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CENTRE DE RECHERCHE ET DE FORMATION DOCTORALE EN SCIENCES, TECHNOLOGIES ET GEOSCIENCES

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POSTGRADUATE SCHOOL OF SCIENCE, TECHNOLOGY AND GEOSCIENCES

OPTIMAL CONTROL PROBLEM AND INHOMOGENEOUS MINIMAX VISCOSITY SOLUTION IN ∞ FOR RELATIVISTIC VLASOV EQUATION

THIS THESIS IS SUBMITTEED IN FULFILLMENT OF THE ACADEMIC REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS

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OPTIMAL CONTROL PROBLEM AND INHOMOGENEOUS MINIMAX VISCOSITY SOLUTION IN L^∞ FOR THE RELATIVISTIC VLASOV EQUATION

This thesis is submitted in fulfillment of the academic requirements for the degree of
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Dedication

To MVOGO Léon, my father, and NDJE Ékani, my mother, for everything...

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Abstract

In this work, using an important result of Chen Guiqiang and Su Bo [7], we set a theorem about a global in finite time and local in space existence and uniqueness of a minimax viscosity solution in L^∞ of the relativistic Vlasov equation in Yang-Mills charged time oriented four dimensional curved space-time with non-zero mass, therefore derive from it an optimal control problem. To our knowledge, the method used here to derive an existence theorem is original and totally different from the ones used to solve similar problems.

Keywords: relativistic Vlasov equation, viscosity solution, minimax solution, L^∞ solution, optimal control problem

Résumé

Partant d'un important résultat de Chen Guiqiang et Su Bo établi dans [7], nous donnons un théorème d'existence globale en temps fini et local dans l'espace d'une solution minimax de viscosité, avec des données initiales dans L^∞, de l'équation relativiste de Vlasov, dans un espace temps courbe avec une charge de Yang-Mills, pour des particules de masse non nulle. De ce résultat d'existence, on en déduit un problème de contrôle optimal. Cette approche est nouvelle relativement à d'autres approches utilisées pour résoudre des problèmes similaires.

Mots clés: équation relativiste de Vlasov, solution de viscosité, solution minimax, solution L^∞, problème de contrôle optimal

Introduction

The main purpose of this study is to give a global in finite time and local in space existence and uniqueness result about a generalized solution in L^∞, minimax viscosity solution, of the relativistic inhomogeneous Vlasov equation in a curved space time in which a Yang-Mills potential is given, and then we derive from it an optimal control problem. All these results are based on the transformation of the relativistic Vlasov equation into a Hamilton-Jacobi type equation.

The main purpose of this work comes from our will to bring forward another method to study the relativistic Vlasov equation and also for the requirements to have a valid solution for all positive times and at all points of space or at least in a given domain, and be doing so it may open a door to other use of the relativistic Vlasov equation. Let us remark that the above requirements are not generally ensured by classical methods. Classical methods are generally possible in a local sense, and then the domain of existence is very restricted. On the other hand, it is yet impossible to range all the applications of Hamilton-Jacobi equations. But the Hamilton-Jacobi equations are intimately related to the problem of calculus of variations through the Hamiltonian or Lagrangian, control theory, to numerical methods and artificial viscosity, refer to [6]; with this work all theses possibilities are now offered for the relativistic Vlasov equation.

It may be possible now to define an optimal control problem in which the value problem is the solution of the relativistic Vlasov equation in order to increase or reduce the value of the probability of presence of particles in a given volume.

Many studies have been made by several authors about a similar topic, with different significant results. Let us recall some of these contributions. Choquet-Bruhat and Noutchegueme in [9] studied the Yang-Mills Vlasov system using the method of characteristics, they obtained a local in time existence result; the method was complicated due to the introduction of weighted functional spaces that required many energy estimates. In [10], Choquet-Bruhat and Noutchegueme studied the Yang-Mills-Vlasov system only for the zero mass particles case, using a conformal invariant of system to prove a global existence theorem only in Minkowski space-time for small initial data. Noutchegueme and Noundjeu in [32] proved a local in time and global in space existence theorem for the Yang-Mills-Vlasov system in temporal gauge with current generated by a distribution function that satisfies the Vlasov equation, but still using the method of characteristics and where many energy estimates were also required. Wolfgang in [41] obtained a local existence result and uniqueness of solution of the Vlasov equation, but in the absence of the electromagnetic field, still using the method of characteristics. The natural question which may come up is why another method among all this relevant results?

Many reasons have motivated the present work. Comparatively to the methods used above, the present method is particular by his simplicity, and the approach is original. We study this problem by a global method. In contrary to classical methods, for instance the method of characteristics, which are local in nature and in which the domain of definition of the solution is generally severely restricted by the nature of the problem, as is the case for this work, global methods produce solutions defined in the whole given domain, and present sometimes interesting properties. In this work we are interested by three particular generalized methods: the viscosity method of first order equations of Hamilton-Jacobi type in Section 2.1, the minimax solution of first order equations of Hamilton-Jacobi type in Section 2.2, and discontinuous solutions in L^∞ for Hamilton-Jacobi equations in chapter III. All these methods present many interesting properties, among them the possibility to obtain existence and uniqueness criterion. In particular the viscosity method for first order equation of Hamilton-Jacobi type, initiated by M. Crandall and Lions [18, 17] and the influential monograph [29], provides an extremely convenient partial differential tools for dealing with the lack of smoothness of the valued problem arising in the domain of optimization problems.

The second motivation is to extend the result of Ayissi and Noutchegueme in [2] to the inhomogeneous relativistic Vlasov equation.

This last point brings out another motivation of this work : the introduction of an optimal control problem. The method adopted in this work is intimately related to the viscosity method of first order equation of Hamilton-Jacobi type, and this one permits to deduce an optimal control problem.

This work is made of five chapters. The method of characteristics of first order partial differential equation is presented in Chapter I. In Chapter II, we present the viscosity method of first order equation of Hamilton-Jacobi type in Section 2.1, the minimax solution of first order equation of Hamilton-Jacobi type in Section 2.2. The research on minimax solution employs methods of nonsmooth analysis, Lyapunov functions, dynamical optimization and the theory of differential games. In order to present the minimax solution theory, we have given some important definitions and issues. The purpose here is to cover all the methods of investigation of solution used in this work.

In Chapter III, made of four sections, the theory of discontinuous L^∞ solutions for Hamilton-Jacobi equations is presented. This theory originated from ideas of Chen Quiqiang and Su Bo [7], contains an important result, which is in the core of this work.

Chapter IV is devoted to the relativistic Vlasov equation, and is organized into ten sections. Here an effort is made to present gradually all the concepts involved in this equation. In Section 4.8 all the assumptions made in this work are displayed. In Section 4.9 we show how the relativistic Vlasov equation is transformed into an Hamilton-Jacobi type equation in order to use it further, this feature is new, because it has never been done in another work mentioned here, except partially in [2]. With this transformation, we give a time and space existence theorem, which is a new result and enlarges the usefulness of this one.

In Chapter V the corresponding Hamiltonian related to the relativistic Vlasov equation, obtained in Chapter IV, is studied to verify some assumptions, denoted (B) of this work. Note that the method of study of this Hamiltonian is presented in Section 3, based on [7]. The main existence theorem of this work is given in Section 5.3 . The optimal control problem and its outline are settled in the last sections.

CHAPTER 1

Preliminaries

The purpose of this chapter is firstly to recall, without proofs, some important and fundamental theorems that will be used in the sequel. Secondly we give a brief explanation of the method of characteristics, which is a classical method to solve first order nonlinear partial differential equations (PDE) by converting them into an appropriate system of ordinary differential equations (ODE). The following ideas come from [14], [15], [11] and [8].

1.1 Some important definitions and theorems

In order to make this work self-contained, we give here without proof some classical results.

Let $U \subset \mathbb{R}^n$ be open and bounded, and let ∂U its boundary, $k \in \{1, 2, \dots\}$.

Definition 1.1. We say that ∂U is C^k if for each point $x^0 \in \partial U$ there exists $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that - upon relabeling and reorienting the coordinates axes if necessary- we have

$$U \cap B\left(x^0, r\right) = \left\{x \in B\left(x^0, r\right) | x_n > \gamma\left(x_1, \ldots, x_{n-1}\right)\right\}.$$

Assume ∂U be C^k. We will need to change coordinates near a point of ∂U so as to flatten out the boundary. Fix $x^0 \in \partial U$, and choose r, and γ as above. Define then

$$\begin{cases} y_i = x_i := \Phi^i(x) \qquad (i = 1, \ldots, n-1) \\ y_n = x_n - \gamma\left(x_1, \ldots, x_{n-1}\right) := \Phi^n(x), \end{cases}$$

and write

$$y = \left(y_1, \ldots, y_n\right) := \Phi(x).$$

Similarly, we set

$$\begin{cases} x_i = y_i := \Psi^i(y) \qquad (i = 1, \ldots, n-1) \\ x_n = y_n - \gamma\left(y_1, \ldots, y_{n-1}\right) := \Psi^n(y), \end{cases}$$

and write

$$y = \Psi(x).$$

Then $\Phi = \Psi^{-1}$, and the mapping $x \mapsto \Phi(x) = y$ straightens out ∂U near x^0. Observe also that $\det\Phi = \det\Psi = 1$.

Now let $U \subset \mathbb{R}^n$ be an open set and suppose $f : U \longrightarrow \mathbb{R}^n$ is C^1, $f = \left(f^1, \ldots, f^n\right)$. Assume $x_0 \in U$, $z_0 = f(x_0)$.

Notation 1.1. We write

$$Df = \begin{pmatrix} f^1_{x_1} & \cdots & f^1_{x_n} \\ & \ddots & \\ f^n_{x_1} & \cdots & f^n_{x_n} \end{pmatrix},$$

$$Jf(x_0) = \det Df_{|x=x_0} = \left|\frac{\partial\left(f^1, \ldots, f^n\right)}{\partial\left(x_1, \ldots, x_n\right)}\right|_{|x=x_0}.$$

Theorem 1.1 (Inverse Function Theorem)**.** *Assume* $f \in C^1(U;\mathbb{R}^n)$ *and*

$$Jf(x_0) \neq 0.$$

Then there exist an open set $V \subset U$*, with* $x_0 \in V$*, and open set* $W \subset \mathbb{R}^n$*, with* $z_0 \in W$*, such that*
(i) the mapping

$$f : V \longrightarrow W$$

is one-to-one and onto, and
(ii) the inverse function

$$f^{-1} : W \longrightarrow V$$

in C^1*.*
(iii) If $f \in C^k$*, then* $f^{-1} \in C^k$ $(k = 2, \dots)$.

Proof. See [14]. □

Notation 1.2. Let n, m be positive integers.
We write a typical point in $\mathbb{R}^{n+m}$ as

$$(x, y) = (x_1, \dots, x_n, y_1, \dots, y_m)$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.
Let $U \subset \mathbb{R}^{n+m}$ be an open set and suppose $f : U \longrightarrow \mathbb{R}^m$ is C^1, $f = \left(f^1, \dots, f^m\right)$. Assume $(x_0, y_0) \in U$, $z_0 = f(x_0, y_0)$.

Theorem 1.2 (Implicit Function Theorem)**.** *Assume* $f \in C^1(U;\mathbb{R}^m)$ *and*

$$J_y f(x_0, y_0) \neq 0.$$

Then there exists an open set $V \subset U$*, with* $(x_0, y_0) \in V$*, an open set* $W \subset \mathbb{R}^n$*, with* $x_0 \in W$*, and a* C^1 *mapping* $g : W \longrightarrow \mathbb{R}^m$ *such that :*

i) $g(x_0) = y_0$,

ii) $f(x, g(x)) = z_0 \quad (x \in W)$,

iii) if $(x, y) \in V$ *and* $f(x, y) = z_0$*, then* $y = g(x)$*,*

iv) if $f \in C^k$*, then* $g \in C^k \quad (k = 2, \dots)$.

The function g *is implicitly defined near* x_0 *by equation* $f(x, y) = z_0$*.*

Proof. See [14]. □

Theorem 1.3 (Rademacher's Theorem). *Let u be a locally Lipschitz continuous function in U. Then u is differentiable almost everywhere in U.*

Proof. See [14]. □

Theorem 1.4 (Arzela-Ascoli Compactness Criterion). *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of real-valued functions defined on $\mathbb{R}^n$, such that*

$$|f_k(x)| \leq M, \quad \forall x \in \mathbb{R}^n$$

for some constant M, and $\{f_k\}_{k=1}^{\infty}$ are uniformly equicontinuous. Then there exist a subsequence $\left\{f_{k_j}\right\}_{j=1}^{\infty} \subseteq \{f_k\}_{k=1}^{\infty}$ and a continuous function f, such that

$$f_{k_j} \longrightarrow f \qquad \text{uniformly on compact subsets of } \mathbb{R}^n.$$

Proof. See [14]. □

Theorem 1.5 (Lebesgue Density Theorem). *Let $\mu : \mathcal{B}(\mathbb{R}^n) \longrightarrow [0,\infty]$ be a Radon measure and let $E \subset \mathbb{R}^n$ be a Borel set, where $\mathcal{B}(\mathbb{R}^n)$ is the Borel $\sigma-$algebra on $\mathbb{R}^n$. Then there exists a Borel set $M \subset \mathbb{R}^n$, with $\mu(M) = 0$, such that for every $x \in \mathbb{R}^n \setminus M$,*

$$\lim_{r \longrightarrow 0^+} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} = \chi_E(x),$$

where χ_E is the characteristic function of E.

Proof. See [15]. □

Definition 1.2. A point $x \in E$ for which the previous limit is 1 is called *a point of density* 1 for E.

More generally, for any $t \in [0,1]$ a point $x \in \mathbb{R}^n$ such that

$$\lim_{r \longrightarrow 0^+} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} = t,$$

is called *a point of density t* for E.

Theorem 1.6 (Lebesgue Differentiation Theorem). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally summable.*

(i) Then for a.e. point $x_0 \in \mathbb{R}^n$,

$$\lim_{r \longrightarrow 0} \frac{1}{B(x_0, r)} \int_{B(x_0,r)} f\, dx = f(x_0).$$

(ii) In fact, for a.e. $x_0 \in \mathbb{R}^n$,

$$\lim_{r \longrightarrow 0} \frac{1}{B(x_0, r)} \int_{B(x_0,r)} |f(x) - f(x_0)|\, dx = 0. \tag{1.1.1}$$

Proof. See [14]. □

Definition 1.3. A point x_0 at which (1.1.1) holds is called a *Lebesgue point* of f.

Lemma 1.1 (Gronwall's inequality -differential form). *(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\, \eta(t) + \psi(t) \tag{1.1.2}$$

where ϕ and ψ are nonnegative, summable functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)\, ds} \left[\eta(0) + \int_0^t \psi(s)\, ds\right] \tag{1.1.3}$$

for all $0 \leq t \leq T$.

(ii) In particular, if

$$\eta' \leq \phi\eta \quad \text{and } \eta(0) = 0,$$

then

$$\eta = 0 \text{ on } [0, T].$$

Proof. See [14]. □

Lemma 1.2 (Gronwall's inequality−integral form). *(i) Let ξ be a nonnegative, summable function on $[0, T]$ which satisfies for a.e. t the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s)\, ds + C_2 \tag{1.1.4}$$

for constants $C_1, C_2 \geq 0$. *Then*

$$\xi(t) \leq C_2 \left(1 + C_1 t e^{C_1 t}\right)$$

for a.e. $0 \leq t \leq T$.

(ii) In particular , if

$$\xi(t) \leq C_1 \int_0^t \xi(s)\, ds$$

for a.e. $0 \leq t \leq T$, *then*

$$\xi(t) = 0 \text{ a.e.}$$

Proof. See [14]. □

Lemma 1.3 (Gronwall's Lemma)**.** *Let us consider a function* $x : [a, b] \longrightarrow \mathbb{R}^n$ *satisfying*

$$|\dot{x}(t)| \leq \gamma |x(t)| + c(t) \ \ a.e., \quad t \in [a, b],$$

where γ *is a nonnegative constant and where* $c(\cdot) \in L^1[a, b]$, *the space of integrable function from* $[a, b]$ *to* $\mathbb{R}$. *Then, for all* $t \in [a, b]$, *we have*

$$|x(t) - x(a)| \leq \left(e^{\gamma(t-a)} - 1\right) |x(a)| + \int_a^t e^{\gamma(t-s)} c(s)\, ds.$$

If in particular the function c is constant and $\gamma > 0$, *then*

$$|x(t) - x(a)| \leq \left(e^{\gamma(t-a)} - 1\right) (|x(a)| + c/\gamma).$$

Proof. See [11]. □

Theorem 1.7 (Cauchy-Lipschitz theorem or Picard's Theorem)**.** *Let us consider the ordinary differential equation* (E):

$$(E) \qquad \begin{cases} \frac{d}{dt} x(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

where
$f : U \longrightarrow \mathbb{R}^n$ *is continuous and locally Lipschitz with respect to its variable belonging to* $\mathbb{R}^n$, $U \subset \mathbb{R} \times \mathbb{R}^n$ is an open subset, $(t_0, x_0) \in U$.

Then (E) *admits a unique maximal solution of class* C^1.

Proof. See [8]. □

We consider now a similar case to the previous one, where $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Let us consider the following ordinary differential equation

$$(CE) \qquad \begin{cases} \frac{d}{dt}x(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

We consider that a solution $x(\cdot)$ of (CE) is an absolutely continuous function $x : [a, b] \longrightarrow \mathbb{R}^n$, in which the derivative with respect to t, satisfies (CE).

Theorem 1.8. *Suppose that f is continuous, and let $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ be given. Then the following hold :*

1. *there exists a solution of (CE) on an open interval $]t_0 - \delta, t_0 + \delta[$, for some $\delta > 0$ satisfying $x(t_0) = x_0$,*
2. *if in addition, we assume that there exist non negative constants λ and θ such that*
$$|f(t, x)| \leq \lambda |x| + \theta \quad \forall (t, x),$$
then there exists a solution of (CE) in $\mathbb{R}$ such that $x(t_0) = x_0$,
3. *moreover if f is locally Lipschitz, then there exists a unique solution of (CE) on $\mathbb{R}$ such that $x(t_0) = x_0$.*

Proof. See [11]. □

A first-order nonlinear partial differential equation (PDE) is an expression of the form

$$F(x, u, Du) = 0 \tag{1.1.5}$$

where $x \in U$ and U is an open subset of $\mathbb{R}^n$. Here

$$F : \overline{U} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

is given, the function $u : \overline{U} \longrightarrow \mathbb{R}$, in $C^1(U)$, is the unknown, $u = u(x)$ and

$$Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$$

the gradient.

Let us write

$$F = F(x, z, p) = F(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$$

for $x \in U$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$.

Thus z is the variable for which we substitute $u(x)$,and p is the name of the variable for which we substitute the gradient $Du(x)$. We also assume that F is smooth, and we set

$$\begin{cases} D_p F = \left(F_{p_1}, \ldots, F_{p_n}\right) \\ D_z F = F_z \\ D_x F = \left(F_{x_1}, \ldots, F_{x_n}\right) \end{cases}$$

where $F_{p_i} = \frac{\partial F}{\partial p_i}$, $F_{x_i} = \frac{\partial F}{\partial x_i}$ and $F_z = \frac{\partial F}{\partial z}$ $\qquad (i = 1, 2, \ldots n)$.

We study here the nonlinear first-order PDE. (1.1.5) is subject to the boundary condition

$$u = g \quad \text{on } \Gamma, \tag{1.1.6}$$

where $\Gamma \subseteq \partial U$ and $g : \Gamma \longrightarrow \mathbb{R}$ are given. We suppose that F and g are smooth functions.

1.2 Derivation of characteristic ODE

We develop here the method of characteristics, which allows to study (1.1.5), (1.1.6) by converting the PDE into an appropriate system of ordinary differential equations. We would like to calculate $u(x)$ by finding some curve lying within U, connecting x with a point $x^0 \in \Gamma$ and along which we can compute u. Since (1.1.6) says $u = g$ on Γ, we know the value of u at the one end x^0.

Let us suppose that this curve is described parametrically by the function $x(s) = \left(x^1(s), \ldots, x^n(s)\right)$, the parameter s being in some subinterval of $\mathbb{R}$. Assuming also that u is a C^2 solution of (1.1.5), we define

$$z(s) = u(x(s)). \tag{1.2.1}$$

In addition, we set

$$p(s) := Du(x(s)); \tag{1.2.2}$$

that is, $p(s) = \left(p^1(s), \dots, p^n(s)\right)$, where

$$p^i(s) = \frac{\partial u(x(s))}{\partial x_i} := u_{x_i}(x(s)) \qquad (i = 1, \dots, n). \tag{1.2.3}$$

We must choose the function $x(\cdot)$ in such a way that we can compute $z(\cdot)$ and $p(\cdot)$. Then we differentiate (1.2.3)

$$\dot{p}^i(s) = \frac{dp^i(s)}{ds} = \sum_{j=1}^{n} u_{x_i x_j}(x(s))\, \dot{x}^j(s). \tag{1.2.4}$$

Now we differentiate the PDE (1.1.5) with respect to x_i:

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_i}(x, u, Du)\, u_{x_i x_j} + \frac{\partial F}{\partial z}(x, u, Du)\, u_{x_i} + \frac{\partial F}{\partial x_i}(x, u, Du) = 0. \tag{1.2.5}$$

To get rid of the second order derivative terms in (1.2.4) we set

$$\dot{x}^j(s) = \frac{\partial F}{\partial p_j}(x(s), z(s), p(s)) \qquad (j = 1, \dots, n). \tag{1.2.6}$$

Assuming now that (1.2.6) holds, we evaluate (1.2.5) at $x = x(s)$, obtaining thereby from (1.2.1), (1.2.2) the identity:

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_i}(x(s), z(s), p(s))\, u_{x_i x_j}(x(s)) \\ + \frac{\partial F}{\partial z}(x(s), z(s), p(s))\, p^i(s) + \frac{\partial F}{\partial x_i}(x(s), z(s), p(s)) = 0.$$

Substitute this expression and (1.2.6) into (1.2.4):

$$\dot{p}^i(s) = -\frac{\partial F}{\partial z}(x(s), z(s), p(s))\, p^i(s) \\ - \frac{\partial F}{\partial x_i}(x(s), z(s), p(s)) \qquad (i = 1, \dots, n). \tag{1.2.7}$$

Finally we differentiate (1.2.1):

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(x(s))\,\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s)\,\frac{\partial F}{\partial p_j}(x(s), z(s), p(s)), \quad (1.2.8)$$

the second equality follows by (1.2.3) and (1.2.6).

We summarize by writing equations $(1.2.6)-(1.2.8)$ in vector notation:

$$\begin{cases} \dot{p}(s) = -D_z F(x(s), z(s), p(s)) - D_z F(x(s), z(s), p(s))\,p(s) \\ \dot{z}(s) = D_p F(x(s), z(s), p(s)) \cdot p(s) \\ \dot{x}(s) = D_p F(x(s), z(s), p(s)). \end{cases} \quad (1.2.9)$$

This system of $2n+1$ first-order ordinary differential equations is *the characteristic equations* of the nonlinear first-order PDE (1.1.5). We have:

Theorem 1.9. *Let $u \in C^2(U)$ solves the nonlinear first-order partial differential equation* (1.1.5) *in U. Assume $x(\cdot)$ solves the ordinary differential equation* ((1.2.9), *where $p(\cdot) = Du(x(\cdot))$, $z(\cdot) = u(x(\cdot))$. Then $p(\cdot)$ solves the ODE* (1.2.9) *and $z(\cdot)$ solves the ODE* (1.2.9), *for those s such that $x(s) \in U$.*

Proof. See [14]. □

We will need to find appropriate initial conditions for the system of ODE (1.2.9), in order that this theorem be useful.

1.3 Boundary conditions

1.3.1 Straightening the boundary

We use the characteristic ODE (1.2.9) to solve the boundary-value problem (1.1.5), (1.1.6), at least in a small region near an appropriate portion Γ of ∂U. In order to facilitate the relevant calculation, it is convenient first to change variables. To accomplish this, we first fix any point $x^0 \in \partial U$. Then using notation from the section 1.1, we find smooth mappings Φ, $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\Phi = \Psi^{-1}$ and Φ straightens out ∂U near x^0.

Given any function $u : U \longrightarrow \mathbb{R}$, let us write $V := \Phi(U)$ and set

$$v(y) := u(\Psi(y)), \quad y \in V. \quad (1.3.1)$$

Then

$$u(x) = v(\Phi(x)), \quad x \in U. \tag{1.3.2}$$

Now suppose that u is a C^1 solution of our boundary-value problem (1.1.5), (1.1.6) in U.

According to (1.3.2) we see that

$$Du(x) = Dv(\Phi(x))\, D\Phi(x).$$

Thus (1.1.5) implies

$$\begin{aligned} 0 &= F(x, u(x), Du(x)) \\ &= F(\Psi(y), v(y), Dv(y)\, D\Phi(x)). \end{aligned}$$

This is an expression having the form

$$G(y, v(y), Dv(y)) = 0 \text{ in } V.$$

In addition $v = h$ on Δ, where $\Delta := \Phi(\Gamma)$ and $h(y) := g(\Psi(y))$.

In summary, our problem (1.1.5)-(1.1.6) can be transformed to read as

$$\begin{cases} G(y, v, Dv) &= 0 \qquad \text{in } V \\ v &= h \qquad \text{on } \Delta, \end{cases}$$

for G, h as above. Then the boundary-value problem can be transformed into a problem having the same form, if we change variables to flat the boundary near x^0.

1.3.2 Compatibility conditions on boundary data

In view of the previous results, if we are given a point $x^0 \in \Gamma$ we may as well assume at the outset that Γ is flat near x^0, lying in the plane $\{x_n = 0\}$.

We intend now to construct a solution of PDE (1.1.5), (1.1.6), using the characteristic ODE, and for this we must find the appropriate initial conditions

$$x(0) = x^0, \quad z(0) = z^0, \quad p(0) = p^0. \tag{1.3.3}$$

It is clear that if $x(\cdot)$ passes through x^0, we should require that

$$z^0 = g\left(x^0\right). \tag{1.3.4}$$

Since (1.1.6) implies $u\left(x_1, \ldots, x_{n-1}, 0\right) = g\left(x_1, \ldots, x_{n-1}\right)$ near x^0, we may differentiate to find

$$u_{x_i}\left(x^0\right) = g_{x_i}\left(x^0\right) \qquad (i = 1, \ldots, n-1).$$

As we also want the PDE (1.1.5) to hold, we should require $p^0 = \left(p_1^0, \ldots, p_n^0\right)$ to satisfy the relations:

$$\begin{cases} p_i^0 = g_{x_i}\left(x^0\right) & (i = 1, \ldots, n-1) \\ F\left(x^0, z^0, p^0\right) = 0. \end{cases} \tag{1.3.5}$$

We call (1.3.4) and (1.3.5) the compatibility conditions.

Definition 1.4. A triplet $\left(x^0, z^0, p^0\right) \in \mathbb{R}^{2n+1}$ satisfying (1.3.4), (1.3.5) is admissible.

Note that a vector satisfying (1.3.5) may not exist or may not be unique.

1.3.3 Noncharacteristic boundary data

So now assume as above that $x^0 \in \Gamma$, that Γ near x^0 lies in the plane $\{x_n = 0\}$, and that the triplet $\left(x^0, z^0, p^0\right)$ is admissible. We are planning to construct a solution u of (1.1.5), (1.1.6) in U near x^0 by integrating the characteristic ODE (1.2.9). So far we have ascertained $x(0) = x^0$, $z(0) = z^0$, $p(0) = p^0$ are appropriate boundary conditions for the characteristic ODE, with $x(\cdot)$ intersecting Γ at x^0. But we need to solve these ODE for nearby initial points as well, and must consequently ask if we can somehow appropriately perturb $\left(x^0, z^0, p^0\right)$, keeping the compatibility conditions.

In order words, given a point $y = \left(y_1, \ldots, y_{n-1}, 0\right) \in \Gamma$, with y close to x^0, we intend to solve the characteristic ODE

$$\begin{cases} \dot{p}(s) = -D_z F(x(s), z(s), p(s)) - D_z F(x(s), z(s), p(s))\, p(s) \\ \dot{z}(s) = D_p F(x(s), z(s), p(s)) \cdot p(s) \\ \dot{x}(s) = D_p F(x(s), z(s), p(s)), \end{cases} \tag{1.3.6}$$

with the initial conditions

$$x(0) = y, \qquad z(0) = g(y), \qquad p(0) = q(y). \tag{1.3.7}$$

Our task then is to find a function $q(\cdot) = \left(q^1(\cdot), \ldots, q^n(\cdot)\right)$, so that

$$q\left(x^0\right) = p^0 \tag{1.3.8}$$

and $(y, g(y), q(y))$ is admissible; that is, the compatibility conditions

$$\begin{cases} q^i(y) = g_{x_i}(y) & (i = 1, \ldots, n-1) \\ F(y, g(y), q(y)) = 0 \end{cases} \tag{1.3.9}$$

hold for all $y \in \Gamma$ close to x^0.

Lemma 1.4. *There exists a unique solution $q(\cdot)$ of (1.3.8), (1.3.9) for all $y \in \Gamma$ sufficiently close to x^0, provided*

$$F_{p_n}\left(x^0, z^0, p^0\right) \neq 0. \tag{1.3.10}$$

Proof. See [14]. □

1.4 Local solution

Remember that our aim is to use the characteristic ODE to build a solution u of (1.3.10) and (1.1.6), at least near Γ. So as before we choose a point $x^0 \in \Gamma$ and, as shown in subsection 1.3, we may as well assume that near x^0 the surface Γ is flat, lying in the plane $\{x_n = 0\}$. Suppose further that $\left(x^0, z^0, p^0\right)$ is an admissible triplet of boundary data, which is noncharacteristic. According to lemma 1.4 there is a solution $q(\cdot)$ so that $p^0 = q\left(x^0\right)$ and the triplet $(y, g(y), q(y))$ is admissible, for all y sufficiently close to x^0.

Given any such point $y = (y_1, \ldots, y_{n-1}, 0)$, we solve the characteristic ODE (1.3.6), subject to initial conditions (1.3.7).

Notation 1.3. Let us write

$$\begin{cases} p\,(s) = p\,(y,s) = p\,(y_1,\ldots,y_{n-1},s) \\ z\,(s) = z\,(y,s) = z\,(y_1,\ldots,y_{n-1},s) \\ x\,(s) = x\,(y,s) = x\,(y_1,\ldots,y_{n-1},s) \end{cases}$$

to display the dependence of the solution of (1.3.6) and (1.3.7) on s and y.

Lemma 1.5. *Assume we have the noncharacteristic condition $F_{p_n}\left(x^0,z^0,p^0\right) \neq 0$. Then there exist an open interval $I \subset \mathbb{R}$ containing 0, a neighborhood W of x^0 in $\Gamma \subset \mathbb{R}^{n-1}$, and a neighborhood V of x^0 in $\mathbb{R}^n$, such that for each $x \in V$ there exist unique $s \in I$, $y \in W$ such that*

$$x = x\,(y,s)\,.$$

The mappings $x \mapsto s, y$ are of class C^2.

