REPUBLIQUE DU CAMEROUN
Paix – Travail – Patrie
****
UNIVERSITE DE YAOUNDE I
FACULTE DES SCIENCES
DEPARTEMENT DE Mathématiques
****
Laboratoire de Mathématiques Appliquées

REPUBLIC OF CAMEROUN
Peace – Work – Fatherland
****
UNIVERSITY OF YAOUNDE I
FACULTY OF SCIENCE
DEPARTMENT OF Mathematics
****
Laboratory of Applied Mathematics

Decision Criteria on Fuzzy Variables for Portfolio Selection with Fuzzy Returns.

PhD. THESIS

Par : **TASSAK DEFFO CHRISTIAN**
Master of Sciences Mathematics

Sous la direction de
Pr. FONO LOUIS AIME
Associate Professor
University of Douala
Pr. NICOLAS GABRIEL ANDJIGA
Professor
University of Yaoundé I

Année Académique : 2016 - 2017

DEDICATION

TO

- My dear father: Mr DEFFO Jean Seraphin.

- My dear mother: Mrs DJUIDJE Jacqueline.

- My dear sisters and brothers: Therese, Christine, Celestin, Augustin, and Joseph to who I wish a lot of success in the sacred ministry of God.

ACKNOWLEDGEMENT

- I exceedingly thank the Almighty God for his guidance and support for the successful realization of this work.

- I am highly thankful to Pr. Louis Aimé FONO, Pr. Jules SADEFO and Pr. Nicolas Gabriel ANDJIGA, for their potential skills, their good management, suggestions and orientation which lead to a best achievement of this work.

- I am grateful and thankful to the members of Jury for accepting to evaluate this work.

- I express gratitude to all the members of the Applied Mathematics Laboratory and the Research team MASS, for the good ideas and experience they provided me during our work in seminary sessions.

- I give many thanks to all members of my family for the great and permanent attention they gave to me.

- My sincere thanks and gratitude go to my friends, especially EPOH Edvrad, Nlend Serges, Dzuche Justin for all their encouragements and moral support which inspired me confidence and a will to continue this work.

RESUME

Zadeh [40] a proposé la mesure de possibilité et la mesure de nécessité pour décrire les expressions vagues. Ces mesures ont permis d'étudier les caractéristiques d'une variable floue et d'appliquer les résultats obtenus dans divers domaines (agriculture, médécine, finance,...). Cependant, ces deux premières mesures n'étant pas duales, Liu [20] a proposé la mesure de crédibilité qui est la moyenne arithmétique des deux mesures précédentes. A l'aide de cette mesure de crédibilité, il a défini les deux premiers moments d'une variable floue: l'espérance mathématique et la variance. De plus, Huang [11] et Li et al. [16] ont proposé respectivement la semi-variance et le coefficient d'asymétrie d'une variable floue et les ont utilisés pour la détermination d'un portefeuille optimal dans un ensemble de portfeuilles d'un nombre fini d'actifs dont les rendements sont exprimés par des expressions vagues. Peng et al. [27] ont introduit deux relations de dominance sur les variables floues et ils les ont utilisées pour l'analyse du risque.

Dans cette thèse, nous introduisons, à l'aide de la mesure de crédibilité, les moments et les semi-moments d'ordre k (k est un entier naturel non nul) ainsi qu'une nouvelle relation de dominance. Nous déterminons les propriétés de ces moments et semi-moments, nous caractérisons chacune de ces trois dominances et nous déterminons leurs propriétés.

Les résultats théoriques obtenus sont appliqués à la détermination d'un portefeuille optimal d'actifs dont les rendements sont vagues et représentés par des nombres flous triangulaires selon deux approches: l'approche basée sur les quatre premiers moments et l'approche basée sur les portefeuilles non dominés.

Mots clés: Variable floue, Mesure de crédibilité, Moments, Relation de dominance, Portefeuille optimal.

ABSTRACT

In the literature, three measures were proposed to deal with imprecision and uncertainty in phenomenons. Zadeh [40] proposed the two first measures, namely possibility and necessity measures, and they enable to determine and study fuzzy variable's parameters and to apply theoretical results in some research areas (medical diagnosis, robot control, strategic decision, games,...). The third measure, namely credibility measure and proposed by Liu [20], is a dual measure and the average of the two first measures. Following that, scholars (Liu [20], Huang [11], Li et al. [16]) determined the three first moments (mean, variance, skewness) and the first semi-moment (semi-variance) of a fuzzy variable. They used the obtained results to solve portfolio selection problem with fuzzy returns by means of the mean-semi-variance model and the mean-variance-skewness model. Furthermore, Peng et al. [27] introduced two dominance relations on fuzzy variables, namely the first and the second order dominances, and they used them to analyze risk in fuzzy context.

In this thesis, we introduce moments and semi-moments of order k ($k \in \mathbb{N}$) of a fuzzy variable and we study their properties. We introduce a new dominance relation on fuzzy variables, we characterize three dominance relations (the two previous ones and the new one) and determine their properties.

The obtained theoretical results are applied to solve the main problem of portfolio selection with fuzzy returns described by triangular fuzzy numbers by means of two approaches: the first approach based on four first moments (mean, variance, skewness, kurtosis) and the second approach based on the core of portfolios of a finite family of assets, that is, the subset of non dominated portfolios.

Keywords: Fuzzy variable, Credibility measure, Moments, Dominance relation, Portfolio selection.

Contents

List of Figures

INTRODUCTION

Since many decades, portfolio selection theory had been used to solve some problems in Finance and it contributed to the development of financial market. The main problem is to invest a given capital on a finite number of assets so that the future return obtained from that investment has a maximum expected benefit and provides less loss (risk). Notice that future return is a convex linear combination of futures returns of assets where the scalars of that combination are the percentages of the capital invested on assets. In that sense, scholars developed theoretical tools in order to solve portfolio selection question.

In the literature, there are two branches to formalize futures returns in portfolio selection theory. In the first branch introduced by Markowitz [22], many scholars (Sharpe [33], Stone [35], Sengupta [32], Grauer [9], Rom and Ferguson [28], Krauss [14] and Konno [13]) described future returns by random variables, studied ramdom variables and proposed optimization models of portfolios based on characteristics (parameters) of those variables such as mean, variance, semi-variance and skewness. We notice that one of those four parameters defines the objective function of the optimization model and the others define its constraints. We do not consider this branch of literature in this thesis.

In the second branch that we consider, empirical studies (Tanaka et al. [37], Carlsson et al. [6], Huang [10], Smimou et al. [34]) proved that in some situations, future returns cannot be described by random variables due to lack of information or historical data or human being feelings. This can be explained by at least two reasons: (i) data bases do not exist or are incomplete or contain wrong information and values and (ii) in some cases, investors ask experts' advice to estimate returns: in fact, an expert can express assets future returns as follows: "around 20F", "between 15F and 25F", "approximately 20F", "no more than 25F and no less than 15F". Following that direction, we develop theoretical tools on fuzzy variables in order to solve portfolio selection question with vague returns.

To introduce such theoretical tools, we need a measure for fuzzy variables which plays the similar role as the probability measure for random variables. Zadeh [40] introduced possibility and necessity measures. These two measures are not dual, that means, even if the chance for an event to be realized is known, it is not easy to deduce the chance of this event to not be realized. Recently, Liu [20] introduced credibility measure as a self-dual measure which is the average of possibility and necessity measures. Based on credibility measure, many scholars (Liu [20], Huang [11], Li et al. [16]) introduced and studied first parameters of a fuzzy variable such as mean, variance, semi-variance, skewness. To solve portfolio selection question, Huang [11] proposed the mean-semi-variance deterministic model and implemented his model to determine a best portfolio on a set of seven assets with returns described by triangular fuzzy variables. More later, Li et al. ([16]) proposed the mean-variance-skewness deterministic model with skewness as objective function. They implemented their model on the same set of assets and obtained a better sharing of capital on those assets, that is, a portfolio with best parameters (greater mean, greater skewness, less variance and less semi-variance) than the one obtained by Huang. As we noticed in the two branches of the literature, expected benefits (resp. risks or losses) of a future return are formalized by the mean (resp. variance, semi-variance and skewness) of the (random or fuzzy) variable representing the return. However, Peng et al. [27] introduced two binary relations on fuzzy variables, namely the first and the second order dominance relations, to compare fuzzy variables. They characterize the first order dominance for triangular fuzzy variables and give some properties of those dominance relations. Consequently, there is a need to extend such studies based on parameters and dominance relations of fuzzy variables in order to improve the determination of best portfolios with fuzzy returns.

The aim of this thesis is to study, by means of the credibility measure, parameters of fuzzy variables and dominance relations on fuzzy variables in order to tackle the question of portfolio optimization with fuzzy returns. Our modest contribution to the development of uncertainty theory and its application in Finance is made up through the following aspects: introduction of moments and semi-moments of order k of a fuzzy variable, determination of their properties, introduction of a new dominance relation on fuzzy variables, characterization and determination of properties of three dominance relations, application in portfolio selection in Finance by the determination of best portfolios through the mean-variance-skewness-kurtosis model and by the determination of some non dominated portfolios with respect to the first order dominance.

This thesis contains five chapters and an appendix which contains some useful notions, their proofs and two published papers. Chapter one presents some basic notions on fuzzy sets, fuzzy numbers and fuzzy arithmetic. It recalls definitions and properties of possibility and necessity measures. It ends with illustration of some parameters of fuzzy numbers based on the possibility measure introduced by Saeidifar and Pasha [30].

Chapter two recalls definition and properties of credibility measure and its link with the membership function of a fuzzy variable. We deduce the credibility that a fuzzy event occurs. We recall first parameters of a fuzzy variable based on the credibility measure such as mean, variance, semi-variance and skewness. We deduce some basic properties of mean and variance.

In Chapter three, we introduce moments and semi-moments of a fuzzy variable and determine their properties. We characterize moments for symmetric fuzzy variables. We compare moment and semi-moment of a fuzzy variable and determine necessary and sufficient condition under which even moments of a fuzzy variable are null. The particular cases of kurtosis, semi-kurtosis, normalized kurtosis and normalized semi-kurtosis of a fuzzy variable are studied. We compute parameters of a convex linear combination of independent fuzzy variables, which represents a description of a portfolio with fuzzy returns.

In Chapter four, we introduce a new dominance relation on fuzzy variables, namely the mean-risk dominance, through the fuzzy lower partial moment of a fuzzy variable. We characterize that new dominance relation and, the first and second order dominance relations. Comparisons between those dominance relations and some of their properties are established.

Chapter five proposes some new deterministic portfolio optimization models whose objective function is either kurtosis or semi-kurtosis of portfolios with fuzzy returns. In addition, we introduce the core of a portfolio of a finite number of assets with respect to the first order dominance. We establish that it is non empty and is a union of the set of best portfolios and the set of incomparable portfolios. We implement with Matlab, on the set of portfolios of the seven assets introduced by Huang [11], our two optimization models and the set of best portfolios. We display optimal portfolios with respect to our deterministic models and best portfolios with respect to the first order dominance.

Finally, we give some concluding remarks and perspectives. The appendix presents some details on Fuzzy Lower Moments and some proofs.

Chapter One

Fuzzy sets and possibility theory

In this chapter, we present basic and useful notions on fuzzy sets, fuzzy numbers and fuzzy arithmetic. We also present some well-known concepts and results obtained in possibility theory.

Throughout this thesis, X is a nonempty set namely the universal set and $\mathcal{P}(X)$ is the power set of X (set of subsets of X). If X is finite, $card(X)$ is its cardinal.

1.1 Fuzzy numbers

1.1.1 Fuzzy numbers and its characteristics

Definition 1.1.1. *A fuzzy subset A of X is defined by its membership function: $\mu_A : X \to [0,1]$ such that, to each $x \in X$, is associated $\mu_A(x)$.*

Let x be an element of X. $\mu_A(x)$ represents the membership grade of x to A.

If $\forall x \in A, \mu_A(x) \in \{0,1\}$, then A becomes a crisp subset of X. A fuzzy subset A of X is denoted by $\{(x, \mu_A(x)), x \in X\}$.

Let us recall some useful characteristics of a fuzzy subset.

Definition 1.1.2. *Let A be a fuzzy subset of X and $\alpha \in]0,1]$.*

1. *The kernel of A is the crisp subset of X denoted by $Ker(A)$ and defined by:*

$$Ker(A) = \{x \in X/\mu_A(x) = 1\}.$$

2. *The support of A is the crisp subset of X denoted by $Supp(A)$ and defined by:*

$$Supp(A) = \overline{\{x \in X/\mu_A(x) > 0\}}.$$

3. *The height of A is the real number defined by:* $\sup_{x\in X} \mu_A(x)$.

4. *A is a normalized fuzzy subset if* $\sup_{x\in X} \mu_A(x) = 1$.

5. *A is a fuzzy quantity if A is a normalized fuzzy subset of $\mathbb{R}$.*

6. *The α-level set (α-cut) of A is a crisp subset of X, denoted by A_α and defined by:*

$$A_\alpha = \{x \in X/\mu_A(x) \geq \alpha\}.$$

Let us recall definition of a fuzzy number and some usual examples.

Definition 1.1.3. *Let A be a fuzzy subset of $\mathbb{R}$ and μ_A its membership function.*

A is a fuzzy number if the following conditions are satisfied:

- $\sup_{x\in\mathbb{R}} \mu_A(x) = 1$.

- *μ_A is convex, that means, $\forall x, y \in \mathbb{R}, \forall\lambda \in [0, 1], \mu_A(\lambda x + (1-\lambda y)) \geq \min(\mu_A(x), \mu_A(y))$.*

- *μ_A is upper semi-continuous, that is, $\forall\alpha \in]0, 1]$, A_α is a closed subset of $\mathbb{R}$.*

- *$Supp(A)$ is a compact subset of $\mathbb{R}$.*

Remark 1.1.1. *The notions of compact and closure are relative to the usual topology defined on $\mathbb{R}$.*

Example 1.1.1. *1. A trapezoidal fuzzy number denoted by* (a, b, c, d) *with* $a < b < c < d$ *is defined by the following membership function:*

$$\forall x \in \mathbb{R}, \mu(x) = \begin{cases} (\frac{x-a}{b-a}), & if\ a \leq x \leq b \\ 1, & if\ b \leq x \leq c \\ (\frac{x-d}{c-d}), & if\ c \leq x \leq d \\ 0, & elsewhere \end{cases} .$$

In this case, $Supp(A) = [a, d]$ *and* $Ker(A) = [b, c]$.

2. When $b = c$*, we obtain the triangular fuzzy number* (a, b, d).

Figures 1.1 and 1.2 display the trapezoidal fuzzy number $(1, 2, 3, 4)$ and the triangular fuzzy number $(1, 3.5, 4)$.

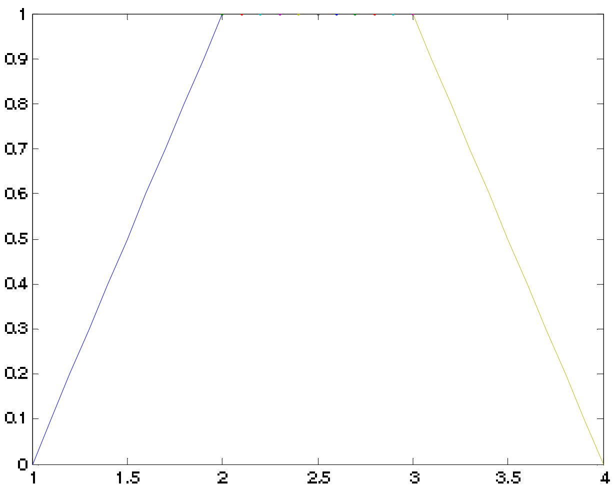

Figure 1.1: Trapezoidal fuzzy number $(1, 2, 3, 4)$.

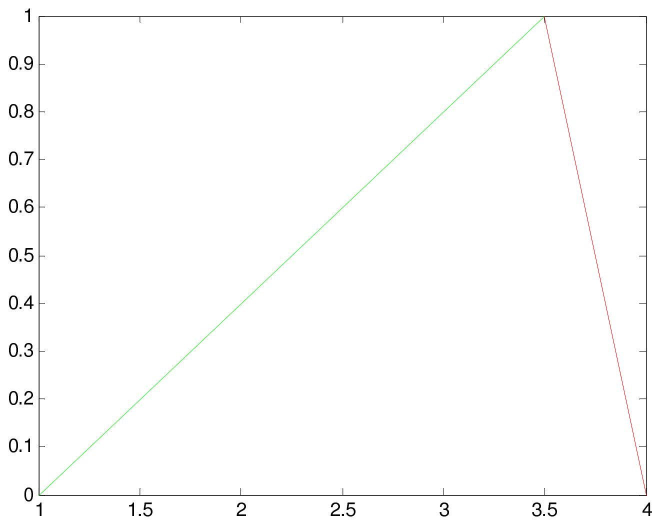

Figure 1.2: Triangular fuzzy number $(1, 3.5, 4)$.

Throughout this thesis, $\mathcal{F}$ is the set of fuzzy numbers of $\mathbb{R}$.

Let us end this paragraph by introducing a well-known family of fuzzy numbers, namely parametric fuzzy numbers.

