m(x)\big) \in B$; thus, $Sg(x) \subseteq B$ and, $Sg(x) = B$. Hence, $Sg(x)$ is an atom of $\mathbb{S}ub(\mathcal{A})$. □

Theorem 2.3.5. *Atoms of* $\mathbb{F}s(\mathcal{A}, L)$ *are exactly the* L*-fuzzy subalgebras* $Sg(x)_r$*, where* r *is an atom of* $\mathcal{L}$ *and* x *is a cyclic element of* $\mathcal{A}$*.*

Proof. Immediate consequence of Theorem 2.1.6 and Lemma 2.3.4. □

Theorem 2.3.6. *Co-atoms of* $\mathbb{F}s(\mathcal{A}, L)$ *are exactly the* L*-fuzzy subalgebras* B^s*, where* s *and* B *are co-atoms of* $\mathcal{L}$ *and* $\mathcal{S}ub(\mathcal{A})$*, respectively.*

Proof. Immediate consequence of Theorem 2.1.8 and the fact that without distributivity of $\mathcal{L}$ the following holds: for any r in L and any L-fuzzy subalgebra μ of $\mathcal{A}$, $\underline{r} \vee \mu$ is also an L-fuzzy subalgebra of $\mathcal{A}$. □

Lemma 2.3.7. *Let* c *be a compact element of* $\mathcal{L}$ *and* $a \in A$*. Then* $Sg(a)_c$ *is a compact element of* $\mathbb{F}s(\mathcal{A}, L)$*.*

Proof. Let $\{\mu_i\}_{i\in I} \subseteq Fs(\mathcal{A}, L)$ such that $Sg(a)_c \leq \bigvee_{i\in I} \mu_i$. Since $c \leq \bigvee_{i\in I} \mu_i(a)$, there is a finite subset I_0 of I such that $c \leq \bigvee_{i\in I_0} \mu_i(a)$. For any $x \in Sg(a)$, we have $Sg(a)_c(x) = c \leq \bigvee_{i\in I_0} \mu_i(a) \leq \bigvee_{i\in I_0} \mu_i(x) = \big(\bigvee_{i\in I_0} \mu_i\big)(x)$; thus, $Sg(a)_c \leq \bigvee_{i\in I_0} \mu_i$. Hence, $Sg(a)_c$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$. □

Theorem 2.3.8. *If* $\mathcal{L}$ *is algebraic, then* $\mathbb{F}s(\mathcal{A}, L)$ *is algebraic and its compact elements are exactly finite suprema of compact principal* L*-fuzzy subalgebras of* $\mathcal{A}$*.*

Proof. Similar to the proof of Theorem 2.1.12. □

2.3.2 Residuated lattice of L-fuzzy subalgebras

Let $\mathcal{A} := (A;\ f)$ be a mono-unary algebra. $Fs(\mathcal{A}, L)$ is closed under the binary operation $\ominus$ of the residuated lattice $\mathcal{F}u(A, L)$ of L-fuzzy subsets of A, but the binary operations $\twoheadrightarrow$ and $\multimap$ are not necessarily well defined on $Fs(\mathcal{A}, L)$ as the following example shows.

Example 2.3.9. *Let* $L = \{0,\ \alpha,\ \beta,\ \gamma,\ 1\}$ *be a lattice such that* $0 < \alpha < \beta, \gamma < 1$*; where,* β, γ *are incomparable. Consider the binary operations* $\ominus, \twoheadrightarrow, \multimap$ *given by the following Cayley tables:*

$\ominus$	0	α	β	γ	1
0	0	0	0	0	0
α	0	0	0	α	α
β	0	α	β	α	β
γ	0	0	0	γ	γ
1	0	α	β	γ	1

$\twoheadrightarrow$	0	α	β	γ	1
0	1	1	1	1	1
α	γ	1	1	1	1
β	γ	γ	1	γ	1
γ	0	β	β	1	1
1	0	α	β	γ	1

$\multimap$	0	α	β	γ	1
0	1	1	1	1	1
α	β	1	1	1	1
β	0	γ	1	γ	1
γ	β	β	β	1	1
1	0	α	β	γ	1

.

Then $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ *is a residuated lattice. Consider the Peano algebra* $\mathcal{N} = (\mathbb{N};\ \sigma)$*, given by* $\sigma(x) = x + 1$ *for all* $x \in \mathbb{N}$*, and the* L*-fuzzy subalgebras* μ *and* ν *of* $\mathcal{N}$ *defined for any* $x \in \mathbb{N}$ *by:*

$$\mu(x) = \begin{cases} 0 \text{ if } x = 0, \\ \beta \text{ if not.} \end{cases} \text{ and } \nu(x) = \begin{cases} 0 \text{ if } x = 0, \\ \gamma \text{ if not.} \end{cases}.$$

The L*-fuzzy subset* $\mu \twoheadrightarrow \nu$ *of* $\mathbb{N}$ *is not an* L*-fuzzy subalgebra of* $\mathcal{N}$*, since*

$$\big(\mu \twoheadrightarrow \nu\big)\big(\sigma(0)\big) = \big(\mu \twoheadrightarrow \nu\big)(1) = \beta \twoheadrightarrow \gamma = \gamma \ngeq 1 = 0 \twoheadrightarrow 0 = \big(\mu \twoheadrightarrow \nu\big)(0).$$

Theorem 2.3.10. *Let* μ *be an* L*-fuzzy subset of* A*. The* L*-fuzzy subset* $\mu^\star$ *of* A*, given by*

$$\mu^\star(x) = \bigwedge_{k \in \mathbb{N}} \mu\big(f^k(x)\big) \text{ for all } x \in A,$$

is the biggest L*-fuzzy subalgebra of* $\mathcal{A}$ *contained in* μ*.*

Proof. We have $\mu^\star \leq \mu$, since $\mu^\star(x) \leq \mu\big(f^0(x)\big) = \mu(x)$ for all $x \in A$. We next show that $\mu^\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.
For any $x \in A$, we have

$$\mu^\star\big(f(x)\big) = \bigwedge_{k \in \mathbb{N}} \mu\big(f^{k+1}(x)\big) \geq \mu\big(f^0(x)\big) \wedge \bigwedge_{k \in \mathbb{N}} \mu\big(f^{k+1}(x)\big) = \mu^\star(x).$$

Hence, $\mu^\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.
Finally, let ν be an L-fuzzy subalgebra of $\mathcal{A}$ contained in μ. For any $x \in A$, we have $\nu(x) \leq \nu\big(f^k(x)\big) \leq \mu\big(f^k(x)\big)$ for all $k \in \mathbb{N}$; thus,

$$\nu(x) \leq \bigwedge_{k \in \mathbb{N}} \mu\big(f^k(x)\big) = \mu^\star(x).$$

Hence, $\nu \leq \mu^\star$. Therefore, $\mu^\star$ is the biggest L-fuzzy subalgebra of $\mathcal{A}$ contained in μ. □

Theorem 2.3.11. *For any* $\mu, \nu \in Fs(\mathcal{A}, L)$*, set* $\mu \hookrightarrow \nu := (\mu \twoheadrightarrow \nu)^\star$ *and* $\mu \looparrowright \nu := (\mu \multimap \nu)^\star$*. Then* $\mathcal{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \vee,\ \ominus,\ \hookrightarrow,\ \looparrowright;\ \underline{0},\ \underline{1}\big)$ *is a complete residuated lattice.*

Proof. We only have to show that $\mu \hookrightarrow \nu = \max\{\delta \in Fs(\mathcal{A}, L) :\ \delta \ominus \mu \leq \nu\}$ and $\mu \looparrowright \nu = \max\{\delta \in Fs(\mathcal{A}, L) :\ \mu \ominus \delta \leq \nu\}$ for all $\mu, \nu \in Fs(\mathcal{A}, L)$. So, let $\mu, \nu \in Fs(\mathcal{A}, L)$. We have

$$(\mu \hookrightarrow \nu) \ominus \mu = (\mu \twoheadrightarrow \nu)^\star \ominus \mu \leq (\mu \twoheadrightarrow \nu) \ominus \mu \leq \nu.$$

Moreover, for any $\delta \in Fs(\mathcal{A}, L)$ such that $\delta \ominus \mu \leq \nu$, we have $\delta \leq \mu \twoheadrightarrow \nu$; thus, $\delta \leq (\mu \twoheadrightarrow \nu)^\star = \mu \hookrightarrow \nu$. Hence, $\mu \hookrightarrow \nu = \max\{\delta \in Fs(\mathcal{A}, L) :\ \delta \ominus \mu \leq \nu\}$ and, $\mu \looparrowright \nu = \max\{\delta \in Fs(\mathcal{A}, L) :\ \mu \ominus \delta \leq \nu\}$ by similar arguments. Therefore, $\mathcal{F}s(\mathcal{A}, L)$ is a complete residuated lattice. □

Theorem 2.3.12. *The map* $\phi : Sub(\mathcal{A}) \to Fs(\mathcal{A}, L)$*, given by* $\phi(B) = B_1$ *for all* $B \in Sub(\mathcal{A})$*, is a complete residuated lattice embedding.*

Proof. By Proposition 2.1.3, ϕ is a complete lattice embedding of $\mathbb{S}ub(\mathcal{A})$ into $\mathbb{F}s(\mathcal{A}, L)$. Since we have $\phi(B \cap C) = (B \cap C)_1 = B_1 \ominus C_1 = \phi(B) \ominus \phi(C)$ for all $B, C \in Sub(\mathcal{A})$, it suffices to show that $\phi(B) \hookrightarrow \phi(C) = \phi(B \Rightarrow C) = \phi(B) \looparrowright \phi(C)$. So, let $B, C \in Sub(\mathcal{A})$. For any $x \notin B \Rightarrow C$, we have $Sg(x) \cap B \nsubseteq C$; thus, $f^{k_0}(x) \in B$ and $f^{k_0}(x) \notin C$ for some $k_0 \in \mathbb{N}$; so,

$$\begin{aligned}
\big(\phi(B) \hookrightarrow \phi(C)\big)(x) &= \big(B_1 \hookrightarrow C_1\big)(x) \\
&= \big[B_1\big(f^{k_0}(x)\big) \twoheadrightarrow C_1\big(f^{k_0}(x)\big)\big] \wedge \\
&\qquad \big[\bigwedge_{k \in \mathbb{N}} B_1\big(f^{k+1}(x)\big) \twoheadrightarrow C_1\big(f^{k+1}(x)\big)\big] \\
&= (1 \twoheadrightarrow 0) \wedge \big[\bigwedge_{k \in \mathbb{N}} B_1\big(f^{k+1}(x)\big) \twoheadrightarrow C_1\big(f^{k+1}(x)\big)\big] \\
&= 0 \\
&= (B \Rightarrow C)_1(x) \\
&= \phi(B \Rightarrow C)(x).
\end{aligned}$$

Now, let $x \in B \Rightarrow C$ and $D \in Sub(\mathcal{A})$ such that $D \cap B \subseteq C$ and $x \in D$.

- For any $n \in \Omega(B) := \{k \in \mathbb{N} : \ f^k(x) \in B\}$, we have $f^n(x) \in D \cap B \subseteq C$; thus, $f^n(x) \in B$ and $f^n(x) \in C$; so,

$$B_1\big(f^n(x)\big) \twoheadrightarrow C_1\big(f^n(x)\big) = 1 \twoheadrightarrow 1 = 1.$$

- For any $n \notin \Omega(B)$, we have

$$B_1\big(f^n(x)\big) \twoheadrightarrow C_1\big(f^n(x)\big) = 0 \twoheadrightarrow C_1\big(f^n(x)\big) = 1.$$

Thus,

$$\begin{aligned}
\big(\phi(B) \hookrightarrow \phi(C)\big)(x) &= \big(B_1 \hookrightarrow C_1\big)(x) \\
&= \big[\bigwedge_{k \in \Omega(B)} B_1\big(f^k(x)\big) \twoheadrightarrow C_1\big(f^k(x)\big)\big] \wedge \\
&\qquad \big[\bigwedge_{k \notin \Omega(B)} B_1\big(f^k(x)\big) \twoheadrightarrow C_1\big(f^k(x)\big)\big] \\
&= \big(\bigwedge_{k \in \Omega(B)} 1\big) \wedge \big(\bigwedge_{k \notin \Omega(B)} 1\big) \\
&= 1 \wedge 1 \\
&= 1 \\
&= (B \Rightarrow C)_1(x) \\
&= \phi(B \Rightarrow C)(x).
\end{aligned}$$

Hence, $\phi(B \Rightarrow C) = \phi(B) \hookrightarrow \phi(C)$ and, $\phi(B \Rightarrow C) = \phi(B) \looparrowright \phi(C)$ by similar arguments. Therefore, ϕ is a complete residuated lattice embedding of $\mathcal{S}ub(\mathcal{A})$ into $\mathcal{F}s(\mathcal{A}, L)$. □

Theorem 2.3.13. *The map $\psi : L \to Fs(\mathcal{A}, L)$, given by $\psi(r) = \underline{r}$ for all $r \in L$, is a complete residuated lattice embedding.*

Proof. By Proposition 2.1.4, ψ is a complete lattice embedding of the lattice of $\mathcal{L}$ into $\mathbb{F}s(\mathcal{A}, L)$. Now, let $r, s \in L$. For any $x \in A$, we have

$$\begin{aligned} \psi(r \ominus s)(x) &= r \ominus s \\ &= \underline{r}(x) \ominus \underline{s}(x) \\ &= \psi(r)(x) \ominus \psi(s)(x) \\ &= \big(\psi(r) \ominus \psi(s)\big)(x). \end{aligned}$$

Thus, $\psi(r \ominus s) = \psi(r) \ominus \psi(s)$.
For any $x \in A$, we have

$$\begin{aligned} \psi(r \twoheadrightarrow s)(x) &= r \twoheadrightarrow s \\ &= \bigwedge_{k \in \mathbb{N}} \underline{r}\big(f^k(x)\big) \twoheadrightarrow \underline{s}\big(f^k(x)\big) \\ &= \bigwedge_{k \in \mathbb{N}} \psi(r)\big(f^k(x)\big) \twoheadrightarrow \psi(s)\big(f^k(x)\big) \\ &= \big(\psi(r) \hookrightarrow \psi(s)\big)(x). \end{aligned}$$

Thus, $\psi(r \twoheadrightarrow s) = \psi(r) \hookrightarrow \psi(s)$ and, $\psi(r \multimap s) = \psi(r) \looparrowright \psi(s)$ by similar arguments.
Hence, ψ is a complete residuated lattice embedding of $\mathcal{L}$ into $\mathcal{F}s(\mathcal{A}, L)$. □

2.3.3 Mono-unary algebras and MV-algebras

Let $\mathcal{A} := (A;\ f)$ be a mono-unary algebra. Since $\wedge$, $\vee$ and $\ominus$ are defined componentwise on $Fs(\mathcal{A}, L)$, $\mathcal{L}$ and $\mathcal{F}s(\mathcal{A}, L)$ satisfy the same bounded lattice-ordered monoid identities.

Proposition 2.3.14. *(See, [5]) Let $Sym(A)$ be the permutation group of A. The following hold:*
(1) *If $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice, then $f \in Sym(A)$.*
(2) *If f is of finite order in $Sym(A)$, then $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice.*
(3) *If A is finite, then $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice if and only if $f \in Sym(A)$.*

Lemma 2.3.15. *The following statements are equivalent:*
(a) *For any $\mu \in Fu(A, L)$, $\mu \in Fs(\mathcal{A}, L)$ iff $\mu\big(f(x)\big) = \mu(x)$ for all $x \in A$.*
(b) *$\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice.*

Proof. Suppose that **(a)** is satisfied. Let $B \in Sub(\mathcal{A})$. For any $x \in \overline{B}$, we have $B_1\big(f(x)\big) = B_1(x) = 0$ and, $f(x) \in \overline{B}$. Thus, $\overline{B} \in Sub(\mathcal{A})$. Hence, $\mathcal{S}ub(\mathcal{A})$ is a Boolean lattice.
Conversely, suppose that **(b)** is satisfied. Let $\mu \in Fs(\mathcal{A}, L)$. For any $x \in A$, we have $f(x) \in U\big[\mu, \mu\big(f(x)\big)\big] \in Sub(\mathcal{A})$; thus, $x \in U\big[\mu, \mu\big(f(x)\big)\big]$; so,

$$\mu(x) \geq \mu\big(f(x)\big) \text{ and, } \mu\big(f(x)\big) = \mu(x).$$

Whence the result. □

Theorem 2.3.16. *$\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$ if and only if $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice.*

Proof. Assume that $\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$. Let B be a subuniverse of $\mathcal{A}$. For any $x \in \overline{B}$, we have

$$\begin{aligned} B_1\big(f(x)\big) \twoheadrightarrow 0 &= B_1\big(f(x)\big) \twoheadrightarrow \underline{0}\big(f(x)\big) \\ &\geq (B_1 \hookrightarrow \underline{0})(x) \\ &= (B_1 \twoheadrightarrow \underline{0})(x) \\ &= B_1(x) \twoheadrightarrow \underline{0}(x) \\ &= 0 \twoheadrightarrow 0 \\ &= 1; \end{aligned}$$

thus, $B_1\big(f(x)\big) \twoheadrightarrow 0 = 1$ and, $B_1\big(f(x)\big) = 0$; *i.e.*, $f(x) \notin B$ and, $f(x) \in \overline{B}$. So, $\overline{B}$ is a subuniverse of $\mathcal{A}$. Hence, $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice.
Conversely, assume that $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice. Let μ and ν be two L-fuzzy subalgebras of $\mathcal{A}$. For any $x \in A$, we have

$$\begin{aligned} (\mu \hookrightarrow \nu)(x) &= \bigwedge_{k \in \mathbb{N}} \mu\big(f^k(x)\big) \twoheadrightarrow \nu\big(f^k(x)\big) \\ &= \bigwedge_{k \in \mathbb{N}} \mu(x) \twoheadrightarrow \nu(x) \\ &= \mu(x) \twoheadrightarrow \nu(x) \\ &= (\mu \twoheadrightarrow \nu)(x). \end{aligned}$$

Thus, $\mu \hookrightarrow \nu = \mu \twoheadrightarrow \nu$. Hence, $\hookrightarrow$ is the restriction of $\twoheadrightarrow$ to $Fs(\mathcal{A}, L)$. A similar reasoning shows that $\looparrowright$ is the restriction of $\multimap$ to $Fs(\mathcal{A}, L)$. Therefore, $\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$. □

Let $\mathcal{K}$ be a class of residuated lattices such that

$$Mod\big(Id(\mathcal{K}) \cup \{x \otimes y = x \wedge y\}\big)$$

is included in the class of Boolean algebras; for example, the class of MV-algebras.

Theorem 2.3.17. *$\mathcal{F}s(\mathcal{A}, L) \models Id(\mathcal{K})$ if and only if $\mathcal{L} \models Id(\mathcal{K})$ and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra.*

Proof. If $\mathcal{F}s(\mathcal{A}, L) \models Id(\mathcal{K})$, then $\mathcal{S}ub(\mathcal{A}) \models Id(\mathcal{K})$ and $\mathcal{L} \models Id(\mathcal{K})$ by Theorem 2.3.12 and Theorem 2.3.13, respectively; thus, $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra and $\mathcal{L} \models Id(\mathcal{K})$.
Conversely, assume that $\mathcal{L} \models Id(\mathcal{K})$ and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra. Then $\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$ by Theorem 2.3.16. Consequently, $\mathcal{F}s(\mathcal{A}, L) \models Id(\mathcal{K})$, since $\mathcal{L} \models Id(\mathcal{K})$. □

If $\mathcal{F}s(\mathcal{A}, L)$ is an RL-monoid, then $\mathcal{L}$ is an RL-monoid by Theorem 2.3.13; but the converse is not necessarily true as the following example shows.

Example 2.3.18. *Let $L = \{0,\ \alpha,\ \beta,\ 1\}$ be a lattice such that $0 < \alpha < \beta < 1$. Define the binary operations $\ominus$ and $\twoheadrightarrow$ on L as follows:*

$\ominus$	0	α	β	1
0	0	0	0	0
α	0	0	α	α
β	0	α	β	β
1	0	α	β	1

$\twoheadrightarrow$	0	α	β	1
0	1	1	1	1
α	α	1	1	1
β	0	α	1	1
1	0	α	β	1

.

Then $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \twoheadrightarrow;\ 0,\ 1)$ is an RL-monoid. Consider the mono-unary algebra $\mathcal{A}$ given by the table

	$\overset{f}{\curvearrowright}$
0	0
a	a
b	a
c	b

, and the L-fuzzy subalgebras σ and τ of $\mathcal{A}$ defined for any $x \in A$ by:

$$\sigma(x) = \begin{cases} 1 & if\ \ x = 0, \\ \beta & if\ \ x \in \{a,\ b\}, \\ \alpha & if\ \ x = c. \end{cases} \quad \text{and } \tau(x) = \begin{cases} 1 & if\ \ x = 0, \\ \alpha & if\ \ x \in \{a,\ b,\ c\}. \end{cases}.$$

Since $\sigma \hookrightarrow \tau = \tau$, we have $\big((\sigma \hookrightarrow \tau) \ominus \sigma\big)(c) = \big(\tau \ominus \sigma\big)(c) = \alpha \ominus \alpha = 0 \neq \alpha = \big(\sigma \wedge \tau\big)(c)$; thus, $(\sigma \hookrightarrow \tau) \ominus \sigma \neq \sigma \wedge \tau$. It follows that $\mathcal{F}s(\mathcal{A}, L)$ is not an RL-monoid.

Proposition 2.3.19. *If $\mathcal{F}s(\mathcal{A}, L)$ is an RL-monoid, then for any $x \in A$ and $B \in Sub(\mathcal{A})$, we have $x \in \overline{B}$ if and only if $Sg(x) \cap \overline{B} \neq \emptyset$.*

Proof. Assume that $\mathcal{F}s(\mathcal{A}, L)$ is an RL-monoid. Let $x \in A$ and $B \in Sub(\mathcal{A})$. It is clear that $Sg(x) \cap \overline{B} \neq \emptyset$ for all $x \in \overline{B}$.
Conversely, assume that $Sg(x) \cap \overline{B} \neq \emptyset$. Since there is $k_0 \in \mathbb{N}$ such that $f^{k_0}(x) \in \overline{B}$, we have

$$\begin{aligned} B_1(x) &= \big(\underline{1} \wedge B_1\big)(x) \\ &= \big((\underline{1} \hookrightarrow B_1) \ominus \underline{1}\big)(x) \\ &= \big(\underline{1} \hookrightarrow B_1\big)(x) \\ &\leq \underline{1}\big(f^{k_0}(x)\big) \twoheadrightarrow B_1\big(f^{k_0}(x)\big) \\ &= 1 \twoheadrightarrow 0 \\ &= 0; \end{aligned}$$

thus, $B_1(x) = 0$ and, $x \in \overline{B}$. □

If $\mathcal{F}s(\mathcal{A}, L)$ is a MTL-algebra, then $\mathcal{S}ub(\mathcal{A})$ and $\mathcal{L}$ are MTL-algebras by Theorem 2.3.12 and Theorem 2.3.13, respectively; but the converse is not necessarily true as the following example shows.

Example 2.3.20. *Let $L = \{0,\ \alpha,\ \beta,\ \gamma,\ 1\}$ be a lattice such that $0 < \alpha < \beta, \gamma < 1$; where, β, γ are incomparable. Define the binary operations $\ominus$, $\twoheadrightarrow$ and $\multimap$ on L as follows:*

$\ominus$	0	α	β	γ	1
0	0	0	0	0	0
α	0	0	α	0	α
β	0	0	β	0	β
γ	0	α	α	γ	γ
1	0	α	β	γ	1

$\twoheadrightarrow$	0	α	β	γ	1
0	1	1	1	1	1
α	β	1	1	1	1
β	0	γ	1	γ	1
γ	β	β	β	1	1
1	0	α	β	γ	1

$\multimap$	0	α	β	γ	1
0	1	1	1	1	1
α	γ	1	1	1	1
β	γ	γ	1	γ	1
γ	0	β	β	1	1
1	0	α	β	γ	1

.

Then $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a MTL-algebra. Consider the unar $\mathcal{A}$ given in Example 2.3.18. The subuniverses of $\mathcal{A}$ are $B_1 = \emptyset$, $B_2 = \{a\}$, $B_3 = \{a,\ b\}$, $B_4 = \{a,\ b,\ c\}$, $B_5 = \{0\}$, $B_6 = \{0,\ a\}$, $B_7 = \{0,\ a,\ b\}$ and $B_8 = A$. The binary operation $\Rightarrow$ of $\mathcal{S}ub(\mathcal{A})$ is given by

$\Rightarrow$	$\emptyset$	B_2	B_3	B_4	B_5	B_6	B_7	A
$\emptyset$	A	A	A	A	A	A	A	A
B_2	B_5	A	A	A	B_5	A	A	A
B_3	B_5	B_6	A	A	B_5	A	A	A
B_4	B_5	B_6	B_7	A	B_5	B_6	B_7	A
B_5	B_4	B_4	B_4	B_4	A	A	A	A
B_6	$\emptyset$	B_4	B_4	B_4	B_5	A	A	A
B_7	$\emptyset$	B_2	B_3	B_4	B_5	B_6	A	A
A	$\emptyset$	B_2	B_3	B_4	B_5	B_6	B_7	A

.

It is easy to check that $\mathcal{S}ub(\mathcal{A})$ is a MTL-algebra. Consider the L-fuzzy subalgebras σ and τ of $\mathcal{A}$ defined for any $x \in A$ by:

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 0, \\ \beta & \text{if } x \in \{a,\ b\}, \\ \alpha & \text{if } x = c. \end{cases} \quad \text{and } \tau(x) = \begin{cases} 1 & \text{if } x = 0, \\ \gamma & \text{if } x \in \{a,\ b,\ c\}, \\ 0 & \text{if } x = c. \end{cases}.$$

Then $\sigma \hookrightarrow \tau = 0_1 \vee \{a,\ b\}_\gamma \vee c_\alpha$ and $\tau \hookrightarrow \sigma = 0_1 \vee \{a,\ b,\ c\}_\beta$; thus, $\big((\sigma \hookrightarrow \tau) \vee (\tau \hookrightarrow \sigma)\big)(c) = \beta \neq 1$. So, $(\sigma \hookrightarrow \tau) \vee (\tau \hookrightarrow \sigma) \neq \underline{1}$. Hence, $\mathcal{F}s(\mathcal{A}, L)$ is not a MTL-algebra.

Proposition 2.3.21. *If $\mathcal{F}s(\mathcal{A}, L)$ is a MTL-algebra, then for any $x \in A$ and $B, D \in Sub(\mathcal{A})$, we have $Sg(x) \subseteq \overline{B} \cup D$ or $Sg(x) \subseteq B \cup \overline{D}$.*

Proof. Assume that $\mathcal{F}s(\mathcal{A}, L)$ is a MTL-algebra. Let $x \in A$ and $B, D \in Sub(\mathcal{A})$. Suppose that $Sg(x) \nsubseteq \overline{B} \cup D$. Since there is $k_0 \in \mathbb{N}$ such that $f^{k_0}(x) \notin \overline{B} \cup D$, we have

$$\big(B_1 \hookrightarrow D_1\big)(x) \leq B_1\big(f^{k_0}(x)\big) \twoheadrightarrow D_1\big(f^{k_0}(x)\big) = 1 \twoheadrightarrow 0 = 0$$

and, $\big(B_1 \hookrightarrow D_1\big)(x) = 0$. Since

$$\big(D_1 \hookrightarrow B_1\big)(x) = 0 \vee \big(D_1 \hookrightarrow B_1\big)(x) = \big(B_1 \hookrightarrow D_1\big)(x) \vee \big(D_1 \hookrightarrow B_1\big)(x) = 1,$$

we have

$$D_1\big(f^k(x)\big) \twoheadrightarrow B_1\big(f^k(x)\big) = 1 \text{ for all } k \in \mathbb{N};$$

thus,

$$f^k(x) \notin D \cap \overline{B} \text{ for all } k \in \mathbb{N};$$

i.e., $f^k(x) \in \overline{D} \cup B$ for all $k \in \mathbb{N}$ and, $Sg(x) \subseteq \overline{D} \cup B$. □

Chapter Three

Residual transfer in fuzzy rings

In this chapter, unless otherwise specified, $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a complete meet-distributive residuated lattice (See, Definition 1.2.13) and $\mathcal{A} := (A;\ +,\ \cdot;\ -;\ 0)$ is a unital ring with unity 1. The binary operation $\cdot$ will be denoted by juxtaposition.

In Section 3.1, we define a residuated lattice structure $\mathcal{F}id(\mathcal{A}, L)$ on the set of L-fuzzy ideals of $\mathcal{A}$ which is both an extension of $\mathcal{L}$ and the residuated lattice $\mathcal{I}d(\mathcal{A})$ on the set of ideals of $\mathcal{A}$. Furthermore, we show that $\mathcal{F}id(\mathcal{A}, L)$ is commutative (resp., a Brouwerian algebra) if and only if so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$. In Section 3.2, we characterize prime elements of $\mathcal{F}id(\mathcal{A}, L)$ and investigate some embedding properties of the lattice of its filters. In Section 3.3, we show that $\mathcal{F}id(\mathcal{A}, L)$ is a Boolean algebra if and only if so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$. Furthermore, we introduce the concept of Łukasiewicz rings under $\mathcal{L}$ and establish its connection with rings whose L-fuzzy ideals form an MV-algebra.

3.1 Residuated lattice of L-fuzzy ideals of a ring

3.1.1 Lattice of L-fuzzy ideals

Remark 3.1.1. *The complete residuated lattice of ideals of $\mathcal{A}$ is defined by:*

$$\mathcal{I}d(\mathcal{A}) := \big(Id(\mathcal{A});\ \cap,\ +,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ \{0\},\ A\big);$$

where, for any $I, J \in Id(\mathcal{A})$, $I + J = \{x + y :\ x \in I$ and $y \in J\}$,

$$I \odot J := IJ = \{\sum_{i=1}^{n} x_i y_i :\ x_1, \ldots, x_n \in I \text{ and } y_1, \ldots, y_n \in J\},$$
$$I \rightarrow J = \{x \in A :\ xI \subseteq J\} \text{ and } I \rightsquigarrow J = \{x \in A :\ Ix \subseteq J\}.$$

Recall that for any $\{I_\lambda\}_{\lambda\in\Lambda} \subseteq Id(\mathcal{A})$, we have

$$\bigsqcup_{\lambda\in\Lambda} I_\lambda = \{\sum_{\lambda\in\Omega} a_\lambda :\ \Omega \text{ is a finite subset of } \Lambda \text{ and } a_\lambda \in I_\lambda \text{ for all } \lambda\in\Omega\}.$$

Remark 3.1.2. *According to Definition 1.3.6 and **(c)** of Remark 1.1.7, an L-fuzzy subset μ of A is an L-fuzzy ideal of $\mathcal{A}$ if and only if $\mu(0) = 1$, $\mu(x-y) \geq \mu(x)\wedge\mu(y)$ and $\mu(xy) \geq \mu(x)\vee\mu(y)$ for all $x, y \in A$.*

The set $Fid(\mathcal{A}, L)$ of L-fuzzy ideals of $\mathcal{A}$ forms a lattice $\mathbb{F}id(\mathcal{A}, L) := \big(Fid(\mathcal{A}, L);\ \wedge,\ +;\ \chi_0,\ \underline{1}\big)$; where, for any $\mu, \nu \in Fid(\mathcal{A}, L)$, $\mu+\nu := Fidg(\mu\vee\nu)$ is the L-fuzzy ideal of $\mathcal{A}$ generated by $\mu\vee\nu$. U.M. Swamy and K.L.N. Swamy showed in [37] that: for any $\mu, \nu \in Fid(\mathcal{A}, L)$, $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fid(\mathcal{A}, L)$ and $x \in A$,

$$\big(\mu+\nu\big)(x) = \bigvee\{\mu(a)\wedge\nu(b) :\ x = a+b\}$$
$$\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda\big)(x) = \bigvee\{\bigwedge_{\lambda\in\Omega}\mu_\lambda(a_\lambda) :\ \Omega \text{ is a finite subset of } \Lambda \text{ and } x = \sum_{\lambda\in\Omega} a_\lambda\}.$$

According to Subsection 2.1.2, atoms of $\mathbb{F}id(\mathcal{A}, L)$ are exactly of the form $(Idg(a)_r)_*$; where, r and $Idg(a)$ are atoms of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, respectively. Co-atoms of $\mathbb{F}id(\mathcal{A}, L)$ are exactly of the form M^r; where, r and M are co-atoms of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, respectively. If $\mathcal{L}$ is algebraic, then $\mathbb{F}id(\mathcal{A}, L)$ is algebraic; moreover, its compact elements are exactly finite suprema of compact principal L-fuzzy ideals of $\mathcal{A}$. I. Jahan showed that $\mathbb{F}id(\mathcal{A}, L)$ is modular (See, [19], Theorem 3.5.).

3.1.2 Residuated lattice of L-fuzzy ideals

This subsection outlines the construction of the residuated lattice $\mathcal{F}id(\mathcal{A}, L)$ of L-fuzzy ideals of $\mathcal{A}$.

Definition 3.1.3. *For any $\mu, \nu \in Fu(A, L)$, $\mu\circ\nu$ denotes the L-fuzzy subset of A defined by: $\big(\mu\circ\nu\big)(x) = \bigvee\{\mu(a)\ominus\nu(b) :\ x = ab\}$ for all $x \in A$.*

Lemma 3.1.4. *The binary operation $\circ$ on $Fu(A, L)$ is associative.*

Proof. Let $\mu, \nu, \delta \in Fu(A, L)$ and $x \in A$. Let $a, b \in A$ such that $x = ab$. For any $u_b, v_b \in A$ such that $b = u_b v_b$, we have $x = a(u_b v_b) = (au_b)v_b$ and,

$$\begin{aligned}\mu(a)\ominus\big(\nu(u_b)\ominus\delta(v_b)\big) &= \big(\mu(a)\ominus\nu(u_b)\big)\ominus\delta(v_b)\\ &\leq \big(\mu\circ\nu\big)(au_b)\ominus\delta(v_b)\\ &\leq \big((\mu\circ\nu)\circ\delta\big)(x).\end{aligned}$$

Thus, $\mu(a)\ominus\big(\nu\circ\delta\big)(b) \leq \big((\mu\circ\nu)\circ\delta\big)(x)$. So, $\big(\mu\circ(\nu\circ\delta)\big)(x) \leq \big((\mu\circ\nu)\circ\delta\big)(x)$ and, $\big((\mu\circ\nu)\circ\delta\big)(x) \leq \big(\mu\circ(\nu\circ\delta)\big)(x)$ by similar arguments. It follows that $\big(\mu\circ(\nu\circ\delta)\big)(x) = \big((\mu\circ\nu)\circ\delta\big)(x)$. Hence, $\mu\circ(\nu\circ\delta) = (\mu\circ\nu)\circ\delta$. Therefore, $\circ$ is associative. □

Proposition 3.1.5. *For any $n \geq 2$, $\mu_1, \mu_2, \ldots, \mu_n \in Fu(A, L)$ and $x \in A$,*

$\big(\mu_1 \circ \mu_2 \circ \ldots \circ \mu_n\big)(x) = \bigvee\{\mu_1(a_1) \ominus \mu_2(a_2) \ominus \ldots \ominus \mu_n(a_n) : \ x = a_1 a_2 \ldots a_n\}.$

Proof. We proceed by induction on the number of L-fuzzy subsets of A.
If $n = 2$, then the result follows from Definition 3.1.3.
Now, let $n \geq 2$ such that for any $\mu_1, \ldots, \mu_n \in Fu(A, L)$ and $x \in A$, we have $\big(\mu_1 \circ \mu_2 \circ \ldots \circ \mu_n\big)(x) = \bigvee\{\mu_1(a_1) \ominus \mu_2(a_2) \ominus \ldots \ominus \mu_n(a_n) : \ x = a_1 a_2 \ldots a_n\}$.
Let $\mu_1, \ldots, \mu_{n+1} \in Fu(A, L)$ and $x \in A$. Let $a, b \in A$ such that $x = ab$.
For any $a_1, \ldots, a_n \in A$ such that $a = a_1 \ldots a_n$, we have $x = a_1 \ldots a_n b$ and $\big(\mu_1(a_1) \ominus \ldots \ominus \mu_n(a_n)\big) \ominus \mu_{n+1}(b) = \mu_1(a_1) \ominus \ldots \ominus \mu_n(a_n) \ominus \mu_{n+1}(b)$; thus, $\big(\mu_1(a_1) \ominus \ldots \ominus \mu_n(a_n)\big) \ominus \mu_{n+1}(b) \leq \bigvee\limits_{x = a_1 \ldots a_{n+1}} \mu_1(a_1) \ominus \ldots \ominus \mu_{n+1}(a_{n+1})$. So, $\big(\mu_1 \circ \ldots \circ \mu_n\big)(a) \ominus \mu_{n+1}(b) \leq \bigvee\limits_{x = a_1 \ldots a_{n+1}} \mu_1(a_1) \ominus \ldots \ominus \mu_{n+1}(a_{n+1})$ and,

$$\begin{aligned} \big(\mu_1 \circ \ldots \circ \mu_{n+1}\big)(x) &= \big[\big(\mu_1 \circ \ldots \circ \mu_n\big) \circ \mu_{n+1}\big](x) \\ &\leq \bigvee_{x = a_1 \ldots a_{n+1}} \mu_1(a_1) \ominus \ldots \ominus \mu_{n+1}(a_{n+1}). \end{aligned}$$

For any $a_1, \ldots, a_{n+1} \in A$ such that $x = a_1 \ldots a_{n+1}$, we have $x = (a_1 \ldots a_n)a_{n+1}$; thus,

$$\begin{aligned} \mu_1(a_1) \ominus \ldots \ominus \mu_{n+1}(a_{n+1}) &= \big(\mu_1(a_1) \ominus \ldots \ominus \mu_n(a_n)\big) \ominus \mu_{n+1}(a_{n+1}) \\ &\leq \big[\big(\mu_1 \circ \ldots \circ \mu_n\big) \circ \mu_{n+1}\big](x) \\ &= \big(\mu_1 \circ \ldots \circ \mu_{n+1}\big)(x). \end{aligned}$$

So, $\bigvee\limits_{x = a_1 a_2 \ldots a_{n+1}} \mu_1(a_1) \ominus \ldots \ominus \mu_{n+1}(a_{n+1}) \leq \big(\mu_1 \circ \ldots \circ \mu_{n+1}\big)(x)$. It follows that $\big(\mu_1 \circ \ldots \circ \mu_{n+1}\big)(x) = \bigvee\limits_{x = a_1 a_2 \ldots a_{n+1}} \mu_1(a_1) \ominus \ldots \ominus \mu_{n+1}(a_{n+1})$.
Hence, the desired result follows. □

Proposition 3.1.6. *Let $r, s \in L$, $x \in A$, $I \in Id(\mathcal{A})$ and $\mu, \nu \in Fid(\mathcal{A}, L)$.*
(1) $\mu \circ \nu \leq \mu \ominus \nu$.
(2) $x_r \circ \mu \leq \nu$ *if and only if* $(x_r)_* \circ \mu \leq \nu$.
(3) $x_r \circ (I_s)_* = (xI)_{r \ominus s} \vee 0_r$.

Proof. **(1)** Let $y \in A$. For any $a, b \in A$ such that $y = ab$, we have

$$\mu(a) \ominus \nu(b) \leq \mu(ab) \ominus \nu(ab) = \mu(y) \ominus \nu(y) = \big(\mu \ominus \nu\big)(y).$$

Thus, $\big(\mu \circ \nu\big)(y) \leq \big(\mu \ominus \nu\big)(y)$. So, $\mu \circ \nu \leq \mu \ominus \nu$.
(2) Assume that $x_r \circ \mu \leq \nu$. Let $a \neq 0$ in A. For any $u, v \in A$ such that $a = uv$, we have $u \neq 0$; thus,

$$(x_r)_*(u) \ominus \mu(v) = x_r(u) \ominus \mu(v) \leq \big(x_r \circ \mu\big)(a) \leq \nu(a).$$

So, $\big((x_r)_* \circ \mu\big)(a) \leq \nu(a)$. Hence, $(x_r)_* \circ \mu \leq \nu$. The converse is obvious, since $\circ$ is order-preserving.
(3) $\big(x_r \circ (I_s)_*\big)(0) = r$, since

$$r = r \ominus 1 = x_r(x) \ominus (I_s)_*(0) \leq \big(x_r \circ (I_s)_*\big)(0) \leq r.$$

Let $a \in xI \setminus \{0\}$. For any $v \in A$ such that $a = xv$, we have

$$r \ominus (I_s)_*(v) = \begin{cases} r \ominus s & \text{if } v \in I, \\ r \ominus 0 = 0 & \text{if } v \notin I. \end{cases}.$$

Thus, $\big(x_r \circ (I_s)_*\big)(a) \leq r \ominus s = x_r(x) \ominus (I_s)_*(u) \leq \big(x_r \circ (I_s)_*\big)(a)$ for some $u \neq 0$ in I such that $a = xu$. So, $\big(x_r \circ (I_s)_*\big)(a) = r \ominus s$.
Now, let $a \notin xI$. For any $v \in A$ such that $a = xv$, we have $v \notin I$; thus, $r \ominus (I_s)_*(v) = r \ominus 0 = 0$. So, $\big(x_r \circ (I_s)_*\big)(a) = \bigvee\{0\} = 0$.
Hence, $x_r \circ (I_s)_* = (xI)_{r\ominus s} \vee 0_r$. □

For any L-fuzzy ideal μ of $\mathcal{A}$, $x \in A$ and $r \in L$, one can easily verify that $x_0 \circ \mu = \underline{0} = \mu \circ x_0$ and $0_r \circ \mu = 0_r = \mu \circ 0_r$.

Proposition 3.1.7. *Let $\mu, \nu \in Fid(\mathcal{A}, L)$. Then the L-fuzzy subset $\mu \otimes \nu$ of A, given for any $x \in A$ by*

$$\big(\mu \otimes \nu\big)(x) = \bigvee\{\bigwedge_{1\leq i\leq n} \mu(a_i) \ominus \nu(b_i) : \; x = \sum_{i=1}^{n} a_i b_i \text{ and } a_1, \dots, a_n \in A\},$$

is the smallest L-fuzzy ideal of $\mathcal{A}$ containing $\mu \circ \nu$; i.e., $Fidg(\mu \circ \nu) = \mu \otimes \nu$.