Proof. See [14]. □

Using Lemma 1.5 for each $x \in V$, we can locally uniquely solve the equation

$$\begin{cases} v = x\,(y,s)\,, \\ \text{for } y = y\,(v)\,, \quad s = s\,(v)\,. \end{cases} \tag{1.4.1}$$

Finally, let us define

$$\begin{cases} u\,(v) := z\,(y(v), s\,(v)) \\ p\,(v) := p\,(y(v), s\,(v)) \end{cases} \tag{1.4.2}$$

for $v \in V$ and s, y as in (1.4.1).

We come finally to our principal assertion, namely, we can locally weave together the solutions of the characteristic ODE into a solution of the PDE.

Theorem 1.10. *The function u defined above is C^2 and solves the PDE*

$$F\,(x, u\,(x), Du\,(x)) = 0 \quad (x \in V)\,,$$

with the boundary condition

$$u\,(x) = g\,(x) \quad (x \in \Gamma \cap V)\,.$$

Proof. See [14]. □

CHAPTER 2

Minimax and viscosity solutions of Hamilton-Jacobi equations

In this chapter we present the viscosity method of continuous solutions of Hamilton-Jacobi equations in Section 2.1, inspired by [3], and the minimax method of solutions of Hamilton-Jacobi equations in Section 2.2, [38]. My contribution has been to elaborate a clear and concise presentation of the different subjects.

2.1 Continuous viscosity solutions of Hamilton -Jacobi equations

This section is devoted to some aspects of the basic theory of continuous viscosity solutions of first order partial differential equations of Hamilton-Jacobi type, also called Hamilton-Jacobi equations

$$H\left(x, u\left(x\right), Du\left(x\right)\right) = 0, \qquad x \in \Omega \tag{2.1.1}$$

where Ω is an open domain of $\mathbb{R}^N$, the called Hamiltonian $H = H(x, r, p)$ is a continuous real-valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, N is a nonnegative integer.

Remark 2.1. In the pioneering paper [19], Grandall and Lions called Hamilton-Jacobi equation the problem (2.1.2) or (2.1.3) which are global nonlinear first order partial differential equations; this concept became a generic name for each differential equation of this type, not only associated to the well-known Hamilton-Jacobi equation in optic and mechanic:

$$\begin{cases} H(x, u, Du) = 0 & \text{in } \Omega \\ u = Z & \text{on } \partial\Omega \end{cases} \tag{2.1.2}$$

or

$$\begin{cases} u_t + H(x, t, u, Du) = 0 & \text{in } \Omega \\ u = Z & \text{on } \partial\Omega \times]0, T[\end{cases} \tag{2.1.3}$$

In (2.1.2) we have a stationary partial differential equation, and in (2.1.3) an evolution partial differential equation.

2.1.1 Definitions and basic properties

We introduce two equivalent definitions of viscosity solution of (2.1.1) and present some of their basic properties.

Definition 2.1. A function $u \in C(\Omega)$ is a viscosity subsolution of (2.1.1) if for any $\varphi \in C^1(\Omega)$

$$H(x_0, u(x_0), D\varphi(x_0)) \leq 0 \tag{2.1.4}$$

at any local maximum point $x_0 \in \Omega$ of $u - \varphi$.

Similarly a function $u \in C(\Omega)$ is a viscosity supersolution of (2.1.1) if for any
$\varphi \in C^1(\Omega)$

$$H(x_1, u(x_1), D\varphi(x_1)) \geq 0 \tag{2.1.5}$$

at any local minimum point $x_1 \in \Omega$ of $u - \varphi$.

Finally a function $u \in C(\Omega)$ is a viscosity solution of (2.1.1) if it is simultaneously a viscosity sub- and supersolution.

Remark 2.2. Let us mention explicitly that the definition 2.1 also apply to evolution Hamilton-Jacobi equations of the form

$$u_t + H(y,t,u,Du) = 0 \quad (y,t) \in D\times]0,T[\,.$$

Indeed the equation above is reduced to the form (2.1.1) by

$$x = (y,t) \in D\times]0,T[\subset \mathbb{R}^{N+1}, \tilde{H}(x,u,q) = q_{N+1} + H(x,u,q_1,\ldots,q_N)$$

with $q = (q_1,\ldots,q_{N+1}) \in \mathbb{R}^{N+1}$.

Remark 2.3. In the definition of subsolution we can always assume that x_0 is a strict local maximum point for $u - \varphi$ (otherwise replace $\varphi(x)$ by $\varphi(x) + |x - x_0|^2$). Moreover since (2.1.4) depends only on the value of $D\varphi$ at x_0, it is not restrictive to assume that $u(x_0) = \varphi(x_0)$. Similar remarks apply of course to the definition of supersolution.

We note also that the space $C^1(\Omega)$ of test functions of definition 2.1 can be replaced by $C^\infty(\Omega)$: this can be shown by a density argument.

The following proposition explains the local character of the notion of viscosity solution and its consistency with the classical pointwise definition.

Proposition 2.1.

a) *If $u \in C(\Omega)$ is a viscosity solution of (2.1.1) in Ω then u is a viscosity solution of (2.1.1) in $\mathcal{O}$ for any open set $\mathcal{O} \subset \Omega$.*

b) *If u is a classical solution of (2.1.1), that is, u is differentiable at any point $x \in \Omega$ and*

$$H(x,u(x),Du(x)) = 0\,, \forall x \in \Omega\,, \tag{2.1.6}$$

then u is a viscosity solution of (2.1.1) in Ω.

c) *If $u \in C^1(\Omega)$ is a viscosity solution of (2.1.1), then u is a classical solution (2.1.1).*

Proof. See [3]. □

The definition of viscosity solution is closely related to two properties that are typical in the theory of elliptic and parabolic equations,[3], namely the *maximum principle* (MP) and the *comparison principle* (CP). For the equation (2.1.1) these properties can be respectively formulated as follows.

Definition 2.2. A function $u \in C(\Omega)$ satisfies the comparison principle with smooth strict supersolutions, briefly (CP), if for any $\varphi \in C^1(\Omega)$ and $\mathcal{O}$ open subset of Ω,

$$\begin{cases} H(x, u(x), D\varphi(x)) > 0 & \text{in } \mathcal{O} \\ u \leq \varphi & \text{on } \partial\mathcal{O}, \end{cases}$$

implies

$$u \leq \varphi \text{ in } \mathcal{O}.$$

We say that $u \in C(\Omega)$ satisfies the maximum principle (MP) if for any $\varphi \in C^1(\Omega)$ and $\mathcal{O}$ open subset of Ω the inequality

$$H(x, u(x), D\varphi(x)) > 0 \text{ in } \mathcal{O}$$

implies that

$$u - \varphi \text{ cannot have a nonnegative maximum in } \mathcal{O}.$$

It is quite clear that the maximum principle implies the comparison principle. The connection with the viscosity subsolution of (2.1.1) is expressed by the following result.

Proposition 2.2. *If $u \in C(\Omega)$ satisfies the (CP) then u is a viscosity subsolution of (2.1.1). Conversely if u is a viscosity subsolution of (2.1.1) and $r \mapsto H(x, r, p)$ is nondecreasing for all x,p, then u satisfies (MP) and (CP).*

Proof. See [3]. □

Remark 2.4. A similar result holds for supersolutions, provided all inequalities are reversed in (CP), (MP) and nonnegative maximum is replaced by nonpositive minimum.

Example 2.1. The function $u(x) = |x|$ is a viscosity solution of 1-dimensional equation

$$-\left|u'(x)\right| + 1 = 0 \quad x \in]-1, 1[\,.$$

To check this, notice that if $x \neq 0$ is a local extremum for $u - \varphi$, then $u'(x) = \varphi'(x)$. Therefore, at those points both the supersolution and the subsolution conditions are trivially satisfied. Also, if 0 is a local minimum for $u - \varphi$, a simple calculation shows that $|\varphi'(0)| \leq 1$ and the supersolution condition holds. To conclude is enough to observe that 0 cannot be a local maximum for $u - \varphi$ with $\varphi \in C^1(]-1, 1[)$(this would imply $-1 \geq \varphi'(0) \geq 1$).

On the other hand, $u(x) = |x|$ is not a viscosity solution of

$$\left|u'(x)\right| - 1 = 0 \quad x \in]-1,1[\,.$$

Actually, the supersolution condition is not fulfilled at $x_0 = 0$ which is a local minimum for $|x| - \left(-x^2\right)$.

We describe now an alternative way of defining viscosity solution of equation (HJ) and prove the equivalence of the new definition with the one given previously.

Definition 2.3. Let us associate with a function $u \in C(\Omega)$ the sets

$$D^+u(x) = \left\{ p \in \mathbb{R}^N : \limsup_{y \longrightarrow x,\, y\in\Omega} \frac{u(y) - u(x) - p\cdot(y-x)}{|y-x|} \leq 0 \right\},$$

$$D^-u(x) = \left\{ p \in \mathbb{R}^N : \liminf_{y \longrightarrow x,\, y\in\Omega} \frac{u(y) - u(x) - p\cdot(y-x)}{|y-x|} \geq 0 \right\}.$$

These sets are called respectively the super- and the subdifferential of u at x. The next lemma in [3] provides a description of $D^+u(x)$ and $D^-u(x)$ in terms of test functions.

Lemma 2.1. *Let $u \in C(\Omega)$. Then:*

a) $p \in D^+u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at x.

b) $p \in D^-u(x)$ if and only if there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local minimum at x.

Proof. See [3]. □

The following lemma in [3] collects some properties of super- and subdifferential.

Lemma 2.2. *Let $u \in C(\Omega)$ and $x \in \Omega$. Then*

a) $D^+u(x)$ and $D^-u(x)$ are closed convex (possibly empty) subsets of $\mathbb{R}^N$.

b) If u is differentiable at x, then $\{Du(x)\} = D^+u(x) = D^-u(x)$.

c) *If for some x both $D^+u(x)$ and $D^-u(x)$ are nonempty; then*

$$D^+u(x) = D^-u\,(x) = \{Du(x)\}\ .$$

Proof. See [3]. □

As a direct consequence of lemma 2.1 the following new definition of viscosity solution turns out to be equivalent to the initial one.

Definition 2.4. A function $u \in C\,(\Omega)$ is
a viscosity subsolution of (2.1.1) if

$$H(x, u(x), p) \leq 0 \qquad \forall x \in \Omega\,, \qquad \forall p \in D^+u(x), \tag{2.1.7}$$

a viscosity supersolution of (2.1.1) if

$$H(x, u(x), p) \geq 0 \qquad \forall x \in \Omega\,, \qquad \forall p \in D^-u(x). \tag{2.1.8}$$

u is a viscosity solution if (2.1.7) and (2.1.8) hold simultaneously.

The above definition, which is more in the spirit of nonsmooth analysis, is sometimes more easier to handle than the previous one; it is generally employed in the proofs of some important properties of viscosity solutions.

Now we present a consistency result that improves Proposition 2.2.

Proposition 2.3.

a) *If u is a viscosity solution of (2.1.1) then*

$$H\,(x, u\,(x), Du(x)) = 0$$

at any point where u is differentiable;

b) *If u is locally Lipschitz continuous and if it is viscosity solution of (2.1.1), then*

$$H\,(x, u\,(x), Du(x)) = 0$$

almost everywhere in Ω.

Proof. See [3]. □

Remark 2.5. Part (b) of Proposition 2.3 says that any viscosity solution of (2.1.1) is also a generalized solution i.e locally Lipschitz continuous function such that

$$H\left(x, u\left(x\right), Du(x)\right) = 0 \quad \text{a.e in } \Omega.$$

The next result is on the stability with respect to the lattice operations on $C\left(\Omega\right)$,

$$\begin{aligned} \left(u \vee v\right)\left(x\right) &= \max\left\{u\left(x\right), v\left(x\right)\right\}, \\ \left(u \wedge v\right)\left(x\right) &= \min\left\{u\left(x\right), v\left(x\right)\right\}. \end{aligned}$$

Proposition 2.4.

a) Let $u, v \in C(\Omega)$ be viscosity subsolutions of (2.1.1)*; then $u \vee v$ is a viscosity subsolution of (2.1.1);*

b) Let $u, v \in C(\Omega)$ be viscosity supersolutions of (HJ); then $u \wedge v$ is a viscosity supersolution of (2.1.1);

c) Let $u \in C\left(\Omega\right)$ be a viscosity subsolution of (2.1.1) such that $u \geq v$ for any viscosity subsolution v of (2.1.1), then u is a viscosity supersolution and therefore a viscosity solution of (2.1.1).

Proof. See [3]. □

The next result is a stability result in the topology of $C\left(\Omega\right)$; a particularity of viscosity solution. The next lemma is useful to establish this result.

Lemma 2.3. *Let $v \in C\left(\Omega\right)$ and suppose that $x_0 \in \Omega$ is a strict maximum point for v in $B\left(x_0; \delta\right) \subset \Omega$. If $v_n \in C\left(\Omega\right)$ converges locally uniformly to v in Ω, then there exists a sequence $\{x_n\}$ such that*

$$x_n \longrightarrow x \quad v_n\left(x_n\right) \geq v\left(x\right) \quad \forall x \in \overline{B}\left(x_0, \delta\right).$$

Proof. See [3]. □

Proposition 2.5. *Let $u_n \in C\left(\Omega\right)$ $(n \in \mathbb{N})$ be a viscosity solution of*

$$(HJ)_n \qquad\qquad H_n(x, u_n(x), Du_n(x)) = 0 \text{ in } \Omega.$$

Assume that

$$u_n \longrightarrow u \quad \text{locally uniformly in } \Omega$$
$$H_n \longrightarrow H \quad \text{locally uniformly in } \Omega \times \mathbb{R} \times \mathbb{R}^N .$$

Then u is a viscosity solution of (2.1.1) in Ω .

Proof. See [3]. □

2.1.2 Comparison and uniqueness results

Here we address the problem of comparison of viscosity subsolution and viscosity supersolution, and uniqueness of viscosity solutions. There is in fact no general results about comparison and uniqueness of viscosity solutions: these results depend generally of the type of Hamilton-Jacobi equation. The result selected here is not the most general; it is given to show the main ideas involved in the proofs of other comparison and uniqueness results.

We restrict our attention to the case $H(x,r,p) = r + F(x,p)$; the result holds however for a general H provided that $r \mapsto H(x,r,p)$ is strictly increasing and some special H independent of r .

Theorem 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$.*

Assume that $u_1, u_2 \in C(\Omega)$ are , respectively, viscosity sub- and supersolution of

$$u(x) + F(x, Du(x) = 0 \quad , x \in \Omega \tag{2.1.9}$$

and

$$u_1 \leq u_2 \text{ on } \partial\Omega . \tag{2.1.10}$$

Assume also that F satisfies

$$(F_1) \qquad |F(x,p) - F(y,p)| \leq \omega_1 (|x-y| (1+|p|)) \ ,$$

for $x, y \in \Omega, p \in \mathbb{R}^N$, where $\omega_1 : [0,\infty[\longrightarrow [0,\infty[$ is continuous nondecreasing with $\omega_1(0) = 0$. Then

$$u_1 \leq u_2$$

in Ω.

Proof. See [3]. □

Remark 2.6. If u_1, u_2 are both viscosity solutions of (2.1.9) with $u_1 = u_2$ in $\partial\Omega$, from Theorem 2.1 it follows that $u_1 = u_2$ in $\overline{\Omega}$.

Remark 2.7. In this section, we have not given any existence result, similar to comparison and uniqueness results, these ones depend also on the type of Hamilton-Jacobi equation involved. In fact, there is no a general result of this type. The best approach to the existence result is the so called Perron's method, established by Ishii in [23].

2.2 Minimax solutions of Hamilton-Jacobi equations

This section is devoted to some aspects of the basic theory of continuous minimax solutions of first order partial differential equations of Hamilton-Jacobi equation

$$H(x, u(x), Du(x)) = 0, \qquad x \in \Omega \tag{2.2.1}$$

where Ω is an open domain of $\mathbb{R}^N$ and the Hamiltonian $H = H(x, r, p)$ is a continuous real-valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, N is a nonnegative integer. Here $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$ is the gradient.

Remark 2.8. Particularly the Hamilton-Jacobi equation has the form

$$\frac{\partial u}{\partial t} + H(t, x, u, Du) = 0, \quad (t, x) \in G :=]0, \theta[\times \mathbb{R}^n . \tag{2.2.2}$$

We assume that the function $(t, x, z, s) \mapsto H(t, x, z, s)$ is continuous on $]0, \theta[\times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and satisfies the Lipschitz conditions in the variable p

$$|H(t, x, z, p) - H(t, x, z, q)| \leq \rho(t, x, z)\, |p - q| \tag{2.2.3}$$

for all $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n$. Here the function ρ is continuous on $G \times \mathbb{R}$. It is supposed also that the function $z \mapsto H(t, x, z, s)$ is nonincreasing.

In subsections 2.2.1-2.2.6, some useful notions, coming from [38], are given in order to introduce of the notion of minimax solution.

2.2.1 Multifunctions

Let X and Y be metric spaces with both metrics denoted by dist$(\cdot,\cdot)$. For subset $C \subset Y$ and a point $y \in Y$, we let dist $(y;C)$ denote the number $\inf\{\text{dist }(y,\xi) : \xi \in C\}$. For $\varepsilon \geq 0$, we denote by C^{ε} the closed neighborhood of C, defined by

$$C^{\varepsilon} = \{y \in Y : \text{dist}\,(y;C) \leq \varepsilon\}\,.$$

Definition 2.5. A multifunction $\Phi : X \longrightarrow 2^{Y}$(set-valued mapping or multivalued function) is a mapping which assigns to each point $x \in X$ a set $\Phi(x) \subset Y$.

Consider the set

$$\begin{aligned} \text{gr}\,(\Phi) &= \{(x,y) : x \in X, y \in \Phi(x)\}\,, \\ \text{dom}\,\Phi &= \{x \in X : \Phi(x) \neq \emptyset\}\,, \end{aligned}$$

which are called respectively the *graph* and the *effective set* of the multifunction Φ . For a subset $D \subset X$, we let $\Phi(D)$ denote the set $\cup_{y\in D}\Phi(y)$.

Definition 2.6. Let $\Phi : X \longrightarrow 2^{Y}$ be a given multifunction and $x_0 \in X$. An upper topological limit is the set denoted by

$$\limsup_{x \longrightarrow x_0} \Phi(x) = \left\{ y \in Y : \liminf_{x \longrightarrow x_0} \text{dist }(y;\Phi(x)) = 0 \right\}.$$

A lower topological limit is the set

$$\liminf_{x \longrightarrow x_0} \Phi(x) = \left\{ y \in Y : \limsup_{x \longrightarrow x_0} \text{dist }(y;\Phi(x)) = 0 \right\}.$$

Remark 2.9. The upper and lower topological limits are closed sets. One also has:

$$\liminf_{x \longrightarrow x_0} \Phi(x) \subset \limsup_{x \longrightarrow x_0} \Phi(x)\,.$$

Definition 2.7. A multifunction Φ is called upper (resp. lower) semicontinuous at a point $x_0 \in X$ if the inclusion (2.2.4)[resp. (2.2.5)]

$$\limsup_{x \longrightarrow x_0} \Phi(x) \subset \Phi(x_0), \tag{2.2.4}$$

$$\left(\text{resp.} \liminf_{x \longrightarrow x_0} \Phi(x) \supset \Phi(x_0)\right) \tag{2.2.5}$$

holds.

Definition 2.8. A multifunction Φ is called closed if it is upper semicontinuous at any point $x_0 \in X$.

Definition 2.9. A multifunction is called continuous at a point x if it is simultaneously upper and lower semi-continuous at this point .

Definition 2.10. A multifunction Φ is said to be upper (resp. lower) semicontinuous in the Hausdorff sense at a point $x_0 \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in B(x_0, \delta)$ the inclusion (2.2.6) [resp. (2.2.7)]

$$\Phi(x) \subset \Phi^{\varepsilon}(x_0), \tag{2.2.6}$$

$$(\text{resp. } \Phi(x_0) \supset \Phi^{\varepsilon}(x)) . \tag{2.2.7}$$

A multifunction Φ is called continuous in the Hausdorff sense if it is simultaneously upper and lower semicontinuous in the Hausdorff sense.

Definition 2.11. A multifunction $\Phi : X \longrightarrow 2^Y$ is said to be locally compact at a point $x_0 \in X$ if there exists $\delta > 0$ such that the set $\Phi(B(x_0; \delta))$ is compact in Y.

Remark 2.10. Multifunctions considered in this work are locally compacts, and sets $\Phi(x)$ are closed for each $x \in X$. Therefore for each multifunction the definition of semicontinuity by means of topological limits and the Hausdorff sense turn out to be equivalent. Taking into account this remark, usually we do not distinguish between the above notions of semicontinuity.

The following result in [1] is important.

Theorem 2.2. *Let a multifunction $\Phi : X \longrightarrow 2^Y$ be upper semicontinuous in the Hausdorff sense and $\Phi(x)$ be compact for each $x \in X$. Then for any compact set $D \subset X$ its image $\Phi(D)$ is a compact subset of Y.*

2.2.2 Semicontinuous functions

Let $\varphi : X \longrightarrow \overline{\mathbb{R}}$ be an extended valued function, X be a metric space, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be the extended real line. For such a function the symbol

$$gr\, \varphi = \{(x, z) \in X \times \mathbb{R} : z = \varphi(x)\},$$

is called the graph of the function φ, we denote by epi φ the set

$$\{(x, z) \in X \times \mathbb{R} : z \geq \varphi(x)\}$$

called the epigraph of φ and we let hypo φ denote the set

$$\{(x, z) \in X \times \mathbb{R} : z \leq \varphi(x)\}$$

which is called the hypograph of the function φ.

Definition 2.12. A function $\varphi : X \longrightarrow \overline{\mathbb{R}}$ is said to be lower semicontinuous at a point $x_\star \in X$ if for any $r < \varphi(x_\star)$, there exists δ>0 such that dist $(x_\star, x) < \delta$ implies $\varphi(x) \geq r$.
A function φ is said to lower semicontinuous if it is lower semicontinuous at every point $x \in X$.

Remark 2.11. Along with the above definition, any one of the following equivalent conditions can be used, [38]:
1) $\liminf_{x \longrightarrow x_\star} \varphi(x) \geq \varphi(x_\star)$;
2) for any $r \in \mathbb{R}$ the set $\{x \in X : \varphi(x) \leq r\}$ is closed;
3) epi φ is a closed set.

Definition 2.13. A function $\varphi : X \longrightarrow \overline{\mathbb{R}}$ is said to be upper semicontinuous if the function $-\varphi$ is lower semicontinuous.
A function $\varphi : X \longrightarrow \overline{\mathbb{R}}$ is said to be upper semicontinuous if it is upper semicontinuous at every point $x_\star \in X$.
A function $\varphi : X \longrightarrow \overline{\mathbb{R}}$ is said to continuous if and only φ it is both lower and upper semicontinuous.

Definition 2.14. A real-valued function $\varphi : X \longrightarrow \mathbb{R}$ is locally bounded if $\sup_{x \in D} |\varphi(x)| < \infty$ for any bounded subset $D \subset X$.

For a locally bounded function φ one can define its lower closure $\underline{\varphi} : X \longrightarrow \mathbb{R}$ as follows

$$\underline{\varphi}(x) := \sup_{\varepsilon>0} \inf \{\varphi(\xi) : \xi \in B(x;\varepsilon)\}. \tag{2.2.8}$$

Remark 2.12. Let $x_\star$ be an arbitrary point in X, and let $r < \varphi(x_\star)$. Choose $\delta > 0$ such that $\inf\{\varphi(\xi) : \xi \in B(x_\star; 2\delta)\} > r$. Then for any $x \in B(x_\star;\delta)$ we have

$$\underline{\varphi}(x) \geq \inf\{\varphi(\xi) : \xi \in B(x;\delta)\} \geq \inf\{\varphi(\xi) : \xi \in B(x_\star;2\delta)\} > r.$$

Therefore the lower closure $\underline{\varphi}$ is a lower semicontinuous function.

Analogously, for any locally bounded function $\varphi : X \longrightarrow \mathbb{R}$, its upper closure defined by

$$\overline{\varphi}(x) = \inf_{\varepsilon>0} \sup\{\varphi(\xi) : \xi \in B(x;\varepsilon)\} \tag{2.2.9}$$

is an upper semicontinuous function.

2.2.3 Contingent tangent cones, Directional derivatives, Subdifferentials

2.2.3.1 Contingent tangent cone

Let W be a nonempty set in $\mathbb{R}^m$. For a point $y \in \mathbb{R}^m$, the distance symbol dist $(y;W)$ denotes the distance from the point y to the set W and is defined by

$$\text{dist}\,(y;W)= \inf_{w\in W} |y-w|.$$

Note that

$$|\text{dist}\,(x;W) - \text{dist}\,(y;W)| \leq |x-y| \qquad \forall x,y \in \mathbb{R}^m. \tag{2.2.10}$$

Definition 2.15. The set

$$T(w;W) = \left\{ h \in \mathbb{R}^m : \liminf_{\delta \longrightarrow 0} \frac{\text{dist}\,(w+\delta h;W)}{\delta} = 0 \right\} \tag{2.2.11}$$

is called the contingent tangent cone to the set W at the point ω (or the upper tangent cone).

Lemma 2.4. *$T(\omega;W)$ is a closed cone.*

Proof. See [38]. □

2.2.3.2 Directional derivatives

Let $x \in G$ and $x \mapsto u(x) \in \mathbb{R}$ be a real valued function defined on an open domain $G \subset \mathbb{R}^n$. We use the following definitions and notations for lower and upper derivatives of the function u at a point $x \in G$ in direction $f \in \mathbb{R}^n$.

Definition 2.16.

$$d^- u(x; f) := \liminf_{\varepsilon \longrightarrow 0} \left\{ \frac{u(x + \delta g) - u(x)}{\delta} : (\delta, g) \in \Delta_\varepsilon(x, f) \right\} \quad (2.2.12)$$

$$d^+ u(x; f) := \limsup_{\varepsilon \longrightarrow 0} \left\{ \frac{u(x + \delta g) - u(x)}{\delta} : (\delta, g) \in \Delta_\varepsilon(x, f) \right\} \quad (2.2.13)$$

where

$$\Delta_\varepsilon(x, f) := \{(\delta, g) \in]0, \varepsilon[\times \mathbb{R}^n : |f - g| \leq \varepsilon, x + \delta g \in G\}.$$

The quantities $d^- u(x; f)$ and $d^+ u(x; f)$ (possibly infinite) are also called lower and upper Dini directional derivatives or Hadamard directional derivatives.

Below we introduce an equivalent definition of these derivatives with the help of contingent tangent cones to the epigraph and hypograph of the function u.

By the definition 2.15 of a contingent tangent cone we have

$$(f, g) \in T((x, u(x)); \operatorname{epi} u) \Longleftrightarrow \exists \{(\delta_k, f_k, z_k)\}_1^\infty \subset]0, 1[\times \mathbb{R}^n \times \mathbb{R}$$

such that

$$z_k \geq u(x + \delta_k f_k), \delta_k \longrightarrow 0, f_k \longrightarrow f, \frac{z_k - u(x)}{\delta_k} \longrightarrow g \quad as \quad k \longrightarrow \infty.$$

Using this remark, we obtain that definition (2.2.12) is equivalent to the following one:

Definition 2.17.

$$d^- u(x; f) := \inf\{g \in \mathbb{R} : (f, g) \in T((x, u(x)); \operatorname{epi} u)\}. \quad (2.2.14)$$

Note that we set $d^- u(x; f) = \infty$ if

$$\{g \in \mathbb{R} : (f, g) \in T((x, u(x)); \text{epi}\, u)\} = \emptyset.$$

Similarly, we have

Definition 2.18.

$$d^+ u(x; f) := \sup\{g \in \mathbb{R} : (f, g) \in T((x, u(x)); \text{hypo}\ u)\}. \tag{2.2.15}$$

We also note that

$$[(f, g) \in T((x, z); \text{epi}\ u), g' \geq g] \Longrightarrow (f, g') \in T((x, z); \text{epi}\ u), \tag{2.2.16}$$

$$[(f, g) \in T((x, z); \text{hypo}\ u), g' \leq g] \Longrightarrow (f, g') \in T((x, z); \text{hypo}\ u). \tag{2.2.17}$$

It is clear that the inequality $z'' \geq z'$ implies

$$\text{dist}((x, z'); \text{epi}\ u) \geq \text{dist}((x, z''); \text{epi}\ u),$$

$$\text{dist}((x, z'); \text{hypo}\ u) \leq \text{dist}((x, z''); \text{hypo}\ u).$$

Therefore for any $x \in G$ and $-\infty < z' \leq z'' < +\infty$ the following relations

$$\begin{aligned} T((x, z'); \text{epi}\ u) &\subset T((x, z''); \text{epi}\ u) \\ T((x, z'); \text{hypo}\ u) &\supset T((x, z''); \text{hypo}\ u) \end{aligned} \tag{2.2.18}$$

hold.

Consider the mapping $\mathbb{R}^n \ni f \longmapsto d^- u(x; f) \in [-\infty, \infty]$. In view of (2.2.14) and (2.2.16), we have

$$\begin{aligned} \text{epi}\, d^- u(x; .) &:= \{(g, f) \in \mathbb{R} \times \mathbb{R}^n : g \geq d^- u(x; f)\} \\ &= T((x, u(x)); \text{epi}\ u). \end{aligned} \tag{2.2.19}$$

Similarly,

$$\begin{aligned} \text{hypo}\ d^+ u(x; .) &:= \{(g, f) \in \mathbb{R} \times \mathbb{R}^n : g \leq d^+ u(x; f)\} \\ &= T((x, u(x)); \text{hypo}\ u). \end{aligned} \tag{2.2.20}$$

Since $T((x, u(x)); \text{epi}\ u)[T((x, u(x)); \text{hypo}\ u)]$ is closed cone, we deduce that the function $d^- u(x; .)$ [resp. the function $d^+ u(x; .)$] is lower [resp. upper] semicontinuous.

2.2.3.3 Some relations between directional derivatives and contingent tangent cones to the graph function

We will use the following notation:

$$T(u)(x) := T((x, u(x)); \text{gr } u). \tag{2.2.21}$$

Thus

$$T(u)(x) := \{(f, g) \in \mathbb{R}^n \times \mathbb{R} : \lim_{\delta \longrightarrow 0} \inf \frac{\text{dist}((x + \delta f, u(x) + \delta g); \text{gr } u)}{\delta} = 0\}. \tag{2.2.22}$$

By the definitions of the cone $T(u)(x)$ and directional derivatives $d^- u(x; f)$, $d^+ u(x; f)$ we have

$$[-\infty < d^- u(x; f) < +\infty] \Longrightarrow [(f, d^- u(x; f)) \in T(u)(x)] \tag{2.2.23}$$

$$[-\infty < d^+ u(x; f) < +\infty] \Longrightarrow [(f, d^+ u(x; f)) \in T(u)(x)] \tag{2.2.24}$$

$$[(f, g) \in T(u)(x)] \Longrightarrow [d^- u(x; f) \leq g \leq d^+ u(x; f)]. \tag{2.2.25}$$

The following result is used.