Definition 1.1.4. *Let* $A \in \mathcal{F}$*. The parametric form of* A *is defined by its* α*-level sets by:*

$$\forall \alpha \in [0, 1], [A]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)].$$

Example 1.1.2. *1) Let A be a trapezoidal fuzzy number denoted by (a, b, c, d).*

Its parametric form is given by:

$$\forall\alpha \in [0,1], [A]_\alpha = [a + (b - a)\alpha, d - (d - c)\alpha].$$

2) The parametric form of a triangular fuzzy number $A = (a, b, d)$ is given by:

$$\forall\alpha \in [0,1], [A]_\alpha = [a + (b - a)\alpha, d - (d - b)\alpha].$$

In the next paragraph, we recall some operations made on fuzzy numbers similarly to those made on real numbers.

1.1.2 Fuzzy arithmetic

We recall the well-known Zadeh's Extension Principle which is the basis of fuzzy arithmetic.

Definition 1.1.5. *Let Y be a nonempty set and $\Phi : X \to \mathcal{P}(Y)$ be a mapping that corresponds to each element x of X, one or many elements of Y.*

A fuzzy subset B of Y compatible with Φ and associated with A is defined by:

$$\forall y \in Y, \mu_B(y) = \left\{ \begin{array}{l} \sup_{\{x \in X, y = \Phi(x)\}} \mu_A(x) \ \textit{if}\ \{x \in X, y = \Phi(x)\} \neq \emptyset \\ 0, \ \textit{otherwise} \end{array} \right. .$$

Let us apply that principle in an example.

Example 1.1.3. *Let us set: $X = \{a, b, c\}$, $Y = \{p, q\}$ two universal sets and $A = \{(a; 0.4), (b; 0.7), (c; 0.2)\}$ a fuzzy subset of X. $\Phi : X \to Y$ is a mapping defined by: $\Phi(a) = \Phi(c) = q$ and $\Phi(b) = p$.*

Let us define the fuzzy subset B of Y compatible with Φ and associated with A. Its membership function is given by:

$\mu_B(p) = \sup_{\{x \in X, p = \Phi(x)\}} \mu_A(x) = \sup\{\mu_A(b)\} = 0.7$ *and* $\mu_B(q) = \sup_{\{x \in X, q = \Phi(x)\}} \mu_A(x) = \sup\{\mu_A(a), \mu_A(c)\} = 0.4$. *Thus, we have:* $B = \{(p; 0.7), (q; 0.4)\}$.

The first application of Zadeh's Extension principle is the definition of a unary operation on $\mathcal{F}$. In the following, we recall such operation and some of its usual examples.

Definition 1.1.6. *(Bouchon-Meunier [3]) Let δ be an unary operation defined on $\mathbb{R}$. A unary operation Δ defined on $\mathcal{F}$ associated with δ is a mapping from $\mathcal{F}$ to $\mathcal{F}$ that corresponds to each fuzzy number A, another fuzzy number ΔA whose membership function is defined by:*

$$\forall z \in \mathbb{R}, \mu_{(\Delta A)}(z) = \sup\{\mu_A(x), x \in \mathbb{R} \text{ and } z = \delta(x)\}.$$

Let us recall three well-known types of unary operations Δ defining opposite of a fuzzy number, inverse of a fuzzy number, and product of a fuzzy number by a real number.

Definition 1.1.7. *1) In the case where $\delta : \mathbb{R} \to \mathbb{R}$, such that $\delta(x) = -x$, and $A \in \mathcal{F}$, ΔA is the opposite of A denoted by $-A$ and whose membership function is defined by:*

$$\forall z \in \mathbb{R}, \mu_{(-A)}(z) = \mu_A(-z).$$

2) In the case where $\delta : \mathbb{R}^* \to \mathbb{R}$, such that $\delta(x) = \frac{1}{x}$, and $A \in \mathcal{F}$, ΔA is the inverse of A denoted by $\frac{1}{A}$ and whose membership function is defined by:

$$\forall z \in \mathbb{R}^*, \mu_{(\frac{1}{A})}(z) = \mu_A(\frac{1}{z}).$$

3) In the case where $\delta : \mathbb{R} \to \mathbb{R}$, such that $\delta(x) = \lambda x$, and $A \in \mathcal{F}$, ΔA is the product of A by the real number λ denoted by λA and whose membership function is defined by:

$$\forall \lambda \in \mathbb{R}, \forall z \in \mathbb{R}, \mu_{(\lambda A)}(z) = \begin{cases} \mu_A(\lambda^{-1}z) \text{ if } \lambda \neq 0 \\ 0 \text{ if } \lambda = 0 \text{ and } z \neq 0 \\ \sup\{\mu_A(x), x \in \mathbb{R}\} \text{ if } \lambda = 0 \text{ and } z = 0 \end{cases}.$$

In the following Example, we apply those three unary operations on a trapezoidal fuzzy number.

Example 1.1.4. *Let* $A = (2, 4, 7, 8)$ *be a trapezoidal fuzzy number. Then* $-A = (-8, -7, -4, -2)$, $3A = (3, 12, 21, 24)$ *and* $\frac{1}{A} = (\dfrac{1}{8}, \dfrac{1}{7}, \dfrac{1}{4}, \dfrac{1}{2})$.

Another application of Zadeh's Extension Principle is the definition of a binary operation on $\mathcal{F}$. In what follows, we recall such operation and some of its four usual cases.

Definition 1.1.8. *Let* φ *be a binary operation defined on* $\mathbb{R}$.
A binary operation ϕ *defined on* $\mathcal{F}$ *and associated to* φ *is a mapping defined from* $\mathcal{F} \times \mathcal{F}$ *to* $\mathcal{F}$ *that corresponds to two fuzzy numbers* A *and* A'*, another fuzzy number* $A\phi A'$ *whose membership function is defined by:*

$$\forall z \in \mathbb{R}, \mu_{(A\phi A')}(z) = \sup\{\min(\mu_A(x), \mu_A(y)), (x, y) \in \mathbb{R}^2 \text{ and } z = \varphi(x, y)\}.$$

Let us recall four well-known types of binary operations ϕ defining sum, product, difference and quotient of two fuzzy numbers.

Definition 1.1.9. *1) In the case where* $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, such that* $\varphi(x, y) = x + y$*,* $A \in \mathcal{F}$ *and* $A' \in \mathcal{F}$*,* $A\phi A'$ *is the sum of* A *and* A' *denoted by* $A + A'$ *and whose membership function is defined by:*

$$\forall z \in \mathbb{R}, \mu_{(A+A')}(z) = \sup\{\min(\mu_A(x), \mu_A(y)), (x, y) \in \mathbb{R}^2 \text{ and } z = x + y\}.$$

2) In the case where $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, such that* $\varphi(x, y) = xy$*,* $A \in \mathcal{F}$ *and* $A' \in \mathcal{F}$*,* $A\phi A'$ *is the product of* A *and* A' *denoted by* $A \times A'$ *and whose membership function is defined by:*

$$\forall z \in \mathbb{R}, \mu_{(A\times A')}(z) = \sup\{\min(\mu_A(x), \mu_A(y)), (x, y) \in \mathbb{R}^2 \text{ and } z = xy\}.$$

3) In the case where $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*, such that* $\varphi(x, y) = x - y$*,* $A \in \mathcal{F}$ *and* $A' \in \mathcal{F}$*,* $A\phi A'$ *is the difference of* A *and* A' *denoted by* $A - A'$ *and whose membership function is defined by:*

$$\forall z \in \mathbb{R}, \mu_{(A-A')}(z) = \sup\{\min(\mu_A(x), \mu_A(y)), (x, y) \in \mathbb{R}^2 \text{ and } z = x - y\}.$$

4) In the case where $\varphi : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$*, such that* $\varphi(x, y) = \frac{x}{y}$*, for* $A \in \mathcal{F}$ *and* $A' \in \mathcal{F}$*,* $A\phi A'$ *is the quotient of* A *and* A' *denoted by* $\frac{A}{A'}$ *and whose membership function is defined by:*

$$\forall z \in \mathbb{R}, \mu_{(\frac{A}{A'})}(z) = \sup\{\min(\mu_A(x), \mu_A(y)), (x, y) \in \mathbb{R} \times \mathbb{R}^* \text{ and } z = \frac{x}{y}\}.$$

In the following Example, we apply the sum on two trapezoidal fuzzy numbers.

Example 1.1.5. *Let* $A = (a, b, c, d)$ *and* $A' = (a', b', c', d')$ *be two trapezoidal fuzzy numbers. The sum of* A *and* A' *is the trapezoidal fuzzy number:* $A + A' = (a + a', b + b', c + c', d + d')$.

The following example presents a concrete situation where we can illustrate the Extension Principle and some of the previous operations.

Example 1.1.6. *John was approximately twenty years old when he arrived in Cameroon. He left this country roughly two years later after living there for one year and 6 months approximately.*

- How can we characterize the age of John?

- How can we characterize the age of all individuals who are younger than John?

By considering that the universal set $X = [0, +\infty[$ *as a time space expressed in years, we propose answers to these questions in three steps.*

1) We assume that we represent:

- the expression "approximately 20 years old" by the triangular fuzzy number

$A_1 = (20 - \frac{1}{3}, 20, 20 + \frac{1}{3}) = (\frac{59}{3}, 20, \frac{61}{3})$*;*

- the expression"approximately 2 years old" by the triangular fuzzy number

$A_2 = (2 - \frac{1}{12}, 2, 2 + \frac{1}{12}) = (\frac{23}{12}, 2, \frac{25}{12})$ *and*

- the expression "approximately 1 year old and 6 months" by the triangular fuzzy number

$A_3 = (\frac{3}{2} - \frac{1}{12}, \frac{3}{2}, \frac{3}{2} + \frac{1}{12}) = (\frac{17}{12}, \frac{3}{2}, \frac{19}{12})$.

2) By applying the sum on $\mathcal{F}$, the age of John is represented by the triangular fuzzy number $A = A_1 + A_2 + A_3 = (23, \frac{47}{2}, 24)$. In order words, John is around 23 and half years old.

3) By applying the Extension Principle where φ is the mapping defined by $\varphi(x) = \{y/y \leq x\}$, the age of individuals younger than John is defined by the fuzzy subset B of X whose membership function μ_B is linked to the membership function of A by the relation $\mu_B(y) = \sup_{\{x/y \leq x\}} \mu_A(x)$. More precisely, we obtain:

$$\mu_B(y) = \begin{cases} 1, \text{ if } y \in [0, \frac{47}{2}] \\ 2(24 - y), \text{ if } y \in [\frac{47}{2}, 24] \\ 0, \text{ if } y \in [24, +\infty[\end{cases}.$$

That means, if you are less than 23 and half years old, you are younger than John, if you are at least 24 years old, you are not younger than John. If you are between the two ages, you are younger than John with the degree $2(24 - y)$ where y is your age.

Let us end this subsection with operations on parametrical fuzzy numbers.

Proposition 1.1.1. *Let A, B be two fuzzy numbers given by their respective parametric forms $[A]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$, $[B]_\alpha = [\underline{b}(\alpha), \overline{b}(\alpha)]$ and $\lambda \in \mathbb{R}$.*

- *The parametric form of the fuzzy number $A + B$ is given by:*

$$[A + B]_\alpha = [\underline{a}(\alpha) + \underline{b}(\alpha), \overline{a}(\alpha) + \overline{b}(\alpha)].$$

- *The parametric form of the fuzzy number λA is given by:*

$$[\lambda A]_\alpha = [\lambda\underline{a}(\alpha), \lambda\overline{a}(\alpha)].$$

In the following Subsection, we recall the two first measures introduced by Zadeh [40] and some parameters of a parametrical fuzzy variable with respect to the possibility measure. For that, we review notions on σ-algebra.

1.2 Possibility and necessity measures

1.2.1 σ-algebra

Definition 1.2.1. *1) A collection $\mathcal{A}$ consisting of subsets of X is called an algebra over X if the three following conditions are satisfied: (a) $X \in \mathcal{A}$; (b) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$; (c) if $A_1, A_2, ..., A_n \in \mathcal{A}$, then $\bigcup_{i=1}^{n} \in \mathcal{A}$.*

2) The collection $\mathcal{A}$ is a σ-algebra over X if the two conditions (b) and (c) below are satisfied and $\mathcal{A}$ is closed under countable union, that means, if $A_1, A_2, ... \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} \in \mathcal{A}$.

Let us recall some usual examples.

Example 1.2.1. *1) The collection $\{\emptyset, X\}$ is the smallest σ-algebra over X and the $\mathcal{P}(X)$, is the largest σ-algebra over X.*

2) Let A be a subset of X such that $A \neq \emptyset$ and $A \neq X$. Then $\mathcal{A} = \{\emptyset, A, A^c, X\}$ is a σ-algebra generated by A over X.

3) The smallest σ-algebra $\mathcal{B}_{\mathbb{R}}$ containing all open intervals is called the Borel algebra over the set of real numbers. We have: $\mathcal{B}(\mathbb{R}) \subset \mathcal{P}_{\mathbb{R}}$.

Remark 1.2.1. *1)Each element in $\mathcal{P}(X)$ is called an event.*

2) When X is finite, we have: card $\mathcal{P}(X) = 2^{cardX}$.

In the following Subsection, we study possibility and necessity measures introduced by Zadeh [40].

1.2.2 Possibility and necessity measures on fuzzy variables

Definition 1.2.2. *(Zadeh, [40]) 1) A function $Pos : \mathcal{P}(X) \to [0,1]$ is called possibility measure if (i) $Pos(X) = 1$, (ii) $Pos(\emptyset) = 0$ and (iii) $Pos(\cup_{i\in I} A_i) = \sup_{i\in I} A_i$ for any* *collection* $(A_i)_{i\in I}$ *in* $\mathcal{P}(X)$.

2) The necessity measure is a function $Nec : \mathcal{P}(X) \to [0,1]$ *defined by:*

$$\forall A \in \mathcal{P}(X), Nec(A) = 1 - Pos(A^c).$$

3) The triplet $(X, \mathcal{P}(X), Pos)$ *is called a possibility space.*

4) The triplet $(X, \mathcal{P}(X), Nec)$ *is called a necessity space.*

5) A possibility distribution on X *is a function* $\pi : X \to [0,1]$ *that satisfies the following normalization condition:* $\sup_{x\in X} \pi(x) = 1$.

Remark 1.2.2. *1) A possibility measure* Pos *can be defined by means of a possibility distribution* π *as follows:* $\forall A \in \mathcal{P}(X),\ Pos(A) = \sup_{x\in A} \pi(x)$.

2) The necessity measure Nec *satisfies the following conditions:* $\forall (A,B) \in \mathcal{P}(X)^2$, *(i)* $Nec(A \cap B) = \min(Nec(A), Nec(B))$ *and (ii)* $Nec(A) + Nec(A^c) \leq 1$.

In the following, $(X, \mathcal{P}(X), Pos)$ is a possibility space.

We give two examples of possibility measure based on possibility distribution when X is finite.

Example 1.2.2. *1) Let* $X = \{a, b, c, d, e, f\}$ *and* π *the possibility distribution on* X *defined by* $\pi(a) = \pi(b) = 1, \pi(c) = \pi(d) = \frac{1}{3}$ *and* $\pi(e) = \pi(f) = \frac{1}{4}$.

The possibility measure associated with π *is defined by:* $Pos(\{a, b\}) = Pos(\{a, c\}) = Pos(\{c, b\}) = 1$, $Pos(\{d, f\}) = \frac{1}{3}$ *and* $Pos(\{e, f\}) = \frac{1}{4}$.

2) Let $X = \{x_1, x_2, ..., x_n\}$ *and and* π *the possibility distribution on* X *defined by* $\forall x_i \in X, \pi(x_i) = y_i$.

The possibility measure associated with π *is defined by:* $Pos(\{x_{i_1}, x_{i_2}, ..., x_{i_k}\}) = \max_{1\leq j\leq k} y_{i_j}$.

Let us recall basic notions of fuzzy variables on a possibility space.

Definition 1.2.3. *1) A fuzzy variable ξ is a function from the $(X, \mathcal{P}(X), Pos)$ to $\mathbb{R}$.*

2) Let ξ be a fuzzy variable with membership function μ.

ξ is symmetric if $\exists a \in \mathbb{R}, \forall r \in \mathbb{R}, \mu(a-r) = \mu(a+r)$.

3) Let $r \in \mathbb{R}$. A fuzzy event $\{\xi \leq r\}$ associated to a fuzzy variable ξ with membership function μ, is a subset of X defined by:

$$\{\xi \leq r\} = \{x \in X, \mu(x) \leq r\}.$$

4) Let ξ be is a fuzzy variable with membership function μ and $r \in \mathbb{R}$. The possibility measure of a fuzzy event $\{\xi \leq r\}$ is defined as:

$$Pos(\{\xi \leq r\}) = \sup_{x \leq r} \mu(x).$$

We give some usual examples of fuzzy variables.