Proof. It is clear that $\mu \otimes \nu$ contains $\mu \circ \nu$. Next we show that $\mu \otimes \nu$ is an L-fuzzy ideal of $\mathcal{A}$.
We have $(\mu \otimes \nu)(0) = 1$, since $(\mu \otimes \nu)(0) \geq \mu(0) \ominus \nu(0) = 1 \ominus 1 = 1$.
Now, let $x, y \in A$. Set $X := \{(a_i, b_i)_{1\leq i\leq m+n} : \; x = \sum_{i=1}^{m} a_i b_i$ and $-y = \sum_{i=m+1}^{m+n} a_i b_i\}$ and $Y := \{(u_j, v_j)_{1\leq j\leq p} : \; x - y = \sum_{j=1}^{p} u_j v_j\}$. Then $X \subseteq Y$.
Furthermore, for any $(a_i, b_i)_{1\leq i\leq m+n} \in X$, we have

$$\begin{aligned}\big(\bigwedge_{1\leq i\leq m} \mu(a_i) \ominus \nu(b_i)\big) \wedge \big(\bigwedge_{m+1\leq i\leq m+n} \mu(a_i) \ominus \nu(b_i)\big) &= \bigwedge_{1\leq i\leq m+n} \mu(a_i) \ominus \nu(b_i) \\ &\leq \big(\mu \otimes \nu\big)(x - y).\end{aligned}$$

Thus, $\big(\mu \otimes \nu\big)(x) \wedge \big(\mu \otimes \nu\big)(y) \leq \big(\mu \otimes \nu\big)(x - y)$. For any $a_1, b_1, \dots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{m} a_i b_i$, we have $xy = \sum_{i=1}^{m} a_i (b_i y)$ and,

$$\bigwedge_{1\leq i\leq m} \mu(a_i) \ominus \nu(b_i) \leq \bigwedge_{1\leq i\leq m} \mu(a_i) \ominus \nu(b_i y) \leq \big(\mu \otimes \nu\big)(xy).$$

Thus, $\big(\mu \otimes \nu\big)(x) \leq \big(\mu \otimes \nu\big)(xy)$. Similarly, we obtain $\big(\mu \otimes \nu\big)(y) \leq \big(\mu \otimes \nu\big)(xy)$. So, $\big(\mu \otimes \nu\big)(xy) \geq \big(\mu \otimes \nu\big)(x) \vee \big(\mu \otimes \nu\big)(y)$. Hence, $\mu \otimes \nu$ is an L-fuzzy ideal of $\mathcal{A}$.
Finally, let δ be an L-fuzzy ideal of $\mathcal{A}$ containing $\mu \circ \nu$. For any $a_1, b_1, \dots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$, we have

$$\bigwedge_{1\leq i\leq n} \mu(a_i)\ominus\nu(b_i) \leq \bigwedge_{1\leq i\leq n} \big(\mu\circ\nu\big)(a_ib_i) \leq \bigwedge_{1\leq i\leq n} \delta(a_ib_i) \leq \delta\big(\sum_{i=1}^{n} a_ib_i\big) = \delta(x).$$

Thus, $\big(\mu\otimes\nu\big)(x) \leq \delta(x)$. Hence, $\mu\otimes\nu \leq \delta$. Therefore, $\mu\otimes\nu$ is the smallest L-fuzzy ideal of $\mathcal{A}$ containing $\mu\circ\nu$. □

Remark 3.1.8. *For any $\mu_1,\ldots,\mu_n \in Fid(\mathcal{A},L)$ and $x\in A$, we have*

$$Fidg(\mu_1\circ\ldots\circ\mu_n)(x) = \bigvee\{\bigwedge_{1\leq j\leq p} \mu_1(a_j^1)\ominus\ldots\ominus\mu_n(a_j^n) :\ x=\sum_{j=1}^{p} a_j^1\ldots a_j^n\}.$$

Proposition 3.1.9. *The binary operation $\otimes$ on $Fid(\mathcal{A},L)$ is associative.*

Proof. Let $\mu,\nu,\delta \in Fid(\mathcal{A},L)$. Let $x\in A$. Let $a_1,b_1,\ldots,a_n,b_n \in A$ such that $x=\sum_{i=1}^{n} a_ib_i$. Let $1\leq i\leq n$. For any $c_{i_1},d_{i_1},\ldots,c_{i_p},d_{i_p}\in A$ such that $b_i=\sum_{j=1}^{p} c_{i_j}d_{i_j}$, we have for each $1\leq k\leq p$,

$$\begin{aligned}
\mu(a_i)\ominus\big(\bigwedge_{1\leq j\leq p}\nu(c_{i_j})\ominus\delta(d_{i_j})\big) &\leq \mu(a_i)\ominus\big(\nu(c_{i_k})\ominus\delta(d_{i_k})\big)\\
&= \big(\mu(a_i)\ominus\nu(c_{i_k})\big)\ominus\delta(d_{i_k})\\
&\leq \big(\mu\otimes\nu\big)(a_ic_{i_k})\ominus\delta(d_{i_k})\\
&\leq \big((\mu\otimes\nu)\otimes\delta\big)\big((a_ic_{i_k})d_{i_k}\big)\\
&= \big((\mu\otimes\nu)\otimes\delta\big)\big(a_i(c_{i_k}d_{i_k})\big);
\end{aligned}$$

thus,

$$\begin{aligned}
\mu(a_i)\ominus\big(\bigwedge_{1\leq j\leq p}\nu(c_{i_j})\ominus\delta(d_{i_j})\big) &\leq \bigwedge_{1\leq j\leq p}\big((\mu\otimes\nu)\otimes\delta\big)\big(a_i(c_{i_j}d_{i_j})\big)\\
&\leq \big((\mu\otimes\nu)\otimes\delta\big)\big(\sum_{j=1}^{p} a_i(c_{i_j}d_{i_j})\big)\\
&= \big((\mu\otimes\nu)\otimes\delta\big)\big(a_i\sum_{j=1}^{p} c_{i_j}d_{i_j}\big)\\
&= \big((\mu\otimes\nu)\otimes\delta\big)(a_ib_i).
\end{aligned}$$

So, $\mu(a_i)\ominus\big(\nu\otimes\delta\big)(b_i) \leq \big((\mu\otimes\nu)\otimes\delta\big)(a_ib_i)$. It follows that

$$\bigwedge_{1\leq i\leq n}\mu(a_i)\ominus\big(\nu\otimes\delta\big)(b_i) \leq \bigwedge_{1\leq i\leq n}\big((\mu\otimes\nu)\otimes\delta\big)(a_ib_i) \leq \big((\mu\otimes\nu)\otimes\delta\big)(x).$$

Hence, $\big(\mu\otimes(\nu\otimes\delta)\big)(x) \leq \big((\mu\otimes\nu)\otimes\delta\big)(x)$ and, $\big((\mu\otimes\nu)\otimes\delta\big)(x) \leq \big(\mu\otimes(\nu\otimes\delta)\big)(x)$ by similar arguments. Therefore, $\mu\otimes(\nu\otimes\delta) = (\mu\otimes\nu)\otimes\delta$. □

Corollary 3.1.10. $\mathbf{F}id(\mathcal{A},L) := \big(Fid(\mathcal{A},L);\ \otimes,\ \underline{1}\big)$ *is a monoid.*

Proof. Since $\otimes$ is associative by Proposition 3.1.9, it suffices to show that $\underline{1}$ is the unity of $Fid(\mathcal{A}, L)$. So, let μ be an L-fuzzy ideal of $\mathcal{A}$. Let $x \in A$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$, we have

$$\bigwedge_{1\le i\le n} \mu(a_i) \ominus \underline{1}(b_i) = \bigwedge_{1\le i\le n} \mu(a_i) \ominus 1 = \bigwedge_{1\le i\le n} \mu(a_i) \le \bigwedge_{1\le i\le n} \mu(a_i b_i) \le \mu(x).$$

Thus, $\big(\mu\otimes\underline{1}\big)(x) \le \mu(x)$. Furthermore, $\big(\mu\otimes\underline{1}\big)(x) \ge \mu(x)\ominus\underline{1}(1) = \mu(x)\ominus 1 = \mu(x)$. So, $\big(\mu \otimes \underline{1}\big)(x) = \mu(x)$. Hence, $\mu \otimes \underline{1} = \mu$; and, $\underline{1} \otimes \mu = \mu$ by similar arguments. Therefore, $\underline{1}$ is the unity of $Fid(\mathcal{A}, L)$. $\square$

Proposition 3.1.11. *For any $n \ge 2$, we have $\mu_1\otimes\ldots\otimes\mu_n = Fidg(\mu_1\circ\ldots\circ\mu_n)$ for all $\mu_1, \ldots, \mu_n \in Fid(\mathcal{A}, L)$.*

Proof. We proceed by induction on the number of L-fuzzy ideals of $\mathcal{A}$.
If $n = 2$, then the result follows from Proposition 3.1.7.
Now, let $n \ge 2$ such that $\mu_1 \otimes \ldots \otimes \mu_n = Fidg(\mu_1 \circ \ldots \circ \mu_n)$ for all $\mu_1, \ldots, \mu_n \in Fid(\mathcal{A}, L)$. Let $\mu_1, \ldots, \mu_{n+1} \in Fid(\mathcal{A}, L)$. Since $\circ$ is order-preserving, we have

$$\begin{aligned} \mu_1 \otimes \ldots \otimes \mu_{n+1} &= Fidg\big[(\mu_1 \otimes \ldots \otimes \mu_n) \circ \mu_{n+1}\big] \\ &= Fidg\big[Fidg(\mu_1 \circ \ldots \circ \mu_n) \circ \mu_{n+1}\big] \\ &\ge Fidg(\mu_1 \circ \ldots \circ \mu_{n+1}). \end{aligned}$$

Finally, we show that $\mu_1 \otimes \ldots \otimes \mu_{n+1} \le Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})$. So, let $x \in A$. Let $r \in L$ such that $x \in Idg\big[U\big((\mu_1 \otimes \ldots \otimes \mu_n) \circ \mu_{n+1}, r\big)\big]$. There are $a_1, b_1, \ldots, a_p, b_p \in A$ and $u_1, \ldots, u_p \in U\big((\mu_1 \otimes \ldots \otimes \mu_n) \circ \mu_{n+1}, r\big)$ such that $x = \sum_{k=1}^{p} a_k u_k b_k$. Since $r \le \big[(\mu_1 \otimes \ldots \otimes \mu_n) \circ \mu_{n+1}\big](u_k)$ for all $1 \le k \le p$, we have $r \le \bigwedge_{1\le k\le p} \big[(\mu_1 \otimes \ldots \otimes \mu_n) \circ \mu_{n+1}\big](u_k)$.
Let $v_1, w_1, \ldots, v_p, w_p \in A$ such that $u_1 = v_1 w_1, \ldots, u_p = v_p w_p$. Let $1 \le k \le p$. For any $z_{k1}^1, \ldots, z_{k1}^n, \ldots, z_{kq}^1, \ldots, z_{kq}^n \in A$ such that $v_k = \sum_{j=1}^{q} z_{kj}^1 \ldots z_{kj}^n$, we have

$$\begin{aligned} &\Big(\bigwedge_{1\le j\le q} \mu_1(z_{kj}^1) \ominus \ldots \ominus \mu_n(z_{kj}^n)\Big) \ominus \mu_{n+1}(w_k) \\ &\qquad \le \bigwedge_{1\le j\le q} \mu_1(z_{kj}^1) \ominus \ldots \ominus \mu_n(z_{kj}^n) \ominus \mu_{n+1}(w_k) \\ &\qquad \le Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})\Big[\sum_{j=1}^{q} z_{kj}^1 \ldots z_{kj}^n w_k\Big] \\ &\qquad = Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})\Big[\Big(\sum_{j=1}^{q} z_{kj}^1 \ldots z_{kj}^n\Big) w_k\Big] \\ &\qquad = Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(v_k w_k) \\ &\qquad = Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(u_k). \end{aligned}$$

Thus, $\big(\mu_1 \otimes \ldots \otimes \mu_n\big)(v_k) \ominus \mu_{n+1}(w_k) \leq Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(u_k)$. So,

$$\begin{aligned} \bigwedge_{1\leq k\leq p} \big(\mu_1 \otimes \ldots \otimes \mu_n\big)(v_k) \ominus \mu_{n+1}(w_k) &\leq \bigwedge_{1\leq k\leq p} Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(u_k) \\ &\leq \bigwedge_{1\leq k\leq p} Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(a_k u_k b_k) \\ &\leq Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(\sum_{k=1}^{p} a_k u_k b_k) \\ &= Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(x). \end{aligned}$$

Consequently, $\bigwedge_{1\leq k\leq p} \big[(\mu_1 \otimes \ldots \otimes \mu_n) \circ \mu_{n+1}\big](u_k) \leq Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(x)$ and, $r \leq Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(x)$. Thus,

$$\begin{aligned} \big(\mu_1 \otimes \ldots \otimes \mu_{n+1}\big)(x) &= Fidg\big[(\mu_1 \otimes \ldots \otimes \mu_n) \circ \mu_{n+1}\big](x) \\ &\leq Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})(x). \end{aligned}$$

So, $\mu_1 \otimes \ldots \otimes \mu_{n+1} \leq Fidg(\mu_1 \circ \ldots \circ \mu_{n+1})$ and,

$$\mu_1 \otimes \ldots \otimes \mu_{n+1} = Fidg(\mu_1 \circ \ldots \circ \mu_{n+1}).$$

Hence, the desired result follows. □

Definition 3.1.12. *For any L-fuzzy subsets μ and ν of A, $\mu \hookrightarrow \nu$ and $\mu \looparrowright \nu$ denote the L-fuzzy subsets of A defined for any $x \in A$ by:*

$$\begin{aligned} \big(\mu \hookrightarrow \nu\big)(x) &= \bigvee\{r \in L: \ x_r \circ \mu \leq \nu\} \\ \big(\mu \looparrowright \nu\big)(x) &= \bigvee\{r \in L: \ \mu \circ x_r \leq \nu\}. \end{aligned}$$

Proposition 3.1.13. *Let μ and ν be two L-fuzzy ideals of $\mathcal{A}$. Then $\mu \hookrightarrow \nu$ and $\mu \looparrowright \nu$ are L-fuzzy ideals of $\mathcal{A}$.*

Proof. Since $0_1 \circ \mu = \chi_0 \leq \nu$, we have $1 \leq \big(\mu \hookrightarrow \nu\big)(0)$ and, $\big(\mu \hookrightarrow \nu\big)(0) = 1$. Now, let $x, y \in A$. Let $r, s \in L$ such that $x_r \circ \mu \leq \nu$ and $y_s \circ \mu \leq \nu$. Let $a \in A$. Let $b, c \in A$ such that $a = bc$.

- If $b \neq x - y$, then $(x-y)_{r\wedge s}(b) \ominus \mu(c) = 0 \ominus \mu(c) = 0 \leq \nu(a)$.
- If $b = x - y$, then

$$\begin{aligned} (x-y)_{r\wedge s}(b) \ominus \mu(c) &= (r \wedge s) \ominus \mu(c) \\ &\leq \big(r \ominus \mu(c)\big) \wedge \big(s \ominus \mu(c)\big) \\ &= \big(x_r(x) \ominus \mu(c)\big) \wedge \big(y_s(y) \ominus \mu(c)\big) \\ &\leq \big(x_r \circ \mu\big)(xc) \wedge \big(y_s \circ \mu\big)(yc) \\ &\leq \nu(xc) \wedge \nu(yc) \\ &\leq \nu(xc - yc) = \nu(a). \end{aligned}$$

Thus, $\big((x-y)_{r\wedge s}\circ\mu\big)(a)\leq\nu(a)$. So,

$$(x-y)_{r\wedge s}\circ\mu\leq\nu \text{ and, } r\wedge s\leq(\mu\hookrightarrow\nu)(x-y).$$

It follows that $(\mu\hookrightarrow\nu)(x)\wedge(\mu\hookrightarrow\nu)(y)\leq(\mu\hookrightarrow\nu)(x-y)$. Now, let $r\in L$ such that $x_r\circ\mu\leq\nu$. Let $a\in A$. Let $b,c\in A$ such that $a=bc$.

- If $b\neq xy$, then $(xy)_r(b)\ominus\mu(c)=0\ominus\mu(c)=0\leq\nu(a)$.
- If $b=xy$, then

$$\begin{aligned}(xy)_r(b)\ominus\mu(c) &= r\ominus\mu(c)\\ &= x_r(x)\ominus\mu(c)\\ &\leq x_r(x)\ominus\mu(yc)\\ &\leq (x_r\circ\mu)\big(x(yc)\big)\\ &\leq \nu\big(x(yc)\big)=\nu(a).\end{aligned}$$

Thus, $\big((xy)_r\circ\mu\big)(a)\leq\nu(a)$. So, $(xy)_r\circ\mu\leq\nu$ and, $r\leq\big(\mu\hookrightarrow\nu\big)(xy)$. It follows that $\big(\mu\hookrightarrow\nu\big)(x)\leq\big(\mu\hookrightarrow\nu\big)(xy)$.
Now, let $r\in L$ such that $y_r\circ\mu\leq\nu$. Let $a\in A$. Let $b,c\in A$ such that $a=bc$.

- If $b\neq xy$, then $(xy)_r(b)\ominus\mu(c)=0\ominus\mu(c)=0\leq\nu(a)$.
- Suppose that $b=xy$. For any $z\in U\big(\mu,\mu(c)\big)$, we have

$$\begin{aligned}\nu(yz) &\geq \big(y_r\circ\mu\big)(yz)\\ &\geq y_r(y)\ominus\mu(z)\\ &= r\ominus\mu(z)\\ &\geq r\ominus\mu(c)\end{aligned}$$

and, $yz\in U\big(\nu,r\ominus\mu(c)\big)$. Thus,

$$yU\big(\mu,\mu(c)\big)\subseteq U\big(\nu,r\ominus\mu(c)\big) \text{ and, } y\in U\big(\mu,\mu(c)\big)\rightarrow U\big(\nu,r\ominus\mu(c)\big).$$

So, $xy\in U\big(\mu,\mu(c)\big)\rightarrow U\big(\nu,r\ominus\mu(c)\big)$ and, $xyU\big(\mu,\mu(c)\big)\subseteq U\big(\nu,r\ominus\mu(c)\big)$.
Since $a=xyc\in U\big(\nu,r\ominus\mu(c)\big)$, we have $(xy)_r(b)\ominus\mu(c)=r\ominus\mu(c)\leq\nu(a)$.
Thus, $\big((xy)_r\circ\mu\big)(a)\leq\nu(a)$. So, $(xy)_r\circ\mu\leq\nu$ and, $r\leq\big(\mu\hookrightarrow\nu\big)(xy)$. It follows that $\big(\mu\hookrightarrow\nu\big)(y)\leq\big(\mu\hookrightarrow\nu\big)(xy)$.
Consequently, $\big(\mu\hookrightarrow\nu\big)(x)\vee\big(\mu\hookrightarrow\nu\big)(y)\leq\big(\mu\hookrightarrow\nu\big)(xy)$.
Hence, $\mu\hookrightarrow\nu$ is an L-fuzzy ideal of $\mathcal{A}$. A similar reasoning shows that $\mu\looparrowright\nu$ is an L-fuzzy ideal of $\mathcal{A}$. □

Theorem 3.1.14. $\mathcal{F}id(\mathcal{A},L):=\big(Fid(\mathcal{A},L);\ \wedge,\ +,\ \otimes,\ \hookrightarrow,\ \looparrowright;\ \chi_0,\ \underline{1}\big)$ *is a complete residuated lattice.*

Proof. Since $\mathbb{F}id(\mathcal{A},L)$ is a complete lattice and $\mathbf{F}id(\mathcal{A},L)$ is a monoid, it suffices to show that: for any $\mu,\nu,\delta\in Fid(\mathcal{A},L)$, $\mu\otimes\nu\leq\delta$ iff $\mu\leq\nu\hookrightarrow\delta$ iff $\nu\leq\mu\looparrowright\delta$. So, let $\mu,\nu,\delta\in Fid(\mathcal{A},L)$.

Assume that $\mu \otimes \nu \leq \delta$. Let $x \in A$. Let $a \in A$. For any $v \in A$ such that $a = xv$, we have

$$x_{\mu(x)}(x) \ominus \nu(v) = \mu(x) \ominus \nu(v) \leq \big(\mu \otimes \nu\big)(a) \leq \delta(a).$$

Thus, $\big(x_{\mu(x)} \circ \nu\big)(a) \leq \delta(a)$. So, $x_{\mu(x)} \circ \nu \leq \delta$ and, $\mu(x) \leq \big(\nu \hookrightarrow \delta\big)(x)$. Hence, $\mu \leq \nu \hookrightarrow \delta$.

Conversely, assume that $\mu \leq \nu \hookrightarrow \delta$. Let $x \in A$. Let $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$. Let $1 \leq i \leq n$. For any $r_i \in L$ such that $(a_i)_{r_i} \circ \nu \leq \delta$, we have $\big(\mu(a_i) \wedge r_i\big) \ominus \nu(b_i) \leq r_i \ominus \nu(b_i) = (a_i)_{r_i}(a_i) \ominus \nu(b_i) \leq \big((a_i)_{r_i} \circ \nu\big)(a_i b_i) \leq \delta(a_i b_i)$. Since $\mathcal{L}$ is meet-distributive, we have

$$\mu(a_i) \ominus \nu(b_i) = \big(\mu(a_i) \wedge \big(\nu \hookrightarrow \delta\big)(a_i)\big) \ominus \nu(b_i) \leq \delta(a_i b_i).$$

Thus,

$$\bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \nu(b_i) \leq \bigwedge_{1 \leq i \leq n} \delta(a_i b_i) \leq \delta(x).$$

So, $(\mu \otimes \nu)(x) \leq \delta(x)$. It follows that $\mu \otimes \nu \leq \delta$.

Hence, $\mu \otimes \nu \leq \delta$ iff $\mu \leq \nu \hookrightarrow \delta$. A similar reasoning shows that: $\mu \otimes \nu \leq \delta$ iff $\nu \leq \mu \looparrowright \delta$. $\square$

3.1.3 Embeddings

In this subsection, we embed $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ into $\mathcal{F}id(\mathcal{A}, L)$.

Proposition 3.1.15. *Let $I, J \in Id(\mathcal{A})$ and $r, s \in L$. Then the following hold:*
(1) $(I_r)_* \otimes (J_s)_* = \big((I \odot J)_{r \ominus s}\big)_*$.
(2) $I^r \otimes J^s = (I \odot J)^{r \ominus s} + (I_s)_* + (J_r)_*$.
(3) $(I_r)_* + (J_s)_* = \big[(I + J \setminus I \cup J)_{r \wedge s} \vee (I \setminus J)_r \vee (J \setminus I)_s \vee (I \cap J \setminus \{0\})_{r \vee s}\big]_*$.

Proof. **(1)** Let $x \in I \odot J \setminus \{0\}$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$, there is $1 \leq i_0 \leq n$ such that $a_{i_0} \neq 0$ and $b_{i_0} \neq 0$; thus, $\bigwedge_{1 \leq i \leq n} (I_r)_*(a_i) \ominus (J_s)_*(b_i) \leq (I_r)_*(a_{i_0}) \ominus (J_s)_*(b_{i_0}) \leq r \ominus s$. So, $\big((I_r)_* \otimes (J_s)_*\big)(x) \leq r \ominus s$. Since there are $a_1, \ldots, a_n \in I \setminus \{0\}$ and $b_1, \ldots, b_n \in J \setminus \{0\}$ such that $x = \sum_{i=1}^{n} a_i b_i$, we have $r \ominus s = \bigwedge_{1 \leq i \leq n} (I_r)_*(a_i) \ominus (J_s)_*(b_i) \leq \big((I_r)_* \otimes (J_s)_*\big)(x)$ and, $\big((I_r)_* \otimes (J_s)_*\big)(x) = r \ominus s$.

Now, let $x \notin I \odot J$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$, there is $1 \leq i_0 \leq n$ such that $a_{i_0} \notin I$ or $b_{i_0} \notin J$; *i.e.*, $(I_r)_*(a_{i_0}) = 0$ or $(J_s)_*(b_{i_0}) = 0$; thus, $\bigwedge_{1 \leq i \leq n} (I_r)_*(a_i) \ominus (J_s)_*(b_i) \leq (I_r)_*(a_{i_0}) \ominus (J_s)_*(b_{i_0}) = 0$ and,

$\bigwedge_{1\le i\le n} (I_r)_*(a_i) \ominus (J_s)_*(b_i) = 0$. So, $\big((I_r)_* \otimes (J_s)_*\big)(x) = \bigvee\{0\} = 0$.
Hence, $(I_r)_* \otimes (J_s)_* = \big((I \odot J)_{r\ominus s}\big)_*$.
(2) We first show that $I^r = I_1 + (A_r)_*$. For any $x \in I$, we have $\big(I_1 + (A_r)_*\big)(x) \ge I_1(x) = 1$ and, $\big(I_1 + (A_r)_*\big)(x) = 1$. Now, let $x \notin I$. Let $a, b \in A$ such that $x = a + b$. If $b = 0$, then $a \notin I$ and, $I_1(a) \wedge (A_r)_*(b) = 0 \wedge 1 = 0$. If $b \neq 0$, then $I_1(a) \wedge (A_r)_*(b) \le (A_r)_*(b) = r$. Thus, $r = (A_r)_*(x) \le \big(I_1 + (A_r)_*\big)(x) \le r$ and, $\big(I_1 + (A_r)_*\big)(x) = r$. So, $I^r = I_1 + (A_r)_*$. A similar reasoning shows that $J^s = J_1 + (A_s)_*$. Finally, we have

$$\begin{aligned} I^r \otimes J^s &= \big(I_1 + (A_r)_*\big) \otimes \big(J_1 + (A_s)_*\big) \\ &= (I \odot J)_{1\ominus 1} + \big((I \odot A)_{1\ominus s}\big)_* + \big((A \odot J)_{r\ominus 1}\big)_* + \big((A \odot A)_{r\ominus s}\big)_* \\ &= (I \odot J)_1 + (I_s)_* + (J_r)_* + (A_{r\ominus s})_* \\ &= (I \odot J)^{r\ominus s} + (I_s)_* + (J_r)_*. \end{aligned}$$

(3) • Let $x \notin I + J$. For any $a, b \in A$ such that $x = a + b$, we have $a \notin I$ or $b \notin J$; *i.e.*, $(I_r)_*(a) = 0$ or $(J_s)_*(b) = 0$; thus, $(I_r)_*(a) \wedge (J_s)_*(b) = 0$. So, $\big((I_r)_* + (J_s)_*\big)(x) = \bigvee\{0\} = 0$.
• Let $x \in I + J \setminus I \cup J$. For any $a, b \in A$ such that $x = a + b$, we have

$$(I_r)_*(a) \wedge (J_s)_*(b) = \begin{cases} r \wedge s & \text{if } a \in I \text{ and } b \in J, \\ 0 & \text{if } a \notin I \text{ or } b \notin J. \end{cases}.$$

Thus, $r \wedge s = (I_r)_*(u) \wedge (J_s)_*(v) \le \big((I_r)_* + (J_s)_*\big)(x) \le r \wedge s$ for some $u \in I \setminus \{0\}$ and $v \in J \setminus \{0\}$ such that $x = u + v$; so, $\big((I_r)_* + (J_s)_*\big)(x) = r \wedge s$.
• Let $x \in I \setminus J$. For any $a, b \in A$ such that $x = a + b$, we have

$$(I_r)_*(a) \wedge (J_s)_*(b) = \begin{cases} r \wedge (J_s)_*(b) & \text{if } a \in I \text{ and } b \in J, \\ 0 & \text{if } a \notin I \text{ or } b \notin J. \end{cases}.$$

Thus, $r = (I_r)_*(x) \wedge (J_s)_*(0) \le \big((I_r)_* + (J_s)_*\big)(x) \le r$ and, $\big((I_r)_* + (J_s)_*\big)(x) = r$. A similar reasoning shows that $\big((I_r)_* + (J_s)_*\big)(x) = s$ for all $x \in J \setminus I$.
• Let $x \in (I \cap J) \setminus \{0\}$. For any $a, b \in A$ such that $x = a + b$, we have $a \neq 0$ or $b \neq 0$; thus, $(I_r)_*(a) \wedge (J_s)_*(b) \le r \vee s$. So, $r \vee s = \big((I_r)_* \vee (J_s)_*\big)(x) \le \big((I_r)_* + (J_s)_*\big)(x) \le r \vee s$ and, $\big((I_r)_* + (J_s)_*\big)(x) = r \vee s$.
Hence, $(I_r)_* + (J_s)_* = \big[(I + J \setminus I \cup J)_{r\wedge s} \vee (I \setminus J)_r \vee (J \setminus I)_s \vee (I \cap J \setminus \{0\})_{r\vee s}\big]_*$. □

For any $I, J \in Id(\mathcal{A})$ and $r \in L$, one can easily verify that:
• $(I_r)_* + (J_r)_* = \big((I + J)_r\big)_*$ and $I_1 + (J_r)_* = \big(I_1 \vee (I + J)_r\big)_*$.
• If $r^2 = r$, then $I^r \otimes J^r = (I \odot J)^r$.

Proposition 3.1.16. *Let $r, s \in L$ and $I, J \in Id(\mathcal{A})$. Then*

$$\big((I \rightarrow J)_{r\twoheadrightarrow s}\big)_* \le (I_r)_* \hookrightarrow (J_s)_* \text{ and } \big((I \rightsquigarrow J)_{r\multimap s}\big)_* \le (I_r)_* \looparrowright (J_s)_*.$$

Proof. Since $\big((I \to J)_{r\twoheadrightarrow s}\big)_* \otimes (I_r)_* = \big[\big((I \to J) \odot I\big)_{(r\twoheadrightarrow s)\ominus r}\big]_* \le (J_s)_*$, we have $\big((I \to J)_{r\twoheadrightarrow s}\big)_* \le (I_r)_* \hookrightarrow (J_s)_*$. Similarly, $\big((I \rightsquigarrow J)_{r\multimap s}\big)_* \le (I_r)_* \hookleftarrow (J_s)_*$. $\square$

As the following example shows, the previous inequalities are not necessarily equalities.

Example 3.1.17. *Consider the ring $\mathcal{Z}_6$. Let $L = \{0,\ a,\ b,\ c,\ d,\ 1\}$ be a lattice such that $0 < a, b < c < 1$ and $0 < b < d < 1$; where, a, b and c, d are incomparable, respectively. Define the binary operations $\ominus$ and $\twoheadrightarrow$ by the two tables below:*

$\ominus$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\twoheadrightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

.

Then $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow;\ 0,\ 1)$ is an MV-algebra (See, [34], Example 1.9.). For any $r \in \{a,\ c,\ 1\}$, we have $1_r \circ \big((\frac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_* \not\le \big((\frac{3\mathbb{Z}}{6\mathbb{Z}})_b\big)_*$, since

$$\big[1_r \circ \big((\tfrac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_*\big](2) = r \ominus \big((\tfrac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_*(2) = r \ominus a = a \not\le 0 = \big((\tfrac{3\mathbb{Z}}{6\mathbb{Z}})_b\big)_*(2).$$

For any $r \in \{b,\ d\}$, we have for each $x \in \mathbb{Z}_6 \setminus \{0\}$,

$$\big[1_r \circ \big((\tfrac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_*\big](x) = r \ominus \big((\tfrac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_*(x) = \begin{cases} r \ominus a & \text{if } x \in \{2,\ 4\}, \\ r \ominus 0 & \text{if } x \in \{1,\ 3,\ 5\}. \end{cases} = 0;$$

thus, $1_r \circ \big((\frac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_* \le \big((\frac{3\mathbb{Z}}{6\mathbb{Z}})_b\big)_*$. So, $\big[\big((\frac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_* \hookrightarrow \big((\frac{3\mathbb{Z}}{6\mathbb{Z}})_b\big)_*\big](1) = \bigvee\{0,\ b,\ d\} = d$. Since $\big((\frac{2\mathbb{Z}}{6\mathbb{Z}} \to \frac{3\mathbb{Z}}{6\mathbb{Z}})_{a\twoheadrightarrow b}\big)_*(1) = \big((\frac{3\mathbb{Z}}{6\mathbb{Z}})_d\big)_*(1) = 0$, we have $\big((\frac{2\mathbb{Z}}{6\mathbb{Z}} \to \frac{3\mathbb{Z}}{6\mathbb{Z}})_{a\twoheadrightarrow b}\big)_* < \big((\frac{2\mathbb{Z}}{6\mathbb{Z}})_a\big)_* \hookrightarrow \big((\frac{3\mathbb{Z}}{6\mathbb{Z}})_b\big)_*$.

Theorem 3.1.18. *The function $\phi : Id(\mathcal{A}) \to Fid(\mathcal{A}, L)$, given by $\phi(I) = I_1$ for all $I \in Id(\mathcal{A})$, is a complete residuated lattice embedding.*

Proof. By Proposition 2.1.3 and the fact that

$\phi(I \odot J) = (I \odot J)_1 = (I \odot J)_{1\ominus 1} = I_1 \otimes J_1 = \phi(I) \otimes \phi(J)$ for all $I, J \in Id(\mathcal{A})$,

we only have to prove that ϕ preserves the residues. So, let $I, J \in Id(\mathcal{A})$. Let $x \notin I \to J$. There is $a \in I$ such that $xa \notin J$. For any $r \in L$ such that $x_r \circ I_1 \le J_1$, we have $r = r \ominus 1 = x_r(x) \ominus I_1(a) \le (x_r \circ I_1)(xa) \le J_1(xa) = 0$ and, $r = 0$. Thus, $(I_1 \hookrightarrow J_1)(x) = \bigvee\{0\} = 0$. So, $I_1 \hookrightarrow J_1 \le (I \to J)_1$ and, $(I \to J)_1 = I_1 \hookrightarrow J_1$. Hence, $\phi(I \to J) = (I \to J)_1 = I_1 \hookrightarrow J_1 = \phi(I) \hookrightarrow \phi(J)$. A similar reasoning shows that $\phi(I \rightsquigarrow J) = \phi(I) \hookleftarrow \phi(J)$. Therefore, ϕ is a complete residuated lattice embedding of $\mathcal{I}d(\mathcal{A})$ into $\mathcal{F}id(\mathcal{A}, L)$. $\square$

Theorem 3.1.19. *The function $\psi : L \to Fid(\mathcal{A}, L)$, given by $\psi(r) = (\underline{r})_*$ for all $r \in L$, is a complete residuated lattice embedding.*

Proof. By Proposition 2.1.4 and the fact that for any $r, s \in L$, we have

$\psi(r \ominus s) = \big(A_{r\ominus s}\big)_* = \big((A \odot A)_{r\ominus s}\big)_* = (A_r)_* \otimes (A_s)_* = (\underline{r})_* \otimes (\underline{s})_* = \psi(r) \otimes \psi(s),$

we only have to prove that ψ preserves the residues. So, let $r, s \in L$. Let $x \neq 0$ in A. For any $t \in L$ such that $x_t \circ (\underline{r})_* \leq (\underline{s})_*$, we have

$$t \ominus r = x_t(x) \ominus (\underline{r})_*(1) \leq \big(x_t \circ (\underline{r})_*\big)(x) \leq (\underline{s})_*(x) = s \text{ and, } t \leq r \twoheadrightarrow s.$$

Thus, $\big((\underline{r})_* \hookrightarrow (\underline{s})_*\big)(x) \leq r \twoheadrightarrow s = (\underline{r \twoheadrightarrow s})_*(x)$. So, $(\underline{r})_* \hookrightarrow (\underline{s})_* \leq (\underline{r \twoheadrightarrow s})_*$ and, $(\underline{r \twoheadrightarrow s})_* = (\underline{r})_* \hookrightarrow (\underline{s})_*$. Hence,

$$\psi(r \twoheadrightarrow s) = (\underline{r \twoheadrightarrow s})_* = (\underline{r})_* \hookrightarrow (\underline{s})_* = \psi(r) \hookrightarrow \psi(s).$$

A similar reasoning shows that $\psi(r \multimap s) = \psi(r) \looparrowright \psi(s)$. Therefore, ψ is a complete residuated lattice embedding of $\mathcal{L}$ into $\mathcal{F}id(\mathcal{A}, L)$. □

Corollary 3.1.20. *1. $\mathcal{F}id(\mathcal{A}, L)$ is commutative iff so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$.*
2. $\mathcal{F}id(\mathcal{A}, L)$ is a Brouwerian algebra iff so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$.

Proof. **1.** Since $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ can be embedded into $\mathcal{F}id(\mathcal{A}, L)$, it is clear that they are commutative when $\mathcal{F}id(\mathcal{A}, L)$ is commutative.
Conversely, assume that $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ are commutative. Let $\mu, \nu \in Fid(\mathcal{A}, L)$. Let $x \in A$. Let $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$. For any $1 \leq i \leq n$, we have

$$a_i b_i \in U\big(\mu, \mu(a_i)\big) \odot U\big(\nu, \nu(b_i)\big) = U\big(\nu, \nu(b_i)\big) \odot U\big(\mu, \mu(a_i)\big);$$

thus, there are $v_{i1}, \ldots, v_{im_i} \in U\big(\nu, \nu(b_i)\big)$ and $u_{i1}, \ldots, u_{im_i} \in U\big(\mu, \mu(a_i)\big)$ such that $a_i b_i = \sum_{k_i=1}^{m_i} v_{ik_i} u_{ik_i}$. Since $x = \sum_{i=1}^{n} \sum_{k_i=1}^{m_i} v_{ik_i} u_{ik_i}$, we have

$$\begin{aligned}
\big(\nu \otimes \mu\big)(x) &\geq \bigwedge_{1\leq i\leq n} \bigwedge_{1\leq k_i\leq m_i} \nu(v_{ik_i}) \ominus \mu(u_{ik_i}) \\
&\geq \bigwedge_{1\leq i\leq n} \bigwedge_{1\leq k_i\leq m_i} \nu(b_i) \ominus \mu(a_i) \\
&= \bigwedge_{1\leq i\leq n} \nu(b_i) \ominus \mu(a_i) \\
&= \bigwedge_{1\leq i\leq n} \mu(a_i) \ominus \nu(b_i).
\end{aligned}$$

So, $\big(\nu \otimes \mu\big)(x) \geq \big(\mu \otimes \nu\big)(x)$. It follows that $\mu \otimes \nu \leq \nu \otimes \mu$. A similar reasoning shows that $\nu \otimes \mu \leq \mu \otimes \nu$. Hence, $\mu \otimes \nu = \nu \otimes \mu$. Therefore, $\mathcal{F}id(\mathcal{A}, L)$ is commutative.
2. Since $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ can be embedded into $\mathcal{F}id(\mathcal{A}, L)$, it is clear that they

are Brouwerian algebras when $\mathcal{F}id(\mathcal{A}, L)$ is a Brouwerian algebra.
Conversely, assume that $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ are Brouwerian algebras. Let $x \in A$. Since $x \in U\big(\mu, \mu(x)\big) = U\big(\mu, \mu(x)\big) \odot U\big(\mu, \mu(x)\big)$, there are $a_1, b_1, \ldots, a_n, b_n \in U\big(\mu, \mu(x)\big)$ such that $x = \sum_{i=1}^{n} a_i b_i$; thus,

$$\mu^2(x) \geq \bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \mu(b_i) \geq \bigwedge_{1 \leq i \leq n} \mu(x) \ominus \mu(x) = \big(\mu(x)\big)^2 = \mu(x)$$

and, $\mu^2(x) = \mu(x)$. So, $\mu^2 = \mu$. Hence, $\mathcal{F}id(\mathcal{A}, L)$ is a Brouwerian algebra. □

Definition 3.1.21. *(See, [39]) A ring $\mathcal{A}$ is called a Von Neumann Regular ring (or VNR-ring for short) if for any $x \in A$, there is $a \in A$ such that $x = xax$.*

Proposition 3.1.22. *Suppose that $\mathcal{A}$ is a VNR-ring. If $\mathcal{L}$ is a Brouwerian algebra, then $\mathcal{F}id(\mathcal{A}, L)$ is a Brouwerian algebra.*

Proof. Assume that $\mathcal{L}$ is a Brouwerian algebra. Let $\mu, \nu \in Fid(\mathcal{A}, L)$. For any $x \in A$, there is $a \in A$ such that $x = xax$; thus,

$$\big(\mu \wedge \nu\big)(x) = \mu(x) \wedge \nu(x) \leq \mu(xa) \wedge \nu(x) \leq \big(\mu \otimes \nu\big)(xax) = \big(\mu \otimes \nu\big)(x).$$

So, $\mu \wedge \nu \leq \mu \otimes \nu$ and, $\mu \otimes \nu = \mu \wedge \nu$. Hence, $\mathcal{F}id(\mathcal{A}, L)$ is a Brouwerian algebra. □

Theorem 3.1.23. *Suppose that $\mathcal{A}$ is a commutative ring. Then $\mathcal{F}id(\mathcal{A}, L)$ is a Brouwerian algebra if and only if $\mathcal{A}$ is a VNR-ring and $\mathcal{L}$ is a Brouwerian algebra.*

Proof. If $\mathcal{F}id(\mathcal{A}, L)$ is a Brouwerian algebra, then $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ are Brouwerian algebras; thus $\mathcal{A}$ is a VNR-ring by Proposition 3.2. in [3] and $\mathcal{L}$ is a Brouwerian algebra. The other direction follows immediately from Proposition 3.1.22. □

3.2 Prime elements and filters of the set of L-fuzzy ideals of a ring

3.2.1 Prime elements

Theorem 3.2.1. *Prime elements (See, Definition 1.2.18) of $\mathcal{F}id(\mathcal{A}, L)$ are exactly of the form P^r, where r and P are prime elements of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, respectively.*

Proof. ($\Rightarrow$) Let μ be a prime element of $\mathcal{F}id(\mathcal{A}, L)$. For any $a, b \notin U(\mu, 1)$, we have

$$Idg(a)^0 \otimes U(\mu,1)^{\mu(a)} \leq Fidg\big[Idg(a)^0 \ominus U(\mu,1)^{\mu(a)}\big] \leq Fidg(\mu) = \mu$$
$$Idg(b)^0 \otimes U(\mu,1)^{\mu(b)} \leq Fidg\big[Idg(b)^0 \ominus U(\mu,1)^{\mu(b)}\big] \leq Fidg(\mu) = \mu;$$

thus,

$$U(\mu,1)^{\mu(a)} \leq \mu \text{ and } U(\mu,1)^{\mu(b)} \leq \mu;$$

so,

$$\mu(a) = U(\mu,1)^{\mu(a)}(b) \leq \mu(b) \text{ and } \mu(b) = U(\mu,1)^{\mu(b)}(a) \leq \mu(a);$$

consequently, $\mu(a) = \mu(b)$. Hence, $\mu = U(\mu,1)^r$ for some $r \in L$.
Since $\mu \neq \underline{1}$, we have $r \neq 1$ and $U(\mu,1) \neq A$.
For any $t, s \in L$ such that $t \ominus s \leq r$, we have

$$\begin{aligned} U(\mu,1)^t \otimes U(\mu,1)^s \ &\leq Fidg\big[U(\mu,1)^t \ominus U(\mu,1)^s\big] \\ &= Fidg\big(U(\mu,1)^{t\ominus s}\big) \\ &\leq Fidg\big(U(\mu,1)^r\big) \\ &= U(\mu,1)^r; \end{aligned}$$

thus, $U(\mu,1)^t \leq U(\mu,1)^r$ or $U(\mu,1)^s \leq U(\mu,1)^r$; *i.e.*, $t \leq r$ or $s \leq r$. Hence, r is a prime element of $\mathcal{L}$.
For any $I, J \in Id(\mathcal{A})$ such that $I \odot J \subseteq U(\mu,1)$, we have

$$I^0 \otimes J^0 = (I \odot J)^0 \leq U(\mu,1)^r = \mu;$$

thus, $I^0 \leq \mu$ or $J^0 \leq \mu$; so, $I = U(I^0,1) \subseteq U(\mu,1)$ or $J = U(J^0,1) \subseteq U(\mu,1)$. Hence, $U(\mu,1)$ is a prime element of $\mathcal{I}d(\mathcal{A})$.
($\Leftarrow$) Let r and P be prime elements of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, respectively. We have $P^r \neq \underline{1}$, since $P \neq A$ and $r \neq 1$. Now, let ν and δ be two L-fuzzy ideals of $\mathcal{A}$ such that $\nu \otimes \delta \leq P^r$ and $\nu \nleq P^r$. Let $y \notin P$. Since there is $x \notin P$ such that $\nu(x) \nleq r$, we have $Idg(x) \nsubseteq P$ and $Idg(y) \nsubseteq P$; thus, $Idg(x) \odot Idg(y) \nsubseteq P$; so, there is $u \in Idg(x) \odot Idg(y)$ such that $u \notin P$. Since there are $a_1, \ldots, a_n \in Idg(x)$ and $b_1, \ldots, b_n \in Idg(y)$ such that $u = \sum_{i=1}^{n} a_i b_i$, we have

$$\nu(x) \ominus \delta(y) \leq \nu(a_i) \ominus \delta(b_i) \text{ for all } 1 \leq i \leq n;$$

thus,

$$\nu(x) \ominus \delta(y) \leq \bigwedge_{1\leq i\leq n} \nu(a_i) \ominus \delta(b_i) \leq \big(\nu \otimes \delta\big)(u) \leq P^r(u) = r$$

and, $\delta(y) \leq r$. So, $\delta \leq P^r$. Hence, P^r is a prime element of $\mathcal{F}id(\mathcal{A}, L)$. □

Lemma 3.2.2. *Let $r \in L$ and $P \in Id(\mathcal{A})$. Then $\sqrt{P^r} = \sqrt{P}^{\sqrt{r}}$.*

Proof. If $r = 1$ or $P = A$, then $\sqrt{r} = 1$ or $\sqrt{P} = A$; thus, $\sqrt{P^r} = \underline{1} = \sqrt{P}^{\sqrt{r}}$. Now, suppose that $r \neq 1$ and $P \neq A$.
Let $x \in \sqrt{P}$. For any $Q^s \in Spec\big(\mathcal{F}id(\mathcal{A}, L)\big)$ such that $P^r \leq Q^s$, we have $x \in Q$, since $P = U(P^r, 1) \subseteq U(Q^s, 1) = Q$ and $Q \in Spec\big(\mathcal{I}d(\mathcal{A})\big)$; thus, $Q^s(x) = 1$. So, $\sqrt{P^r}(x) = \bigwedge\{1\} = 1$.
Let $x \notin \sqrt{P}$. There is $R \in Spec\big(\mathcal{I}d(\mathcal{A})\big)$ such that $P \subseteq R$ and $x \notin R$.
• For any $t \in Spec(\mathcal{L})$ such that $r \leq t$, we have $R^t \in Spec\big(\mathcal{F}id(\mathcal{A}, L)\big)$ and $P^r \leq R^t$; thus, $\sqrt{P^r}(x) \leq R^t(x) = t$. So, $\sqrt{P^r}(x) \leq \sqrt{r}$.
• For any $Q^s \in Spec\big(\mathcal{F}id(\mathcal{A}, L)\big)$ such that $P^r \leq Q^s$, we have $\sqrt{r} \leq s \leq Q^s(x)$, since $r \leq s$ and $s \in Spec(\mathcal{L})$. Thus, $\sqrt{r} \leq \sqrt{P^r}(x)$.
So, $\sqrt{P^r}(x) = \sqrt{r}$. Hence, $\sqrt{P^r} = \sqrt{P}^{\sqrt{r}}$. □

Proposition 3.2.3. *Let $r \in L$ and $P \in Id(\mathcal{A})$ such that $\sqrt{P^r} \neq \underline{1}$. Then P^r is a primary element of $\mathcal{F}id(\mathcal{A}, L)$ if and only if r and P are primary elements of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, respectively.*

Proof. $(\Rightarrow)$ Assume that P^r is a primary element of $\mathcal{F}id(\mathcal{A}, L)$. Since $P^r \neq \underline{1}$, we have $r \neq 1$ and $P \neq A$.
For any $s, t \in L$ such that $s \ominus t \leq r$, we have $P^s \otimes P^t \leq Fidg(P^s \ominus P^t) = Fidg(P^{s\ominus t}) = P^{s\ominus t} \leq P^r$; thus, $P^s \leq P^r$ or $P^t \leq \sqrt{P^r} = \sqrt{P}^{\sqrt{r}}$; so, $s \leq r$ or $t \leq \sqrt{r}$. Hence, r is a primary element of $\mathcal{L}$.
For any $I, J \in Id(\mathcal{A})$ such that $I \odot J \subseteq P$, we have $I^0 \otimes J^0 = (I \odot J)^0 \leq P^r$; thus, $I^0 \leq P^r$ or $J^0 \leq \sqrt{P^r} = \sqrt{P}^{\sqrt{r}}$; so, $I = U(I^0, 1) \subseteq U(P^r, 1) = P$ or $J = U(J^0, 1) \subseteq U(\sqrt{P}^{\sqrt{r}}, 1) = \sqrt{P}$.
Hence, P is a primary element of $\mathcal{I}d(\mathcal{A})$.
$(\Leftarrow)$ Assume that r and P are primary elements of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, respectively. Let $\nu, \delta \in Fid(\mathcal{A}, L)$ such that $\nu \otimes \delta \leq P^r$ and $\nu \nleq P^r$. Let $y \notin \sqrt{P}$. Since there is $x \notin P$ such that $\nu(x) \nleq r$, we have $Idg(x) \nsubseteq P$ and $Idg(y) \nsubseteq \sqrt{P}$; thus, $Idg(x) \odot Idg(y) \nsubseteq P$; so, there is $u \in Idg(x) \odot Idg(y)$ such that $u \notin P$. Since there are $a_1, \dots, a_n \in Idg(x)$ and $b_1, \dots, b_n \in Idg(y)$ such that $u = \sum_{i=1}^{n} a_i b_i$, we have $\nu(x) \ominus \delta(y) \leq \bigwedge_{1\leq i\leq n} \nu(a_i) \ominus \delta(b_i) \leq \big(\nu \otimes \delta\big)(u) \leq P^r(u) = r$; thus, $\delta(y) \leq \sqrt{r} = \sqrt{P}^{\sqrt{r}}(y)$. So, $\delta \leq \sqrt{P}^{\sqrt{r}} = \sqrt{P^r}$. Hence, P^r is a primary element of $\mathcal{F}id(\mathcal{A}, L)$. □

Even though the elements of the form P^r, where r and P are primary elements of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, are primary, they do not necessarily constitute the complete list of primary elements of $\mathcal{F}id(\mathcal{A}, L)$ as the following example shows.