Proposition 2.6. *Let $u : G \longmapsto \mathbb{R}$ be a lower semicontinuous function. For some $x \in G$ and $f \in \mathbb{R}$, let $d^- u(x; f) = -\infty$.*

Then

$$\{\bar{0}\} \times] - \infty, 0] \subset T(u)(x).$$

Similarly, let $u : G \longmapsto \mathbb{R}$ be an upper semicontinuous function.
For some $x \in G$ and $f \in \mathbb{R}$, let $d^+ u(x; f) = \infty$. Then

$$\{\bar{0}\} \times [0, \infty[\subset T(u)(x).$$

Here $\bar{0}$ is the zero vector in $\mathbb{R}^n$.

Proof. See [38]. □

2.2.3.4 Directional derivatives of the upper envelope of family of smooth functions

Definition 2.19. For a real valued function $\mathbb{R}^n \ni x \mapsto h(x) \in \mathbb{R}$ and a set $X \subset \mathbb{R}^n$ we let

$$\operatorname{Arg}\min_{x\in X} h(x) = \{x_0 \in X : h(x_0) \leq h(x), \forall x \in X\},$$

$$\operatorname{Arg}\max_{x\in X} h(x) = \{x_0 \in X : h(x_0) \geq h(x), \forall x \in X\}.$$

Definition 2.20. A function $u : G \longmapsto \mathbb{R}$ is said to be differentiable at a point $x \in G \subset \mathbb{R}^n$ in the direction of $f \in \mathbb{R}^n$ provided the limit

$$du(x;f) = \lim_{\delta \longrightarrow 0} \frac{u(x+\delta f) - u(x)}{\delta} \tag{2.2.26}$$

exists and is finite.

It is obvious that the equality $d^- u(x;f) = d^+ u(x;f) \in]-\infty, \infty[$ implies the existence of the directional derivative $du(x;f)$.

Let us consider a useful class of function u possessing the directional derivative $du(x;f)$ at any point x and in any direction $f \in \mathbb{R}^n$.

Let a function $G \ni x \longmapsto u(x) \in \mathbb{R}$ be defined by the equality

$$u(x) := \sup_{s\in\mathbb{R}^m} \varphi(x,s), \tag{2.2.27}$$

we shall assume that the following conditions are fulfilled:

(a) $\varphi(x,s) = \varphi_1(x,s) - \varphi_2(s)$;

(b) the function $\varphi_1 : G \times \mathbb{R}^m \longmapsto \mathbb{R}$ is continuous; for any $s \in \mathbb{R}^m$ the function $\varphi_1(.,s)$ is differentiable on G; the function $(x,s) \longmapsto D_x\varphi_1(x,s)$ is continuous on $G \times \mathbb{R}^m$;

(c) the function $\varphi_2 : \mathbb{R}^m \longmapsto [a,\infty]$ is lower semicontinuous (here a is a finite number); the effective domain of function φ_2

$$\operatorname{dom} \varphi_2 := \{s \in \mathbb{R}^m : \varphi_2(s) < \infty\}$$

is nonempty and bounded.
It is clear that for any $x \in G$, $u(x)$ is a finite number. Let us define the set

$$S_0(x) = \text{Arg} \max_{s \in \mathbb{R}^m} \varphi(x,s) \tag{2.2.28}$$

and show that $S_0(x) \neq \emptyset$ for any $x \in G$.

Let $s_i \in \mathbb{R}^m (i = 1, 2, ...)$, $\lim_i \varphi(x, s_i) = u(x)$. It can be assumed that

$$s_i \in \text{dom } \varphi_2 (i = 1, 2, ...)$$

and $\lim_{i \longrightarrow \infty} s_i = s_*$. Since the function $s \longmapsto \varphi(x,s)$ is upper semicontinuous, we have $u(x) = \lim_{i \longrightarrow \infty} \varphi(x, s_i) \leq \varphi(x, s_*)$. The strict inequality $u(x) < \varphi(x, s_*)$ contradicts equality (2.2.27). We thus have obtained that $\varphi(x, s_*) = u(x)$, i.e., $s_* \in S_0(x)$.

Similarly, one can verify that the set $S_0(x)$ is closed and the multifunction $x \longmapsto S_0(x)$ is upper semicontinuous.

Proposition 2.7. *Let conditions $(a) - (c)$ above be fulfilled. Then the function $u : G \longmapsto \mathbb{R}$ defined by the equality (2.2.27) satisfies the Lipschitz condition in every bounded convex domain $X \subset G$. At every point $x \in G$ the function u is directionally differentiable. More than that, for any $x \in G$ and any $f \in \mathbb{R}^n$ the following relations*

$$d^- u(x; f) = d^+ u(x; f) = du(x; f) = \max_{s \in S_0(x)} \langle D_x \varphi_1(x,s), f \rangle \tag{2.2.29}$$

are valid.

Proof. See [38]. □

2.2.3.5 Subdifferentials and Superdifferentials

Now we recall the definition of subdifferential and superdifferential. Let

$$D^- u(x) := \{p \in \mathbb{R}^n : \langle p, f \rangle - d^- u(x; f) \leq 0, \quad \forall f \in \mathbb{R}^n\} \tag{2.2.30}$$

$$D^+ u(x) := \{p \in \mathbb{R}^n : \langle p, f \rangle - d^+ u(x; f) \geq 0, \quad \forall f \in \mathbb{R}^n\}, \tag{2.2.31}$$

where $x \in G$ and $x \mapsto u(x) \in \mathbb{R}$ be a real valued function defined on an open domain $G \subset \mathbb{R}^n$.

Definition 2.21. The set $D^-u(x)$ [resp. the set $D^+u(x)$] is called subdifferential (resp. superdifferential) of function u at point $x \in G$.

The elements of $D^-u(x)$ (resp. $D^+u(x)$) are called subgradients (resp. supergradients).

We note that $D^-u(x)$ and $D^+u(x)$ are closed and convex sets (which may be empty). If a function u is differentiable at a point $x \in G$, then one can easily verify that

$$D^-u(x) = D^+u(x) = \{Du(x)\},$$

where $Du(x)$ is the gradient of u at x.

2.2.4 On a property of Subdifferentials

The proof of the equivalence of minimax and viscosity solutions is based on the following assertion.

Theorem 2.3. *Let $\mathbb{R}^m \ni y \longmapsto v(y) \in]-\infty, \infty]$ be a lower semicontinuous function and let Q be a convex compact set in $\mathbb{R}^m$. Suppose for some point $y_0 \in \mathbb{R}^m$ and some $\alpha > 0$, the function v is bounded below on $[y_0, Q] + B_\alpha$ (the symbol $[y_0, Q]$ denotes the convex hull of $\{y_0\} \bigcup Q$). Then for any*

$$r < r_0 := \min_{y \in Q} v(y) - v(y_0) \tag{2.2.32}$$

and $\beta > 0$ there exist $z_* \in [y_0, Q] + B_\beta$ and $s_* \in D^-v(z_*)$ such that

$$r < \langle s_*, y - y_0 \rangle \qquad \text{for all} \qquad y \in Q. \tag{2.2.33}$$

Proof. See [38]. □

Theorem 2.4. *Let $Y \ni y \longmapsto v(y) \in \mathbb{R}$ be a lower semicontinuous function defined on an open set $Y \subset \mathbb{R}^m$, and let H be a convex compact set in $\mathbb{R}^m$. Suppose for some point $y_0 \in Y$ we have*

$$d^-v(y_0; h) > 0, \qquad \forall h \in H. \tag{2.2.34}$$

Then for any $\varepsilon > 0$ there exists a point $y_\varepsilon \in Y$ and a subgradient $s_\varepsilon \in D^-v(y_\varepsilon)$ such that

$$|y_0 - y_\varepsilon| < \varepsilon, \qquad \langle s_\varepsilon, h \rangle > 0, \qquad \forall h \in H, \tag{2.2.35}$$

$$|v(y_0) - v(y_\varepsilon)| < \varepsilon. \tag{2.2.36}$$

Proof. See [38]. □

2.2.5 Differential Inclusions

Consider a multifunction $[0, \theta] \times \mathbb{R}^m \ni (t, y) \longmapsto E(t, y) \subset \mathbb{R}^m$ satisfying the following conditions:

(a) $E(t, y)$ is a convex compact set in $\mathbb{R}^m$ for all $(t, y) \in [0, \theta] \times \mathbb{R}^m$;

(b) the multifunction E is upper semicontinuous in the Hausdorff sense.

Let $(t_0, y_0) \in [0, \theta] \times \mathbb{R}^m$. For $\nu > 0$ we let

$$I_\nu(t_0) := [t_0 - \nu, t_0 + \nu] \cap [0, \theta].$$

Below we will prove that there exist a positive number ν and an absolutely continuous function $y(\cdot) : I_\nu(t_0) \longmapsto \mathbb{R}^m$, which satisfies the condition $y(t_0) = y_0$ and the differential inclusion

$$\dot{y}(t) \in E(t, y(t)) \tag{2.2.37}$$

for almost all $t \in I_\nu(t_0)$.

For this purpose we need to introduce some notations. For $\varepsilon \in]0, 2]$ and $(t, y) \in [0, \theta] \times \mathbb{R}^m$ we let $O_\varepsilon(t, y)$ denote the set

$$\{(\tau, \eta) \in [0, \theta] \times \mathbb{R}^m : |t - \tau| \leq \varepsilon, |y - \eta| \leq \varepsilon\}.$$

Define the set

$$E(t, y; \varepsilon) := \text{co}\{e + b : e \in E(\tau, \eta), (\tau, \eta) \in O_\varepsilon(t, y), |b| \leq \varepsilon\},$$

the convex hull of the set

$$\{e + b : e \in E(\tau, \eta), (\tau, \eta) \in O_\varepsilon(t, y), |b| \leq \varepsilon\}.$$

Choose and fix a point $(t_0, y_0) \in [0, \theta] \times \mathbb{R}^m$. Let

$$E^\# := E(t_0, y_0; 2), \qquad \lambda := max\{|e| : e \in E^\#\}, \qquad \nu := \lambda^{-1}.$$

Denote by Sol_ε the set of absolutely continuous functions $y(\cdot)$: $I_\nu(t_0) \mapsto \mathbb{R}^m$ which satisfy the condition $y(t_0) = y_0$ and the differential inclusion

$$\dot{y}(t) \in E(t, y(t); \varepsilon) \qquad \text{for almost all} \qquad t \in I_\nu(t_0). \tag{2.2.38}$$

Proposition 2.8. *Let $\varepsilon_k \in]0,1]$, $y_k(.) \in Sol_{\varepsilon_k}$ $(k = 1,2,...)$, and $\varepsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$. Then one can extract from the sequence $\{y_k(.)\}_1^\infty$ a convergent subsequence whose limit $y(\cdot)$ satisfies (2.2.37) for almost all $t \in I_\nu(t_0)$.*

Proof. See [38]. □

Using Proposition 2.8, we can prove the existence of solutions of differential inclusion (2.2.37). Let a point $(t_0, y_0) \in [0,\theta] \times \mathbb{R}^m$ be given. As in proposition 2.8 we define the time interval $I_\nu(t_0)$. Choose a sequence of positive numbers $\delta_k (k = 1,2,...)$ which converges to 0 as $k \longrightarrow \infty$. Let us introduce a sequence of piecewise-linear functions $y_k(.) : I_\nu(t_0) \longrightarrow \mathbb{R}^m$ as follows: $y_k(t_0) = y_0$,

$$y_k(t) = y_k(t_0 + i\delta_k) + e^{(i,k)}(t - t_0 - i\delta_k), t \in [t_0 + i\delta_k, t_0 + (i+1)\delta_k[,$$

where $e^{(i,k)} \in E(t_0 + i\delta_k, y_k(t_0 + i\delta_k))$, and

$$y_k(t) = y_k(t_0 - i\delta_k) + e_{i,k}(t - t_0 + i\delta_k), t \in]t_0 - (i+1)\delta_k, t_0 - i\delta_k],$$

where $e_{i,k} \in E(t_0 - i\delta_k, y_k(t_0 - i\delta_k))$. As in the proof of proposition 2.8, we obtain

$$\left|y_k(t') - y_k(t'')\right| \leq \lambda |t' - t''| \text{ for any } t', t'' \in I_\nu(t_0).$$

By definition for $t \in]t_0 + i\delta_k, t_0 + (i+1)\delta_k[$ we have

$$\dot{y}_k(t) = e^{(i,k)} \in E(t_0 + i\delta_k, y_k(t_0 + i\delta_k)) \subset E(t, y_k(t); \varepsilon_k),$$

where $\varepsilon_k = \lambda\delta_k$ (note that $\lambda \geq 2$). Similarly, $\dot{y}_k(t) \in E(t, y_k(t); \varepsilon_k)$ for almost all $t \in [t_0 - \nu, t_0] \cap [0, t_0]$. In view of proposition 2.8, the sequence $y_k(\cdot)$ contains a convergent subsequence whose limit $y(\cdot)$ satisfies (2.2.37) for almost all $t \in I_\nu(t_0)$.

Thus we can derive the following.

Proposition 2.9. *Let a multifunction $(t,y) \longmapsto E(t,y)$ satisfy conditions (a) and (b). Then for any point $(t_0, y_0) \in [0,\theta] \times \mathbb{R}^m$ there exist a positive number ν and an absolutely continuous function $y(\cdot) : I_\nu(t_0) \longmapsto \mathbb{R}^m$ which satisfies the condition $y(t_0) = y_0$ and differential inclusion (2.2.37) for almost all $t \in I_\nu(t_0)$.*

Now we assume that in addition to conditions (a), (b) the multifunction E satisfies the following estimate

$$max\{|e| : e \in E(t,y)\} \leq r(t)(1 + |y|), \tag{2.2.39}$$

where $[0,\theta] \ni t \longmapsto r(t) \in \mathbb{R}^+$ is an integrable function. Note that this condition assures that the solutions of differential inclusion (2.2.37) can be extended over the whole interval $[0,\theta]$. This condition may be formulated in various ways. For example, inequality (2.2.39) can be replaced by the inequality

$$\max\{\langle x, e\rangle : e \in E(t,y)\} \leq r(t)(1+|y|^2),$$

where as above $r(\cdot)$ is an integrable function.

Let $(t_0, y_0) \in [0,\theta] \times \mathbb{R}^m$. Denote by $\mathrm{Sol}(t_0, y_0)$ the set of absolutely continuous functions $y(.) : [0,\theta] \longmapsto \mathbb{R}^m$ which satisfies the condition $y(t_0) = y_0$ and the differential inclusion (2.2.37) for almost all $t \in [0,\theta]$. From the above propositions one can obtain the following proposition.

Proposition 2.10. *Let a multivalued mapping E satisfy conditions (a), (b) and estimate (2.2.39). Then for any point* $(t_0, y_0) \in [0,\theta] \times \mathbb{R}^m$ *the set* $Sol(t_0, y_0)$ *is nonempty and compact in* $C([0,\theta];\mathbb{R}^m)$. *Let* $(\tau_k, \eta_k) \in [0,\theta] \times \mathbb{R}^m$, $y_k(\cdot) \in Sol(\tau_k, \eta_k)(k = 1, 2, ...)$ *and* $(\tau_k, \eta_k) \longrightarrow (t_0, x_0)$ *as* $k \longrightarrow \infty$. *Then one can extract from the sequence of functions* $y_k(\cdot)(k = 1, 2, ...)$ *a convergent subsequence whose limit is contained in* $Sol(t_0, y_0)$.
Let $M \subset [0,\theta] \times \mathbb{R}^m$ *be a compact set. Then the set*

$$Sol(M) := \cup_{(\tau,\eta)\in M} Sol(\tau, \eta) \tag{2.2.40}$$

is compact in $C([0,\theta];\mathbb{R}^m)$.

Proof. See [38]. □

2.2.6 Criteria for Weak Invariance

Let W be a locally compact nonempty set in $\mathbb{R}^m$, that is, for any $w \in W$ there exists a number $\varepsilon > 0$ such that the intersection $W \cap B(w;\varepsilon)$ is closed in $\mathbb{R}^m$. Consider a multifunction

$$W \ni y \longmapsto P(y) \subset W$$

with locally compact graph

$$\mathrm{gr}P := \{(y, w) : w \in P(y), y \in W\}.$$

We assume that the multifunction P is both reflexive and transitive, that is, it satisfies the following conditions:

$$x \in P(x) \subset W,\ \forall x \in M, \tag{2.2.41}$$

Thus,

$$P(y) \subset P(x),\ \forall x \in W,\ \forall y \in P(x). \tag{2.2.42}$$

The multifunction P induces on W a preorder $\succeq$ as follows

$$w \succeq y \Longleftrightarrow w \in P(y).$$

Let a multifunction

$$W \ni y \longmapsto Y(y) \in conv(\mathbb{R}^m)$$

be upper semicontinuous in the Hausdorff sense, that is, for any $y \in W$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that $Y(y') \subset Y(y) + B_\varepsilon$ for all $y' \in B(y;\delta) \cap W$. The set $conv(\mathbb{R}^m)$ is the totality of nonempty convex and compact sets in $\mathbb{R}^m$. Consider the differential inclusion

$$\dot{y}(t) \in Y(y(t)). \tag{2.2.43}$$

We will use the following notions.

Definition 2.22. A set $W \subset \mathbb{R}^m$ is called weakly invariant with respect to differential inclusion (2.2.43) if for any point $y_0 \in W$ there exist a positive number θ and an absolutely continuous function (viable trajectory) $y(.)$: $[0,\theta] \longmapsto W$ such that $y(0) = y_0$ and (2.2.43) is satisfied for almost all $t \in [0,\theta]$. (In this case it is said also that the set W enjoys the viability property.)

Definition 2.23. An absolutely continuous function $y(.) : [0,\theta] \longmapsto W$ is said to be a monotone trajectory of (2.2.43) if it satisfies the differential inclusion (2.2.43) for almost all $t \in [0,\theta]$, and the following property of monotonicity: $y(t) \succeq y(s)$ for all $(s,t) \in \Theta^+$;that is,

$$y(t) \in P(y(s)), \qquad \forall (s,t) \in \Theta^+, \tag{2.2.44}$$

where

$$\Theta^+ = \{(s,t) \in [0,\theta] \times [0,\theta] : s \leq t\}. \tag{2.2.45}$$

Definition 2.24. Let G be a subset of $\mathbb{R}^m$. The convex hull of the set G is denoted by co$G$.

Below we shall prove the following theorem.

Theorem 2.5. *Let $W \subset \mathbb{R}^m$ be a locally compact set. Let a multifunction $Y : W \longmapsto conv(\mathbb{R}^m)$ be upper semicontinuous in the Hausdorff sense.*
Let $W \ni y \longmapsto P(y) \subset W$ be a multifunction whose graph is locally compact. Assume also that conditions (2.2.41) and (2.2.42) are fulfilled. Then the following three conditions are equivalent:

(a_1)

$$T(y; P(y)) \cap Y(y) \neq \emptyset \qquad \forall y \in W, \tag{2.2.46}$$

where $T(y; P(y))$ is the contingent tangent cone to $P(y)$ at y, that is,

$$T(y; P(y)) := \left\{ h : \liminf_{\delta \downarrow 0} \frac{dist(y + \delta h; P(y))}{\delta} = 0 \right\};$$

(b_1)

$$coT(y; P(y)) \cap Y(y) \neq \emptyset \qquad \forall y \in W; \tag{2.2.47}$$

(c_1) *for any point $y_0 \in W$ there exist a number $\theta > 0$ and a monotone trajectory $y(.) : [0, \theta] \longmapsto W$ of differential inclusion (2.2.43) satisfying the initial condition $y(0) = y_0$.*

Proof. See [38]. □

If we take $P(y) = W$ for all $y \in W$, then:

Corollary 2.1. *Let $Y : W \longrightarrow conv(\mathbb{R}^m)$ be an upper semicontinuous multifunction and let $W \subset \mathbb{R}^m$ be a locally compact set. Then the following three conditions are equivalent:*

(a_2)

$$T(y; W) \cap Y(y) \neq \emptyset; \qquad \forall y \in W; \tag{2.2.48}$$

(b_2)

$$coT(y; W) \cap Y(y) \neq \emptyset; \qquad \forall y \in W; \tag{2.2.49}$$

(c_2) *the set W is weakly invariant with respect to differential inclusion (2.2.43).*

Remark 2.13. Let $\mathbb{R}^m \ni y \longmapsto E(y) \in conv(\mathbb{R}^m)$ be an upper semicontinuous multifunction. Consider the differential inclusion

$$\dot{y}(t) \in E(y(t)). \tag{2.2.50}$$

According to the Definition 2.22, a locally compact set $W \subset \mathbb{R}^m$ is said to be weakly invariant with respect to the differential inclusion (2.2.50) if for every $y_0 \in W$ there exists a number $\theta > 0$, an absolutely continuous solution $y(.) : [0, \theta] \longmapsto \mathbb{R}^m$ of inclusion (2.2.50) such that $y(0) = y_0$ and $y(t) \in W$ for all $t \in [0, \theta]$. It is clear that this definition is equivalent to the following one: For every $y_0 \in W$ there exist a number $\theta > 0$ such that for any $\tau \in]0, \theta]$ an absolutely continuous function $y(\cdot) : [0, \tau] \longmapsto \mathbb{R}^m$ can be found that satisfies (2.2.50) and conditions $y(0) = y_0$ and $y(\tau) \in W$.

Remark 2.14. Assumptions are known which assure the extendibility of locally viable trajectories. For example, one of these conditions can be formulated as follows.

Let the requirements of corollary 2.1 be fulfilled, and let the multifunction $W \ni y \longmapsto Y(y) \in conv(\mathbb{R}^m)$ satisfy the estimate

$$|h| \leq c(1 + |y|), \qquad \forall \in W, \qquad h \in Y(y), \tag{2.2.51}$$

where c is a positive number. Then for any point $y_0 \in W$ there exists at least one viable trajectory $y(.) : [0, \theta[\longmapsto W$ of differential inclusion (2.2.50) which satisfies the initial condition $y(0) = y_0$ and is defined on a time interval $[0, \theta[$, where either $\theta = \infty$ or $y(\theta) \notin W$. If W is a closed subset of $\mathbb{R}^m$ and estimate (2.2.51) holds, then any viable trajectory can be extended on the whole interval $[0, \infty[$.

In conclusion we formulate necessary and sufficient conditions for a set $W \subset [0, t^\#[\times \mathbb{R}^m$ to be weakly invariant with respect to a time dependent differential inclusion of the form

$$\dot{y}(t) \in Y(t, y(t)). \tag{2.2.52}$$

We assume that the set W is locally compact in $[0, t^\#[\times \mathbb{R}^m$ and the multifunction $W \ni (t, y) \longmapsto Y(t, y) \in conv(\mathbb{R}^m)$ is upper semicontinuous in the Hausdorff sense. Here $t^\#$ is either a positive number or $+\infty$. We note that the set W can be considered as the graph of the multifunction

$$[0, t^\#[\ni t \longmapsto W(t) \subset \mathbb{R}^m,$$

where $W(t) := \{w \in \mathbb{R}^m : (t, w) \in W\}$.

Definition 2.25. A set $W \subset \mathbb{R}^m$ is called weakly invariant with respect to differential inclusion (2.2.50) if for any point $(t_0, y_0) \in W$ there exist a number $\tau \in]t_0, t^\#[$ and an absolutely continuous function (viable trajectory) $y(\cdot) : [t_0, \tau] \longmapsto \mathbb{R}^m$ such that

$$y(t_0) = y_0,$$

$$y(t) \in W(t)$$

for all $t \in [t_0, \tau]$, and (2.2.50) is satisfied for almost all $t \in [t_0, \tau]$.

In order to formulate a criterion for weak invariance we define a derivative of the multifunction $t \longmapsto W(t)$ as follows:

$$(D_t W)(t, w) = \{h \in \mathbb{R}^m : \liminf_{\delta \downarrow 0} \frac{\text{dist}(w + \delta h; W(t + \delta))}{\delta} = 0\}.$$

Definition 2.26. The set $(D_t W)(t, w)$ is called the right-hand derivative of the multifunction $t \longmapsto W(t)$ at the point $(t, w) \in W$.

Proposition 2.11. *A locally compact set $W \subset [0, t^\#[\times \mathbb{R}^m$ is weakly invariant with respect to differential inclusion (2.2.52) if and only if the condition*

$$(D_t W)(t, w) \cap Y(t, w) \neq \emptyset, \qquad \forall\ (t, w) \in W \tag{2.2.53}$$

is fulfilled. This criterion is equivalent to the condition

$$co(D_t W)(t, w) \cap Y(t, w) \neq \emptyset, \qquad \forall\ (t, w) \in W. \tag{2.2.54}$$

Proof. See [38]. □

2.2.7 Characteristic inclusions for Hamilton-Jacobi equations

An essential property of a generalized (minimax) solution is the invariance of its graph with respect to some system of differential inclusions. We call these inclusions characteristic inclusions. In this section we present the characteristic inclusions considered in the case of Hamilton-Jacobi equations and give properties that describe them.

The characteristic differential inclusions are defined as follows

$$E(t,x,z,s) = \{(f,g) \in \mathbb{R}^n \times \mathbb{R} : |f| \leq \rho(x,z), \\ g = \langle f,s\rangle - H(t,x,z,s)\} \quad (2.2.55)$$

for $(t,x,z,s) \in G \times \mathbb{R} \times \mathbb{R}^n$.

Consider the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in E(t,x,z,s). \quad (2.2.56)$$

The analysis of minimax solutions of Cauchy problems for Hamilton-Jacobi equations is based on the properties of weak invariance of the graphs of the solutions of this differential inclusions. For Ψ a nonempty set of $\mathbb{R}^n$, note that the multifunction E defined as in [38]

$$]0,\theta[\times\mathbb{R}^n \times \mathbb{R} \times \Psi \ni (t,x,z,\psi) \mapsto E(t,x,z,\psi) \subset \mathbb{R}^n \times \mathbb{R}$$

satisfies all the following properties:

(i) for all $(t,x) \in G = (0,\theta) \times \mathbb{R}^n, z \in \mathbb{R}, \psi \in \Psi$ the set $E(t,x,z,\psi)$ is convex and compact in $\mathbb{R}^n \times \mathbb{R}$;

(ii) for any $\psi \in \Psi$, the multifunction $(t,x,z) \mapsto E(t,x,z,\psi)$ is upper semicontinuous;

(iii$^+$) for any $\psi \in \Psi$, $(t,x) \in G, z' \leq z''$, and $(f,g') \in E(t,x,z',\psi)$, there exists $(f,g'') \in E(t,x,z'',\psi)$ such that $g'' \geq g'$;

(iii$^-$) for any $\psi \in \Psi$, $(t,x) \in G, z' \leq z''$, and $(f,g'') \in E(t,x,z'',\psi)$, there exists $(f,g') \in E(t,x,z',\psi)$ such that $g'' \geq g'$;

(iv$^+$) for any $(t,x) \in G, z \in \mathbb{R},$ and $s \in \mathbb{R}^n$, there exists $\psi^0 \in \Psi$ such that

$$H(t,x,z,s) = \min\left\{\langle f,s\rangle - g : (f,g) \in E\left(t,x,z,\psi^0\right)\right\} \\ \geq \min\{\langle f,s\rangle - g : (f,g) \in E(t,x,z,\psi)\}$$

for all $\psi \in \Psi$;

(iv)$^-$ for any $(t,x) \in G, z \in \mathbb{R},$ and $s \in \mathbb{R}^n$, there exists $\psi_0 \in \Psi$ such that

$$H(t,x,z,s) = \max\{\langle f,s\rangle - g : (f,g) \in E(t,x,z,\psi_0)\} \\ \leq \max\{\langle f,s\rangle - g : (f,g) \in E(t,x,z,\psi)\}$$

for all $\psi \in \Psi$.

2.2.8 Criteria for minimax solutions of Hamilton-Jacobi equations

The upper, lower and minimax solutions of Hamilton-Jacobi equations can be defined in several equivalent forms. Let us give these definitions.

2.2.8.1 Upper solutions

First we consider conditions (U1)-(U5), which give definitions of the upper solution of equation (2.2.2). The function $]0,\theta[\times\mathbb{R}^n \ni (t,x) \mapsto u(t,x) \in \mathbb{R}$ is assumed to be lower semicontinuous.

(U1) For any $(t_0,x_0) \in G =]0,\theta[\times\mathbb{R}^n, z_0 \geq u(t_0,x_0), s \in \mathbb{R}^n$ there exist $\tau \in]t_0,\theta[$ and a Lipschitz continuous function $(x(\cdot), z(\cdot)) : [t_0,\tau] \mapsto \mathbb{R}^n \times \mathbb{R}$ which satisfies the equality $(x(t_0), z(t_0)) = (x_0, z_0)$, the equation

$$\dot{z}(t) = \langle \dot{x}(t), s\rangle - H(t, x(t), z(t), s) \qquad (2.2.57)$$

for almost all $t \in [t_0,\tau]$, and the inequality $z(t) \geq u(t, x(t))$ for all $t \in [t_0,\tau]$.

(U2) For any choice $\psi \in \Psi$, the epigraph of the function u is weakly invariant with respect to the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in E^+(t, x(t), z(t), \psi). \qquad (2.2.58)$$

Here and in condition (U3) the symbol E^+ stands for an arbitrary multifunction which satisfies (i), (ii), (iii$^+$), (iv$^+$) formulated in Subsection 2.2.7.

(U3) $\inf\{d^-u(t,x;1;f) - g : (f,g) \in E^+(t,x(t),z(t),\psi)\} \leq 0$ for all $(t,x) \in G$ and $\psi \in \Psi$.

(U4) $$a + H(t,x,u(t,x),s) \leq 0$$
for all $(t,x) \in G$ and $(a,s) \in D^-u(t,x)$.

(U5) $\inf\{d^-u(t,x;1;f) - \langle s,f\rangle + H(t,x,u(t,x),s) : f \in \mathbb{R}^n\} \leq 0$ for all $(t,x) \in G$ and $s \in \mathbb{R}^n$.