Example 1.2.3. *Let a, b, c, d be four real numbers such that $a < b < c < d$.*

1) A fuzzy number ξ is an equipossible fuzzy variable if its membership function satisfies: $\exists a, b \in \mathbb{R}$, such that $a < b$ and

$$\forall r \in \mathbb{R}, \mu(r) = \begin{cases} 1, & \textit{if } a \leq r \leq b \\ 0, otherwise. \end{cases}$$

We denote it by $\xi = (a, b)$.

2) A fuzzy variable ξ is a trapezoidal fuzzy variable if its membership function satisfies: $\exists a, b, c, d \in \mathbb{R}$, such that $a < b < c < d$ and

$$\mu(r) = \begin{cases} (\frac{r-a}{b-a}), & \textit{if } a \leq r \leq b \\ 1, & \textit{if } b \leq r \leq c \\ (\frac{r-d}{c-d}), & \textit{if } c \leq r \leq d \\ 0, & otherwise. \end{cases}$$

We denote it by $\xi = (a, b, c, d)$.

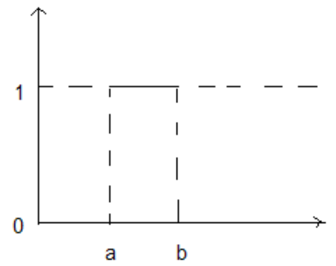

Figure 1.3: Equipossible variable (a,b).

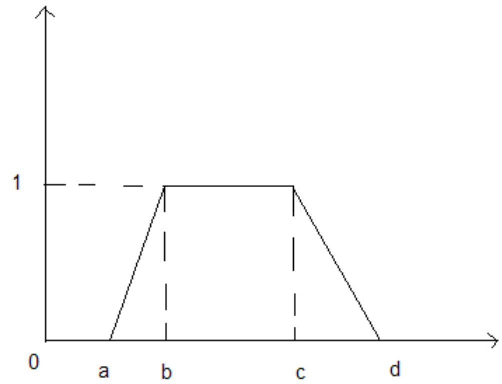

Figure 1.4: Trapezoidal variable (a,b,c,d).

3) A trapezoidal fuzzy variable $\xi = (a, b, c, d)$ *is symmetric when* $b - a = d - c$ *and a triangular fuzzy variable* $\xi = (a, b, d)$ *is symmetric when* $b - a = d - b$.

Let us notice that when $b = c$, $\xi = (a, b, c, d)$ becomes a triangular fuzzy variable $\xi = (a, b, d)$.

In the following example, we determine possibility and necessity of some fuzzy events of a fuzzy variable by means of the level sets of the fuzzy variable.

Example 1.2.4. *Let* ξ *be a fuzzy variable,* $\alpha \in]0, 1]$ *and* $[a_\alpha, b_\alpha]$ *the* α*- level set of* ξ. *Then, we have:*

$Pos(\{\xi \leq a_\alpha\}) = \alpha$ *and* $Pos(\{\xi \geq b_\alpha\}) = \alpha$, $Nec(\{\xi \geq a_\alpha\}) = 1 - \alpha$ *and* $Nec(\{\xi \leq b_\alpha\}) =$

$1-\alpha$.

Let us end this section by displaying in the following table some similarities between probability and possibility measures when the universe X is finite. For that, π, Pos, p and P are possibility distribution, possibility measure, probability distribution and probability measure respectively.

Let $A, B \in \mathcal{P}(X)$.

Possibility theory	Probability theory
$\sup_{x\in X} \pi(x) = 1$	$\sum_{x\in X} p(x) = 1$
$Pos(A\cup B) = \max(Pos(A), Pos(B))$	$P(A\cup B) = P(A) + P(B)$ if $A\cap B = \emptyset$
$Pos(A) = \sup_{x\in A} \pi(x)$	$P(A) = \sum_{x\in A} p(x)$
$\max(Pos(A), Pos(A^c)) = 1$ and $Pos(A) + Pos(A^c) \geq 1$	$P(A) + P(A^c) = 1$

In the next paragraph, we recall some known parameters of a parametrical fuzzy variable in a possibility space, namely possibility distance quantity, interval approximation, mean and variance.

1.2.3 Some characteristics of a fuzzy variable based on the possibility measure

The nearest weighted possibilistic interval

Let us recall the possibility distance quantity of a fuzzy number.

Definition 1.2.4. *(Saeidifar and Pasha, [30]) Let A be a parametrical fuzzy number defined by $\forall\alpha \in [0,1]$, $[A]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and let $C_L, C_U \in \mathbb{R}$ such that $Supp(A) = [C_L, C_U]$.*

A possibilistic distance quantity of A is the positive real number defined by:

$d(A, Supp(A)) = [\int_0^1 Pos(A \leq \underline{a}(\alpha))(\underline{a}(\alpha) - C_L)^2\, d\alpha + \int_0^1 Pos(A \geq \overline{a}(\alpha))(\overline{a}(\alpha) - C_U)^2\, d\alpha]^{\frac{1}{2}}$,

that is,

$$d(A, Supp(A)) = \sqrt{\int_0^1 \alpha(\underline{a}(\alpha) - C_L)^2\, d\alpha + \int_0^1 \alpha(\overline{a}(\alpha) - C_U)^2\, d\alpha}. \tag{1.1}$$

Interpretation 1.2.1. *Relation (1.1) is a type expected distance between the endpoints of its level sets and the two endpoints of its support.*

For the three other parameters, we need the following function.

Definition 1.2.5. *A function $f : [0, 1] \to \mathbb{R}$ is a weighting function if it is non-negative, increasing and satisfies the following normalization condition $\int_0^1 f(\alpha)\, d\alpha = 1$.*

Example 1.2.5. *The following functions $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ defined respectively as $f(x) = 2x$ and $g(x) = 3x^2$ are weighting functions.*

We now recall the interval approximation of a parametrical fuzzy number.

Definition 1.2.6. *(Saeidifar and Pasha, [30]) Let $[A]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ be a fuzzy number and f be a weighting function.*

1) The nearest lower weighted possibilistic point (NLWPP) of A associated with f is the real number defined by: $NLWPP_f(A) = \int_0^1 f(\alpha)\underline{a}(\alpha)\, d\alpha$.

2) The nearest upper weighted possibilistic point (NUWPP) of A associated with f is the real number defined by: $NUWPP_f(A) = \int_0^1 f(\alpha)\overline{a}(\alpha)\, d\alpha$.

3) The interval approximation of A or the nearest $f-$weighted possibilistic interval of A is the real interval defined by: $NWPI_f(A) = [NLWPP_f(A), NUWPP_f(A)]$.

Let us recall the nearest weighted possibilistic point of a parametrical fuzzy number.

The nearest weighted possibilistic point

Definition 1.2.7. *(Saeidifar and Pasha, [30]) Let $[A]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ be a fuzzy number and f be a weighting function. The $f-$weighted possibilistic mean (WPM) of fuzzy number A is the real number defined by: $\overline{M}_f(A) = \int_0^1 f(\alpha)\frac{\underline{a}(\alpha)+\overline{a}(\alpha)}{2}\, d\alpha$.*

When $f(\alpha)=2\alpha$, $\overline{M}_f(A)$ is simply denoted by $\overline{M}(A)$ and its becomes:

$$\overline{M}_f(A)=\overline{M}(A)=\int_0^1 \alpha(\underline{a}(\alpha)+\overline{a}(\alpha))\,d\alpha.$$

In that case, $\overline{M}(A)$ is called the weighted possibilistic mean value of fuzzy number A.

Therefore, we have the following result.

Theorem 1.2.1. *(Saeidifar and Pasha, [30]) Let $[A]_\alpha=[\underline{a}(\alpha),\overline{a}(\alpha)]$ be a fuzzy number and f be a weighting function. Then $\overline{M}_f(A)$ is the nearest weighted possibilistic point to A which is unique.*

Let us give an application of the previous notions on a trapezoidal fuzzy number.

Example 1.2.6. *1) Let $A=(-2,-1,1,3)$ be a trapezoidal fuzzy number and $f(\alpha)=2\alpha$.*

(i) The parametric form of A is given by $\forall\alpha\in[0,1],[A]_\alpha=[\underline{a}(\alpha),\overline{a}(\alpha)]=[-2+\alpha,3-2\alpha]$.

(ii) The nearest weighted possibilistic interval to A is $NWPI_f(A)=[-\frac{2}{3},\frac{5}{6}]$.

The nearest weighted possibilistic point to fuzzy number A is $\overline{M}(A)=\frac{1}{12}$.

2) If $A=(a,b,c,d)$ is a trapezoidal fuzzy number, then: $\overline{M}(A)=\frac{a+b+c+d}{4}$.

We end with possibilistic variance of a parametrical fuzzy number.

Possibilistic variance

Definition 1.2.8. *(Saeidifar and Pasha, see ([30])) Let $[A]_\alpha=[\underline{a}(\alpha),\overline{a}(\alpha)]$ be a fuzzy number and $\overline{M}(A)$ its weighted possibilistic mean value.*

The possibilistic variance of a fuzzy number A is the real number defined by:

$$\sigma_A^2=Var(A)=\int_0^1 \alpha[(\underline{a}(\alpha)-\overline{M}(A))^2+(\overline{a}(\alpha)-\overline{M}(A))^2]\,d\alpha.$$

Interpretation 1.2.2. *1) The possibilistic variance of a fuzzy number A explains the variation of this fuzzy number with respect to its possibilistic mean value $\overline{M}(A)$.*

2) Similarly to probability theory, one can define possibilistic skwewness and kurtosis.

We apply the previous notion on the trapezoidal number of the previous example.

Example 1.2.7. *Let A be the trapezoidal fuzzy number of the previous example where $\overline{M}(A) = \frac{1}{12}$.*

The possibilistic variance of A is obtained as follows:

$$Var(A) = \int_0^1 \alpha[(-2+\alpha-\frac{1}{12})^2 + (3-2\alpha-\frac{1}{12})^2]\, d\alpha = \frac{345}{144} \simeq 3.4$$

The possibility measure gives the possibility of occurrence of fuzzy (imprecise) events such as: "around 10", "approximately 2", "between 3 and 4", "almost young", and so on... That is the reason why this measure deals with these types of uncertainty: imprecision and vagueness. Nevertheless, this measure is not dual and that is a significant inconvenience for the description of uncertain financial markets, in that sense it doesn't make decisions consistent with the law of contradiction and excluded middle.

Chapter Two

First parameters of a fuzzy variable based on the credibility measure

In this chapter, we present the credibility measure introduced earlier by Liu [20]. We recall some well-known parameters of a fuzzy variable such as mean, variance, semi-variance and skewness. We deduce some basic properties of the mean and the variance.

2.1 Credibility measure and membership function

2.1.1 Credibility measure: definitions and examples

Definition 2.1.1. *Liu[20] Let $Cr : \mathcal{P}(X) \to [0,1]$ be a function and $I \subseteq \mathbb{N}$.*

1) Cr is a credibility measure if:

- Axiom 1 (Normality): $Cr(X) = 1$;

- Axiom 2 (Monotonicity): $\forall A, B \in \mathcal{P}(X)$, $Cr(A) \leq Cr(B)$ whenever $A \subset B$;

- Axiom 3 (Self-duality): $\forall A \in \mathcal{P}(X), Cr(A) + Cr(A^c) = 1$;

- Axiom 4 (Maximality): $\forall (A_i)_{i\in I} \subseteq \mathcal{P}(X)$ with $\sup_i Cr(A_i) < \frac{1}{2}$, $Cr(\cup_i A_i) = \sup_i Cr(A_i)$.

2) The triplet $(X, \mathcal{P}(X), Cr)$ is called a credibility space.

Let us recall some usual examples of credibility measure.

Example 2.1.1. *1) Let $X = \{X_1, X_2\}$.*

i) There are four events: $\emptyset, \{X_1\}, \{X_2\}$ *and* X.

ii) The set function Cr *is defined by:* $Cr(\emptyset) = 0, Cr(\{X_1\}) = 0.3, Cr(\{X_2\}) = 0.7$ *and* $Cr(X) = 1$.

Cr *is a credibility measure because it satisfies the four axioms.*

2) Let $X = \{X_1, X_2, X_3\}$.

i) There are eight events: $\emptyset, \{X_1\}, \{X_2\}, \{X_3\}, \{X_1, X_2\}, \{X_1, X_3\}, \{X_2, X_3\}$ *and* X.

ii) The set function Cr *is defined by:* $Cr(\emptyset) = 0, Cr(\{X_1\}) = 0.3, Cr(\{X_2\}) = 0.4, Cr(\{X_3\}) = 0.6, Cr(\{X_1, X_2\}) = 0.4, Cr(\{X_1, X_3\}) = 0.6, Cr(\{X_2, X_3\}) = 0.7, Cr(X) = 1$.

Cr *is a credibility measure because it satisfies the four axioms.*

3) Let $X = \mathbb{R}$. *The set function* Cr *is defined by:*

$$Cr(\theta) = \begin{cases} 1 \ \text{if} \ \ \theta = \mathbb{R} \\ 0 \ \text{if} \ \ \theta = \emptyset \\ \frac{1}{2} \ \text{otherwise} \end{cases}.$$

Cr *is a credibility measure because it satisfies the four axioms.*

Let us end this paragraph with the link between Cr, Pos and Nec.

Remark 2.1.1. *A useful link between* Cr, Pos *and* Nec *is*

$\forall A \in \mathcal{P}(X),\ Cr(A) = \frac{1}{2}[1 + Pos(A) - Pos(A^c)] = \frac{1}{2}[Pos(A) + Nec(A)]$.

Let us notice that, Definition 1.2.3 defines events associated with a fuzzy variable on a possibility space. Since the work of Liu [20], a new approach defined it on a credibility space. Thereby, throughout this thesis, a fuzzy variable is defined on the credibility space $(X, \mathcal{P}(X), Cr)$.

2.1.2 Link between credibility measure and the membership function of a fuzzy variable

Definition 2.1.2. *(Liu [20], B. Liu and Y. Liu [21]) 1) A fuzzy variable is defined as a function from a credibility space* $(X, \mathcal{P}(X), Cr)$ *to* $\mathbb{R}$.

2) A fuzzy variable ξ *is nonnegative, denoted by* $\xi \geq 0$, *if* $C(\{\xi < 0\}) = 0$.

3) Let ξ_1 *and* ξ_2 *be two fuzzy variables defined on the credibility space* $(X, \mathcal{P}(X), Cr)$.

i) $\xi_1 = \xi_2$ *if* $\xi_1(x) = \xi_2(x)$ *for almost* $x \in X$, *that means,* $Cr(\{x \in X, \xi_1(x) \neq \xi_2(x)\}) = 0$.

ii) ξ_1 *and* ξ_2 *are independent fuzzy variables if for any sets* B_1, B_2 *of* $\mathbb{R}$, $Cr(\{\xi_1 \in B_1\} \cap \{\xi_2 \in B_2\}) = \min(Cr(\{\xi_1 \in B_1\}), Cr(\{\xi_2 \in B_2\}))$.

Definition 2.1.3. *(B. Liu and Y.Liu, [20]) Let* ξ *be a fuzzy variable defined on* $(X, \mathcal{P}(X), Cr)$. *Then its membership function is derived from the credibility measure by:*

$$\forall r \in \mathbb{R}, \mu(r) = (2Cr(\{\xi = r\})) \wedge 1. \tag{2.1}$$

The following result, established by Liu, gives the credibility measure of events with respect to a fuzzy variable ξ by means of its membership function.

Theorem 2.1.1. *(Credibility Inversion Theorem, B. Liu and Y.Liu [20], p.445) Let* ξ *be a fuzzy variable with membership function* μ. *Then for any set* A *of reals numbers, we have:*

$$Cr(\{\xi \in A\}) = \frac{1}{2}(\sup_{t \in A} \mu(t) + 1 - \sup_{t \in A^c} \mu(t)). \tag{2.2}$$

Let us calculate the credibility measure of some usual events by applying the previous result.

Example 2.1.2. *1) In the usual cases where* $A =]-\infty; r]$ *or* $A = [r; +\infty[$ *with* $r \in \mathbb{R}$, *then*

$Cr(\{\xi \in A\})$ becomes

$$\begin{cases} Cr(\{\xi \leq r\}) = \frac{1}{2}(\sup_{x\in]-\infty,r]} \mu(x) + 1 - \sup_{x\in]r,+\infty[} \mu(x)) \\ Cr(\{\xi \geq r\}) = \frac{1}{2}(\sup_{x\in[r,+\infty[} \mu(x) + 1 - \sup_{x\in]-\infty,r[} \mu(x)) \end{cases}.$$

2) For an equipossible fuzzy variable $\xi = (a,b)$, we have:

$$Cr(\{\xi \leq r\}) = \begin{cases} 0, \ \textit{if } r \leq a \\ \frac{1}{2} \ \textit{if } a \leq r \leq b \\ 1, \ b \leq r \end{cases} \tag{2.3}$$

and

$$Cr(\{\xi \geq r\}) = \begin{cases} 1, \ \textit{if } r \leq a \\ \frac{1}{2} \ \textit{if } a \leq r \leq b \\ 0, \ b \leq r \end{cases}. \tag{2.4}$$

A proof of those results:

a) Let us take $r \in \mathbb{R}$:

- If $r \leq a$, then $Cr(\{\xi \leq r\}) = \frac{1}{2}(0+1-1) = 0$.