Example 3.2.4. *Let $\mathcal{L}$ be the Łukasiewicz structure (See, Example 1.2.10). Since $Spec(\mathcal{L}) = \emptyset$, we have $Spec\big(\mathcal{F}id(\mathcal{A}, L)\big) = \emptyset$ and, $\underline{1}$ is the only radical of $\mathcal{F}id(\mathcal{A}, L)$. Hence, every proper element of $\mathcal{F}id(\mathcal{A}, L)$ is primary.*

3.2.2 Filters

Lemma 3.2.5. *Let $\mu, \nu \in Fid(\mathcal{A}, L)$. Then $U(\mu, 1) \odot U(\nu, 1) \subseteq U(\mu \otimes \nu, 1)$.*

Proof. For any $x \in U(\mu, 1) \odot U(\nu, 1)$, we have $x = \sum_{i=1}^{n} a_i b_i$ with $a_i \in U(\mu, 1)$ and $b_i \in U(\nu, 1)$ for all $1 \leq i \leq n$; thus, $\big(\mu \otimes \nu\big)(x) \geq \bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \nu(b_i) = \bigwedge_{1 \leq i \leq n} 1 \ominus 1 = 1$ and, $x \in U(\mu \otimes \nu, 1)$. Hence, $U(\mu, 1) \odot U(\nu, 1) \subseteq U(\mu \otimes \nu, 1)$. □

Proposition 3.2.6. *Let $\emptyset \neq F \subseteq Id(\mathcal{A})$. Then F is a filter (See, Definition 1.2.27) of $\mathcal{I}d(\mathcal{A})$ if and only if $\dot{F} := \{\mu \in Fid(\mathcal{A}, L) : \ U(\mu, 1) \in F\}$ is a filter of $\mathcal{F}id(\mathcal{A}, L)$.*

Proof. Assume that F is a filter of $\mathcal{I}d(\mathcal{A})$.
• For any $\mu, \nu \in \dot{F}$, we have $U(\mu, 1) \odot U(\nu, 1) \in F$ and $U(\mu, 1) \odot U(\nu, 1) \subseteq U(\mu \otimes \nu, 1)$; thus, $U(\mu \otimes \nu, 1) \in F$ and, $\mu \otimes \nu \in \dot{F}$.
• For any $\mu, \nu \in Fid(\mathcal{A}, L)$ such that $\mu \in \dot{F}$ and $\mu \leq \nu$, we have $U(\mu, 1) \in F$ and $U(\mu, 1) \subseteq U(\nu, 1)$; thus, $U(\nu, 1) \in F$ and, $\mu \in \dot{F}$. Hence, $\dot{F}$ is a filter of $\mathcal{F}id(\mathcal{A}, L)$.
Conversely, assume that $\dot{F}$ is a filter of $\mathcal{F}id(\mathcal{A}, L)$. For any $I, J \in F$, we have $I_1, J_1 \in \dot{F}$; thus, $(I \odot J)_1 = I_1 \otimes J_1 \in \dot{F}$; so, $I \odot J = U\big((I \odot J)_1, 1\big) \in F$. For any $I, J \in Id(\mathcal{A})$ such that $I \in F$ and $I \subseteq J$, we have $I_1 \in \dot{F}$ and $I_1 \leq J_1$; thus, $J_1 \in \dot{F}$; so, $J = U(J_1, 1) \in F$. Hence, F is a filter of $\mathcal{I}d(\mathcal{A})$. □

Theorem 3.2.7. *The function $\phi : Fil\big(\mathcal{I}d(\mathcal{A})\big) \rightarrow Fil\big(\mathcal{F}id(\mathcal{A}, L)\big)$, given by $\phi(F) = \dot{F}$ for all $F \in Fil\big(\mathcal{I}d(\mathcal{A})\big)$, is a complete lattice embedding.*

Proof. For any $F, G \in Fil\big(\mathcal{I}d(\mathcal{A})\big)$ such that $\phi(F) = \phi(G)$, we have $I \in F$ iff $I_1 \in \dot{F}$ iff $I_1 \in \dot{G}$ iff $I \in G$, for all $I \in Id(\mathcal{A})$; thus, $F = G$. Hence, ϕ is one-to-one. Now, let $\{F_\lambda\}_{\lambda \in \Lambda} \subseteq Fil\big(\mathcal{I}d(\mathcal{A})\big)$. Clearly, $\phi(\bigcap_{\lambda \in \Lambda} F_\lambda) = \bigcap_{\lambda \in \Lambda} \phi(F_\lambda)$.
We next show that $\phi(\bigsqcup_{\lambda \in \Lambda} F_\lambda) = \bigsqcup_{\lambda \in \Lambda} \phi(F_\lambda)$.
Let $\mu \in \phi(\bigsqcup_{\lambda \in \Lambda} F_\lambda)$. Since $U(\mu, 1) \in \bigsqcup_{\lambda \in \Lambda} F_\lambda$, there are $I_1, \ldots, I_n \in \bigcup_{\lambda \in \Lambda} F_\lambda$ such that $I_1 \odot I_2 \odot \ldots \odot I_n \subseteq U(\mu, 1)$; thus,

$$(I_1)_1 \otimes \ldots \otimes (I_n)_1 = (I_1 \odot I_2 \odot \ldots \odot I_n)_{1 \ominus \ldots \ominus 1} = (I_1 \odot I_2 \odot \ldots \odot I_n)_1 \leq U(\mu, 1)_1 \leq \mu.$$

Since $U\big((I_i)_1, 1\big) = I_i$ for all $1 \leq i \leq n$, we have $(I_1)_1, \ldots, (I_n)_1 \in \bigcup_{\lambda \in \Lambda} \phi(F_\lambda)$; thus, $\mu \in \bigsqcup_{\lambda \in \Lambda} \phi(F_\lambda)$. So, $\phi(\bigsqcup_{\lambda \in \Lambda} F_\lambda) \subseteq \bigsqcup_{\lambda \in \Lambda} \phi(F_\lambda)$ and, $\phi(\bigsqcup_{\lambda \in \Lambda} F_\lambda) = \bigsqcup_{\lambda \in \Lambda} \phi(F_\lambda)$, since ϕ is order-preserving. Hence, ϕ is a complete lattice embedding of the lattice of $\mathcal{F}il\big(\mathcal{I}d(\mathcal{A})\big)$ into the lattice of $\mathcal{F}il\big(\mathcal{F}id(\mathcal{A}, L)\big)$. □

Corollary 3.2.8.
1. For any $F, G \in Fil\big(\mathcal{I}d(\mathcal{A})\big)$, we have $\phi(F \Rightarrow G) \subseteq \phi(F) \Rightarrow \phi(G)$.

2. If $L \setminus \{1\}$ has a maximum, then ϕ is an embedding of $\mathcal{F}il\big(\mathcal{I}d(\mathcal{A})\big)$ into $\mathcal{F}il\big(\mathcal{F}id(\mathcal{A}, L)\big)$.

Proof. **1.** Let $F, G \in Fil\big(\mathcal{I}d(\mathcal{A})\big)$. Since ϕ is order-preserving, we have

$$\phi(F \Rightarrow G) \cap \phi(F) = \phi\big((F \Rightarrow G) \cap F\big) \subseteq \phi(G);$$

thus, $\phi(F \Rightarrow G) \subseteq \phi(F) \Rightarrow \phi(G)$.
2. Assume that $L \setminus \{1\}$ has a maximum p. Let $F, G \in Fil\big(\mathcal{I}d(\mathcal{A})\big)$. Let $\mu \in \phi(F) \Rightarrow \phi(G)$. Let $I \in \big[U(\mu, 1)\big) \cap F$. Since $U(I^p, 1) = I \in F$, we have $I^p \in \phi(F)$. We next show that $I^p \in [\mu)$.
Since $I \in \big[U(\mu, 1)\big)$, there is $n \geq 1$ such that $U(\mu, 1)^n \subseteq I$. Now, let $x \notin I$. For any $a_1, \dots, a_n \in A$ such that $x = a_1 \dots a_n$, there is $1 \leq i_0 \leq n$ such that $a_{i_0} \notin U(\mu, 1)$; thus, $\mu(a_1) \ominus \dots \ominus \mu(a_n) \leq \mu(a_{i_0}) \leq p$. So, $\big(\underbrace{\mu \circ \dots \circ \mu}_{n\ times}\big)(x) \leq p$.
It follows that $\underbrace{\mu \circ \dots \circ \mu}_{n\ times} \leq I^p$; *i.e.*, $\mu^n \leq I^p$. Consequently, $I^p \in [\mu)$.
Since $I^p \in [\mu) \cap \phi(F)$, we have $I^p \in \phi(G)$; thus, $U(I^p, 1) \in G$; *i.e.*, $I \in G$. So, $\big[U(\mu, 1)\big) \cap F \subseteq G$; *i.e.*, $U(\mu, 1) \in F \Rightarrow G$; *i.e.*, $\mu \in \phi(F \Rightarrow G)$. Hence,

$$\phi(F) \Rightarrow \phi(G) \subseteq \phi(F \Rightarrow G) \text{ and, } \phi(F \Rightarrow G) = \phi(F) \Rightarrow \phi(G).$$

Therefore, ϕ is an embedding of $\mathcal{F}il\big(\mathcal{I}d(\mathcal{A})\big)$ into $\mathcal{F}il\big(\mathcal{F}id(\mathcal{A}, L)\big)$. □

Proposition 3.2.9. *Let $\emptyset \neq F \subseteq L$. Then F is a filter of $\mathcal{L}$ if and only if $F' := \{\mu \in Fid(\mathcal{A}, L) : \ Im(\mu) \subseteq F\}$ is a filter of $\mathcal{F}id(\mathcal{A}, L)$.*

Proof. Assume that F is a filter of $\mathcal{L}$.
• Let $\mu, \nu \in F'$. Let $x \in A$. Since $\mu(x) \in Im(\mu) \subseteq F$ and $\nu(1) \in Im(\nu) \subseteq F$, we have $\mu(x) \ominus \nu(1) \in F$; thus, $\big(\mu \otimes \nu\big)(x) \in F$, since $\mu(x) \ominus \nu(1) \leq \big(\mu \otimes \nu\big)(x)$. So, $Im(\mu \otimes \nu) \subseteq F$ and, $\mu \otimes \nu \in F'$.
• Let $\mu, \nu \in Fid(\mathcal{A}, L)$ such that $\mu \in F'$ and $\mu \leq \nu$. For any $x \in A$, we have $\mu(x) \leq \nu(x)$ and $\mu(x) \in F$; thus, $\nu(x) \in F$. So, $Im(\nu) \subseteq F$ and, $\nu \in F'$. Hence, F' is a filter of $\mathcal{F}id(\mathcal{A}, L)$.
Conversely, assume that F' is a filter of $\mathcal{F}id(\mathcal{A}, L)$. For any $r, s \in F$, we have $(\underline{r})_*, (\underline{s})_* \in F'$; thus, $(\underline{r \ominus s})_* = (\underline{r})_* \otimes (\underline{s})_* \in F'$; so, $Im\big((\underline{r \ominus s})_*\big) \subseteq F$ and, $r \ominus s \in F$. For any $r, s \in L$ such that $r \in F$ and $r \leq s$, we have $(\underline{r})_* \leq (\underline{s})_*$ and $(\underline{r})_* \in F'$; thus, $(\underline{s})_* \in F'$; so, $Im\big((\underline{s})_*\big) \subseteq F$ and, $s \in F$. Hence, F is a filter of $\mathcal{L}$. □

Theorem 3.2.10. *Suppose that A or L is finite. Then the function $\psi : Fil(\mathcal{L}) \to Fil\big(\mathcal{F}id(\mathcal{A}, L)\big)$, given by $\psi(F) = F'$ for all $F \in Fil(\mathcal{L})$, is a complete lattice embedding.*

Proof. For any $F, G \in Fil(\mathcal{L})$ such that $\psi(F) = \psi(G)$, we have $r \in F$ iff $(\underline{r})_* \in F'$ iff $(\underline{r})_* \in G'$ iff $r \in G$, for all $r \in L$; thus, $F = G$. Hence, ψ is

one-to-one. Now, let $\{F_\lambda\}_{\lambda\in\Lambda} \subseteq Fil(\mathcal{L})$. Clearly, $\psi(\bigcap\limits_{\lambda\in\Lambda} F_\lambda) = \bigcap\limits_{\lambda\in\Lambda} \psi(F_\lambda)$. We next show that $\psi(\bigsqcup\limits_{\lambda\in\Lambda} F_\lambda) = \bigsqcup\limits_{\lambda\in\Lambda} \psi(F_\lambda)$.
Let $\mu \in \psi(\bigsqcup\limits_{\lambda\in\Lambda} F_\lambda)$. Let $r_1, \ldots, r_p \in L$ such that $Im(\mu) = \{r_1, \ldots, r_p\}$.
Since $r_1, \ldots, r_p \in \bigsqcup\limits_{\lambda\in\Lambda} F_\lambda$, there are $r_1^1, \ldots, r_1^{n_1}, \ldots, r_p^1, \ldots, r_p^{n_p} \in \bigcup\limits_{\lambda\in\Lambda} F_\lambda$ such that $r_1^1 \ominus \ldots \ominus r_1^{n_1} \le r_1, \ldots, r_p^1 \ominus \ldots \ominus r_p^{n_p} \le r_p$; thus,

$$\bigotimes_{1\le i\le p} (\underline{r_i^1})_* \otimes \ldots \otimes (\underline{r_i^{n_i}})_* = (\underline{r_1^1 \ominus \ldots \ominus r_1^{n_1} \ominus \ldots \ominus r_p^1 \ominus \ldots \ominus r_p^{n_p}})_* \le \mu.$$

Since $(\underline{r_1^1})_*, \ldots, (\underline{r_1^{n_1}})_*, \ldots, (\underline{r_p^1})_*, \ldots, (\underline{r_p^{n_p}})_* \in \bigcup\limits_{\lambda\in\Lambda} \psi(F_\lambda)$, we have $\mu \in \bigsqcup\limits_{\lambda\in\Lambda} \psi(F_\lambda)$.
So, $\psi(\bigsqcup\limits_{\lambda\in\Lambda} F_\lambda) \subseteq \bigsqcup\limits_{\lambda\in\Lambda} \psi(F_\lambda)$ and, $\psi(\bigsqcup\limits_{\lambda\in\Lambda} F_\lambda) = \bigsqcup\limits_{\lambda\in\Lambda} \psi(F_\lambda)$, since ψ is order-preserving.
Hence, ψ is a complete lattice embedding of the lattice of $\mathcal{F}il(\mathcal{L})$ into the lattice of $\mathcal{F}il\big(\mathcal{F}id(\mathcal{A}, L)\big)$. □

3.3 Rings and MV-algebras

3.3.1 Rings and Boolean algebras

The left and right annihilators in $\mathcal{F}id(\mathcal{A}, L)$ of an L-fuzzy ideal μ of $\mathcal{A}$ will be denoted by μ^- and $\mu^\sim$, respectively; *i.e.*, $\mu^- = \mu \hookrightarrow \chi_0$ and $\mu^\sim = \mu \looparrowright \chi_0$.

Proposition 3.3.1. *Let I be a proper ideal of $\mathcal{A}$ and $r, s \in L$ such that $r \le s$. Then $\big((I_r^s)_*\big)^- = \big((I^-)_{\overline{s}}^{\overline{r}}\big)_*$ and $\big((I_r^s)_*\big)^\sim = \big((I^\sim)_{\widetilde{s}}^{\widetilde{r}}\big)_*$; where, I^- and $I^\sim$ denote the left and right annihilator of I in $\mathcal{I}d(\mathcal{A})$, respectively.*

Proof. We first show that $\big((I_r^s)_*\big)^- = \big((I^-)_{\overline{s}}^{\overline{r}}\big)_*$.
• Let $x \in I^- \setminus \{0\}$. For any $t \in L$ such that $x_t \circ (I_r^s)_* \le \chi_0$, we have

$$t \ominus r = x_t(x) \ominus (I_r^s)_*(1) \le \big(x_t \circ (I_r^s)_*\big)(x) \le \chi_0(x) = 0;$$

thus, $t \le \overline{r}$. So, $\big((I_r^s)_*\big)^-(x) \le \overline{r}$. Now, let $a \ne 0$ in A. For any $v \in A$ such that $a = xv$, we have $v \notin I$; thus, $\overline{r} \ominus (I_r^s)_*(v) = \overline{r} \ominus r = 0$. So,

$$\big(x_{\overline{r}} \circ (I_r^s)_*\big)(a) = \bigvee\{0\} = 0.$$

It follows that

$$x_{\overline{r}} \circ (I_r^s)_* \le \chi_0 \text{ and, } \overline{r} \le \big((I_r^s)_*\big)^-(x).$$

Consequently, $\big((I_r^s)_*\big)^-(x) = \overline{r}$.
• Let $x \notin I^-$. For any $t \in L$ such that $x_t \circ (I_r^s)_* \le \chi_0$, we have

$$t \ominus s = x_t(x) \ominus (I_r^s)_*(v) \le \big(x_t \circ (I_r^s)_*\big)(xv) \le \chi_0(xv) = 0$$

for some $v \in I$ such that $xv \ne 0$; thus, $t \le \overline{s}$. So, $\big((I_r^s)_*\big)^-(x) \le \overline{s}$. Now, let $a \ne 0$ in A. For any $v \in A$ such that $a = xv$, we have

$$\overline{s} \ominus (I_r^s)_*(v) = \begin{cases} \overline{s} \ominus s & \text{if } v \in I \\ \overline{s} \ominus r & \text{if } v \notin I \end{cases} \leq \overline{s} \ominus s = 0;$$

thus, $\overline{s} \ominus (I_r^s)_*(v) = 0$. So,

$$\big(x_{\overline{s}} \circ (I_r^s)_*\big)(a) = \bigvee\{0\} = 0.$$

It follows that

$$x_{\overline{s}} \circ (I_r^s)_* \leq \chi_0 \text{ and, } \overline{s} \leq \big((I_r^s)_*\big)^-(x).$$

Consequently, $\big((I_r^s)_*\big)^-(x) = \overline{s}$.
Hence, $\big((I_r^s)_*\big)^- = \big((I^-)_{\overline{s}}^{\overline{r}}\big)_*$ and, $\big((I_r^s)_*\big)^\sim = \big((I^\sim)_{\widetilde{s}}^{\widetilde{r}}\big)_*$ by similar arguments. □

Lemma 3.3.2. *Suppose that $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ are Boolean algebras. Then for any $r \in L$ and $I \in Id(\mathcal{A})$, we have $(I_r)_* + \big((I_r)_*\big)^- = \underline{1}$.*

Proof. Let $r \in L$ and $I \in Id(\mathcal{A})$. For any $x \in I^-$, we have

$$\big[(I_r)_* + \big((I_r)_*\big)^-\big](x) \geq \big((I_r)_*\big)^-(x) = (I^-)^{\overline{r}}(x) = 1$$

and,

$$\big[(I_r)_* + \big((I_r)_*\big)^-\big](x) = 1.$$

Now, let $x \notin I^-$.
• If $x \in I$, then

$$\begin{aligned} \big[(I_r)_* + \big((I_r)_*\big)^-\big](x) &\geq (I_r)_*(x) \vee \big((I_r)_*\big)^-(x) \\ &= (I_r)_*(x) \vee (I^-)^{\overline{r}}(x) \\ &= r \vee \overline{r} \\ &= 1; \end{aligned}$$

thus, $\big[(I_r)_* + \big((I_r)_*\big)^-\big](x) = 1$.
• If $x \notin I$, then $x = a + b$ for some $a \in I \setminus \{0\}$ and $b \in I^- \setminus \{0\}$; thus,

$$\begin{aligned} \big[(I_r)_* + \big((I_r)_*\big)^-\big](x) &\geq (I_r)_*(a) \wedge \big((I_r)_*\big)^-(b) \\ &= (I_r)_*(a) \wedge (I^-)^{\overline{r}}(b) \\ &= r \wedge 1 \\ &= r \end{aligned}$$

and

$$\big[(I_r)_* + \big((I_r)_*\big)^-\big](x) \geq \big((I_r)_*\big)^-(x) = (I^-)^{\overline{r}}(x) = \overline{r};$$

so,

$$\big[(I_r)_* + \big((I_r)_*\big)^-\big](x) \geq r \vee \overline{r} = 1 \text{ and, } \big[(I_r)_* + \big((I_r)_*\big)^-\big](x) = 1.$$

Hence, $(I_r)_* + \big((I_r)_*\big)^- = \underline{1}$. □

Theorem 3.3.3. *$\mathcal{F}id(\mathcal{A}, L)$ is a Boolean algebra if and only if so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$.*

Proof. Since $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ can be embedded into $\mathcal{F}id(\mathcal{A}, L)$, it is clear that they are Boolean algebras when $\mathcal{F}id(\mathcal{A}, L)$ is a Boolean algebra.
Conversely, assume that $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ are Boolean algebras. Let $\mu \in Fid(\mathcal{A}, L)$. Since $Id(\mathcal{A})$ is finite (See, [21], Proposition 4.7.), there is a finite subset $\mathbb{A}$ of A such that

$$\begin{aligned}
\mu^- + \mu &= \big[\bigsqcup_{a\in A} \big(Idg(a)_{\mu(a)}\big)_*\big]^- + \mu \\
&= \big[\bigsqcup_{a\in \mathbb{A}} \big(Idg(a)_{\mu(a)}\big)_*\big]^- + \mu \\
&= \big[\bigwedge_{a\in \mathbb{A}} \big(Idg(a)_{\mu(a)}\big)_*^-\big] + \mu \\
&= \bigwedge_{a\in \mathbb{A}} \big[\big(Idg(a)_{\mu(a)}\big)_*^- + \mu\big] \\
&\geq \bigwedge_{a\in \mathbb{A}} \big[\big(Idg(a)_{\mu(a)}\big)_*^- + \big(Idg(a)_{\mu(a)}\big)_*\big] \\
&= \bigwedge_{a\in \mathbb{A}} \underline{1} \\
&= \underline{1}
\end{aligned}$$

and, $\mu^- + \mu = \underline{1}$. Hence, $\mathcal{F}id(\mathcal{A}, L)$ is a Boolean algebra. □

3.3.2 Łukasiewicz rings under an MV-algebra

Definition 3.3.4. *$\mathcal{A}$ is called a Łukasiewicz ring under $\mathcal{L}$ if it satisfies the following conditions for any $\mu, \nu \in Fid(\mathcal{A}, L)$:*
(i) $\big((\mu^- \otimes \nu)^- \otimes \mu^-\big)^\sim = \mu + \nu = \big(\mu^- \otimes (\nu \otimes \mu^\sim)^-\big)^\sim$.
(ii) $(\mu^- \otimes \nu^-)^\sim = (\mu^\sim \otimes \nu^\sim)^-$.

Theorem 3.3.5. *The following are equivalent:*
(1) *$\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$.*
(2) *$\mathcal{M}(\mathcal{A}, L) := \big(Fid(\mathcal{A}, L);\ \oplus,\ \otimes;\ ^\sim,\ ^-;\ \chi_0,\ \underline{1}\big)$, where $\mu \oplus \nu = (\nu^- \otimes \mu^-)^\sim$ for all $\mu, \nu \in Fid(\mathcal{A}, L)$, is an MV-algebra.*

Proof. **(1) ⇒ (2)** Assume that $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$. By Proposition 1.2.33, it suffices to show that $\big(Fid(\mathcal{A}, L);\ +,\ \otimes;\ ^\sim,\ ^-;\ \chi_0,\ \underline{1}\big)$ is a

Łukasiewicz semi-ring. Since $\mathcal{F}id(\mathcal{A}, L)$ is a residuated lattice, it is clear that $\big(Fid(\mathcal{A}, L);\ +,\ \otimes\big)$ is an additively idempotent semi-ring with χ_0 as additive identity and $\underline{1}$ as multiplicative identity. Furthermore, we have $\mu \otimes \nu = \chi_0$ iff $\mu \leq \nu \hookrightarrow \chi_0 = \nu^-$ iff $\nu \leq \mu \looparrowright \chi_0 = \mu^\sim$ (for all $\mu, \nu \in Fid(\mathcal{A}, L)$). Thus, conditions (**LS1**) and **(LS2)(i)** of Proposition 1.2.31 are satisfied. Conditions **(LS2)(ii)** and **(LS2)(iii)** are immediate consequences of Definition 3.3.4.
(2) $\Rightarrow$ **(1)** If $\mathcal{M}(\mathcal{A}, L)$ is an MV-algebra, then $\big(Fid(\mathcal{A}, L);\ +,\ \otimes;\ ^\sim,\ ^-;\ \chi_0,\ \underline{1}\big)$ is a Łukasiewicz semi-ring by Proposition 1.2.31; thus, $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$ by **(LS2)(ii)** and **(LS2)(iii)**. □

Proposition 3.3.6. *Suppose that $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$. Then the associated lattice of $\mathcal{M}(\mathcal{A}, L)$ is the complete lattice $\mathbb{F}id(\mathcal{A}, L)$.*

Proof. Since $\big(Fid(\mathcal{A}, L);\ +,\ \otimes;\ ^\sim,\ ^-;\ \chi_0,\ \underline{1}\big)$ is a Łukasiewicz semi-ring, Remark 1.2.32 shows that $\big(Fid(\mathcal{A}, L);\ \sqcap,\ +;\ \chi_0,\ \underline{1}\big)$ is a bounded lattice; where, $\mu \sqcap \nu = (\mu^- + \nu^-)^\sim$ for all $\mu, \nu \in Fid(\mathcal{A}, L)$. Since $\mathcal{F}id(\mathcal{A}, L)$ is a residuated lattice, for any $\mu, \nu \in Fid(\mathcal{A}, L)$, we have $\mu \sqcap \nu = (\mu^- + \nu^-)^\sim = \mu^{-\sim} \wedge \nu^{-\sim} = \mu \wedge \nu$. Thus, $\sqcap = \wedge$. Hence, the associated lattice of $\mathcal{M}(\mathcal{A}, L)$ is $\mathbb{F}id(\mathcal{A}, L)$. □

Corollary 3.3.7. *If $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$, then $\mathcal{M}(\mathcal{A}, L)$ is a commutative MV-algebra.*

Proof. Straightforward, since $\mathcal{M}(\mathcal{A}, L)$ is a complete MV-algebra. □

Theorem 3.3.8. *$\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$ if and only if the following conditions are satisfied:*
(CO) *$\mathcal{F}id(\mathcal{A}, L)$ is a commutative residuated lattice.*
(LR) *For any $\mu, \nu \in Fid(\mathcal{A}, L)$, $\mu + \nu = \big(\mu^- \otimes (\nu \otimes \mu^-)^-\big)^-$.*

Proof. ($\Rightarrow$) If $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$, then $\oplus$ is commutative by Corollary 3.3.7; thus, $\otimes$ is commutative; so, $\mathcal{F}id(\mathcal{A}, L)$ is commutative. Condition **(LR)** is an immediate consequence of Definition 3.3.4.
($\Leftarrow$) Assume that conditions **(CO)** and **(LR)** are satisfied. Since $\mathcal{F}id(\mathcal{A}, L)$ is commutative, the unary operations $^-$ and $^\sim$ are confused. Hence, conditions **(i)** and **(ii)** of Definition 3.3.4 are satisfied. □

If $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$, then $\mathcal{L}$ is an MV-algebra and $\mathcal{A}$ is a usual Łukasiewicz ring (See, [21], Definition 3.2.), but the converse is not necessarily true as the following example shows.

Example 3.3.9. *Consider the MV-algebra $\mathcal{L}$ of Example 3.1.17 and the L-fuzzy ideals μ and ν of the Łukasiewicz ring $\mathcal{Z}_4$ defined for any $x \in \frac{\mathbb{Z}}{4\mathbb{Z}}$ by:*

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ c & \text{if } x = 2, \\ a & \text{if } x \in \{1,\ 3\}. \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} 1 & \text{if } x = 0, \\ c & \text{if } x = 2, \\ b & \text{if } x \in \{1,\ 3\}. \end{cases} \quad \textit{. We have}$$

$$
\begin{aligned}
\big[\mu^- \otimes (\nu \otimes \mu^-)^-\big]^- &= \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^c_a\big)^-_* \otimes \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^c_b\big)_* \otimes \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^c_a\big)^-_*\Big]^-\Big]^- \\
&= \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^d_b\big)_* \otimes \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^c_b\big)_* \otimes \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^d_b\big)_*\Big]^-\Big]^- \\
&= \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^d_b\big)_* \otimes \Big[[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_c\big)_* + (\underline{b})_*] \otimes [\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_d\big)_* + (\underline{b})_*]\Big]^-\Big]^- \\
&= \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^d_b\big)_* \otimes \Big[\chi_0 + \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_{c\ominus b}\big)_* + \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_{b\ominus d}\big)_* + (\underline{b \ominus b})_*\Big]^-\Big]^- \\
&= \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^d_b\big)_* \otimes \Big[\chi_0 + \chi_0 + \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_b\big)_* + \chi_0\Big]^-\Big]^- \\
&= \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^d_b\big)_* \otimes \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_b\big)^-_*\Big]^- \\
&= \Big[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^d_b\big)_* \otimes (\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^c\Big]^- \\
&= \Big[[\big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_d\big)_* + (\underline{b})_*] \otimes [(\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_1 + (\underline{c})_*]\Big]^- \\
&= \Big[\chi_0 + \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_{d\ominus c}\big)_* + \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_{b\ominus 1}\big)_* + (\underline{b \ominus c})_*\Big]^- \\
&= \Big[\chi_0 + \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_b\big)_* + \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_b\big)_* + \chi_0\Big]^- \\
&= \big((\tfrac{2\mathbb{Z}}{4\mathbb{Z}})_b\big)^-_* \\
&= (\tfrac{2\mathbb{Z}}{4\mathbb{Z}})^c \\
&> (\underline{c})_* \\
&= \mu + \nu.
\end{aligned}
$$

Hence, $\mathcal{Z}_4$ is not a Łukasiewicz ring under $\mathcal{L}$.

Proposition 3.3.10. *If $\mathcal{A}$ is a field and $\mathcal{L}$ is an MV-algebra, then $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$.*

Proof. Let μ be an L-fuzzy ideal of $\mathcal{A}$. For any $x, y \in A \setminus \{0\}$, we have

$$\mu(x) = \mu(yy^{-1}x) \geq \mu(y) = \mu(xx^{-1}y) \geq \mu(x);$$

thus, $\mu(x) = \mu(y)$. Hence, L-fuzzy ideals of $\mathcal{A}$ are only of the form $(\underline{r})_*$, where $r \in L$.

Now, assume that $\mathcal{A}$ is a field and $\mathcal{L}$ is an MV-algebra.

(CO) Since $\mathcal{I}d(\mathcal{A})$ and $\mathcal{L}$ are commutative residuated lattices, $\mathcal{F}id(\mathcal{A}, L)$ is a commutative residuated lattice.

(LR) For any $r, s \in L$, we have

$$\begin{aligned}\left[(\underline{r})^{-}_{*}\otimes\left((\underline{s})_{*}\otimes(\underline{r})^{-}_{*}\right)^{-}\right]^{-} &= \left[(A_r)^{-}_{*}\otimes\left((A_s)_{*}\otimes(A_r)^{-}_{*}\right)^{-}\right]^{-}\\ &= \left[(A_{\overline{r}})_{*}\otimes\left((A_s)_{*}\otimes(A_{\overline{r}})_{*}\right)^{-}\right]^{-}\\ &= \left[(A_{\overline{r}})_{*}\otimes\left((A_{s\ominus\overline{r}})_{*}\right)^{-}\right]^{-}\\ &= \left[(A_{\overline{r}})_{*}\otimes(A_{\overline{s\ominus\overline{r}}})_{*}\right]^{-}\\ &= \left[(A_{\overline{r}\ominus\overline{s\ominus\overline{r}}})_{*}\right]^{-}\\ &= (A_{\overline{\overline{r}\ominus\overline{s\ominus\overline{r}}}})_{*}\\ &= (A_{r\vee s})_{*}\\ &= (A_r)_{*}+(A_s)_{*}\\ &= (\underline{r})_{*}+(\underline{s})_{*}.\end{aligned}$$

Hence, $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$. □

Lemma 3.3.11. *(See, [21], Proposition 4.7.) If $\mathcal{A}$ is a Łukasiewicz ring, then $Id(\mathcal{A})$ is finite.*

Proposition 3.3.12. *If $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$, then $Fid(\mathcal{A}, L)$ is finite if and only if L is finite.*

Proof. Assume that $\mathcal{A}$ is a Łukasiewicz ring under $\mathcal{L}$. If $Fid(\mathcal{A}, L)$ is finite, then L is finite, since $\mathcal{L}$ can be embedded into $\mathcal{F}id(\mathcal{A}, L)$. Conversely, assume that L is finite. Let ρ be the map from $Fid(\mathcal{A}, L)$ to $Id(\mathcal{A})^L$ defined by:

$$\rho(\mu)(r) = U(\mu, r) \text{ for all } \mu \in Fid(\mathcal{A}, L) \text{ and } r \in L.$$

Let $\mu, \nu \in Fid(\mathcal{A}, L)$ such that $\rho(\mu) = \rho(\nu)$. For any $x \in A$, we have

$$x \in U\big(\mu, \mu(x)\big) = \rho(\mu)\big(\mu(x)\big) = \rho(\nu)\big(\mu(x)\big) = U\big(\nu, \mu(x)\big);$$

thus, $\nu(x) \geq \mu(x)$ and, $\mu(x) \geq \nu(x)$ by similar arguments; so, $\mu(x) = \nu(x)$. It follows that $\mu = \nu$. Consequently, ρ is one-to-one. Hence, $Fid(\mathcal{A}, L)$ is finite, since $Id(\mathcal{A})^L$ is finite. □

Definition 3.3.13. *$\mathcal{A}$ is said to be special primary if it has a unique maximal ideal M, and every proper ideal of $\mathcal{A}$ is a power of M.*

Proposition 3.3.14. *(See, [21], Proposition 4.1.) Every special primary ring is a Łukasiewicz ring.*

Theorem 3.3.15. *(See, [21], Theorem 4.10.) A ring is a Łukasiewicz ring if and only if it is isomorphic to a direct sum of special primary rings.*

Corollary 3.3.16. *A Łukasiewicz ring under $\mathcal{L}$ is isomorphic to a direct sum of special primary rings.*

Chapter Four

Some properties of quotients and images

In this chapter, unless otherwise specified, $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a complete meet-distributive residuated lattice (See, Definition 1.2.13), and $\mathcal{A}$ is a unital ring with unity 1 (sometimes simply called ring).

In Section 4.1, L-fuzzy ideals of a quotient ring are characterized, and some of their properties are investigated. In Sections 4.2 and 4.3, some functors from the category of unital rings to the category of po-monoids are studied.

4.1 L-fuzzy ideals of quotients

Lemma 4.1.1. *Let $I \in Id(\mathcal{A})$ and $\mu \in Fid(\mathcal{A}, L)$. The L-fuzzy subset $\frac{\mu}{I}$ of $\frac{A}{I}$, given by $\big(\frac{\mu}{I}\big)(\frac{a}{I}) = \bigvee\{\mu(x) :\ x \in \frac{a}{I}\}$ for all $a \in A$, is an L-fuzzy ideal of $\frac{\mathcal{A}}{I}$.*

Proof. Since $0 \in I$, we have $\big(\frac{\mu}{I}\big)(I) = \big(\frac{\mu}{I}\big)(\frac{0}{I}) \geq \mu(0) = 1$ and, $\big(\frac{\mu}{I}\big)(I) = 1$. Furthermore, for any $a, b \in A$, we have

$$\begin{aligned}\Big(\frac{\mu}{I}\Big)\Big(\frac{a}{I} - \frac{b}{I}\Big) &= \bigvee\{\mu(u) :\ u \in \tfrac{a-b}{I}\}\\ &\geq \bigvee\{\mu(x-y) :\ x \in \tfrac{a}{I} \text{ and } y \in \tfrac{b}{I}\}\\ &\geq \bigvee\{\mu(x) \wedge \mu(y) :\ x \in \tfrac{a}{I} \text{ and } y \in \tfrac{b}{I}\}\\ &= \Big(\frac{\mu}{I}\Big)\Big(\frac{a}{I}\Big) \wedge \Big(\frac{\mu}{I}\Big)\Big(\frac{b}{I}\Big)\end{aligned}$$

and

$$\begin{aligned}\Big(\frac{\mu}{I}\Big)\Big(\frac{a}{I}\frac{b}{I}\Big) &= \bigvee\{\mu(u) :\ u \in \tfrac{ab}{I}\}\\ &\geq \Big(\bigvee\{\mu(vb) :\ v \in \tfrac{a}{I}\}\Big) \vee \Big(\bigvee\{\mu(aw) :\ w \in \tfrac{b}{I}\}\Big)\\ &\geq \Big(\bigvee\{\mu(v) :\ v \in \tfrac{a}{I}\}\Big) \vee \Big(\bigvee\{\mu(w) :\ w \in \tfrac{b}{I}\}\Big)\\ &= \Big(\frac{\mu}{I}\Big)\Big(\frac{a}{I}\Big) \vee \Big(\frac{\mu}{I}\Big)\Big(\frac{b}{I}\Big).\end{aligned}$$

Hence, $\frac{\mu}{I}$ is an L-fuzzy ideal of $\frac{\mathcal{A}}{I}$. $\square$

For any $r \in L$ and $J \in Id(\mathcal{A})$ such that $I \subseteq J$, we have $\frac{(J_r)_*}{I} = \big((\frac{J}{I})_r\big)_*$; indeed,

- for any $\frac{a}{I} \in \frac{J}{I} \setminus \{I\}$, we have $a \in J \setminus I$ and, $\big(\frac{(J_r)_*}{I}\big)(\frac{a}{I}) = \bigvee\{r\} = r$;
- for any $\frac{a}{I} \notin \frac{J}{I}$, we have $a \notin J$ and, $\big(\frac{(J_r)_*}{I}\big)(\frac{a}{I}) = \bigvee\{0\} = 0$.

In particular, we have $\frac{\chi_J}{I} = \chi_{\frac{J}{I}}$ for all $J \in Id(\mathcal{A})$ such that $I \subseteq J$.

Lemma 4.1.2. *Let I be an ideal of $\mathcal{A}$ and ν an L-fuzzy ideal of $\frac{\mathcal{A}}{I}$. Then the L-fuzzy subset ν_I of A, given by $\nu_I(a) = \nu(\frac{a}{I})$ for all $a \in A$, is an L-fuzzy ideal of $\mathcal{A}$.*

Proof. We have $\nu_I(0) = \nu(\frac{0}{I}) = \nu(I) = 1$. For any $a, b \in A$, we have

$$\nu_I(a-b) = \nu(\tfrac{a-b}{I}) = \nu(\tfrac{a}{I} - \tfrac{b}{I}) \geq \nu(\tfrac{a}{I}) \wedge \nu(\tfrac{b}{I}) = \nu_I(a) \wedge \nu_I(b)$$

and

$$\nu_I(ab) = \nu(\tfrac{ab}{I}) = \nu(\tfrac{a}{I}\tfrac{b}{I}) \geq \nu(\tfrac{a}{I}) \vee \nu(\tfrac{b}{I}) = \nu_I(a) \vee \nu_I(b).$$

Hence, ν_I is an L-fuzzy ideal of $\mathcal{A}$. $\square$

For any ideal I of $\mathcal{A}$, define $Fid(\mathcal{A}, L, I) := \{\mu \in Fid(\mathcal{A}, L) : \ I \subseteq U(\mu, 1)\}$.

Lemma 4.1.3. *Let $a \in A$, $I \in Id(\mathcal{A})$ and $\mu \in Fid(\mathcal{A}, L, I)$. Then the following hold:*
(a) *For any $x \in \frac{a}{I}$, we have $\mu(x) = \mu(a)$.*
(b) *$\big(\frac{\mu}{I}\big)(\frac{a}{I}) = \mu(a)$.*

Proof. **(a)** For any $x \in \frac{a}{I}$, we have $\mu(x) = \mu(x - a + a) \geq \mu(x-a) \wedge \mu(a) = 1 \wedge \mu(a) \geq \mu(a - x + x) \geq \mu(a-x) \wedge \mu(x) = 1 \wedge \mu(x) = \mu(x)$ and, $\mu(x) = \mu(a)$.
(b) We have $\big(\frac{\mu}{I}\big)(\frac{a}{I}) = \bigvee\{\mu(x) : \ x \in \frac{a}{I}\} = \bigvee\{\mu(a)\} = \mu(a)$. $\square$

Theorem 4.1.4. *Let I be an ideal of $\mathcal{A}$. Then L-fuzzy ideals of $\frac{\mathcal{A}}{I}$ are of the form $\frac{\mu}{I}$, where $\mu \in Fid(\mathcal{A}, L, I)$.*

Proof. Consider the maps $\pi : Fid(\mathcal{A}, L, I) \to Fid(\frac{\mathcal{A}}{I}, L)$ and $\tau : Fid(\frac{\mathcal{A}}{I}, L) \to Fid(\mathcal{A}, L, I)$ defined by:

$$\pi(\mu) = \tfrac{\mu}{I} \text{ for all } \mu \in Fid(\mathcal{A}, L, I) \text{ and } \tau(\nu) = \nu_I \text{ for all } \nu \in Fid(\tfrac{\mathcal{A}}{I}, L).$$

- For any $\nu \in Fid(\frac{\mathcal{A}}{I}, L)$, we have $\nu_I(x) = \nu(\frac{x}{I}) = \nu(I) = 1$ for all $x \in I$; thus, $I \subseteq U(\nu_I, 1)$; *i.e.*, $\nu_I \in Fid(\mathcal{A}, L, I)$. So, π and τ are well-defined.
- For any $\nu \in Fid(\frac{\mathcal{A}}{I}, L)$ and $a \in A$, we have

$$\big(\pi \circ \tau\big)(\nu)(\tfrac{a}{I}) = \bigvee\{\nu_I(x) : \ x \in \tfrac{a}{I}\} = \bigvee\{\nu(\tfrac{x}{I}) : \ x \in \tfrac{a}{I}\} = \bigvee\{\nu(\tfrac{a}{I})\} = \nu(\tfrac{a}{I}).$$

Thus, $\pi \circ \tau = Id_{Fid(\frac{\mathcal{A}}{I}, L)}$.

- For any $\mu \in Fid(\mathcal{A}, L, I)$ and $a \in A$, we have

$$\big(\tau \circ \pi\big)(\mu)(a) = \tau\big[\pi(\mu)\big](a) = \tau\big(\tfrac{\mu}{I}\big)(a) = \big(\tfrac{\mu}{I}\big)_I(a) = \big(\tfrac{\mu}{I}\big)\big(\tfrac{a}{I}\big) = \mu(a).$$

Thus, $\tau \circ \pi = Id_{Fid(\mathcal{A},L,I)}$.
Hence, the desired result follows. □

Proposition 4.1.5. *Let $I \in Id(\mathcal{A})$ and $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fid(\mathcal{A}, L, I)$. Then*

(1) $\bigwedge\limits_{\lambda\in\Lambda} \mu_\lambda \in Fid(\mathcal{A}, L, I)$ *and* $\bigwedge\limits_{\lambda\in\Lambda} \frac{\mu_\lambda}{I} = \frac{\bigwedge\limits_{\lambda\in\Lambda} \mu_\lambda}{I}$.

(2) $\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda \in Fid(\mathcal{A}, L, I)$ *and* $\bigsqcup\limits_{\lambda\in\Lambda} \frac{\mu_\lambda}{I} = \frac{\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda}{I}$.

Proof. **(1)** Since $I \subseteq U(\mu_\lambda, 1)$ for all $\lambda \in \Lambda$, we have $I \subseteq \bigcap\limits_{\lambda\in\Lambda} U(\mu_\lambda, 1)$; *i.e.*, $I \subseteq U(\bigwedge\limits_{\lambda\in\Lambda} \mu_\lambda, 1)$; *i.e.*, $\bigwedge\limits_{\lambda\in\Lambda} \mu_\lambda \in Fid(\mathcal{A}, L, I)$.
For any $a \in A$, we have

$$\big(\bigwedge_{\lambda\in\Lambda} \tfrac{\mu_\lambda}{I}\big)\big(\tfrac{a}{I}\big) = \bigwedge_{\lambda\in\Lambda} \big(\tfrac{\mu_\lambda}{I}\big)\big(\tfrac{a}{I}\big) = \bigwedge_{\lambda\in\Lambda} \mu_\lambda(a) = \big(\bigwedge_{\lambda\in\Lambda} \mu_\lambda\big)(a) = \Big(\frac{\bigwedge\limits_{\lambda\in\Lambda} \mu_\lambda}{I}\Big)\big(\tfrac{a}{I}\big).$$

Hence, $\bigwedge\limits_{\lambda\in\Lambda} \frac{\mu_\lambda}{I} = \frac{\bigwedge\limits_{\lambda\in\Lambda} \mu_\lambda}{I}$.
(2) Since $I \subseteq U(\mu_{\lambda_0}, 1) \subseteq U(\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda, 1)$ for some $\lambda_0 \in \Lambda$, we have $\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda \in Fid(\mathcal{A}, L, I)$.
Since $\mu_\lambda \leq \bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda$ for all $\lambda \in \Lambda$, we have $\frac{\mu_\lambda}{I} \leq \frac{\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda}{I}$ for all $\lambda \in \Lambda$; *i.e.*, $\bigsqcup\limits_{\lambda\in\Lambda} \frac{\mu_\lambda}{I} \leq \frac{\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda}{I}$. Now, let $x \in A$. For any finite subset Ω of Λ and $\{a_\lambda\}_{\lambda\in\Omega} \subseteq A$ such that $x = \sum\limits_{\lambda\in\Omega} a_\lambda$, we have

$$\bigwedge_{\lambda\in\Omega} \mu_\lambda(a_\lambda) = \bigwedge_{\lambda\in\Omega} \big(\tfrac{\mu_\lambda}{I}\big)\big(\tfrac{a_\lambda}{I}\big) \leq \big(\bigsqcup_{\lambda\in\Lambda} \tfrac{\mu_\lambda}{I}\big)\big(\sum_{\lambda\in\Omega} \tfrac{a_\lambda}{I}\big) = \big(\bigsqcup_{\lambda\in\Lambda} \tfrac{\mu_\lambda}{I}\big)\Big(\frac{\sum\limits_{\lambda\in\Omega} a_\lambda}{I}\Big) = \big(\bigsqcup_{\lambda\in\Lambda} \tfrac{\mu_\lambda}{I}\big)\big(\tfrac{x}{I}\big).$$

Thus, $\Big(\frac{\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda}{I}\Big)\big(\frac{x}{I}\big) = \big(\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda\big)(x) \leq \big(\bigsqcup\limits_{\lambda\in\Lambda} \frac{\mu_\lambda}{I}\big)\big(\frac{x}{I}\big)$. So, $\frac{\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda}{I} \leq \bigsqcup\limits_{\lambda\in\Lambda} \frac{\mu_\lambda}{I}$. Hence, $\bigsqcup\limits_{\lambda\in\Lambda} \frac{\mu_\lambda}{I} = \frac{\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda}{I}$. □

Example 4.1.6. *Let $L = \{0,\ a,\ b,\ c,\ d,\ 1\}$ be a lattice such that $0 < a < b, c < d < 1$ and b, c are incomparable. Define the binary operations $\ominus$, $\twoheadrightarrow$ and $\multimap$ by the three tables below:*

$\ominus$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	a	a
b	0	0	b	0	b	b
c	0	a	a	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\twoheadrightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	1	1	1	1
b	0	c	1	c	1	1
c	b	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\multimap$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	c	c	1	c	1	1
c	0	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

.