Let us explain the notation $d^{-}u\,(t,x;1,f)$ and $D^{-}u\,(t,x)$ used in (U3)-(U5). According to the general definition of lower derivative, we have

$$d^{-}u\,(t,x;\alpha,f) := \lim_{\varepsilon \longrightarrow 0} \inf_{(\delta,\alpha',f')\in\Delta_{\varepsilon}(t,x,\alpha,f)} \left[\frac{u\,(t+\delta\alpha',x+\delta f')-u\,(t,x)}{\delta}\right], \quad (2.2.59)$$

here $(t,x)\in G$, $(\alpha,f)\in\mathbb{R}\times\mathbb{R}^n$ and

$$\Delta_{\varepsilon}\,(t,x,\alpha,f) := \left\{(\delta,\alpha',f')\in]0,\varepsilon[\times\mathbb{R}\times\mathbb{R}^n : \left|\alpha-\alpha'\right|+\left|f-f'\right|\le\varepsilon, t+\alpha'\delta\in]0,\theta[\right\}. \quad (2.2.60)$$

Assume $\alpha = 1$ .The quantity $d^{-}u\,(t,x;1,f)$ is the lower derivative of the function u in the direction $(1,f)$. According to the definition, we have

$$D^{-}u\,(t,x) := \{(a,s)\in\mathbb{R}\times\mathbb{R}^n : a\alpha+\langle s,f\rangle+d^{-}u\,(t,x;\alpha,f)\le 0 \quad \forall\,(\alpha,f)\in\mathbb{R}\times\mathbb{R}^n\}. \quad (2.2.61)$$

2.2.8.2 Lower solutions

Now we consider conditions (L1)-(L5), which give definitions of the lower solution of equation (2.2.2). The function $]0,\theta[\times\mathbb{R}^n \ni (t,x) \mapsto u\,(t,x)\in\mathbb{R}$ is assumed to be upper semicontinuous.

(L1) For any $(t_0,x_0)\in G=]0,\theta[\times\mathbb{R}^n, z_0\ge u\,(t_0,x_0), s\in\mathbb{R}^n$ there exist $\tau\in]t_0,\theta[$ and a Lipschitz continuous function $(x\,(\cdot),z\,(\cdot)) : [t_0,\tau]\mapsto\mathbb{R}^n\times\mathbb{R}$ which satisfies the equality $(x\,(t_0),z\,(t_0)) = (x_0,z_0)$, the equation

$$\dot{z}\,(t) = \langle\dot{x}\,(t)\,,s\rangle - H\,(t,x\,(t),z\,(t)\,,s) \quad (2.2.62)$$

for almost all $t\in[t_0,\tau]$, and the inequality $z\,(t)\le u\,(t,x\,(t))$ for all $t\in[t_0,\tau]$.

(L2) For any choice $\psi\in\Psi$, the hypograph of the function u is weakly invariant with respect to the differential inclusion

$$(\dot{x}\,(t)\,,\dot{z}\,(t))\in E^{-}\,(t,x\,(t),z\,(t)\,,\psi)\,. \quad (2.2.63)$$

Here and in condition (L3) the symbol E^{-} stands for an arbitrary multifunction which satisfies (i), (ii), (iii^{-}), (iv^{-}) formulated in Subsection 2.2.7

(L3) $\sup\{d^+u(t,x;1;f) - g : (f,g) \in E^-(t,x(t),z(t),\psi)\} \geq 0$ for all $(t,x) \in G$ and $\psi \in \Psi$.

(L4)
$$a + H(t,x,u(x),s) \geq 0$$
for all $(t,x) \in G$ and $(a,s) \in D^+u(t,x)$.

(L5) $\sup\{d^+u(t,x;1;f) - \langle s,f\rangle + H(t,x,u(t,x),s) : f \in \mathbb{R}^n\} \geq 0$ for all $(t,x) \in G$ and $s \in \mathbb{R}^n$.

Let us explain the notation $d^+u(t,x;1,f)$ and $D^+u(t,x)$ used in (L3)-(L5). According to the general definition of upper derivative, we have

$$d^+u(t,x;\alpha,f) := \lim_{\varepsilon \longrightarrow 0} \sup_{(\delta,\alpha',f')\in\Delta_\varepsilon(t,x,\alpha,f)} \left[\frac{u(t+\delta\alpha',x+\delta f') - u(t,x)}{\delta}\right], \quad (2.2.64)$$

here $(t,x) \in G$, $(\alpha,f) \in \mathbb{R} \times \mathbb{R}^n$ and

$$\Delta_\varepsilon(t,x,\alpha,f) := \left\{(\delta,\alpha',f') \in]0,\varepsilon[\times\mathbb{R}\times\mathbb{R}^n : \left|\alpha-\alpha'\right| + \left|f-f'\right| \leq \varepsilon, t+\alpha'\delta \in]0,\theta[\right\}. \quad (2.2.65)$$

Assume $\alpha = 1$ .The quantity $d^+u(t,x;1,f)$ is the upper derivative of the function u in the direction $(1,f)$. According to the definition, we have

$$D^+u(t,x) := \left\{(a,s) \in \mathbb{R}\times\mathbb{R}^n : a\alpha + \langle s,f\rangle + d^+u(t,x;\alpha,f) \geq 0 \ \forall(\alpha,f) \in \mathbb{R}\times\mathbb{R}^n\right\}. \quad (2.2.66)$$

Note that

$$d^+u(t,x;\alpha,f) = -d^-(-u(t,x;\alpha,f)),$$

$$D^+u(t,x) = -D^-(-u(t,x)).$$

2.2.8.3 Definitions of minimax solutions

Let us consider now conditions $(M1)-(M3)$, which gives definitions of the minimax solution of equation (2.2.2). We suppose that the function $u(t,x)$ is continuous in these conditions.

(M1) For any $(t_0, x_0, z_0) \in \operatorname{gr} u$ and $s \in \mathbb{R}^n$ there exist some number $\tau \in]0, \theta[$ and a Lipschitz continuous function $(x(\cdot), z(\cdot)) : [t_0, \tau] \mapsto \mathbb{R}^n \times \mathbb{R}$ which satisfy the initial condition $(x(t_0), z(t_0)) = (x_0, z_0)$,the equation (2.2.57) and the equality $z(t) = u(t, x(t))$ for all $t \in [t_0, \tau]$.

(M2) For any choice of element $\psi \in \Psi$, the graph of the function u is weakly invariant with respect to the differential inclusion

$$(\dot{x}(t), \dot{z}(t)) \in E(t, x(t), z(t), \psi), \tag{2.2.67}$$

where E is an arbitrary multifunction which satisfies all conditions (i)-(iv), formulated in Subsection 2.2.7.

(M3) The function u is simultaneously an upper and a lower solution of equation (2.2.2), that is, u satisfies pair of conditions (Ui), (Lj) for some $i, j = 1, 2, \ldots, 5$.

Theorem 2.6. *For a lower semicontinuous function*

$$]0, \theta[\times \mathbb{R}^n \ni (t, x) \mapsto u(t, x) \in \mathbb{R}$$

the conditions $(U1) - (U5)$ are equivalent. Analogously, for an upper semicontinuous function u the conditions $(L1) - (L5)$ are equivalent. For a continuous function u the conditions $(M1) - (M3)$ are equivalent.

Proof. See [38]. □

Based on Theorem 2.6 we introduce the following definition of upper, lower and minimax solution.

Definition 2.27. A lower semicontinuous function

$$]0, \theta[\times \mathbb{R}^n \ni (t, x) \mapsto u(t, x) \in \mathbb{R}$$

that satisfies one of the above (equivalent) conditions $(U1) - (U5)$ is said to be an upper solution of equation (2.2.2) .

Definition 2.28. An upper semicontinuous function $u(\cdot, \cdot)$ that satisfies one of the conditions $(L1) - (L5)$ is said to be a lower solution of equation (2.2.2).

Definition 2.29. A continuous function u that satisfies one of the conditions $(M1) - (M3)$ is called a minimax solution of equation (2.2.2).

Remark 2.15. According to Theorem 2.6, minimax and viscosity solutions are equivalent. More than that, if upper (resp. lower) solutions are considered in the class of continuous functions, then they are equivalent to viscosity supersolutions (resp. subsolutions).

2.2.9 Existence and uniqueness of minimax solution of the Cauchy problem for Hamilton-Jacobi equation

In this subsection we give some results which imply the existence and uniqueness of solutions of the Cauchy Problem for Hamilton-Jacobi equations. The proofs of these results can be found in [38], the purpose here is just to present some aspects of the minimax theory of solution of Hamilton-Jacobi equation.

Definition 2.30. A continuous (respectively, lower or upper semicontinuous) function $(t,x) \mapsto u(t,x) :]0,T[\times\mathbb{R}^n \mapsto \mathbb{R}$ is called a minimax (respectively, upper or lower) solution of the Cauchy problem

$$\frac{\partial u}{\partial t} + \mathrm{H}(t,x,u,D_x u) = 0 \qquad (t,x) \in G =]0,T[\times\mathbb{R}^n \tag{2.2.68}$$

$$u(\theta,x) = \sigma(x) \qquad x \in \mathbb{R}^n \tag{2.2.69}$$

if it satisfies condition (2.2.69) and if the restriction of u to $G =]0,T[\times\mathbb{R}^n$ is a minimax (respectively, upper or lower) solution of (2.2.68).

Let us suppose that the Hamiltonian H and the boundary function σ satisfy the following conditions:

(H1) the Hamiltonian $H(t,x,z,p)$ is continuous on $D =]0,T[\times\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, the function $z \mapsto H(t,x,z,p)$ is nonincreasing;

(H2) the Lipschitz condition in the variable s is fulfilled

$$\left|H\left(t,x,z,s^{(1)}\right) - H\left(t,x,z,s^{(2)}\right)\right| \leq \rho(x)\left|s^{(1)} - s^{(2)}\right|$$

for all $\left(t,x,z,s^{(i)}\right) \in D$, $i = 1,2$, and the following estimate holds

$$|H(t,x,z,0)| \leq (1+|x|+|z|)\,c \qquad \forall\,(t,x,z) \in]0,\theta[\times\mathbb{R}^n \times \mathbb{R},$$

where $\rho(x) := (1+|x|)\,c$; and the number c is nonnegative;

(H3) for any bounded set $M \subset \mathbb{R}^n$ there exists a constant $\lambda(M)$ such that

$$\left|H\left(t, x', z, p\right) - H\left(t, x'', z, p\right)\right| \leq \lambda(M)\left(1 + |p|\right)\left|x' - x''\right|$$

for all $x', x'' \in M$, $(t, z, p) \in]0, \theta[\times \mathbb{R} \times \mathbb{R}^n$;

$(H4)$ the function $x \mapsto \sigma(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is continuous.

The following results are proved in [38].

Theorem 2.7. *For any upper solution u of the Cauchy problem* $(2.2.68) - (2.2.69)$ *and any lower solution v of this problem the inequality* $u \geq v$ *is valid.*

Theorem 2.8. *There exists a unique solution of the Cauchy problem* $(2.2.68) - (2.2.69)$.

Theorem 2.9. *There exist an upper solution u and a lower solution v of the Cauchy problem* $(2.2.68) - (2.2.69)$ *such that* $u \geq v$.

CHAPTER 3

Discontinuous solutions in L^∞ for Hamilton-Jacobi Equations

The original idea of this chapter comes from the paper of Chen Quiqiang and Su Bo [7]. Our contribution here has been to adapt these results with all the proofs in order to use them to attain the purpose of this study.

An approach is proposed to construct global discontinuous solutions in L^∞ for Hamilton-Jacobi equations; that will be useful in the sequel of this work. This approach allows the initial data only in L^∞ and may be applied to non convex Hamiltonian.

We are concerned with the global in finite time and local in space discontinuous solutions in L^∞ of the Cauchy problem for the Hamilton-Jacobi equations:

$$u_t + H(t, x, u, Du) = 0, \qquad x \in \mathbb{R}^n,\ 0 \le t \le T \tag{3.0.1}$$

$$u(0, x) = \varphi(x), \qquad x \in \mathbb{R}^n. \tag{3.0.2}$$

where $T > 0$ and $\varphi(\cdot)$ is a locally bounded measurable function are given.

3.1 Profit functions and their regularity

The following assumptions are made on the Hamiltonian $H(\cdot,\cdot,\cdot,\cdot)$ of the Cauchy problem (3.0.1)-(3.0.2) :

(B1) $H(\cdot,\cdot,\cdot,\cdot)$ is continuous in (t,x,z,p) and increasing in z;

(B2) $|H(t,x,z,p_1)-H(t,x,z,p_2)|\leq C_0(1+|x|)|p_1-p_2|$, and
$|H(t,x,z,0)|\leq C_0(1+|x|+|z|)$, for all $t\in]0,T]$;

(B3) $|H(t,x_1,z,p)-H(t,x_2,z,p)|\leq \lambda(L)(1+|p|)|x_1-x_2|$
where $|x_1|,|x_2|\leq L$;

(B4) $|H(t,x,z_1,p)-H(t,x,z_2,p)|\leq C_0(1+|x|+|p|)|z_1-z_2|$.

where $C_0>0$ and $L>0$ and $\lambda(L)>0$.

Definition 3.1. We define the essential infimum and supremum of an $L^\infty_{loc}(\mathbb{R}^d)$ function $v(\cdot)$ at every point $x\in\mathbb{R}^d$:

$$I(v)(x)\equiv \sup_{A\in S_x}\operatorname{ess\,inf}_{y\in A} v(y),\quad S(v)(x)\equiv \inf_{A\in S_x}\operatorname{ess\,sup}_{y\in A} v(y),$$

where

$$B^d(x,r)=\left\{y\in\mathbb{R}^n\,\middle|\,\left(\Sigma_{i=1}^d (y_i-x_i)^2\right)^{\frac{1}{2}}<r\right\}$$

$$S_x=\left\{A\subset\mathbb{R}^d \text{ measurable} \mid \lim_{r\longrightarrow 0}\frac{m(A\cap B^d(x,r))}{m(B^d(x,r))}=1\right\},$$

and $m:\mathcal{B}(\mathbb{R}^n)\longrightarrow[0,+\infty[$ is a Radon measure.

The definition of $v(\cdot)$ implies that $I(v)(x)$ and $S(v)(x)$ are well defined at every point $x\in\mathbb{R}^d$, and $I(v)(x)=S(v)(x)$ almost everywhere.

Now we introduce the winning and losing functions.

Definition 3.2. Fix $\tau\in[0,T]$ and $p(t,x)\in C([0,T]\times\mathbb{R}^n;\mathbb{R}^n)$. Given a measurable function v and a position (or value) function f, we define the winning and the losing functions:

$$\Lambda_{-}^{v}(t,x,(\tau,f,p)) = \inf\{S(v)(x(\tau)) - z(\tau) | (x(\cdot),z(\cdot)) \in \mathrm{Sol}(t,f(t,x),p)\}, \tag{3.1.1}$$

$$\Lambda_{+}^{v}(t,x,(\tau,f,p)) = \sup\{I(v)(x(\tau)) - z(\tau) | (x(\cdot),z(\cdot)) \in \mathrm{Sol}(t,f(t,x),p)\}, \tag{3.1.2}$$

where $\mathrm{Sol}(t,f(t,x),p)$ denotes the set of solutions:

$$(x(\cdot),z(\cdot)) : [\tau,t] \longrightarrow \mathbb{R}^n \times \mathbb{R}, \quad \text{for } t \geq \tau$$

of the characteristic inclusions $(\dot{x}(\cdot),\dot{z}(\cdot)) \in E(t,x,z,p)$ satisfying the conditions: $x(t) = x, z(t) = f(t,x)$, where

$$E(t,x,z,p) = \{(h,g) \in \mathbb{R}^n \times \mathbb{R}\ |h| \leq C_0(1+|x|), g = \langle h,p\rangle - H(t,x,z,p)\},$$

and $\langle \cdot,\cdot \rangle$ is the usual inner product in the Euclidean space $(\mathbb{R}^n,+,\cdot)$.

Remark 3.1. Note that the set $E(t,x,z,p)$ is

- a compact set in $\mathbb{R}^n \times \mathbb{R}$ for all $(t,x,z,p) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$;
- upper semicontinuous in the Hausdorff sense,

then, according to Subsection 2.2.5, the set $\mathrm{Sol}(t,f(t,x),p)$ is always non empty, and this ensures the validity of the definition 3.2.

Remark 3.2. Note that for $(h,g) \in E(t,x,z,p)$, estimates $(B2)$ and $(B3)$ imply

$$|h| \leq C_0 (1+|x|) \tag{3.1.3}$$

$$|g| \leq 2C_0 (1+|x|+|z|)(1+|p|), \tag{3.1.4}$$

where the nonnegative constant C_0 is the same one in assumptions $(B1)-(B4)$.

Lemma 3.1. *Fix $\tau \in [0,T]$ and $p(t,x) \in C([0,T] \times \mathbb{R}^n;\mathbb{R}^n)$. Then, for any nonnegative locally measurable function $h(t,x)$ and any point $x \in B(0,r)$,*

$$h(t,x) \leq \Lambda_{-}^{v}(t,x,(\tau,f,p)) - \Lambda_{-}^{v}(t,x,(\tau,f+h,p)) \leq e^{C(t-\tau)} h(t,x) \tag{3.1.5}$$

where C depends only on $C_0, T,$ and $|p|_c$, with $|p|_c = \sup_{s\in]\tau,t[} |p(s,x_h(s))|$.

Proof. See [7]. □

Remark 3.3. Similarly, for Λ_+^v, we have

$$h(t,x) \leq \Lambda_+^v(t,x,(\tau,f,p)) - \Lambda_+^v(t,x,(\tau,f+h,p)) \leq e^{C(t-\tau)} h(t,x) \tag{3.1.6}$$

where C depends only on $C_0, T,$ and $|p|_c$.

Before we study the properties of winning and losing profit functions, we first state the following simple fact which can be proved by the Gronwall inequality.

Suppose that $(x_j(\cdot), z_j(\cdot)), j = 1,2$, are two solutions of the characteristic inclusions:

$$|\dot{x}_j| \leq C(1 + |x_j|), \qquad \dot{z}_j = \langle \dot{x}_j, p \rangle - H(t, x_j, z_j, p)$$

with $x_1(\cdot) = x_2(\cdot), |z_1(t_0) - z_2(t_0)| \leq \epsilon, |x_1(t_0)| \leq M$, where $p(t,x) \in C([0,T] \times \mathbb{R}^n; \mathbb{R}^n)$ and $0 \leq \tau \leq t_0 \leq T$. Then

$$|z_1(\tau) - z_2(\tau)| \leq C\epsilon \tag{3.1.7}$$

$$|z_1(\tau) - z_2(t_0)| \leq C|\tau - t_0| \tag{3.1.8}$$

where C depends only on $M, T,$ and p.

Now we check whether our definition of winning and losing profit functions is well-defined; that is, given a measurable position function, whether the associated profit functions are measurable. For this purpose, we introduce a useful lemma from measure theory.

Lemma 3.2. *Suppose that $A \subset B^d(0,M) \subset \mathbb{R}^d$ enjoys the pointwise nondegenerate density property: for each $x \in A$, there exists a measurable subset $A_x \subset A, x \in A_x$, such that*

$$\limsup_{r \longrightarrow 0} \frac{m(A_x \cap B^d(x,r))}{m(B^d(x,r))} > 0. \tag{3.1.9}$$

Then A is measurable.

Proof. See [7]. □

We here mention a necessary and sufficient condition of measurability of a given set in $\mathbb{R}^d$ deduced from lemma 3.2.

Lemma 3.3. *A set $A \subset \mathbb{R}^d$ is measurable if and only if there is a zero-measure set $B \subset A$ such that every point $x \in A \setminus B$ satisfies nondegenerate density property.*

Proof. See [7]. □

Lemma 3.4. *Suppose $v \in L^{\infty}_{loc}\left(\mathbb{R}^d\right)$ .Then, for a fixed point x and any $\epsilon > 0$*

$$\limsup_{r \longrightarrow 0} \frac{m\left(\left\{y \in \mathbb{R}^d | v\left(y\right) \geq S\left(v\right)\left(x\right) - \epsilon\right\} \cap B^d\left(x,r\right)\right)}{m\left(B^d\left(x,r\right)\right)} > 0, \tag{3.1.10}$$

$$\limsup_{r \longrightarrow 0} \frac{m\left(\left\{y \in \mathbb{R}^d | v\left(y\right) \leq S\left(v\right)\left(x\right) - \epsilon\right\} \cap B^d\left(x,r\right)\right)}{m\left(B^d\left(x,r\right)\right)} > 0. \tag{3.1.11}$$

Proof. See [7]. □

Lemma 3.5. *Suppose U and V are open sets in $\mathbb{R}^d$. Let $f : U \longrightarrow V$ be a bi-Lipschitz homeomorphism. If $x \in A \subset \overline{A} \subset U$ with*

$$\limsup_{r \longrightarrow 0} \frac{m\left(A \cap B^d\left(x,r\right)\right)}{m\left(B^d\left(x,r\right)\right)} > 0,$$

then $f\left(x\right)$ is a point in $f\left(A\right) \subset f\left(\overline{A}\right) \subset V$ with

$$\limsup_{r \longrightarrow 0} \frac{m\left(f\left(A\right) \cap B^d\left(f\left(x\right),r\right)\right)}{m\left(B^d\left(f\left(x\right),r\right)\right)} > 0.$$

Proof. See [7]. □

Lemma 3.6. *Suppose that v is a locally bounded measurable function and $p(t,x)$ is continuous. Then, for any positive function $f\left(t,x\right) \in C\left(\left[0,T\right] \times \mathbb{R}^n\right)$, the corresponding profit function, as a function of x, $\Lambda^v_-\left(t,x,\left(\tau,f,p\right)\right) \in L^1_{loc}\left(\mathbb{R}^n\right)$ for any $\tau \leq t \leq T$.*

Proof. See [7]. □

Now we prove that $\Lambda^v_-\left(t,x,\left(\tau,f,p\right)\right)$ is measurable in both time and space variables if the position function is continuous.

Lemma 3.7. *Suppose that v is a locally bounded measurable function and $p\left(t,x\right)$ is continuous. Then, for any position function $f\left(t,x\right) \in C\left(\left[0,T\right] \times \mathbb{R}^n\right)$, the corresponding profit function $\Lambda^v_-\left(t,x,\left(\tau,f,p\right)\right) \in L^1_{loc}\left(\left[0,T\right] \times \mathbb{R}^n\right)$.*

Proof. See [7]. □

Indeed, there is an intrinsic regularity relation between the position function $f(t,x)$ and the profit function $\Lambda_{-}^{v}(t,x,(\tau,f,p))$, which is stated in the following lemma.

Lemma 3.8. *Let $p(t,x) \in C(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$. Let $v(x)$ be a locally bounded measurable function. Then*

(i) For any position function $f(t,x) \in L^1_{loc}([\tau,T]\times\mathbb{R}^n)$ satisfying $\sup_{[\tau,T]} ||f(t,\cdot)||_{L^\infty(A)} < \infty$, with any bounded measurable set A, $\Lambda_{-}^{v}(t,x,(\tau,f,p)) \in L^1_{loc}([\tau,T]\times\mathbb{R}^n)$.

(ii) Suppose that $g(t,x) \in L^1_{loc}([\tau,T]\times\mathbb{R}^n)$ satisfying $\sup_{[\tau,T]} ||g(t,\cdot)||_{L^\infty(A)} < \infty$, with any bounded measurable set A. Then there exists a unique $f(t,x) \in L^1_{loc}([\tau,T]\times\mathbb{R}^n)$ with

$$\sup_{[\tau,T]} ||f(t,\cdot)||_{L^\infty(A)} < \infty$$

for any bounded measurable set A such that

$$g(t,x) = \Lambda_{-}^{v}(t,x,(\tau,f,p)), \quad \text{for all } t \in [\tau,T],$$

and, in particular, if $g(t,x) \equiv 0$, then $f(\tau,x) = v(x)$ a.e.

Proof. See [7]. □

Similarly, for the losing profit function, we have

Lemma 3.9. *Let $p(t,x) \in C(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$. Let $v(x)$ be a locally bounded measurable function. Then*

(i) For any position function $f(t,x) \in L^1_{loc}([\tau,T]\times\mathbb{R}^n)$ satisfying $\sup_{[\tau,T]} ||f(t,\cdot)||_{L^\infty(A)} < \infty$, with any bounded measurable set A, $\Lambda_{+}^{v}(t,x,(\tau,f,p)) \in L^1_{loc}([\tau,T]\times\mathbb{R}^n)$.

(ii) *Suppose that* $g(t,x) \in L^1_{loc}([\tau,T] \times \mathbb{R}^n)$ *satisfying* $\sup_{[\tau,T]} ||g(t,\cdot)||_{L^{\infty}(A)} < \infty$, *with any bounded measurable set* A. *Then there exists a unique* $f(t,x) \in L^1_{loc}([\tau,T] \times \mathbb{R}^n)$ *with* $\sup_{[\tau,T]} ||f(t,\cdot)||_{L^{\infty}(A)} < \infty$ *for any bounded measurable set* A *such that*

$$g(t,x) = \Lambda^v_+(t,x,(\tau,f,p)), \quad \text{for all } t \in [\tau,T],$$

and, in particular, if $g(t,x) \equiv 0$,*then* $f(\tau,x) = v(x)$ *a.e.*

It follows from Lemma 3.8 (Lemma 3.9, respectively) that there is unique locally bounded measurable function $u^{\varphi}_{-}((t,x),p)$ ($u^{\varphi}_{+}((t,x),p)$, respectively) satisfying

$$\Lambda^v_-\left(t,x,\left(0,u^{\varphi}_-((t,x),p),p\right)\right) = 0 \tag{3.1.12}$$

$$\Lambda^v_+\left(t,x,\left(0,u^{\varphi}_+((t,x),p),p\right)\right) = 0 \tag{3.1.13}$$

respectively, for $(t,x) \in [0,T] \times \mathbb{R}^n$, where $\varphi(\cdot)$ is a locally bounded measurable function. It is easy to see that

$$u^{\varphi}_-((0,x),p) = \varphi(x) = u^{\varphi}_+((0,x),p). \tag{3.1.14}$$

3.2 Existence of Discontinuous Solution in L^{∞}

First we define the supsolution set and the subsolution set for the Cauchy problem (3.0.1)-(3.0.2) in terms of profit functions. Then we present the existence proof.

Let

$$W = \left\{u(t,x) \in L^{\infty}_{loc}([0,T] \times \mathbb{R}^n) \,|\, u(t,\cdot) \in L^{\infty}_{loc}(\mathbb{R}^n) \text{ for every } t \in [0,T]\right\}.$$

Denote by S^u the set of supsolutions $w(t,x) \in W$ which satisfy

(i) For any $p(t,x) \in C(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$,

$$\Lambda^{\varphi}_-(t,x,(0,w,p)) \leq 0, \tag{3.2.1}$$

for almost everywhere $(t,x) \in [0;T] \times \mathbb{R}^n$. Additionally, for every $t \in [0,T]$, (3.2.1) holds for almost every $x \in \mathbb{R}^n$.

(ii) The semigroup property : for every $\tau \in [0, T]$,

$$\Lambda_{-}^{w(\tau,x)}\left(t, x, (0, w, p)\right) \leq 0 \tag{3.2.2}$$

for almost every $(t, x) \in [\tau, T] \times \mathbb{R}^n$. Additionally, for every $t \in [0, T]$, (3.2.2) holds for almost every $x \in \mathbb{R}^n$.

Similarly, S^l denotes the set of subsolutions $w \in W$ which satisfy

(i) For any $p\,(t, x) \in C\left(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n\right)$,

$$\Lambda_{+}^{\varphi}\left(t, x, (0, w, p)\right) \geq 0, \tag{3.2.3}$$

for almost everywhere $(t, x) \in [0; T] \times \mathbb{R}^n$. Additionally, for every $t \in [0, T]$, (3.2.3) holds for almost every $x \in \mathbb{R}^n$.

(ii) Furthermore, for $\tau \in [0, T]$,

$$\Lambda_{+}^{w(\tau,x)}\left(t, x, (0, w, p)\right) \geq 0 \tag{3.2.4}$$

for almost everywhere $(t, x) \in [\tau, T] \times \mathbb{R}^n$. Additionally, for every $t \in [0, T]$, (3.2.4) holds for almost every $x \in \mathbb{R}^n$.

It implies from the definition of S^u with the aid of (3.1.5) that, for any $w \in S^u$ and

$p\,(t, x) \in C\left(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n\right), w\,(t, x) \geq u_{-}^{\varphi}\left((t, x), p\right)$ almost everywhere in $[0, T] \times \mathbb{R}^n$.

Similarly, for any $w \in S^l$ and

$p\,(t, x) \in C\left(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n\right), w\,(t, x) \leq u_{+}^{\varphi}\left((t, x), p\right)$ almost everywhere in $[0, T] \times \mathbb{R}^n$.

Definition 3.3. A function u is a solution of the Cauchy problem (3.0.1)-(3.0.2) if u belongs to S^u and S^l simultaneously.

Condition (i) of S^u and S^l contains the exact information how the solution u is determined by the initial data $\varphi\,(\cdot)$.

To study the perturbation of characteristics paths, we first recall the definition of weak isotropy.

Definition 3.4. Suppose that $O \subset \mathbb{R}^d$ is a domain. A Lipschitz continuous map $x : [0, T] \times O \longrightarrow \mathbb{R}^d$ is weak isotropy if

(i) $x(0) = I$;

(ii) $x(\tau)$ is a bi-Lipschitz continuous homeomorphism for any $\tau \in [0, T]$ with uniform Lipschitz constant independent of τ.

A typical example of weak isotropy is given by the following Lemma, which can be proven by the Theorem 1.8.

Lemma 3.10. *Suppose that $f_i : [0, T] \longrightarrow \mathbb{R}^n, i = 1, 2$ are bounded measurable functions and $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is Lipschitz continuous. Then the following differential equation generates a weak isotropy over $[0, T]$*

$$\dot{x}(t) = g(x) f_1(t) + f_2(t).$$

Proof. See [7]. □

The following lemma establishes a nice property of weak isotropy, which is the preservation of nondegenerate measure.

Lemma 3.11. *Suppose $x_0 \in B \subset x(T) O$ and $\limsup_{r \longrightarrow 0} \frac{m\left(B \cap B^d(x_0, r)\right)}{m\left(B^d(x_0, r)\right)} > 0$ where $O \subset \mathbb{R}^d$ is a domain. Then*

$$\limsup_{r \longrightarrow 0} \frac{m\left(x^{-1}(]0, T[, B) \cap B^{d+1}\left(x^{-1}(T) x_0, r\right)\right)}{m\left(B^{d+1}\left(x^{-1}(T) x_0, r\right)\right)} > 0 \tag{3.2.5}$$

where $x^{-1}(]0, T[, B) = \left\{(t, y) \in]0, T[\times \mathbb{R}^d | x^{-1}(t) y \in B\right\}$.

Proof. See [7]. □

We now show that S^u is not empty. In the proof later on, we denote by $L(f)$ and $L(B)$ the Lebesgue set of measurable function f and the subset of points of density 1 of measurable set B, respectively.