- If $a \leq r \leq b$, then $Cr(\{\xi \leq r\}) = \frac{1}{2}(1+1-1) = \frac{1}{2}$.

- If $b \leq r$, $Cr(\{\xi \leq r\}) = \frac{1}{2}(1+1-0) = 1$.

b) $Cr(\{\xi \geq r\})$ is obtained by using the self-duality axiom.

3) For a trapezoidal fuzzy variable $\xi = (a,b,c,d)$, we have:

$$Cr(\{\xi \leq r\}) = \begin{cases} 0, \ \textit{if } r < a \\ \frac{1}{2}(\frac{r-a}{b-a}), \ \textit{if } a \leq r < b \\ \frac{1}{2}, \ \textit{if } b \leq r < c \\ 1 - \frac{1}{2}(\frac{r-d}{c-d}), \ \textit{if } c \leq r < d \\ 1, \ \textit{if } d \leq r \end{cases} \tag{2.5}$$

and

$$Cr(\{\xi \geq r\}) = \begin{cases} 1, \ \textit{if } r < a \\ 1 - \frac{1}{2}(\frac{r-a}{b-a}), \ \textit{if } a \leq r < b \\ \frac{1}{2}, \ \textit{if } b \leq r < c \\ \frac{1}{2}(\frac{r-d}{c-d}), \ \textit{if } c \leq r < d \\ 0, \ \textit{if } d \leq r \end{cases} \tag{2.6}$$

A proof of those results:

a) Let us take $r \in \mathbb{R}$:

- If $r \leq a$, then $Cr(\{\xi \leq r\}) = \frac{1}{2}(0+1-1) = 0$.

- *If* $a \le r \le b$, *then* $Cr(\{\xi \le r\}) = \frac{1}{2}(\frac{r-a}{b-a} + 1 - 1) = \frac{1}{2}(\frac{r-a}{b-a})$.

- *If* $b \le r \le c$, *then* $Cr(\{\xi \le r\}) = \frac{1}{2}(1 + 1 - \frac{r-d}{c-d}) = 1 - \frac{1}{2}(\frac{r-d}{c-d})$.

- *If* $d \le r$, $Cr(\{\xi \le r\}) = \frac{1}{2}(1 + 1 - 0) = 1$.

b) $Cr(\{\xi \ge r\})$ *is obtained by using the self-duality axiom.*

4) For a triangular fuzzy variable (a, b, d)*, we just set* $b = c$ *in the expressions of* $Cr(\{\xi \le r\})$ *and* $Cr(\{\xi \ge r\})$ *for a trapezoidal variable* (a, b, c, d)*.*

Let us end this section by introducing the credibility distribution of a fuzzy variable.

Definition 2.1.4. *(Liu,[17]) 1) The credibility distribution of a fuzzy variable* ξ *is an application* $\Phi : \mathbb{R} \to [0, 1]$ *defined by:* $\forall r \in \mathbb{R}$,

$$\Phi(r) = Cr(\{\xi \le r\}). \tag{2.7}$$

2) Let Φ *be the distribution function of* ξ*.*

Φ *is a degenerate distribution function if* $\exists t_0 \in \mathbb{R}$ *such that* $\forall t \in \mathbb{R}, t \ge t_0$, $\Phi(t) = 1$ *and* $\forall t \in \mathbb{R}, t < t_0$, $\Phi(t) = 0$.

3) The density credibility function of ξ*, when it exists, is the function defined such that:* $\forall r \in \mathbb{R}, \Phi(r) = \int_{-\infty}^{t} \phi(t)\, dt$.

Example 2.1.3. *The distribution function* Φ *of a trapezoidal fuzzy variable* $\xi = (a, b, c, d)$ *is given by* (2.5) *and the distribution function* Φ *of an equipossible fuzzy variable* $\xi = (a, b)$ *is given by* (2.3)*.*

Proposition 2.1.1. *Let* Φ *be a distribution function.*

Φ *is an increasing function.*

Proof: The result is obtained by using the fact that Cr is an increasing function. □

2.2 First parameters of a fuzzy variable

2.2.1 Expected value: definitions and examples

Definition 2.2.1. *(B. Liu and Y.Liu [20], p.446) Let ξ be a fuzzy variable.*

The expected value of ξ is the real number defined by:

$$E[\xi] = \int_{0}^{+\infty} Cr(\{\xi \geq r\})dr - \int_{-\infty}^{0} Cr(\{\xi \leq r\})dr \tag{2.8}$$

provided that at least one of the two integrals is finite.

Let us calculate expected values of some well-known fuzzy variables.

Example 2.2.1. *Let us consider an equipossible fuzzy variable $\xi = (a, b)$:*

We distinguish three cases: $0 \leq a$, $a < 0 \leq b$ and $b < 0$.

1) If $0 \leq a$, we have $Cr(\{\xi \leq r\}) = 0$ when $r < 0$. Then, according to relations (2.8) and (2.4):

$$E[\xi] = \int_{0}^{a} 1\, dr + \int_{a}^{b} \frac{1}{2}\, dr = \frac{a+b}{2}.$$

2) If $a < 0 \leq b$, then, according to relations (2.8),(2.3) and (2.4):

$$E[\xi] = -\int_{a}^{0} \frac{1}{2}\, dr + \int_{0}^{b} \frac{1}{2}\, dr = \frac{a+b}{2}.$$

3) If $b < 0$, we have $Cr(\{\xi \geq r\}) = 0$ when $r > 0$. Then, according to relations (2.8) and (2.3):

$$E[\xi] = -\int_{a}^{b} \frac{1}{2}\, dr - \int_{b}^{0} 1\, dr = \frac{a+b}{2}.$$

Therefore, the expected value of the equipossible fuzzy variable $\xi = (a, b)$ is:

$$E[\xi] = \frac{a+b}{2}.$$

Example 2.2.2. *I) Let us consider a trapezoidal fuzzy variable* $\xi = (a, b, c, d)$*:*

We distinguish five cases: $0 \leq a < b < c < d$, $a < 0 \leq b < c < d$, $a < b < 0 \leq c < d$, $a < b < c \leq 0 < d$ *and* $a < b < c < d \leq 0$.

1) If $0 \leq a < b < c < d$, *we have* $Cr(\{\xi \leq r\}) = 0$ *when* $r < 0$ *Then, according to relations (2.8) and* (2.6)*:*

$$E[\xi] = \int_0^a 1\ dr + \int_a^b 1 - \frac{1}{2}(\frac{r-a}{b-a})\ dr + \int_b^c \frac{1}{2}\ dr + \int_c^d \frac{1}{2}(\frac{r-d}{c-d})\ dr = \frac{a+b+c+d}{4}.$$

2) If $a < 0 \leq b < c < d$, *then, according to relations (2.8),*(2.5) *and* (2.6)*:*

$$E[\xi] = -\int_a^0 \frac{1}{2}(\frac{r-a}{b-a})\ dr + \int_0^b 1 - \frac{1}{2}(\frac{r-a}{b-a})\ dr + \int_b^c \frac{1}{2}\ dr + \int_c^d \frac{1}{2}(\frac{r-d}{c-d})\ dr = \frac{a+b+c+d}{4}.$$

3) If $a < b < 0 \leq c < d$, *then, according to relations (2.8),*(2.5) *and* (2.6)*:*

$$E[\xi] = -\int_a^b \frac{1}{2}(\frac{r-a}{b-a})\ dr - \int_b^0 \frac{1}{2}\ dr + \int_0^c \frac{1}{2}\ dr + \int_c^d \frac{1}{2}(\frac{r-d}{c-d})\ dr = \frac{a+b+c+d}{4}.$$

4) If $a < b < c \leq 0 < d$, *then, according to relations (2.8),*(2.5) *and* (2.6)*:*

$$E[\xi] = -\int_a^b \frac{1}{2}(\frac{r-a}{b-a})\ dr - \int_b^c \frac{1}{2}\ dr - \int_c^0 1 - \frac{1}{2}(\frac{r-d}{c-d})\ dr + \int_0^d \frac{1}{2}(\frac{r-d}{c-d})\ dr = \frac{a+b+c+d}{4}.$$

5) If $a < b < c < d \leq 0$, *we have* $Cr(\{\xi \geq r\}) = 0$ *when* $r > 0$. *Then, according to relations (2.8) and* (2.5)*:*

$$E[\xi] = -\int_a^b \frac{1}{2}(\frac{r-a}{b-a})\ dr - \int_b^c \frac{1}{2}\ dr - \int_c^d 1 - \frac{1}{2}(\frac{r-d}{c-d})\ dr - \int_d^0 1\ dr = \frac{a+b+c+d}{4}.$$

Therefore, the expected value of the trapezoidal fuzzy variable $\xi = (a, b, c, d)$ *is:*

$$E[\xi] = \frac{a+b+c+d}{4}.$$

II) We deduce from those results that, when $b = c$, *the expected value of the triangular fuzzy variable* $\xi = (a, b, d)$ *is:* $E[\xi] = \frac{a+2b+d}{4}$.

Remark 2.2.1. *According to relations* (2.7) *and* (2.8)*, the expected value of* ξ *can be defined by means of credibility distribution* Φ *of* ξ *as follows:*

$$E[\xi] = \int_0^{+\infty} [1 - \Phi(r)]\, dr - \int_{-\infty}^{0} \Phi(r)\, dr. \tag{2.9}$$

We end with some properties of the expected value of fuzzy variables.

The following result establishes the expected value of a fuzzy variable ξ by means of a credibility distribution function.

2.2.2 Some basic properties

Proposition 2.2.1. *Let* ξ *be a fuzzy variable with a bijective credibility distribution function* Φ*. The expected value of* ξ *is defined by:*

$$E[\xi] = \int_0^1 \Phi^{-1}(u)\, du \tag{2.10}$$

Proof: According to the definition, $E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r\}\, dr \; - \; \int_{-\infty}^{0} Cr\{\xi \leq r\}\, dr$. By using the fact that $\forall r \in \mathbb{R}, \Phi(r) = Cr\{\xi \leq r\}$, we have:

$E[\xi] = \int_0^{+\infty} [1 - \Phi(r)]\, dr \; - \; \int_{-\infty}^{0} \Phi(r)\, dr$. By setting $u = \Phi(r)$, it follows that $r = \Phi^{-1}(u)$ and $dr = \dfrac{du}{\Phi' \circ \Phi^{-1}(u)}$. The new integral becomes:

$E[\xi] = \int_{\Phi(0)}^{1} d\Phi^{-1}(u) - \int_0^1 u d\Phi^{-1}(u)\, du$. By the two following equalities:

$\int_{\Phi(0)}^{1} d\Phi^{-1}(u) = \Phi^{-1}(1)$ and $\int_0^1 u d\Phi^{-1}(u) \; = \Phi^{-1}(1) - \int_0^1 \Phi^{-1}(u)\, du$,

it follows that $E[\xi] = \int_0^1 \Phi^{-1}(u)\, du$. $\square$

The following result establishes that the expected value is a linear operator.

Proposition 2.2.2. *Let* ξ *be a fuzzy variable with finite expected value. Then,*

$$\forall a, b \in \mathbb{R}, E[a\xi + b] = aE[\xi] + b. \tag{2.11}$$

Proof: Let us consider $a, b \in \mathbb{R}$. According to relation (2.8) :

$E[a\xi+b] = \int_0^{+\infty} Cr(a\xi+b \geq r)dr - \int_{-\infty}^0 Cr(a\xi+b \leq r)dr.$

We distinguish two cases:

<u>1st case</u>: If $a = 0$. We have: $E[a\xi+b] = E[b] = b = aE[\xi]+b$.

<u>2nd case</u>: If $a \neq 0$.

We have: $a\xi+b \geq r \Leftrightarrow \xi \geq \frac{r-b}{a}$ and $a\xi+b \leq r \Leftrightarrow \xi \leq \frac{r-b}{a}$.

- Let us evaluate $\int_0^{\infty} Cr(\{a\xi+b \geq r\})dr$.

We set: $r' = \frac{r-b}{a}$. We have: $r \in [0,+\infty[\Leftrightarrow r' \in [-\frac{b}{a},+\infty[$ and $dr' = \frac{1}{a}dr$. Thus: $\int_0^{+\infty} Cr(\{a\xi+b\} \geq r)dr = a\int_{-\frac{b}{a}}^{+\infty} Cr(\{\xi \geq r'\})dr'$.

- Let us evaluate $\int_{-\infty}^0 Cr(\{a\xi+b \leq r\})dr$.

We set: $r' = \frac{r-b}{a}$

We have: $r \in]-\infty,0] \Leftrightarrow r' \in]-\infty,-\frac{b}{a}]$ and $dr' = \frac{1}{a}dr$. Thus:

$\int_{-\infty}^0 Cr(\{a\xi+b \leq r\})dr = a\int_{-\infty}^{-\frac{b}{a}} Cr(\{\xi \leq r'\})dr'$.

- Let us evaluate $E[a\xi+b]$

a) If a and b have the same sign, then $-\frac{b}{a} < 0$ and we have:

$$\begin{aligned}
E[a\xi+b] &= \int_0^{+\infty} Cr(\{a\xi+b \geq r\})dr - \int_{-\infty}^0 Cr(\{a\xi+b \leq r\})dr \\
&= a\int_{-\frac{b}{a}}^{+\infty} Cr(\{\xi \geq r'\})dr' - a\int_{-\infty}^{-\frac{b}{a}} Cr(\{\xi \leq r'\})dr' \\
&= a\int_{-\frac{b}{a}}^{+\infty} Cr(\{\xi \geq r'\})dr' - a\int_{-\infty}^{0} Cr(\{\xi \leq r'\})dr' + a\int_{-\frac{b}{a}}^{0} Cr(\{\xi \leq r'\})dr' \\
&= a(\int_0^{+\infty} Cr(\{\xi \geq r'\})dr' - \int_{-\infty}^{0} Cr(\{\xi \leq r'\})dr') + a\int_{-\frac{b}{a}}^{0} (Cr(\{\xi \geq r'\})dr' + Cr(\{\xi \leq r'\})dr').
\end{aligned}$$

Finally, we obtain: $E[a\xi+b] = aE[\xi]+b$.

- If a and b have opposite signs, then $-\frac{b}{a} > 0$ and by a similar way, we add and remove the term $a\int_0^{-\frac{b}{a}} Cr(\{\xi \geq r'\})dr'$ to obtain the result. □

Remark 2.2.2. *(B. Liu and Y. Liu [21]) When ξ_1 and ξ_2 are independent fuzzy variables with finite expected values, a and b are two reals numbers, then:*

$$E[a\xi_1 + b\xi_2] = aE[\xi_1] + bE[\xi_2].$$

In the following subsection, we recall definitions, examples and properties of the variance and the semi-variance of a fuzzy variable.

2.2.3 Variance and Semi-variance of a fuzzy variable: Definition, Examples and Properties

Variance

Definition 2.2.2. *(B. Liu and Y.Liu [20]) Let ξ be a fuzzy variable with finite expected value e. The variance of ξ is the real number defined by:*

$$V[\xi] = E[(\xi - e)^2]. \tag{2.12}$$

Let us determine variance of an equipossible fuzzy variable.

Example 2.2.3. *Let ξ be an equipossible fuzzy variable (a, b) with $E[\xi] = e = \frac{a+b}{2}$. Then for any positive real number r, we can easily check that:*

$$Cr(\{(\xi - e)^2)\} \geq r) = \begin{cases} \frac{1}{2}, & \text{if } r \leq \frac{(b-a)^2}{4} \\ 0, & \text{if } r > \frac{(b-a)^2}{4} \end{cases}.$$

Thus, according to relations (2.12) and (2.8) the variance of ξ is:

$$V[\xi] = \int_0^{+\infty} Cr(\{(\xi - e)^2)\} \geq r)\, dr = \int_0^{\frac{(b-a)^2}{4}} \frac{1}{2}\, dr = \frac{(b-a)^2}{8}.$$

The following result determines variance of a trapezoidal fuzzy variable.

Proposition 2.2.3. *1) Let ξ be a trapezoidal fuzzy variable (a, b, c, d) with $E[\xi] = e = \frac{a+b+c+d}{4}$. We set: $\alpha_1 = \max(b - a, d - c)$, $\beta_1 = \min(b - a, d - c)$ and $\gamma = c - b$.*

The variance of ξ is defined by:

$$\frac{33\alpha_1^3+21\alpha_1^2\beta_1+11\alpha_1\beta_1^2-\beta_1^3+60\alpha_1\gamma^2+66\alpha_1^2\gamma-12\beta_1\gamma^2-6\beta_1^2\gamma+36\alpha_1\beta_1\gamma-8\gamma^3}{384\alpha_1}.$$

2) Let ξ be a triangular fuzzy variable (a,b,c) with $E[\xi]=e=\frac{a+2b+c}{4}$. We set: $\alpha_1=\max(b-a,c-b)$ and $\beta_1=\min(b-a,c-b)$.