Then $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a complete meet-distributive residuated lattice (See, [22], Example 9). Consider the L-fuzzy ideal μ of $\mathcal{Z}_6$ defined by: $\mu(x) = \begin{cases} 1 & \textit{if } x = 0, \\ c & \textit{if } x = 3, \\ a & \textit{elsewhere.} \end{cases}$ *for all $x \in \frac{\mathbb{Z}}{6\mathbb{Z}}$. We have*

$$\begin{aligned}
\frac{\mu}{\frac{2\mathbb{Z}}{6\mathbb{Z}}} &= \frac{\big((\frac{3\mathbb{Z}}{6\mathbb{Z}})_c\big)_* + \big((\frac{\mathbb{Z}}{6\mathbb{Z}})_a\big)_*}{\frac{2\mathbb{Z}}{6\mathbb{Z}}} \\
&= \frac{\big((\frac{3\mathbb{Z}}{6\mathbb{Z}})_c\big)_*}{\frac{2\mathbb{Z}}{6\mathbb{Z}}} + \frac{\big((\frac{\mathbb{Z}}{6\mathbb{Z}})_a\big)_*}{\frac{2\mathbb{Z}}{6\mathbb{Z}}} \\
&= \Big(\big(\frac{\frac{3\mathbb{Z}}{6\mathbb{Z}} + \frac{2\mathbb{Z}}{6\mathbb{Z}}}{\frac{2\mathbb{Z}}{6\mathbb{Z}}}\big)_c\Big)_* + \Big(\big(\frac{\frac{\mathbb{Z}}{6\mathbb{Z}} + \frac{2\mathbb{Z}}{6\mathbb{Z}}}{\frac{2\mathbb{Z}}{6\mathbb{Z}}}\big)_a\Big)_* \\
&= \big((\frac{\mathbb{Z}}{2\mathbb{Z}})_c\big)_* + \big((\frac{\mathbb{Z}}{2\mathbb{Z}})_a\big)_* \\
&= \big((\frac{\mathbb{Z}}{2\mathbb{Z}})_c\big)_*.
\end{aligned}$$

Proposition 4.1.7. *Let $I \in Id(\mathcal{A})$ and $\mu, \nu \in Fid(\mathcal{A}, L, I)$. Then $\frac{\mu}{I} \otimes \frac{\nu}{I} = \frac{\mu\otimes\nu}{I}$.*

Proof. Let $x \in A$. Let $a \in \frac{x}{I}$. For any $u_1, v_1, \ldots, u_n, v_n \in A$ such that $a = \sum_{i=1}^{n} u_i v_i$, we have

$$\begin{aligned}
\bigwedge_{1\leq i\leq n} \mu(u_i) \ominus \nu(v_i) &= \bigwedge_{1\leq i\leq n} \big(\tfrac{\mu}{I}\big)\big(\tfrac{u_i}{I}\big) \ominus \big(\tfrac{\nu}{I}\big)\big(\tfrac{v_i}{I}\big) \\
&\leq \big(\tfrac{\mu}{I} \otimes \tfrac{\nu}{I}\big)\Big(\sum_{i=1}^{n} \frac{u_i}{I}\frac{v_i}{I}\Big) \\
&= \big(\tfrac{\mu}{I} \otimes \tfrac{\nu}{I}\big)\big(\tfrac{a}{I}\big) \\
&= \big(\tfrac{\mu}{I} \otimes \tfrac{\nu}{I}\big)\big(\tfrac{x}{I}\big).
\end{aligned}$$

Thus, $\big(\mu \otimes \nu\big)(a) \leq \big(\frac{\mu}{I} \otimes \frac{\nu}{I}\big)\big(\frac{x}{I}\big)$. So, $\big(\frac{\mu\otimes\nu}{I}\big)\big(\frac{x}{I}\big) \leq \big(\frac{\mu}{I} \otimes \frac{\nu}{I}\big)\big(\frac{x}{I}\big)$.

Now, let $a_1, b_1, \ldots, a_n, b_n \in A$ such that $\frac{x}{I} = \sum_{i=1}^{n} \frac{a_i}{I}\frac{b_i}{I}$. For any $u_1 \in \frac{a_1}{I}, v_1 \in \frac{b_1}{I}, \ldots, u_n \in \frac{a_n}{I}, v_n \in \frac{b_n}{I}$, we have

$$\frac{\sum_{i=1}^{n} u_i v_i}{I} = \sum_{i=1}^{n} \frac{u_i}{I}\frac{v_i}{I} = \sum_{i=1}^{n} \frac{a_i}{I}\frac{b_i}{I} = \frac{x}{I} \text{ and, } \sum_{i=1}^{n} u_i v_i \in \frac{x}{I};$$

thus,

$$\bigwedge_{1\leq i\leq n} \mu(u_i) \ominus \nu(v_i) \leq \big(\mu \otimes \nu\big)(\sum_{i=1}^{n} u_i v_i) \leq (\frac{\mu \otimes \nu}{I})(\frac{x}{I}).$$

So,

$$\bigwedge_{1\leq i\leq n} \big(\tfrac{\mu}{I}\big)(\tfrac{a_i}{I}) \ominus \big(\tfrac{\nu}{I}\big)(\tfrac{b_i}{I}) \leq \big(\tfrac{\mu\otimes\nu}{I}\big)(\tfrac{x}{I}).$$

Consequently, $\big(\frac{\mu}{I} \otimes \frac{\nu}{I}\big)(\frac{x}{I}) \leq \big(\frac{\mu\otimes\nu}{I}\big)(\frac{x}{I})$ and, $\big(\frac{\mu}{I} \otimes \frac{\nu}{I}\big)(\frac{x}{I}) = \big(\frac{\mu\otimes\nu}{I}\big)(\frac{x}{I})$.
Hence, $\frac{\mu}{I} \otimes \frac{\nu}{I} = \frac{\mu\otimes\nu}{I}$. □

Proposition 4.1.8. *Let $I \in Id(\mathcal{A})$ and $\mu, \nu \in Fid(\mathcal{A}, L, I)$. Then the following hold:*
(1) $\mu \hookrightarrow \nu, \mu \looparrowright \nu \in Fid(\mathcal{A}, L, I)$.
(2) $\frac{\mu}{I} \hookrightarrow \frac{\nu}{I} = \frac{\mu\hookrightarrow\nu}{I}$ *and* $\frac{\mu}{I} \looparrowright \frac{\nu}{I} = \frac{\mu\looparrowright\nu}{I}$.
(3) $(\frac{\mu}{I})^{-} = \frac{\mu\hookrightarrow\chi_I}{I}$ *and* $(\frac{\mu}{I})^{\sim} = \frac{\mu\looparrowright\chi_I}{I}$.

Proof. **(1)** Since $\nu \leq \mu \hookrightarrow \nu$, we have $I \subseteq U(\nu, 1) \subseteq U(\mu \hookrightarrow \nu, 1)$ and, $\mu \hookrightarrow \nu \in Fid(\mathcal{A}, L, I)$. A similar reasoning shows that $\mu \looparrowright \nu \in Fid(\mathcal{A}, L, I)$.
(2) Since $\frac{\mu\hookrightarrow\nu}{I} \otimes \frac{\mu}{I} = \frac{(\mu\hookrightarrow\nu)\otimes\mu}{I} \leq \frac{\nu}{I}$, we have $\frac{\mu\hookrightarrow\nu}{I} \leq \frac{\mu}{I} \hookrightarrow \frac{\nu}{I}$. Now, let $a \in A$. Let $r \in L$ such that $(\frac{a}{I})_r \circ (\frac{\mu}{I}) \leq \frac{\nu}{I}$. Let $x \in A$. For any $v \in A$ such that $x = av$, we have

$$\begin{aligned}
r \ominus \mu(v) &= (\tfrac{a}{I})_r(\tfrac{a}{I}) \ominus (\tfrac{\mu}{I})(\tfrac{v}{I}) \\
&\leq \big((\tfrac{a}{I})_r \circ \tfrac{\mu}{I}\big)(\tfrac{a}{I}\tfrac{v}{I}) \\
&= \big((\tfrac{a}{I})_r \circ \tfrac{\mu}{I}\big)(\tfrac{x}{I}) \\
&\leq \big(\tfrac{\nu}{I}\big)(\tfrac{x}{I}) \\
&= \nu(x).
\end{aligned}$$

Thus, $\big(a_r \circ \mu\big)(x) \leq \nu(x)$. So, $a_r \circ \mu \leq \nu$ and, $r \leq \big(\mu \hookrightarrow \nu\big)(a) = \big(\frac{\mu\hookrightarrow\nu}{I}\big)(\frac{a}{I})$. It follows that $\big(\frac{\mu}{I} \hookrightarrow \frac{\nu}{I}\big)(\frac{a}{I}) \leq \big(\frac{\mu\hookrightarrow\nu}{I}\big)(\frac{a}{I})$.
Hence, $\frac{\mu}{I} \hookrightarrow \frac{\nu}{I} \leq \frac{\mu\hookrightarrow\nu}{I}$ and, $\frac{\mu\hookrightarrow\nu}{I} = \frac{\mu}{I} \hookrightarrow \frac{\nu}{I}$. A similar reasoning shows that $\frac{\mu\looparrowright\nu}{I} = \frac{\mu}{I} \looparrowright \frac{\nu}{I}$.
(3) We have $(\frac{\mu}{I})^{-} = \frac{\mu}{I} \hookrightarrow \chi_{\{I\}} = \frac{\mu}{I} \hookrightarrow \chi_{\frac{I}{I}} = \frac{\mu}{I} \hookrightarrow \frac{\chi_I}{I} = \frac{\mu\hookrightarrow\chi_I}{I}$ and, $(\frac{\mu}{I})^{\sim} = \frac{\mu\looparrowright\chi_I}{I}$ by similar arguments. □

Theorem 4.1.9. *$Id(\mathcal{A}, I) := \{J \in Id(\mathcal{A}) : \ I \subseteq J\}$ is a subresiduated lattice-ordered monoid of $\mathcal{I}(\mathcal{A})$ if and only if $Fid(\mathcal{A}, L, I)$ is a subresiduated lattice-ordered monoid of $\mathcal{F}id(\mathcal{A}, L)$.*

Proof. It suffices to show that $Id(\mathcal{A}, I)$ is closed under $\odot$ iff $Fid(\mathcal{A}, L, I)$ is closed under $\otimes$.
Assume that $Id(\mathcal{A}, I)$ is closed under $\odot$. For any $\mu, \nu \in Fid(\mathcal{A}, L, I)$, we have $U(\mu, 1), U(\nu, 1) \in Id(\mathcal{A}, I)$; thus, $U(\mu, 1) \odot U(\nu, 1) \in Id(\mathcal{A}, I)$; so,

$$I \subseteq U(\mu, 1) \odot U(\nu, 1) \subseteq U(\mu \otimes \nu, 1) \text{ and, } \mu \otimes \nu \in Fid(\mathcal{A}, L, I).$$

Conversely, assume that $Fid(\mathcal{A}, L, I)$ is closed under $\otimes$. For any $J, K \in Id(\mathcal{A}, I)$, we have $\chi_J, \chi_K \in Fid(\mathcal{A}, L, I)$; thus, $\chi_J \otimes \chi_K \in Fid(\mathcal{A}, L, I)$; so, $I \subseteq U(\chi_J \otimes \chi_K, 1) = U(\chi_{J \odot K}, 1) = J \odot K$ and, $J \odot K \in Id(\mathcal{A}, I)$. □

Lemma 4.1.10. *Let $r \in L$, $a \in A$, $I \in Id(\mathcal{A})$ and $\mu \in Fid(\mathcal{A}, L)$. Then*
(a) *$(\frac{a}{I})_r \circ \frac{\mu}{I} \leq \chi_{\{I\}}$ if and only if $a_r \circ \mu \leq \chi_I$.*
(b) *$(\frac{\mu}{I}) \circ (\frac{a}{I})_r \leq \chi_{\{I\}}$ if and only if $\mu \circ a_r \leq \chi_I$.*

Proof. **(a)** Assume that $(\frac{a}{I})_r \circ \frac{\mu}{I} \leq \chi_{\{I\}}$. Let $x \notin I$. For any $v \in A$ such that $x = av$, we have $r \ominus \mu(v) \leq (\frac{a}{I})_r(\frac{a}{I}) \ominus (\frac{\mu}{I})(\frac{v}{I}) \leq \big((\frac{a}{I})_r \circ (\frac{\mu}{I})\big)(\frac{x}{I}) \leq \chi_{\{I\}}(\frac{x}{I}) = 0$ and, $r \ominus \mu(v) = 0$. Thus,

$$\big(a_r \circ \mu\big)(x) = \bigvee\{0\} = 0.$$

Hence, $a_r \circ \mu \leq \chi_I$.
Conversely, assume that $a_r \circ \mu \leq \chi_I$. We have

$$\big((\tfrac{a}{I})_r \circ (\tfrac{\mu}{I})\big)(I) \leq 1 = \chi_{\{I\}}(I).$$

Now, let $x \notin I$. Let $v \in A$ such that $\frac{x}{I} = \frac{a}{I}\frac{v}{I}$. For any $w \in \frac{v}{I}$, we have $aw \notin I$; thus, $r \ominus \mu(w) = a_r(a) \ominus \mu(w) \leq \big(a_r \circ \mu\big)(aw) \leq \chi_I(aw) = 0$ and, $r \ominus \mu(w) = 0$. So,

$$r \ominus (\tfrac{\mu}{I})(\tfrac{v}{I}) = \bigvee\{0\} = 0.$$

It follows that

$$\big((\tfrac{a}{I})_r \circ (\tfrac{\mu}{I})\big)(\tfrac{x}{I}) = \bigvee\{0\} = 0 = \chi_{\{I\}}(\tfrac{x}{I}).$$

Hence, $(\frac{a}{I})_r \circ (\frac{\mu}{I}) \leq \chi_{\{I\}}$.
(b) Similar to **(a)**. □

Lemma 4.1.11. *Suppose that $\mathcal{L}$ is product-distributive (See, Definition 1.2.15) and $xJ \in Id(\mathcal{A})$ for all $x \in A$ and $J \in Id(\mathcal{A})$. Let $r \in L$, $a \in A$, $I \in Id(\mathcal{A})$ and $\mu, \nu \in Fid(\mathcal{A}, L)$. Then the following hold:*
(a) *$(a_r)_* \circ \mu$ and $\mu \circ (a_r)_*$ are L-fuzzy ideals of $\mathcal{A}$.*
(b) *$(a_r)_* \circ (\mu \otimes \nu) = \big((a_r)_* \circ \mu\big) \otimes \nu$ and $(\mu \otimes \nu) \circ (a_r)_* = \mu \otimes \big(\nu \circ (a_r)_*\big)$.*
(c) *If $I^{\sim -} = I$ (resp., $I^{-\sim} = I$), then*

$$(\tfrac{\mu}{I})^{-}(\tfrac{a}{I}) = (\mu \otimes \chi_{I^{\sim}})^{-}(a) \text{ (resp., } (\tfrac{\mu}{I})^{\sim}(\tfrac{a}{I}) = (\chi_{I^{-}} \otimes \mu)^{\sim}(a)\text{)}.$$

Proof. **(a)** $\big((a_r)_* \circ \mu\big)(0) \geq (a_r)_*(0) \ominus \mu(0) = 1 \ominus 1 = 1$ and, $\big((a_r)_* \circ \mu\big)(0) = 1$. Now, let $x, y \in A$.

Let $u_1, v_1, u_2, v_2 \in A$ such that $x = u_1 v_1$ and $y = u_2 v_2$. Set

$$\Gamma(u,v) := \big((a_r)_*(u_1) \ominus \mu(v_1)\big) \wedge \big((a_r)_*(u_2) \ominus \mu(v_2)\big).$$

• If $u_1 \notin \{0,\ a\}$ or $u_2 \notin \{0,\ a\}$, then $(a_r)_*(u_1) = 0$ or $(a_r)_*(u_2) = 0$; thus,

$$\Gamma(u,v) = 0 \leq \big((a_r)_* \circ \mu\big)(x - y).$$

• If $u_1 = 0$ and $u_2 = 0$, then

$$\Gamma(u,v) \leq 1 = \big((a_r)_* \circ \mu\big)(0) = \big((a_r)_* \circ \mu\big)(x - y).$$

• If $u_1 = a$ and $u_2 = 0$, then

$$\Gamma(u,v) \leq (a_r)_*(a) \ominus \mu(v_1) \leq \big((a_r)_* \circ \mu\big)(av_1) = \big((a_r)_* \circ \mu\big)(x - y).$$

• If $u_1 = a$ and $u_2 = a$, then

$$\begin{aligned}
\Gamma(u,v) &= \big(r \ominus \mu(v_1)\big) \wedge \big(r \ominus \mu(v_2)\big)\\
&= r \ominus \big(\mu(v_1) \wedge \mu(v_2)\big)\\
&\leq r \ominus \mu(v_1 - v_2)\\
&= (a_r)_*(a) \ominus \mu(v_1 - v_2)\\
&\leq \big((a_r)_* \circ \mu\big)\big(a(v_1 - v_2)\big)\\
&= \big((a_r)_* \circ \mu\big)(x - y).
\end{aligned}$$

• If $u_1 = 0$ and $u_2 = a$, then

$$\Gamma(u,v) \leq (a_r)_*(a) \ominus \mu(-v_2) \leq \big((a_r)_* \circ \mu\big)\big(-av_2\big) = \big((a_r)_* \circ \mu\big)(x - y).$$

It follows that $\big((a_r)_* \circ \mu\big)(x - y) \geq \big((a_r)_* \circ \mu\big)(x) \wedge \big((a_r)_* \circ \mu\big)(y)$.

• For any $u, v \in A$ such that $x = uv$, we have

$$(a_r)_*(u) \ominus \mu(v) \leq (a_r)_*(u) \ominus \mu(vy) \leq \big((a_r)_* \circ \mu\big)\big(u(vy)\big) = \big((a_r)_* \circ \mu\big)(xy).$$

Thus,

$$\big((a_r)_* \circ \mu\big)(x) \leq \big((a_r)_* \circ \mu\big)(xy).$$

• For any $u, v \in A$ such that $y = uv$, we have

$$xy = xuv \in x\big[uU\big(\mu, \mu(v)\big)\big] \subseteq uU\big(\mu, \mu(v)\big)$$

and, $xy = uc$ for some $c \in U\big(\mu, \mu(v)\big)$; thus,

$$(a_r)_*(u) \ominus \mu(v) \leq (a_r)_*(u) \ominus \mu(c) \leq \big((a_r)_* \circ \mu\big)(uc) = \big((a_r)_* \circ \mu\big)(xy).$$

Thus,

$$\big((a_r)_* \circ \mu\big)(y) \leq \big((a_r)_* \circ \mu\big)(xy).$$

So, $\big((a_r)_* \circ \mu\big)(xy) \geq \big((a_r)_* \circ \mu\big)(x) \vee \big((a_r)_* \circ \mu\big)(y)$.
Hence, $(a_r)_* \circ \mu$ is an L-fuzzy ideal of $\mathcal{A}$. A similar reasoning shows that $\mu \circ (a_r)_*$ is an L-fuzzy ideal of $\mathcal{A}$.
(b) Let $x \in A$. Let $u, v \in A$ such that $x = uv$. For any $b_1, c_1, \ldots, b_n, c_n \in A$ such that $v = \sum_{i=1}^{n} b_i c_i$, we have $\sum_{i=1}^{n} (ub_i)c_i = u \sum_{i=1}^{n} b_i c_i = uv = x$ and,

$$\begin{aligned} (a_r)_*(u) \ominus \bigwedge_{1 \leq i \leq n} \mu(b_i) \ominus \nu(c_i) &= \bigwedge_{1 \leq i \leq n} \big((a_r)_*(u) \ominus \mu(b_i)\big) \ominus \nu(c_i) \\ &\leq \bigwedge_{1 \leq i \leq n} \big((a_r)_* \circ \mu\big)(ub_i) \ominus \nu(c_i) \\ &\leq \big[\big((a_r)_* \circ \mu\big) \otimes \nu\big](x). \end{aligned}$$

Thus, $(a_r)_*(u) \ominus \big(\mu \otimes \nu\big)(v) \leq \big[\big((a_r)_* \circ \mu\big) \otimes \nu\big](x)$. So, $\big[(a_r)_* \circ (\mu \otimes \nu)\big](x) \leq \big[\big((a_r)_* \circ \mu\big) \otimes \nu\big](x)$. Now, let $u_1, v_1, \ldots, u_n, v_n \in A$ such that $x = \sum_{i=1}^{n} u_i v_i$. Let $w_1, z_1, \ldots, w_n, z_n \in A$ such that $u_1 = w_1 z_1, \ldots, u_n = w_n z_n$.

- If there is $1 \leq i_0 \leq n$ such that $w_{i_0} \notin \{0,\ a\}$, then $(a_r)_*(w_{i_0}) = 0$ and,

$$\bigwedge_{1 \leq i \leq n} \big((a_r)_*(w_i) \ominus \mu(z_i)\big) \ominus \nu(v_i) = 0 \leq \big[(a_r)_* \circ (\mu \otimes \nu)\big](x).$$

- If there are $i_1, \ldots, i_p \in \{1, \ldots, n\}$ such that $w_{i_k} = a$ for all $1 \leq k \leq p$ and $w_j = 0$ for all $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$, then $x = a\big(\sum_{k=1}^{p} z_{i_k} v_{i_k}\big)$ and,

$$\begin{aligned} \bigwedge_{1 \leq i \leq n} \big((a_r)_*(w_i) \ominus \mu(z_i)\big) \ominus \nu(v_i) &\leq \bigwedge_{1 \leq k \leq p} \big((a_r)_*(w_{i_k}) \ominus \mu(z_{i_k})\big) \ominus \nu(v_{i_k}) \\ &= \bigwedge_{1 \leq k \leq p} \big((a_r)_*(a) \ominus \mu(z_{i_k})\big) \ominus \nu(v_{i_k}) \\ &= (a_r)_*(a) \ominus \big(\bigwedge_{1 \leq k \leq p} \mu(z_{i_k}) \ominus \nu(v_{i_k})\big) \\ &\leq (a_r)_*(a) \ominus \big(\mu \otimes \nu\big)\big(\sum_{k=1}^{p} z_{i_k} v_{i_k}\big) \\ &\leq \big[(a_r)_* \circ (\mu \otimes \nu)\big](x). \end{aligned}$$

It follows that $\bigwedge_{1 \leq i \leq n} \big((a_r)_* \circ \mu\big)(u_i) \ominus \nu(v_i) \leq \big[(a_r)_* \circ (\mu \otimes \nu)\big](x)$. Consequently,

$$\big[\big((a_r)_* \circ \mu\big) \otimes \nu\big](x) \leq \big[(a_r)_* \circ (\mu \otimes \nu)\big](x) \text{ and,}$$

$$\big[(a_r)_* \circ (\mu \otimes \nu)\big](x) = \big[\big((a_r)_* \circ \mu\big) \otimes \nu\big](x).$$

Hence, $(a_r)_* \circ (\mu \otimes \nu) = \big((a_r)_* \circ \mu\big) \otimes \nu$. A similar reasoning shows that $(\mu \otimes \nu) \circ (a_r)_* = \mu \otimes \big(\nu \circ (a_r)_*\big)$.
(c) Assume that $I^{\sim -} = I$. Since $\chi_I \otimes (\chi_I)^{\sim} = \chi_0$ and $\big((\chi_I)^{\sim}\big)^{-} = \chi_I$, we have

$$\begin{aligned}
\left(\tfrac{\mu}{I}\right)^{-}\left(\tfrac{a}{I}\right) &= \bigvee\{r \in L :\ (\tfrac{a}{I})_r \circ (\tfrac{\mu}{I}) \leq \chi_{\{I\}}\} \\
&= \bigvee\{r \in L :\ a_r \circ \mu \leq \chi_I\} \\
&= \bigvee\{r \in L :\ (a_r)_* \circ \mu \leq \chi_I\} \\
&= \bigvee\{r \in L :\ \big((a_r)_* \circ \mu\big) \otimes (\chi_I)^{\sim} \leq \chi_0\} \\
&= \bigvee\{r \in L :\ (a_r)_* \circ \big(\mu \otimes (\chi_I)^{\sim}\big) \leq \chi_0\} \\
&= \bigvee\{r \in L :\ a_r \circ \big(\mu \otimes (\chi_I)^{\sim}\big) \leq \chi_0\} \\
&= \big(\mu \otimes (\chi_I)^{\sim}\big)^{-}(a) \\
&= \big(\mu \otimes \chi_{I^{\sim}}\big)^{-}(a).
\end{aligned}$$

A similar reasoning shows the second implication. □

As the following example shows, the result **(a)** of the previous Lemma is not true in general.

Example 4.1.12. *Let $\mathcal{L}$ be the residuated lattice of Example 1.2.17 and μ the L-fuzzy ideal of $\mathcal{Z}_{12}$ defined by:* $\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ a & \text{if } x = 6, \\ b & \text{if } x \in \{4,\ 8\}, \\ n & \text{if } x \in \{2,\ 10\}, \\ 0 & \text{if not.} \end{cases}$ *for all $x \in \mathbb{Z}_{12}$.*
$(7_m)_* \circ \mu$ is not an L-fuzzy ideal of $\mathcal{Z}_{12}$, since

$$\begin{aligned}
\big((7_m)_* \circ \mu\big)(6-4) &= \big((7_m)_* \circ \mu\big)(2) = \bigvee\{m \ominus \mu(2)\} = m \ominus n \\
&= 0 \\
&\ngeq n \\
&= a \wedge b \\
&= (m \ominus a) \wedge (m \ominus b) \\
&= \big(\bigvee\{m \ominus \mu(6)\}\big) \wedge \big(\bigvee\{m \ominus \mu(4)\}\big) \\
&= \big((7_m)_* \circ \mu\big)(6) \wedge \big((7_m)_* \circ \mu\big)(4).
\end{aligned}$$

Proposition 4.1.13. *Suppose that $\mathcal{L}$ is product-distributive and $xJ \in Id(\mathcal{A})$ for all $x \in A$ and $J \in Id(\mathcal{A})$. Let $I \in Id(\mathcal{A})$ and $\mu \in Fid(\mathcal{A}, L, I)$. Then the*

following hold:
(1) $(\mu \otimes \chi_{I^\sim})^-, (\chi_{I^-} \otimes \mu)^\sim \in Fid(\mathcal{A}, L, I)$.
(2) *If* $I^{\sim-} = I$ *(resp.,* $I^{-\sim} = I$*), then*

$$(\tfrac{\mu}{I})^- = \tfrac{(\mu\otimes\chi_{I^\sim})^-}{I} \text{ (resp., } (\tfrac{\mu}{I})^\sim = \tfrac{(\chi_{I^-}\otimes\mu)^\sim}{I}\text{)}.$$

Proof. **(1)** Since $\mu \otimes \chi_{I^\sim} \leq \chi_{I^\sim} = (\chi_I)^\sim$, we have $\chi_I \leq (\chi_I)^{\sim-} \leq (\mu \otimes \chi_{I^\sim})^-$; thus, $I = U(\chi_I, 1) \subseteq U\big((\mu \otimes \chi_{I^\sim})^-, 1\big)$ and, $(\mu \otimes \chi_{I^\sim})^- \in Fid(\mathcal{A}, L, I)$. A similar reasoning shows that $(\chi_{I^-} \otimes \mu)^\sim \in Fid(\mathcal{A}, L, I)$.
(2) Assume that $I^{\sim-} = I$. For any $a \in A$, we have

$$\big(\tfrac{(\mu\otimes\chi_{I^\sim})^-}{I}\big)(\tfrac{a}{I}) = (\mu \otimes \chi_{I^\sim})^-(a) = (\tfrac{\mu}{I})^-(\tfrac{a}{I}).$$

Hence, $(\frac{\mu}{I})^- = \frac{(\mu\otimes\chi_{I^\sim})^-}{I}$. A similar reasoning shows the second implication. □

4.2 L-preimage functor

Let $\mathbb{R}ing$ be the subcategory of the category of rings, with unital rings as objects and homomorphisms of unital rings as arrows.

Lemma 4.2.1. *Let* $\mathcal{A} \xrightarrow{f} \mathcal{B}$ *in* $\mathbb{R}ing$. *The arrow* $Fid(\mathcal{B}, L) \xrightarrow{Fid_L^{-1}(f)} Fid(\mathcal{A}, L)$, *given by* $Fid_L^{-1}(f)(\mu) = \mu \circ f$ *for all* $\mu \in Fid(\mathcal{B}, L)$, *is well-defined.*

Proof. Let $\mu \in Fid(\mathcal{B}, L)$. We have

$$Fid_L^{-1}(f)(\mu)(0_A) = \mu\big(f(0_A)\big) = \mu(0_B) = 1.$$

For any $a, b \in A$, we have

$$\begin{aligned} Fid_L^{-1}(f)(\mu)(a-b) &= \mu\big(f(a-b)\big) \\ &= \mu\big(f(a) - f(b)\big) \\ &\geq \mu\big(f(a)\big) \wedge \mu\big(f(b)\big) \\ &= Fid_L^{-1}(f)(\mu)(a) \wedge Fid_L^{-1}(f)(\mu)(b) \end{aligned}$$

$$\begin{aligned} Fid_L^{-1}(f)(\mu)(ab) &= \mu\big(f(ab)\big) \\ &= \mu\big(f(a)f(b)\big) \\ &\geq \mu\big(f(a)\big) \vee \mu\big(f(b)\big) \\ &= Fid_L^{-1}(f)(\mu)(a) \vee Fid_L^{-1}(f)(\mu)(b). \end{aligned}$$

Hence, $Fid_L^{-1}(f)(\mu)$ is an L-fuzzy ideal of $\mathcal{A}$. □

Lemma 4.2.2. ***(a)*** *For any* $\mathcal{A}$ *in* $\mathbb{R}ing$, $Fid_L^{-1}(Id_A) = Id_{Fid(\mathcal{A},L)}$.
(b) *For any* $\mathcal{A} \xrightarrow{f} \mathcal{B}$ *and* $\mathcal{B} \xrightarrow{g} \mathcal{C}$ *in* $\mathbb{R}ing$, $Fid_L^{-1}(g \circ f) = Fid_L^{-1}(f) \circ Fid_L^{-1}(g)$.

Proof. **(a)** Let $\mathcal{A}$ in $\mathbb{R}ing$. For any $\mu \in Fid(\mathcal{A}, L)$, we have

$$Fid_L^{-1}(Id_A)(\mu)(x) = \mu\big(Id_A(x)\big) = \mu(x) \text{ for all } x \in A;$$

thus, $Fid_L^{-1}(Id_A)(\mu) = \mu$. So, $Fid_L^{-1}(Id_A) = Id_{Fid(\mathcal{A},L)}$.
(b) Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ and $\mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathbb{R}ing$. For any $\mu \in Fid(\mathcal{C}, L)$, we have

$$\begin{aligned}\big[Fid_L^{-1}(f) \circ Fid_L^{-1}(g)\big](\mu) &= Fid_L^{-1}(g)(\mu) \circ f \\ &= (\mu \circ g) \circ f \\ &= \mu \circ (g \circ f) \\ &= Fid_L^{-1}(g \circ f)(\mu).\end{aligned}$$

Hence, $Fid_L^{-1}(g \circ f) = Fid_L^{-1}(f) \circ Fid_L^{-1}(g)$. □

Definition 4.2.3. *[26] A function* $\sigma : L_1 \to L_2$ *from a lattice-ordered monoid to a lattice-ordered monoid is said to be submultiplicative if* $\sigma(e_1) = e_2$ *and* $\sigma(x) \ominus \sigma(y) \leq \sigma(x \ominus y)$ *for all* $x, y \in L_1$.

Theorem 4.2.4. Fid_L^{-1} *is a contravariant functor, called* L*-preimage functor, from* $\mathbb{R}ing$ *to the category* $\mathbb{P}oMod$, *whose objects are partially ordered monoids and arrows are submultiplicative order-preserving functions.*

Proof. From the above lemmas, it suffices to show that Fid_L^{-1} is well-defined. So, let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$. It is easy to check that $Fid_L^{-1}(f)$ is order-preserving. Since $Fid_L^{-1}(f)(\underline{1})(x) = \underline{1}\big(f(x)\big) = 1$ for all $x \in A$, we have $Fid_L^{-1}(f)(\underline{1}) = \underline{1}$. Now, let $\mu, \nu \in Fid(\mathcal{B}, L)$. Let $x \in A$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$, we have $f(x) = f(\sum_{i=1}^{n} a_i b_i) = \sum_{i=1}^{n} f(a_i) f(b_i)$; thus,

$$\begin{aligned}\bigwedge_{1 \leq i \leq n} Fid_L^{-1}(f)(\mu)(a_i) \ominus Fid_L^{-1}(f)(\nu)(b_i) &= \bigwedge_{1 \leq i \leq n} \mu\big(f(a_i)\big) \ominus \nu\big(f(b_i)\big) \\ &\leq \big(\mu \otimes \nu\big)\big(f(x)\big) \\ &= Fid_L^{-1}(f)(\mu \otimes \nu)(x).\end{aligned}$$

So, $\big[Fid_L^{-1}(f)(\mu) \otimes Fid_L^{-1}(f)(\nu)\big](x) \leq Fid_L^{-1}(f)(\mu \otimes \nu)(x)$. It follows that $Fid_L^{-1}(f)(\mu) \otimes Fid_L^{-1}(f)(\nu) \leq Fid_L^{-1}(f)(\mu \otimes \nu)$. Hence, Fid_L^{-1} is a functor. □

Proposition 4.2.5. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$. Then the following hold:*
(1) For any $I \in Id(\mathcal{B})$, we have $Fid_L^{-1}(f)(\chi_I) = \chi_{f^{-1}(I)}$.
(2) $Fid_L^{-1}(f)(\chi_{0_B}) = \chi_{0_A}$ if and only if f is one-to-one.
(3) For any $\mu \in Fid(\mathcal{B}, L)$, we have $Fid_L^{-1}(f)(\mu) \in [\chi_{f^{-1}(\{0_B\})},\ \underline{1}]$.
(4) $\big([\chi_{f^{-1}(\{0_B\})},\ \underline{1}];\ \wedge,\ +,\ \otimes,\ \hookrightarrow,\ \looparrowright;\ \underline{1}\big)$ is a subresiduated lattice ordered monoid of $\mathcal{F}id(\mathcal{A}, L)$ if and only if $\big([f^{-1}(\{0_B\}),\ A];\ \cap,\ +,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ A\big)$ is a subresiduated lattice ordered monoid of $\mathcal{I}d(\mathcal{A})$.
(5) $\big([\chi_{f^{-1}(\{0_B\})},\ \underline{1}];\ \wedge,\ +,\ \otimes,\ \hookrightarrow,\ \looparrowright;\ \chi_{f^{-1}(\{0_B\})},\ \underline{1}\big)$ is a residuated lattice if and only if $\big([f^{-1}(\{0_B\}),\ A];\ \cap,\ +,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ f^{-1}(\{0_B\}),\ A\big)$ is a residuated lattice.

Proof. **(1)** Let $I \in Id(\mathcal{B})$. For any $x \in f^{-1}(I)$, we have $Fid_L^{-1}(f)(\chi_I)(x) = \chi_I\big(f(x)\big) = 1$. For any $x \notin f^{-1}(I)$, we have $Fid_L^{-1}(f)(\chi_I)(x) = \chi_I\big(f(x)\big) = 0$. Hence, $Fid_L^{-1}(f)(\chi_I) = \chi_{f^{-1}(I)}$.
(2) Straightforward, since $Fid_L^{-1}(f)(\chi_{0_B}) = \chi_{f^{-1}(\{0_B\})}$ by **(1)**.
(3) Let $\mu \in Fid(\mathcal{B}, L)$. For any $x \in f^{-1}(\{0_B\})$, we have

$$Fid_L^{-1}(f)(\mu)(x) = \mu\big(f(x)\big) = \mu(0_B) = 1.$$

Thus, $\chi_{f^{-1}(\{0_B\})} \leq Fid_L^{-1}(f)(\mu) \leq \underline{1}$; *i.e.*, $Fid_L^{-1}(f)(\mu) \in [\chi_{f^{-1}(\{0_B\})},\ \underline{1}]$.
(4) Since $\mathcal{I}d(\mathcal{A})$ can be embedded into $\mathcal{F}id(\mathcal{A}, L)$, it suffices to show the second implication. So, assume that $\big([f^{-1}(\{0_B\}),\ A];\ \cap,\ +,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ A\big)$ is a subresiduated lattice ordered monoid of $\mathcal{I}d(\mathcal{A})$. Let $\mu, \nu \in [\chi_{f^{-1}(\{0_B\})},\ \underline{1}]$.
• Since $f^{-1}(\{0_B\}) = f^{-1}(\{0_B\}) \odot f^{-1}(\{0_B\}) \subseteq U(\mu, 1) \odot U(\nu, 1) \subseteq U(\mu \otimes \nu, 1)$, we have $\big(\mu \otimes \nu\big)(x) = 1$ for all $x \in f^{-1}(\{0_B\})$; thus, $\chi_{f^{-1}(\{0_B\})} \leq \mu \otimes \nu \leq \underline{1}$; *i.e.*, $\mu \otimes \nu \in [\chi_{f^{-1}(\{0_B\})},\ \underline{1}]$.
• Since $f^{-1}(\{0_B\}) \subseteq U(\nu, 1) \subseteq U(\mu \hookrightarrow \nu, 1)$, we have $\big(\mu \hookrightarrow \nu\big)(x) = 1$ for all $x \in f^{-1}(\{0_B\})$; thus, $\chi_{f^{-1}(\{0_B\})} \leq \mu \hookrightarrow \nu \leq \underline{1}$; *i.e.*, $\mu \hookrightarrow \nu \in [\chi_{f^{-1}(\{0_B\})},\ \underline{1}]$. A similar reasoning shows that $\mu \looparrowright \nu \in [\chi_{f^{-1}(\{0_B\})},\ \underline{1}]$.
Hence, $\big([\chi_{f^{-1}(\{0_B\})},\ \underline{1}];\ \wedge,\ +,\ \otimes,\ \hookrightarrow,\ \looparrowright;\ \underline{1}\big)$ is a subresiduated lattice ordered monoid of $\mathcal{F}id(\mathcal{A}, L)$.
(5) Immediate consequence of **(4)**. □

Proposition 4.2.6. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$. For any $\{\mu_\lambda\}_{\lambda \in \Lambda} \subseteq Fid(\mathcal{B}, L)$, we have $\bigwedge\limits_{\lambda \in \Lambda} Fid_L^{-1}(f)(\mu_\lambda) = Fid_L^{-1}(f)(\bigwedge\limits_{\lambda \in \Lambda} \mu_\lambda)$.*

Proof. Let $\{\mu_\lambda\}_{\lambda \in \Lambda} \subseteq Fid(\mathcal{B}, L)$. For any $x \in A$, we have $Fid_L^{-1}(f)(\bigwedge\limits_{\lambda \in \Lambda} \mu_\lambda)(x) = \big(\bigwedge\limits_{\lambda \in \Lambda} \mu_\lambda\big)\big(f(x)\big) = \bigwedge\limits_{\lambda \in \Lambda} \mu_\lambda\big(f(x)\big) = \bigwedge\limits_{\lambda \in \Lambda} Fid_L^{-1}(f)(\mu_\lambda)(x) = \big[\bigwedge\limits_{\lambda \in \Lambda} Fid_L^{-1}(f)(\mu_\lambda)\big](x)$.
Hence, $\bigwedge\limits_{\lambda \in \Lambda} Fid_L^{-1}(f)(\mu_\lambda) = Fid_L^{-1}(f)(\bigwedge\limits_{\lambda \in \Lambda} \mu_\lambda)$. □

Proposition 4.2.7. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$. Then the following are equivalent:*
(1) For any $\{\mu_\lambda\}_{\lambda \in \Lambda} \subseteq Fid(\mathcal{B}, L)$, $\bigsqcup\limits_{\lambda \in \Lambda} Fid_L^{-1}(f)(\mu_\lambda) = Fid_L^{-1}(f)(\bigsqcup\limits_{\lambda \in \Lambda} \mu_\lambda)$.

(2) For any $\{I_\lambda\}_{\lambda\in\Lambda} \subseteq Id(\mathcal{B})$, $\bigsqcup\limits_{\lambda\in\Lambda} f^{-1}(I_\lambda) = f^{-1}(\bigsqcup\limits_{\lambda\in\Lambda} I_\lambda)$.

Proof. Since $\mathcal{I}d(\mathcal{B})$ can be embedded into $\mathcal{F}id(\mathcal{B}, L)$, it suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. Let $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fid(\mathcal{B}, L)$. For any $x \in A$ such that $f(x) = 0_B$, we have

$$Fid_L^{-1}(f)(\bigsqcup_{\lambda\in\Lambda} \mu_\lambda)(x) = 1 = Fid_L^{-1}(f)(\mu_{\lambda_0})(x) \leq \big[\bigsqcup_{\lambda\in\Lambda} Fid_L^{-1}(f)(\mu_\lambda)\big](x)$$

for some $\lambda_0 \in \Lambda$. Now, let $x \in A$ such that $f(x) \neq 0_B$. For any finite subset Ω of Λ such that $f(x) = \sum\limits_{\lambda\in\Omega} a_\lambda$, we have

$$x \in f^{-1}\big[\bigsqcup_{\lambda\in\Omega} U\big(\mu_\lambda, \mu_\lambda(a_\lambda)\big)\big] = \bigsqcup_{\lambda\in\Omega} f^{-1}\big[U\big(\mu_\lambda, \mu_\lambda(a_\lambda)\big)\big];$$

thus, $x = \sum\limits_{\lambda\in\Omega} u_\lambda$ for some $u_\lambda \in f^{-1}\big[U\big(\mu_\lambda, \mu_\lambda(a_\lambda)\big)\big]$ $(\lambda \in \Omega)$; so,

$$\bigwedge_{\lambda\in\Omega} \mu_\lambda(a_\lambda) \leq \bigwedge_{\lambda\in\Omega} \mu_\lambda\big(f(u_\lambda)\big) = \bigwedge_{\lambda\in\Omega} Fid_L^{-1}(f)(\mu_\lambda)(u_\lambda) \leq \big[\bigsqcup_{\lambda\in\Lambda} Fid_L^{-1}(f)(\mu_\lambda)\big](x).$$

Hence, $Fid_L^{-1}(f)(\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda)(x) = \big(\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda\big)\big(f(x)\big) \leq \big[\bigsqcup\limits_{\lambda\in\Lambda} Fid_L^{-1}(f)(\mu_\lambda)\big](x)$. Therefore, $Fid_L^{-1}(f)(\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda) \leq \bigsqcup\limits_{\lambda\in\Lambda} Fid_L^{-1}(f)(\mu_\lambda)$. Since $Fid_L^{-1}(f)$ is order-preserving, we have $\bigsqcup\limits_{\lambda\in\Lambda} Fid_L^{-1}(f)(\mu_\lambda) = Fid_L^{-1}(f)(\bigsqcup\limits_{\lambda\in\Lambda} \mu_\lambda)$. □

One can remark that f^{-1} preserves $\bigsqcup$ if and only if f^{-1} preserves $+$ if and only if $Fid_L^{-1}(f)$ preserves $\bigsqcup$ if and only if $Fid_L^{-1}(f)$ preserves $+$. Furthermore, the L-preimage of any projection (resp., natural) homomorphism preserves $\bigsqcup$.