Lemma 3.12. *For fixed $p'(t, x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $u_+^\varphi((t, x), p') \in S^u$. More precisely u_+^φ satisfies that, for any $p(t, x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $0 \le \tau \le T$, and for every point $(t, x) \in L\left(u_+^\varphi\right)$ which is the set of Lebesgue points of u_+^φ,*

$$\Lambda_-^{u_+^\varphi}\left(t, x, \left(\tau, u_+^\varphi, p\right)\right) \le 0. \tag{3.2.6}$$

And, for every $t \ge \tau$,(3.2.6) holds for almost every $x \in \mathbb{R}^n$.

Proof. See [7]. □

Based on a given element $w \in S^u$, we can produce another one in S^u.

Lemma 3.13. *Given $w(t,x) \in S^u$ and $p(t,x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, we define*

$$\hat{w}(t,x) = \begin{cases} w(t,x) & \text{if } (t,x) \in [0,s] \times \mathbb{R}^n, \\ u_+^{w(s,x)}(t,x) & \text{if } (t,x) \in: [s,T] \times \mathbb{R}^n, \end{cases}$$

where $u_+^{w(s,x)}$ satisfies

$$\Lambda_+^{w(s,x)}\left(t,x,\left(s,u_+^{w(s,x)},p\right)\right) = 0.$$

Then $\hat{w} \in S^u$.

Proof. See [7]. □

It is easy to show that $\hat{w} \in S^u$, with the help of the proof of Lemma 3.12 and by the definition of Λ_+^v.

Now we are ready to prove the main result of this chapter.

Theorem 3.1. *Given a locally bounded measurable function $\varphi(\cdot)$, there exists a unique minimal elements of S^u, that is, the solution of the Cauchy problem (3.0.1)-(3.0.2).*

Proof. See [7]. □

3.3 Consistency

It has been shown in Theorem 2.6 that the minimax solutions are equivalent to the viscosity solutions, provided that the initial data are continuous. In this section we show that our solutions coincide with the minimax solutions, provided that the initial data are continuous.

Let the following functions

$$(t,x,y) \longrightarrow p_{\pm}(t,x,y) :]0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

$$(t,x,y) \longrightarrow p(t,x,y) :]0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

be locally Lipschitz continuous.

For the purpose of the proof of consistency, we need to establish the following lemma on the existence of mutual tracking trajectories of the characteristic inclusions.

Lemma 3.14. *Suppose $\varphi(\cdot)$ is continuous. Let $(w_\pm)$ and u be L^∞ supsolution (subsolution) and minimax solution of (3.0.1) - (3.0.2), respectively. Then, for every point $(t_0, x_0) \in L(w_\pm)$,there exist solutions of the systems of differential inclusions*

$$(\dot{x}, \dot{z}_\pm) \in E(t, x, z_\pm, p_\pm(t, x, y)), \quad (\dot{y}, \dot{z}) \in E(t, y, z, p(t, x, y)),$$

that satisfy the initial conditions:

$$(x(t_0), z_\pm(t_0)) = (x_0, w_\pm(t_0, x_0)), (y(t_0), z(t_0)) = (x_0, u(t_0, x_0))$$

and the inequalities $\pm(z_\pm(0)) - \varphi(x(0)) \geq 0 \quad \pm(z_\pm(0)) - \varphi(y(0)) \leq 0$, respectively.

Proof. See in [7]. □

Based upon Lemma 3.14, we can prove the following theorem.

Theorem 3.2. *Assume that $\varphi(\cdot)$ is continuous. Let u be L^∞ supsolution and v the continuous minimax solution of (3.0.1) - (3.0.2) respectively. Then $u \geq v$ almost everywhere.*

Proof. See [7]. □

Similarly, with the help of Lemma 3.14, we have

Theorem 3.3. *Assume that $\varphi(\cdot)$ is continuous. Let u be the L^∞ subsolution and v the continuous minimax solution of (3.0.1)-(3.0.2) respectively. Then $u \leq v$ almost everywhere.*

Proof. See [7]. □

Therefore, L^∞ solutions coincide with the continuous minimax solutions when the initial data are continuous. Consequently, the L^∞ solutions coincide with the continuous viscosity solutions.

CHAPTER 4

The relativistic Vlasov equation in the (HJ) form

This chapter presents the relativistic Vlasov equation in a time oriented four dimensional $\left(\mathbb{R}^{3+1}, g\right)$ curved space time of class C^{∞}, which has local coordinates (x^{α}), such that x^0 or t on $\mathbb{R}$ is time, $\left(x^i\right)$, $i = 1, 2; 3$, on $\mathbb{R}^3$ are space coordinates. The given metric tensor g is of hyperbolic signature $(-, +, +, +)$. We assume that in $\left(\mathbb{R}^{3+1}, g\right)$:

A_1 - the hypersurfaces $\mathbb{R}^3_t = \{t\} \times \mathbb{R}^3$ are spatial, and the lines $\mathbb{R} \times \{x\}$, $x \in \mathbb{R}^3$, are temporal,

A_2 - the time lines are orthogonal to space sections $\mathbb{R}^3_t$, it means that if (e_α) is a base of $\mathbb{R}^4$, then $g_{0i} = g\left(e_0, e_i\right) = 0$.

Greek indices $\alpha, \beta, \ldots$ range from 0 to 3, and the Latin indices from 1 to 3. We adopt the Einstein summation convention $a_\alpha b^\alpha = \sum_\alpha a_\alpha b^\alpha$.

The assumptions (A_1) and (A_2) imply that in local coordinates x=(x^α) of $\mathbb{R}^{3+1}$ the metric tensor is defined by:

$$g = g_{00}(x)\left(dx^0\right)^2 + g_{ij}(x)\,dx^i dx^j$$

where $g_{0i}(x) = 0, g_{ij}(x) > 0$, $i, j = 1, 2, 3$, $g_{00}(x) < 0$.

Different aspects of the relativistic Vlasov equation are presented through Section 4.1-4.7, this contribution is adapted from [8]. In Section 4.8 all the assumptions adopted in this study are given. In Section 4.9 the relativistic Vlasov equation is given and the transformation of this one into an Hamilton-Jacobi type equation is exposed. In this chapter, we give mathematical definitions of all the quantities appearing in the Vlasov equation, we transform steps by steps the relativistic Vlasov equation into a Hamilton-Jacobi equation, and present different steps which lead to the definitive form of relativistic Vlasov equation studied in this work.

4.1 Fibres bundles

Definition 4.1. A bundle (E, X, π) is a pair of two topological spaces E (the total space) and X (the total base), together with a continuous surjective map $\pi : E \longrightarrow X$.

Definition 4.2. A fiber bundle space (E, X, π, F, G) is a bundle (E, X, π) together with a space F, called the typical fibre, a topological group G of homomorphism of F into itself and a covering of X by a family of open sets $\{U_i; i \in J\}$ such that :

a) locally the bundle is a trivial bundle : i.e. it exists an homeomorphism $\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times F, \quad i \in J$, such that $\varphi_i(p) = (\pi(p), \tilde{\varphi}_i(p))$ with $\pi_{|\pi^{-1}(U_i)} = P_1 \circ \varphi_i$ where P_1 is the first projection;

b) $\forall x \in U_i$, $\pi^{-1}(x)$ is called the fiber at x, denoted F_x, and F is called the typical fibre ;

c) $\forall x \in U_i, \quad \tilde{\varphi}_{i,x} := \tilde{\varphi}_{|F_x} : F_x \longrightarrow F$ is an homeomorphism;

d) $\tilde{\varphi}_{i,x} \circ \tilde{\varphi}_{j,x}^{-1} : F \longrightarrow F$ is an element of the topological group G for all $x \in U_i \cap U_j$ and all $i, j \in J$;

e) the induced mapping $g_{ij} : U_i \cap U_j \longrightarrow G$ by $x \mapsto g_{ij}(x) = \tilde{\varphi}_{i,x} \circ \tilde{\varphi}^{-1}_{j,x}$ are continuous. They are called the transition functions. The transition functions satisfy the relation

$$g_{ik}(x) g_{kj}(x) = g_{ij}(x).$$

Remark 4.1. If F is a vector space and the group G is the linear group, the fiber bundle space is called a vector bundle.

Definition 4.3. A principal fibre bundle (E, X, π, G) is a fibre bundle (E, X, π) in which the typical fibre and the structural group are simultaneously G and G acts on G by left translation $\left(i.e\; R_g : G \longrightarrow G \text{ by } R_g(h) = hg, \text{ with } g \in G\right)$.

We shall need the following definition of the right action of G on the principal fibre bundle (E, X, π, G). Let $\{U_i : i \in J\}$ be the covering of X used to define the principal fibre structure. We first define the mapping $\tilde{R}_g$ on $\pi^{-1}(U_i)$ and then show that it can be defined coherently in the whole bundle E.

Let $p \in F_x$, $x \in U_i$, define g_i by

$$g_i = \tilde{\varphi}_{i,x}(p)$$

where $\tilde{\varphi}_{i,x}$ is the homomorphism from F_x to G. By definition

$$\left(\tilde{R}_g p\right)_i = \tilde{\varphi}^{-1}_{i,x}(g_i g), p \in \pi^{-1}(U_i).$$

Remark 4.2. Clearly $\tilde{R}_{g_1} \tilde{R}_{g_2} p = \tilde{R}_{g_1 g_2} p$, that is $\left\{\tilde{R}_g, g \in G\right\}$ is a group (anti) isomorphic to G which acts on the right on $\pi^{-1}(U_i)$.

p and $\tilde{R}_g$ belong to the same fibre. The group $\left\{\tilde{R}_g, g \in G\right\}$ acts transitively in each fibre.

Theorem 4.1. *For* $p \in \pi^{-1}\left(U_i \cap U_j\right)$

$$\left(\tilde{R}_g p\right)_i = \left(\tilde{R}_g p\right)_j.$$

Proof. See [8]. □

Remark 4.3. Since the mapping $\tilde{R}_g$ is independent of the choice of the open set U_i containing $\pi(p)$ it is well defined over all of E and we can write

$$\tilde{R}_g(p) = \tilde{\varphi}^{-1}_{i,x} \circ R_g \circ \tilde{\varphi}_{i,x}, \quad x = \pi(p).$$

One also note the simplified notation pg instead of $\tilde{R}_g(p)$.

The definition of cross section will be useful in the sequel of this work, we recall it here for its importance.

Definition 4.4. A cross-section of the bundle (E, X, π) is a mapping $f : X \longrightarrow E$ such that $\pi \circ f = id_E$.

The following theorem shows a particular relation between the bundle structure and a cross-section.

Theorem 4.2. *A principal fibre bundle (E, X, π, G) is trivial if and only if it has a continuous cross-section.*

Proof. See [8]. □

4.2 Lie group of transformations and adjoint representation

Let us consider a Lie group G and $\mathcal{G}$ its associated Lie algebra.

4.2.1 Lie group of transformations

We consider the finite dimensional group of transformation $\{\sigma_g; g \in G\}$ where G is a Lie group of dimension n, X is a smooth manifold of dimension n.

Definition 4.5. $\{\sigma_g : g \in G\}$ is a Lie group of transformation if the mapping

$$\sigma : G \times X \longrightarrow G \text{ by } (g, x) \mapsto \sigma(g, x)$$

is differentiable and if the set of transformation $\{\sigma_g : X \longrightarrow X; \sigma_g(x) = \sigma(g, x)\}$ together with the composition mapping follows the group properties:

$$\begin{cases} \sigma_{gh} = \sigma_g \circ \sigma_h \\ \sigma_e = id_X. \end{cases}$$

Remark 4.4. It follows that $\sigma_{g^{-1}} = \sigma_g^{-1}$.

Definition 4.6. A Lie group operates G operates effectively on X if $\sigma_g(x) = x$ for any $x \in X$ implies $g = e$.

G operates freely on X if $\sigma_g(x) \neq x$ unless $g = e$.

G operates transitively on X if for every $x \in X$ and $y \in X$ there exists $g \in G$ such that $\sigma_g(x) = y$.

Definition 4.7. A one-parameter subgroup of a Lie group G is a differentiable curve

$$g : \mathbb{R} \longrightarrow G \text{ by } t \mapsto g(t)$$

such that

$$\begin{cases} g(t)\, g(s) = g(t+s) \\ g(0) = e. \end{cases}$$

Definition 4.8. A Killing vector field on X relative to the action of G is the vector field with generators the group of transformation $\left\{\sigma_{g(t)} : t \in \mathbb{R}\right\}$.

Remark 4.5. The integral curve going through the Killing vector field v satisfies the equation

$$\begin{cases} \frac{d}{dt}\sigma_x(g(t)) = v(\sigma_x(g(t)) \\ \sigma_x(e) = x. \end{cases}$$

The following Theorem proved in [8] establishes that a one parameter subgroup is defined by its tangent vector γ on e.

Theorem 4.3. *The one parameter subgroup of G is the integral curve going through the origin e of left invariant vector field.*

Hence we can label the Killing vector field with generator

$$\left\{\sigma_{g(t)}; dg(t)\,/dt_{|t=0} = \gamma, t \in \mathbb{R}\right\}$$

by γ,

$$v^{\gamma}(x) = \frac{d\sigma_x(g(t))}{dt}_{|t=0} = \sigma'_x(e)\,\gamma\,.$$

The next result still in [8] proves that the set of left invariant vector fields on G forms a vector space of same dimension as G.

Theorem 4.4. *There is a bijective correspondence between the set of left invariant vector fields and the set of vectors tangent to G at e, namely the tangent space denoted $T_e(G)$.*

An element $\gamma \in T_e$ defines the generator v^L of a one parameter subgroup of transformation

$$v^L(h) = R_h'(e)\,\gamma$$

where $R_h(g) = gh$. The element γ also define the Killing vector fields v^K on X which generates the group of transformation $\left\{\sigma_{g(t)}\right\}$ of X as follows

$$v^K(x) = \sigma_x'(e)\,\gamma.$$

Then the dimension of the space $\{v^K\}$ of Killing vectors fields is equal to the rang r of the mapping $\sigma_x'(e)$; r is equal or small than the dimension p of $T_e(G)$. The following result in [8] characterizes the relation between the set of Killing vector fields and $T_e(G)$.

Theorem 4.5. *The four following statements are equivalent.*

1. $r = p$.
2. $v^K = 0$ *if and only if* $\gamma = 0$.
3. *G acts effectively on X.*
4. *the space of Killing vector fields on X is isomorphic to $T_e(G)$.*

4.2.2 The adjoint representation

Let us consider g an element of the Lie group G. The map $L_g : G \longrightarrow G$, $L_g(h) = gh$ is called the left transformation and the map $R_g : G \longrightarrow G$, $R_g(h) = hg$ is called the right transformation. By the definition of the Lie group G the maps L_g and R_g are both differentiable maps.

The map

$$L_g \circ R_g^{-1} : G \longrightarrow G, h \mapsto ghg^{-1}$$

is an inner automorphism of G that defines a linear isomorphism $\left(L_g \circ R_g^{-1}\right)'(e)$ from $T_e(G)$ into itself, and denoted $Ad(g)$. Since the Lie algebra and the tangent space $T_e(G)$ are identified, we deduce the following definition.

Definition 4.9. The mapping $Ad : G \longrightarrow \mathcal{L}(\mathcal{G}, \mathcal{G})$ such that

$$Ad(g) = \left(L_g \circ R_g^{-1}\right)'(e)$$

is called the adjoint representation of G on $\mathcal{G}$.

Definition 4.10. Let X and Y be two differentiable $\left(C^k\right)$ manifolds. Let $f : X \longrightarrow Y$ be a differentiable mapping.

The reciprocal image of a covariant vector θ_y under a differentiable mapping f, denoted $f^\star$, is defined by

$$\left(f^\star\theta\right)_x v_x = \theta_y \left(f'v\right)_y, \quad y = f(x).$$

4.3 The canonical form: Maurer-Cartan form

Definition 4.11. The canonical form or the Maurer-Cartan form ω on a Lie group is a one form with values in the Lie algebra $\mathcal{G}$ of G defined through the relation

$$\omega\left(v_g\right) = \gamma \text{ where } \gamma = L_g^{-1'} v_g \in T_e(G).$$

Theorem 4.6. *The Maurer-Cartan form is left invariant vector field, its reciprocal image under a right translation satisfies the relation*

$$R_g^\star \omega = Ad(g^{-1}) \circ \omega.$$

Proof. See [8]. □

4.4 Connections on a principal fibre bundle

Definition 4.12. A connection on the principal fibre bundle (P, X, π, G) is a mapping $\sigma_p : T_x(X) \longrightarrow T_p(P), \quad x = \pi(p)$ for each $p \in P$ such that
1. σ_p is linear,
2. $\pi'\sigma_p$ is the identity mapping on $T_x(X)$,
3. σ_p depends differentially on p,
4. $\sigma_{\tilde{R}_g p} = \tilde{R}'_g \sigma_p, \quad g \in G.$

Remark 4.6. Since σ_p is linear, the space of horizontal vector at p, denoted H_p, is defined by

$$H_p = \sigma_p \left(T_x \left(X \right) \right), \quad x = \pi \left(p \right).$$

The space H_p is a vector subspace of $T_p \left(P \right)$. Due to property 2 of Definition 4.12 we have also

$$\pi' H_p = T_x \left(X \right), \quad x = \pi \left(p \right)$$

thus H_p is isomorphic to $T_x \left(X \right)$, by the linear mapping π'. The Definition 4.12 can be expressed in terms of these horizontal vector spaces H_p as follows.

Definition 4.13. A connection on the principal fibre bundle (P, X, π, G) is a field of vector spaces H_p , $H_p \subset T_p \left(P \right)$, such that
1. $\pi' : H_p \longrightarrow T_x \left(X \right)$, $x = \pi \left(p \right)$, is an isomorphism of vector spaces;
2. H_p depends differentially on p,
3. $H_{\tilde{R}_g p} = \tilde{R}'_g H_p$.

Remark 4.7. The elements of the tangent space $V_p := T_p \left(G_x \right)$ to the fibre bundle G_x at p are called vertical vectors. Since $\pi' V_p = 0$ then

$$T_p \left(P \right) = H_p \oplus V_p, \tag{4.4.1}$$

that is any $v \in T_p \left(P \right)$ can be written uniquely

$$v = v_H + v_V, v_H \in H_p, v_V \in V_p$$

v_V depends like v_H on the choice of H_p.

4.4.1 Canonical isomorphism between the Lie algebra $\mathcal{G}$ and the vertical space V_p

Since G acts effectively on P by $\tilde{R}_g$, there is a natural vector isomorphism between the Lie algebra $\mathcal{G}$ of G and the space of Killing vector fields $\left\{ v^K \right\}$ on P relative to G defined by $\hat{v}_{(\alpha)} \leftrightarrow v^K_{(\alpha)}$, where $\hat{v}_{(\alpha)} \in \mathcal{G}$ and $v^K_{(\alpha)} \in \left\{ v^K \right\}$ are both generated by $v_{(\alpha)} \left(e \right) = dg \left(s \right) / ds_{|_{s=0}} \in T_e \left(G \right)$

$$\hat{v}_{(\alpha)} \left(g \right) = L'_g \left(e \right) v_\alpha \left(e \right), \quad v^K_{(\alpha)} \left(p \right) = d \left(\tilde{R}_{g(s)} p \right) / ds_{|_{s=0}}.$$

Next because p and $\tilde{R}_g p$ lie in the same fibre, any Killing vector $v^K(p)$ is a vertical vector. In addition a Killing vector field does not vanish at any point unless it corresponds to the zero element of the Lie algebra according to Theorem 4.5. Since the correspondence is linear, the dimension of the space $\{v^K(p)\}$ is equal to the dimension of the space $\{v^K\}$, which is equal to the dimension of the Lie group G and of V_p. In conclusion let $\gamma \in \mathcal{G}$ corresponds to $dg(s)/ds_{|_{s=0}} \in T_e(G)$; the equation

$$v(p) = d\left(\tilde{R}_{g(s)} p\right)/ds|_{s=0}$$

defines the canonical isomorphism between $\mathcal{G}$ and V_p

$$v(p) \leftrightarrow \hat{v} \quad v(p) \in V_p. \tag{4.4.2}$$

Definition 4.14. An (exterior differential) p-form φ with values in a given finite dimensional real vector space V on a manifold X is an application $x \mapsto \varphi_x, x \in X, \varphi_x$ is p-form at x with values in V.

It can be written, if e_α is a basis of V

$$\varphi = \varphi^\alpha \otimes e_\alpha$$

where the φ^α are the scalar valued p forms. φ is of class C^k on X if the φ^α are of class C^k.

Given an element of $\mathcal{G}$, the canonical isomorphism defines a vector field $\widehat{v_\gamma(p)} = \gamma$. When we are given the field of horizontal subspaces H_p we have for each $p \in P$ a well defined family of linear mapping

$$T_p(P) \longrightarrow \mathcal{G} \text{ by } v \mapsto \widehat{ver\, v} \quad p \in P, \tag{4.4.3}$$

a direct consequence of (4.4.1) and (4.4.2).

In agreement with previous definition we call the family of mapping (4.4.3) 1-form ω in P with values in the vector space $\mathcal{G}$, the Lie algebra of G:

$$\omega(v) = \widehat{ver\, v} \text{ and thus } \omega(hor\, v) = 0, \quad \forall v \in T_p(P).$$

Note that if (e_α) is a basis for $\mathcal{G}$ and if $\left(\theta^i\right)$ is a basis for $T_p^\star(P)$, then ω can be written

$$\omega = \omega^\alpha \otimes e_\alpha = \omega_i^\alpha \theta^i \otimes e_\alpha$$

where the ω^α are 1-form on P.

It results from the property 2 of Definition 4.13 that the ω^α are differentiable 1-form on P, and ω is a differentiable 1-form on P with values on $\mathcal{G}$. The equivariance (propertiy 3 of Definition 4.13) of the horizontal subspaces H_p insures that $\tilde{R}'_g$ preserves the decomposition of any tangent vector space to P into a horizontal and vertical part

$$\tilde{R}'_g v = \left(\tilde{R}'_g v\right)_H + \left(\tilde{R}'_g v\right)_V = \tilde{R}'_g v_H + \tilde{R}'_g v_V \text{ if } v = v_H + v_V.$$

If we compute the pull-back of ω by $\tilde{R}_g$ we find

$$\left(\tilde{R}^\star_g \omega\right)(v) = \omega\left(\tilde{R}'_g v\right) = \omega\left(\left(\tilde{R}'_g v\right)_V\right) = \omega\left(\tilde{R}'_g v_V\right).$$

The restriction of ω to a fibre $G_x = \pi^{-1}(x)$ defines a 1-form on G_x (which we shall also call ω) by

$$\omega(v_V) = \hat{v}_V, \quad v_V \in T_p(G_x) \equiv V_p.$$

This form can be identified with the Maurer-Cartan canonical 1-form on G through the identification $\mathcal{I} : G \longrightarrow G_x$ obtained by choosing a point in G_x and setting $\mathcal{I}(h) = p$ if and only if $p = \tilde{R}_h p$. We deduce then from the transformation law of this canonical 1-form that

$$\left(\tilde{R}'_g \omega\right)(v) = Ad\left(g^{-1}\right)\omega(v)$$

where Ad is the adjoint representation. We arrive thus to the third definition.

Definition 4.15. A connection in the principal fibre bundle (P, X, π, G) is a 1-form on P with values in the vector space $\mathcal{G}$ such that
1. $\omega_p(u) = \hat{u}$ where $u \in V_p$ and $\hat{u} \in \mathcal{G}$ are related by the canonical isomorphism,
2. ω_p depends differentiably on p,
3. $\omega_{\tilde{R}_g p}\left(\tilde{R}'_g \omega\right) = Ad\left(g^{-1}\right)\omega_p(v)$.

Remark 4.8. If a connection is given by the Definition 4.15 we define the horizontal subspaces by the kernels of the mapping $\omega_p : T_p(P) \longrightarrow \mathcal{G}$, namely

$$H_p = \left\{v \in T_p(P); \omega_p(v) = 0\right\},$$

it is easy to verify that these spaces verify the properties of Definition 4.13 and thus the equivalence of the two definitions.

4.4.2 Local connection 1-form on the base manifold

For a given connection ω we shall now associate with each differentiable local section of $\pi^{-1}(U) \subset P$, $U \subset X$, a 1-form with values in $\mathcal{G}$.

Let

$$f : U \subset X \longrightarrow f(U) \subset P, \; \pi \circ f = id_X$$

be a local section of P, we define a 1-form $f^{\star}\omega$ on U with values in $\mathcal{G}$ by the pull-back of ω by f : that is if $u \in T_x X$, $u \in U$

$$\left(f^{\star}\omega\right)_x (u) = \omega_{f(x)}\left(f' u\right).$$

Conversely

Theorem 4.7. *Given a differentiable 1-form $\overline{w}$ on U with values in $\mathcal{G}$, and a differentiable section f of $\pi(U)^{-1}$, there exists one and only one connection ω on $\pi(U)^{-1}$ such that $f^{\star}\omega = \overline{\omega}$.*

Proof. See [8]. □

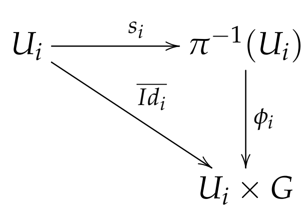
This construction can be extended to the case where $\overline{\omega}$ is a differentiable 1-form on the whole base X and leads to the following theorem.

Theorem 4.8. *There exists in each principal bundle with compact base X infinitely many connections.*

Proof. See [8]. □

In the next paragraph we shall prove the converse of the above theorem, namely given a trivialization $\{U_i, \phi_i\}$ of the bundle P and a connection ω on P, there corresponds a unique family $\{\overline{\omega}_i\}$ of connection 1-form on the base of manifold.

First we define the section s_i of $\pi^{-1}(U_i)$ canonically associated with the trivialization ϕ_i.



Let $\overline{Id_i} : U_i \longrightarrow U_i \times G$ by $x \mapsto (x, e)$. A trivialization defines a section s_i and vice-versa, through the equation

$$s_i = \phi_i^{-1} \circ \overline{Id_i}.$$

Let $\overline{\omega}_i = s_i^{\star}\omega$, the form $\overline{\omega}_i$ on U_i is called the connection form in the local trivialization ϕ_i.

Potentials. In the Yang-Mills theory of physics, the 1-forms $\overline{\omega}_i$ are usually called potentials (gauge potentials) and the trivialization ϕ_i are called local gauges. The $\overline{\omega}_i$ are related to the traditional potential A by a multiplicative constants.

The next Theorem [8] gives the gauge transform relation.

Theorem 4.9. *At a point $x \in U_i \cap U_j$ the connection forms $\overline{\omega}_i$ and $\overline{\omega}_j$ in the local gauges ϕ_i and ϕ_j corresponding to the same connection on P are linked by the relation*

$$\overline{\omega}_{i,x} = Ad\left(g_{ij}^{-1}\left(x\right)\right)\overline{\omega}_{j,x} + \left(g_{ji}^{\star}\theta_{MC}\right)_x$$

where g_{ij} is the transition mapping

$$g_{ij} : U_i \cap U_j \longrightarrow G \quad by \quad x \mapsto g_{ij}\left(x\right) = \tilde{\phi}_{i,x} \circ \tilde{\phi}_{j,x} \in G$$

and $g_{ij}^{\star}\theta_{MC}$ denotes the pull back on $U_i \cap U_j$ of the Maurer-Cartan 1-form on G by this transition mapping.

4.5 Curvature

4.5.1 Curvature

Definition 4.16. Let (P, X, π, G) be a principal bundle with connection H_P defined by a 1-form ω on P with values in $\mathcal{G}$. Let $h : T_p(P) \longrightarrow H_p$ by $v \mapsto v_H$.

The exterior covariant derivative $D\phi$ of a 1-form $\phi = \phi^{\alpha} \otimes e_{\alpha}$ on P with values in some vector space with basis (e_{α}) is defined by the relation

$$D\phi\left(v_1, \ldots, v_{r+1}\right) = d\phi\left(hv_1, \ldots, hv_{r+1}\right)$$

where $d\phi = (d\phi^{\alpha}) \otimes e_{\alpha}$.

Definition 4.17. The 2-form $\Omega = D\omega$ with values in $\mathcal{G}$ is called the curvature form of the connection ω (curvature form of the connection H_p).

Definition 4.18. A differentiable r-form α on P with values in a vector space E is said to be of type (ρ, E) if

$$\tilde{R}_g^\star \alpha = \rho\left(g^{-1}\right)\alpha, \forall g \in G.$$

where ρ is a representative of G in E.

One also says that α is equivariant under the right action R_g by the representation ρ.

A differentiable r-form α on a principal fibre bundle P is said to be horizontal form if $\alpha\left(v_1, \ldots, v_r\right) = 0$ whenever at least one of the vectors $v_1, \ldots, v_r$ is vertical.

If in addition α is horizontal, it is said to be tensorial of type (ρ, E).

Lemma 4.1. *The curvature form Ω is a tensorial form of type $(Ad, \mathcal{G})$:*

$$\left(\tilde{R}_g'\Omega\right)(u, v) = Ad\left(g^{-1}\right)\Omega\,(u, v)\,.$$

Proof. See [8]. □

Theorem 4.10 (Cartan structural equation)**.** *If ω is a connection on P and $D\omega = \Omega$, then*

$$\Omega\,(u, v) = d\omega\,(u, v) + \left[\omega\,(u)\,, \omega\,(v)\right]\,.$$

Proof. See [8]. □

4.5.2 Local curvature on the manifold, coordinate expressions of potential and field strength

In a local trivialization (U_i, ϕ_i) the 2-form Ω on $\pi^{-1}(U_i)$ is represented by the 2-form $\overline{\Omega}_i$ on U_i defined through the corresponding cross section s_i by

$$\overline{\Omega}_i = s_i^\star \Omega\,.$$

It results from the Cartan structural equation and the commutation of the pull back with d that

$$\overline{\Omega}_i = d\overline{\omega}_i + \left[\overline{\omega}_i, \overline{\omega}_j\right]\,.$$

In Yang-Mills theory a 2-form $\overline{\Omega}_i$ is called a field strength or a Yang-Mills field in the gauge ϕ_i and is usually labeled $\boldsymbol{F_i}$ up to a multiplicative constant.

Let (e_α) denote a basis of $\mathcal{G}$ and c^a_{bc} the structure constant given by $[e_a, e_b] = c^c_{ab} e_c$. Let (e_μ) denote a basis of T_xX for $x \in U$. Then the components $\overline{\omega}^a_\mu$ and $\overline{\Omega}^a_{\mu\nu}$ are defined by $\overline{\omega}\left(e_\mu\right) = \overline{\omega}^a_\mu e_a\,, \quad \overline{\Omega}\left(e_\mu, e_\nu\right) = \overline{\Omega}^a_{\mu\nu} e_a$, while the structure equation gives

$$\overline{\Omega}^a_{\mu\nu} = \partial_\mu \overline{\omega}^a_\nu - \partial_\nu \overline{\omega}^a_\mu + c^a_{bc} \overline{\omega}^b_\mu \overline{\omega}^c_\nu.$$

4.6 Phase space of particles, Yang-Mills charge

Definition 4.19. The **Yang-Mills charge** is a C^∞ function

$$q : \mathbb{R}^4 \longrightarrow \mathcal{G} \tag{4.6.1}$$

such that $q = q^a \varepsilon_a$, and of Ad type by change gauge transform whose given norm is e.