The variance of ξ is defined by:

$$V[\xi]=\frac{33\alpha_1^3+21\alpha_1^2\beta_1+11\alpha_1\beta_1^2-\beta_1^3}{384\alpha_1}.$$

Proof: 1) Let ξ be a trapezoidal fuzzy variable (a,b,c,d) with $E[\xi]=e=\frac{a+b+c+d}{4}$. Let us set: $\alpha=b-a$, $\beta=d-c$, $\gamma=c-b$, $A=\frac{\alpha+2\gamma-\beta}{4}$, $B=\frac{\alpha+2\gamma-\beta}{4}$, $C=\frac{\alpha+2\gamma+2\beta}{4}$, $D=\frac{3\alpha+2\gamma+\beta}{4}$, and $X=\frac{\alpha+3\gamma+\beta}{4}$.

For the calculation of the variance of ξ, we distinguish two cases: $\alpha>\beta$ and $\alpha<\beta$. The case where $\alpha=\beta$ will be study in the case of symmetric fuzzy variables.

For any positive number r, we obtain:

In the case where $\alpha>\beta$,

$$Cr(\{(\xi-e)^2)\}\geq r)=\begin{cases}\frac{2b-a-e-\sqrt{r}}{2\alpha}, \text{ if } 0<r\leq(A)^2\\ \frac{1}{2}, \text{ if } A^2<r\leq B^2\\ \frac{d-e-\sqrt{r}}{2\beta}, \text{ if } B^2<r\leq X^2\\ \frac{e-a-\sqrt{r}}{2\alpha}, \text{ if } X^2<r\leq C^2\\ \frac{e-a-\sqrt{r}}{2\alpha}, \text{ if } C^2<r\leq D^2\\ 0, \text{ if } r>D^2\end{cases} \quad \text{if } e<b$$

and

$$Cr(\{(\xi-e)^2)\}\geq r)=\begin{cases}\frac{1}{2}, \text{ if } 0<r\leq B^2\\ \frac{d-e-\sqrt{r}}{2\beta}, \text{ if } B^2<r\leq X^2\\ \frac{e-\sqrt{r}}{2\alpha}, \text{ if } X^2<r\leq C^2\\ \frac{e-\sqrt{r}}{2\alpha}, \text{ if } C^2<r\leq D^2\\ 0, \text{ if } r>D^2\end{cases} \quad \text{if } e\geq b\ .$$

<u>First case</u>: $\alpha>\beta$ If $e<b$.

According to relations (2.12) and (2.8), the variance of ξ is:

$V[\xi]=\int_0^{+\infty}Cr(\{(\xi-e)^2)\}\geq r)\ dr=\int_0^{A^2}\frac{2b-a-e-\sqrt{r}}{2\alpha}\ dr+\int_{A^2}^{B^2}\frac{1}{2}\ dr+\int_{B^2}^{X^2}\frac{d-e-\sqrt{r}}{2\beta}\ dr+$

$\int_{X^2}^{C^2} \frac{e-a-\sqrt{r}}{2\alpha}\, dr + \int_{C^2}^{D^2} \frac{e-a-\sqrt{r}}{2\alpha}\, dr$

That is, $V[\xi] = \dfrac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3 + 60\alpha\gamma^2 + 66\alpha^2\gamma - 12\beta\gamma^2 - 6\beta^2\gamma + 36\alpha\beta\gamma - 8\gamma^3}{384\alpha}$.

If $e \geq b$.

Then $V[\xi] = \int_0^{+\infty} Cr(\{(\xi - e)^2)\} \geq r)\, dr = \int_0^{B^2} \frac{1}{2}\, dr + \int_{B^2}^{X^2} \frac{d-e-\sqrt{r}}{2\beta}\, dr + \int_{X^2}^{C^2} \frac{e-a-\sqrt{r}}{2\alpha}\, dr +$
$\int_{C^2}^{D^2} \frac{e-a-\sqrt{r}}{2\alpha}\, dr$

that is, $V[\xi] = \dfrac{32\alpha^3 + 24\alpha^2\beta + 8\alpha\beta^2 + 48\alpha\gamma^2 + 72\alpha^2\gamma + 24\alpha\beta\gamma - 8\gamma^3}{384\alpha}$.

Second case: $\alpha < \beta$

By the same way, we obtain:

$V[\xi] = \dfrac{33\beta^3 + 21\beta^2\alpha + 11\beta\alpha^2 - \alpha^3 + 60\beta\gamma^2 + 66\beta^2\gamma - 6\alpha^2\gamma - 12\alpha\gamma^2 + 36\alpha\beta\gamma - 8\gamma^3}{384\beta}$.

The last case where $\alpha = \beta$, is studied in the particular case of symmetric variable.

2) In the particular case of a triangular fuzzy variable $\xi = (a, b, c)$, by setting: $\alpha_1 = \max\{b - a, c - b\}$ and $\beta_1 = \min\{b - a, c - b\}$, we get: $V[\xi]= \dfrac{33\alpha_1^3 + 21\alpha_1^2\beta_1 + 11\alpha_1\beta_1^2 - \beta_1^3}{384\alpha_1}$. $\square$

Let us recall a useful property on the linearity of the variance of a fuzzy variable.

Theorem 2.2.1. *(B. Liu, [18]) Let a and b be reals numbers and ξ a fuzzy variable whose variance exists. Then:*

$$V[a\xi + b] = a^2 V[\xi].$$

Remark 2.2.3. *1) Variance is a parameter which evaluates the spread or the deviation of values taken by a fuzzy variable from its expected value.*

2) Variance can be used to distinguish two fuzzy variables which have the same expected value. For example, let us consider two triangular fuzzy variables $\xi_1 = (1, 3, 5)$ and $\xi_2 = (0, 3, 6)$. We have: $E[\xi_1] = E[\xi_2] = 3$, $V[\xi_1] = \frac{2}{3}$ and $V[\xi_2] = \frac{3}{2}$.

Let us recall definition and properties of the Semi-variance which is the parameter which allows to distinguish the low part deviation from the expected value and the high part deviation. In finance, low part deviation means a possible loss of investment and high part deviation means a potential return of investment. For that, we introduce the fuzzy variable

$$(\xi - e)^- = \begin{cases} \xi - e, \text{ if } \xi \leq e \\ 0, \text{ if } \xi > e \end{cases} \tag{2.13}$$

associated to the fuzzy variable ξ with expected value e. It defines the low part deviation of a fuzzy variable ξ from its expected value e.

Semi-variance

Definition 2.2.3. *(Huang [11], page 3) Let ξ be a fuzzy variable with finite expected value e. The semi-variance of ξ is the real number defined by:*

$$SV[\xi] = E[[(\xi - e)^-]^2]. \tag{2.14}$$

Remark 2.2.4. $SV[\xi] = E[[(\xi - e)^-]^2] = \int_0^{+\infty} Cr(\{[(\xi - e)^-]^2 \geq r\})\, dr = \int_0^{+\infty} Cr(\{(\xi - e)^- \leq -\sqrt{r}\})\, dr = \int_0^{+\infty} Cr(\{\xi \leq e - \sqrt{r}\})\, dr.$

Let us recall semi-variance of some usual fuzzy variables.

Example 2.2.4. *1) For an equipossible variable $\xi = (a, b)$, we have:*

$$SV[\xi] = \int_0^{(e-b)^2} dr + \int_{(e-b)^2}^{(e-a)^2} \frac{1}{2}\, dr = \frac{(b-a)^2}{4}.$$

2) For a trapezoidal fuzzy variable $\xi = (a, b, c, d)$, we obtain:

$$SV[\xi] = \frac{1}{6(b-a)}[(e-a)^3 + \min(0, (b-e)^3)] + \frac{1}{6(d-c)} \max(0, (e-c)^3).$$

3) In the particular case of a triangular fuzzy variable $\xi = (a, b, c)$ with expected value e, we have:

$$SV[\xi] = \frac{1}{6(b-a)}[(e-a)^3 + \frac{4}{(b-c)}(b-e)^3 \min(0, (b-e)^3)].$$

Let us recall the result which establishes that the variance of ξ is greater than the semi-variance and the two parameters are equal if ξ is symmetric.

Theorem 2.2.2. *(Huang [11], page 3)*

Let ξ be a fuzzy variable with finite expected value e, $SV[\xi]$ and $V[\xi]$ the semi-variance and variance of ξ respectively.

1) $0 \leq SV[\xi] \leq V[\xi]$.

2) If ξ has a symmetric membership function then $SV[\xi] = V[\xi]$.

The following Subsection recalls definition, examples and properties of the skewness of a fuzzy variable.

2.2.4 Skewness of a fuzzy variable: Definition, Examples and Properties

Definition 2.2.4. *(Li et al. [16], page 240) Let ξ be a fuzzy variable with finite expected value e. The skewness of ξ is the real number defined by:*

$$S[\xi] = E[(\xi - e)^3]. \tag{2.15}$$

Remark 2.2.5. $S[\xi] = E[(\xi-e)^3] = \int_0^{+\infty} Cr(\{(\xi-e)^3 \geq r\})\, dr - \int_{-\infty}^{0} Cr(\{(\xi-e)^3 \leq r\})\, dr = 3\int_0^{+\infty} r^2 Cr(\{\xi \geq e+r\})\, dr - 3\int_0^{+\infty} r^2 Cr(\{\xi \leq e-r\})\, dr.$

Let us recall skewness of some usual fuzzy variables.

Example 2.2.5. *1) For an equipossible fuzzy variable $\xi = (a, b)$, we obtain:*

$$S[\xi] = 3\int_0^{(\frac{b-a}{2})^3} \frac{r^2}{2}\, dr - 3\int_0^{(\frac{b-a}{2})^3} \frac{r^2}{2}\, dr = 0.$$

2) For a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ with expected value e, we obtain:

$$S[\xi] = \frac{1}{8(b-a)}[(b-e)^4 - (a-e)^4] + \frac{1}{8(c-d)}[(c-e)^4 - (d-e)^4].$$

Remark 2.2.6. *In the particular case of a triangular fuzzy variable $\xi = (a, b, c)$ with expected value e, we have:*

$$S[\xi] = \frac{1}{8(b-a)}[(b-e)^4 - (a-e)^4] + \frac{1}{8(b-c)}[(b-e)^4 - (c-e)^4] = \frac{(c-a)^2}{32}(c+a-2b).$$

We end this chapter with some properties of the skewness.

Theorem 2.2.3. *(Li et al. [16], pages 240 et 241) Let a and b be two reals numbers and ξ a fuzzy variable with finite expected value.*

1) $S[a\xi + b] = a^3 S[\xi]$.

2) If ξ is a symmetric fuzzy variable, then $S[\xi] = 0$.

Remark 2.2.7. *1) Skewness is a parameter which describes the asymmetry of fuzzy variables.*

2) Skewness can be used to distinguish two fuzzy variables which have the same expected value and the same variance (or the same semi-variance).

For example, let us consider two triangular fuzzy variables $\xi_1 = (1, 2, 4)$ and $\xi_2 = (\frac{1}{2}, \frac{5}{2}, \frac{7}{2})$. We have: $E[\xi_1] = E[\xi_2] = \frac{9}{4}$, $V[\xi_1] = V[\xi_2] = \frac{179}{384}$, $S[\xi_1] = \frac{27}{32}$ and $S[\xi_2] = -\frac{27}{32}$.

According to what precedes, parameters like mean, variance, semi-variance, skewness, describe fuzzy variables. But they are first moments and semi-moments of a fuzzy variable. A main question is to study moments and semi-moments of order k ($k \in \mathbb{N}^*$) of fuzzy variables. That is the focus of the next Chapter.

Chapter Three

Moments and Semi-moments of fuzzy variables based on credibility measure

In this Chapter, we generalize the entire family of parameters describing a fuzzy variable by introducing its moments and semi-moments. Some characterizations and useful properties of those parameters are established. Many results of this Chapter are in our first article Sadefo, Tassak and Fono [29].

3.1 Kurtosis and semi-kurtosis of a fuzzy variable

In the next Section, we introduce the kurtosis of a fuzzy variable. We study some of its properties and give some examples.

3.1.1 Kurtosis: Definitions, Examples and Properties

Definition 3.1.1. *(Sadefo et al., [29], Definition 4 P520) Let ξ be a fuzzy variable such that $E[\xi] = e < \infty$.*

1. *The kurtosis of ξ is the real number denoted by $K[\xi]$ and defined by:*

$$K[\xi] = E[(\xi - e)^4].$$

2. The normalized kurtosis of ξ *is the real number denoted by* $K^1[\xi]$ *and defined by:*

$$K_1[\xi] = \frac{E[(\xi - e)^4]}{(\sigma[\xi])^4}.$$

Let us rewrite $K[\xi]$ and $K_1[\xi]$ by means of a credibility measure. Let ξ be a fuzzy variable such that $E[\xi] = e < \infty$.

- The kurtosis $K[\xi]$ is given by:

$$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}\, dr. \tag{3.1}$$

- The normalized kurtosis $K_1[\xi]$ is given by:

$$K_1[\xi] = \frac{\int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\} dr}{[\int_0^{+\infty} Cr\{(\xi - e)^2 \geq r\} dr]^2}. \tag{3.2}$$

Remark 3.1.1. *1) Kurtosis is a parameter used to describe a fuzzy variable's tail, such as fat-tail or thin-tail. In finance, investors prefer portfolio return described by fuzzy variables with smaller kurtosis indicating the fat tail.*

2) Kurtosis allows to distinguish two fuzzy variables with the same mean, the same variance and the same skewness as it is proved in this next example.

Example 3.1.1. *Let* $\xi_1 = (2, \frac{23+\sqrt{73}}{4}, \frac{19+\sqrt{73}}{2})$ *and* $\xi_2 = (4, 5, \frac{13+\sqrt{73}}{2}, \frac{15+\sqrt{73}}{2})$ *be two fuzzy variables.*

We have $E[\xi_1] = E[\xi_2] = \frac{46+\sqrt{73}}{4}, V[\xi_1] = V[\xi_2] = \frac{298+30\sqrt{73}}{96}$ *and* $SK[\xi_1] = SK[\xi_2] = 0$.

But $K[\xi_1] \simeq 120.027$ *and* $K[\xi_2] \simeq 68.6$.

The following result establishes some properties of the kurtosis.

Proposition 3.1.1. *Let* ξ *be a fuzzy variable such that* $E[\xi] = e$.

1. *The kurtosis of ξ is defined by*

$$K[\xi] = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\} dr. \tag{3.3}$$

2. *The normalized kurtosis of ξ is defined by*

$$K^1[\xi] = \frac{\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\} dr}{[\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[2]{r}\} \vee Cr\{\xi - e \leq \sqrt[2]{r}\} dr]^2}. \tag{3.4}$$

3. $\forall a, b \in \mathbb{R}, K[a\xi + b] = a^4 K[\xi]$.

4. $\forall a, b \in \mathbb{R}, K^1[a\xi + b] = K^1[\xi]$.

Proof: 1) It is easy to show that: $Cr\{(\xi - e)^4 \geq r\} = Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\}$. Hence we have the following equality:

$$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\} dr = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\} dr.$$

2) We deduce the second result from the definition of $K^1[\xi]$ and by using the fact that:

$$V[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^2 \geq r\} dr = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[2]{r}\} \vee Cr\{\xi - e \leq \sqrt[2]{r}\} dr.$$

3) i) Let $a, b \in \mathbb{R}$. We have $K[a\xi + b] = E[(a\xi + b - E[a\xi + b])^4]$. Since $E[a\xi + b] = aE[\xi] + b$, we deduce that $K[a\xi + b] = E[(a\xi + b - aE[\xi] - b)^4] = E[(a\xi - aE[\xi])^4] = a^4 E[(\xi - E[\xi])^4] = a^4 K[\xi]$.
ii) Since $V[a\xi + b] = a^2 V[\xi]$, we deduce $K^1[a\xi + b] = K^1[\xi]$. □

The following result rewrites the previous formulae when ξ becomes a symmetric fuzzy variable.

Corollary 3.1.1. *If ξ is a symmetric fuzzy variable, then*

1. *(3.3) becomes*

$$K[\xi] = \int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\}dr. \tag{3.5}$$

2. *(3.4) becomes*

$$K^1[\xi] = \frac{\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[4]{r}\}dr}{[\int_0^{+\infty} Cr\{\xi - e \geq \sqrt[2]{r}\}dr]^2}. \tag{3.6}$$

Proof: When ξ is a symmetric fuzzy variable, we have:

$Cr\{(\xi - e)^4 \geq r\}dr = Cr\{\xi - e \geq \sqrt[4]{r}\}$ and $Cr\{(\xi - e)^2 \geq r\}dr = Cr\{\xi - e \geq \sqrt[2]{r}\}$ and the proof is complete. □

Let us end this Subsection with the following result which determines the kurtosis and normalized kurtosis of trapezoidal and triangular fuzzy variables.