Lemma 4.2.8. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$. Then the following are equivalent:*
(a) *For any $I, J \in Id(\mathcal{B})$ and $r, s \in L$,*

$$Fid_L^{-1}(f)\big((I_r)_*\big) \otimes Fid_L^{-1}(f)\big((J_s)_*\big) = Fid_L^{-1}(f)\big[\big((I \odot J)_{r\ominus s}\big)_*\big].$$

(b) *For any $I, J \in Id(\mathcal{B})$, $f^{-1}(I) \odot f^{-1}(J) = f^{-1}(I \odot J)$.*

Proof. Since $\mathcal{I}d(\mathcal{B})$ can be embedded into $\mathcal{F}id(\mathcal{B}, L)$, it suffices to show that **(b)** implies **(a)**. So, assume that **(b)** is satisfied. Let $I, J \in Id(\mathcal{B})$ and $r, s \in L$. For any $x \in A$ such that $f(x) \notin I \odot J$, we have

$$\begin{aligned} Fid_L^{-1}(f)\big[\big((I \odot J)_{r\ominus s}\big)_*\big](x) &= \big((I \odot J)_{r\ominus s}\big)_*\big(f(x)\big) \\ &= 0 \\ &\leq \big[Fid_L^{-1}(f)\big((I_r)_*\big) \otimes Fid_L^{-1}(f)\big((J_s)_*\big)\big](x). \end{aligned}$$

For any $x \in A$ such that $f(x) = 0_B$, we have

$$x \in f^{-1}(\{0_B\} \odot \{0_B\}) = f^{-1}(\{0_B\}) \odot f^{-1}(\{0_B\});$$

thus, $x = \sum_{i=1}^{n} a_i b_i$ for some $a_1, b_1, \ldots, a_n, b_n \in f^{-1}(\{0_B\})$; so,

$$\begin{aligned} Fid_L^{-1}(f)\big[\big((I \odot J)_{r\ominus s}\big)_*\big](x) &= 1 \\ &= \bigwedge_{1\le i\le n} 1 \ominus 1 \\ &= \bigwedge_{1\le i\le n} (I_r)_*\big(f(a_i)\big) \ominus (J_s)_*\big(f(b_i)\big) \\ &= \bigwedge_{1\le i\le n} Fid_L^{-1}(f)\big((I_r)_*\big)(a_i) \ominus Fid_L^{-1}(f)\big((J_s)_*\big)(b_i) \\ &\le \big[Fid_L^{-1}(f)\big((I_r)_*\big) \otimes Fid_L^{-1}(f)\big((J_s)_*\big)\big](x). \end{aligned}$$

For any $x \in A$ such that $f(x) \in (I \odot J) \setminus \{0_B\}$, we have

$$x \in f^{-1}(I \odot J) = f^{-1}(I) \odot f^{-1}(J);$$

thus, $x = \sum_{i=1}^{n} a_i b_i$ for some $a_1, \ldots, a_n \in f^{-1}(I)$ and $b_1, \ldots, b_n \in f^{-1}(J)$; so,

$$r \le (I_r)_*\big(f(a_i)\big) = Fid_L^{-1}(f)\big((I_r)_*\big)(a_i) \text{ and } s \le Fid_L^{-1}(f)\big((J_s)_*\big)(b_i)$$

for all $1 \le i \le n$ and, $r \ominus s \le Fid_L^{-1}(f)\big((I_r)_*\big)(a_i) \ominus Fid_L^{-1}(f)\big((J_s)_*\big)(b_i)$ for all $1 \le i \le n$; consequently,

$$\begin{aligned} Fid_L^{-1}(f)\big[\big((I \odot J)_{r\ominus s}\big)_*\big](x) &= r \ominus s \\ &\le \bigwedge_{1\le i\le n} Fid_L^{-1}(f)\big((I_r)_*\big)(a_i) \ominus Fid_L^{-1}(f)\big((J_s)_*\big)(b_i) \\ &\le \big[Fid_L^{-1}(f)\big((I_r)_*\big) \otimes Fid_L^{-1}(f)\big((J_s)_*\big)\big](x). \end{aligned}$$

Hence, $Fid_L^{-1}(f)\big[\big((I \odot J)_{r\ominus s}\big)_*\big] \le Fid_L^{-1}(f)\big((I_r)_*\big) \otimes Fid_L^{-1}(f)\big((J_s)_*\big)$ and, $Fid_L^{-1}(f)\big((I_r)_*\big) \otimes Fid_L^{-1}(f)\big((J_s)_*\big) = Fid_L^{-1}(f)\big[\big((I \odot J)_{r\ominus s}\big)_*\big]$. □

Proposition 4.2.9. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$ such that f^{-1} preserves $+$. Then the following are equivalent:*
(1) *For any $\mu, \nu \in Fid(\mathcal{B}, L)$, $Fid_L^{-1}(f)(\mu) \otimes Fid_L^{-1}(f)(\nu) = Fid_L^{-1}(f)(\mu \otimes \nu)$.*
(2) *For any $I, J \in Id(\mathcal{B})$, $f^{-1}(I) \odot f^{-1}(J) = f^{-1}(I \odot J)$.*

Proof. Since $\mathcal{I}d(\mathcal{B})$ can be embedded into $\mathcal{F}id(\mathcal{B}, L)$, it suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. For any $\mu, \nu \in Fid(\mathcal{B}, L)$,

$$\begin{aligned}
Fid_L^{-1}(f)(\mu\otimes\nu) &= Fid_L^{-1}(f)\big[\big(\bigsqcup_{a\in B}\big(Idg(a)_{\mu(a)}\big)_*\big)\otimes\big(\bigsqcup_{b\in B}\big(Idg(b)_{\nu(b)}\big)_*\big)\big]\\
&= Fid_L^{-1}(f)\big[\bigsqcup_{a\in B}\bigsqcup_{b\in B}\big(Idg(a)_{\mu(a)}\big)_*\otimes\big(Idg(b)_{\nu(b)}\big)_*\big]\\
&= \bigsqcup_{a\in B}\bigsqcup_{b\in B}Fid_L^{-1}(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\otimes\big(Idg(b)_{\nu(b)}\big)_*\big]\\
&= \bigsqcup_{a\in B}\bigsqcup_{b\in B}Fid_L^{-1}(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\big]\otimes Fid_L^{-1}(f)\big[\big(Idg(b)_{\nu(b)}\big)_*\big]\\
&= \big(\bigsqcup_{a\in B}Fid_L^{-1}(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\big]\big)\otimes\\
&\qquad\qquad \big(\bigsqcup_{b\in B}Fid_L^{-1}(f)\big[\big(Idg(b)_{\nu(b)}\big)_*\big]\big)\\
&= Fid_L^{-1}(f)\big[\bigsqcup_{a\in B}\big(Idg(a)_{\mu(a)}\big)_*\big]\otimes Fid_L^{-1}(f)\big[\bigsqcup_{b\in B}\big(Idg(b)_{\nu(b)}\big)_*\big]\\
&= Fid_L^{-1}(f)(\mu)\otimes Fid_L^{-1}(f)(\nu).
\end{aligned}$$

□

Proposition 4.2.10. *Let $\mathcal{A}\xrightarrow{f}\mathcal{B}$ in $\mathbb{R}ing$. If $\mathcal{L}$ is a Brouwerian algebra, then the following are equivalent:*
(1) *For any $\mu,\nu\in Fid(\mathcal{B},L)$, $Fid_L^{-1}(f)(\mu\otimes\nu)=Fid_L^{-1}(f)(\mu)\otimes Fid_L^{-1}(f)(\nu)$.*
(2) *For any $I,J\in Id(\mathcal{B})$, $f^{-1}(I\odot J)=f^{-1}(I)\odot f^{-1}(J)$.*

Proof. Assume that $\mathcal{L}$ is a Brouwerian algebra. Since $\mathcal{I}d(\mathcal{B})$ can be embedded into $\mathcal{F}id(\mathcal{B},L)$, it suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. Let $\mu,\nu\in Fid(\mathcal{B},L)$. For any $x\in A$ such that $f(x)=0_B$, we have $x\in f^{-1}(\{0_B\}\odot\{0_B\})=f^{-1}(\{0_B\})\odot f^{-1}(\{0_B\})$; thus, $x=\sum_{i=1}^{n}a_ib_i$ for some $a_1,b_1,\ldots,a_n,b_n\in f^{-1}(\{0_B\})$; so,

$$\begin{aligned}
Fid_L^{-1}(f)(\mu\otimes\nu)(x) &= \big(\mu\otimes\nu\big)\big(f(x)\big)\\
&= 1\\
&= \bigwedge_{1\leq i\leq n} 1\ominus 1\\
&= \bigwedge_{1\leq i\leq n} \mu\big(f(a_i)\big)\ominus\nu\big(f(b_i)\big)\\
&= \bigwedge_{1\leq i\leq n} Fid_L^{-1}(f)(\mu)(a_i)\ominus Fid_L^{-1}(f)(\nu)(b_i)\\
&\leq \big[Fid_L^{-1}(f)(\mu)\otimes Fid_L^{-1}(f)(\nu)\big](x).
\end{aligned}$$

Now, let $x \in A$ such that $f(x) \neq 0_B$. Let $a_1, b_1, \dots, a_n, b_n \in B$ such that $f(x) = \sum_{i=1}^{n} a_i b_i$. Since $x \in f^{-1}\big[U\big(\mu, \bigwedge_{1\leq i\leq n} \mu(a_i)\big) \odot U\big(\nu, \bigwedge_{1\leq i\leq n} \nu(b_i)\big)\big]$, we have $x \in f^{-1}\big[U\big(\mu, \bigwedge_{1\leq i\leq n} \mu(a_i)\big)\big] \odot f^{-1}\big[U\big(\nu, \bigwedge_{1\leq i\leq n} \nu(b_i)\big)\big]$; thus, there are $u_1, \dots, u_m \in f^{-1}\big[U\big(\mu, \bigwedge_{1\leq i\leq n} \mu(a_i)\big)\big]$ and $v_1, \dots, v_m \in f^{-1}\big[U\big(\nu, \bigwedge_{1\leq i\leq n} \nu(b_i)\big)\big]$ such that $x = \sum_{j=1}^{m} u_j v_j$; so,

$$\begin{aligned}
\bigwedge_{1\leq i\leq n} \mu(a_i) \ominus \nu(b_i) &= \big(\bigwedge_{1\leq i\leq n} \mu(a_i)\big) \ominus \big(\bigwedge_{1\leq i\leq n} \nu(b_i)\big) \\
&\leq \big[\bigwedge_{1\leq j\leq m} \mu\big(f(u_j)\big)\big] \ominus \big[\bigwedge_{1\leq j\leq m} \nu\big(f(v_j)\big)\big] \\
&= \bigwedge_{1\leq j\leq m} \mu\big(f(u_j)\big) \ominus \nu\big(f(v_j)\big) \\
&= \bigwedge_{1\leq j\leq m} Fid_L^{-1}(f)(\mu)(u_j) \ominus Fid_L^{-1}(f)(\nu)(v_j) \\
&\leq \big[Fid_L^{-1}(f)(\mu) \otimes Fid_L^{-1}(f)(\nu)\big](x).
\end{aligned}$$

It follows that $Fid_L^{-1}(f)(\mu \otimes \nu)(x) \leq \big[Fid_L^{-1}(f)(\mu) \otimes Fid_L^{-1}(f)(\nu)\big](x)$. Hence, $Fid_L^{-1}(f)(\mu \otimes \nu) \leq Fid_L^{-1}(f)(\mu) \otimes Fid_L^{-1}(f)(\nu)$. Therefore, $Fid_L^{-1}(f)(\mu \otimes \nu) = Fid_L^{-1}(f)(\mu) \otimes Fid_L^{-1}(f)(\nu)$. □

One can verify that the L-preimage of any projection homomorphism preserves $\otimes$; but, the L-preimage of a natural homomorphism does not necessarily preserve $\otimes$. Indeed, considering the natural homomorphism $\phi : \mathbb{Z} \to \frac{\mathbb{Z}}{4\mathbb{Z}}$ from $\mathcal{Z}$ to $\mathcal{Z}_4$, given by $\phi(x) = x + 4\mathbb{Z}$ for all $x \in \mathbb{Z}$, we have

$$\phi^{-1}(\{0\}) \odot \phi^{-1}(\tfrac{2\mathbb{Z}}{4\mathbb{Z}}) = 4\mathbb{Z} \odot 2\mathbb{Z} = 8\mathbb{Z} \subset 4\mathbb{Z} = \phi^{-1}(\{0\}) = \phi^{-1}\big(\{0\} \odot \tfrac{2\mathbb{Z}}{4\mathbb{Z}}\big).$$

Lemma 4.2.11. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$ and $\mu, \nu \in Fid(\mathcal{B}, L)$. Then $Fid_L^{-1}(f)(\mu \hookrightarrow \nu) \leq Fid_L^{-1}(f)(\mu) \hookrightarrow Fid_L^{-1}(f)(\nu)$ and $Fid_L^{-1}(f)(\mu \not\hookrightarrow \nu) \leq Fid_L^{-1}(f)(\mu) \not\hookrightarrow Fid_L^{-1}(f)(\nu)$.*

Proof. Since $Fid_L^{-1}(f)(\mu \hookrightarrow \nu) \otimes Fid_L^{-1}(f)(\mu) \leq Fid_L^{-1}(f)\big((\mu \hookrightarrow \nu) \otimes \mu\big) \leq Fid_L^{-1}(f)(\nu)$, we have $Fid_L^{-1}(f)(\mu \hookrightarrow \nu) \leq Fid_L^{-1}(f)(\mu) \hookrightarrow Fid_L^{-1}(f)(\nu)$. Similarly, we have $Fid_L^{-1}(f)(\mu \not\hookrightarrow \nu) \leq Fid_L^{-1}(f)(\mu) \not\hookrightarrow Fid_L^{-1}(f)(\nu)$. □

Proposition 4.2.12. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$. Then the following (and their mirror images) are equivalent:*
(1) *For any $\mu \in Fid(\mathcal{B}, L)$, $Fid_L^{-1}(f)(\mu) \hookrightarrow \chi_{f^{-1}(\{0_B\})} = Fid_L^{-1}(f)(\mu^-)$.*
(2) *For any $I \in Id(\mathcal{B})$, $f^{-1}(I) \to f^{-1}(\{0_B\}) = f^{-1}(I^-)$.*

Proof. Since $\mathcal{I}d(\mathcal{B})$ can be embedded into $\mathcal{F}id(\mathcal{B},L)$, it suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. Let $\mu \in Fid(\mathcal{B},L)$. Let $x \in A$. Let $r \in L$ such that $x_r \circ Fid_L^{-1}(f)(\mu) \leq \chi_{f^{-1}(\{0_B\})}$. Let $y \neq 0_B$ in B. Let $b \in B$ such that $y = f(x)b$. Since $f(x)U\big(\mu,\mu(b)\big) \nsubseteq \{0_B\}$, we have $f(x) \notin U\big(\mu,\mu(b)\big)^-$ and,

$$x \notin f^{-1}\big[U\big(\mu,\mu(b)\big)^-\big] = f^{-1}\big[U\big(\mu,\mu(b)\big)\big] \to f^{-1}(\{0_B\});$$

thus, $xf^{-1}\big[U\big(\mu,\mu(b)\big)\big] \nsubseteq f^{-1}(\{0_B\})$; so, $f(a) \in U\big(\mu,\mu(b)\big)$ for some $a \in A$ such that $f(xa) \neq 0_B$. It follows that

$$\begin{aligned} r \ominus \mu(b) &\leq r \ominus \mu\big(f(a)\big) \\ &= x_r(x) \ominus Fid_L^{-1}(f)(\mu)(a) \\ &\leq \big(x_r \circ Fid_L^{-1}(f)(\mu)\big)(xa) \\ &\leq \chi_{f^{-1}(\{0_B\})}(xa) \\ &= 0 \end{aligned}$$

and, $r \ominus \mu(b) = 0$. Consequently, $\big(f(x)_r \circ \mu\big)(y) = \bigvee\{0\} = 0$. Thus,

$$f(x)_r \circ \mu \leq \chi_{0_B} \text{ and, } r \leq \mu^-\big(f(x)\big) = Fid_L^{-1}(f)(\mu^-)(x).$$

So, $\big(Fid_L^{-1}(f)(\mu) \hookrightarrow \chi_{f^{-1}(\{0_B\})}\big)(x) \leq Fid_L^{-1}(f)(\mu^-)(x)$. Hence,

$$Fid_L^{-1}(f)(\mu) \hookrightarrow \chi_{f^{-1}(\{0_B\})} \leq Fid_L^{-1}(f)(\mu^-)$$

and, $Fid_L^{-1}(f)(\mu) \hookrightarrow \chi_{f^{-1}(\{0_B\})} = Fid_L^{-1}(f)(\mu^-)$.
A similar reasoning shows the mirror equivalence. □

Lemma 4.2.13. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$. Then the following (and their mirror images) are equivalent:*
(a) *For any $I, J \in Id(\mathcal{B})$ and $r, s \in L$,*

$$Fid_L^{-1}(f)\big((I_r)_*\big) \hookrightarrow Fid_L^{-1}(f)\big((J_s)_*\big) = Fid_L^{-1}(f)\big((I_r)_* \hookrightarrow (J_s)_*\big).$$

(b) *For any $I, J \in Id(\mathcal{B})$, $f^{-1}(I) \to f^{-1}(J) = f^{-1}(I \to J)$.*

Proof. Since $\mathcal{I}d(\mathcal{B})$ can be embedded into $\mathcal{F}id(\mathcal{B},L)$, it suffices to show that **(b)** implies **(a)**. So, assume that **(b)** is satisfied. Let $I, J \in Id(\mathcal{B})$ and $r, s \in L$. Let $x \neq 0_A$ in A. Let $t \in L$ such that $x_t \circ Fid_L^{-1}(f)\big((I_r)_*\big) \leq Fid_L^{-1}(f)\big((J_s)_*\big)$. If $t \ominus r = 0$, then

$$\big(f(x)_t \circ (I_r)_*\big)(b) = \bigvee\{0\} = 0 \leq (J_s)_*(b) \text{ for all } b \neq 0_B \text{ in } B;$$

thus, $f(x)_t \circ (I_r)_* \leq (J_s)_*$ and,

$$t \leq \big((I_r)_* \hookrightarrow (J_s)_*\big)\big(f(x)\big) = Fid_L^{-1}(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)(x).$$

Now, suppose that $t \ominus r \neq 0$. For any $a \in f^{-1}(I)$, we have

$$\begin{aligned}
(J_s)_*\big(f(xa)\big) &= Fid_L^{-1}(f)\big((J_s)_*\big)(xa) \\
&\geq \big[x_t \circ Fid_L^{-1}(f)\big((I_r)_*\big)\big](xa) \\
&\geq x_t(x) \ominus Fid_L^{-1}(f)\big((I_r)_*\big)(a) \\
&= t \ominus (I_r)_*\big(f(a)\big) \\
&= \begin{cases} t & \text{if } f(a) = 0_B, \\ t \ominus r & \text{if } f(a) \in I \setminus \{0_B\}. \end{cases} \\
&\geq t \ominus r;
\end{aligned}$$

thus, $f(xa) \in J$ and, $xa \in f^{-1}(J)$. So,

$$xf^{-1}(I) \subseteq f^{-1}(J) \text{ and, } x \in f^{-1}(I) \to f^{-1}(J) = f^{-1}(I \to J).$$

It follows that $f(x) \in I \to J$. We now wish to show that $f(x)_t \circ (I_r)_* \leq (J_s)_*$. So, let $y \neq 0_B$ in B. For any $v \notin I$ such that $y = f(x)v$, we have

$$t \ominus (I_r)_*(v) = t \ominus 0 = 0 \leq (J_s)_*(y).$$

Now, let $v \in I$ such that $y = f(x)v$. Since $f(x)U\big((I_r)_*, (I_r)_*(v)\big) \nsubseteq \{0_B\}$, we have $f(x) \notin U\big((I_r)_*, (I_r)_*(v)\big)^-$ and,

$$x \notin f^{-1}\big[U\big((I_r)_*, (I_r)_*(v)\big)^-\big] = f^{-1}\big[U\big((I_r)_*, (I_r)_*(v)\big)\big] \to f^{-1}(\{0_B\});$$

thus, $xf^{-1}\big[U\big((I_r)_*, (I_r)_*(v)\big)\big] \nsubseteq f^{-1}(\{0_B\})$; so, $f(b) \in U\big((I_r)_*, (I_r)_*(v)\big)$ for some $b \in A$ such that $f(xb) \neq 0_B$. Since $y \in f(x)I \subseteq J$, we have $y \in J \setminus \{0_B\}$; thus,

$$\begin{aligned}
t \ominus (I_r)_*(v) &\leq t \ominus (I_r)_*\big(f(b)\big) \\
&= x_t(x) \ominus Fid_L^{-1}(f)\big((I_r)_*\big)(b) \\
&\leq \big[x_t \circ Fid_L^{-1}(f)\big((I_r)_*\big)\big](xb) \\
&\leq Fid_L^{-1}(f)\big((J_s)_*\big)(xb) \\
&= (J_s)_*\big(f(xb)\big) \\
&\leq s \\
&= (J_s)_*(y).
\end{aligned}$$

So, $\big(f(x)_t \circ (I_r)_*\big)(y) \leq (J_s)_*(y)$. It follows that $f(x)_t \circ (I_r)_* \leq (J_s)_*$. Consequently, $t \leq \big((I_r)_* \hookrightarrow (J_s)_*\big)\big(f(x)\big) = Fid_L^{-1}(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)(x)$. Hence,

$$\big[Fid_L^{-1}(f)\big((I_r)_*\big) \hookrightarrow Fid_L^{-1}(f)\big((J_s)_*\big)\big](x) \leq Fid_L^{-1}(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)(x).$$

Therefore, $Fid_L^{-1}(f)\big((I_r)_*\big) \hookrightarrow Fid_L^{-1}(f)\big((J_s)_*\big) \leq Fid_L^{-1}(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)$ and, $Fid_L^{-1}(f)\big((I_r)_*\big) \hookrightarrow Fid_L^{-1}(f)\big((J_s)_*\big) = Fid_L^{-1}(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)$.
A similar reasoning shows the mirror equivalence. □

Proposition 4.2.14. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbb{R}ing$ such that f^{-1} preserves $+$. If $\mathcal{F}id(\mathcal{B}, L)$ is completely join-implicative (See, Definition 1.2.16), then the following (and their mirror images) are equivalent:*
(1) *For any $\mu, \nu \in Fid(\mathcal{B}, L)$,*

$$Fid_L^{-1}(f)(\mu) \hookrightarrow Fid_L^{-1}(f)(\nu) = Fid_L^{-1}(f)(\mu \hookrightarrow \nu).$$

(2) *For any $I, J \in Id(\mathcal{B})$, $f^{-1}(I) \to f^{-1}(J) = f^{-1}(I \to J)$.*

Proof. Assume that $\mathcal{F}id(\mathcal{B}, L)$ is completely join-implicative. Since $\mathcal{I}d(\mathcal{B})$ can be embedded into $\mathcal{F}id(\mathcal{B}, L)$, it suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. For any $\mu, \nu \in Fid(\mathcal{B}, L)$, we have

$$\begin{aligned}
Fid_L^{-1}(f)(\mu \hookrightarrow \nu) &= Fid_L^{-1}(f)\big[\big(\bigsqcup_{a\in B}\big(Idg(a)_{\mu(a)}\big)_*\big) \hookrightarrow \big(\bigsqcup_{b\in B}\big(Idg(b)_{\nu(b)}\big)_*\big)\big] \\
&= Fid_L^{-1}(f)\big[\bigwedge_{a\in B}\bigsqcup_{b\in B}\big[\big(Idg(a)_{\mu(a)}\big)_* \hookrightarrow \big(Idg(b)_{\nu(b)}\big)_*\big]\big] \\
&= \bigwedge_{a\in B}\bigsqcup_{b\in B} Fid_L^{-1}(f)\big[\big(Idg(a)_{\mu(a)}\big)_* \hookrightarrow \big(Idg(b)_{\nu(b)}\big)_*\big] \\
&= \bigwedge_{a\in B}\bigsqcup_{b\in B} \big(Fid_L^{-1}(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\big] \hookrightarrow \\
&\qquad\qquad Fid_L^{-1}(f)\big[\big(Idg(b)_{\nu(b)}\big)_*\big]\big) \\
&\quad \text{by the above lemma;} \\
&= \bigwedge_{a\in B} \big(Fid_L^{-1}(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\big] \hookrightarrow \\
&\qquad\qquad \big(\bigsqcup_{b\in B} Fid_L^{-1}(f)\big[\big(Idg(b)_{\nu(b)}\big)_*\big]\big)\big) \\
&= \big(\bigsqcup_{a\in B} Fid_L^{-1}(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\big]\big) \hookrightarrow \\
&\qquad\qquad \big(\bigsqcup_{b\in B} Fid_L^{-1}(f)\big[\big(Idg(b)_{\nu(b)}\big)_*\big]\big) \\
&= Fid_L^{-1}(f)\big[\bigsqcup_{a\in B}\big(Idg(a)_{\mu(a)}\big)_*\big] \hookrightarrow \\
&\qquad\qquad Fid_L^{-1}(f)\big[\bigsqcup_{b\in B}\big(Idg(b)_{\nu(b)}\big)_*\big] \\
&= Fid_L^{-1}(f)(\mu) \hookrightarrow Fid_L^{-1}(f)(\nu).
\end{aligned}$$

A similar reasoning shows the mirror equivalence. □

4.3 L-image functor

Let $\mathbf{R}ing$ be the subcategory of the category of rings, with unital rings as objects, and onto homomorphisms of unital rings as arrows.

Lemma 4.3.1. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbf{R}ing$. The arrow $Fid(\mathcal{A}, L) \xrightarrow{Fid_L(f)} Fid(\mathcal{B}, L)$, given by $Fid_L(f)(\mu)(y) = \bigvee_{f(a)=y} \mu(a)$ for all $\mu \in Fid(\mathcal{A}, L)$ and $y \in B$, is well-defined.*

Proof. Let $\mu \in Fid(\mathcal{A}, L)$. Since $f(0_A) = 0_B$, we have $Fid_L(f)(\mu)(0_B) \geq \mu(0_A) = 1$ and, $Fid_L(f)(\mu)(0_B) = 1$. Now, let $y, z \in B$. For any $b, c \in A$ such that $f(b) = y$ and $f(c) = z$, we have $f(b - c) = f(b) - f(c) = y - z$ and $\mu(b) \wedge \mu(c) \leq \mu(b - c)$; thus, $\mu(b) \wedge \mu(c) \leq Fid_L(f)(\mu)(y - z)$. So,

$$Fid_L(f)(\mu)(y) \wedge Fid_L(f)(\mu)(z) \leq Fid_L(f)(\mu)(y - z).$$

For any $a \in A$ such that $y = f(a)$, we have $yz = f(a)f(b) = f(ab)$ and $\mu(a) \leq \mu(ab)$ for some $b \in A$; thus, $\mu(a) \leq Fid_L(f)(\mu)(yz)$. So, $Fid_L(f)(\mu)(y) \leq Fid_L(f)(\mu)(yz)$ and, $Fid_L(f)(\mu)(z) \leq Fid_L(f)(\mu)(yz)$ by similar arguments. It follows that $Fid_L(f)(\mu)(y) \vee Fid_L(f)(\mu)(z) \leq Fid_L(f)(\mu)(yz)$. Hence, $Fid_L(f)(\mu)$ is an L-fuzzy ideal of $\mathcal{B}$. $\square$

Lemma 4.3.2. ***(a)*** *For any $\mathcal{A}$ in $\mathbf{R}ing$, $Fid_L(Id_A) = Id_{Fid(\mathcal{A},L)}$.*
(b) *For any $\mathcal{A} \xrightarrow{f} \mathcal{B}$ and $\mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathbf{R}ing$, $Fid_L(g \circ f) = Fid_L(g) \circ Fid_L(f)$.*

Proof. **(a)** Let $\mathcal{A}$ in $\mathbf{R}ing$. For any $\mu \in Fid(\mathcal{A}, L)$, we have

$$Fid_L(Id_A)(\mu)(y) = \bigvee_{Id_A(a)=y} \mu(a) = \bigvee\{\mu(y)\} = \mu(y) \text{ for all } y \in A;$$

thus, $Fid_L(Id_A)(\mu) = \mu$. So, $Fid_L(Id_A) = Id_{Fid(\mathcal{A},L)}$.
(b) Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ and $\mathcal{B} \xrightarrow{g} \mathcal{C}$ in $\mathbf{R}ing$. Let $\mu \in Fid(\mathcal{A}, L)$ and $y \in C$. For any $a \in A$ such that $(g \circ f)(a) = y$, we have $g\big(f(a)\big) = y$; thus,

$$\begin{aligned} \mu(a) &\leq Fid_L(f)(\mu)\big(f(a)\big) \\ &\leq Fid_L(g)\big[Fid_L(f)(\mu)\big](y) \\ &= \big[Fid_L(g) \circ Fid_L(f)\big](\mu)(y). \end{aligned}$$

So, $Fid_L\big(g \circ f\big)(\mu)(y) \leq \big[Fid_L(g) \circ Fid_L(f)\big](\mu)(y)$. Now, let $x \in B$ such that $y = g(x)$. For any $a \in A$ such that $x = f(a)$, we have $y = g\big(f(a)\big) = \big(g \circ f\big)(a)$; thus, $\mu(a) \leq Fid_L(g \circ f)(\mu)(y)$. So, $Fid_L(f)(\mu)(x) \leq Fid_L(g \circ f)(\mu)(y)$. Hence, $\big[Fid_L(g) \circ Fid_L(f)\big](\mu)(y) = Fid_L(g)\big[Fid_L(f)(\mu)\big](y) \leq Fid_L(g \circ f)(\mu)(y)$ and, $Fid_L\big(g \circ f\big)(\mu)(y) = \big[Fid_L(g) \circ Fid_L(f)\big](\mu)(y)$. Therefore, $Fid_L(g \circ f) = Fid_L(g) \circ Fid_L(f)$. $\square$

Theorem 4.3.3. *Fid_L is a covariant functor, called L-image functor, from* $\mathbf{R}ing$ *to the category* $\mathbf{P}oMod$ *of partially ordered monoids.*

Proof. From the above lemmas, it suffices to show that Fid_L is well-defined. So, let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in $\mathbf{R}ing$.
Let $\mu, \nu \in Fid(\mathcal{A}, L)$ such that $\mu \leq \nu$. For any $y \in B$ and $a \in A$ such that $f(a) = y$, we have $\mu(a) \leq \nu(a) \leq Fid_L(f)(\nu)(y)$; thus, $Fid_L(f)(\mu)(y) \leq Fid_L(f)(\nu)(y)$. So, $Fid_L(f)(\mu) \leq Fid_L(f)(\nu)$. Hence, $Fid_L(f)$ is order-preserving.
For any $y \in B$, we have $Fid_L(f)(\underline{1})(y) \geq \underline{1}(a) = 1$ for some $a \in A$ such that $f(a) = y$; thus, $Fid_L(f)(\underline{1})(y) = 1$. So, $Fid_L(f)(\underline{1}) = \underline{1}$. We finally show that $Fid_L(f)(\mu \otimes \nu) = Fid_L(f)(\mu) \otimes Fid_L(f)(\nu)$ for all $\mu, \nu \in Fid(\mathcal{A}, L)$.
So, let $\mu, \nu \in Fid(\mathcal{A}, L)$. Let $y \in B$. Let $x \in A$ such that $y = f(x)$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum\limits_{i=1}^{n} a_i b_i$, we have

$$\begin{aligned} \bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \nu(b_i) &\leq \bigwedge_{1 \leq i \leq n} Fid_L(f)(\mu)\big(f(a_i)\big) \ominus Fid_L(f)(\nu)\big(f(b_i)\big) \\ &\leq \big[Fid_L(f)(\mu) \otimes Fid_L(f)(\nu)\big]\Big(\sum_{i=1}^{n} f(a_i) f(b_i)\Big) \\ &= \big[Fid_L(f)(\mu) \otimes Fid_L(f)(\nu)\big](y). \end{aligned}$$

It follows that $(\mu \otimes \nu)(x) \leq \big[Fid_L(f)(\mu) \otimes Fid_L(f)(\nu)\big](y)$. Consequently, $Fid_L(f)(\mu \otimes \nu)(y) \leq \big[Fid_L(f)(\mu) \otimes Fid_L(f)(\nu)\big](y)$. Let $u_1, v_1, \ldots, u_n, v_n \in B$ such that $y = \sum\limits_{i=1}^{n} u_i v_i$. For any $w_1, t_1, \ldots, w_n, t_n \in A$ such that $u_1 = f(w_1), v_1 = f(t_1), \ldots, u_n = f(w_n), v_n = f(t_n)$, we have $y = f(\sum\limits_{i=1}^{n} w_i t_i)$; thus,

$$\bigwedge_{1 \leq i \leq n} \mu(w_i) \ominus \nu(t_i) \leq \big(\mu \otimes \nu\big)(\sum_{i=1}^{n} w_i t_i) \leq Fid_L(f)(\mu \otimes \nu)(y).$$

So, $\bigwedge\limits_{1 \leq i \leq n} Fid_L(f)(\mu)(u_i) \ominus Fid_L(f)(\nu)(v_i) \leq Fid_L(f)(\mu \otimes \nu)(y)$. It follows that $\big[Fid_L(f)(\mu) \otimes Fid_L(f)(\nu)\big](y) \leq Fid_L(f)(\mu \otimes \nu)(y)$. Consequently, $Fid_L(f)(\mu \otimes \nu) = Fid_L(f)(\mu) \otimes Fid_L(f)(\nu)$. Hence, $Fid_L(f)$ is a monoid homomorphism. Therefore, Fid_L is a functor. □

Proposition 4.3.4. *Let $\mathcal{A} \overset{f}{\to} \mathcal{B}$ in* $\mathbf{R}ing$*. For any $I \in Id(\mathcal{A})$ and $r, s \in L$ such that $r \leq s$, we have $Fid_L(f)\big((I_r^s)_*\big) = \big(f(I)_r^s\big)_*$. In particular, we have $Fid_L(f)(\chi_{0_A}) = \chi_{0_B}$.*

Proof. Let $I \in Id(\mathcal{A})$ and $r, s \in L$ such that $r \leq s$. Let $y \neq 0_B$ in $f(I)$. For any $x \in A$ such that $y = f(x)$, we have $x \neq 0_A$; thus, $\big(I_r^s\big)_*(x) \leq s$.

So, $Fid_L(f)\big((I_r^s)_*\big)(y) \leq s$. Since $Fid_L(f)\big((I_r^s)_*\big)(y) \geq (I_r^s)_*(x) = s$ for some $x \neq 0_A$ in I such that $y = f(x)$, we have $Fid_L(f)\big((I_r^s)_*\big)(y) = s$. Now, let $y \notin f(I)$. For any $x \in A$ such that $y = f(x)$, we have $x \notin I$; thus, $(I_r^s)_*(x) = r$. So, $Fid_L(f)\big((I_r^s)_*\big)(y) = \bigvee\{r\} = r$. Hence, $Fid_L(f)\big((I_r^s)_*\big) = \big(f(I)_r^s\big)_*$. □

Proposition 4.3.5. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in* **R***ing. For any $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fid(\mathcal{A}, L)$, we have $\bigsqcup_{\lambda\in\Lambda} Fid_L(f)(\mu_\lambda) = Fid_L(f)\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda\big)$.*

Proof. Let $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fid(\mathcal{A}, L)$. Let $y \in B$. Let $x \in A$ such that $y = f(x)$. For any finite subset Ω of Λ such that $x = \sum_{\lambda\in\Omega} a_\lambda$, we have $y = f\big(\sum_{\lambda\in\Omega} a_\lambda\big) = \sum_{\lambda\in\Omega} f(a_\lambda)$; thus, $\bigwedge_{\lambda\in\Omega}\mu_\lambda(a_\lambda) \leq \bigwedge_{\lambda\in\Omega} Fid_L(f)(\mu_\lambda)\big(f(a_\lambda)\big) \leq \big[\bigsqcup_{\lambda\in\Lambda} Fid_L(f)(\mu_\lambda)\big](y)$. So, $\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda\big)(x) \leq \big[\bigsqcup_{\lambda\in\Lambda} Fid_L(f)(\mu_\lambda)\big](y)$. It follows that $Fid_L(f)\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda\big)(y) \leq \big[\bigsqcup_{\lambda\in\Lambda} Fid_L(f)(\mu_\lambda)\big](y)$. Hence, $Fid_L(f)\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda\big) \leq \bigsqcup_{\lambda\in\Lambda} Fid_L(f)(\mu_\lambda)$. Therefore, $\bigsqcup_{\lambda\in\Lambda} Fid_L(f)(\mu_\lambda) = Fid_L(f)\big(\bigsqcup_{\lambda\in\Lambda}\mu_\lambda\big)$, since $Fid_L(f)$ is order-preserving. □

Proposition 4.3.6. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in* **R***ing. Then the following are equivalent:*
(1) *For any $\mu, \nu \in Fid(\mathcal{A}, L)$, $Fid_L(f)(\mu) \wedge Fid_L(f)(\nu) = Fid_L(f)(\mu \wedge \nu)$.*
(2) *For any $I, J \in Id(\mathcal{A})$, $f(I) \cap f(J) = f(I \cap J)$.*

Proof. Since $\mathcal{I}d(\mathcal{A})$ can be embedded into $\mathcal{F}id(\mathcal{A}, L)$, it suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. Let $\mu, \nu \in Fid(\mathcal{A}, L)$. Let $y \in B$. For any $a, b \in A$ such that $f(a) = y$ and $f(b) = y$, we have

$$y \in f\big[U\big(\mu, \mu(a)\big)\big] \cap f\big[U\big(\nu, \nu(b)\big)\big] \subseteq f\big[U\big(\mu, \mu(a)\big) \cap U\big(\nu, \nu(b)\big)\big];$$

thus, $y = f(c)$ for some $c \in U\big(\mu, \mu(a)\big) \cap U\big(\nu, \nu(b)\big)$; so,

$$\mu(a) \wedge \nu(b) \leq \mu(c) \wedge \nu(c) = \big(\mu \wedge \nu\big)(c) \leq Fid_L(f)(\mu \wedge \nu)(y).$$

Consequently, $\big[Fid_L(f)(\mu) \wedge Fid_L(f)(\nu)\big](y) \leq Fid_L(f)(\mu \wedge \nu)(y)$. Hence, $Fid_L(f)(\mu) \wedge Fid_L(f)(\nu) \leq Fid_L(f)(\mu \wedge \nu)$ and, $Fid_L(f)(\mu) \wedge Fid_L(f)(\nu) = Fid_L(f)(\mu \wedge \nu)$, since $Fid_L(f)$ is order-preserving. □

Lemma 4.3.7. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in* **R***ing and $\mu, \nu \in Fid(\mathcal{A}, L)$. Then $Fid_L(f)(\mu \hookrightarrow \nu) \leq Fid_L(f)(\mu) \hookrightarrow Fid_L(f)(\nu)$ and $Fid_L(f)(\mu \looparrowright \nu) \leq Fid_L(f)(\mu) \looparrowright Fid_L(f)(\nu)$.*

Proof. Since $Fid_L(f)(\mu \hookrightarrow \nu) \otimes Fid_L(f)(\mu) = Fid_L(f)\big((\mu \hookrightarrow \nu) \otimes \mu\big) \leq Fid_L(f)(\nu)$, we have $Fid_L(f)(\mu \hookrightarrow \nu) \leq Fid_L(f)(\mu) \hookrightarrow Fid_L(f)(\nu)$. A similar reasoning shows that $Fid_L(f)(\mu \looparrowright \nu) \leq Fid_L(f)(\mu) \looparrowright Fid_L(f)(\nu)$. □

Proposition 4.3.8. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in* **R***ing such that $Fid_L(f)$ preserves $\bigwedge$. Then the following (and their mirror images) are equivalent:*
(1) *For any $\mu \in Fid(\mathcal{A}, L)$, $Fid_L(f)(\mu)^- = Fid_L(f)(\mu^-)$.*
(2) *For any $I \in Id(\mathcal{A})$, $f(I)^- = f(I^-)$.*

Proof. Since $\mathcal{I}d(\mathcal{A})$ can be embedded into $\mathcal{F}id(\mathcal{A}, L)$, it suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. For any $\mu \in Fid(\mathcal{A}, L)$, we have

$$\begin{aligned}
Fid_L(f)(\mu^-) &= Fid_L(f)\big[\big(\bigsqcup_{a\in A}\big(Idg(a)_{\mu(a)}\big)_*\big)^-\big] \\
&= Fid_L(f)\big[\bigwedge_{a\in A}\big(\big(Idg(a)_{\mu(a)}\big)_*\big)^-\big] \\
&= \bigwedge_{a\in A} Fid_L(f)\big[\big(Idg(a)^-\big)^{\overline{\mu(a)}}\big] \\
&= \bigwedge_{a\in A}\big[f\big(Idg(a)^-\big)\big]^{\overline{\mu(a)}} \\
&= \bigwedge_{a\in A}\big[f\big(Idg(a)\big)^-\big]^{\overline{\mu(a)}} \\
&= \bigwedge_{a\in A}\big[\big(f\big(Idg(a)\big)_{\mu(a)}\big)_*\big]^- \\
&= \big[\bigsqcup_{a\in A}\big(f\big(Idg(a)\big)_{\mu(a)}\big)_*\big]^- \\
&= \big[\bigsqcup_{a\in A} Fid_L(f)\big(\big(Idg(a)_{\mu(a)}\big)_*\big)\big]^- \\
&= \big[Fid_L(f)\big(\bigsqcup_{a\in A}\big(Idg(a)_{\mu(a)}\big)_*\big)\big]^- \\
&= Fid_L(f)(\mu)^-.
\end{aligned}$$

A similar reasoning shows the mirror equivalence. □

Lemma 4.3.9. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in* **R***ing. Then the following (and their mirror images) are equivalent:*
(a) *For any $r, s \in L$ and $I, J \in Id(\mathcal{A})$,*
$Fid_L(f)\big((I_r)_*\big) \hookrightarrow Fid_L(f)\big((J_s)_*\big) = Fid_L(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)$.
(b) *For any $I, J \in Id(\mathcal{A})$, $f(I) \to f(J) = f(I \to J)$.*

Proof. It suffices to show that **(b)** implies **(a)**. So, assume that **(b)** is satisfied. Let $r, s \in L$ and $I, J \in Id(\mathcal{A})$. Let $y \neq 0_B$ in B. Let $t \in L$ such that $y_t \circ Fid_L(f)\big((I_r)_*\big) \leq Fid_L(f)\big((J_s)_*\big)$.
If $t \ominus r = 0$, then for some $x \in A$ such that $y = f(x)$, we have

$$\big(x_t \circ (I_r)_*\big)(a) \leq t \ominus r = 0 \leq (J_s)_*(a) \text{ for all } a \neq 0_A \text{ in } A;$$

thus, $x_t \circ (I_r)_* \leq (J_s)_*$ and, $t \leq \big((I_r)_* \hookrightarrow (J_s)_*\big)(x)$; so,

$$t \leq Fid_L(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)(y).$$

Now, suppose that $t \ominus r \neq 0$.
• If $y \in f(I)^-$, then $y \in f(I) \to f(J)$, since $f(I)^- \subseteq f(I) \to f(J)$.
• If $y \notin f(I)^-$, then for any $a \in I$ such that $yf(a) \neq 0_B$, we have

$$\begin{aligned}
\big(f(J)_s\big)_*\big(yf(a)\big) &= Fid_L(f)\big((J_s)_*\big)\big(yf(a)\big) \\
&\geq \big[y_t \circ Fid_L(f)\big((I_r)_*\big)\big]\big(yf(a)\big) \\
&\geq y_t(y) \ominus Fid_L(f)\big((I_r)_*\big)\big(f(a)\big) \\
&= y_t(y) \ominus \big(f(I)_r\big)_*\big(f(a)\big) \\
&= t \ominus r
\end{aligned}$$

and, $yf(a) \in f(J)$; thus, $yf(I) \subseteq f(J)$ and, $y \in f(I) \to f(J)$.
It follows that $y = f(x)$ for some $x \in I \to J$. Since

$$x_t \circ (I_r)_* = (xI)_{t\ominus r} \vee 0_t \leq J_{t\ominus r} \vee 0_t \leq J_s \vee 0_t \leq (J_s)_*,$$

we have $t \leq \big((I_r)_* \hookrightarrow (J_s)_*\big)(x) \leq Fid_L(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)(y)$.
Hence,

$$\big[Fid_L(f)\big((I_r)_*\big) \hookrightarrow Fid_L(f)\big((J_s)_*\big)\big](y) \leq Fid_L(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)(y).$$

It follows that

$$Fid_L(f)\big((I_r)_*\big) \hookrightarrow Fid_L(f)\big((J_s)_*\big) \leq Fid_L(f)\big((I_r)_* \hookrightarrow (J_s)_*\big)$$

and,

$$Fid_L(f)\big((I_r)_*\big) \hookrightarrow Fid_L(f)\big((J_s)_*\big) = Fid_L(f)\big((I_r)_* \hookrightarrow (J_s)_*\big).$$

A similar reasoning shows the mirror equivalence. □

Proposition 4.3.10. *Let $\mathcal{A} \xrightarrow{f} \mathcal{B}$ in* **R***ing such that $Fid_L(f)$ preserves $\bigwedge$. If $\mathcal{F}id(\mathcal{A}, L)$ is completely join-implicative, then the following (and their mirror images) are equivalent:*
(1) *For any $\mu, \nu \in Fid(\mathcal{A}, L)$, $Fid_L(f)(\mu) \hookrightarrow Fid_L(f)(\nu) = Fid_L(f)(\mu \hookrightarrow \nu)$.*
(2) *For any $I, J \in Id(\mathcal{A})$, $f(I) \to f(J) = f(I \to J)$.*

Proof. Assume that $\mathcal{F}id(\mathcal{A}, L)$ is completely join-implicative. It suffices to show that **(2)** implies **(1)**. So, assume that **(2)** is satisfied. For any $\mu, \nu \in Fid(\mathcal{A}, L)$, we have

$$\begin{aligned}
Fid_L(f)(\mu \hookrightarrow \nu) &= Fid_L(f)\big[\big(\bigsqcup_{a\in A}\big(Idg(a)_{\mu(a)}\big)_*\big) \hookrightarrow \big(\bigsqcup_{b\in A}\big(Idg(b)_{\nu(b)}\big)_*\big)\big] \\
&= Fid_L(f)\big[\bigwedge_{a\in A}\bigsqcup_{b\in A}\big(\big(Idg(a)_{\mu(a)}\big)_* \hookrightarrow \big(Idg(b)_{\nu(b)}\big)_*\big)\big] \\
&= \bigwedge_{a\in A}\bigsqcup_{b\in A}\big(Fid_L(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\big] \hookrightarrow \\
&\qquad\qquad Fid_L(f)\big[\big(Idg(b)_{\nu(b)}\big)_*\big]\big) \\
&= \big(\bigsqcup_{a\in A} Fid_L(f)\big[\big(Idg(a)_{\mu(a)}\big)_*\big]\big) \hookrightarrow \\
&\qquad\qquad \big(\bigsqcup_{b\in A} Fid_L(f)\big[\big(Idg(b)_{\nu(b)}\big)_*\big]\big) \\
&= Fid_L(f)\big[\bigsqcup_{a\in A}\big(Idg(a)_{\mu(a)}\big)_*\big] \hookrightarrow Fid_L(f)\big[\bigsqcup_{b\in A}\big(Idg(b)_{\nu(b)}\big)_*\big] \\
&= Fid_L(f)(\mu) \hookrightarrow Fid_L(f)(\nu).
\end{aligned}$$

A similar reasoning shows the mirror equivalence. □

Proposition 4.3.11. *Let I be an ideal of $\mathcal{A}$, and $\pi_I : A \to \frac{A}{I}$ the natural homomorphism from $\mathcal{A}$ to $\frac{A}{I}$. Then the following hold:*
(1) *For any $\mu \in Fid(\mathcal{A}, L)$, $Fid_L(\pi_I)(\mu) = \frac{\mu}{I}$.*
(2) *The restriction of $Fid_L(\pi_I)$ to $Fid(\mathcal{A}, L, I)$ preserves $\bigwedge$.*
(3) *The restriction of $Fid_L(\pi_I)$ to $Fid(\mathcal{A}, L, I)$ preserves $\hookrightarrow$ and $\looparrowright$.*

Proof. **(1)** Let $\mu \in Fid(\mathcal{A}, L)$. For any $y \in A$, we have

$$\begin{aligned}
Fid_L(\pi_I)(\mu)(\tfrac{y}{I}) &= \bigvee\{\mu(x) : \ \pi_I(x) = \tfrac{y}{I}\} \\
&= \bigvee\{\mu(x) : \ \tfrac{x}{I} = \tfrac{y}{I}\} \\
&= \bigvee\{\mu(x) : \ x \in \tfrac{y}{I}\} \\
&= \big(\tfrac{\mu}{I}\big)\big(\tfrac{y}{I}\big).
\end{aligned}$$

Hence, $Fid_L(\pi_I)(\mu) = \frac{\mu}{I}$.
(2) For any $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fid(\mathcal{A}, L, I)$, we have

$$Fid_L(\pi_I)\big(\bigwedge_{\lambda\in\Lambda}\mu_\lambda\big) = \frac{\bigwedge_{\lambda\in\Lambda}\mu_\lambda}{I} = \bigwedge_{\lambda\in\Lambda}\frac{\mu_\lambda}{I} = \bigwedge_{\lambda\in\Lambda} Fid_L(\pi_I)(\mu_\lambda).$$

(3) For any $\mu, \nu \in Fid(\mathcal{A}, L, I)$, we have

$$Fid_L(\pi_I)(\mu \hookrightarrow \nu) = \frac{\mu\hookrightarrow\nu}{I} = \frac{\mu}{I} \hookrightarrow \frac{\nu}{I} = Fid_L(\pi_I)(\mu) \hookrightarrow Fid_L(\pi_I)(\nu)$$

and, $Fid_L(\pi_I)(\mu \looparrowright \nu) = Fid_L(\pi_I)(\mu) \looparrowright Fid_L(\pi_I)(\nu)$ by similar arguments. □

Conclusion

In this thesis, given a residuated lattice $\mathcal{L}$ and a universal algebra $\mathcal{A}$ of type $\mathcal{F}$ with a residuated lattice $\mathcal{S}ub(\mathcal{A})$ on the set of its subuniverses, we investigated possibilities of building a residuated lattice $\mathcal{F}s(\mathcal{A}, L)$, on the set of L-fuzzy subalgebras of $\mathcal{A}$, which extends both $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$. It appeared that this construction is always possible when $\mathcal{L}$ is a finite linearly ordered Brouwerian algebra.