One defines $\mathcal{O}$ the sphere on $\mathcal{G}$ defined by

$$\mathcal{O} : q \cdot q = |q|^2 = e. \tag{4.6.2}$$

The phase space of particles denoted $\mathcal{P}_V$ with Yang-Mills charge is defined by

$$\mathcal{P}_\mathcal{V} = T\mathbb{R}^4 \times \mathcal{G} \equiv \mathbb{R}^4 \times \mathbb{R}^4 \times \mathcal{G}$$

with local coordinates $(x^\alpha, p^\alpha, q^a), \alpha = 0, 1, 2, 3,\ a = 1, 2, \ldots, N$. Here $x = (x^\alpha)$ is the particle position, $p = (p^\alpha)$ the particle momentum, $q = (q^a)$ the Yang-Mills charge of particles.

Remark 4.9. The trajectories of particles of momentum $p = (p^\alpha)$ and charge $q = q^a \varepsilon_a$ are a solution of the differential system, [31],

$$\frac{dx^\alpha}{ds} = p^\alpha \tag{4.6.3}$$

$$\frac{dp^\alpha}{ds} = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + q \cdot F^\alpha_\beta p^\beta = P^\alpha \tag{4.6.4}$$

$$\frac{dq^a}{ds} = -p^\alpha c^a_{bc} q^c A^b_\alpha = Q^a. \tag{4.6.5}$$

Remark 4.10. According to (4.6.3-4.6.5) the local coordinates of the tangent vector X in the trajectory of a particle is given by

$$X = (p^{\alpha}, P^{\alpha}, Q^{a}) . \tag{4.6.6}$$

The scalar $g(p, p)$ is constant along the orbit of X in $\mathcal{P}_V$, for a particle of rest mass normalized to unity $m = 1$ one has

$$\forall x \in \mathbb{R}^4 \; g_{\alpha\beta}(x)\, p^{\alpha} p^{\beta} = -1. \tag{4.6.7}$$

For a fixed x, equation (4.6.7) defines an hyperboloid $\mathcal{P}_x \subset \mathcal{P}_V$. By (4.6.7) we obtain

$$\mathcal{P}_x : p^0 = \sqrt{(g_{00}(x^{\alpha}))^{-1}\left(-1 - g_{ij}(x^{\alpha})\, p^i p^j\right)} \tag{4.6.8}$$

where $p^0 > 0$ symbolizes the fact that particles eject towards the future.

Observing that $\frac{dx^0}{ds} = p^0$, we deduce that x^0 is an increasing parameter, hereafter it will be denoted $x^0 = t \in [0, +\infty[$.

The following proposition expresses that particles with rest mass $m = 1$ lie in $\mathcal{P}_{\mathcal{V}}$.

Proposition 4.1. *The trajectories of particles of the rest mass m=1 lie in $\mathcal{P}_V$.*

Proof. See [31]. □

Remark 4.11. In [31] the proof of global existence and uniqueness of the trajectory of the system (4.6.3)-(4.6.5) is settled if initial conditions are given.

In the kinetic theory, the matter is composed of the collection of particles whose size is negligible at the considered scale. It is assumed that the state of matter, in a space-time $(\mathbb{R}^4; g)$, is represented by a particle distribution function. The distribution is interpreted as density of particles at a point x which has associated momentum $p \in T_x(\mathbb{R}^4)$. We state a definition.

Definition 4.20. A distribution function f is a positive scalar function on the space phase $\mathcal{P}_{\mathcal{V}}$ of a Yang-Mills charge i.e

$$f : \mathcal{P}_{\mathcal{V}} \times \mathcal{G} \longrightarrow \mathbb{R}_+ , \quad (x, p, q) \mapsto f(x, p, q).$$

4.7 Yang-Mills potential and field

Definition 4.21. A **Yang-Mills potential** A is a $\mathcal{G}-$valued 1-form in $\mathbb{R}^4$.

$$A : \mathbb{R}^4 \longrightarrow \mathcal{G}\,, A = \left(A_\mu\right) \text{ in local coordinates.}$$

Now $\mathbb{R}^4$ can be consider as a base of principal fibre bundle η. In fact, let $\eta = \mathbb{R}^4 \times G$, then G operates freely in the right on η by

$$\psi : \eta \times G \longrightarrow \eta\,, ((n,g)\,,h) \mapsto (n,gh)\,.$$

One deduces that $\left(\eta, \mathbb{R}^4, \pi, G\right)$ is a principal fibre bundle with base $\mathbb{R}^4$ and structural group G. Let S be a global section of η (if it exists), one can find a 1-form connection $\omega : \eta \longrightarrow \mathcal{G}$, called Yang-Mills connection such that

$$A = S^\star \omega.$$

Let Ω be the curvature of ω, then one deduces from the Cartan structural equation that

$$\Omega = d\omega + [\omega, \omega]\,.$$

Let F be the $\mathcal{G}-$valued 2-form on $\mathbb{R}^4$, the **Yang-Mills field**, define by

$$F = S^\star \Omega\,,$$

then

$$\begin{aligned} F &= S^\star \Omega = S^\star \left(d\omega + [\omega,\omega]\right) \\ &= S^\star d\omega + [S^\star\omega, S^\star\omega] \\ &= dA + [A, A]\,. \end{aligned}$$

It follows that in local coordinates x^μ of $\mathbb{R}^4$ and in the basis (ε_a) of $\mathcal{G}$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + c^a_{bc} A^b_\mu A^c_\nu\,, \tag{4.7.1}$$

where c^a_{bc} are the structure constants of the Lie algebra $\mathcal{G}$. One observes that (4.7.1) can also be denoted

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \left[A_\mu, A_\nu\right]^a\,. \tag{4.7.2}$$

By the antisymmetry of the Lie bracket, it follows that $F^a_{\mu\nu}$ is antisymmetric with respect to the indexes μ and ν , thus $F_{ii} = 0$. Consequently c^a_{bc} is also antisymmetric in b and c. Note that by (4.8.1), one deduces that

$$c^a_{bc} = 0. \tag{4.7.3}$$

In fact by (4.8.1), setting $a = \varepsilon_a\,, \quad b = \varepsilon_b\,, \quad c = \varepsilon_c$

$$\varepsilon_a \cdot [\varepsilon_b, \varepsilon_c] = [\varepsilon_a, \varepsilon_b] \cdot \varepsilon_c$$

i.e

$$c^d_{bc} \varepsilon_a \cdot \varepsilon_d = c^d_{ab} \varepsilon_d \cdot \varepsilon_c. \tag{4.7.4}$$

One has $\varepsilon_a \cdot \varepsilon_d = \delta^a_d\,, \quad \varepsilon_d \cdot \varepsilon_c = \delta^d_c$, then (4.7.4) becomes

$$c^a_{bc} = c^c_{ab}\,. \tag{4.7.5}$$

Setting $a = c$ in (4.7.5), one obtains

$$c^a_{bc} = c^a_{cb} = -c^a_{bc}\,.$$

Then (4.7.3) follows easily.

Definition 4.22. The gauge covariant derivative, denoted $\hat{\nabla}$, of a function $\lambda : \mathbb{R}^4 \longrightarrow \mathcal{G}$ is defined by

$$\hat{\nabla}_\beta \lambda = \nabla_\beta \lambda + \left[A_\beta, \lambda\right].$$

4.8 Main Assumptions

We present here the assumptions of this work.

1. We assume that G is a Lie group with $\mathcal{G}$ the associated Lie algebra. We consider that $\mathcal{G}$ is the euclidean space $\mathbb{R}^N$ embedded with an Ad-invariant scalar product, which is denoted by the dot $\cdot$. This scalar product is such that

$$[u, v] \cdot w = u \cdot [v, w]\,,\ u, v, w \in \mathcal{G} \tag{4.8.1}$$

where $[\cdot, \cdot]$ is the Lie bracket. The Lie algebra $\mathcal{G}$ is assumed to have a fixed basis denoted (ε_a) $a = 1, \ldots, N$.

2. The $F_{\alpha\beta}$ and A_α are given in $C_0^\infty\left([0,\infty[\times\mathbb{R}^3\right)$ the space of restriction to $[0,\infty[\times\mathbb{R}^3$ of C^∞ functions or $\mathcal{G}-$values tensors with compact support on $\mathbb{R}^3$.

3. We impose on the Yang-Mills potential $A = (A_\alpha)$ the temporal gauge
$$A_0 = 0\,. \tag{4.8.2}$$

4. We assume that the metric tensor $g = \left(g_{\alpha\beta}\right)$ has a Lorentzian signature $(-,+,+,+)$ and expresses in local coordinates
$$g = -dt^2 + g_{ij}\left(x^\alpha\right)dx^i dx^j \tag{4.8.3}$$
in which g_{ij} are given differentiable functions of the the time t and space $\overline{x} = \left(x^i\right)$, $i = 1,2,3$. One then has $g_{0j} = 0, \quad g_{ij} > 0 \;\; i,j = 1,2,3$.
One assumes the $\frac{\partial_\alpha g_{ij}}{g_{ij}}$ are bounded. This implies that there exists a constant $C > 0$ such that
$$\left|\frac{\partial_\alpha g_{ij}}{g_{ij}}\right| \leq C,\; i,j = 1,2,3,\; \alpha = 0,1,2,3. \tag{4.8.4}$$
The assumption (4.8.4) is for instance satisfies by the Minkowski tensor metric and also by an inhomogeneous tensor metric of the Szekeres-Szafron family of solution of Einstein equation [25] .

5. We assume that the non-abelian charge q of the Yang-Mills particles is a function of class C^∞ from $\mathbb{R}^4$ to $\mathcal{G}$ whose given norm is $e > 0$. One also supposes that $q^N \geq 0$.

Remark 4.12. If we use the relations (4.7.1) and (4.8.2) we obtain that
$$F_{0i} = \partial_0 A_i\,, \quad i = 1,2,3. \tag{4.8.5}$$
The Christoffel symbols $\Gamma^\lambda_{\alpha\beta}$ of the Levi-Cevita connection ∇, associated with g are
$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}\left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right),$$
are computed to be
$$\begin{cases} \Gamma^0_{ij} = \frac{1}{2}\partial_0 g_{ij}\,, & \Gamma^i_{0j} = \frac{1}{2}g^{il}\partial_0 g_{il}\,, \\ \Gamma^0_{0\alpha} = 0\,, & \Gamma^i_{pj} = \frac{1}{2}g^{ik}\left(\partial_p g_{kj} + \partial_j g_{pk} - \partial_k g_{pj}\right). \end{cases} \tag{4.8.6}$$

4.9 Transformation of the relativistic Vlasov equation

The Vlasov equation, also called Liouville -Vlasov equation, is the equation satisfies by the distribution function in the space-time $\left(\mathbb{R}^{3+1}, g\right)$. In the Vlasov model, one assumes that a gas is so rarefied that the particles trajectories do not cross. The equation is the cancellation of the Lie derivative of f with respect to vector field tangent to the trajectories of particles; it follows that

$$\mathcal{L}_X f = 0. \tag{4.9.1}$$

In the local coordinates, (4.9.1) writes

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} + Q^a \frac{\partial f}{\partial q^a} = 0. \tag{4.9.2}$$

We recall that

$$X = \left(p^\alpha, P^\alpha, Q^a\right).$$

. By (4.6.2) and (4.6.7), q^N can be expressed with q^a ,
$a = 1, \ldots, N-1$, and p^0 expressed with $x^\alpha, p^i \quad i = 1,2,3$. Then we obtain that the distribution function f of Yang-Mills particles is defined as function of independent variables $\left(t, x^i, p^i, q^a\right) \quad i = 1,2,3;\ a = 1, \ldots, N-1$, denote by $(t, \overline{x}, \overline{p}, \overline{q})$. So $f = f\,(t, \overline{x}, \overline{p}, \overline{q})\,;\, t \in \mathbb{R}, \overline{x} \in \mathbb{R}^3, \overline{p} \in \mathbb{R}^3, \overline{q} \in \mathbb{R}^{N-1}$. Using the fact that $q^N \geq 0$ and $p^0 > 0$, we deduce from (4.9.2) the following equivalent form of relativistic Vlasov equation

$$-\frac{\partial f}{\partial t} = \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} + \frac{Q^a}{p^0}\frac{\partial f}{\partial q^a}. \tag{4.9.3}$$

Now we will transform (4.9.3) into another equivalent form. Let us denote H the function defined at the right hand side of equation (4.9.3) that is

$$H(t, \bar{x}, \bar{p}, \bar{q}, f, \nabla_{\bar{x},\bar{p},\bar{q}} f(t, \bar{x}, \bar{p}, \bar{q})) = \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} + \frac{Q^a}{p^0}\frac{\partial f}{\partial q^a}, \tag{4.9.4}$$

where using the relations (4.6.3)-(4.6.5)

$$\frac{P^i}{p^0} = -2\Gamma^i_{0j}p^j - \Gamma^i_{jk}p^j\frac{p^k}{p^0} + q\cdot\left(F^i_0 + F^i_j\frac{p^j}{p^0}\right), \frac{Q^a}{p^0} = -\frac{p^k}{p^0}C^a_{bc}q^c A^b_k. \quad (4.9.5)$$

Setting

$$u_i = \frac{\partial f}{\partial x_i}, v_i = \frac{\partial f}{\partial p^i}, w_a = \frac{\partial f}{\partial q^a}$$

in the relation (4.9.4) and then using relations (4.9.5), the function H becomes:

$$H : [0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\longrightarrow\mathbb{R}$$

with

$$H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) = \frac{p^i}{p^0}u_i + \left(q\cdot\left(F^i_0 + F^i_j\frac{p^j}{p^0}\right) - 2\Gamma^i_{0j}p^j - \Gamma^i_{jk}p^j\frac{p^k}{p^0}\right)v_i - \frac{p^k}{p^0}C^a_{bc}q^c A^b_k w_a. \quad (4.9.6)$$

Then we obtain the following Hamilton-Jacobi equation, where a Lipschitz continuous function $f_0 : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\longrightarrow\mathbb{R}$ and a real number $T > 0$ are given:

$$\begin{cases} f_t(t,\bar{x},\bar{p},\bar{q}) + \overline{H}(t,\bar{x},\bar{p},\bar{q}) = 0 & \text{in }]0,T[\times B_{\mathbb{R}^3}(O,T)\times\mathbb{R}^3\times\mathbb{R}^{N-1} \\ f(0,\bar{x},\bar{p},\bar{q}) = f_0(\bar{x},\bar{p},\bar{q}) & \text{on } B_{\mathbb{R}^3}(\text{O,T})\times\mathbb{R}^3\times\mathbb{R}^{N-1}. \end{cases} \quad (4.9.7)$$

with

$$\overline{H}(t,\bar{x},\bar{p},\bar{q}) = H(t,\bar{x},\bar{p},\bar{q},f,\nabla_{\bar{x},\bar{p},\bar{q}}f(t,\bar{x},\bar{p},\bar{q})).$$

CHAPTER 5

Existence results and optimal control problem

In this chapter, we address the issues of a global existence theorem in finite time to the main problem of this work, and the construction of an optimal control problem. Firstly, we establish some energy estimates, based in particular on the previous chapters and the use of some classical results. Secondly, we prove the main existence theorem of this work. Finally, we establish that the viscosity solution of the relativistic Vlasov equation, for which the existence is proved in this chapter, is a solution of an optimal control problem.

5.1 Fundamental estimates

The next lemma will be useful to bound some quantities.

Lemma 5.1. *All* $\Gamma^{\lambda}_{\alpha\beta}$ *and* $\frac{p^i}{p^0}$ *are bounded over* $[0, T] \times \mathsf{B}_{\mathbb{R}^3}(\mathsf{O}, T)$.

Proof. Let $(t, \bar{x}) \in [0, T] \times \mathsf{B}_{\mathbb{R}^3}(\mathsf{O}, T)$ and $1 \leq i, j \leq 3$. By (4.8.4) and integrating over $[0, t]$ one deduces from

$$-C \leq \frac{\frac{d}{dt} g_{ij}\left(t, \bar{x}\right)}{g_{ij}\left(t, \bar{x}\right)} \leq C,$$

that

$$g_{ij}\left(0, \bar{x}\right) e^{-Ct} \leq g_{ij}\left(t, \bar{x}\right) \leq g_{ij}\left(0, \bar{x}\right) e^{Ct}.$$

The function g_{ij} is continuous on the compact set $[0, T] \times \overline{\mathsf{B}}_{\mathbb{R}^3}(\mathsf{O}, T)$, then for $0 \leq t \leq T$ and $\bar{x} \in B_{\mathbb{R}^3}(O, T)$

$$e^{-CT} g^{I_0} \leq e^{-Ct} g_{jk}^{I_0} \leq g_{jk}\left(t, \bar{x}\right) \leq e^{Ct} g_{jk}^{S_0} \leq e^{CT} g^{S_0}, \tag{5.1.1}$$

where $g_{jk}^{I_0} = \inf\limits_{\bar{x} \in \bar{B}_{\mathbb{R}^3}(O,T)} g_{jk}(0, \bar{x})$ and $g_{jk}^{S_0} = \sup\limits_{\bar{x} \in \bar{B}_{\mathbb{R}^3}(0,T)} g_{jk}(0, \bar{x})$

and

$$g^{s_0} = \max_{i,j} g_{ij}^{S_0}, g^{I_0} = \inf_{i,j} g_{ij}^{I_0}.$$

Note that $g_{ij}^{S_0}$ and $g_{ij}^{I_0}$ are not vanished because because $g_{ij} > 0$ in the compact set $\bar{B}_{\mathbb{R}^3}(0, T)$.

Using (4.8.6), (4.8.4) and (5.1.1) and we obtain:

$$\begin{aligned}
\left|\Gamma_{0j}^{i}\right| &= \frac{1}{2}\left|g^{ik}\partial_0 g_{ik}\right| \\
&= \frac{1}{2}\left|g^{ik} g_{ik} \frac{\partial_0 g_{ik}}{g_{ik}}\right| \\
&\leq \frac{1}{2}\left|g^{ik} g_{ik}\right| \left|\frac{\partial_0 g_{ik}}{g_{ik}}\right| && (5.1.2) \\
&\leq \frac{C}{2}, && (5.1.3)
\end{aligned}$$

$$\begin{aligned}
\left|\Gamma_{ij}^{0}\right| &= \left|\frac{1}{2}\partial_0 g_{ij}\right| \\
&\leq \frac{1}{2}\left|g_{ij}\right| \left|\frac{\partial_0 g_{ij}}{g_{ij}}\right| \\
&\leq \frac{1}{2} e^{CT} g^{S_0}, && (5.1.4)
\end{aligned}$$

$$\left|\frac{p^i}{p^0}\right| \leq \frac{1}{\sqrt{g_{ii}\left(t,\bar{x}\right)}} \leq e^{CT}\frac{1}{\sqrt{g^{I_0}}}, \tag{5.1.5}$$

and

$$\left|\Gamma^i_{jk}\right| \leq \frac{1}{2}g^{im}\left(\left|\partial_k g_{mj}\right| + \left|\partial_j g_{mk}\right| + \left|\partial_m g_{jk}\right|\right) \leq 10C^2e^{2CT}\frac{g^{S_0}}{g^{I_0}},$$

that is

$$\left|\Gamma^i_{jk}\right| \leq 10C^2e^{2CT}\frac{g^{S_0}}{g^{I_0}}. \tag{5.1.6}$$

Consequently, we conclude that all $\Gamma^\lambda_{\alpha\beta}$ and $\frac{p^i}{p^0}$ are bounded over $[0,T] \times B_{\mathbb{R}^3}(0,T)$. □

Lemma 5.2. *The map $x^\alpha \mapsto \bar{p}(x^\alpha)$ is uniformly bounded over $[0,T] \times \mathsf{B}_{\mathbb{R}^3}(0,T)$.*

Proof. Using (4.6.4), (4.9.3), and (4.9.5), we have

$$\frac{dp^i}{dt} = -2\Gamma^i_{0j}p^j - \Gamma^i_{jk}p^j\frac{p^k}{p^0} + q\cdot\left(F^i_0 + F^i_j\frac{p^j}{p^0}\right). \tag{5.1.7}$$

So using (4.6.2), inequalities (5.1.5), (5.1.6) and the fact that $F \in C^\infty_0([0,+\infty[\times\mathbb{R}^3)$ we get

$$\begin{aligned}\left|\frac{dp^i}{dt}\right| \leq &\left(6e^{CT}g^{S_0} + 3e^{C\frac{T}{2}}\frac{1}{\sqrt{g^{I_0}}} \times 10C^2e^{2CT}\frac{g^{S_0}}{g^{I_0}}\right)\sum_{j=1}^{3}|p^j| \\ &+ e\,|F|\left(1 + e^{C\frac{T}{2}}\frac{1}{\sqrt{g^{I_0}}}\right).\end{aligned} \tag{5.1.8}$$

So

$$\left|\frac{d\bar{p}}{dt}\right| \leq A|\bar{p}| + B \tag{5.1.9}$$

where $A = \left(6e^{CT}g^{S_0} + 30C^2e^{\frac{5T}{2}C}\frac{g^{S_0}}{g^{\frac{3}{2}I_0}}\right)$ and $B = 3e\,|F|\left(1 + e^{C\frac{T}{2}}\frac{1}{\sqrt{g^{I_0}}}\right)$.

Integrating relation (5.1.9) over $[0,t]$, $0 \leq t \leq T$ and using the inequality

$$\left| \int_0^t \frac{d\overline{p}\left(\tau, \bar{x}\left(\tau\right)\right)}{d\tau} d\tau \right| \leq \int_0^t \left| \frac{d\overline{p}\left(\tau, x\left(\tau\right)\right)}{d\tau} \right| d\tau,$$

one obtains

$$|\bar{p}(t, \overline{x}\,(t)) \leq |\bar{p}(0, \overline{x}\,(0))| + Bt + A \int_0^t |\bar{p}(s, \bar{x}\,(s))| ds.$$

By the Gronwall Lemma 1.2, one obtains the following inequality

$$|\bar{p}(t, \bar{x})| \leq (|\bar{p}(0, \bar{x}\,(0))| + Bt) \left(1 + Ate^{At}\right), (t, \bar{x}) \in [0, T] \times B_{\mathbb{R}^3}(O, T)$$

which completes the proof of Lemma 5.1. □

5.2 Estimates on the Hamiltonian

In Section 4.9, we have obtained the following function called the Hamiltonian, after the transformation of the relativistic Vlasov equation,

$$H(t, \bar{x}, \bar{p}, \bar{q}, z, \bar{u}, \bar{v}, \bar{w}) = \frac{p^i}{p^0} u_i + \left(q \cdot \left(F_0^i + F_j^i \frac{p^j}{p^0} \right) - 2\Gamma_{0j}^i p^j - \Gamma_{jk}^i p^j \frac{p^k}{p^0} \right) v_i - \frac{p^k}{p^0} C_{bc}^a q^c A_k^b w_a \quad (5.2.1)$$

with $(t, \bar{x}, \bar{p}, \bar{q}, z) \in [0, +\infty[\times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \times \mathbb{R}$.

Now we propose to verify that the Hamiltonian H satisfies all the assumptions (B1)-(B4) of the Section 3.1, in order to use this one further. This is done with the following Proposition.

Proposition 5.1. *Let $T > 0$ be given. The Hamiltonian*

$$H : [0, T] \times B_{\mathbb{R}^3}(O, T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$$

$$(t, \bar{x}, \bar{p}, \bar{q}, z, \bar{u}, \bar{v}, \bar{w}) \mapsto H(t, \bar{x}, \bar{p}, \bar{q}, z, \bar{u}, \bar{v}, \bar{w})$$

defined by (5.2.1) satisfies the following properties :

(B1) *H is continuous in* $(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})$

(B2)

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) - H(t,\bar{x},\bar{p},\bar{q},z,\bar{m},\bar{n},\bar{r})| \leq C_0(1+|\bar{x}|+|\bar{p}|+|\bar{q}|)(|\bar{u}-\bar{m}|+|\bar{v}-\bar{n}|)+|\bar{w}-\bar{r}|) \quad (5.2.2)$$

and

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{0},\bar{0},\bar{0})| \leq C_0(1+|\bar{x}|+|\bar{p}|+|\bar{q}|)\,, \quad (5.2.3)$$

for some $C_0 > 0$, $(t,\bar{x},\bar{p},\bar{q},z) \in [0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\times\mathbb{R}$; $(\bar{u},\bar{v},\bar{w}),(\bar{m},\bar{n},\bar{r}) \in \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}$.

(B3)

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) - H(t,\bar{y},\bar{r},\bar{s},z,\bar{u},\bar{v},\bar{w})| \leq \lambda(L)(1+|\bar{u}|+|\bar{v}|+|\bar{w}|)(|\bar{x}-\bar{y}|+|\bar{p}-\bar{r}|)+|\bar{q}-\bar{s}|) \quad (5.2.4)$$

with $|\bar{x}|+|\bar{p}|+|\bar{q}| \leq L$, $|\bar{y}|+|\bar{r}|+|\bar{s}| \leq L$ *for some* $\lambda\,(L)$ *where* $L > 0$ *is given.* $(\bar{x},\bar{p},\bar{q})\,,(\bar{y},\bar{r},\bar{s})\,,\bar{u},\bar{v},\bar{w} \in \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}$, $(t,z) \in [0,\infty[\times\mathbb{R}$.

(B4)

$$|H(t,\bar{x},\bar{p},\bar{q},z_1,\bar{u},\bar{v},\bar{w}) - H(t,\bar{x},\bar{p},\bar{q},z_2,\bar{u},\bar{v},\bar{w})| \leq C_0(1+|\bar{x}|+|\bar{p}|+|\bar{q}|+|\bar{u}|+|\bar{v}|+|\bar{w}|)\,|z_1-z_2|\,. \quad (5.2.5)$$

for some $C_0 > 0$, $(t,\bar{x},\bar{p},\bar{q},\bar{u},\bar{v},\bar{w}) \in [0;+\infty[\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}$, $z_1,z_2 \in \mathbb{R}$.

Proof. Consider that $T > 0$ is given.

- *For assertion (B1):* Since $p^0 > 0$, H is obviously continuous in $(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})$.
- *For assertion (B2):* One has by definition of H:

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{0},\bar{0},\bar{0})| = 0,$$

which implies that

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{0},\bar{0},\bar{0})| \leq C_0(1+|\bar{x}|+|\bar{p}|+|\bar{q}|),$$

$(t,\bar{x},\bar{p},\bar{q},z) \in [0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\times\mathbb{R}$.

Using the definition of H, one has

$$H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) - H(t,\bar{x},\bar{p},\bar{q},z,\bar{m},\bar{n},\bar{r}) =$$
$$\frac{p^i}{p^0}(u_i - m_i) + \left(q \cdot \left(F^i_0 + F^i_j \frac{p^j}{p^0}\right) - 2\Gamma^i_{0j} p^j - \Gamma^i_{jk} p^j \frac{p^k}{p^0}\right)(v_i - n_i)$$
$$+ \frac{p^k}{p^0} C^a_{bc} q^c A^b_k (r_a - w_a). \quad (5.2.6)$$

Using the Lemmas 5.1 and 5.2 and the hypotheses $|q| = e$, $A, F \in C^\infty_0([0,+\infty[\times \mathbb{R}^3)$ which allow to bound $\Gamma^\lambda_{\alpha\beta}$, $q \cdot \left(F^i_0 + F^i_j \frac{p^j}{p^0}\right)$ and $\frac{p^i}{p^0}$ over $[0,T] \times B_{\mathbb{R}^3}(O,T)$, one easily obtains from (5.2.6) the following inequality

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) - H(t,\bar{x},\bar{p},\bar{q},z,\bar{m},\bar{n},\bar{r})|$$
$$\leq C_0(1 + |\bar{x}| + |\bar{p}| + |\bar{q}|)(|\bar{u} - \bar{m}| + |\bar{v} - \bar{n}|) + |\bar{w} - \bar{r}|) \quad (5.2.7)$$

in which $C_0 = C_0\left(e, g^{S_0}_{ij}, g^{I_0}_{ij}, T, |A|, |F|\right)$.

- *Assertion (B3):* Let $L > 0$, such that $|\bar{x}| + |\bar{p}| + |\bar{q}| \leq L$, $|\bar{y}| + |\bar{r}| + |\bar{s}| \leq L$. Using the definition of H one obtains

$$H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) - H(t,\bar{y},\bar{r},\bar{s},z,\bar{u},\bar{v},\bar{w}) =$$
$$\left(\frac{p^i}{p^0} - \frac{r^i}{r^0}\right) u_i + \left((q-s)\cdot F^i_0 + F^i_j \cdot \left(q\frac{p^j}{p^0} - s\frac{r^j}{r^0}\right) + 2\Gamma^i_{0j}(r^j - p^j)\right.$$
$$\left. + \Gamma^i_{jk}\left(r^j \frac{r^k}{r^0} - p^j \frac{p^k}{p^0}\right)\right) v_i + \left(s^c \frac{r^k}{r^0} - q^c \frac{p^k}{p^0}\right) C^a_{bc} q^c A^b_k w_a. \quad (5.2.8)$$

But

$$\begin{cases} \frac{p^i}{p^0} - \frac{r^i}{r^0} = \frac{p^i}{p^0}\frac{1}{r^0}\left(r^0 - p^0\right) + \frac{1}{r^0}(p^i - r^i) \\ s^c \frac{r^k}{r^0} - q^c \frac{p^k}{p^0} = \frac{r^k}{r^0}\left(s^c - q^c\right) + \frac{q^c}{r^0}\left(r^k - p^k\right) - \frac{p^k}{p^0}\frac{1}{r^0}\left(p^0 - r^0\right) q^c \\ q\frac{p^j}{p^0} - s\frac{r^j}{r^0} = \frac{p^j}{p^0}\left(q - s\right) + \frac{s}{p^0}\left(p^j - r^j\right) - \frac{r^j}{r^0}\frac{1}{p^0}\left(r^0 - p^0\right) s \\ r^j \frac{r^k}{r^0} - p^j \frac{p^k}{p^0} = \frac{r^k}{r^0}\left(r^j - p^j\right) + \frac{p^j}{r^0}\left(r^k - p^k\right) - \frac{p^k}{p^0}\frac{1}{r^0}\left(p^0 - r^0\right) p^j. \end{cases} \quad (5.2.9)$$

Due to the fact that $\frac{1}{p^0}, \frac{1}{r^0} \leq 1$, the Lemmas 5.1 and 5.2 which allow to bound $\Gamma^{\lambda}_{\alpha\beta}$, $\frac{p^i}{p^0}$ and $\frac{r^k}{r^0}$ over $[0,T] \times B_{\mathbb{R}^3}(O,T)$, also using the hypotheses $|q| = |s| = e, A, F \in C_0^{\infty}\left([0,+\infty[\times\mathbb{R}^3\right)$, one obtains from (5.2.8) and (5.2.9) the following inequality

$$\begin{aligned} &|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) - H(t,\bar{y},\bar{r},\bar{s},z,\bar{u},\bar{v},\bar{w})| \\ &\qquad \leq C(1+|\bar{u}|+|\bar{v}|+|\bar{w}|)(|\bar{x}-\bar{y}|+|\bar{p}-\bar{r}|)+|\bar{q}-\bar{s}|) \end{aligned}$$

where $C = C\left(e, g_{ij}^{S_0}, g_{ij}^{I_0}, T, |A|, |F|\right) = \lambda\,(L)$.