Proposition 3.1.2. *Let $\xi = (a, b, c, d)$ be a fuzzy trapezoidal variable with expected value $E[\xi] = e$. We set: $\alpha = b - a$, $\beta = d - c$, $l_s(\xi)$ and $l_c(\xi)$ are respectively the length of the support and the kernel of ξ.*

1. *The kurtosis $K[\xi]$ of ξ is given by:*

$$K[\xi] = -[\frac{1}{4}(l_s(\xi) + l_c(\xi))]^5(\frac{|\alpha - \beta|}{5\alpha\beta}) + \max(\frac{(\frac{|\alpha-\beta|}{4} - \frac{1}{2}l_c(\xi))^5}{10\alpha \vee \beta}, 0) + \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_s(\xi))^5}{10\alpha \vee \beta}$$

$$\frac{|\alpha - \beta|}{2\alpha\beta}[\frac{1}{2}l_s(\xi) - \frac{(\alpha + \beta)}{4}][\frac{1}{4}(l_s(\xi) + l_c(\xi))]^4 - \frac{(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_c(\xi))^5}{10\alpha \wedge \beta}.$$

2. *If $\xi = (a, b, c, d)$ is symmetric, then*

   - *the previous expression of $K[\xi]$ becomes:*

$$K[\xi] = \frac{5[l_c(\xi) + \beta]^4 + 10\beta^2[l_c(\xi) + \beta]^2 + \beta^4}{160}. \tag{3.7}$$

- *its normalized Kurtosis* $K_1[\xi]$ *is*

$$K_1[\xi] = \frac{5[l_c(\xi)+\beta]^4 + 10\beta^2[l_c(\xi)+\beta]^2 + \beta^4}{160[\frac{3[l_c(\xi)+\beta]^2+\beta^2}{24}]^2}.$$

3. *Let* $\xi = (a,b,c)$ *be a triangular fuzzy variable such that* $E[\xi] = \frac{a+2b+c}{4} = e$. *We set:*

$\alpha_1 = \max(b-a, c-b)$, $\gamma = \min(b-a, c-b)$.

The kurtosis $K[\xi]$ *of* ξ *is given by:*

$$K[\xi] = \frac{253\alpha_1^5 + 395\alpha_1^4\gamma + 17\alpha_1\gamma^4 + 290\alpha_1^3\gamma^2 + 70\alpha_1^2\gamma^3 - \gamma^5}{10.240\alpha_1}.$$

Proof: 1) Let $\xi = (a,b,c,d)$ be a trapezoidal fuzzy variable such that $E[\xi] = e, \alpha = b-a, \beta = d-c$.

By using the fact that $Cr\{(\xi-e)^4 \geq r\} = Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\}$, we can easily obtain the following results:

i)When $\alpha > \beta$, then $e < c$. We can distinguish the two following cases as follows:

1^{st}case: $e < b$

$$Cr\{(\xi-e)^4 \geq r\} = \begin{cases} 1 - \frac{\sqrt[4]{r}+e-a}{2\alpha}, \text{ if } 0 \leq r \leq (b-e)^4 \\ \frac{1}{2}, \text{ if } (b-e)^4 \leq r \leq (c-e)^4 \\ -\frac{\sqrt[4]{r}+e-d}{2\beta}, \text{ if } (c-e)^4 \leq r \leq (e-\frac{a+b}{2})^4 \\ \frac{-\sqrt[4]{r}+e-a}{2\alpha}, \text{ if } (e-\frac{a+b}{2})^4 \leq r \leq (e-a)^4 \\ 0, \text{ if } r \geq (e-a)^4. \end{cases}$$

and finally we get:

$K[\xi] = \int_0^{+\infty} Cr\{(\xi-e)^4 \geq r\}dr = (\frac{(e-a)+(e-b)}{2})^5.(\frac{\beta-\alpha}{5\alpha\beta}) + (\frac{(e-a)+(e-b)}{2})^4.(\frac{\alpha(d-e)+\beta(e-a)}{2\alpha\beta}) + \frac{(e-a)^5}{10\alpha} + \frac{(b-e)^5}{10\alpha} - \frac{(c-e)^5}{10\beta}$.

2^{nd}case: $e > b$

$$Cr\{(\xi-e)^4 \geq r\} = \begin{cases} \frac{1}{2}, \text{ if } 0 \leq r \leq (c-e)^4 \\ -\frac{\sqrt[4]{r}+e-d}{2\beta}, \text{ if } (c-e)^4 \leq r \leq (e-\frac{a+b}{2})^4 \\ \frac{-\sqrt[4]{r}+e-a}{2\alpha}, \text{ if } (e-\frac{a+b}{2})^4 \leq r \leq (e-a)^4 \\ 0, \text{ if } r \geq (e-a)^4. \end{cases}$$

and finally we get:

$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}dr = (\frac{(e-a)+(e-b)}{2})^5.(\frac{\beta-\alpha}{5\alpha\beta}) + (\frac{(e-a)+(e-b)}{2})^4.(\frac{\alpha(d-e)+\beta(e-a)}{2\alpha\beta}) + \frac{(e-a)^5}{10\alpha} - \frac{(c-e)^5}{10\beta}$.

ii) When $\alpha < \beta$, we use a similar way to calculate $K[\xi]$.

iii)When $\alpha = \beta$, we have:

$$Cr\{(\xi - e)^4 \geq r\} = \begin{cases} \frac{1}{2}, \text{ if } 0 \leq r \leq (\frac{c-b}{2})^4 \\ -\frac{\sqrt[4]{r}}{2\beta} + \frac{c-b}{4\beta} + \frac{1}{2}, \text{ if } (\frac{c-b}{2})^4 \leq r \leq (\frac{c-b}{2} + \beta)^4 \\ 0, \text{ if } r \geq (\frac{c-b}{2} + \beta)^4 \end{cases}$$

$\alpha = d - c = b - a$

and this result implies that:

$$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}dr = \frac{5[(c-b)+\beta]^4 + 10\beta^2[(c-b)+\beta]^2 + \beta^4}{160}.$$

2) Let $\xi = (a, b, c)$ be a triangular fuzzy variable such that $E[\xi] = e, \alpha = b - a, \beta = c - b$. By using the fact that $Cr\{(\xi - e)^4 \geq r\} = Cr\{\xi - e \geq \sqrt[4]{r}\} \vee Cr\{\xi - e \leq \sqrt[4]{r}\}$, we can easily obtain the following results:

i)When $\alpha > \beta$, then $e < b$ and

$$Cr\{(\xi - e)^4 \geq r\} = \begin{cases} 1 - \frac{\sqrt[4]{r}+e-a}{2\alpha}, \text{ if } 0 \leq r \leq (b-e)^4 \\ -\frac{\sqrt[4]{r}+e-c}{2\beta}, \text{ if } (b-e)^4 \leq r \leq (\frac{\alpha+\beta}{4})^4 \\ \frac{-\sqrt[4]{r}+e-a}{2\alpha}, \text{ if } (\frac{\alpha+\beta}{4})^4 \leq r \leq (e-a)^4 \\ 0, \text{ if } r \geq (e-a)^4 \end{cases}$$

and finally we get:

$$K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}dr = \frac{253\alpha^5 + 395\alpha^4\beta + 17\alpha\beta^4 + 290\alpha^3\beta^2 + 70\alpha^2\beta^3 - \beta^5}{10.240\alpha}.$$

ii) When $\alpha < \beta$, we use a similar way to calculate $K[\xi]$.

iii)When $\alpha = \beta$, we have:

$$Cr\{(\xi - e)^4 \geq r\} = \begin{cases} \frac{\alpha - \sqrt[k]{4}}{2\alpha}, \text{ if } 0 \leq r \leq \alpha^k \\ 0, \text{ if } r \geq \alpha^4. \end{cases}$$

where $\alpha = c-b = b-a$ and this result implies that: $K[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^4 \geq r\}dr = \frac{\alpha^4}{10}$. □

From the previous formulae, we deduce the normalized kurtosis of some examples of trapezoidal fuzzy variables.

Example 3.1.2. $K^1[(-1,2,3,4)] = \frac{27414}{8405}$, $K^1[(1,2,3,4)] = \frac{2178}{845}$, $K^1[(-2,-1,3,4)] = \frac{3798}{1805}$ *and* $K^1[(1,2,2,4)] = \frac{90928}{25215}$.

Remark 3.1.2. *We notice that: for a triangular fuzzy number $\xi = (a,b,c)$, we have:*

- if b=a, then $K[\xi] = \frac{253}{10.240}\gamma^4$ *with* $E[\xi] = \frac{3b+c}{4}$.

- if b=c, then $K[\xi] = \frac{253}{10.240}\alpha^4$ *with* $E[\xi] = \frac{a+3b}{4}$.

In the following, we introduce semi-kurtosis and establish some of its properties. We display some usual examples.

3.1.2 Semi-kurtosis: Definitions, Examples and Properties

Definition 3.1.2. *Let ξ be a fuzzy variable with finite expected value e. Then the semi-kurtosis of ξ is the real number denoted by K^S and defined by:*

$$K^S[\xi] = E[[(\xi - e)^-]^4] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^4 \geq r\}dr. \tag{3.8}$$

Let us determine the semi-kurtosis of trapezoidal and triangular fuzzy numbers.

Example 3.1.3. *1. The semi-kurtosis of a trapezoidal fuzzy variable $\xi = (a,b,c,d)$ with expected value $e = \frac{a+b+c+d}{4}$ is given by:*

$$K^S[\xi] = \frac{1}{10(b-a)}[(e-a)^5 + \min(0,(b-e)^5)] + \frac{1}{10(d-c)}\max(0,(e-c)^5).$$

2. *The semi-kurtosis of a triangular fuzzy number* $\xi = (a, b, c)$ *with expected value* $e = \frac{a+2b+c}{4}$ *is deduced from the semi-kurtosis of a trapezoidal one by this way:*

$$K^S[\xi] = \frac{1}{10(b-a)}[(e-a)^5 + \frac{4}{(b-c)}(b-e)^5 \min(0,(b-e))].$$

Let us end this Subsection by introducing normalized semi-kurtosis.

Definition 3.1.3. *Let* ξ *a fuzzy variable with expected value* e.

The normalized semi-kurtosis of ξ *is the real number denoted by* $K_1^S[\xi]$ *and defined by:*

$$K_1^S[\xi] = \frac{K^S[\xi]}{(V^S[\xi])^2}.$$

Example 3.1.4. 1. *The normalized semi-kurtosis of a trapezoidal fuzzy variable* $\xi = (a, b, c, d)$ *with expected value* e *is defined as follows:*

$$K_1^S[\xi] = \frac{\frac{1}{10(b-a)}[(e-a)^5 + \min(0,(b-e)^5)] + \frac{1}{10(d-c)}\max(0,(e-c)^5)}{[\frac{1}{6(b-a)}[(e-a)^3 + \min(0,(b-e)^3)] + \frac{1}{6(d-c)}\max(0,(e-c)^3)]]^2}.$$

2. *The normalized semi-kurtosis of a triangular fuzzy variable* $\xi = (a, b, c)$ *with expected value* e *is defined as follows:*

$$K_1^S[\xi] = \frac{\frac{1}{10(b-a)}[(e-a)^5 + \frac{4}{(b-c)}(b-e)^5 \min(0,(b-e))]}{[\frac{1}{6(b-a)}[(e-a)^3 + \frac{4}{(b-c)}(b-e)^3 \min(0,(b-e))]]^2}.$$

In the next Section, we introduce moments and semi-moments of a fuzzy variable and study their properties. Thereby, those notions are generalizations of the new parameters (kurtosis and semi-kurtosis) and the known ones (expected value, variance, semi-variance, skewness).

3.2 Moments and semi-moments of fuzzy variables

In the following subsection, we determine, for an integer $k > 1$, the k-moment of a symmetric trapezoidal fuzzy variable.

3.2.1 Moments of symmetric trapezoidal and triangular fuzzy variables

Proposition 3.2.1. *Let $\xi = (a, b, c, d)$ be a symmetric trapezoidal fuzzy variable with expected value $E[\xi] = e$. For an integer $k > 1$, the k-moment $m_k[\xi] = E[(\xi - e)^k]$ is given by:*

$$m_k[\xi] = \begin{cases} 0, & \textit{if } k \textit{ is odd} \\ \frac{\sum_{i=0}^{\frac{k}{2}} C_{k+1}^{2i+1}[(c-b)+\alpha]^{k-2i}}{2^{k+1}(k+1)}, & \textit{if } k \textit{ is even} \end{cases}$$

Proof: For a symmetric trapezoidal fuzzy variable $\xi = (a, b, c, d)$, we can easily prove the following result:

$Cr\{(\xi - e)^k \geq r\} = Cr\{\xi - e \geq \sqrt[k]{r}\} \vee Cr\{\xi - e \geq \sqrt[k]{r}\}$.

$$Cr\{(\xi - e)^k \geq r\} = \begin{cases} \frac{1}{2}, \text{ if } 0 \leq r \leq (\frac{c-b}{2})^k \\ -\frac{\sqrt[k]{r}}{2\beta} + \frac{c-b}{4\beta} + \frac{1}{2}, \text{ if } (\frac{c-b}{2})^k \leq r \leq (\frac{c-b}{2} + \beta)^k \\ 0, \text{ if } r \geq (\frac{c-b}{2} + \beta)^k \end{cases}$$

where $\alpha = d - c = b - a$.

So, we can conclude that:

$m_k[\xi] = \int_0^{(\frac{c-b}{2}+\beta)^k} Cr\{(\xi - e)^k \geq r\} = \frac{\sum_{i=0}^{k}\sum_{j=0}^{k-i} C_{k-i}^{j}(2\beta)^j(c-b)^{k-j}}{2^{k+1}(k+1)} = \frac{\sum_{j=0}^{k} C_{k+1}^{j+1}(2\beta)^j(c-b)^{k-j}}{2^{k+1}(k+1)} = \frac{\sum_{i=0}^{\frac{k}{2}} C_{k+1}^{2i+1}[(c-b)+\alpha]^{k-2i}}{2^{k+1}(k+1)}$. The proof is complete. □

From the previous result, we deduce moments and semi-moments of a symmetric triangular fuzzy variable.

Corollary 3.2.1. *Let $\xi = (a, b, c)$ be a symmetric triangular fuzzy variable with expected value $E[\xi] = e$. For an integer $k \geq 1$, the k-moment $m_k[\xi] = E[(\xi - e)^k]$ is given by:*

- *If $k = 2p + 1$, then*

$$m_{2p+1}[\xi] = m_k[\xi] = 0 \tag{3.9}$$

- *If $k = 2p$, then*

$$m_{2p}[\xi] = \frac{\alpha^k}{2k+2}. \tag{3.10}$$

Proof: We prove that, for a symmetric fuzzy variable $\xi, m_k[\xi]$ is null when k is an odd number.

By definition, we have:

$m_k[\xi] = E[(\xi - E[\xi])^k] = \int_0^{+\infty} Cr\{(\xi - E[\xi])^k \geq r\}dr - \int_{-\infty}^0 Cr\{(\xi - E[\xi])^k \leq r\}dr, \forall k \in \mathbb{N}^*.$

In ([16]), X. Li has already proved that for a symmetric fuzzy variable ξ, $E[\xi] = e$ and $Cr\{\xi - e \geq r\} = Cr\{\xi - e \leq -r\}$, where e is a real number such that $\mu(e-r) = \mu(e+r), \forall r \in \mathbb{R}$ and μ is the membership function of ξ.

Furthermore, we have:

$m_k[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^k \geq r\}dr - \int_{-\infty}^0 Cr\{(\xi - e)^k \leq r\}dr = \int_0^{+\infty} kr^{k-1}Cr\{\xi - e \geq r\}dr - \int_{-\infty}^0 kr^{k-1}Cr\{\xi - e \leq r\}dr = \int_0^{+\infty} kr^{k-1}Cr\{\xi - e \leq -r\}dr - \int_0^{+\infty} kr^{k-1}Cr\{\xi - e \leq r\}dr = 0.$

Now, we assume that k is an even integer.

For a symmetric triangular fuzzy variable $\xi = (a, b, c)$, we can easily show the following result:

Since $Cr\{(\xi - e)^k \geq r\} = Cr\{\xi - e \geq \sqrt[k]{r}\} \vee Cr\{\xi - e \leq \sqrt[k]{r}\}$, we have:

$$Cr\{(\xi - e)^k \geq r\} = \begin{cases} \frac{\alpha - \sqrt[k]{r}}{2\alpha}, & \text{if } 0 \leq r \leq \alpha^k \\ 0, & \text{if } r \geq \alpha^k \end{cases}$$

where $\alpha = c - b = b - a$. Then, we have $m_k[\xi] = \int_0^{\alpha^k} \frac{\alpha - \sqrt[k]{r}}{2\alpha}dr = \frac{1}{2k+2}\alpha^k$. $\square$

We end this Section by introducing semi-moment and by establishing a link between a moment and a semi-moment of a fuzzy variable.

3.2.2 Semi-moment of fuzzy variables and link between moments and semi-moments

Let ξ be a fuzzy variable with finite expected value e.

Definition 3.2.1. *Let $p \in \mathbb{N}^*$.*

1. *The semi-moment of order $n = 2p$ of ξ is the real number denoted by M_{2p}^S and defined*

by:

$$M_{2p}^S[\xi] = M_n^S[\xi] = E[[(\xi - e)^-]^{2p}] = \int_0^{+\infty} Cr\{[(\xi - e)^-]^{2p} \geq r\}dr. \tag{3.11}$$

2. *The normalized semi-moment of ξ is the real number denoted by $M_{2p}^{S,1}$ and defined by:*

$$M_{2p}^{S,1}[\xi] = \frac{M_{2p}^S[\xi]}{(M_2^S[\xi])^p}.$$

In the case where $p = 1$, we obtain the well-known semi-variance of ξ and for $p = 2$, we obtain the semi-kurtosis of ξ.