We have generalized the preceding result in the classes of mono-unary algebras and rings. We have also established that the residuated lattice $\mathcal{F}s(\mathcal{A}, L)$ of L-fuzzy subalgebras of a mono-unary algebra $\mathcal{A}$ is an MV-algebra (resp., a Boolean algebra) if and only if $\mathcal{L}$ is an MV-algebra (resp., a Boolean algebra) and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra, and the residuated lattice $\mathcal{F}id(\mathcal{A}, L)$ of L-fuzzy ideals of a ring $\mathcal{A}$ is commutative (a Brouwerian algebra, a Boolean algebra) if and only if so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$. Furthermore, we have introduced the concept of Łukasiewicz rings under $\mathcal{L}$ and established its connection with rings whose L-fuzzy ideals form an MV-algebra.

As future work on this research line, we are going to look for other classes of residuated lattices (or residuated multilattices [8]) and algebras (or hyperalgebras [1]) for which the previous generalizations remain possible. Since it appeared that mono-unary algebras (resp., rings) do not necessarily transfer some specific classes of residuated lattices, it would be interesting to study the classes of mono-unary algebras (resp., rings) for which some transfers are satisfied.

This dissertation has shown the importance of two varieties of residuated lattices, called product-distributive and join-implicative, that would be interesting to study in detail. Since in the literature the arithmetic study of residuated lattices is still superficial, it would be also interesting to deepen it latter.

We mention below a number of open problems that have come up from this work. We believe that some of them need a serious study.

1. Is there a mimetic description of primary elements of $\mathcal{F}id(\mathcal{A}, L)$?

2. Is $\mathcal{F}id(\mathcal{A}, L)$ primary decomposable if and only if so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$?

3. Is there a nice embedding of $\mathcal{F}il\big(\mathcal{I}d(\mathcal{A})\big)$ $\big($resp., $\mathcal{F}il(\mathcal{L})\big)$ into $\mathcal{F}il\big(\mathcal{F}id(\mathcal{A}, L)\big)$?

4. Does the L-preimage of a ring homomorphism preserve L-fuzzy ideals products (resp., residues) if and only if the ring homorphism preserves ideals products (resp., residues)?

5. Does the L-image of a ring epimorphism preserve L-fuzzy ideals residues if and only if the ring epimorphism preserves ideals residues?

6. Is $\mathcal{F}id(\mathcal{A}, L)$ product-distributive (resp., join-implicative) if and only if so are $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$?

Index

Symbols

Bibliography

[1] R. Ameri and T. Nozari: Fuzzy hyperalgebras, Computers and Mathematics with Applications, 61 (2) (2011) 149-154.

[2] M. Anderson and T. Feil: Lattice-ordered groups, An introduction, D. Reidel Texts in the Mathematical Sciences, 1988.

[3] L.P. Belluce, A. Di Nola and E. Marchioni: Rings and Gödel algebras, Algebra Universalis, 64 (1-2) (2010) 103-116.

[4] R. Bělohlávek: Some properties of residuated lattices, Czechoslovak Mathematical Journal, 53 (1) (2003) 161-171.

[5] M. Bianchi, A. Gillio and L. Verardi: Monounary simple algebras, Contemporary Mathematics, 402 (2006) 119-132.

[6] S. Burris and H.P. Sankappanavar: A course in universal algebra, Springer-Verlag, 1981.

[7] D. Buşneag, D. Piciu and A. Jeflea: Archimedean residuated lattices, Annals of the Alexandru Ioan Cuza University - Mathematics, 56 (1) (2010) 227-252.

[8] I.P. Cabrera, P. Cordero, G. Gutiérrez, J. Martinez and M. Ojega-Aciego: On residuation in multilattices: Filters, congruences and homomorphisms, Fuzzy Sets and Systems, 234 (2014) 1-21.

[9] C.C. Chang: Algebraic analysis of many valued logics, Transactions of the American Mathematical Society, 88 (2) (1958) 467-490.

[10] L.C. Ciungu: Non-commutative multiple-valued logic algebras, Springer, Cham, Heidelberg, New York, Dordrecht, 2014.

[11] R. Dedekind: Supplement XI of Dirichlet's Vorlesungen, 4th edition (1894), $\oint 170$, Collected Works, Vol. III, p. 71.

[12] R.P. Dilworth: Non-commutative residuated lattices, Transactions of the American Mathematical Society, 46 (3) (1939) 426-444.

[13] A. Dvurečenskij: Pseudo MV-algebras are intervals in l-groups, Journal of the Australian Mathematical Society, 72 (2002) 427-445.

[14] A. Dvurečenskij and S. Pulmannová: New trends in quantum structures, kluwer Academic Publishers, Mathematics and its applications, Vol. 516, 2000.

[15] F. Esteva and L. Godo: Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems, 124 (3) (2001) 271-288.

[16] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono: Residuated Lattices: An algebraic glimpse at substructural logics, Studies in Logic and the Foundations of Mathematics, Volume 151, Elsevier, Amsterdam, 2007.

[17] G. Georgescu and A. Iorgulescu: Pseudo-MV algebras, Multiple Valued Logic, 6 (1-2) (2001) 95-135.

[18] J.A. Goguen: L-fuzzy sets, Journal of Mathematical Analysis and Applications, 18 (1) (1967) 145-174.

[19] I. Jahan: The lattice of L-ideals of a ring is modular, Fuzzy Sets and Systems, 199 (2012) 121-129.

[20] P. Jipsen and C. Tsinakis: A survey of residuated lattices, Ordered algebraic structures (J. Martinez, ed.), Kluwer academic publisher, Dordrecht, (2002) 19-56.

[21] A. Kadji, C. Lele and J.B. Nganou: A non-commutative generalization of Łukasiewicz rings, Journal of Applied Logic, 16 (2016) 1-13.

[22] A. Kadji, C. Lele and M. Tonga: Some classes of pseudo-residuated lattices, Afrika Matematika, 27 (2016) 1147-1178.

[23] A. Kadji and M. Tonga: Some new axiomatic extensions of residuated logics, Journal of Intelligent and Fuzzy Systems, 31 (2016) 1-12.

[24] T. Kuraoka and N.Y. Suzuki: Lattice of fuzzy subalgebras in universal algebra, Algebra Universalis, 47 (2002) 223-237.

[25] T.Y. Lam: A first course in noncommutative rings, Springer-Verlag, Volume 131, Berlin/Heidelberg, New York, 1991.

[26] J. Lambek, M. Barr, J.F. Kennison and R. Raphael: Injective hulls of partially ordered monoids, Theory and Applications of Categories, 26 (13) (2012) 338-348.

[27] H. Liviu-Constantin, N. Luisa-Maria and C. Gilena: Distributive residuated lattices, Annals of University of Craiova, 39 (1) (2012) 100-109.

[28] E.G. Manes: A class of fuzzy theories, Journal of Mathematical Analysis and Applications, 85 (2) (1982) 409-451.

[29] V. Murali: Lattice of fuzzy subalgebras and closure systems in I^X, Fuzzy Sets and Systems, 41 (1) (1991) 101-111.

[30] K. Murata: Primary decomposition of elements in compactly generated integral multiplicative lattices, Osaka Journal of Mathematics, 7 (1) (1970) 97-115.

[31] V. Novák: On the syntactico-semantical completeness of first-order fuzzy logic I, Kybernetika, 26 (1) (1990) 47-66.

[32] V. Novák: On the syntactico-semantical completeness of first-order fuzzy logic II, Kybernetika, 26 (1) (1990) 134-154.

[33] H. Ono: Substructural logics and residuated lattices - an introduction, Trends in Logic, Kluwer, Dordrecht, 21 (2003) 177-212.

[34] D. Piciu: Algebras of fuzzy logic, Ed. Universitaria, Craiova, 2007.

[35] B. Šešelja: Lattice of partially ordered fuzzy subalgebras, Fuzzy Sets and Systems, 81 (2) (1996) 265-269.

[36] H. Simmons: An introduction to category theory, Cambridge University Press, Cambridge/New York, 2011.

[37] U.M. Swamy and K.L.N. Swamy: Fuzzy prime ideals of rings, Journal of Mathematical Analysis and Applications, 134 (1) (1988) 94-103.

[38] M. Tonga: Maximality on fuzzy filters of lattices, Afrika Matematika, 22 (2) (2011) 105-114.

[39] J. Von Neumann: On regulars rings, Proceedings of the National Academy of Sciences, USA, 22 (12) (1936) 707-713.

[40] M. Ward and R.P. Dilworth: Residuated lattices, Transactions of the American Mathematical Society, 45 (1939) 335-354.

[41] L.A. Zadeh: Fuzzy sets, Information and Control, 8 (3) (1965) 338-353.

Published papers

During this research, some of our contributions have been published in journals as listed below:

1. S.V. Tchoffo Foka and M. Tonga: *A residuated lattice of L-fuzzy subalgebras of a mono-unary algebra*, New Mathematics and Natural Computation, 15 (3) (2019) 539-551.

2. S.V. Tchoffo Foka and M. Tonga: *A note on the algebraicity of L-fuzzy subalgebras in universal algebra*, Soft Computing, 24 (2) (2020) 895-899.

3. S.V. Tchoffo Foka and M. Tonga: *Residuated lattice of L-fuzzy ideals of a ring*, Soft Computing, 24 (12) (2020) 8717-8724.

New Mathematics and Natural Computation
Vol. 15, No. 3 (2019) 539–551

DOI: 10.1142/S1793005719500315

A Residuated Lattice of L-Fuzzy Subalgebras of a Mono-Unary Algebra

S. V. Tchoffo Foka* and Marcel Tonga†

Department of Mathematics, University of Yaounde 1
P.O. Box 812, Yaounde, Cameroon
**tchoffofoka88@yahoo.fr*
†tongamarcel@yahoo.fr

Given a complete residuated lattice $\mathcal{L} := (L; \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; 0, 1)$ and a mono-unary algebra $\mathcal{A} := (A;\ f)$, it is well known that $\mathcal{L}$ and the residuated lattice $\mathcal{F}u(A, L) := (\mathrm{Fu}(A, L); \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; \underline{0}, \underline{1})$ of L-fuzzy subsets of A satisfy the same residuated lattice identities. In this paper, we show that $\mathcal{L}$ and the residuated lattice $\mathcal{F}s(\mathcal{A}, L) := (\mathrm{Fs}(\mathcal{A}, L); \wedge, \vee, \ominus, \hookrightarrow, \nleftrightarrow; \underline{0}, \underline{1})$ of L-fuzzy subalgebras of $\mathcal{A}$ satisfy the same residuated lattice identities if and only if the Heyting algebra $\mathcal{S}ub(\mathcal{A}) := (\mathrm{Sub}(\mathcal{A}); \cap, \cup, \Rightarrow; \emptyset, A)$ of subuniverses of $\mathcal{A}$ is a Boolean algebra. We also show that $\mathcal{F}s(\mathcal{A}, L)$ is a Boolean algebra (respectively, an *MV*-algebra) if and only if $\mathcal{L}$ is a Boolean algebra (respectively, an *MV*-algebra) and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra.

Keywords: Residuated lattice; *MV*-algebra; Boolean algebra; mono-unary algebra; subuniverse; L-fuzzy subset; L-fuzzy subalgebra.

1. Introduction

In 1965, Zadeh[1] defined the notion of fuzzy subset of a set, which led to a revision of mathematics, to formalize the concept of set membership under uncertainty. In order to satisfy the needs of fuzzy reasoning, several kinds of algebraic structures were then considered. In 1967, Goguen[2] generalized the Zadeh's concept of fuzzy subset to L-fuzzy subset, replacing the unit interval [0, 1] of real numbers by the underlying set L of an appropriate structure of truth values. In 1996, Šešelja[3] introduced the concept of L-fuzzy subalgebra of a universal algebra, where L is the underlying set of a partially ordered set $\mathcal{L}$, by considering compatibility on levels sets.

In 1894, Dedekind[4] introduced the idea of residuation. Since then, many applications appeared in various algebraic theories (see, Refs. 5–7). In 1939, Ward and Dilworth[8] developed a systematic theory of lattices over which an auxiliary operation of multiplication or residuation is defined, called residuated lattices. In 1990,

*Corresponding author.

Novák[9,10] introduced first-order fuzzy logic and proved that the algebra of this logic is a residuated lattice. In 2002, Blount and Tsinakis[11] established for the first time a general structural theory of the class of residuated lattices.

In this paper, given a complete residuated lattice $\mathcal{L}$ and a mono-unary algebra $\mathcal{A}$, we set up a construction of the L-fuzzy subalgebra of $\mathcal{A}$ generated by an L-fuzzy subset of A, characterize atoms and co-atoms of the lattice $\mathbb{F}s(\mathcal{A}, L)$ of L-fuzzy subalgebras of $\mathcal{A}$ and show that the latter is algebraic, when $\mathcal{L}$ is algebraic. We also define a residuated lattice $\mathcal{F}s(\mathcal{A}, L)$ on the set of L-fuzzy subalgebras of $\mathcal{A}$ which is both an extension of $\mathcal{L}$ and the Heyting algebra $\mathcal{S}ub(\mathcal{A})$ on the set of subuniverses of $\mathcal{A}$. Furthermore, we show that $\mathcal{F}s(\mathcal{A}, L)$ is an MV-algebra (respectively, a Boolean algebra) if and only if $\mathcal{L}$ is an MV-algebra (respectively, a Boolean algebra) and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra.

2. Preliminaries

2.1. *Residuated lattices*

Definition 2.1. An algebra $(L; \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ is called a residuated lattice if it satisfies the following conditions:

(**RL1**) $(L; \wedge, \vee)$ is a bounded lattice with a partial order $\leq$;
(**RL2**) $(L; \ominus, 1)$ is a monoid;
(**RL3**) for any $x, y, z \in L$, $x \ominus y \leq z$ iff $x \leq y \twoheadrightarrow z$ iff $y \leq x \multimap z$.

An algebra $(L; \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ is a residuated lattice if and only if $(L; \wedge, \vee; 0, 1)$ is a bounded lattice, $(L; \ominus, 1)$ is a monoid, $\ominus$ is order-preserving in each argument and the inequality $x \ominus y \leq z$ has a largest solution for x (namely $y \twoheadrightarrow z$) and for y (namely $x \multimap z$). For any $x \in L$ and a nonnegative integer n, x^n is defined inductively by $x^0 = 1$ and $x^{n+1} = x^n \ominus x$.

Proposition 2.1 (Refs. 11–13). *In a residuated lattice, the following hold (whenever $\wedge$ and $\vee$ exist) for any $z \in L$, $X, Y \subseteq L$ and $\dashrightarrow \in \{\twoheadrightarrow, \multimap\}$:*

(1) $(\vee X) \ominus (\vee Y) = \bigvee_{x\in X, y\in Y} x \ominus y$,
(2) $z \dashrightarrow (\wedge X) = \bigwedge_{x\in X}(z \dashrightarrow x)$ *and* $(\vee X) \dashrightarrow z = \bigwedge_{x\in X}(x \dashrightarrow z)$,
Furthermore, the following (quasi-)identities and their mirror images (obtained by replacing $x \ominus y$ by $y \ominus x$ and interchanging $x \twoheadrightarrow y$ with $x \multimap y$) also hold:
(3) *If $x \leq y$, then $x \ominus z \leq y \ominus z$, $y \twoheadrightarrow z \leq x \twoheadrightarrow z$ and $z \twoheadrightarrow x \leq z \twoheadrightarrow y$,*
(4) $x \ominus y \leq x \wedge y$,
(5) $x \ominus 0 = 0 = 0 \ominus x$,
(6) $1 \twoheadrightarrow x = x$,
(7) *If $x \leq y$, then $x \twoheadrightarrow y = 1$.*

Proposition 2.2 (Refs. 11 and 12). *The class of residuated lattices is a variety.*

A residuated lattice is called complete (respectively, algebraic, distributive, modular,...) if so is its lattice. In a residuated lattice $\mathcal{L}$, for any $x \in L$,

$$\overline{x} := x \twoheadrightarrow 0 \quad \text{and} \quad \tilde{x} := x \multimap 0 \text{ (mirror image of } \overline{x})$$

are called left annihilator and right annihilator of x, respectively.

Definition 2.2. A residuated lattice $\mathcal{L}$ is called

- a Heyting algebra if $x \ominus y = x \wedge y$ for all $x, y \in L$;
- an RL-monoid if $(x \twoheadrightarrow y) \ominus x = x \wedge y = x \ominus (x \multimap y)$ for all $x, y \in L$;
- a MTL-algebra if $(x \twoheadrightarrow y) \vee (y \twoheadrightarrow x) = 1 = (x \multimap y) \vee (y \multimap x)$ for all $x, y \in L$;
- a BL-algebra if it is both an RL-monoid and a MTL-algebra;
- an MV-algebra if it is a BL-algebra satisfying $\tilde{\overline{x}} = x = \overline{\tilde{x}}$ for all $x \in L$;
- a Boolean algebra if it is both an MV-algebra and a Heyting algebra.

2.2. *L-fuzzy subsets of a set*

Definition 2.3. Let A be a nonempty set. A fuzzy subset of A under $\mathcal{L}$, or an L-fuzzy subset of A, is a map from A to L.

For any $B \subseteq A$, $a \in A$ and $r, s \in L$, the following functions from A to L are L-fuzzy subsets of A:

$$B_r^s(x) := \begin{cases} s & \text{if } x \in B, \\ r & \text{if not.} \end{cases} \quad \text{for all } x \in A,$$

$B_r := B_0^r$, $B^r := B_r^1$, $a_r^s := \{a\}_r^s$, $a_r := a_0^r$ (L-fuzzy point of A), $B_1 =: \chi_B := B^0$ (characteristic function of B), $\chi_a := \chi_{\{a\}}$ and $A_r =: \underline{r} := \emptyset^r$ (constant L-fuzzy subset of A with value r). For any L-fuzzy subset μ of A and $r \in L$, the sets

$$\begin{aligned} \mathrm{Supp}(\mu) &:= \{x \in A : \mu(x) \neq 0\}, \\ \mathrm{Im}(\mu) &:= \{\mu(x) : x \in A\}, \\ U(\mu, r) &:= \{a \in A : \mu(a) \geq r\}, \end{aligned}$$

are called the support, the image and the r-level set (or r-cut) of μ, respectively. The order relation $\leq$ on the set $\mathrm{Fu}(A, L)$ of L-fuzzy subsets of A is defined as follows: for any $\mu, \nu \in \mathrm{Fu}(A, L)$,

$$\mu \leq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

The relation $<$ on $\mathrm{Fu}(A, L)$ is defined as follows: for any $\mu, \nu \in \mathrm{Fu}(A, L)$,

$$\mu < \nu \text{ if and only if } \mu \leq \nu \text{ and there is } a \in A \text{ such that } \mu(a) < \nu(a).$$

The set $\mathrm{Fu}(A, L)$ forms a bounded lattice $\mathbb{F}u(A, L) := (\mathrm{Fu}(A, L); \wedge, \vee; \underline{0}, \underline{1})$ and a residuated lattice $\mathcal{F}u(A, L) := (\mathrm{Fu}(A, L); \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; \underline{0}, \underline{1})$; where the binary operations $\wedge, \vee, \ominus, \twoheadrightarrow, \multimap$ are defined componentwise. Since the class of residuated lattices is a variety, $\mathcal{L}$ and $\mathcal{F}u(A, L)$ satisfy the same residuated lattice identities. Furthermore, $\mathcal{L}$ is a complete Brouwerian residuated lattice if and only if so is $\mathcal{F}u(A, L)$.

2.3. *Mono-unary algebras*

Definition 2.4 (see, Refs. 14 and 15). A mono-unary algebra or a unary is an algebra with a single unary operation, that is an algebra of type $\langle 1\rangle$.

Let $\mathcal{A} := (A; f)$ be a mono-unary. A subset B of A is a subuniverse of $\mathcal{A}$ if and only if $f(x) \in B$ for all $x \in B$.

Remark 2.1. The set of subuniverses of $\mathcal{A}$ forms a Heyting algebra

$$\mathcal{S}ub(\mathcal{A}) := (\mathrm{Sub}(\mathcal{A}); \cap, \cup, \Rightarrow; \emptyset, A),$$

where the binary operation $\Rightarrow$ is defined by: $B \Rightarrow C := \bigcup\{D \in \mathrm{Sub}(\mathcal{A}) : D \cap B \subseteq C\}$ for all $B, C \in \mathrm{Sub}(\mathcal{A})$.

For any nonnegative integer n, f^n is defined inductively by: $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $x \in A$.

Definition 2.5. An element x of A is said to be cyclic if there is some integer $p \geq 1$ such that $f^p(x) = x$.

Remark 2.2. The subuniverse of $\mathcal{A}$ generated by an element x of A is given by $\mathrm{Sg}(x) = \{f^k(x) : k \in \mathbb{N}\}$.

In the rest of this paper, unless otherwise specified, $\mathcal{A} = (A; f)$ is a mono-unary algebra and $\mathcal{L} = (L; \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; 0, 1)$ is a complete residuated lattice.

3. Lattice of *L*-Fuzzy Subalgebras

Definition 3.1. An L-fuzzy subset μ of A is an L-fuzzy subalgebra of $\mathcal{A}$ if $\mu(f(x)) \geq \mu(x)$ for all $x \in A$.

Note that an L-fuzzy subset of A is an L-fuzzy subalgebra of $\mathcal{A}$ if and only if all its levels sets are subuniverses of $\mathcal{A}$. Furthermore, for any $B \in \mathrm{Sub}(\mathcal{A})$, the L-fuzzy subsets B_r and B^r are L-fuzzy subalgebras of $\mathcal{A}$.

Remark 3.1. The set of L-fuzzy subalgebras of $\mathcal{A}$ forms a complete lattice $\mathbb{F}s(\mathcal{A}, L) := (\mathrm{Fs}(\mathcal{A}, L); \wedge, \vee; \underline{0}, \underline{1})$.

Now, set $C_x^k := \{a \in A : f^k(a) = x\}$ for all $x \in A$ and $k \in \mathbb{N}$.

Theorem 3.1. *Let μ be an L-fuzzy subset of A. The L-fuzzy subalgebra of $\mathcal{A}$ generated by μ is defined by: $\mu_\star(x) = \vee_{k\in\mathbb{N}} \vee_{a\in C_x^k} \mu(a)$ for all $x \in A$.*

Proof. Since $\mu_\star(x) \geq \vee_{a\in C_x^0}\mu(a) = \vee\{\mu(x)\} = \mu(x)$ for all $x \in A$, we have $\mu \leq \mu_\star$. We next show that $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$. For any $x \in A$, we have

$$\mu_\star(f(x)) = \bigvee_{k\in\mathbb{N}} \bigvee_{a\in C^k_{f(x)}} \mu(a) = \left[\bigvee_{a\in C^0_{f(x)}} \mu(a)\right] \vee \left[\bigvee_{k\in\mathbb{N}} \bigvee_{a\in C^{k+1}_{f(x)}} \mu(a)\right];$$

since, $C_x^k \subseteq C_{f(x)}^{k+1}$, we have $\mu_\star(f(x)) \geq \vee_{k\in\mathbb{N}} \vee_{a\in C^{k+1}_{f(x)}} \mu(a) \geq \vee_{k\in\mathbb{N}} \vee_{a\in C^k_x} \mu(a) = \mu_\star(x)$. Hence, $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.

Finally, let ν be an L-fuzzy subalgebra of $\mathcal{A}$ which contains μ. Let $x \in A$. For any $k \in \mathbb{N}$ and $a \in C_x^k$, we have $\nu(x) = \nu(f^k(a)) \geq \nu(a) \geq \mu(a)$. Thus, $\nu(x) \geq \vee_{a\in C_x^k}\mu(a)$ for all $k \in \mathbb{N}$; i.e. $\nu(x) \geq \vee_{k\in\mathbb{N}} \vee_{a\in C_x^k} \mu(a)$; i.e. $\nu(x) \geq \mu_\star(x)$. So, $\mu \leq \nu$. Hence, $\mu_\star = \mathrm{Fsg}(\mu)$. □

Lemma 3.1. *Let $x \in A$. Then $\mathrm{Sg}(x)$ is an atom of $\mathbb{S}ub(\mathcal{A})$ if and only if x is cyclic.*

Proof. Assume that $\mathrm{Sg}(x)$ is an atom of $\mathbb{S}ub(\mathcal{A})$. Since $\emptyset \subset \mathrm{Sg}(f(x)) \subseteq \mathrm{Sg}(x)$, we have $\mathrm{Sg}(f(x)) = \mathrm{Sg}(x)$; thus, there is $n \in \mathbb{N}$ such that $f^n(f(x)) = x$; so, $f^{n+1}(x) = x$. Hence, x is cyclic.

Conversely, assume that x is cyclic of order n. Let B be a subuniverse of $\mathcal{A}$ such that $\emptyset \subset B \subseteq \mathrm{Sg}(x)$. Since there is $m \leq n$ such that $f^m(x) \in B$, we have $x = f^n(x) = f^{n-m}(f^m(x)) \in B$; thus, $\mathrm{Sg}(x) \subseteq B$ and, $\mathrm{Sg}(x) = B$. Hence, $\mathrm{Sg}(x)$ is an atom of $\mathbb{S}ub(\mathcal{A})$. □

Theorem 3.2. *Atoms of $\mathbb{F}s(\mathcal{A}, L)$ are only the L-fuzzy subalgebras $\mathrm{Sg}(a)_r$, where r is an atom of $\mathcal{L}$ and a is a cyclic element of $\mathcal{A}$.*

Proof. $(\Rightarrow)$ Let μ be an atom of $\mathrm{Fs}(\mathcal{A}, L)$. Since there is $a \in A$ such that $\mu(a) \neq 0$, we have $\underline{0} < \mathrm{Sg}(a)_{\mu(a)} \leq \mu$; thus, $\mu = \mathrm{Sg}(a)_{\mu(a)}$. We next show that $\mathrm{Sg}(a)$ is an atom of $\mathcal{S}ub(\mathcal{A})$. So, for any subuniverse B of $\mathcal{A}$ such that $\emptyset \subset B \subseteq \mathrm{Sg}(a)$, we have $\underline{0} < B_{\mu(a)} \leq \mu$; thus, $\mu = B_{\mu(a)}$; i.e. $B = \mathrm{Sg}(a)$. Hence, $\mathrm{Sg}(a)$ is an atom of $\mathcal{S}ub(\mathcal{A})$. Finally, for any $s \in L$ such that $0 < s \leq \mu(a)$, we have $\underline{0} < \mathrm{Sg}(a)_s \leq \mu$; thus, $\mathrm{Sg}(a)_s = \mu$; i.e. $s = \mu(a)$. Hence, $\mu(a)$ is an atom of $\mathcal{L}$.

$(\Leftarrow)$ Let r be an atom of $\mathcal{L}$ and $a \in A$ such that $\mathrm{Sg}(a)$ is an atom of $\mathcal{S}ub(\mathcal{A})$. It is clear that $\mathrm{Sg}(a)_r \neq \underline{0}$. Now, let $\mu \in \mathrm{Fs}(\mathcal{A}, L)$ such that $\underline{0} < \mu \leq \mathrm{Sg}(a)_r$. Since there is $b \in \mathrm{Sg}(a)$ such that $0 \leq \mu(b) < r$, we have $\mu(b) = 0$ and $\emptyset \subset \mathrm{Sg}(b) \subseteq \mathrm{Sg}(a)$; thus, $\mu(b) = 0$ and $\mathrm{Sg}(a) = \mathrm{Sg}(b)$; so, $\mu(a) = 0$. Hence, $\mu(x) = 0$ for all $x \in \mathrm{Sg}(a)$; i.e. $\mu = \underline{0}$. Therefore, $\mathrm{Sg}(a)_r$ is an atom of $\mathbb{F}s(\mathcal{A}, L)$. □

Theorem 3.3. *Co-atoms of $\mathbb{F}s(\mathcal{A}, L)$ are only the L-fuzzy subalgebras B^s, where s and B are co-atoms of $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$, respectively.*

Proof. $(\Rightarrow)$ Let μ be a co-atom of $\mathbb{F}s(\mathcal{A}, L)$. For any $a, b \notin U(\mu, 1)$, we have $\mu \leq \underline{\mu(a)} \vee \mu < \underline{1}$ and $\mu \leq \underline{\mu(b)} \vee \mu < \underline{1}$; thus, $\underline{\mu(a)} \vee \mu = \mu = \underline{\mu(b)} \vee \mu$ and, $\mu(a) = \mu(b)$. It follows that $\mu = (U(\mu, 1))^s$ for some $s \in L$. Since $\mu \neq \underline{1}$, we have $s \neq 1$ and

$U(\mu, 1) \neq A$.

- For any $r \in L$ such that $s < r \leq 1$, we have $\mu < \underline{r} \vee \mu \leq \underline{1}$ and, $\underline{r} \vee \mu = \underline{1}$; thus, $r = r \vee s = 1$. Hence, s is a co-atom of $\mathcal{L}$.
- For any $D \in \mathrm{Sub}(\mathcal{A})$ such that $U(\mu, 1) \subset D \subseteq A$, we have $\mu < D^s \leq \underline{1}$ and, $D^s = \underline{1}$; thus, $D = A$. Hence, $U(\mu, 1)$ is a co-atom of $\mathcal{S}ub(\mathcal{A})$.

($\Leftarrow$) Let s and B be co-atoms of $\mathcal{L}$ and $\mathcal{S}ub(\mathcal{A})$, respectively. We have $B^s \neq \underline{1}$, since $s \neq 1$ and $B \neq A$. For any $\mu \in \mathrm{Fs}(\mathcal{A}, L)$ such that $B^s < \mu \leq \underline{1}$, we have $B = U(B^s, 1) \subseteq U(\mu, 1) \subseteq A$ and $a \notin B$ such that $s < \mu(a) \leq 1$; thus, $B \subseteq U(\mu, 1) \subseteq A$ and $a \in U(\mu, 1) \backslash B$; so, $B \subset U(\mu, 1) \subseteq A$ and, $U(\mu, 1) = A$; i.e. $\mu = \underline{1}$. Hence, B^s is a co-atom of $\mathbb{F}s(\mathcal{A}, L)$. □

Lemma 3.2. *Let c be a compact element of $\mathcal{L}$ and $a \in A$. Then $\mathrm{Sg}(a)_c$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$.*

Proof. Let $\{\mu_i\}_{i\in I} \subseteq \mathrm{Fs}(\mathcal{A}, L)$ such that $\mathrm{Sg}(a)_c \leq \vee_{i\in I}\mu_i$. Since $c \leq \vee_{i\in I}\mu_i(a)$, there is a finite subset I_0 of I such that $c \leq \vee_{i\in I_0}\mu_i(a)$. For any $x \in \mathrm{Sg}(a)$, we have $\mathrm{Sg}(a)_c(x) = c \leq \vee_{i\in I_0}\mu_i(a) \leq \vee_{i\in I_0}\mu_i(x) = (\vee_{i\in I_0}\mu_i)(x)$; thus, $\mathrm{Sg}(a)_c \leq \vee_{i\in I_0}\mu_i$. Hence, $\mathrm{Sg}(a)_c$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$. □

For any compact element c of $\mathcal{L}$ and $a \in A$, $\mathrm{Sg}(a)_c$ will be called a compact principal L-fuzzy subalgebra of $\mathcal{A}$.

Theorem 3.4. *Suppose that $\mathcal{L}$ is algebraic. Then the following hold*:

(1) *Compact elements of $\mathbb{F}s(\mathcal{A}, L)$ are only finite suprema of compact principal L-fuzzy subalgebras of $\mathcal{A}$.*
(2) *$\mathbb{F}s(\mathcal{A}, L)$ is an algebraic lattice.*

Proof. (1) A finite supremum of compact principal L-fuzzy subalgebras of $\mathcal{A}$ is a finite supremum of compact elements of $\mathbb{F}s(\mathcal{A}, L)$ by Lemma 3.2; so, it is a compact element of $\mathbb{F}s(\mathcal{A}, L)$.

Conversely, let μ be a compact element of $\mathbb{F}s(\mathcal{A}, L)$. Since $\mu = \vee_{a\in A}\mathrm{Sg}(a)_{\mu(a)}$, there are $a_1, \ldots, a_n \in A$ such that $\mu = \vee_{1\leq i\leq n}\mathrm{Sg}(a_i)_{\mu(a_i)}$. Since $\mathcal{L}$ is algebraic, for any $1 \leq i \leq n$, there is a family $\{c_j\}_{j\in I_i}$ of compact elements of $\mathcal{L}$ such that $\mu(a_i) = \vee_{j\in I_i}c_j$. It follows that

$$\begin{aligned} \mu &= \bigvee_{1\leq i\leq n} \mathrm{Sg}(a_i)_{\bigvee_{j\in I_i} c_j} \\ &= \bigvee_{1\leq i\leq n} \bigvee_{j\in I_i} \mathrm{Sg}(a_i)_{c_j} \\ &= \bigvee_{(j_1,\ldots,j_n)\in \prod_{1\leq i\leq n} I_i} \bigvee_{1\leq i\leq n} \mathrm{Sg}(a_i)_{c_{j_i}}. \end{aligned}$$

Since μ is compact, there is a family $\{K_i\}_{1\leq i\leq n}$ of finite sets such that $K_i \subseteq I_i$ for all $1 \leq i \leq n$ and $\mu = \vee_{(j_1,\ldots,j_n)\in \prod_{1\leq i\leq n} K_i} \vee_{1\leq i\leq n} \mathrm{Sg}(a_i)_{c_{j_i}}$. Hence, by Proposition 3.2, μ is a finite supremum of compact principal L-fuzzy subalgebras of $\mathcal{A}$.

(2) Since $\mathbb{F}s(\mathcal{A}, L)$ is complete, it suffices to show that it is compactly generated. So, let $\mu \in \mathrm{Fs}(\mathcal{A}, L)$. Since $\mathcal{L}$ is algebraic, for any $a \in A$, there is a family $\{c_{i,a}\}_{i\in I_a}$ of compact elements of $\mathcal{L}$ such that $\mu(a) = \vee_{i\in I_a} c_{i,a}$. Hence, $\mu = \vee_{a\in A} \vee_{i\in I_a} \mathrm{Fsg}(a_{c_{i,a}})$, and for each $a \in A$ and $i \in I_a$, $\mathrm{Fsg}(a_{c_{i,a}})$ is compact by Proposition 3.2. Therefore, $\mathbb{F}s(\mathcal{A}, L)$ is algebraic. □

4. Residuated Lattice of *L*-Fuzzy Subalgebras

$\mathrm{Fs}(\mathcal{A}, L)$ is closed under the binary operation $\ominus$ of the residuated lattice $\mathcal{F}u(A, L)$ of L-fuzzy subsets of A, but the binary operations $\twoheadrightarrow$ and $\multimap$ are not necessarily well defined on $\mathrm{Fs}(\mathcal{A}, L)$ as the following example shows.

Example 4.1. Let's take $L = \{0, \alpha, \beta, \gamma, 1\}$, where $0 < \alpha < \beta, \gamma < 1$ and β, γ are incomparable. Consider the binary operations $\ominus$, $\twoheadrightarrow$, $\multimap$ given by the following Cayley tables:

$\ominus$	0	α	β	γ	1
0	0	0	0	0	0
α	0	0	0	α	α
β	0	α	β	α	β
γ	0	0	0	γ	γ
1	0	α	β	γ	1

$\twoheadrightarrow$	0	α	β	γ	1
0	1	1	1	1	1
α	γ	1	1	1	1
β	γ	γ	1	γ	1
γ	0	β	β	1	1
1	0	α	β	γ	1

$\multimap$	0	α	β	γ	1
0	1	1	1	1	1
α	β	1	1	1	1
β	0	γ	1	γ	1
γ	β	β	β	1	1
1	0	α	β	γ	1

.

Then $\mathcal{L} = (L; \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; 0, 1)$ is a residuated lattice. Consider the Peano algebra $\mathcal{N} = (\mathbb{N}; \sigma)$, given by $\sigma(x) = x + 1$ for all $x \in \mathbb{N}$, and the L-fuzzy subalgebras μ and ν of $\mathcal{N}$ defined by

$$\mu(x) = \begin{cases} 0 & \text{if } x = 0, \\ \beta & \text{if not.} \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} 0 & \text{if } x = 0, \\ \gamma & \text{if not.} \end{cases} \quad \text{for all } x \in \mathbb{N},$$

The L-fuzzy subset $\mu \twoheadrightarrow \nu$ of $\mathbb{N}$ is not an L-fuzzy subalgebra of $\mathcal{N}$, since

$$(\mu \twoheadrightarrow \nu)(\sigma(0)) = (\mu \twoheadrightarrow \nu)(1) = \beta \twoheadrightarrow \gamma = \gamma < 1 = 0 \twoheadrightarrow 0 = (\mu \twoheadrightarrow \nu)(0).$$

Theorem 4.1. *Let μ be an L-fuzzy subset of A. The L-fuzzy subset $\mu^\star$ of A, given by $\mu^\star(x) = \wedge_{k\in\mathbb{N}} \mu(f^k(x))$ for all $x \in A$, is the biggest L-fuzzy subalgebra of $\mathcal{A}$ contained in μ.*

Proof. We have $\mu^\star \leq \mu$, since for any $x \in A$, $\mu^\star(x) \leq \mu(f^0(x)) = \mu(x)$. We next show that $\mu^\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.

For any $x \in A$, we have

$$\mu^\star(f(x)) = \bigwedge_{k\in\mathbb{N}} \mu(f^{k+1}(x)) \geq \mu(f^0(x)) \wedge \bigwedge_{k\in\mathbb{N}} \mu(f^{k+1}(x)) = \mu^\star(x).$$

Hence, $\mu^\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.

Finally, let ν be an L-fuzzy subalgebra of $\mathcal{A}$ contained in μ. For any $x \in A$, we have $\nu(x) \leq \nu(f^k(x)) \leq \mu(f^k(x))$ for all $k \in \mathbb{N}$; thus, $\nu(x) \leq \wedge_{k\in\mathbb{N}}\mu(f^k(x)) = \mu^\star(x)$. Hence, $\nu \leq \mu^\star$. Therefore, $\mu^\star$ is the biggest L-fuzzy subalgebra of $\mathcal{A}$ contained in μ. □

Theorem 4.2. *For any $\mu, \nu \in \mathrm{Fs}(\mathcal{A}, L)$, set $\mu \hookrightarrow \nu := (\mu \twoheadrightarrow \nu)^\star$ and $\mu \looparrowright \nu := (\mu \multimap \nu)^\star$. Then $\mathcal{F}s(\mathcal{A}, L) := (\mathrm{Fs}(\mathcal{A}, L); \wedge, \vee, \ominus, \hookrightarrow, \looparrowright; \underline{0}, \underline{1})$ is a complete residuated lattice.*

Proof. We only have to show that $\mu \hookrightarrow \nu = \max\{\delta \in \mathrm{Fs}(\mathcal{A}, L) : \delta \ominus \mu \leq \nu\}$ and $\mu \looparrowright \nu = \max\{\delta \in \mathrm{Fs}(\mathcal{A}, L) : \mu \ominus \delta \leq \nu\}$ for all $\mu, \nu \in \mathrm{Fs}(\mathcal{A}, L)$. So, let $\mu, \nu \in \mathrm{Fs}(\mathcal{A}, L)$. We have $(\mu \hookrightarrow \nu) \ominus \mu = (\mu \twoheadrightarrow \nu)^\star \ominus \mu \leq (\mu \twoheadrightarrow \nu) \ominus \mu \leq \nu$. Moreover, for any $\delta \in \mathrm{Fs}(\mathcal{A}, L)$ such that $\delta \ominus \mu \leq \nu$, we have $\delta \leq \mu \twoheadrightarrow \nu$; thus, $\delta \leq (\mu \twoheadrightarrow \nu)^\star = \mu \hookrightarrow \nu$. Hence, $\mu \hookrightarrow \nu = \max\{\delta \in \mathrm{Fs}(\mathcal{A}, L) : \delta \ominus \mu \leq \nu\}$ and, $\mu \looparrowright \nu = \max\{\delta \in \mathrm{Fs}(\mathcal{A}, L) : \mu \ominus \delta \leq \nu\}$ by similar arguments. Therefore, $\mathcal{F}s(\mathcal{A}, L)$ is a complete residuated lattice. □

Theorem 4.3. *The map $\phi : \mathrm{Sub}(\mathcal{A}) \to \mathrm{Fs}(\mathcal{A}, L)$, given by $\phi(B) = B_1$ for all $B \in \mathrm{Sub}(\mathcal{A})$, is a complete residuated lattice embedding.*

Proof. It is clear that ϕ is a complete lattice embedding of $\mathbb{S}ub(\mathcal{A})$ into $\mathbb{F}s(\mathcal{A}, L)$. Since we have $\phi(B \cap C) = (B \cap C)_1 = B_1 \ominus C_1 = \phi(B) \ominus \phi(C)$ for all $B, C \in \mathrm{Sub}(\mathcal{A})$, it suffices to show that $\phi(B) \hookrightarrow \phi(C) = \phi(B \Rightarrow C) = \phi(B) \looparrowright \phi(C)$. So, let $B, C \in \mathrm{Sub}(\mathcal{A})$. For any $x \notin B \Rightarrow C$, we have $\mathrm{Sg}(x) \cap B \not\subseteq C$; thus, $f^{k_0}(x) \in B$ and $f^{k_0}(x) \notin C$ for some $k_0 \in \mathbb{N}$; so,

$$\begin{aligned}
(\phi(B) \hookrightarrow \phi(C))(x) &= (B_1 \hookrightarrow C_1)(x) \\
&= \left[B_1(f^{k_0}(x)) \twoheadrightarrow C_1(f^{k_0}(x))\right] \\
&\quad \wedge \left[\bigwedge_{k\in\mathbb{N}} B_1(f^{k+1}(x)) \twoheadrightarrow C_1(f^{k+1}(x))\right] \\
&= (1 \twoheadrightarrow 0) \wedge \left[\bigwedge_{k\in\mathbb{N}} B_1(f^{k+1}(x)) \twoheadrightarrow C_1(f^{k+1}(x))\right] \\
&= 0 \\
&= (B \Rightarrow C)_1(x) \\
&= \phi(B \Rightarrow C)(x).
\end{aligned}$$

Now, let $x \in B \Rightarrow C$ and $D \in \mathrm{Sub}(\mathcal{A})$ such that $D \cap B \subseteq C$ and $x \in D$.

- For any $n \in \Omega(B) := \{k \in \mathbb{N} : f^k(x) \in B\}$, we have $f^n(x) \in D \cap B \subseteq C$; thus, $f^n(x) \in B$ and $f^n(x) \in C$; so, $B_1(f^n(x)) \twoheadrightarrow C_1(f^n(x)) = 1 \twoheadrightarrow 1 = 1$.
- For any $n \notin \Omega(B)$, we have $B_1(f^n(x)) \twoheadrightarrow C_1(f^n(x)) = 0 \twoheadrightarrow C_1(f^n(x)) = 1$.

Thus,

$$
\begin{aligned}
(\phi(B)\hookrightarrow\phi(C))(x) &= (B_1\hookrightarrow C_1)(x)\\
&= \left[\bigwedge_{k\in\Omega(B)} B_1(f^k(x))\twoheadrightarrow C_1(f^k(x))\right]\\
&\quad \wedge \left[\bigwedge_{k\notin\Omega(B)} B_1(f^k(x))\twoheadrightarrow C_1(f^k(x))\right]\\
&= \left(\bigwedge_{k\in\Omega(B)} 1\right) \wedge \left(\bigwedge_{k\notin\Omega(B)} 1\right)\\
&= 1 \wedge 1\\
&= 1\\
&= (B \Rightarrow C)_1(x)\\
&= \phi(B \Rightarrow C)(x).
\end{aligned}
$$

Hence, $\phi(B \Rightarrow C) = \phi(B)\hookrightarrow\phi(C)$ and, $\phi(B \Rightarrow C) = \phi(B)\looparrowright\phi(C)$ by similar arguments. Therefore, ϕ is a complete residuated lattice embedding of $\mathcal{S}ub(\mathcal{A})$ into $\mathcal{F}s(\mathcal{A}, L)$. ◻

Theorem 4.4. *The map* $\psi : L \to \mathrm{Fs}(\mathcal{A}, L)$, *given by* $\psi(r) = \underline{r}$ *for all* $r \in L$, *is a complete residuated lattice embedding.*

Proof. It is clear that ψ is a complete lattice embedding of the lattice of $\mathcal{L}$ into $\mathbb{F}s(\mathcal{A}, L)$. Now, let $r, s \in L$. For any $x \in A$, we have

$$
\begin{aligned}
\psi(r \ominus s)(x) &= r \ominus s\\
&= \underline{r}(x) \ominus \underline{s}(x)\\
&= \psi(r)(x) \ominus \psi(s)(x)\\
&= (\psi(r) \ominus \psi(s))(x).
\end{aligned}
$$

Thus, $\psi(r \ominus s) = \psi(r) \ominus \psi(s)$. For any $x \in A$, we have

$$
\begin{aligned}
\psi(r\twoheadrightarrow s)(x) &= r\twoheadrightarrow s\\
&= \bigwedge_{k\in\mathbb{N}} \underline{r}(f^k(x))\twoheadrightarrow\underline{s}(f^k(x))\\
&= \bigwedge_{k\in\mathbb{N}} \psi(r)(f^k(x))\twoheadrightarrow\psi(s)(f^k(x))\\
&= (\psi(r)\hookrightarrow\psi(s))(x).
\end{aligned}
$$

Thus, $\psi(r\twoheadrightarrow s) = \psi(r)\hookrightarrow\psi(s)$ and, $\psi(r\multimap s) = \psi(r)\looparrowright\psi(s)$ by similar arguments. Hence, ψ is a complete residuated lattice embedding of $\mathcal{L}$ into $\mathcal{F}s(\mathcal{A}, L)$. ◻

5. Residuated Lattice Theoretic Properties of $\mathcal{F}s(\mathcal{A},L)$

Since $\wedge$, $\vee$ and $\ominus$ are defined componentwise on $\mathrm{Fs}(\mathcal{A}, L)$, $\mathcal{L}$ and $\mathcal{F}s(\mathcal{A}, L)$ satisfy the same bounded lattice-ordered monoid identities.