- *Assertion (B4):* In (5.2.8) take $\overline{x} = \overline{y}$, $\overline{p} = \overline{r}$ and $\overline{s} = \overline{q}$, then *(B4) follows.* □

5.3 Global in finite time existence theorem

In what follows let $T > 0$ we denote

$$\mathcal{K}_T = [0,T] \times B_{\mathbb{R}^3}(O,T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1}.$$

Let us assume that a Lipschitz continuous function

$$f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \longrightarrow \mathbb{R}$$

and a real number $T > 0$ are a given. Let

$$H_T =]0,T[\times B_{\mathbb{R}^3}(O,T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1},$$

and consider the following Cauchy problem:

$$\begin{cases} f_t(t,\bar{x},\bar{p},\bar{q}) + \overline{H}(t,\bar{x},\bar{p},\bar{q}) = 0 & \text{in } H_T \\ f(0,\bar{x},\bar{p},\bar{q}) = f_0(\bar{x},\bar{p},\bar{q}) & \text{on } B_{\mathbb{R}^3}(\text{O,T})\times\mathbb{R}^3 \times \mathbb{R}^{N-1}, \end{cases} \tag{5.3.1}$$

where

$$\overline{H}(t,\bar{x},\bar{p},\bar{q}) = H(t,\bar{x},\bar{p},\bar{q},f,\nabla_{\bar{x},\bar{p},\bar{q}}f(t,\bar{x},\bar{p},\bar{q})).$$

Our main purpose will be to prove using a result of [7] that the Cauchy problem (5.3.1) has a unique L^{∞} minimax viscosity solution $f \in C(\mathcal{K}_T)$.

We are now able to give the existence theorem of this work, which is deduced from Theorem 3.1.

Theorem 5.1. *Let us assume that a Lipschitz continuous function*

$$f_0 : [0, +\infty[\times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \mapsto \mathbb{R}$$

and $T > 0$ *are given. Then the Cauchy problem*

$$\begin{cases} f_t(t, \bar{x}, \bar{p}, \bar{q}) + \overline{H}(t, \bar{x}, \bar{p}, \bar{q}) = 0 & \text{on } H_T \\ f(0, \bar{x}, \bar{p}, \bar{q}) = f_0(\bar{x}, \bar{p}, \bar{q}) & \text{in } B_{\mathbb{R}^3}(O,T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \end{cases}$$

admits a unique continuous L^∞ *minimax viscosity solution.*

Proof. It is proved in Proposition 5.1 that the Hamiltonian H satisfies the properties (B1), (B2), (B3) and (B4). Consequently theorem 3.1 and theorems 3.2 imply that the Cauchy problem admits a unique continuous L^∞ minimax viscosity solution $f \in C(\mathcal{K}_T)$ for $T > 0$. □

Corollary 5.1. *The relativistic Vlasov equation*

$$\frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} + \frac{Q^a}{p^0}\frac{\partial f}{\partial q^a} = 0$$

in Yang-Mills charged models has a unique continuous L^∞ *minimax viscosity solution* $f = f(t, \bar{x}, \bar{p}, \bar{q})$ *on* $\mathcal{K}_T$ *that satisfies the initial condition* $f(0, \bar{x}, \bar{p}, \bar{q}) = f_0(\bar{x}, \bar{p}, \bar{q})$ *in* $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.

Proof. The proof is a direct consequence of equivalence of the Cauchy problem (5.3.1) and the relativistic Vlasov equation in Section 4.9 with initial condition $f(0, \bar{x}, \bar{p}, \bar{q}) = f_0(\bar{x}, \bar{p}, \bar{q})$ in $B_{\mathbb{R}^3}(O, T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$. □

5.4 Optimal control problem

5.4.1 Optimal control problem

We first describe some general results about the deterministic optimal control problems. To describe these one, we consider a system which state is given by the solution $y_x(\cdot)$ of the following differential equation:

$$\frac{dy_x(t)}{dt} = b(y_x(t), v(t)) \text{ for } t \geq 0, \; y_x(0) = x \in \mathbb{R}^N, \tag{5.4.1}$$

where b maps $\mathbb{R}^N \times V$ into $\mathbb{R}^N$, V is some given closed convex set (or compact) in $\mathbb{R}^N$ which will be called the set of values of control.

The control $v(\cdot)$ is any measurable bounded function from $[0, +\infty[$ to V. We will hereafter assume that b satisfies:

$$\begin{cases} |b(x,v) - b(y,v)| \leq C\,|x-y|\,, \ \forall x,y \in \mathbb{R}^N, \forall v \in V\,; \\ |b(x,v)| \leq C, \quad \forall (x,v) \in \mathbb{R}^N \times V, \\ b(\cdot,\cdot) \text{ is continuous on } \mathbb{R}^N \times V, \end{cases} \tag{5.4.2}$$

for some constant $C > 0$.

Then Theorem 1.8 implies that (5.4.1) has a unique solution for all $x \in \mathbb{R}^N$ denoted by $y_x(\cdot)$.

Definition 5.1. A pay-off function (or cost function) for each given control $v(\cdot)$ is defined as

$$\begin{aligned} J(t,x;v(\cdot)) = &\int_0^t l\left(y_x\left(s\right), v\left(s\right)\right) \exp\left[-\int_0^s c\left(y_x\left(\lambda\right), v\left(\lambda\right)\right)\, d\lambda\right] ds \\ &+ u_0\left(y_x\left(t\right)\right) \exp\left[-\int_0^t c\left(y_x\left(\lambda\right), v\left(\lambda\right)\right)\, d\lambda\right] \end{aligned} \tag{5.4.3}$$

where l, c and u_0 are given functions which satisfy: $\exists C > 0$ such that for $\varphi = l, c$ we have

$$\begin{cases} |\varphi(x,v) - \varphi(y,v)| \leq C\,|x-y| \ \forall x,y \in \mathbb{R}^N, \forall v \in V\,; \\ |\varphi(x,v)| \leq C, \quad \forall (x,v) \in \mathbb{R}^N \times V; \\ \varphi \text{ is continuous on } \mathbb{R}^N \times V, \end{cases} \tag{5.4.4}$$

and

$$\begin{cases} |u_0\left(x\right) - u_0\left(y\right)| \leq C\,|x-y|\,, & \\ |u_0(x)| \leq C, & \forall x,y \in \mathbb{R}^N; \\ u_0 \text{ is continuous on } \mathbb{R}. & \end{cases} \tag{5.4.5}$$

5.4.2 The problem

The problem to solve is to minimize the cost function over all controls $v(\cdot)$, that is to find

$$u(t,x) = \inf_{v(\cdot)} J\left(t,x;v\left(\cdot\right)\right). \tag{5.4.6}$$

Definition 5.2. The problem (5.4.6) is called, in the optimal control theory, finite horizon problem.

The main purpose of optimal control theory is to give a characterization of this optimal cost function and to compute optimal control, eventually in the form called feedback optimal control, namely a control $v^\star$ such that

$$u(t,x) = J\left(t,x;v^\star\left(\cdot\right)\right).$$

The following theorem expresses the dynamic programming principle, an essential tool for the optimal control problem.

Theorem 5.2. *Under assumptions (5.4.2), (5.4.4), we have*

$$\begin{aligned} u(t,x) = \inf_{v(\cdot)} \Bigg\{ & \int_0^s l\left(y_x\left(\lambda\right), v\left(\lambda\right)\right) \exp\left[-\int_0^\lambda c\left(y_x\left(\tau\right), v\left(\tau\right)\right)\, d\tau\right] d\lambda \\ & + u\left(t-s, y_x\left(s\right)\right) \exp\left[-\int_0^s c\left(y_x\left(\tau\right), v\left(\tau\right)\right)\, d\tau\right]\Bigg\}, \end{aligned} \tag{5.4.7}$$

for all $0 \le s \le t$.

Proof. See [29]. □

Now we give a result about the regularity of the cost function.

Proposition 5.2. *Under assumptions (5.4.2), (5.4.4), (5.4.5), the function*

$$u\left(\cdot,\cdot\right):]0,T[\times \mathbb{R}^N \longrightarrow \mathbb{R}$$

is bounded and Lipschitz continuous for all $0 < T < +\infty$ *on* $[0,T] \times \mathbb{R}^N$.

Proof. See [29]. □

5.4.3 Link between Hamilton-Jacobi equation and optimal control

The next result explains a relation between the optimal control problem and the Hamilton-Jacobi equation.

Theorem 5.3. *Under assumptions (5.4.2), (5.4.4) and (5.4.5), we have u is differentiable and uniformly bounded a.e in* $]0, T[\times \mathbb{R}^N$ *for all* $T \in]0, +\infty[$ *and the viscosity solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup_{v \in V} \{b(x,v) \cdot D_x u + c(x,v) u - l(x,v)\} = 0 \ a.e. \ in \ ]0, +\infty[\times \mathbb{R}^N \\ u(0,x) = u_0(x) \ \ in \ \mathbb{R}^N. \end{cases} \tag{5.4.8}$$

Proof. See [29]. □

Remark 5.1. According to the Theorem 5.3, the function u satisfies an Hamilton-Jacobi-Bellman equation, a particular Hamilton-Jacobi equation which Hamiltonian is defined by

$$H(t,x,p) = \sup_{v \in V} \{b(x,v) \cdot p + \lambda t - l(x,v)\}.$$

This Hamiltonian is clearly Lipschitz continuous and convex in (t, p) as a supremum of affine functions.

Conversely if $H(t,x,p)$ is a convex continuous function in (t, p) and Lipschitz continuous at least locally in x then it is possible to write $H(t,x,p)$ as a supremum of affine functions and in this way to write down some associated optimal control problem: indeed let us denote by $L(t,x,p)$ the dual convex function of $H(t,x,p)$, recall that L is given by

$$L(t,x,p) = \sup_{(s,q) \in \mathbb{R} \times \mathbb{R}^N} \{ts + p \cdot q - H(t,x,q)\} \leq +\infty.$$

Now, we know in [13] that

$$H(t,x,p) = \sup_{(s,q) \in Dom\, L(\cdot,x,\cdot)} \{p \cdot q + ts - L(t,x,q)\}. \tag{5.4.9}$$

And this proves that, at least formally, we may define for each convex Hamiltonian some associated optimal control problem in the sense that the corresponding optimal cost function solves the Hamilton-Jacobi equation.

The following proposition gives a result about the feedback control, under some assumptions, of the optimal control problem.

Proposition 5.3. *Assume that $u \in C^1(\bar{Q}_T)$ for some $T > 0$, where $Q_T =]0,T[\times\mathbb{R}^N$, and that there exists a continuous function $\boldsymbol{v}(\boldsymbol{t},\boldsymbol{x})$ defined on $\bar{Q}_T$ such that :*

$$\frac{\partial u}{\partial t}(t,x) + \sup_{v\in V}\{b(x,v)\cdot D_x u(t,x) - l(x,v)\} = \frac{\partial u}{\partial t}(t,x) + b(x,\boldsymbol{v}(t,x))\cdot D_x u(t,x) - l(x,\boldsymbol{v}(t,x)) = 0.$$

Let $y_x(s)$ be a solution for $0 \le s \le t$ of :

$$\begin{cases} \dfrac{dy_x}{ds}(s) + b(y_x(s), \boldsymbol{v}(t-s, y_x(s))) = 0, \\ y_x(0) = x, \end{cases}$$

where $x \in \mathbb{R}^N$.

Then the feedback $v_{t;x}(s) = \boldsymbol{v}(t-s, y_x(s))$ is optimal, that is, we have

$$u(t,x) = J(t,x;v_{t,x}(\cdot)), \qquad \forall x \in \mathbb{R}^N, \qquad \forall t \in [0,T].$$

Proof. See [29]. □

5.4.4 Application to the relativistic Vlasov equation

Remark 5.2. The Hamiltonian (5.2.1), according to the assumptions $(B1)$ and $(B3)$, is continuous, clearly convex in $(t,\bar{u},\bar{v},\bar{w})$ and Lipschitz continuous locally in $(\bar{x},\bar{p},\bar{q})$. We can now state that the L^∞ minimax viscosity solution of the relativistic Vlasov equation is a solution of an optimal control problem.

Proposition 5.4. *Let H be the Hamiltonian (5.2.1) and L its dual convex function. Let us assume that a Lipschitz continuous function $f_0 : [0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\mapsto\mathbb{R}$ is given. Consider the functions*

$$b,\ c : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\times Dom\,L(\cdot,x,\cdot) \longrightarrow \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}.$$

satisfying assumptions (5.4.4) and the function

$$u_0 : \mathbb{R}^N \longrightarrow \mathbb{R}$$

satisfying assumptions (5.4.5).

The unique solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \hat{H} = 0 & \textit{a.e. in } \mathcal{K}_T \\ u\left(0, x\right) = f_0\left(x\right) & \textit{in } \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \end{cases} \tag{5.4.10}$$

where

$$\hat{H} = \sup_{(s,q) \in Dom\, L(\cdot,x,\cdot)} \left\{ b\left(s,x,q\right) \cdot D_x u + \lambda u - L\left(s,x,q\right) \right\}$$

solves the relativistic Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} + \frac{Q^a}{p^0}\frac{\partial f}{\partial q^a} = 0$$

in YangMills charged curved space times in $C\left(\mathcal{K}_T\right)$ *for all* $T > 0$ *with initial condition* $f(0, \bar{x}, \bar{p}, \bar{q}) = f_0(\bar{x}, \bar{p}, \bar{q})$ *in* $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.

Proof. By Theorem 5.1 the relativistic Vlasov equation is a viscosity solution of an Hamilton-Jacobi equation. By (5.4.8), (5.4.9) this Hamilton-Jacobi equation is equivalent to the optimal control problem equivalent to the system (5.4.10), and the proof is completed. □

Conclusion and Prospects

Using the transformation of the Vlasov equation into an Hamilton-Jacobi equation, based on major results stated in [7], a global in finite time existence and local in space uniqueness theorem of a generalized solution of the Cauchy problem for the relativistic Vlasov equation is given, and an optimal control problem is derived from this existence theorem, the initial data is just assumed to be a Lipschitz continuous function. In contrary to the usual methods used for this kind of equation, our approach is totally new and may permit to extend the analysis, in the frame of the vast studies made around the Hamilton-Jacobi equations. In this sense, we have shown that the L^∞ minimax viscosity solution may be seen as a solution of an optimal control problem.

This study permits to deduce easily the following facts. It is possible to give now viscosity solution result of the relativistic Vlasov equation in all the range of time and space, and the properties of viscosity solution permit to see that this one behaves like classical solution when it exists in a domain. The possibility of numerical simulation around the Hamilton-Jacobi equations may be available for the relativistic Vlasov equation. In the base of result in the optimal control problem, it is made possible to control the value of the distribution function of the particles in a Yang-Mills field.

The desire may be to extend this method to the relativistic Boltzmann equation.

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Minimax and Viscosity Solution in L^∞ for the Inhomogeneous Relativistic Vlasov Equation

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Abstract

In this paper, we set a new theorem about existence and uniqueness of L^∞ solution of the inhomogeneous relativistic Vlasov equation in Yang-Mills charged curved space times with non-zero mass. We prove the equivalence between the Vlasov equation and an Hamilton-Jacobi equation and show that the previous solution is also a minimax and a viscosity solution of the same equation. We therefore derive from it an optimal control problem. The methods and techniques used here for the Vlasov equation are original and totally different from the ones used by authors working in the same field.

Mathematics Subject Classification: 35D40, 35F21, 35Q83, 49J21, 83A99

Keywords: Inhomogeneous relativistic Vlasov equation, L^∞ solution, viscosity solution, minimax solution, global existence, optimal control problem

INTRODUCTION

In this paper, we study the existence and uniqueness of a generalized solution of the inhomogeneous relativistic Vlasov equation in which a Yang-Mills potential is given and an optimal control problem in which the value function is the unique solution of the corresponding Hamilton-Jacobi equation.

The Vlasov equation is one of the basic equations of the relativistic kinetic theory. This equation rules the dynamic of the collision-less considered particles, by determining their distribution function, which is a non-negative real-valued function of both the position and the momentum of particles.

Many authors have already studied the relativistic Vlasov equation. Choquet-Bruhat and Noutchegueme in [4] studied the Yang-Mills-Vlasov system using the characteristics method.This method was very complicated because they introduced functional spaces with weight that required many estimates. They obtained a local in time existence result. Choquet-Bruhat and Noutchegueme in [5] also studied the Yang-Mills-Vlasov system only for the zero mass particles case and used the conformal invariance of the system to prove a global existence theorem only in Minkowski space time for small initial data. Nouchegueme and Noundjeu in [7] proved a local in time existence and global in space theorem of the Cauchy problem for the Yang-Mills - Vlasov system in temporal gauge with current generated by a distribution function that satisfied a Vlasov equation, but still using characteristics and many energy estimates.

The main objective of the present work is to extend the result obtained in [2] to the inhomogeneous relativistic Vlasov equation. To achieve this goal, we bring out a new method to justify existence of solution of the inhomogeneous relativistic Vlasov PDE. Our method follows the one used in [2]. But the techniques used and the results obtained here are different. We consider the inhomogeneous Vlasov equation, we find local existence of solutions and we obtain two new types of solutions : L^∞ and minimax solutions , while the solutions obtained in [2] were only in the viscosity sense and for the One-body Liouville equation. Firstly, using the techniques of [2] ,we transform the Vlasov equation and obtain a Hamilton -Jacobi equation. This equivalence allows to introduce an Hamiltonian, which clearly satisfies all the assumptions denoted in this work by (B). Then we apply an important result obtained in [[3], theorem 3.1], which allows to state a time and space existence and uniqueness theorem of L^∞ solution for the Vlasov relativistic equation. Still using [3] and also invoking [2] and [7], we prove that this L^∞ solution is equaly a minimax and a viscosity solution of the same equation. We consider for this study given Yang-Mills charged curved space times with a local symmetry. In the last part of this paper, we introduce an optimal control problem, which is solved by the method of dynamic programming.

The paper is divided as follows:

- in section 1, we give definitions and present some useful results of [3],
- in section 2, we present the space-time and the equation,
- in section 3, we set the main existence theorem,
- in section 4, we display and solve an optimal control problem.

1. PRELIMINARIES

The main purpose of this section is to give some important definitions, and present the theory of global discontinuous solutions in L^∞ of the Cauchy problem for the following Hamilton-Jacobi equation by recalling without giving proofs, some important results belonging to [3]:

$$u_t + H(t, x, u, Du) = 0, \qquad x \in R^n, 0 \le t \le T, \tag{1.1}$$

$$u(0, x) = \varphi(x) \qquad x \in \mathbb{R}^n \tag{1.2}$$

where $T > 0$.

To display our ideas and methods in a clear setting, we make the following assumptions on the Hamiltonian $H(t, x, u, Du)$ of the Cauchy problem (1.1)–(1.2):

(B1): $H(t, x, z, p)$ is continuous in (t, x, z, p) and increasing in z;

(B2): $|H(t, x, z, p_1) - H(t, x, z, p_2)| \le C_0(1 + |x|)|p_1 - p_2|$, and $|H(t, x, z, 0)| \le C_0(1 + |x| + |z|)$, for all $t \in (0, T]$;

(B3): $|H(t, x_1, z, p) - H(t, x_2, z, p)| \le \lambda(L)(1 + |p|)|x_1 - x_2|$ where $|x_1|, |x_2| \le L$

(B4): $|H(t, x, z_1, p) - H(t, x, z_2, p)| \le C_0(1 + |x| + |p|)|z_1 - z_2|$.

We define the essential infimum and supremum of an $L^\infty_{loc}(\mathbb{R}^d)$ function $v(x)$ at every point $x \in \mathbb{R}^d$:

$$I(v)(x) \equiv \sup_{A \in S_x} ess \inf_{y \in A} v(y), \quad S(v)(x) \equiv \inf_{A \in S_x} ess \sup_{y \in A} v(y),$$

where

$$S_x = \left\{ A \subset \mathbb{R}^d \text{ measurable } | \lim_{r \to 0} \frac{m(A \cap B^d(x, r))}{m(B^d(x, r))} = 1 \right\}.$$

Definition 1. Fix $\tau \in [0, T]$ and $p(t, x) \in C([0, T] \times \mathbb{R}^n; \mathbb{R}^n)$. Given a measurable function v and a position (or value) function f, we define the winning and the losing functions :

$$\Lambda^v_-(t, x, (\tau, f, p)) = inf\{S(v)(x(\tau)) - z(\tau) | (x(\cdot), z(\cdot)) \in Sol(t, f(t, x), p)\}, \tag{1.3}$$

$$\Lambda^v_+(t, x, (\tau, f, p)) = sup\{I(v)(x(\tau)) - z(\tau) | (x(\cdot), z(\cdot)) \in Sol(t, f(t, x), p)\}, \tag{1.4}$$

where $Sol(t, f(t, x), p)$ denotes the set of solutions:

$$(x(\cdot), z(\cdot)) : [\tau, t] \to \mathbb{R}^n \times \mathbb{R}, \quad \text{for t} \ge \tau$$

of the characteristic inclusions $(\dot{x}(\cdot), \dot{z}(\cdot)) \in E(t, x, z, p)$ satisfying the conditions: $x(t) = x$, $z(t) = f(t, x)$, where

$$E(t,x,z,p)=\{(h,g)\in\mathbb{R}^n\times\mathbb{R}\ |h|\leq C_0(1+|x|),g=\langle h,p\rangle-H(t,x,z,p)\}.$$

1.1. **Existence of discontinuous solutions in L^∞.** Let

$$W=\{u(t,x)\in L^\infty_{loc}([0,T]\times\mathbb{R}^n)|\ u(t,\cdot)\in L^\infty_{loc}(\mathbb{R}^n)\text{ for every }t\in[0,T]\}.$$

Denote by S^u the set of L^∞ **supsolutions** $w(t,x)\in W$ which satisfy

(i) for any $p(t,x)\in C(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$,

$$\Lambda^\varphi_-(t,x,(0,w,p))\leq 0 \tag{1.5}$$

for almost every $(t,x)\in[0,T]\times\mathbb{R}^n$, and where φ is a locally bounded measurable function, $u^\varphi_-((t,x),p)$ the unique locally bounded measurable function satisfying

$$\Lambda^\varphi_-(t,x,(0,u^\varphi_-((t,x),p),p)=0.$$

Additionally, for every $t\in[0,T]$,(1.5) holds for almost every $x\in\mathbb{R}^n$.

(ii) The semigroup property: for every $\tau\in[0,T]$,

$$\Lambda^{w(\tau,x)}_-(t,x,(0,w,p))\leq 0 \tag{1.6}$$

for almost every $(t,x)\in[\tau,T]\times\mathbb{R}^n$. Additionally, for every $t\in[\tau,T]$,(1.6) holds for almost every $x\in\mathbb{R}^n$.

Denote by S^l the set of L^∞ **subsolutions** $w(t,x)\in W$ which satisfy

(iii) for any $p(t,x)\in C(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$,

$$\Lambda^\varphi_+(t,x,(0,w,p))\geq 0 \tag{1.7}$$

for almost every $(t,x)\in[0,T]\times\mathbb{R}^n$, and where φ is a locally bounded measurable function, $u^\varphi_+((t,x),p)$ the unique locally bounded measurable function satisfying

$$\Lambda^\varphi_+(t,x,(0,u^\varphi_+((t,x),p),p)=0.$$

Additionally, for every $t\in[0,T]$,(1.7) holds for almost every $x\in\mathbb{R}^n$.

(iv) The semigroup property: for every $\tau\in[0,T]$,

$$\Lambda^{w(\tau,x)}_+(t,x,(0,w,p))\geq 0 \tag{1.8}$$

for almost every $(t,x)\in[\tau,T]\times\mathbb{R}^n$. Additionally, for every $t\in[\tau,T]$,(1.8) holds for almost every $x\in\mathbb{R}^n$.

Definition 2. u is a L^∞ ***solution*** of the Cauchy problem (1.1)-(1.2) if u belongs to S^u and S^l simultaneously.

Definition 3. A continuous function $u:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is a **viscosity subsolution** of (1.1)-(1.2) if $u(0,x)=\varphi(x)$ and for every $\mathcal{C}^1$ function $\rho=\rho(t,x)$ such that $u-\rho$ has a local maximum at (t,x), one has

$$\rho_t(t,x)+H(t,x,u,D\rho)\leq 0.$$

A continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a **viscosity supersolution** of (1.1)-(1.2) if $u(0,x) = \varphi(x)$ and for every $\mathcal{C}^1$ function $\rho{=}\rho(t,x)$ such that $u - \rho$ has a local minimum at (t,x), one has

$$\rho_t(t,x) + H(t,x,u,D\rho) \geq 0.$$

A continuous function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a **viscosity solution** of (1.1)-(1.2) if it is both a supersolution and subsolution in the viscosity sense.

Definition 4. A continuous function $u : [0,T] \times \mathbb{R}^n$ is called a **minimax solution** of (1.1)-(1.2) if $u(0,y) = \varphi(y)$, $y \in \mathbb{R}^n$, and for every $(x_0, z_0) \in \{(x, u(x)) : x \in [0,T] \times \mathbb{R}^n\}$ and $s \in [0,T] \times \mathbb{R}^n$ there exist a number $\tau > 0$ and a Lipschitz function $(x(\cdot), z(\cdot)) : [0,\tau] \mapsto [0,T] \times \mathbb{R}^n \times \mathbb{R}$ such that $(x(0), z(0)) = (x_0, z_0)$ for all $t \in [0,\tau]$ and

$$\dot{z}(t) = \langle \dot{x}(t), s \rangle - H(t, x(t), z(t), s)$$

for almost all $t \in [0,\tau]$.

Theorem 5. [[3], p.13] *Given a locally bounded measurable function φ, there exists a unique minimal element of S^u, that is the solution of the Cauchy problem (1.1)-(1.2).*

Remark 6. In [8, 9], provided that initial data are continuous, it is shown that minimax solutions are equivalent to viscosity solutions. The next two theorems prove that L^∞ solutions coincide with minimax solutions when initial data are continuous.

Theorem 7. [[3],p.15] *Assume that $\varphi(x)$ is continuous. Let $u(t,x)$ be an L^∞ supsolution of (1.1)-(1.2) and $v(t,x)$ the continuous minimax solution. Then $u(t,x) \geq v(t,x)$ almost everywhere.*

Theorem 8. [[3],p.17] *Assume that $\varphi(x)$ is continuous. Let $u(t,x)$ be an L^∞ subsolution of (1.1)-(1.2) and $v(t,x)$ the continuous minimax solution. Then $u(t,x) \leq v(t,x)$ almost everywhere.*

Consequently, the L^∞ solutions coincide with the continuous viscosity solutions when initial data are continuous.

2. THE SPACE TIME AND THE EQUATIONS

Greek indexes $\alpha, \beta, \dots$ range from 0 to 3, and the Latin indexes $i, j, \dots$ from 1 to 3. We adopt the Einstein summation convention

$$A^\alpha B_\alpha = \sum_\alpha A^\alpha B_\alpha.$$

We consider the Vlasov equation in temporal gauge of the form

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} + Q^a \frac{\partial f}{\partial q^a} = 0\,. \tag{2.1}$$

Equation (2.1) , generalizes to the non-abelian case the classical Vlasov equation in presence of electromagnetic field. This equation governs the evolution without collisions of a plasma of charged particles, with a non-zero rest mass m in a given Yang-Mills field, and whose unknown distribution function generates this field.

The 4-momentum of particles is denoted by $p = (p^\alpha) = (p^0, p^i) = (p^0, \bar{p})$ and their non-abelian charge is denoted by q. Their distribution function f, solution of the Vlasov equation (2.1), is a positive scalar function defined on the product $T(\mathbb{R}^4) \times \mathcal{G}$ where $(\mathcal{G}, [.])$ is a Lie algebra of a non-abelian Lie group G. We consider that $\mathcal{G}$ is a vector space on $\mathbb{R}$ whose dimension is $N \geq 2$ and whose fixed basis is denoted (ε_a), $a = 1, \ldots, N$. (q^a) will denote the coordinates of $q \in \mathcal{G}$ in (ε_a). The distribution function f, in the sense of kinetic theory, is consequently a function of $(x^\alpha, p^\alpha, q^a)$ where (x^α, p^α) denotes the usual coordinates of the tangent bundle $T(\mathbb{R}^4) = \mathbb{R}^4 \times \mathbb{R}^4$ of $\mathbb{R}^4$.The collisionless particles then evolve in the space-time $(\mathbb{R}^4, g)$ on one hand under the action of their own gravitational field represented by the given metric tensor $g = (g_{\alpha\beta})$ that informs about gravitational effects, and on the other hand under the non-abelian force generated by the Yang-Mills field $F = (F_{\alpha\beta})$, deriving itself from a given Yang-Mills potential $A = (A_\alpha)$.

The $F_{\alpha\beta}$ and A_α are then functions from $[0, \infty[\times \mathbb{R}^3$ on $\mathcal{G}$, linked by the relation

$$F^a_{\alpha\beta} = \nabla_\alpha A^a_\beta - \nabla_\beta A^a_\alpha + C^a_{bc} A^b_\alpha A^c_\beta \tag{2.2}$$

where C^a_{bc} are the structure constants of $\mathcal{G}$ and ∇ the covariant derivative associated with g.

One imposes on the Yang-Mills potential $A = (A_\alpha)$, the temporal gauge

$$A_0 = 0. \tag{2.3}$$

We consider that metric tensor $g = (g_{\alpha\beta})$ is of Lorentzian signature $(-, +, +, +)$ and we also assume that the time lines are orthogonal to the space sections. So g writes:

$$g = g_{00}\,(x^\alpha)\,dt^2 + g_{ij}(x^\alpha) dx^i dx^j \tag{2.4}$$

in which $g_{ij}(x^\alpha) > 0$ are given differentiables functions of the time t and the space $(\bar{x}) = (x^i)$, $i = 1, 2, 3$ and where we take for simplicity $g_{00} = -1$.