In the following, we study the link between moment and semi-moment of a fuzzy variable.

The following result compares semi-moment and moment of a fuzzy variable.

Proposition 3.2.2. *Let ξ be a fuzzy variable with finite expected value e, $p \in \mathbb{N}$ and, $M_{2p}^S[\xi]$ and $M_{2p}[\xi]$ the semi-moment and moment of ξ respectively. Then*

$$0 \leq M_{2p}^S[\xi] \leq M_{2p}[\xi]. \tag{3.12}$$

Proof: Let $\theta \in \Theta$ and $r \in \mathbb{R}$. With (2.13), we have: $[(\xi - e)^-]^{2p} = \left\{ \begin{array}{l} (\xi - e)^{2p} \text{ if } \xi \leq e \\ 0 \text{ if } \xi > e \end{array} \right.$.

Thus we distinguish two cases as follows:

i) If $\xi(\theta) \leq e$, then $[(\xi(\theta) - e)^-]^{2p} = (\xi(\theta) - e)^{2p}$. And $[(\xi(\theta) - e)^-]^{2p} \geq r \Leftrightarrow (\xi(\theta) - e)^{2p} \geq r$.

ii) If $\xi(\theta) > e$, then $[(\xi(\theta) - e)^-]^{2p} = 0$ and $(\xi(\theta) - e)^{2p} \geq [(\xi(\theta) - e)^-]^{2p}$.

For those two cases, we have:

$[(\xi(\theta) - e)^-]^{2p} \geq r$ implies $(\xi(\theta) - e)^{2p} \geq r$. We deduce that $\forall \theta, r$, $\{\theta / [(\xi(\theta) - e)^-]^{2p} \geq r\} \subseteq \{\theta / (\xi(\theta) - e)^{2p} \geq r\}$. Since Cr is monotone, we have: $\forall r, Cr\{[(\xi - e)^-]^{2p} \geq r\} \leq Cr\{(\xi - e)^{2p} \geq r\}$. Hence, $M_{2p}[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^{2p} \geq r\}dr \geq \int_0^{+\infty} Cr\{[(\xi - e)^-]^{2p} \geq r\}dr = M_{2p}^S[\xi]$.

For $p = 2$, we prove (3.14). $\square$

The following result establishes a necessary and sufficient condition under which even moments of a fuzzy variable are null.

Proposition 3.2.3. *Let ξ be a fuzzy variable with finite expected value e. Then*

$$M_{2p}[\xi] = 0 \text{ if and only if } Cr\{\xi = e\} = 1. \tag{3.13}$$

Proof: Let ξ be a fuzzy variable with finite expected value e and $p \in \mathbb{N}^*$.

$(\Leftarrow)$: Assume that $Cr\{\xi = e\} = 1$. Thus we have: $Cr\{\xi - e = 0\} = 1$ if and only if $Cr\{(\xi - e)^{2p} = 0\} = 1$. With the self-duality of Cr, we have $Cr\{(\xi - e)^{2p} \neq 0\} = 0$.

Let $r > 0$. We have: $Cr\{(\xi - e)^{2p} \geq r\} \leq Cr\{(\xi - e)^{2p} > 0\} \leq Cr\{(\xi - e)^{2p} \neq 0\} = 0$. That means $\forall r > 0, Cr\{(\xi - e)^{2p} \geq r\} = 0$. We deduce that: $M_{2p}[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^{2p} \geq r\}dr = 0$.

$(\Rightarrow$:) Assume that $M_{2p}[\xi] = 0$. Since Cr takes values in $[0; 1]$, this equality means $Cr\{(\xi - e)^{2p} \geq r\} = 0, \forall r > 0$. Since Cr is self-dual, we have $Cr\{(\xi - e)^{2p} = 0\} = 1$ and we deduce that $Cr\{\xi - e = 0\} = 1$, that is, $Cr\{\xi = e\} = 1$. $\square$

Furthermore, the following result deduces some interesting links between kurtosis and semi-kurtosis of a fuzzy variable.

Corollary 3.2.2. *Let ξ be a fuzzy variable with finite expected value e, $K^S[\xi]$ and $K[\xi]$ the semi-kurtosis and kurtosis of ξ respectively. Then*

1.
$$0 \leq K^S[\xi] \leq K[\xi]. \tag{3.14}$$

2.
$$K[\xi] = 0 \text{ if and only if } Cr\{\xi = e\} = 1. \tag{3.15}$$

3.

$$K^S[\xi] = 0 \text{ if and only if } Cr\{\xi = e\} = 1, i.e., K[\xi] = 0. \tag{3.16}$$

4.

$$K^S[\xi] = K[\xi] \text{ if } \xi \text{ is symmetric .} \tag{3.17}$$

In the next Section, we characterize moments for a convex linear combination of a finite family of independent triangular fuzzy variables called a portfolio of triangular fuzzy variables.

3.3 Moments of a portfolio of triangular fuzzy variables

Definition 3.3.1. *Let $(\xi_i = (a_i, b_i, c_i))_{i=1,2,\ldots,n}$ be a family of n independent triangular fuzzy variables and $x = (x_1, \ldots, x_n)$ be a family of n positive reals of $[0, 1]$ such that $\sum_{i=1}^n x_i = 1$. The portfolio of the n fuzzy variables is the linear combination of those fuzzy variables defined by $\xi(x) = \sum_{i=1}^n x_i\xi_i = (\sum_{i=1}^n x_i a_i, \sum_{i=1}^n x_i b_i, \sum_{i=1}^n x_i c_i)$.*

Example 3.3.1. *Let $\xi_1 = (2, 4, 5)$, $\xi_1 = (-6, 1, 3)$, $\xi_3 = (7, 11, 16)$ be three independent triangular fuzzy variables and $x, y, z \in [0, 1]$ be three real numbers such that $x + y + z = 1$. Then $\xi = x\xi_1 + y\xi_2 + z\xi_3 = (2x - 6y + 7z, 4x + y + 11z, 5x + 3y + 16z)$ is a portfolio of the three fuzzy variables ξ_1, ξ_2, ξ_3.*

Interpretation 3.3.1. *A portfolio indicates futures returns after investment. The returns of investment of the n assets of the portfolio are described by the fuzzy variables $x_1\xi_1$,...,$x_i\xi_i$,..., $x_n\xi_n$ where the scalars x_1,..., x_i,...,x_n are the proportions of investment on those assets. A portfolio suggests how the investor can share his capital among the different assets of the portfolio.*

Since the portfolio of a finite family of triangular fuzzy variables is a triangular fuzzy variable, we deduce its parameters from previous results as follows.

Corollary 3.3.1. *Let $\xi(x) = \sum_{i=1}^{n} x_i\xi_i$ be a portfolio.*

Then

1. *The mean of $\xi(x)$ is:*

$$E[\xi(x)] = \frac{1}{4}\sum_{i=1}^{n} x_i(a_i + 2b_i + c_i).$$

2. *The variance of $\xi(x)$ is:*

$$V[\xi(x)] = -\frac{1}{192\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \alpha_k \beta_l}[\sum_{k=1}^{n} x_k(l_s(\xi_k))]^3|\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)| +$$
$$(\frac{1}{32\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \alpha_k \beta_l}[\sum_{k=1}^{n} x_k(l_s(\xi_k))]^2|\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)|)\times$$
$$([\frac{1}{4}\sum_{k=1}^{n} x_k(2l_s(\xi_k) - (\alpha_k + \beta_k))]) + \frac{(\frac{|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4} + \frac{1}{2}\sum_{k=1}^{n} x_k l_s(\xi_k))^3}{3\sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)} -$$
$$\frac{(\frac{|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4})^3}{3\sum_{k=1}^{n} x_k(\alpha_k + \beta_k - |\alpha_k - \beta_k|)} + \frac{(\frac{|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4})^3 + |\frac{|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4}|^3}{6\sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)}.$$

3. *The Skewness of $\xi(x)$ is:*

$$SK[\xi(x)] = \frac{1}{32}(\sum_{i=1}^{n} x_i(c_i - a_i))^2 . \sum_{i=1}^{n} x_i(c_i - 2b_i + a_i)$$

4. *The Kurtosis of $\xi(x)$ is:*

$$K[\xi(x)] = -\frac{1}{5120\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \alpha_k \beta_l}[\sum_{k=1}^{n} x_k(l_s(\xi_k))]^5|\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)| +$$
$$(\frac{1}{512\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \alpha_k \beta_l}[\sum_{k=1}^{n} x_k(l_s(\xi_k)]^4|\sum_{k=1}^{n} x_k(\alpha_k - \beta_k)|)\times$$
$$([\frac{1}{4}\sum_{k=1}^{n} x_k(2l_s(\xi_k) - (\alpha_k + \beta_k))]) + \frac{(\frac{|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4} + \frac{1}{2}\sum_{k=1}^{n} x_k l_s(\xi_k))^5}{5\sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)} -$$
$$\frac{(\frac{(|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4})^5}{5\sum_{k=1}^{n} x_k(\alpha_k + \beta_k - |\alpha_k - \beta_k|)} + \frac{(\frac{|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4})^5 + |(\frac{|\sum_{k=1}^{n} x_k(\alpha_k-\beta_k)|}{4})^5|}{10\sum_{k=1}^{n} x_k(\alpha_k + \beta_k + |\alpha_k - \beta_k|)}.$$

Proof: We deduce these results from Proposition 3.1.2 and the expressions of mean, variance and skewness of a fuzzy variable. □

The following result determines the semi-variance and the semi-kurtosis of a portfolio.

Proposition 3.3.1. *Let* $(\xi_k)_{k=1,\ldots,n}$ *be a family of independent trapezoidal fuzzy variables with finite expected values* $(e_k)_{k=1,\ldots,n}$, $(x_k)_{k=1,\ldots,n}$*be a family of* n *positive reals and* $\xi(x) = \sum_{k=1}^{n} x_k\xi_k$. *Then*

1. *The semi-variance of* $\xi(x)$ *is*

$$V^S[\xi(x)] = \frac{1}{6\sum_{k=1}^{n} x_k(b_k - a_k)}[(\sum_{k=1}^{n} x_k(e_k - a_k))^3 + \min(0, (\sum_{k=1}^{n} x_k(b_k - e_k))^3)] +$$

$$\frac{1}{6\sum_{k=1}^{n} x_k(d_k - c_k)} \max(0, (\sum_{k=1}^{n} x_k(e_k - c_k))^3).$$

2. *The semi-kurtosis of* $\xi(x)$ *is*

$$K^S[\xi(x)] = \frac{1}{10\sum_{k=1}^{n} x_k(b_k - a_k)}[(\sum_{k=1}^{n} x_k(e_k - a_k))^5 + \min(0, (\sum_{k=1}^{n} x_k(b_k - e_k))^5)] +$$

$$\frac{1}{10\sum_{k=1}^{n} x_k(d_k - c_k)} \max(0, (\sum_{k=1}^{n} x_k(e_k - c_k))^5).$$

Proof: We deduce these results from Example 3.1.3 and the semi-variance formula of a trapezoidal fuzzy variable. □

Those new concepts about the fuzzy variables, obtained by means of the credibility measure, are part of the quantitative approach for solving the portfolio selection problem. In the next Chapter, we introduce another approach, namely qualitative approach, based upon the pairwise comparison of fuzzy variables.

CHAPTER FOUR

DOMINANCE RELATIONS ON FUZZY VARIABLES BASED ON THE CREDIBILITY MEASURE

In this Chapter, we define and characterize three dominance relations on fuzzy variables. We establish some links between these dominance relations and determine some of their properties. Some results of this Chapter are in our recent article Tassak, Sadefo, Fono and Andjiga [38].

4.1 Mean-risk dominance based on $\text{FLPM}_{\alpha,\tau}$: Definitions, Examples and Characterization

In this Section, we introduce a new dominance relation on fuzzy variables and characterize it in some particular cases. For that, we introduce fuzzy lower partial moment of a fuzzy variable which is studied in details in Appendix. $E(.)$ is the expectation operator based on a credibility measure and ξ is a fuzzy variable.

Definition 4.1.1. *Let $\alpha \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$.*

The fuzzy lower partial moment of ξ with order α and target value τ is the real number denoted by $FLPM_{\alpha,\tau}[\xi]$ and defined by:

$$FLPM_{\alpha,\tau}[\xi] = E[\max(\tau - \xi, 0)^{\alpha}]. \tag{4.1}$$

In the following remark, we express the fuzzy lower partial moment of a fuzzy variable by means of its distribution function or its derivative when it exists and we establish some useful links between fuzzy lower partial moment and semi-moment.

Remark 4.1.1. *Let ξ be a fuzzy variable, $\alpha \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$.*

1. *The FLPM of ξ can be defined by means of its distribution function as follows:*

$$FLPM_{\alpha,\tau}[\xi] \;=\; \int_0^{+\infty} Cr\{\max(\tau-\xi,0)^\alpha \geq r\}\, dr = \alpha \int_{-\infty}^{\tau} (\tau-u)^{\alpha-1}\Phi(u)du \tag{4.2}$$

2. *When Φ has a derivative ϕ and ξ has a lower bounded support, we have:*

$$FLPM_{\alpha,\tau}[\xi] = \int_{-\infty}^{\tau} (\tau-u)^\alpha d\Phi(u) = \int_{-\infty}^{\tau} (\tau-u)^\alpha \phi(u)du. \tag{4.3}$$

3. *If the target value $\tau = E[\xi] = \mu$ and $\alpha \in 2\mathbb{N}\setminus\{0\}$, then $FLPM_{\alpha,\mu}[\xi]$ is the semi-moment of order α of ξ.*

4. *For the particular of $\alpha = 0$, we have the so-called credibility of loss of ξ given by: $FLPM_{0,\tau}[\xi] = Cr\{\xi \leq \tau\}$.*

5. *In the case where $\alpha = 1$, $FLPM_{1,\tau}[\xi] = E[\max(\tau-\xi,0)]$ is called the expected loss of ξ. Here the constant target value τ can be considered as the threshold point separating returns in two parts: downside returns and upside returns relative to the threshold.*

The following result determines necessary and sufficient conditions on the credibility distribution function Φ of ξ under which its fuzzy lower partial moment is null.

Proposition 4.1.1. *Let ξ be a fuzzy variable, Φ its credibility distribution function, $\alpha \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$.*

$$FLPM_{\alpha,\tau}[\xi] = 0 \Leftrightarrow \Phi(\tau^-) = 0. \tag{4.4}$$

Proof: ($\Rightarrow$) Assume that $\mathrm{FLPM}_{\alpha,\tau}[\xi] = 0$, then (4.2) implies $\forall r \in \mathbb{R}, r < \tau \implies \Phi(r) = 0$, that means, $\Phi(\tau^-) = \sup\{\Phi(r), r < \tau\} = 0$.

($\Leftarrow$) If $\Phi(\tau^-) = 0$, then the inequality $\Phi(r) \geq 0$ implies $\forall r \in \mathbb{R}, r < \tau \implies \Phi(r) = 0$. According to the relation (4.2), the previous implication leads to $\mathrm{FLPM}_{\alpha,\tau}[\xi] = 0$. □

In the following, we introduce and study the new dominance relation on fuzzy variables.

Definition 4.1.2. *Let $\alpha \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$.*

The fuzzy mean-risk dominance with order α and target value τ is the binary relation on the set of fuzzy variables denoted by $\succeq_{\alpha,\tau}$ and defined as follows: for two fuzzy variables ξ_1, ξ_2,

$$\xi_1 \succeq_{\alpha,\tau} \xi_2 \text{ if } \begin{cases} E[\xi_1] \geq E[\xi_2] \\ FLPM_{\alpha,\tau}[\xi_1] \leq FLPM_{\alpha,\tau}[\xi_2] \end{cases}.$$

Remark 4.1.2. *1) From the previous definition, we deduce the strict dominance of $\succeq_{\alpha,\tau}$ by:*

$$\xi_1 \succ_{\alpha,\tau} \xi_2 \text{ if } \begin{cases} E[\xi_1] \geq E[\xi_2] \\ FLPM_{\alpha,\tau}[\xi_1] \leq FLPM_{\alpha,\tau}[\xi_2] \end{cases} \text{ with at least one strict inequality }. \tag{4.5}$$

2) In Finance, the choice of parameters α and τ is made by the decision maker (investor) according to the minimum benefit τ he expects to obtain and how he evaluates the risk α to obtain such benefits.

The following result characterizes the new dominance relation $\succeq_{\alpha,\tau}$ in the three following cases: (1) the two fuzzy variables have disjoint supports and τ is less than the minimum of the lower bounds of the two supports, (2) the two fuzzy variables are symmetric and τ is between the lower bounds of the two supports and (3) one of the two fuzzy variables is a crisp number and the other one is a fuzzy variable with τ as its upper bound.

Notice that the three results of this theorem can be interpreted as follows:

1. The first case means that, in absence of risk, the "best" fuzzy variable is the one with greater expected return.

2. According to the second case, when two distributions have equal means, it is more suitable to choose the less risky one.

3. The third case reveals that: if two distributions have the same expected return value which is below to the target, in the most case , the "best" distribution is the one which make "certain" to get this value.