Lemma 5.1. *The following statements are equivalent*:

(a) For any $\mu \in \mathrm{Fu}(A, L)$, $\mu \in \mathrm{Fs}(\mathcal{A}, L)$ iff $\mu(f(x)) = \mu(x)$ for all $x \in A$.
(b) $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice.

Proof. Suppose that (a) is satisfied. Let $B \in \mathrm{Sub}(\mathcal{A})$. For any $x \in \overline{B}$, we have $B_1(f(x)) = B_1(x) = 0$ and, $f(x) \in \overline{B}$. So, $\overline{B} \in \mathrm{Sub}(\mathcal{A})$. Hence, $\mathcal{S}ub(\mathcal{A})$ is a Boolean lattice.

Conversely, suppose that (b) is satisfied. Let $\mu \in \mathrm{Fs}(\mathcal{A}, L)$. For any $x \in A$, $f(x) \in U[\mu, \mu(f(x))] \in \mathrm{Sub}(\mathcal{A})$; thus, $x \in U[\mu, \mu(f(x))]$; so, $\mu(x) \geq \mu(f(x))$ and, $\mu(f(x)) = \mu(x)$. Whence the result. □

Theorem 5.1. *$\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$ if and only if $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice.*

Proof. Assume that $\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$. Let B be a subuniverse of $\mathcal{A}$. For any $x \in \overline{B}$, we have

$$\begin{aligned} B_1(f(x)) \twoheadrightarrow 0 &= B_1(f(x)) \twoheadrightarrow \underline{0}(f(x)) \\ &\geq (B_1 \hookrightarrow \underline{0})(x) \\ &= (B_1 \twoheadrightarrow \underline{0})(x) \\ &= B_1(x) \twoheadrightarrow \underline{0}(x) \\ &= 0 \twoheadrightarrow 0 \\ &= 1; \end{aligned}$$

thus, $B_1(f(x)) \twoheadrightarrow 0 = 1$ and, $B_1(f(x)) = 0$; i.e. $f(x) \notin B$ and, $f(x) \in \overline{B}$. So, $\overline{B}$ is a subuniverse of $\mathcal{A}$. Hence $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice.

Conversely, assume that $\mathbb{S}ub(\mathcal{A})$ is a Boolean lattice. Let μ and ν be two L-fuzzy subalgebras of $\mathcal{A}$. For any $x \in A$, we have

$$\begin{aligned} (\mu \hookrightarrow \nu)(x) &= \bigwedge_{k \in \mathbb{N}} \mu(f^k(x)) \twoheadrightarrow \nu(f^k(x)) \\ &= \bigwedge_{k \in \mathbb{N}} \mu(x) \twoheadrightarrow \nu(x) \\ &= \mu(x) \twoheadrightarrow \nu(x) = (\mu \twoheadrightarrow \nu)(x); \end{aligned}$$

thus, $\mu \hookrightarrow \nu = \mu \twoheadrightarrow \nu$. Hence, $\hookrightarrow$ is the restriction of $\twoheadrightarrow$ to $\mathrm{Fs}(\mathcal{A}, L)$. A similar reasoning shows that $\looparrowright$ is the restriction of $\multimap$ to $\mathrm{Fs}(\mathcal{A}, L)$. Therefore, $\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$. □

Let $\mathcal{K}$ be a class of residuated lattices such that $\mathrm{Mod}(\mathrm{Id}(\mathcal{K}) \cup \{x \otimes y = x \wedge y\})$ is included in the class of Boolean algebras (for example, the class of *MV*-algebras).

Theorem 5.2. *$\mathcal{F}s(\mathcal{A}, L) \models \mathrm{Id}(\mathcal{K})$ if and only if $\mathcal{L} \models \mathrm{Id}(\mathcal{K})$ and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra.*

Proof. If $\mathcal{F}s(\mathcal{A}, L) \models \mathrm{Id}(\mathcal{K})$, then $\mathcal{S}ub(\mathcal{A}) \models \mathrm{Id}(\mathcal{K})$ and $\mathcal{L} \models \mathrm{Id}(\mathcal{K})$ by Theorems 4.3 and 4.4, respectively; thus, $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra and $\mathcal{L} \models \mathrm{Id}(\mathcal{K})$.

Conversely, assume that $\mathcal{L} \models \mathrm{Id}(\mathcal{K})$ and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra. Then $\mathcal{F}s(\mathcal{A}, L)$ is a subresiduated lattice of $\mathcal{F}u(A, L)$ by Theorem 5.1. Consequently, $\mathcal{F}s(\mathcal{A}, L) \models \mathrm{Id}(\mathcal{K})$, since $\mathcal{L} \models \mathrm{Id}(\mathcal{K})$. □

If $\mathcal{F}s(\mathcal{A}, L)$ is an *RL*-monoid, then $\mathcal{L}$ is an *RL*-monoid by Theorem 4.4; but the converse is not necessarily true as the following example shows.

Example 5.1. Let $L = \{0, \alpha, \beta, 1\}$, where $0 < \alpha < \beta < 1$, and define the binary operations $\ominus$ and $\twoheadrightarrow$ on L as follows:

$\ominus$	0	α	β	1
0	0	0	0	0
α	0	0	α	α
β	0	α	β	β
1	0	α	β	1

$\twoheadrightarrow$	0	α	β	1
0	1	1	1	1
α	α	1	1	1
β	0	α	1	1
1	0	α	β	1

.

Then $\mathcal{L} = (L; \wedge, \vee, \ominus, \twoheadrightarrow, \twoheadrightarrow; 0, 1)$ is an *RL*-monoid. Consider the mono-unary algebra $\mathcal{A}$ given by the table

f	
0	0
a	a
b	a
c	b

, and the L-fuzzy subalgebras σ and τ of $\mathcal{A}$ defined for any $x \in A$ by

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 0, \\ \beta & \text{if } x \in \{a, b\}, \\ \alpha & \text{if } x = c. \end{cases} \quad \text{and} \quad \tau(x) = \begin{cases} 1 & \text{if } x = 0, \\ \alpha & \text{if } x \in \{a, b, c\}. \end{cases}$$

Since $\sigma \hookrightarrow \tau = \tau$, we have $((\sigma \hookrightarrow \tau) \ominus \sigma)(c) = (\tau \ominus \sigma)(c) = \alpha \ominus \alpha = 0 \neq \alpha = (\sigma \wedge \tau)(c)$; thus, $(\sigma \hookrightarrow \tau) \ominus \sigma \neq \sigma \wedge \tau$. It follows that $\mathcal{F}s(\mathcal{A}, L)$ is not an *RL*-monoid.

If $\mathcal{F}s(\mathcal{A}, L)$ is a *MTL*-algebra, then $\mathcal{S}ub(\mathcal{A})$ and $\mathcal{L}$ are *MTL*-algebra by Theorems 4.3 and 4.4, respectively; but the converse is not necessarily true as the following example shows.

Example 5.2. Let $L = \{0, \alpha, \beta, \gamma, 1\}$, where $0 < \alpha < \beta, \gamma < 1$, and β, γ are not comparable; define the binary operations $\ominus$, $\twoheadrightarrow$ and $\multimap$ on L as follows:

$\ominus$	0	α	β	γ	1
0	0	0	0	0	0
α	0	0	α	0	α
β	0	0	β	0	β
γ	0	α	α	γ	γ
1	0	α	β	γ	1

$\twoheadrightarrow$	0	α	β	γ	1
0	1	1	1	1	1
α	β	1	1	1	1
β	0	γ	1	γ	1
γ	β	β	β	1	1
1	0	α	β	γ	1

$\multimap$	0	α	β	γ	1
0	1	1	1	1	1
α	γ	1	1	1	1
β	γ	γ	1	γ	1
γ	0	β	β	1	1
1	0	α	β	γ	1

.

Then $\mathcal{L} = (L; \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; 0, 1)$ is a *MTL*-algebra. Consider the unar $\mathcal{A}$ given in Example 5.1. The subuniverses of $\mathcal{A}$ are $B_1 = \emptyset$, $B_2 = \{a\}$, $B_3 = \{a, b\}$, $B_4 = \{a, b, c\}$, $B_5 = \{0\}$, $B_6 = \{0, a\}$, $B_7 = \{0, a, b\}$ and $B_8 = A$. The binary operation $\Rightarrow$ of $\mathcal{S}ub(\mathcal{A})$ is given by

$\Rightarrow$	$\emptyset$	B_2	B_3	B_4	B_5	B_6	B_7	A
$\emptyset$	A	A	A	A	A	A	A	A
B_2	B_5	A	A	A	B_5	A	A	A
B_3	B_5	B_6	A	A	B_5	A	A	A
B_4	B_5	B_6	B_7	A	B_5	B_6	B_7	A
B_5	B_4	B_4	B_4	B_4	A	A	A	A
B_6	$\emptyset$	B_4	B_4	B_4	B_5	A	A	A
B_7	$\emptyset$	B_2	B_3	B_4	B_5	B_6	A	A
A	$\emptyset$	B_2	B_3	B_4	B_5	B_6	B_7	A

.

It is easy to check that $\mathcal{S}ub(\mathcal{A})$ is a *MTL*-algebra. Consider the L-fuzzy subalgebras σ and τ of $\mathcal{A}$ defined for any $x \in A$ by

$$\sigma(x) = \begin{cases} 1 & \text{if } x = 0, \\ \beta & \text{if } x \in \{a, b\}, \\ \alpha & \text{if } x = c. \end{cases} \quad \text{and} \quad \tau(x) = \begin{cases} 1 & \text{if } x = 0, \\ \gamma & \text{if } x \in \{a, b, c\}, \\ 0 & \text{if } x = c. \end{cases}$$

Then $\sigma \hookrightarrow \tau = 0_1 \vee \{a, b\}_\gamma \vee c_\alpha$ and $\tau \hookrightarrow \sigma = 0_1 \vee \{a, b, c\}_\beta$; thus,

$$((\sigma \hookrightarrow \tau) \vee (\tau \hookrightarrow \sigma))(c) = \beta \neq 1.$$

So, $(\sigma \hookrightarrow \tau) \vee (\tau \hookrightarrow \sigma) \neq \underline{1}$. Hence, $\mathcal{F}s(\mathcal{A}, L)$ is not a *MTL*-algebra.

6. Conclusion

In this paper, given a complete residuated lattice $\mathcal{L}$ and a mono-unary algebra $\mathcal{A}$, we have defined a residuated lattice $\mathcal{F}s(\mathcal{A}, L)$ on the set of L-fuzzy subalgebras of $\mathcal{A}$ and showed that the latter is an *MV*-algebra (respectively, a Boolean algebra) if and only if $\mathcal{L}$ is an *MV*-algebra (respectively, a Boolean algebra) and $\mathcal{S}ub(\mathcal{A})$ is a Boolean algebra. Since it appeared that this transfer is not necessarily possible in the class of *BL*-algebras (*MTL*-algebras, *RL*-monoids), it would be interesting to investigate the class of mono-unary algebras for which the transfer remains possible.

References

1. L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965) 338–353.
2. J. A. Goguen, L-fuzzy sets, *Journal of Mathematical Analysis and Applications* **18**(1) (1967) 145–174.
3. B. Šešelja, Lattice of partially ordered fuzzy subalgebras, *Fuzzy Sets and Systems* **81**(2) (1996) 265–269.
4. R. Dedekind, *Supplement XI of Dirichlet's Vorlesungen*, 4th edn., ∮170, Collected Works, Vol. III (1894), p. 71.

5. E. Lasker, Zur theorie der moduln und ideale, *Mathematische Annalen* **60**(1) (1905) 20–116.
6. F. S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge Tracts in Mathematics and Mathematical Physics, Vol. 19 (1916).
7. M. Ward, Some arithmetical applications of residuation, *American Journal of Mathematics* **59**(4) (1937) 921–926.
8. M. Ward and R. P. Dilworth, Residuated lattices, *Transactions of the American Mathematical Society* **45** (1939) 335–354.
9. V. Novák, On the syntactico-semantical completeness of first-order fuzzy logic I, *Kybernetika* **26**(1) (1990) 47–66.
10. V. Novák, On the syntactico-semantical completeness of first-order fuzzy logic II, *Kybernetika* **26**(1) (1990) 134–154.
11. K. Blount and C. Tsinakis, The structure of residuated lattices, *International Journal of Algebra and Computation* **13**(4) (2003) 437–461.
12. P. Jipsen and C. Tsinakis, A survey of residuated lattices, *Ordered algebraic structures* ed. J. Martinez (Kluwer academic publisher, Dordrecht, 2002), pp. 19–56.
13. A. Kadji, C. Lele and M. Tonga, Some classes of pseudo-residuated lattices, *Afrika Matematika* **27**(7–8) (2016) 1147–1178.
14. M. Bianchi, A. Gillio and L. Verardi, Monounary simple algebras, *Contemporary Mathematics* **402** (2006) 119–132.
15. K. Denecke and S. L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science* (Chapman and Hall/CRC, 2002).
16. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra* (Springer-Verlag, 1981).
17. D. Buşneag, D. Piciu and A. Jeflea, Archimedean residuated lattices, *Annals of the Alexandru Ioan Cuza University - Mathematics* **56**(1) (2010) 227–252.
18. C. C. Chang, Algebraic analysis of many valued logics, *Transactions of the American Mathematical Society* **88**(2) (1958) 467–490.
19. C. C. Chang, A new proof of the completeness of the Łukasiewicz axioms, *Transactions of the American Mathematical Society* **93**(1) (1959) 74–80.
20. R. Cignoli, F. Esteva, L. Godo and A. Torrens, Basic fuzzy logic is the logic of continuous t-norms and their residua, *Soft Computing* **4**(2) (2000) 106–112.
21. R. Cignoli, I. M. L. Ottaviano and D. Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Trends in Logic, Vol. 7 (Kluwer Academic Publishers, Dordrecht, 2000).
22. R. P. Dilworth, Abstract residuation over lattices, *Bulletin of the American Mathematical Society* **44**(4) (1937) 262–268.
23. R. P. Dilworth, Non-commutative residuated lattices, *Transactions of the American Mathematical Society* **46**(3) (1939) 426–444.
24. F. Esteva and L. Godo, Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems* **124**(3) (2001) 271–288.
25. P. Hájek, *Metamathematics of Fuzzy Logic*, Trends in Logic, Vol. 4 (Kluwer Academic Publishers, Dordrecht, Boston, and London, 1998).
26. S. Jenei and F. Montagna, A proof of standard completeness for Esteva and Godo's logic *MTL*, *Studia Logica* **70**(2) (2002) 183–192.
27. D. Piciu, ed. *Algebras of Fuzzy Logic* (Universitaria, Craiova, 2007).
28. U. M. Swamy and K. L. N. Swamy, Fuzzy prime ideals of rings, *Journal of Mathematical Analysis and Applications* **134**(1) (1988) 94–103.
29. M. Ward, Residuation in structures over which a multiplication is defined, *Duke Mathematical Journal* **4**(3) (1937) 627–636.

Soft Computing (2020) 24:895–899
https://doi.org/10.1007/s00500-019-04594-z

FOUNDATIONS

A note on the algebraicity of L-fuzzy subalgebras in universal algebra

S. V. Tchoffo Foka[1] · Marcel Tonga[1]

Published online: 11 December 2019

Abstract
Given a universal algebra $\mathcal{A} := (A;\ F^A)$ of type $\mathcal{F}$, it is well known that the lattice $\mathbb{S}ub(\mathcal{A})$ of subuniverses of $\mathcal{A}$ is algebraic and its compact elements are exactly finitely generated subuniverses of $\mathcal{A}$. In this paper, under a distributive algebraic lattice $\mathcal{L} := (L;\ \wedge,\ \vee;\ 0,\ 1)$, we characterize the compact elements of the lattice $\mathbb{F}s(\mathcal{A}, L)$ of L-fuzzy subalgebras of $\mathcal{A}$, which is an extension of $\mathbb{S}ub(\mathcal{A})$ and show that the latter is algebraic.

Keywords Universal algebra · Algebraic lattice · Compact element · L-fuzzy subalgebra

1 Introduction

In 1965, Zadeh (1965) introduced the concept of fuzzy subset, as a function from a nonempty set to the unit interval $[0,\ 1]$ of real numbers, to formalize the concept of set membership under uncertainty. In 1967, Goguen (1967) generalized it to L-fuzzy subset, replacing the unit interval $[0,\ 1]$ of real numbers by the underlying set of an appropriate structure of truth values. In 1988, Swamy and Swamy (1988) used the Goguen's concept to introduce the concept of L-fuzzy ideal of a ring, where L is the underlying set of a complete Brouwerian lattice.

In 1933, the notion of universal algebra (sometimes called algebra for short) was introduced by G. Birkhoff [see Birkhoff (1933, 1935, 1944)], to extract as much as possible the common elements of particular algebraic structures. In 1982, Manes (1982) mentioned the idea of fuzzification of universal algebra, and Murali (1991) in 1991 used it to define a fuzzy subalgebra of a universal algebra $\mathcal{A}$ as a function, from the underlying set A of $\mathcal{A}$ to the closed unit interval $[0,\ 1]$ of real numbers, which is $\wedge$-compatible with the fundamental operations of $\mathcal{A}$. Further, he defined closure systems in fuzzy sets and showed that the set of fuzzy subalgebras form an algebraic closure system. In 1996, Seselja (1996) generalized the Murali's concept to L-fuzzy subalgebras, where L is the underlying set of a partially ordered set, by considering compatibility rather on levels sets. He also characterized classes of algebras for which the partially ordered set of L-fuzzy subalgebras is a lattice and pointed out the fact that its definition coincides with that of V. Murali when $\mathcal{L}$ is a bounded lattice.

In this work, we consider the notion of L-fuzzy subalgebra of a universal algebra, where L is the underlying set of a distributive algebraic lattice. Given a universal algebra $\mathcal{A} := (A;\ F^A)$ of type $\mathcal{F}$, we characterize the compact elements of the lattice $\mathbb{F}s(\mathcal{A}, L)$ of L-fuzzy subalgebras of $\mathcal{A}$ and show that the latter is algebraic. As Example 2.4 shows, this result can also be applied to L-fuzzy ideals of a ring (or a lattice) and fuzzy normal subgroups of a group. Its importance lies in the fact that any algebraic lattice is isomorphic to the lattice of the subuniverses of an algebra [see Theorem 3.5 in Burris and Sankappanavar (1981)].

2 Preliminaries

The notion of algebraic lattice was introduced by Birkhoff (1973) to describe the lattice of subuniverses of an algebra.

Definition 2.1 Let $\mathcal{L} = (L;\ \wedge,\ \vee)$ be a lattice.

1. An element c in $\mathcal{L}$ is compact if: whenever $\bigvee B$ exists and $c \leq \bigvee B$ for a subset B of L, we have $c \leq \bigvee B_0$ for some finite subset B_0 of B.

Communicated by A. Di Nola.

✉ S. V. Tchoffo Foka
tchoffofoka88@yahoo.fr

Marcel Tonga
tongamarcel@yahoo.fr

[1] Department of Mathematics, University of Yaounde 1, P.O.Box 812, Yaoundé, Cameroon

2. $\mathcal{L}$ is algebraic if it is complete and compactly generated (that is, each of its elements is a supremum of compact elements of $\mathcal{L}$).

It is well known that any finite lattice is algebraic and the closed unit interval of real numbers is not algebraic. Furthermore, a distributive algebraic lattice $\mathcal{L}$ is a complete Brouwerian lattice; that is, $a \wedge (\bigvee X) = \bigvee_{x\in X}(a \wedge x)$ for all $a \in L$ and $X \subseteq L$ [see Burris and Sankappanavar (1981)].

Definition 2.2 1. A type (or language) of algebras is a pair $\mathcal{F} := \langle F;\ \sigma\rangle$, where F is a set of function symbols and σ a map from F to the set $\mathbb{N}$ of nonnegative integers.
2. An algebra of type $\mathcal{F}$ is a pair $\mathcal{A} := (A;\ F^A)$; where A is a nonempty set (called universe of $\mathcal{A}$), $F^A := \{f^A :\ f \in F\}$ and each $f^A : A^{\sigma(f)} \to A$ is an $\sigma(f)$-ary operation on A, called a fundamental operation of $\mathcal{A}$.

Note that $F = \bigcup_{n\in\mathbb{N}} F_n$, where each F_n is the set of n-ary function symbols in F.

Definition 2.3 A subset B of A is called a subuniverse of $\mathcal{A}$ if $f^A(a_1, \ldots, a_n) \in B$ for all n-ary f in F and every $a_1, \ldots, a_n \in B$.

Since a nullary operation can be seen as a constant operation of arity greater than or equal to 1, the empty set is a subuniverse of $\mathcal{A}$ if and only if $\mathcal{A}$ does not contain a nullary operation.

Example 2.4 a. The ideals of a lattice $\mathcal{L} = (L;\ \sqcap,\ \sqcup)$ are just subuniverses of the algebra $\overline{\mathcal{L}} = (L;\ \sqcup;\ (m_a)_{a\in L})$, where $m_a(x) = a \sqcap x$ for all $a, x \in L$.
b. The normal subgroups of a group $\mathcal{G} = (G;\ \cdot,\ ^{-1},\ e)$ are just subuniverses of the algebra $\overline{\mathcal{G}} = (G;\ \cdot;\ ^{-1},\ (m_a)_{a\in G};\ e)$, where $m_a(x) = axa^{-1}$ for all $a, x \in G$.
c. The ideals of a ring $\mathcal{R} = (R;\ +,\ \cdot;\ -;\ 0)$ are just subuniverses of the algebra $\overline{\mathcal{R}} = (R;\ +;\ -,\ (l_a)_{a\in R},\ (r_a)_{a\in R};\ 0)$, where $l_a(x) = ax$ and $r_a(x) = xa$ for all $a, x \in R$.

The subuniverse of $\mathcal{A}$ generated by a subset X of A, denoted by $Sg_{\mathcal{A}}(X)$ or simply $Sg(X)$, is the smallest subuniverse of $\mathcal{A}$ containing X. The set $Sub(\mathcal{A})$ of subuniverses of $\mathcal{A}$ forms an algebraic lattice $\mathbb{S}ub(\mathcal{A}) := \big(Sub(\mathcal{A});\ \cap,\ \sqcup;\ Sg(\emptyset),\ A\big)$; where $\cap$ is the intersection of sets and $\sqcup$ is given by $B \sqcup C = Sg(B \cup C)$ for all $B, C \in Sub(\mathcal{A})$. Moreover, compact elements of $\mathbb{S}ub(\mathcal{A})$ are only of the form $Sg(X)$; where X is a finite subset of A.

Throughout the work, unless otherwise specified, $\mathcal{A} := (A;\ F^A)$ is a universal algebra of type $\mathcal{F}$ and $\mathcal{L} := (L;\ \wedge,\ \vee;\ 0,\ 1)$ is a complete Brouwerian lattice.

Definition 2.5 A fuzzy subset of A under $\mathcal{L}$, or an L-fuzzy subset of A, is a map $\mu : A \to L$.

This notion was introduced by Goguen (1967) in 1967 as a generalization of the notion of fuzzy subset defined by Zadeh (1965) in 1965 as a function from a set to $[0,\ 1]$.

For any $a \in A$ and $r \in L$, the L-fuzzy subset a_r of A, given by $a_r(x) = \begin{cases} r & \text{if } x = a, \\ 0 & \text{if not.}\end{cases}$ for all $x \in A$, is called an L-fuzzy point of A. The characteristic function χ_B of a subset B of A is an L-fuzzy subset of A. For any L-fuzzy subset μ of A and $r \in L$, $U(\mu, r) := \{a \in A :\ \mu(a) \geq r\}$ is called the r-level set (or r-cut) of μ. The order relation $\leq$ is defined on the set $Fu(A, L)$ of L-fuzzy subsets of A as follows: for any $\mu, \nu \in Fu(A, L)$, $\mu \leq \nu$ (ν contains μ) if and only if $\mu(x) \leq \nu(x)$ for all $x \in A$. $Fu(A, L)$ forms a complete lattice $\mathbb{F}u(A, L) := \big(Fu(A, L);\ \wedge,\ \vee;\ \underline{0},\ \underline{1}\big)$; where the binary operations $\wedge$, $\vee$ are defined componentwise, and $\underline{0} = \chi_\emptyset$ and $\underline{1} = \chi_A$ are the constant L-fuzzy subsets of A with values 0 and 1, respectively.

Definition 2.6 An L-fuzzy subset μ of A is called an L-fuzzy subalgebra of $\mathcal{A}$ if $\mu(f^A) = 1$ for all $f \in F_0$, and $\mu\big(f^A(a_1, \ldots, a_n)\big) \geq \bigwedge_{1\leq i\leq n}\mu(a_i)$ for all $f \in F_n$ and $a_1, \ldots, a_n \in A$.

Let μ be an L-fuzzy subset of A. If μ is an L-fuzzy subalgebra of $\mathcal{A}$, then all its cuts are empty or subuniverses of $\mathcal{A}$. The converse is true when $U(\mu, 1) \neq \emptyset$. A nonempty subset B of A is a subuniverse of $\mathcal{A}$ if and only if χ_B is an L-fuzzy subalgebra of $\mathcal{A}$. For any L-fuzzy subalgebra μ of $\mathcal{A}$ and $a \in Sg(\emptyset)$, we have $a = t^A(f^A, \ldots, f^A)$ for some term $t(x_1, \ldots, x_n)$ and $f \in F_0$; thus, $\mu(a) \geq \bigwedge_{1\leq i\leq n}\mu(f^A) = \mu(f^A) = 1$ and, $\mu(a) = 1$.

For any L-fuzzy subset μ of A, the L-fuzzy subalgebra of $\mathcal{A}$ generated by μ, denoted by $Fsg(\mu)$, is the smallest L-fuzzy subalgebra of $\mathcal{A}$ containing μ. Moreover, for any $\mu, \nu \in Fu(A, L)$, we have $\mu \leq Fsg(\mu)$, $Fsg\big(Fsg(\mu)\big) = Fsg(\mu)$ and $Fsg(\mu) \leq Fsg(\nu)$ whenever $\mu \leq \nu$. The set $Fs(\mathcal{A}, L)$ of L-fuzzy subalgebras of $\mathcal{A}$ forms a complete lattice $\mathbb{F}s(\mathcal{A}, L) := \big(Fs(\mathcal{A}, L);\ \wedge,\ \sqcup;\ \chi_{Sg(\emptyset)},\ \underline{1}\big)$; where the binary operation $\sqcup$ is given by $\mu \sqcup \nu = Fsg(\mu \vee \nu)$ for all $\mu, \nu \in Fs(\mathcal{A}, L)$. One can easily verify that the lattice of subuniverses of $\mathcal{A}$ can be embedded into the lattice of L-fuzzy subalgebras of $\mathcal{A}$.

3 Results

In this section, first we will set up a mimetic construction of the L-fuzzy subalgebra of $\mathcal{A}$ generated by an L-fuzzy subset of A, then characterize the compact elements of $\mathbb{F}s(\mathcal{A}, L)$ and show that the latter is algebraic.

Lemma 3.1 *Let μ be an L-fuzzy subset of A and $\mu_\star$ the L-fuzzy subset of A defined by:*

$$\mu_\star(x) = \bigvee\{r \in L : x \in Sg\big(U(\mu, r)\big)\} \textit{ for all } x \in A.$$

Then, $\mu_\star$ is the smallest L-fuzzy subalgebra of $\mathcal{A}$ containing μ; i.e., $Fsg(\mu) = \mu_\star$.

Proof For any $a \in A$, we have

$$a \in U\big(\mu, \mu(a)\big) \subseteq Sg\big(U\big(\mu, \mu(a)\big)\big) \text{ and, } \mu(a) \leq \mu_\star(a).$$

Thus, $\mu \leq \mu_\star$. We next show that $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.

- For any $f \in F_0$, we have

  $$\mu_\star(f^A) = \bigvee L = 1.$$

- Let $f \in F_n$ and $a_1, \ldots, a_n \in A$. For any $r_1, \ldots, r_n \in L$ such that $a_i \in Sg\big(U(\mu, r_i)\big)$ for all $1 \leq i \leq n$, we have

  $$a_1, \ldots, a_n \in Sg\left(U(\mu, \bigwedge_{1\leq i\leq n} r_i)\right) \text{ and },$$

  $$f^A(a_1, \ldots, a_n) \in Sg\left(U(\mu, \bigwedge_{1\leq i\leq n} r_i)\right);$$

  thus, $\mu_\star\big(f^A(a_1, \ldots, a_n)\big) \geq \bigwedge_{1\leq i\leq n} r_i$. So, $\mu_\star\big(f^A(a_1, \ldots, a_n)\big) \geq \bigwedge_{1\leq i\leq n} \mu_\star(a_i)$.
  Hence, $\mu_\star$ is an L-fuzzy subalgebra of $\mathcal{A}$.
  Now, let ν be an L-fuzzy subalgebra of $\mathcal{A}$ containing μ. Let $u \in A\backslash Sg(\emptyset)$. For any $r \in L$ such that $u \in Sg\big(U(\nu, r)\big)$, there are a term $t(x_1, \ldots, x_n)$ of type $\mathcal{F}$ and $u_1, \ldots, u_n \in U(\nu, r)$ such that $u = t^A(u_1, \ldots, u_n)$; thus,

  $$r \leq \bigwedge_{1\leq i\leq n} \nu(u_i) \leq \nu\big(t^A(u_1, \ldots, u_n)\big) = \nu(u).$$

  So,

  $$\mu_\star(u) \leq \bigvee\{r \in L : u \in Sg\big(U(\nu, r)\big)\} \leq \nu(u).$$

  Consequently, $\mu_\star \leq \nu$. Hence, $\mu_\star$ is the smallest L-fuzzy subalgebra of $\mathcal{A}$ containing μ. □

Proposition 3.2 *Let $a \in A\backslash Sg(\emptyset)$ and $c \in L$. Then, $Fsg\big(a_c\big)$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$ if and only if c is a compact element of $\mathcal{L}$.*

Proof $(\Rightarrow)$ Assume that $Fsg\big(a_c\big)$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$. Let $\{r_i\}_{i\in I} \subseteq L$ such that $c \leq \bigvee_{i\in I} r_i$. Since

$$Fsg(a_c) \leq Fsg(a_{\bigvee_{i\in I} r_i}) = Fsg(\bigvee_{i\in I} a_{r_i}) = \bigsqcup_{i\in I} Fsg(a_{r_i}),$$

there are $\{i_1, \ldots, i_p\} \subseteq I$ such that

$$Fsg(a_c) \leq \bigsqcup_{1\leq j\leq p} Fsg(a_{r_{i_j}}) = Fsg(a_{\bigvee_{1\leq j\leq p} r_{i_j}});$$

thus, $c = Fsg(a_c)(a) \leq Fsg(a_{\bigvee_{1\leq j\leq p} r_{i_j}})(a) = \bigvee_{1\leq j\leq p} r_{i_j}$.
Hence, c is a compact element of $\mathcal{L}$.
$(\Leftarrow)$ Assume that c is a compact element of $\mathcal{L}$. Let $\{\mu_\lambda\}_{\lambda\in\Lambda} \subseteq Fs(\mathcal{A}, L)$ such that $Fsg(a_c) \leq \bigsqcup_{\lambda\in\Lambda} \mu_\lambda$. Since $c \leq \big(\bigvee_{\lambda\in\Lambda} \mu_\lambda\big)_\star(a)$ and c is a compact element of $\mathcal{L}$, there are $r_1, \ldots, r_n \in L$ such that

$$a \in \bigcap_{1\leq i\leq n} Sg\left(U(\bigvee_{\lambda\in\Lambda} \mu_\lambda, r_i)\right) \text{ and } c \leq \bigvee_{1\leq i\leq n} r_i.$$

For any $1 \leq i \leq n$, there are a term $t_i(x_{i1}, \ldots, x_{ik_i})$ of type $\mathcal{F}$ and $u_{i1}, \ldots, u_{ik_i} \in A$ such that

$$a = t_i^A(u_{i1}, \ldots, u_{ik_i}) \text{ and}$$
$$r_i \leq \bigvee_{\lambda\in\Lambda} \mu_\lambda(u_{ij}) \text{ for all } 1 \leq j \leq k_i;$$

thus,

$$r_i \leq \bigwedge_{1\leq j\leq k_i} \left(\bigvee_{\lambda\in\Lambda} \mu_\lambda(u_{ij})\right) = \bigvee_{(\lambda_{i1},\ldots,\lambda_{ik_i})\in\Lambda^{k_i}} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij}).$$

Hence, $c \leq \bigvee_{1\leq i\leq n} \bigvee_{(\lambda_{i1},\ldots,\lambda_{ik_i})\in\Lambda^{k_i}} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij})$; *i.e.*,

$$c \leq \bigvee_{\left((\lambda_{i1},\ldots,\lambda_{ik_i})\right)_{1\leq i\leq n} \in \prod_{1\leq i\leq n} \Lambda^{k_i}} \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij}).$$

Since c is a compact element of $\mathcal{L}$, there is a finite subset Ω of Λ such that

$$c \leq \bigvee_{\left((\lambda_{i1},\ldots,\lambda_{ik_i})\right)_{1\leq i\leq n} \in \prod_{1\leq i\leq n} \Omega^{k_i}} \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij});$$

thus,

$$\begin{aligned} c &\leq \bigvee_{1\leq i\leq n} \bigvee_{(\lambda_{i1},\ldots,\lambda_{ik_i})\in\Omega^{k_i}} \bigwedge_{1\leq j\leq k_i} \mu_{\lambda_{ij}}(u_{ij}) \\ &= \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)(u_{ij}) \\ &\leq \bigvee_{1\leq i\leq n} \bigwedge_{1\leq j\leq k_i} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star(u_{ij}) \\ &\leq \bigvee_{1\leq i\leq n} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star\big(t^A(u_{i1}, \ldots, u_{ik_i})\big) \end{aligned}$$

$$= \bigvee_{1\leq i\leq n} \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star(a)$$
$$= \big(\bigvee_{\lambda\in\Omega} \mu_\lambda\big)_\star(a)$$
$$= \big(\bigsqcup_{\lambda\in\Omega} \mu_\lambda\big)(a).$$

- For any $u \in Sg(a)\backslash Sg(\emptyset)$, we have

$$Fsg(a_c)(u) = c \leq \big(\bigsqcup_{\lambda\in\Omega}\mu_\lambda\big)(a) \leq \big(\bigsqcup_{\lambda\in\Omega}\mu_\lambda\big)(u).$$

- For any $u \notin Sg(a)$, we have

$$Fsg(a_c)(u) = 0 \leq \big(\bigsqcup_{\lambda\in\Omega}\mu_\lambda\big)(u).$$

It follows that $Fsg(a_c)(u) \leq \big(\bigsqcup_{\lambda\in\Omega}\mu_\lambda\big)(u)$ for all $u \in A$; *i.e.*, $Fsg(a_c) \leq \bigsqcup_{\lambda\in\Omega}\mu_\lambda$. Hence, $Fsg(a_c)$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$. □

For any $a \in Sg(\emptyset)$ and $c \in L$, $Fsg(a_c) = \chi_{Sg(\emptyset)}$ is a compact element of $\mathbb{F}s(\mathcal{A}, L)$. For any compact element c of $\mathcal{L}$ and $a \in A$, $Fsg(a_c)$ will be called a compact principal L-fuzzy subalgebra of $\mathcal{A}$.

Theorem 3.3 *Suppose that $\mathcal{L}$ is a distributive algebraic lattice.*

(1) *Compact elements of $\mathbb{F}s(\mathcal{A}, L)$ are only finite suprema of compact principal L-fuzzy subalgebras of $\mathcal{A}$.*
(2) *$\mathbb{F}s(\mathcal{A}, L)$ is an algebraic lattice.*

Proof (1) A finite supremum of compact principal L-fuzzy subalgebras of $\mathcal{A}$ is a finite supremum of compact elements of $\mathbb{F}s(\mathcal{A}, L)$ by Proposition 3.2; so, it is a compact element of $\mathbb{F}s(\mathcal{A}, L)$.
Conversely, let μ be a compact element of $\mathbb{F}s(\mathcal{A}, L)$. Since $\mu = \bigsqcup_{a\in A} Fsg\big(a_{\mu(a)}\big)$, there are $a_1, \ldots, a_n \in A$ such that $\mu = \bigsqcup_{1\leq i\leq n} Fsg\big((a_i)_{\mu(a_i)}\big)$. Since $\mathcal{L}$ is algebraic, for any $1 \leq i \leq n$, there is a family $\{c_j\}_{j\in I_i}$ of compact elements of $\mathcal{L}$ such that $\mu(a_i) = \bigvee_{j\in I_i} c_j$. It follows that

$$\mu = \bigsqcup_{1\leq i\leq n} Fsg\big((a_i)_{\bigvee_{j\in I_i} c_j}\big)$$
$$= \bigsqcup_{1\leq i\leq n} Fsg\big(\bigvee_{j\in I_i}(a_i)_{c_j}\big)$$
$$= \bigsqcup_{1\leq i\leq n}\bigsqcup_{j\in I_i} Fsg\big((a_i)_{c_j}\big)$$
$$= \bigsqcup_{(j_1,\ldots,j_n)\in\prod_{1\leq i\leq n} I_i}\ \bigsqcup_{1\leq i\leq n} Fsg\big((a_i)_{c_{j_i}}\big).$$

Since μ is compact, there is a family $\{K_i\}_{1\leq i\leq n}$ of finite sets such that $K_i \subseteq I_i$ for all $1 \leq i \leq n$ and

$$\mu = \bigsqcup_{(j_1,\ldots,j_n)\in\prod_{1\leq i\leq n} K_i}\ \bigsqcup_{1\leq i\leq n} Fsg\big((a_i)_{c_{j_i}}\big).$$

So, by Proposition 3.2, μ is a finite supremum of compact principal L-fuzzy subalgebras of $\mathcal{A}$.
(2) Since $\mathbb{F}s(\mathcal{A}, L)$ is complete, it suffices to show that it is compactly generated. So, let $\mu \in Fs(\mathcal{A}, L)$. Since $\mathcal{L}$ is algebraic, for any $a \in A$, there is a family $\{c_{i,a}\}_{i\in I_a}$ of compact elements of $\mathcal{L}$ such that $\mu(a) = \bigvee_{i\in I_a} c_{i,a}$. Hence,

$$\mu = \bigsqcup_{a\in A} Fsg\big(a_{\bigvee_{i\in I_a} c_{i,a}}\big) = \bigsqcup_{a\in A}\bigsqcup_{i\in I_a} Fsg\big(a_{c_{i,a}}\big),$$

and for each $a \in A$ and $i \in I_a$, $Fsg\big(a_{c_{i,a}}\big)$ is compact by Proposition 3.2. Therefore, $\mathbb{F}s(\mathcal{A}, L)$ is algebraic. □

4 Conclusion

In the present paper, we investigated the algebraicity of the lattice of fuzzy subalgebras of an algebra under a distributive algebraic lattice. The distributivity of the algebraic lattice having been used several times in our demonstrations, it would be interesting to check if this data is essential. This problem remains open since we have not yet found the solution.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

References

Birkhoff G (1973) Lattice theory, vol 25, 3rd edn. AMS Colloquium Publications, Providence
Birkhoff G (1933) On the combination of subalgebras. Proc Camb Philos Soc 29:441–464
Birkhoff G (1935) On the structure of abstract algebras. Proc Camb Philos Soc 31:433–454
Birkhoff G (1944) Subdirect unions in universal algebra. Bull Am Math Soc 50:764–768
Burris S, Sankappanavar HP (1981) A course in universal algebra. Springer, Berlin
Cohn PM (1981) Universal algebra. Reidel, Dordrecht
Gratzer G (1968) Universal algebra. Van Nostrand, New York
Goguen JA (1967) L-fuzzy sets. J Math Anal Appl 18(1):145–174
Kuraoka T, Suzuki NY (2002) Lattice of fuzzy subalgebras in universal algebra. Algebra Univers 47:223–237

Manes EG (1982) A class of fuzzy theories. J Math Anal Appl 85(2):409–451

Murali V (1991) Lattice of fuzzy subalgebras and closure systems in I^X. Fuzzy Sets Syst 41:101–111

Rao TR, Rao CP, Solomon D, Abeje D (2013) Fuzzy ideals and filters of lattices. Asian J Current Eng Math 2:297–300

Senapati T, Jana C, Bhowmik M, Pal M (2015) L-fuzzy G-subalgebras of G-algebras. J Egypt Math Soc 23(2):219–223

Seselja B (1996) Lattice of partially ordered fuzzy subalgebras. Fuzzy Sets Syst 81:265–269

Swamy UM, Swamy KLN (1988) Fuzzy prime ideals of rings. J Math Anal Appl 134:94–103

Tchoffo Foka SV, Tonga M (2019) A residuated lattice of L-fuzzy subalgebras of a mono-unary algebra. New Math Natural Comput 15(3):539–551

Whitehead AN (1898) A treatise on universal algebra with applications, vol 1. Cambridge University Press, Cambridge

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353

Soft Computing (2020) 24:8717–8724
https://doi.org/10.1007/s00500-020-04894-9

FOUNDATIONS

Residuated lattice of *L*-fuzzy ideals of a ring

S. V. Tchoffo Foka[1] · **Marcel Tonga**[1]

Published online: 7 April 2020

Abstract
In 1988, given a complete Brouwerian lattice $\mathbb{L} := (L;\ \wedge,\ \vee;\ 0,\ 1)$ and a ring $\mathcal{A} := (A;\ +,\ \cdot;\ -;\ 0)$ with unity 1, Swamy and Swamy (J Math Anal Appl 134:94–103, 1988) built a lattice structure, on the set of L-fuzzy ideals of $\mathcal{A}$, and investigated some of its arithmetic properties. Since the residuation theory is richer than the lattice theory [see, Ciungu (Non-commutative multiple-valued logic algebras, Springer monographs in mathematics, Springer, Berlin, 2014), Galatos et al. (An algebraic glimpse at substructural logics, volume 151 of studies in logic and the foundations of mathematics, Elsevier, Amsterdam, 2007), Jipsen and Tsinakis (in: Martinez (ed) Ordered algebraic structures, Kluwer Academic Publisher, Dordrecht, 2002), Piciu (Algebras of fuzzy logic, Editura Universitaria Craiova, Craiova, 2007)], in this paper, we consider the notion of fuzzy ideals rather under a complete Brouwerian residuated lattice $\mathcal{L} := (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$. A residuated lattice $\mathcal{F}id(\mathcal{A}, L) := \big(Fid(\mathcal{A}, L);\ \wedge,\ +,\ \otimes,\ \hookrightarrow,\ \nrightarrow;\ \chi_0,\ \underline{1}\big)$ is built on the set $Fid(\mathcal{A}, L)$ of L-fuzzy ideals of $\mathcal{A}$ and it is shown that the latter is both an extension of $\mathcal{L}$ and the residuated lattice $\mathcal{I}d(\mathcal{A}) := \big(Id(\mathcal{A});\ \cap,\ +,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ \{0\},\ A\big)$ on the set $Id(\mathcal{A})$ of ideals of $\mathcal{A}$.

Keywords Ring · Ideal · L-fuzzy ideal · Residuated lattice

1 Introduction

Since the introduction of the idea of residuation by Dedekind (1894), several researchers have approached it in a general way. Ward and Dilworth (1939) introduced the notion of residuated lattices, as the lattices on which a multiplication or residuation operation is defined. During the same year, Dilworth (1939) introduced the notion of non-commutative residuated lattices and investigated some of its properties among which decompositions into primary and semi-primary elements. Since then, there has been substantial research regarding some specific classes of residuated lattices as RL-monoids, MTL-algebras, BL-algebras, MV-algebras, Boolean algebras,... (see, Ciungu 2014; Galatos et al. 2007; Jipsen and Tsinakis 2002; Piciu 2007).

Zadeh (1965) introduced the concept of fuzzy subset of a set, as a function from a nonempty set to the closed unit interval $[0,\ 1]$ of real numbers, which led to a revision of mathematics, to formalize the concept of set membership under uncertainty. Because of the inability of the latter to interpret certain situations of our daily life, Goguen (1967) introduced in 1967 the concept of L-fuzzy subset of a set, replacing the unit interval $[0,\ 1]$ of real numbers by the underlying set L of some structures of truth values among which complete Brouwerian lattices and residuated lattices.

Swamy and Swamy (1988) used the Goguen's concept to introduce the concept of L-fuzzy ideals of a ring, where L is the underlying set of a complete Brouwerian lattice, and describe maximal and prime elements of their lattice. Since then, the lattice of L-fuzzy ideals of a ring has been the subject of several other studies (see, Martinez 1999; Yue 1988).

In this work, in order to enrich the structures of truth values, we consider the notion of L-fuzzy ideal of a ring, where L is the underlying set of a complete Brouwerian residuated lattice $\mathcal{L}$. Given a ring $\mathcal{A}$ with unity 1, we define a residuated lattice structure $\mathcal{F}id(\mathcal{A}, L)$ on the set of L-fuzzy ideals of $\mathcal{A}$ which extends both $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$. The paper is organized as follows.

Communicated by A. Di Nola.

✉ S. V. Tchoffo Foka
tchoffofoka88@yahoo.fr

Marcel Tonga
tongamarcel@yahoo.fr

[1] Department of Mathematics, University of Yaounde 1, P.O.Box 812, Yaoundé, Cameroon

In Sect. 2, we recall some known facts about residuated lattices and L-fuzzy ideals of rings. Section 3 outlines the construction of the residuated lattice $\mathcal{F}id(\mathcal{A}, L)$. In Sect. 4, we embed $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ into $\mathcal{F}id(\mathcal{A}, L)$.

2 Preliminaries

2.1 Residuated lattices

We collect here some definitions and results on residuated lattices, most of them being well known (See, Ciungu 2014; Galatos et al. 2007; Jipsen and Tsinakis 2002; Piciu 2007).