The rest mass of particles is normalized to the unity, that is $m = 1$ and really the particles move on the future sheet of the mass hyperboloid $P(\mathbb{R}^4) \subset T(\mathbb{R}^4)$, whose equation is $P_{t,x}(p) : g(p, p) = -1$ or using (2.4) :

$$P_{t,x} : p^0 = \sqrt{1 + g_{ij} p^i p^j}\,, \tag{2.5}$$

where the choice $p^0 > 0$ means that the particles eject towards the future.

In this work, one requires that there exists a constant $C > 0$ such that:

$$\left|\frac{\partial_\alpha g_{ij}}{g_{ij}}\right| \leq C\,. \tag{2.6}$$

One also supposes that the non-abelian charge q of the Yang-Mills particles is a function of class C^∞ from $\mathbb{R}^4$ to $\mathcal{G}$ whose given norm is $e > 0$. This means that in fact $\mathcal{G} \cong \mathbb{R}^4$ endowed with an ad-invariant scalar product , denoted by the dot "$\cdot$ " and that q takes its values in an orbit $\mathcal{O}$ of $\mathcal{G}$, whose equation is

$$(\mathcal{O}) : q \cdot q = e^2. \tag{2.7}$$

Equivalently, $|q| = e$, $|.|$ standing for the norm deduced from the scalar product. Also, this scalar product is such that:

$$u \cdot [v, w] = [u, v] \cdot w,\;\; u, v, w \in \mathcal{G}\,. \tag{2.8}$$

The relation (2.7) allows to express the component q^N of q as a function of $\bar{q} = (q^a), a = 1, ..., N-1$.

Using (2.5) and (2.7) we obtain the fact that the distribution function f of Yang-Mills particles is definitely a function of independent variables $(t, x^i, p^i, q^a) = (t, \bar{x}, \bar{p}, \bar{q}), i = 1, 2, 3; a = 1, 2, ..., N-1$. So $f = f(t, \bar{x}, \bar{p}, \bar{q})$.

The trajectories $s \mapsto (x^\alpha(s), p^\alpha(s), q^a(s))$ of such Yang-Mills charged particles are non-longer geodesics, but solutions of the differential system

$$\frac{dx^\alpha}{ds} = p^\alpha\;;\frac{dp^\alpha}{ds} = P^\alpha\;;\frac{dq^a}{ds} = Q^a \tag{2.9}$$

where

$$P^\alpha = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + p^\beta q \cdot F^\alpha_\beta ; Q^a = -p^\alpha [q, A_\alpha]^a = -C^a_{bc} p^\alpha A^b_\alpha q^c. \tag{2.10}$$

The relations (2.5) and (2.7) also show that the space phase is in fact the subset $P_{t,x} \times \mathcal{O}$ of $T(\mathbb{R}^4) \times \mathcal{O}$.

The relation (2.2) shows that $F_{\alpha\beta}$ is antisymmetric with respect to α and β, thus $F_{ii} = 0$. So by (2.2) and (2.3), we obtain:

$$F_{0i} = \partial_0 A_i\,, i = 1, 2, 3. \tag{2.11}$$

We will suppose that A, F are given in the space $C^\infty_0([0, +\infty[\times\mathbb{R}^3)$.

Choosing $q^N \geq 0$, since $p^0 > 0$, we deduce from (2.1) the following transformed Vlasov equation:

$$-\frac{\partial f}{\partial t} = \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} + \frac{Q^a}{p^0}\frac{\partial f}{\partial q^a}\; i = 1, 2, 3\,, a = 1, 2, ..., N-1\,. \tag{2.12}$$

The Christoffel symbols $\Gamma^\lambda_{\alpha\beta}$ of the Levi-Cevita connection ∇ associated with g are defined by the expression:

$$\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \left(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right) \tag{2.13}$$

Now we establish the main existence theorem of this work.

3. THE MAIN EXISTENCE THEOREM

Let us consider the function H defined by the right hand side of the equation (2.12) as follows

$$H(t,\bar{x},\bar{p},\bar{q},f,\nabla_{\bar{x},\bar{p},\bar{q}}f(t,\bar{x},\bar{p},\bar{q})) = \frac{p^i}{p^0}\frac{\partial f}{\partial x^i} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} + \frac{Q^a}{p^0}\frac{\partial f}{\partial q^a}\ , \quad i=1,2,3,\ a=1,2,...,N-1. \tag{3.1}$$

where using the relations (2.10)

$$\frac{P^i}{p^0} = -2\Gamma^i_{0j}p^j - \Gamma^i_{jk}p^j\frac{p^k}{p^0} + q\cdot\left(F^i_0 + F^i_j\frac{p^j}{p^0}\right)\ ,\ \frac{Q^a}{p^0} = -\frac{p^k}{p^0}C^a_{bc}q^cA^b_k\,. \tag{3.2}$$

Setting $u_i = \frac{\partial f}{\partial x_i}$, $v_i = \frac{\partial f}{\partial p^i}$, $w_a = \frac{\partial f}{\partial q^a}$ in the relation (3.1) and then using relation (3.2), the Hamiltonian H can be rewritten in the form:

$$H : [0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\longrightarrow\mathbb{R}$$

with

$$H(t,\bar{x},\bar{p},\bar{q},f,\bar{u},\bar{v},\bar{w}) = \frac{p^i}{p^0}u_i + \left(q\cdot\left(F^i_0 + F^i_j\frac{p^j}{p^0}\right) - 2\Gamma^i_{0j}p^j - \Gamma^i_{jk}p^j\frac{p^k}{p^0}\right)v_i - \frac{p^k}{p^0}C^a_{bc}q^cA^b_kw_a\,. \tag{3.3}$$

Let us assume that a Lipschitz continuous function $f_0 : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\longrightarrow\mathbb{R}$ and a real number $T>0$ are given and consider the following Cauchy problem:

$$\begin{cases} f_t(t,\bar{x},\bar{p},\bar{q}) + H(t,\bar{x},\bar{p},\bar{q},f,\nabla_{\bar{x},\bar{p},\bar{q}}f(t,\bar{x},\bar{p},\bar{q})) = 0 & \text{in }]0,T[\times B_{\mathbb{R}^3}(O,T)\times\mathbb{R}^3\times\mathbb{R}^{N-1} \\ f(0,\bar{x},\bar{p},\bar{q}) = f_0(\bar{x},\bar{p},\bar{q}) & \text{on } B_{\mathbb{R}^3}(O,T)\times\mathbb{R}^3\times\mathbb{R}^{N-1} \end{cases} \tag{3.4}$$

Our main purpose will be to prove using an important result of ([3]) that the Cauchy problem (3.4) has a unique L^∞ minimax viscosity solution $f\in C([0,T]\times B_{\mathbb{R}^3}(O,T)\times\mathbb{R}^3\times\mathbb{R}^{N-1})$.

Firstly, we state the following important lemma.

Lemma 9. *(Main lemma)* $\Gamma^\lambda_{\alpha\beta}$ *and* $\frac{p^i}{p^0}$ *are bounded over* $[0,T]\times\mathsf{B}_{\mathbb{R}^3}(\mathsf{O},T)$ *and the map* $x^\alpha\mapsto\bar{p}(x^\alpha)$ *is uniformly bounded over* $[0,T]\times\mathsf{B}_{\mathbb{R}^3}(\mathsf{O},T)$.

Proof. Relation (2.6) implies that, for $0\le t\le T$ and $\bar{x}\in B_{\mathbb{R}^3}(O,T)$

$$e^{-Ct}g^{I_0}_{jk}\le g_{jk}(t,\bar{x})\le e^{Ct}g^{S_0}_{jk} \tag{3.5}$$

where $g^{I_0}_{jk} = \inf\limits_{\bar{x}\in\bar{B}_{\mathbb{R}^3}(O,T)} g_{jk}(0,\bar{x})$ and $g^{S_0}_{jk} = \sup\limits_{\bar{x}\in\bar{B}_{\mathbb{R}^3}(O,T)} g_{jk}(0,\bar{x})$.

Using (2.13), (3.5) we obtain:

$$\left|\Gamma^i_{0j}\right| \leq \frac{C}{2}, \left|\Gamma^0_{ij}\right| \leq Ce^{Ct} g^{S_0}_{ij} \tag{3.6}$$

$$\left|\frac{p^i}{p^0}\right| \leq \frac{1}{\sqrt{g_{ii}\left(t,\bar{x}\right)}} \tag{3.7}$$

$$\left|\Gamma^i_{jk}\right| \leq 10C^2 e^{2CT} \frac{g^{S_0}}{g^{I_0}}. \tag{3.8}$$

Consequently, combining (3.5), (3.6), (3.7) ,(2.13) , we conclude that $\Gamma^\lambda_{\alpha\beta}$ and $\frac{p^i}{p^0}$ are bounded over $[0,T] \times B_{\mathbb{R}^3}(0,T)$.
Now using (2.12) and (3.2), we have:

$$\frac{dp^i}{dt} = -2\Gamma^i_{0j}p^j - \Gamma^i_{jk}p^j \frac{p^k}{p^0} + q \cdot \left(F^i_0 + F^i_j \frac{p^j}{p^0}\right). \tag{3.9}$$

So using (2.7), inequalities (3.5), (3.6), (3.7), (3.8) and the fact that $F \in C^\infty_0([0,+\infty[\times\mathbb{R}^3)$ we get:

$$\left|\frac{dp^i(t,\bar{x})}{dt}\right| \leq A \sum_{j=1}^{3} |p^j(t,\bar{x})| + B^i, (t,\bar{x}) \in [0,T] \times B(0,T) \tag{3.10}$$

where $B^i = B(e,T,\sum_{j=1}^3 \frac{1}{\sqrt{g^{I_0}_{jj}}}, |F|)$, $A = A(e,T,\sum_{j=1}^3 \frac{1}{\sqrt{g^{I_0}_{jj}}}, |F|)$ and $|F|$ is the norm of F. So

$$\left|\frac{d\bar{p}(t,\bar{x})}{dt}\right| \leq A|\bar{p}(t,\bar{x})| + B \tag{3.11}$$

with $B = \sum_{i=1}^3 B^i$. Integrating the relation (3.11) over $[0,t]$,and appealing to the Gronwall Lemma, one obtains:

$$|\bar{p}(t,\bar{x})| \leq (|\bar{p}(0,\bar{x})| + BT)e^{At}, (t,\bar{x}) \in [0,T] \times B_{\mathbb{R}^3}(O,T)$$

which completes the proof of Main Lemma 9. □

The next proposition will be useful.

Proposition 10. *Let $T > 0$ be given. The Hamiltonian*
$H : [0,T]\times B_{\mathbb{R}^3}(O,T)\times\mathbb{R}^3\times\mathbb{R}^{N-1}\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1} \longrightarrow \mathbb{R} \quad (t,\bar{x},\bar{p},\bar{q},f,\bar{u},\bar{v},\bar{w}) \mapsto H(t,\bar{x},\bar{p},\bar{q},f,\bar{u},\bar{v},\bar{w})$ *defined by (3.3) satisfies the following properties **(B)**:*
(B1) *$H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})$ is continuous in $(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})$.*
(B2)

$$\begin{aligned} &|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w}) - H(t,\bar{x},\bar{p},\bar{q},z,\bar{m},\bar{n},\bar{r})| \\ &\quad\leq C_0(1 + |\bar{x}| + |\bar{p}| + |\bar{q}|)(|\bar{u}-\bar{m}| + |\bar{v}-\bar{n}|) + |\bar{w}-\bar{r}|) \end{aligned} \tag{3.12}$$

and

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{0},\bar{0},\bar{0})| \leq C_0(1 + |\bar{x}| + |\bar{p}| + |\bar{q}|) \ , t \in [0,T]\,. \tag{3.13}$$

(B3)

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})-H(t,\bar{y},\bar{r},\bar{s},z,\bar{u},\bar{v},\bar{w})| \leq \lambda(L)(1+|\bar{u}|+|\bar{v}|+|\bar{w}|)(|\bar{x}-\bar{y}|+|\bar{p}-\bar{r}|)+|\bar{q}-\bar{s}|) \quad (3.14)$$

where $|\bar{x}|+|\bar{p}|+|\bar{q}|\leq L$, $|\bar{y}|+|\bar{r}|+|\bar{s}|\leq L$.

(B4)

$$|H(t,\bar{x},\bar{p},\bar{q},z_1,\bar{u},\bar{v},\bar{w})-H(t,\bar{x},\bar{p},\bar{q},z_2,\bar{u},\bar{v},\bar{w})| \leq C_0(1+|\bar{x}|+|\bar{p}|+|\bar{q}|+|\bar{u}|+|\bar{v}|+|\bar{w}|)\,|z_1-z_2| \quad (3.15)$$

Proof. Consider that $T>0$ is given.

– *For assertion (**B1**):* Since $p^0>0$, $H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})$ is obviously continuous in $(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})$.
– *For assertion (**B2**):* One has by definition of H: $|H(t,\bar{x},\bar{p},\bar{q},z,\bar{0},\bar{0},\bar{0})|=0$, which implies that

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{0},\bar{0},\bar{0})|\leq C_0(1+|\bar{x}|+|\bar{p}|+|\bar{q}|)\ ,t\in[0,T]\,.$$

– Using definition of H,one has

$$H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})-H(t,\bar{x},\bar{p},\bar{q},z,\bar{m},\bar{n},\bar{r})= \frac{p^i}{p^0}(u_i-m_i)+\left(q\cdot\left(F_0^i+F_j^i\frac{p^j}{p^0}\right)-2\Gamma^i_{0j}p^j-\Gamma^i_{jk}p^j\frac{p^k}{p^0}\right)(v_i-n_i) + \frac{p^k}{p^0}C^a_{bc}q^cA^b_k(r_a-w_a). \quad (3.16)$$

Using the Main lemma 9 and the hypotheses $|q|=e, A,F\in C_0^\infty([0,+\infty[\times\mathbb{R}^3)$ which allow to bound $\Gamma^\lambda_{\alpha\beta}$, $q\cdot\left(F_0^i+F_j^i\frac{p^j}{p^0}\right)$ and $\frac{p^i}{p^0}$ over $[0,T]\times B_{\mathbb{R}^3}(O,T)$, one easily obtains from (3.16) the following inequality:

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})-H(t,\bar{x},\bar{p},\bar{q},z,\bar{m},\bar{n},\bar{r})| \leq C_0(1+|\bar{x}|+|\bar{p}|+|\bar{q}|)(|\bar{u}-\bar{m}|+|\bar{v}-\bar{n}|)+|\bar{w}-\bar{r}|) \quad (3.17)$$

in which $C_0=C_0\left(e,\ g^{I_0}_{ij},g^{S_0}_{ij},T,|A|,|F|\right)$.
– *Assertion (**B3**):* Let $L>0$, such that $|\bar{x}|+|\bar{p}|+|\bar{q}|\leq L$, $|\bar{y}|+|\bar{r}|+|\bar{s}|\leq L$. Using definition of H one obtains

$$H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})-H(t,\bar{y},\bar{r},\bar{s},z,\bar{u},\bar{v},\bar{w})=$$
$$\left(\frac{p^i}{p^0}-\frac{r^i}{r^0}\right)u_i+\left((q-s)\cdot F_0^i+F_j^i\cdot\left(q\frac{p^j}{p^0}-s\frac{r^j}{r^0}\right)+2\Gamma_{0j}^i(r^j-p^j)\right.$$
$$\left.+\Gamma_{jk}^i\left(r^j\frac{r^k}{r^0}-p^j\frac{p^k}{p^0}\right)\right)v_i+\left(s^c\frac{r^k}{r^0}-q^c\frac{p^k}{p^0}\right)C_{bc}^aq^cA_k^bw_a. \quad (3.18)$$

Now

$$\begin{cases}\frac{p^i}{p^0}-\frac{r^i}{r^0}=\frac{p^i}{p^0}\frac{1}{r^0}\left(r^0-p^0\right)+\frac{1}{r^0}(p^j-r^j)\\ s^c\frac{r^k}{r^0}-q^c\frac{p^k}{p^0}=\frac{r^k}{r^0}\left(s^c-q^c\right)+\frac{q^c}{r^0}\left(r^k-p^k\right)-\frac{p^k}{p^0}\frac{1}{r^0}\left(p^0-r^0\right)\\ q\frac{p^j}{p^0}-s\frac{r^j}{r^0}=\frac{p^j}{p^0}\left(q-s\right)+\frac{s}{p^0}\left(p^j-r^j\right)-\frac{r^j}{r^0}\frac{1}{p^0}\left(r^0-p^0\right)\\ r^j\frac{r^k}{r^0}-p^j\frac{p^k}{p^0}=\frac{r^k}{r^0}\left(r^j-p^j\right)+\frac{p^j}{r^0}\left(r^k-p^k\right)-\frac{p^k}{p^0}\frac{1}{r^0}\left(p^0-r^0\right).\end{cases} \quad (3.19)$$

Invoking the fact that $\frac{1}{p^0},\frac{1}{r^0}\leq 1$, utilizing the Main lemma which allows to bound $\Gamma^\lambda_{\alpha\beta}$, $\frac{p^i}{p^0}$ and $\frac{r^k}{r^0}$ over $[0,T]\times B_{\mathbb{R}^3}(O,T)$, also using the hypotheses $|q|=|s|=e, A, F\in C_0^\infty\left([0,+\infty[\times\mathbb{R}^3\right)$, one easily obtains from (3.18) and (3.19) the following inequality

$$|H(t,\bar{x},\bar{p},\bar{q},z,\bar{u},\bar{v},\bar{w})-H(t,\bar{y},\bar{r},\bar{s},z,\bar{u},\bar{v},\bar{w})|$$
$$\leq C(1+|\bar{u}|+|\bar{v}|+|\bar{w}|)(|\bar{x}-\bar{y}|+|\bar{p}-\bar{r}|)+|\bar{q}-\bar{s}|)$$

where $C=C\left(e,\,g_{ij}^{I_0},g_{ij}^{S_0},T,|A|,|F|\right)$.

– *Assertion (**B4**):* It is verified because H does not depend on z and the proposition is established.

We are now able to give the Main Existence Theorem of this work, which is deduced from theorem 5.

Theorem 11. *(Main Existence Theorem) Let us assume that a Lipschitz continuous function $f_0:[0,+\infty[\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^{N-1}\mapsto\mathbb{R}$ and $T>0$ are given. Then:*
1- The Cauchy problem

$$\begin{cases}f_t(t,\bar{x},\bar{p},\bar{q})+H(t,\bar{x},\bar{p},\bar{q},f,\nabla_{\bar{x},\bar{p},\bar{q}}f(t,\bar{x},\bar{p},\bar{q}))=0 & \text{on }]0,T[\times B_{\mathbb{R}^3}(O,T)\times\mathbb{R}^3\times\mathbb{R}^{N-1}\\ f(0,\bar{x},\bar{p},\bar{q})=f_0(\bar{x},\bar{p},\bar{q}) & \text{in } B_{\mathbb{R}^3}\left(O,T\right)\times\mathbb{R}^3\times\mathbb{R}^{N-1}\end{cases}$$

where the Hamiltonian H is defined by (3.3), admits a unique continuous L^∞ minimax viscosity solution.

2- The Vlasov equation $p^\alpha\dfrac{\partial f}{\partial x^\alpha}+P^\alpha\dfrac{\partial f}{\partial p^\alpha}+Q^a\dfrac{\partial f}{\partial q^a}=0$ *in Yang-Mills charged* *curved space times has a unique continuous L^∞ minimax viscosity solution $f = f(t,\bar{x},\bar{p},\bar{q})$ on $[0,T] \times B_{\mathbb{R}^3}(O,T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$ that satisfies the initial condition $f(0,\bar{x},\bar{p},\bar{q}) = f_0(\bar{x},\bar{p},\bar{q})$ in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.*

Proof. 1- It is proved in proposition 10 that the Hamiltonian H satisfies the properties **(*B*1)-(*B*4)**. Consequently theorem 5 and theorems 7 -8 imply that the Cauchy problem admits a unique continuous L^∞ minimax viscosity solution $f \in C([0,T] \times B_{\mathbb{R}^3}(O,T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1})$ for $T > 0$.

2- The conclusion is a direct consequence of equivalence of the Cauchy problem (3.4) and the Vlasov equation with initial condition $f(0,\bar{x},\bar{p},\bar{q}) = f_0(\bar{x},\bar{p},\bar{q})$ in $B_{\mathbb{R}^3}(O,T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$. □

Let us set hereafter for a given $T > 0$, $\mathcal{K}_T = [0,T] \times B_{\mathbb{R}^3}(O,T) \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.

4. OPTIMAL CONTROL PROBLEM

Existence and uniqueness of L^∞ minimax viscosity solution of the inhomogeneous relativistic Vlasov equation are set . In this section the purpose is to establish that this solution can be seen as a solution of an optimal control problem. For this aim, we begin by recalling some results of [6] about optimal control and Hamilton-Jacobi equations.

4.1. Optimal control problem without boundary condition.

We first describe some general results about deterministic optimal control problems. To describe them, we consider a system which state is given by solution $y_x(t)$ of the following differential equation:

$$\frac{dy_x}{dt} = b\left(y_x(t), v(t)\right) \text{ for } t \geq 0\ ,\ y(0) = x \in \mathbb{R}^N. \tag{4.1}$$

where b maps $\mathbb{R}^N \times V$ into $\mathbb{R}^N$, V being some given closed convex set (or compact) in $\mathbb{R}^N$ which will be called the **set of values control of the control. The control** $v(t)$ is any measurable bounded function from $[0,+\infty[$ to V.

We will hereafter assume that $b(x,v)$ satisfies:

$$\begin{cases} |b(x,v) - b(y,v)| \leq C\,|x-y|\ \forall x,y \in \mathbb{R}^N, \forall v \in V\,; \\ |b(x,v)| \leq C\ \forall (x,v) \in \mathbb{R}^N \times V \\ b(x,v) \text{ is continuous on } \mathbb{R}^N \times V \end{cases} \tag{4.2}$$

for some constant $C > 0$.
Hence (4.1) has a unique solution for all $x \in \mathbb{R}^N$ denoted by $y_x(t)$.
We now define a pay-off function (or cost function) for each given control $v(\cdot)$.

$$\begin{aligned} J(t,x;v(\cdot)) = &\int_0^t l\left(y_x(s), v(s)\right) \exp\left\{-\int_0^s c\left(y_x(\lambda), v(\lambda)\right)\, d\lambda\right\} ds \\ &+ u_0\left(y_x(t)\right) \exp\left\{-\int_0^t c\left(y_x(s), v(s)\right)\, ds\right\} \end{aligned} \tag{4.3}$$

where $l(x,v)$, $c\,(x,v)$ are given functions which satisfy: $\exists\, C>0$ such that for $\varphi = l, c$ we have

$$\begin{cases} |\varphi(x,v)-\varphi(y,v)| \leq C\,|x-y| \;\forall x,y \in \mathbb{R}^N \,\forall v \in V\,; \\ |\varphi(x,v)| \leq C \,\forall (x,v) \in \mathbb{R}^N \times V; \\ \varphi(x,v) \text{ is continuous on } \mathbb{R}^N \times V\,. \end{cases} \tag{4.4}$$

The problem to solve is to minimize the cost function over all controls $v(\cdot)$, exactly that is to find

$$u(t,x) = \inf_{v(\cdot)} J\,(t,x;v\,(\cdot))\,. \tag{4.5}$$

The problem (4.5) is called **the finite horizon problem.**
The purpose of optimal control theory is to give a characterization of this optimal cost function and to compute optimal control, eventually in the form called feedback optimal control.

The following theorem expresses the dynamic programming about the optimal control problem.

Theorem 12. [6] *Under assumptions (4.3), (4.4): we have*

$$\begin{aligned} u(t,x) = \inf_{v(\cdot)} \Bigg\{ &\int_0^s b\,(y_x\,(\lambda)\,,v\,(\lambda)) \exp\left\{-\int_0^\lambda c\,(y_x\,(\tau)\,,v\,(\tau))\;d\tau\right\}\,d\lambda \\ &+u\,(y_x\,(s)\,,t-s) \exp\left[-\int_0^s c\,(y_x\,(\tau)\,,v\,(\tau))\;d\tau\right]\Bigg\} \end{aligned} \tag{4.6}$$

for all $0 \leq s \leq t$.

Now we give a result about the regularity of the cost function.

Proposition 13. [6] *Under assumptions (4.3), (4.4), the function* $u\,(\cdot,\cdot) : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ *is Lipschitz continuous for all* $0 < T < +\infty$.

The next result explains a relation between the optimal control problem and the Hamilton-Jacobi equations.

Theorem 14. [6] *Under assumptions (4.3), (4.4), we have* $u \in W^{1,\infty}\left((0,T)\times\mathbb{R}^N\right)$, $\forall T < +\infty$ *and:*

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup_{v\in V}\left\{b\,(x,v)\cdot D_x u + c\,(x,v)\,u - l\,(x,v)\right\} = 0 \text{ a.e. in } (0,+\infty)\times\mathbb{R}^N \\ u\,(0,x) = u_0\,(x) \text{ in } \mathbb{R}^N. \end{cases} \tag{4.7}$$

The next proposition proves the uniqueness of solution of theorem 14.

Proposition 15. [6] *Under assumptions (4.3), (4.4), if* $w \in W^{1,\infty}\left((0,T)\times\mathbb{R}^N\right)$ *for some* $T>0$ *and if* w *satisfies:*

$$\begin{cases} \dfrac{\partial w}{\partial t} + \sup_{v\in V}\left\{b\left(x,v\right)\cdot D_x w + c\left(x,v\right)w - l\left(x,v\right)\right\} \leq 0 \; a.e. \; in \;\; (0,T)\times\mathbb{R}^N \\ w\left(0,x\right) \leq u_0\left(x\right) \;\; in\, \mathbb{R}^N\,, \end{cases}$$

then we have $w\left(t,x\right) \leq u\left(t,x\right)$ *in* $(0,T)\times\mathbb{R}^N$.

Remark 16. According to the theorem 14, u satisfies the **Hamilton-Jacobi-Bellman equation**, a particular Hamilton-Jacobi equation in which the Hamiltonian is defined by

$$H\left(t,x,p\right) = \sup_{v\in V}\left\{b\left(x,v\right)\cdot p + c\left(x,v\right)t - l\left(x,v\right)\right\}.$$

This Hamiltonian is clearly Lipschitz continuous and convex in (t,p) as supremum of affine functions. Conversely if $H\left(t,x,p\right)$ is convex continuous function in (t,p) and Lipschitz continuous at least locally in x then it is possible to write $H\left(t,x,p\right)$ as a supremum of affine functions and in this way to write down some associated optimal control problem: indeed let us denote by $L\left(t,x,p\right)$ the dual convex function of $H\left(t,x,p\right)$, recall that L is given by

$$L\left(t,x,p\right) = \sup_{(s,q)\in\mathbb{R}\times\mathbb{R}^N}\left\{ts + p\cdot q - H\left(t,x,q\right)\right\} \leq +\infty.$$

Now, we know that

$$H\left(t,x,p\right) = \sup_{(s,q)\in Dom\, L(\cdot,x,\cdot)}\left\{p\cdot q + ts - L\left(t,x,q\right)\right\}. \tag{4.8}$$

And this proves that, at least formally, we may define for each convex Hamiltonian some associated control problem in the sense that the corresponding optimal cost function solves the Hamilton-Jacobi equation.

The following proposition gives a result about the feedback control of the optimal control problem.

Proposition 17. [6] *Assume that* $u\in C^1\left(\overline{Q_T}\right)$ *for some* $T>0$, *where* $Q_T = (0,T)\times\mathbb{R}^{\mathbf{N}}$, *and that there exists a continuous function* $\boldsymbol{v}\left(t,x\right)$ *defined on* $\overline{Q_T}$ *such that*

$$0 = \frac{\partial u}{\partial t}\left(t,x\right) + \sup_{v\in V}\left\{b\left(x,\boldsymbol{v}\right)\cdot D_x u\left(t,x\right) - l\left(x,\boldsymbol{v}\right)\right\}.$$

Let $y_x\left(s\right)$ *be a solution for* $0\leq s\leq t$ *of :*

$$\frac{dy_x}{ds}\left(s\right) + b\left(y_x\left(s\right),\boldsymbol{v}\left(t-s,y_x\left(s\right)\right)\right) = 0\;,\; y_x\left(0\right) = x\in\mathbb{R}^N.$$

Then the feedback $v_{t;x}\left(s\right) = \boldsymbol{v}\left(t-s,y_x\left(s\right)\right)$ *is optimal, that is, we have:*

$$u\left(t,x\right) = J\left(t,x;v_{t,x}\left(\cdot\right)\right)\;,\;\forall x\in\mathbb{R}^N\;,\;\forall t\in[0,T].$$

4.2. **Application to the Vlasov equation.**

The Hamiltonian (3.3), according to assumptions (***B*1**) and (***B*3**) , is continuous, clearly convex in $(t, \bar{u}, \bar{v}, \bar{w})$ and Lipschitz continuous locally in $(\bar{x}, \bar{p}, \bar{q})$. We can now state that the L^∞ minimax viscosity solution of the Vlasov equation is a solution of an optimal control problem.

Proposition 18. *Let H be the Hamiltonian (3.3) and L its dual convex function. Let us assume that a Lipschitz continuous function $f_0 : [0, +\infty[\times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \mapsto \mathbb{R}$ is given. Consider the functions*

$$b,\, c : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \times Dom\, L\,(\cdot, x, \cdot) \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1}\,.$$

satisfying assumptions (4.4).

The unique solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup_{(s,q) \in Dom\, L(\cdot,x,\cdot)} \{b\,(s,x,q) \cdot D_x u + c\,(s,x,q)\, u - L\,(s,x,q)\} = 0 \text{ a.e. in } \mathcal{K}_\mathcal{T} \\ u\,(0,x) = f_0\,(x) \;\text{ in } \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1} \end{cases}$$

solves the Vlasov equation $p^\alpha \dfrac{\partial f}{\partial x^\alpha} + P^\alpha \dfrac{\partial f}{\partial p^\alpha} + Q^a \dfrac{\partial f}{\partial q^a} = 0$ for Yang-Mills charged curved space times in $C\,(\mathcal{K}_T)$ for all $T > 0$ with initial condition $f(0, \bar{x}, \bar{p}, \bar{q}) = f_0(\bar{x}, \bar{p}, \bar{q})$ in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{N-1}$.

Proof. It is a direct consequence of (4.7), (4.8) and theorem 11 □

CONCLUSION

In the present paper, we have set a theorem of existence of viscosity minimax and L^∞ solutions of the inhomogeneous relativistic Vlasov equation in Yang-Mills charged curved space times , and brought up an optimal control problem. Introduction was made up to present particularities of this study comparatively to other ones done in the same topic. In the second section, we gave details of mathematics tools used to set the main existence theorem in section 3. In section 3, we have presented the frame of the work, the space times and the inhomogeneous relativistic Vlasov equation. In the forth section, we have proved that the L^∞ minimax viscosity solutions of the Vlasov equation may be expressed as a solution of an optimal control problem. In our future investigations, we will extend the present study to the Boltzmann reltivistic equation.

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