Definition 2.1 An algebra $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ is called a residuated lattice if it satisfies the following conditions:

(RL1) $(L;\ \wedge,\ \vee;\ 0,\ 1)$ is a bounded lattice (with a partial order $\leq$);
(RL2) $(L;\ \ominus,\ 1)$ is a monoid;
(RL3) for any $x, y, z \in L$, $x \ominus y \leq z$ iff $x \leq y \twoheadrightarrow z$ iff $y \leq x \multimap z$.

An algebra $(L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ is a residuated lattice if and only if $(L;\ \wedge,\ \vee;\ 0,\ 1)$ is a bounded lattice, $(L;\ \ominus,\ 1)$ is a monoid, $\ominus$ is order-preserving in each argument and the inequality $x \ominus y \leq z$ has a largest solution for x (namely $y \twoheadrightarrow z$) and for y (namely $x \multimap z$). For any $x \in L$ and a non negative integer n, x^n is defined inductively by $x^0 = 1$ and $x^{n+1} = x^n \ominus x$.

Example 2.2 (a) The Gödel structure is the residuated lattice $\mathcal{L} = (L;\ \wedge,\ \vee,\ \wedge,\ \twoheadrightarrow,\ \twoheadrightarrow;\ 0,\ 1)$ given by $L = [0,\ 1]$, $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$ and

$$x \twoheadrightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases} \quad \text{for all } x, y \in L.$$

(b) The product (or Gaines) structure is the residuated lattice $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \twoheadrightarrow;\ 0,\ 1)$ given by $L = [0,\ 1]$, $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, $x \ominus y = xy$ (the usual multiplication of real numbers) and

$$x \twoheadrightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y/x & \text{otherwise.} \end{cases} \quad \text{for all } x, y \in L.$$

(c) The Łukasiewicz structure of order $p \in \mathbb{N}^\star$ is the residuated lattice $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \twoheadrightarrow;\ 0,\ 1)$ given by $L = [0,\ 1]$, $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$,

$$x \ominus y = \sqrt[p]{\max(0, x^p + y^p - 1)} \text{ and}$$
$$x \twoheadrightarrow y = \min\left(1,\ \sqrt[p]{1 - x^p + y^p}\right) \text{ for all } x, y \in L.$$

If $p = 1$, we obtain the Łukasiewicz structure.

Example 2.3 (See, Kadji et al. 2016, Example 8) Let $L = \{0,\ a,\ b,\ c,\ 1\}$ be a lattice such that $0 < a < b < c < 1$. Define the binary operations $\ominus$, $\twoheadrightarrow$ and $\multimap$ by the three tables below:

$\ominus$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\twoheadrightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\multimap$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

.

Then $\mathcal{L} = (L;\ \wedge,\ \vee,\ \ominus,\ \twoheadrightarrow,\ \multimap;\ 0,\ 1)$ is a residuated lattice.

Proposition 2.4 (Ciungu 2014; Galatos et al. 2007; Jipsen and Tsinakis 2002; Piciu 2007) *In a residuated lattice, the following hold (whenever $\bigwedge$ and $\bigvee$ exist) for any $z \in L$, $X, Y \subseteq L$ and $\dashrightarrow \in \{\twoheadrightarrow,\ \multimap\}$:*

(1) $(\bigvee X) \ominus (\bigvee Y) = \bigvee\limits_{x \in X,\ y \in Y} x \ominus y$.
(2) $z \dashrightarrow (\bigwedge X) = \bigwedge\limits_{x \in X} (z \dashrightarrow x)$ *and* $(\bigvee X) \dashrightarrow z = \bigwedge\limits_{x \in X} (x \dashrightarrow z)$.

Furthermore, the following (quasi-)identities and their mirror images (obtained by replacing $x \ominus y$ by $y \ominus x$ and interchanging $x \twoheadrightarrow y$ with $x \multimap y$) also hold:

(3) *If $x \leq y$, then $x \ominus z \leq y \ominus z$, $y \twoheadrightarrow z \leq x \twoheadrightarrow z$ and $z \twoheadrightarrow x \leq z \twoheadrightarrow y$.*
(4) $x \ominus y \leq x \wedge y$.
(5) $x \ominus 0 = 0 = 0 \ominus x$.
(6) $1 \twoheadrightarrow x = x$.
(7) *If $x \leq y$, then $x \twoheadrightarrow y = 1$.*

Proposition 2.5 (Ciungu 2014; Galatos et al. 2007; Jipsen and Tsinakis 2002; Piciu 2007) *The class of residuated lattices is a variety.*

A residuated lattice $\mathcal{L}$ is called complete if so is its lattice. A residuated lattice $\mathcal{L}$ is called Brouwerian or completely meet distributive if: for any $a \in L$ and $B \subseteq A$, $a \wedge \bigvee B = \bigvee\limits_{b \in B} (a \wedge b)$, whenever both $\bigvee$ exist.

Example 2.6 Gödel, Gaines, Łukasiewicz structures and the residuated lattice of Example 2.3 are complete Brouwerian residuated lattices.

Proposition 2.7 (See, Ciungu 2014; Galatos et al. 2007; Jipsen and Tsinakis 2002; Piciu 2007) *In a residuated lattice $\mathcal{L}$, for any $x \in L$,*

$\overline{x} := x \twoheadrightarrow 0$ *and* $\widetilde{x} := x \multimap 0$*(mirror image of* $\overline{x}$*)*

are called the negations of x. Furthermore, the following (quasi-)identities and their mirror images hold for any x, y in L:

(8) $\overline{0} = 1$.
(9) $x \leq y$ *implies* $\overline{y} \leq \overline{x}$.
(10) $\overline{x} \ominus x = 0$, $x \leq \widetilde{\overline{x}}$ *and* $\overline{\widetilde{\overline{x}}} = \overline{x}$.

2.2 *L*-fuzzy ideal of a ring

Throughout the work, $\mathcal{L} := (L;\ \wedge, \vee, \ominus, \twoheadrightarrow, \multimap;\ 0,\ 1)$ is a complete Brouwerian residuated lattice and $\mathcal{A} := (A;\ +,\ \cdot;\ -;\ 0)$ is a ring with unity 1. The binary operation $\cdot$ is denoted by juxtaposition.

Definition 2.8 A fuzzy subset of A under $\mathcal{L}$, or an L-fuzzy subset of A, is a map from A to L.

Recall that this notion was introduced by Goguen (1967) in 1967 as a generalization of the notion of fuzzy subset defined by Zadeh (1965) in 1965 as a function from a set to $[0,\ 1]$.

Example 2.9 (See, Tchoffo Foka and Tonga 2019) For any $B \subseteq A$, $a \in A$ and $r, s \in L$, the following functions from A to L are fuzzy subsets of A:

$$B_r^s(x) := \begin{cases} s & \text{if } x \in B, \\ r & \text{if not.} \end{cases} \quad \text{for all } x \in A,$$

$B_r := B_0^r$, $B^r := B_r^1$, $a_r^s := \{a\}_r^s$, $a_r := a_0^r$ (fuzzy point of A), $B_1 =: \chi_B := B^0$ (characteristic function of B), $\chi_a := \chi_{\{a\}}$ and $A_r =: \underline{r} := \emptyset^r$ (constant fuzzy subset of A).

Notation 2.10 (See, Tchoffo Foka and Tonga 2019) *For any L-fuzzy subset μ of A and $r \in L$,*

$U(\mu, r) := \{x \in A :\ \mu(x) \geq r\}$

is called the r-level set (or r-cut) of μ.

The order relation $\leq$ is defined on the set $Fu(A, L)$ of L-fuzzy subsets of A as follows: for any $\mu, \nu \in Fu(A, L)$, $\mu \leq \nu$ (ν contains μ) if and only if $\mu(x) \leq \nu(x)$ for all $x \in A$.

Remark 2.11 $Fu(A, L)$ forms a complete Brouwerian residuated lattice $\mathcal{F}u(A, L) := \big(Fu(A, L); \wedge, \vee, \ominus, \twoheadrightarrow, \multimap; \underline{0}, \underline{1}\big)$, where the binary operations $\wedge$, $\vee$, $\ominus, \twoheadrightarrow, \multimap$ are defined componentwise.

Definition 2.12 An L-fuzzy subset μ of A is an L-fuzzy ideal of $\mathcal{A}$ if and only if $\mu(0) = 1$ and for any $x, y \in A$, $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq \mu(x) \vee \mu(y)$.

Remark 2.13 • For any ideal I of $\mathcal{A}$ and $r, s \in L$ such that $r \leq s$, the L-fuzzy subset $(I_r^s)_* := I_r^s \vee \chi_0$ of A, given by

$$(I_r^s)_*(x) = \begin{cases} 1 & \text{if } x = 0, \\ s & \text{if } x \in I \setminus \{0\}, \\ r & \text{elsewhere.} \end{cases} \quad \text{for all } x \in A,$$

is an L-fuzzy ideal of $\mathcal{A}$.

• If μ is an L-fuzzy ideal of $\mathcal{A}$, then all its cuts are ideals of $\mathcal{A}$; the converse holds for any L-fuzzy subset μ of A such that $1 \in Im(\mu)$.

Proposition 2.14 (See, Swamy and Swamy 1988) *The set $Fid(\mathcal{A}, L)$ of L-fuzzy ideals of $\mathcal{A}$ forms a complete lattice $\mathbb{F}id(\mathcal{A}, L) := \big(Fid(\mathcal{A}, L);\ \wedge,\ +;\ \chi_0,\ \underline{1}\big)$, where for any $\mu, \nu \in Fid(\mathcal{A}, L)$ and $x \in A$, $(\mu + \nu)(x) = \bigvee\{\mu(a) \wedge \nu(b) :\ x = a + b\}$.*

3 Residuated lattice of *L*-fuzzy ideals of $\mathcal{A}$

Remark 3.1 The residuated lattice of ideals of $\mathcal{A}$ is given by

$$\mathcal{I}d(\mathcal{A}) := \big(Id(\mathcal{A});\ \cap,\ +,\ \odot,\ \rightarrow,\ \rightsquigarrow;\ \{0\},\ A\big);$$

where, for any $I, J \in Id(\mathcal{A})$,
$I + J = \{a + b :\ a \in I \text{ and } b \in J\}$,

$$I \odot J := IJ = \left\{\sum_{i=1}^{n} a_i b_i :\ a_1, \ldots, a_n \in I \text{ and } b_1, \ldots, b_n \in J\right\},$$

$I \rightarrow J = \{x \in A :\ xI \subseteq J\}$ and
$I \rightsquigarrow J = \{x \in A :\ Ix \subseteq J\}$.

Definition 3.2 For any L-fuzzy subsets μ and ν of A, the L-fuzzy subset $\mu \circ \nu$ of A is defined by:

$$(\mu \circ \nu)(x) = \bigvee\{\mu(a) \ominus \nu(b) :\ x = ab\} \text{ for all } x \in A.$$

Proposition 3.3 *Let $\mu, \nu \in Fid(\mathcal{A}, L)$. Then the L-fuzzy subset $\mu \otimes \nu$ of A, given by*

$$(\mu \otimes \nu)(x) = \bigvee\left\{\bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \nu(b_i) : x = \sum_{i=1}^{n} a_i b_i\right\} \text{ for all } x \in A,$$

is the smallest L-fuzzy ideal of $\mathcal{A}$ containing $\mu \circ \nu$; i.e., $Fidg(\mu \circ \nu) = \mu \otimes \nu$ (L-fuzzy ideal of $\mathcal{A}$ generated by $\mu \circ \nu$).

Proof It is clear that $\mu \otimes \nu$ contains $\mu \circ \nu$. Next we show that $\mu \otimes \nu$ is an L-fuzzy ideal of $\mathcal{A}$.

We have $(\mu \otimes \nu)(0) = 1$, since

$$(\mu \otimes \nu)(0) \geq \mu(0) \ominus \nu(0) = 1 \ominus 1 = 1.$$

Now, let $x, y \in A$. Set $X := \{(a_i, b_i)_{1 \leq i \leq m+n} : x = \sum_{i=1}^{m} a_i b_i$ and $-y = \sum_{i=m+1}^{m+n} a_i b_i\}$ and $Y := \{(u_j, v_j)_{1 \leq j \leq p}: x - y = \sum_{j=1}^{p} u_j v_j\}$. Then $X \subseteq Y$. Furthermore, for any $(a_i, b_i)_{1 \leq i \leq m+n} \in X$, we have

$$\left(\bigwedge_{1 \leq i \leq m} \mu(a_i) \ominus \nu(b_i)\right) \wedge \left(\bigwedge_{m+1 \leq i \leq m+n} \mu(a_i) \ominus \nu(b_i)\right) = \bigwedge_{1 \leq i \leq m+n} \mu(a_i) \ominus \nu(b_i) \leq \big(\mu \otimes \nu\big)(x - y).$$

Thus, $\big(\mu \otimes \nu\big)(x) \wedge \big(\mu \otimes \nu\big)(y) \leq \big(\mu \otimes \nu\big)(x - y)$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{m} a_i b_i$, we have $xy = \sum_{i=1}^{m} a_i (b_i y)$ and,

$$\bigwedge_{1 \leq i \leq m} \mu(a_i) \ominus \nu(b_i) \leq \bigwedge_{1 \leq i \leq m} \mu(a_i) \ominus \nu(b_i y) \leq \big(\mu \otimes \nu\big)(xy).$$

Thus, $\big(\mu \otimes \nu\big)(x) \leq \big(\mu \otimes \nu\big)(xy)$. Similarly, we obtain $\big(\mu \otimes \nu\big)(y) \leq \big(\mu \otimes \nu\big)(xy)$. So, $\big(\mu \otimes \nu\big)(xy) \geq \big(\mu \otimes \nu\big)(x) \vee \big(\mu \otimes \nu\big)(y)$. Hence, $\mu \otimes \nu$ is an L-fuzzy ideal of $\mathcal{A}$.

Finally, let δ be an L-fuzzy ideal of $\mathcal{A}$ containing $\mu \circ \nu$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$, we have

$$\bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \nu(b_i) \leq \bigwedge_{1 \leq i \leq n} (\mu \circ \nu)(a_i b_i) \leq \bigwedge_{1 \leq i \leq n} \delta(a_i b_i) \leq \delta\left(\sum_{i=1}^{n} a_i b_i\right) = \delta(x).$$

Thus, $\big(\mu \otimes \nu\big)(x) \leq \delta(x)$. Hence, $\mu \otimes \nu \leq \delta$. Therefore, $\mu \otimes \nu$ is the smallest L-fuzzy ideal of $\mathcal{A}$ containing $\mu \circ \nu$. □

Proposition 3.4 *The binary operation $\otimes$ on $Fid(\mathcal{A}, L)$ is associative.*

Proof Let $\mu, \nu, \delta \in Fid(\mathcal{A}, L)$. Let $x \in A$. Let $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$. Let $1 \leq i \leq n$. For any $c_{i_1}, d_{i_1}, \ldots, c_{i_p}, d_{i_p} \in A$ such that $b_i = \sum_{j=1}^{p} c_{i_j} d_{i_j}$, we have for each $1 \leq k \leq p$,

$$\begin{aligned}\mu(a_i) \ominus \left(\bigwedge_{1 \leq j \leq p} \nu(c_{i_j}) \ominus \delta(d_{i_j})\right) &\leq \mu(a_i) \ominus \big(\nu(c_{i_k}) \ominus \delta(d_{i_k})\big)\\ &= \big(\mu(a_i) \ominus \nu(c_{i_k})\big) \ominus \delta(d_{i_k})\\ &\leq \big(\mu \otimes \nu\big)(a_i c_{i_k}) \ominus \delta(d_{i_k})\\ &\leq \big((\mu \otimes \nu) \otimes \delta\big)\big((a_i c_{i_k}) d_{i_k}\big)\\ &= \big((\mu \otimes \nu) \otimes \delta\big)\big(a_i (c_{i_k} d_{i_k})\big);\end{aligned}$$

thus,

$$\begin{aligned}\mu(a_i) \ominus \left(\bigwedge_{1 \leq j \leq p} \nu(c_{i_j}) \ominus \delta(d_{i_j})\right) &\leq \bigwedge_{1 \leq j \leq p} \big((\mu \otimes \nu) \otimes \delta\big)\big(a_i (c_{i_j} d_{i_j})\big)\\ &\leq \big((\mu \otimes \nu) \otimes \delta\big)\left(\sum_{j=1}^{p} a_i (c_{i_j} d_{i_j})\right)\\ &= \big((\mu \otimes \nu) \otimes \delta\big)\left(a_i \sum_{j=1}^{p} c_{i_j} d_{i_j}\right)\\ &= \big((\mu \otimes \nu) \otimes \delta\big)(a_i b_i).\end{aligned}$$

So, $\mu(a_i) \ominus \big(\nu \otimes \delta\big)(b_i) \leq \big((\mu \otimes \nu) \otimes \delta\big)(a_i b_i)$. It follows that

$$\bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \big(\nu \otimes \delta\big)(b_i) \leq \bigwedge_{1 \leq i \leq n} \big((\mu \otimes \nu) \otimes \delta\big)(a_i b_i) \leq \big((\mu \otimes \nu) \otimes \delta\big)(x).$$

Thus, $\big(\mu \otimes (\nu \otimes \delta)\big)(x) \leq \big((\mu \otimes \nu) \otimes \delta\big)(x)$ and, $\big((\mu \otimes \nu) \otimes \delta\big)(x) \leq \big(\mu \otimes (\nu \otimes \delta)\big)(x)$ by similar arguments. So, $\big(\mu \otimes (\nu \otimes \delta)\big)(x) = \big((\mu \otimes \nu) \otimes \delta\big)(x)$. Hence, $\mu \otimes (\nu \otimes \delta) = (\mu \otimes \nu) \otimes \delta$. Therefore, $\otimes$ is associative. □

Corollary 3.5 $\mathbf{F}id(\mathcal{A}, L) := \big(Fid(\mathcal{A}, L); \otimes, \underline{1}\big)$ *is a monoid.*

Proof Since $\otimes$ is associative by Proposition 3.4, it suffices to show that $\underline{1}$ is the unity of $Fid(\mathcal{A}, L)$. So, let μ be an L-fuzzy ideal of $\mathcal{A}$. Let $x \in A$. For any $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$, we have

$$\bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus \underline{1}(b_i) = \bigwedge_{1 \leq i \leq n} \mu(a_i) \ominus 1 = \bigwedge_{1 \leq i \leq n} \mu(a_i) \leq \bigwedge_{1 \leq i \leq n} \mu(a_i b_i) \leq \mu(x).$$

Thus, $\big(\mu \otimes \underline{1}\big)(x) \leq \mu(x)$. Furthermore, $\big(\mu \otimes \underline{1}\big)(x) \geq \mu(x) \ominus \underline{1}(1) = \mu(x) \ominus 1 = \mu(x)$. So, $\big(\mu \otimes \underline{1}\big)(x) = \mu(x)$. Hence, $\mu \otimes \underline{1} = \mu$ and, $\underline{1} \otimes \mu = \mu$ by similar arguments. Therefore, $\underline{1}$ is the unity of $Fid(\mathcal{A}, L)$. □

Definition 3.6 For any L-fuzzy subsets μ and ν of A, $\mu \hookrightarrow \nu$ and $\mu \looparrowright \nu$ denote the L-fuzzy subsets of A defined for any $x \in A$ by:

$$\big(\mu \hookrightarrow \nu\big)(x) = \bigvee\{r \in L : x_r \circ \mu \leq \nu\}$$
$$\big(\mu \looparrowright \nu\big)(x) = \bigvee\{r \in L : \mu \circ x_r \leq \nu\}.$$

Proposition 3.7 *Let μ and ν be two L-fuzzy ideals of $\mathcal{A}$. Then $\mu \hookrightarrow \nu$ and $\mu \nrightarrow \nu$ are L-fuzzy ideals of $\mathcal{A}$.*

Proof Since $0_1 \circ \mu = \chi_0 \leq \nu$, we have $1 \leq \big(\mu \hookrightarrow \nu\big)(0)$ and, $\big(\mu \hookrightarrow \nu\big)(0) = 1$. Now, let $x, y \in A$. Let $r, s \in L$ such that $x_r \circ \mu \leq \nu$ and $y_s \circ \mu \leq \nu$. Let $a \in A$. Let $b, c \in A$ such that $a = bc$.

- If $b \neq x - y$, then

$$(x - y)_{r \wedge s}(b) \ominus \mu(c) = 0 \ominus \mu(c) = 0 \leq \nu(a).$$

- If $b = x - y$, then

$$\begin{aligned}(x - y)_{r \wedge s}(b) \ominus \mu(c) &= (r \wedge s) \ominus \mu(c)\\ &\leq \big(r \ominus \mu(c)\big) \wedge \big(s \ominus \mu(c)\big)\\ &= \big(x_r(x) \ominus \mu(c)\big) \wedge \big(y_s(y) \ominus \mu(c)\big)\\ &\leq \big(x_r \circ \mu\big)(xc) \wedge \big(y_s \circ \mu\big)(yc)\\ &\leq \nu(xc) \wedge \nu(yc)\\ &\leq \nu(xc - yc)\\ &= \nu(a).\end{aligned}$$

Thus, $\big((x - y)_{r \wedge s} \circ \mu\big)(a) \leq \nu(a)$. So,

$(x - y)_{r \wedge s} \circ \mu \leq \nu$ and, $r \wedge s \leq (\mu \hookrightarrow \nu)(x - y)$.

It follows that $(\mu \hookrightarrow \nu)(x) \wedge (\mu \hookrightarrow \nu)(y) \leq (\mu \hookrightarrow \nu)(x - y)$.
Now, let $r \in L$ such that $x_r \circ \mu \leq \nu$. Let $a \in A$. Let $b, c \in A$ such that $a = bc$.

- If $b \neq xy$, then

$$(xy)_r(b) \ominus \mu(c) = 0 \ominus \mu(c) = 0 \leq \nu(a).$$

- If $b = xy$, then

$$\begin{aligned}(xy)_r(b) \ominus \mu(c) &= r \ominus \mu(c)\\ &= x_r(x) \ominus \mu(c)\\ &\leq x_r(x) \ominus \mu(yc)\\ &\leq (x_r \circ \mu)\big(x(yc)\big)\\ &\leq \nu\big(x(yc)\big)\\ &= \nu(a).\end{aligned}$$

Thus, $\big((xy)_r \circ \mu\big)(a) \leq \nu(a)$. So, $(xy)_r \circ \mu \leq \nu$ and, $r \leq \big(\mu \hookrightarrow \nu\big)(xy)$. It follows that $\big(\mu \hookrightarrow \nu\big)(x) \leq \big(\mu \hookrightarrow \nu\big)(xy)$.

Now, let $r \in L$ such that $y_r \circ \mu \leq \nu$. Let $a \in A$. Let $b, c \in A$ such that $a = bc$.

- If $b \neq xy$, then

$$(xy)_r(b) \ominus \mu(c) = 0 \ominus \mu(c) = 0 \leq \nu(a).$$

- Suppose that $b = xy$. For any $z \in U\big(\mu, \mu(c)\big)$, we have

$$\begin{aligned}\nu(yz) &\geq \big(y_r \circ \mu\big)(yz)\\ &\geq y_r(y) \ominus \mu(z)\\ &= r \ominus \mu(z)\\ &\geq r \ominus \mu(c)\end{aligned}$$

and, $yz \in U\big(\nu, r \ominus \mu(c)\big)$. Thus,

$$yU\big(\mu, \mu(c)\big) \subseteq U\big(\nu, r \ominus \mu(c)\big) \text{ and, }\\ y \in U\big(\mu, \mu(c)\big) \to U\big(\nu, r \ominus \mu(c)\big).$$

So,

$$xy \in U\big(\mu, \mu(c)\big) \to U\big(\nu, r \ominus \mu(c)\big) \text{ and, }\\ xyU\big(\mu, \mu(c)\big) \subseteq U\big(\nu, r \ominus \mu(c)\big).$$

Since $a = xyc \in U\big(\nu, r \ominus \mu(c)\big)$, we have

$$(xy)_r(b) \ominus \mu(c) = r \ominus \mu(c) \leq \nu(a).$$

Thus, $\big((xy)_r \circ \mu\big)(a) \leq \nu(a)$. So, $(xy)_r \circ \mu \leq \nu$ and, $r \leq \big(\mu \hookrightarrow \nu\big)(xy)$. Thus, $\big(\mu \hookrightarrow \nu\big)(y) \leq \big(\mu \hookrightarrow \nu\big)(xy)$. Consequently, $\big(\mu \hookrightarrow \nu\big)(x) \vee \big(\mu \hookrightarrow \nu\big)(y) \leq \big(\mu \hookrightarrow \nu\big)(xy)$.
Hence, $\mu \hookrightarrow \nu$ is an L-fuzzy ideal of $\mathcal{A}$. A similar reasoning shows that $\mu \nrightarrow \nu$ is an L-fuzzy ideal of $\mathcal{A}$. □

Theorem 3.8 *$\mathcal{F}id(\mathcal{A}, L) := \big(Fid(\mathcal{A}, L);\ \wedge,\ +,\ \otimes,\ \hookrightarrow,\ \nrightarrow;\ \chi_0,\ \underline{1}\big)$ is a complete residuated lattice.*

Proof Since $\mathbb{F}id(\mathcal{A}, L)$ is a complete lattice and $\mathbf{F}id(\mathcal{A}, L)$ is a monoid, it suffices to show that: for any $\mu, \nu, \delta \in Fid(\mathcal{A}, L)$, $\mu \otimes \nu \leq \delta$ iff $\mu \leq \nu \hookrightarrow \delta$ iff $\nu \leq \mu \nrightarrow \delta$. So, let $\mu, \nu, \delta \in Fid(\mathcal{A}, L)$.

Assume that $\mu \otimes \nu \leq \delta$. Let $x \in A$. Let $a \in A$. For any $v \in A$ such that $a = xv$, we have

$$x_{\mu(x)}(x) \ominus \nu(v) = \mu(x) \ominus \nu(v) \leq \big(\mu \otimes \nu\big)(a) \leq \delta(a).$$

Thus, $\big(x_{\mu(x)} \circ \nu\big)(a) \leq \delta(a)$. So, $x_{\mu(x)} \circ \nu \leq \delta$ and, $\mu(x) \leq \big(\nu \hookrightarrow \delta\big)(x)$. Hence, $\mu \leq \nu \hookrightarrow \delta$.

Conversely, assume that $\mu \leq \nu \hookrightarrow \delta$. Let $x \in A$. Let $a_1, b_1, \ldots, a_n, b_n \in A$ such that $x = \sum_{i=1}^{n} a_i b_i$. Let $1 \leq i \leq n$. For any $r_i \in L$ such that $(a_i)_{r_i} \circ \nu \leq \delta$, we have $\big(\mu(a_i) \wedge r_i\big) \ominus \nu(b_i) \leq r_i \ominus \nu(b_i) = (a_i)_{r_i}(a_i) \ominus \nu(b_i) \leq \big((a_i)_{r_i} \circ \nu\big)(a_i b_i) \leq \delta(a_i b_i)$. Since $\mathcal{L}$ is Brouwerian, we have

$$\mu(a_i) \ominus \nu(b_i) = \big(\mu(a_i) \wedge \big(\nu \hookrightarrow \delta\big)(a_i)\big) \ominus \nu(b_i) \leq \delta(a_i b_i).$$

Thus,

$$\bigwedge_{1\le i\le n} \mu(a_i)\ominus\nu(b_i)\le \bigwedge_{1\le i\le n}\delta(a_ib_i)\le\delta(x).$$

So, $(\mu\otimes\nu)(x)\le\delta(x)$. It follows that $\mu\otimes\nu\le\delta$.

Hence, $\mu\otimes\nu\le\delta$ iff $\mu\le\nu\hookrightarrow\delta$. A similar reasoning shows that: $\mu\otimes\nu\le\delta$ iff $\nu\le\mu\nrightarrow\delta$. □

4 Embeddings of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$ into $\mathcal{F}id(\mathcal{A}, L)$

Proposition 4.1 *Let $I, J\in Id(\mathcal{A})$ and $r, s\in L$. Then the following hold:*

(1) $(I_r)_*\otimes(J_s)_*=\big((I\odot J)_{r\ominus s}\big)_*$.
(2) $I^r\otimes J^s=(I\odot J)^{r\ominus s}+(I_s)_*+(J_r)_*$.
(3) $(I_r)_*+(J_s)_*=\big[(I+J\setminus I\cup J)_{r\wedge s}\vee(I\setminus J)_r\vee(J\setminus I)_s\vee(I\cap J\setminus\{0\})_{r\vee s}\big]_*$.

Proof (1) Let $x\in I\odot J\setminus\{0\}$. For any $a_1, b_1,\dots,a_n,b_n\in A$ such that $x=\sum_{i=1}^n a_ib_i$, there is $1\le i_0\le n$ such that $a_{i_0}\ne0$ and $b_{i_0}\ne0$; thus, $\bigwedge_{1\le i\le n}(I_r)_*(a_i)\ominus(J_s)_*(b_i)\le(I_r)_*(a_{i_0})\ominus(J_s)_*(b_{i_0})\le r\ominus s$. So, $\big((I_r)_*\otimes(J_s)_*\big)(x)\le r\ominus s$. Since there are $a_1,\dots,a_n\in I\setminus\{0\}$ and $b_1,\dots,b_n\in J\setminus\{0\}$ such that $x=\sum_{i=1}^n a_ib_i$, we have $r\ominus s=\bigwedge_{1\le i\le n}(I_r)_*(a_i)\ominus(J_s)_*(b_i)\le\big((I_r)_*\otimes(J_s)_*\big)(x)$ and, $\big((I_r)_*\otimes(J_s)_*\big)(x)=r\ominus s$.
Now, let $x\notin I\odot J$. For any $a_1,b_1,\dots,a_n,b_n\in A$ such that $x=\sum_{i=1}^n a_ib_i$, there is $1\le i_0\le n$ such that $a_{i_0}\notin I$ or $b_{i_0}\notin J$; *i.e.*, $(I_r)_*(a_{i_0})=0$ or $(J_s)_*(b_{i_0})=0$; thus, $\bigwedge_{1\le i\le n}(I_r)_*(a_i)\ominus(J_s)_*(b_i)\le(I_r)_*(a_{i_0})\ominus(J_s)_*(b_{i_0})=0$ and, $\bigwedge_{1\le i\le n}(I_r)_*(a_i)\ominus(J_s)_*(b_i)=0$. So, $\big((I_r)_*\otimes(J_s)_*\big)(x)=\bigvee\{0\}=0$.
Hence, $(I_r)_*\otimes(J_s)_*=\big((I\odot J)_{r\ominus s}\big)_*$.
(2) We first show that $I^r=I_1+(A_r)_*$. For any $x\in I$, we have $\big(I_1+(A_r)_*\big)(x)\ge I_1(x)=1$ and, $\big(I_1+(A_r)_*\big)(x)=1$. Now, let $x\notin I$. Let $a,b\in A$ such that $x=a+b$. If $b=0$, then $a\notin I$ and, $I_1(a)\wedge(A_r)_*(b)=0\wedge1=0$. If $b\ne0$, then $I_1(a)\wedge(A_r)_*(b)\le(A_r)_*(b)=r$. Thus, $r=(A_r)_*(x)\le\big(I_1+(A_r)_*\big)(x)\le r$ and, $\big(I_1+(A_r)_*\big)(x)=r$. So, $I^r=I_1+(A_r)_*$. A similar reasoning shows that $J^s=J_1+(A_s)_*$. Finally, we have

$$\begin{aligned}I^r\otimes J^s&=\big(I_1+(A_r)_*\big)\otimes\big(J_1+(A_s)_*\big)\\&=(I\odot J)_{1\ominus1}+\big((I\odot A)_{1\ominus s}\big)_*+\big((A\odot J)_{r\ominus1}\big)_*\\&\quad+\big((A\odot A)_{r\ominus s}\big)_*\\&=(I\odot J)_1+(I_s)_*+(J_r)_*+(A_{r\ominus s})_*\\&=(I\odot J)^{r\ominus s}+(I_s)_*+(J_r)_*.\end{aligned}$$

(3) • Let $x\notin I+J$. For any $a,b\in A$ such that $x=a+b$, we have $a\notin I$ or $b\notin J$; *i.e.*, $(I_r)_*(a)=0$ or $(J_s)_*(b)=0$; thus, $(I_r)_*(a)\wedge(J_s)_*(b)=0$. So, $\big((I_r)_*+(J_s)_*\big)(x)=\bigvee\{0\}=0$.
• Let $x\in I+J\setminus I\cup J$. For any $a,b\in A$ such that $x=a+b$, we have

$$(I_r)_*(a)\wedge(J_s)_*(b)=\begin{cases}r\wedge s & \text{if } a\in I \text{ and } b\in J,\\ 0 & \text{if } a\notin I \text{ or } b\notin J.\end{cases}$$

Thus, $r\wedge s=(I_r)_*(u)\wedge(J_s)_*(v)\le\big((I_r)_*+(J_s)_*\big)(x)\le r\wedge s$ for some $u\in I\setminus\{0\}$ and $v\in J\setminus\{0\}$ such that $x=u+v$; so, $\big((I_r)_*+(J_s)_*\big)(x)=r\wedge s$.
• Let $x\in I\setminus J$. For any $a,b\in A$ such that $x=a+b$, we have

$$(I_r)_*(a)\wedge(J_s)_*(b)=\begin{cases}r\wedge(J_s)_*(b) & \text{if } a\in I \text{ and } b\in J,\\ 0 & \text{if } a\notin I \text{ or } b\notin J.\end{cases}$$

Thus, $r=(I_r)_*(x)\wedge(J_s)_*(0)\le\big((I_r)_*+(J_s)_*\big)(x)\le r$ and, $\big((I_r)_*+(J_s)_*\big)(x)=r$. A similar reasoning shows that $\big((I_r)_*+(J_s)_*\big)(x)=s$ for all $x\in J\setminus I$.
• Let $x\in(I\cap J)\setminus\{0\}$. For any $a,b\in A$ such that $x=a+b$, we have $a\ne0$ or $b\ne0$; thus, $(I_r)_*(a)\wedge(J_s)_*(b)\le r\vee s$. So, $r\vee s=\big((I_r)_*\vee(J_s)_*\big)(x)\le\big((I_r)_*+(J_s)_*\big)(x)\le r\vee s$ and, $\big((I_r)_*+(J_s)_*\big)(x)=r\vee s$.
Hence, $(I_r)_*+(J_s)_*=\big[(I+J\setminus I\cup J)_{r\wedge s}\vee(I\setminus J)_r\vee(J\setminus I)_s\vee(I\cap J\setminus\{0\})_{r\vee s}\big]_*$. □

For any $I, J\in Id(\mathcal{A})$ and $r\in L$, one can easily verify that $(I_r)_*+(J_r)_*=\big((I+J)_r\big)_*$ and $I_1+(J_r)_*=\big(I_1\vee(I+J)_r\big)_*$. For any L-fuzzy ideal μ of $\mathcal{A}$, $\mu^-:=\mu\hookrightarrow\chi_0$ and $\mu^\sim:=\mu\nrightarrow\chi_0$ denote the left and right annihilator of μ in $\mathcal{F}id(\mathcal{A}, L)$, respectively.

Proposition 4.2 (1) *Let $r, s\in L$ and $I, J\in Id(\mathcal{A})$. Then*

$\big((I\to J)_{r\twoheadrightarrow s}\big)_*\le(I_r)_*\hookrightarrow(J_s)_*$ *and*
$\big((I\rightsquigarrow J)_{r\multimap s}\big)_*\le(I_r)_*\nrightarrow(J_s)_*$.

(2) *Let I be a proper ideal of $\mathcal{A}$ and $r, s\in L$ such that $r\le s$. Then $\big((I_r^s)_*\big)^-=\big((I^-)_{\frac{\bar r}{\bar s}}\big)_*$ and $\big((I_r^s)_*\big)^\sim=\big((I^\sim)_{\frac{\tilde r}{\tilde s}}\big)_*$; where, $I^-:=I\to\{0\}$ and $I^\sim:=I\rightsquigarrow\{0\}$ denote the left and right annihilator of I in $\mathcal{I}d(\mathcal{A})$, respectively.*

Proof (1) Since $\big((I \to J)_{r \twoheadrightarrow s}\big)_* \otimes (I_r)_* = \big[\big((I \to J) \odot I\big)_{(r \twoheadrightarrow s) \ominus r}\big]_* \le (J_s)_*$, we have $\big((I \to J)_{r \twoheadrightarrow s}\big)_* \le (I_r)_* \hookrightarrow (J_s)_*$. Similarly, $\big((I \rightsquigarrow J)_{r \multimap s}\big)_* \le (I_r)_* \looparrowright (J_s)_*$.

(2) We first show that $\big((I_r^s)_*\big)^- = \big((I^-)_{\overline{s}}^{\overline{r}}\big)_*$.

• Let $x \in I^- \setminus \{0\}$. For any $t \in L$ such that $x_t \circ (I_r^s)_* \le \chi_0$, we have

$$t \ominus r = x_t(x) \ominus (I_r^s)_*(1) \le \big(x_t \circ (I_r^s)_*\big)(x) \le \chi_0(x) = 0;$$

thus, $t \le \overline{r}$. So, $\big((I_r^s)_*\big)^-(x) \le \overline{r}$. Now, let $a \ne 0$ in A. For any $v \in A$ such that $a = xv$, we have $v \notin I$; thus,

$$\overline{r} \ominus (I_r^s)_*(v) = \overline{r} \ominus r = 0.$$

So,

$$\big(x_{\overline{r}} \circ (I_r^s)_*\big)(a) = \bigvee\{0\} = 0.$$

It follows that

$$x_{\overline{r}} \circ (I_r^s)_* \le \chi_0 \text{ and, } \overline{r} \le \big((I_r^s)_*\big)^-(x).$$

Consequently, $\big((I_r^s)_*\big)^-(x) = \overline{r}$.

• Let $x \notin I^-$. For any $t \in L$ such that $x_t \circ (I_r^s)_* \le \chi_0$, we have

$$t \ominus s = x_t(x) \ominus (I_r^s)_*(v) \le \big(x_t \circ (I_r^s)_*\big)(xv) \le \chi_0(xv) = 0$$

for some $v \in I$ such that $xv \ne 0$; thus, $t \le \overline{s}$. So, $\big((I_r^s)_*\big)^-(x) \le \overline{s}$. Now, let $a \ne 0$ in A. For any $v \in A$ such that $a = xv$, we have

$$\overline{s} \ominus (I_r^s)_*(v) = \begin{cases} \overline{s} \ominus s & \text{if } v \in I \\ \overline{s} \ominus r & \text{if } v \notin I \end{cases} \le \overline{s} \ominus s = 0;$$

thus, $\overline{s} \ominus (I_r^s)_*(v) = 0$. So,

$$\big(x_{\overline{s}} \circ (I_r^s)_*\big)(a) = \bigvee\{0\} = 0.$$

It follows that

$$x_{\overline{s}} \circ (I_r^s)_* \le \chi_0 \text{ and, } \overline{s} \le \big((I_r^s)_*\big)^-(x).$$

Consequently, $\big((I_r^s)_*\big)^-(x) = \overline{s}$.

Hence, $\big((I_r^s)_*\big)^- = \big((I^-)_{\overline{s}}^{\overline{r}}\big)_*$ and, $\big((I_r^s)_*\big)^\sim = \big((I^\sim)_{\tilde{s}}^{\tilde{r}}\big)_*$ by similar arguments. □

Theorem 4.3 *The function $\phi : Id(\mathcal{A}) \to Fid(\mathcal{A}, L)$, given by $\phi(I) = I_1$ for all $I \in Id(\mathcal{A})$, is a complete residuated lattice embedding.*

Proof Since ϕ is clearly a complete lattice embedding, and the fact that

$$\phi(I \odot J) = (I \odot J)_1 = (I \odot J)_{1 \ominus 1} = I_1 \otimes J_1 = \phi(I) \otimes \phi(J) \text{ for all } I, J \in Id(\mathcal{A}),$$

we only have to prove that ϕ preserves the residues. So, let $I, J \in Id(\mathcal{A})$. Let $x \notin I \to J$. There is $a \in I$ such that $xa \notin J$. For any $r \in L$ such that $x_r \circ I_1 \le J_1$, we have

$$r = r \ominus 1 = x_r(x) \ominus I_1(a) \le (x_r \circ I_1)(xa) \le J_1(xa) = 0 \text{ and, } r = 0.$$

Thus, $(I_1 \hookrightarrow J_1)(x) = \bigvee\{0\} = 0$. So, $I_1 \hookrightarrow J_1 \le (I \to J)_1$ and, $(I \to J)_1 = I_1 \hookrightarrow J_1$. Hence, $\phi(I \to J) = (I \to J)_1 = I_1 \hookrightarrow J_1 = \phi(I) \hookrightarrow \phi(J)$. A similar reasoning shows that $\phi(I \rightsquigarrow J) = \phi(I) \looparrowright \phi(J)$. Therefore, ϕ is a complete residuated lattice embedding of $\mathcal{I}d(\mathcal{A})$ into $\mathcal{F}id(\mathcal{A}, L)$. □

Theorem 4.4 *The function $\psi : L \to Fid(\mathcal{A}, L)$, given by $\psi(r) = (\underline{r})_*$ for all $r \in L$, is a complete residuated lattice embedding.*

Proof Since ψ is clearly a complete lattice embedding, and the fact that for any $r, s \in L$, we have

$$\psi(r \ominus s) = \big(A_{r \ominus s}\big)_* = \big((A \odot A)_{r \ominus s}\big)_* = (A_r)_* \otimes (A_s)_* = (\underline{r})_* \otimes (\underline{s})_* = \psi(r) \otimes \psi(s),$$

we only have to prove that ψ preserves the residues. So, let $r, s \in L$. Let $x \ne 0$ in A. For any $t \in L$ such that $x_t \circ (\underline{r})_* \le (\underline{s})_*$, we have

$$t \ominus r = x_t(x) \ominus (\underline{r})_*(1) \le \big(x_t \circ (\underline{r})_*\big)(x) \le (\underline{s})_*(x) = s \text{ and, } t \le r \twoheadrightarrow s.$$

Thus, $\big((\underline{r})_* \hookrightarrow (\underline{s})_*\big)(x) \le r \twoheadrightarrow s = (\underline{r \twoheadrightarrow s})_*(x)$. So, $(\underline{r})_* \hookrightarrow (\underline{s})_* \le (\underline{r \twoheadrightarrow s})_*$ and, $(\underline{r \twoheadrightarrow s})_* = (\underline{r})_* \hookrightarrow (\underline{s})_*$. Hence,

$$\psi(r \twoheadrightarrow s) = (\underline{r \twoheadrightarrow s})_* = (\underline{r})_* \hookrightarrow (\underline{s})_* = \psi(r) \hookrightarrow \psi(s).$$

A similar reasoning shows that $\psi(r \multimap s) = \psi(r) \looparrowright \psi(s)$. Therefore, ψ is a complete residuated lattice embedding of $\mathcal{L}$ into $\mathcal{F}id(\mathcal{A}, L)$. □

5 Conclusion

In this paper, given a complete Brouwerian residuated lattice $\mathcal{L}$ and a ring with unity $\mathcal{A}$, we have built a residuated lattice structure $\mathcal{F}id(\mathcal{A}, L)$, on the set $Fid(\mathcal{A}, L)$ of L-fuzzy ideals of $\mathcal{A}$, which extends both $\mathcal{L}$ and the residuated lattice $\mathcal{I}d(\mathcal{A})$, on the set $Id(\mathcal{A})$ of ideals of $\mathcal{A}$. This construction is also valid for non-normalized fuzzy ideals; *i.e.*, fuzzy ideals μ which do not necessarily satisfy condition $\mu(0) = 1$. But in this case $\mathcal{F}id(\mathcal{A}, L)$ is only an extension of $\mathcal{L}$, since it is rather bounded by $\underline{0}$ and $\underline{1}$.

By a $\ominus$-prime element of a residuated lattice $\mathcal{L}$, we mean an element $p \neq 1$ in L such that: for any $x, y \in L, x \ominus y \leq p$ implies $x \leq p$ or $y \leq p$. This definition is slightly different from that known in lattices, which rather coincides with the definition of $\wedge$-prime element. It would be interesting to establish if there is a better (or nice) link between the prime elements in $\mathcal{F}id(\mathcal{A}, L)$ and those in $\mathbb{F}id(\mathcal{A}, L)$. This will can also be extended to primary elements and primary decompositions.

Another interesting aspect to study following this paper is the relationship between $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$, and $\mathcal{F}id(\mathcal{A}, L)$, depending on the structure of $\mathcal{L}$ and $\mathcal{I}d(\mathcal{A})$.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

References

Ciungu LC (2014) Non-commutative multiple-valued logic algebras. Springer monographs in mathematics. Springer, Berlin

Dedekind R (1894) Supplement XI of Dirichlet's Vorlesungen, vol III, 4th edn. ∮ 170, Collected Works. Vieweg, Brauschweig, p 71

Dilworth RP (1939) Non-commutative residuated lattices. Trans Am Math Soc 46(3):426–444

Galatos N, Jipsen P, Kowalski T, Ono H (2007) Residuated lattices, an algebraic glimpse at substructural logics. Volume 151 of studies in logic and the foundations of mathematics. Elsevier, Amsterdam

Goguen JA (1967) L-fuzzy sets. J Math Anal Appl 18(1):145–174

Jipsen P, Tsinakis C (2002) A survey of residuated lattices. In: Martinez J (ed) Ordered algebraic structures. Kluwer Academic Publisher, Dordrecht, pp 19–56

Kadji A, Lele C, Tonga M (2016) Some classes of pseudo-residuated lattices. Afr Mat 27(7–8):1147–1178

Martinez L (1999) Prime and primary L-fuzzy ideals of L-fuzzy rings. Fuzzy Sets Syst 101:489–494

Piciu D (2007) Algebras of fuzzy logic. Editura Universitaria Craiova, Craiova

Swamy UM, Swamy KLN (1988) Fuzzy prime ideals of rings. J Math Anal Appl 134:94–103

Tchoffo Foka SV, Tonga M (2019) A residuated lattice of L-fuzzy subalgebras of a mono-unary algebra. New Math Nat Comput 15(3):539–551

Tchoffo Foka SV, Tonga M (2020) A note on the algebraicity of L-fuzzy subalgebras in universal algebra. Soft Comput 24(2):895–899

Ward M (1937) Some arithmetical applications of residuation. Am J Math 59:921–926

Ward M, Dilworth RP (1939) Residuated lattices. Trans Am Math Soc 45:335–354

Yue Z (1988) Prime L-fuzzy ideals and primary L-fuzzy ideals. Fuzzy Sets Syst 27:345–350

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353