We now state our result.

Theorem 4.1.1. *Assume that (4.5) holds. Then:*

1. *If* $\Phi_1(\tau^-) = \Phi_2(\tau^-) = 0$, *then* $\xi_1 \succ_{\alpha,\tau} \xi_2$ *if and only if* $E[\xi_1] > E[\xi_2]$.

2. *If* $\begin{cases} E[\xi_1] = E[\xi_2] \\ \Phi_1(\tau^-) = 0 \\ \Phi_2(\tau^-) > 0 \end{cases}$, *then* $\xi_1 \succ_{\alpha,\tau} \xi_2$.

3. *If* $E[\xi_1] = E[\xi_2] = \tau - r$ *(with* $r > 0$*),* Φ_1 *is a degenerate distribution that assigns credibility 1 to* $\tau - r$ *with* $r > 0$, *and* Φ_2 *is a non-degenerate distribution with* $\Phi_2(\tau) = 1$, *then:*

$$\xi_1 \succ_{\alpha,\tau} \xi_2 \text{ if and only if } \alpha > 1.$$

To establish this proof, we recall the Jensens' Inequality for fuzzy variable introduced earlier by Liu [19] (Theorem 1.59, page 68):

"Let ξ be a fuzzy variable and $f : \mathbb{R} \to \mathbb{R}$ a strictly convex function. If $E[\xi]$ and $E[f(\xi)]$ are finite, then $f(E[\xi]) < E[f(\xi)]$."

We now establish the proof of the Theorem.

Proof: 1) Let us assume that $\Phi_1(\tau^-) = \Phi_2(\tau^-) = 0$.

By relation (4.4), we have $\mathrm{FLPM}_{\alpha,\tau}[\xi_1] = \mathrm{FLPM}_{\alpha,\tau}[\xi_2] = 0$.

($\Rightarrow$) Assume on the contrary that $\xi_1 \succ_{\alpha,\tau} \xi_2$ and $E[\xi_1] \leq E[\xi_2]$. This inequality and the equality imply that there is not any strict inequality between the means or the fuzzy lower partial moments of the fuzzy variables ξ_1 and ξ_2. This contradicts $\xi_1 \succeq_{\alpha,\tau} \xi_2$. Therefore, we have: $E[\xi_1] > E[\xi_2]$.

($\Leftarrow$) Assume that $E[\xi_1] > E[\xi_2]$. Thus, the equality $\mathrm{FLPM}_{\alpha,\tau}[\xi_1] = \mathrm{FLPM}_{\alpha,\tau}[\xi_2] = 0$ and the definition of $\succ_{\alpha,\tau}$ imply $\xi_1 \succ_{\alpha,\tau} \xi_2$.

2) Assume that $E[\xi_1] = E[\xi_2], \Phi_1(\tau^-) = 0, \Phi_2(\tau^-) > 0$.

That means $\mathrm{FLPM}_{\alpha,\tau}[\xi_1] = 0$ and $\mathrm{FLPM}_{\alpha,\tau}[\xi_1] > 0$, according to relation (4.4).

3) Let us assume that Φ_1 is a degenerate distribution that assigns credibility 1 to $\tau - r$ with $r > 0$, and Φ_2 is a non-degenerate distribution that has $\Phi_2(\tau) = 1$ and $E[\xi_1] = E[\xi_2] = \tau - r$. Let us set $f(y) = (\tau - y)^\alpha$ for $y \leq \tau$, and $r > 0$.

According to the fact that Φ_1 is a degenerate distribution function that assigns credibility 1 to $\tau - r$, we have $\int_{-\infty}^{\tau}(\tau - y)^\alpha d\Phi_1(y) = r^\alpha$ and $f(E[\xi_1]) = r^\alpha$.

f is strictly convex as $\alpha > 1$. By the Inequality of Jensens and the fact that $E[\xi_1] = E[\xi_2]$, we have: $E[f(\xi_2)] = \alpha \int_{-\infty}^{\tau}(\tau - y)^{\alpha-1}\Phi_2(y)\ dy > f(E[\xi_1]) = r^\alpha$. Finally, we have $\alpha \int_{-\infty}^{\tau}(\tau - y)^{\alpha-1}\Phi_2(y)\ dy > \alpha \int_{-\infty}^{\tau}(\tau - y)^{\alpha-1}\Phi_1(y)\ dy$. Thus $\xi_1 \succ_{\alpha,\tau} \xi_2$.

We can prove the converse case in the same way. $\square$

Let us compare two trapezoidal fuzzy variables by means of the mean-risk dominance.

Example 4.1.1. *Let* $\xi_1 = (-1, -\frac{1}{2}, \frac{3}{2}, 2)$ *and* $\xi_2 = (-2, 0, 1, 3)$ *be two trapezoidal fuzzy variables.*

We have: $E[\xi_1] = E[\xi_2] = \frac{1}{2}$.

By taking $\tau = \frac{1}{2}$ *and* $\alpha = 2$. *We have:* $FLPM_{\alpha,\tau}[\xi_1] = \frac{19}{24} \leq FLPM_{\alpha,\tau}[\xi_2] = \frac{31}{24}$. *It follows that* $\xi_1 \succeq_{2,\frac{1}{2}} \xi_2$.

Let us end this subsection by justifying that $\succeq_{\alpha,\tau}$ is not a complete relation on the set of fuzzy variables.

Remark 4.1.3. *Let* $\xi_1 = (1, 4, 5)$ *and* $\xi_2 = (2, 3, 4)$ *be two fuzzy variables,* $\alpha = 2$ *and* $\tau = 4$. *We have:* $E[\xi_1] = \frac{7}{2}$, $E[\xi_2] = 3$, $FLPM_{2,4}[\xi_1] = \frac{3}{2}$ *and* $FLPM_{2,4}[\xi_2] = \frac{4}{3}$. *Thus,* $E[\xi_1] > E[\xi_2]$ *and* $FLPM_{2,4}[\xi_1] > FLPM_{2,4}[\xi_2]$. *Hence* $\xi_1 \not\succeq_{2,4} \xi_2$ *and* $\xi_2 \not\succeq_{2,4} \xi_1$. *Thereby,* $\succeq_{\alpha,\tau}$ *is not a complete relation.*

In the next Section, we recall the first and second orders dominance relations on the set of fuzzy variables introduced by Peng et al. [27]. We characterize each of those dominance relations and determine some of their first properties. For that, Φ_1 and Φ_2 denote the credibility distribution functions of fuzzy variables ξ_1 and ξ_2 respectively.

4.2 First and second orders dominance relations

The next Subsection focus on the first order dominance relation.

4.2.1 The First Order Dominance Relation: Definition, Examples and Characterization

Definition 4.2.1. *(See Peng et al. [27], page 32, Definition 7) The first order dominance is the binary relation on fuzzy variables denoted* $\succeq_1$ *and defined by:* $\forall \xi_1, \xi_2$,

$$\xi_1 \succeq_1 \xi_2 \text{ if } \forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r).$$

From the previous definition, we deduce the strict dominance of $\succeq_1$ by:

$\xi_1 \succ_1 \xi_2$ if $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$ and $\exists r_0 \in \mathbb{R}, \Phi_1(r_0) < \Phi_2(r_0)$. The indifference is given by:

$\xi_1 \sim_1 \xi_2$ if $\forall r \in \mathbb{R}, \Phi_1(r) = \Phi_2(r)$.

The following result characterizes the first order dominance relation for trapezoidal fuzzy variables.

Theorem 4.2.1. *Let $\xi_1 = (a_1, b_1, c_1, d_1)$ and $\xi_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy variables.*

1.

$$\xi_1 \succeq_1 \xi_2 \Leftrightarrow \begin{cases} a_1 \geq a_2 \\ b_1 \geq b_2 \\ c_1 \geq c_2 \\ d_1 \geq d_2 \end{cases} . \tag{4.6}$$

2. *$\xi_1 \sim_1 \xi_2$ if and only if $\xi_1 = \xi_2$.*

In other words, $\xi_1 \not\succeq_1 \xi_2$ if and only if ($a_1 < a_2$ or $b_1 < b_2$ or $c_1 < c_2$ or $d_1 < d_2$).

Figure 4.1 illustrates that the trapezoidal fuzzy variable $\xi_2 = (a_2, b_2, c_2, d_2)$ dominates $\xi_1 = (a_1, b_1, c_1, d_1)$ by means of $\succeq_1$ while Figure 4.2 illustrates that there is no dominance between the two fuzzy variables by means of $\succeq_1$.

We now established our Theorem.

Proof: 1) ($\Rightarrow$) Assume that $a_1 < a_2$ or $b_1 < b_2$ or $c_1 < c_2$ or $d_1 < d_2$ and let us prove that $\xi_1 \not\succeq_1 \xi_2$, that is, there exists some $r_0 \in \mathbb{R}$ such that $\Phi_1(r_0) > \Phi_2(r_0)$. We distinguish four cases:

- Assume that $a_1 < a_2$. Let $r \in]a_1; a_2[$; $r > a_1 \Rightarrow \Phi_1(r) > \Phi_1(a_1) = 0$ and $r < a_2 \Rightarrow \Phi_2(r) = 0$, Thus $\Phi_1(r) > \Phi_2(r)$.

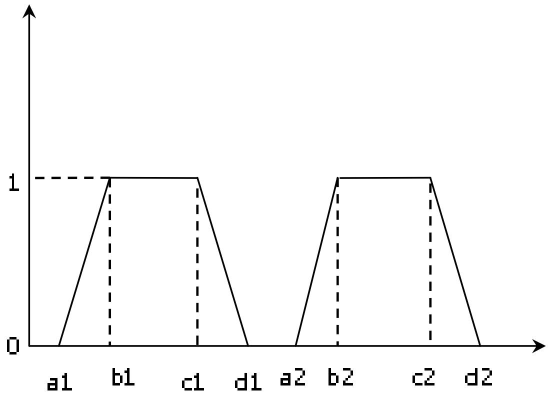

Figure 4.1: Fuzzy variable (a_1, b_1, c_1, d_1) dominated by the other one (a_2, b_2, c_2, d_2).

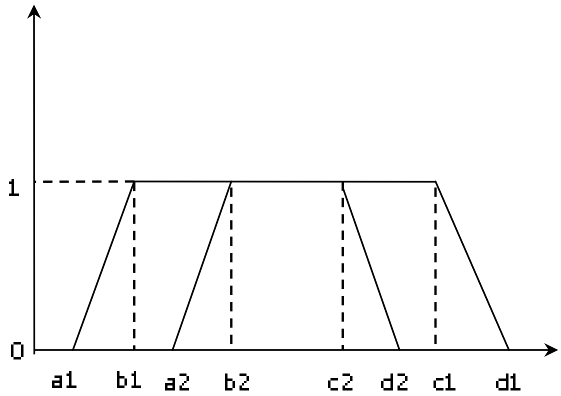

Figure 4.2: Incomparable fuzzy variables.

- Assume that $b_1 < b_2$. By taking $r \in]b_1; b_2[$, we have $\Phi_1(r) > \Phi_2(r)$.

- Assume that $c_1 < c_2$. By taking $r \in]c_1; c_2[$, we have $\Phi_1(r) > \Phi_2(r)$.

- Assume that $d_1 < d_2$. By taking $r \in]d_1; d_2[$, we have $\Phi_1(r) > \Phi_2(r)$.

Finally, we have: $\xi_1 \not\preceq_1 \xi_2$.

($\Leftarrow$) Assume that that $a_1 \geq a_2$,$b_1 \geq b_2$,$c_1 \geq c_2$ and $d_1 \geq d_2$. Let us prove that ξ_2 is dominated by ξ_1, that is, $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$.

We consider the 8 following cases and the results are obtained according to relation (2.5):

i) $\forall r \in]-\infty; a_2]$: $\Phi_2(r) = \Phi_1(r) = 0$ since $r \leq a_2 \leq a_1$. Thus, $\Phi_1(r) \leq \Phi_2(r)$.

ii) $\forall r \in [a_2; a_1]$: $\Phi_2(r) = \frac{r-a_2}{2(b_2-a_2)} \geq 0$ and $\Phi_1(r) = 0$. Thus, $\Phi_1(r) \leq \Phi_2(r)$.

iii) If $a_1 \geq b_2$, then $\forall r \in [b_2; a_1]$,$\Phi_1(r) = 0$ and $\Phi_2(r) = \frac{1}{2}$ that is $\Phi_1(r) \leq \Phi_2(r)$.

Else, that is $a_1 < b_2$: $\forall r \in [a_1; b_2]$, $\Phi_1(r) = \frac{r-a_1}{2(b_1-a_1)}$ and $\Phi_2(r) = \frac{r-a_2}{2(b_2-a_2)}$.

We just have to prove that $\frac{r-a_1}{(b_1-a_1)} \leq \frac{r-a_2}{2(b_2-a_2)}, \forall r \in [a_1; b_2]$.

We set: $f(r) = \frac{r-a_1}{(b_1-a_1)}$ and $g(r) = \frac{r-a_2}{(b_2-a_2)}$.

Let $r_0 \in]a_1; b_2[$. $f(r_0) = g(r_0) \Leftrightarrow r_0 = \frac{a_2(b_1-a_1)-a_1(b_2-a_2)}{(b_1-a_1)-(b_2-a_2)}$. The quantities $r_0 - a_1 = \frac{(b_1-a_1)(a_2-a_1)}{(b_1-a_1)-(b_2-a_2)}$

and $r_0 - b_2 = \frac{(b_2-b_1)(b_2-a_2)}{(b_1-a_1)-(b_2-a_2)}$ have the same sign as $a_1 \geq a_2$ and $b_1 \geq b_2$. This is a contradiction to the fact that $r_0 \in]a_1; b_2[$. By the fact that $f(a_1) \leq g(a_1)$, $f(b_2) \leq g(b_2)$, f and g are strictly non-decreasing on $]a_1; b_2[$ and $f(r) \neq g(r), \forall r \in]a_1; b_2[$, we conclude that $f(r) \leq g(r), \forall r \in [a_1; b_2]$, that is $\frac{r-a_1}{(b_1-a_1)} \leq \frac{r-a_2}{2(b_2-a_2)}$.

iv) $\forall r \in [\max(a_1, b_2); b_1]$: $\Phi_2(r) = \frac{1}{2}$ and $\Phi_1(r) = \frac{r-a_1}{2(b_1-a_1)} \leq \frac{1}{2}$. Thus, $\Phi_1(r) \leq \Phi_2(r)$.

v) $\forall r \in [b_1; c_2]$: $\Phi_2(r) = \Phi_1(r) = \frac{1}{2}$ since $b_2 \leq r \leq b_1$ and $c_2 \leq r \leq c_1$. Thus, $\Phi_1(r) \leq \Phi_2(r)$.

vi) $\forall r \in [c_2; \min(c_1, d_2)]$: $\Phi_2(r) = 1 - \frac{r-d_2}{2(c_2-d_2)} \geq \frac{1}{2}$ and and $\Phi_1(r) = \frac{1}{2}$. Thus, $\Phi_1(r) \leq \Phi_2(r)$.

vii) $\forall r \in [\min(c_1, d_2); d_1]$: By using a similar proof as in iii), we get $\Phi_1(r) \leq \Phi_2(r)$.

viii) $\forall r \in [d_1; +\infty[, \Phi_2(r) = \Phi_1(r) = 1$. So $\Phi_1(r) \leq \Phi_2(r)$.

2) The second result is deduced from the first one. □

Let us state the following example which displays two comparable fuzzy trapezoidal variables by means of $\succeq_1$. It also justifies that $\succeq_1$ is not a complete relation.

Example 4.2.1. *Let us consider the three trapezoidal fuzzy variables:* $\rho_1 = (-2, -1, 4, 9)$, $\rho_2 = (1, 2, 3, 7)$ *and* $\rho_3 = (2, 3, 4, 8)$*. We have the following comparisons:*

$\rho_3 \succeq_1 \rho_2$, $\rho_2 \not\succeq_1 \rho_1$ *since* $3 < 4$ *and* $7 < 9$, $\rho_3 \not\succeq_1 \rho_1$ *since* $8 < 9$.

Let us end this Section by giving two properties of $\succeq_1$ on the particular family of trapezoidal fuzzy variables.

Proposition 4.2.1. *Let* ξ_i *and* ξ_j *be two trapezoidal fuzzy variables.*
If $\xi_i \succeq_1 \xi_j$, *then* $\forall \lambda \in \mathbb{R}^*, \begin{cases} \lambda\xi_i \succeq_1 \lambda\xi_j, & \text{if } \lambda > 0 \\ \lambda\xi_j \succeq_1 \lambda\xi_i, & \text{if } \lambda < 0 \end{cases}$.

Proof: Assume that $\forall t \in \{i, j\}$, $\xi_t = (a_t, b_t, c_t, d_t)$ and $\xi_i \succeq_1 \xi_j$. According to the Extension Principle of Zadeh, if $\lambda > 0$, then $\lambda\xi_t = (\lambda a_t, \lambda b_t, \lambda c_t, \lambda d_t)$ and if $\lambda < 0$, then

$\lambda\xi = (\lambda d_t, \lambda c_t, \lambda b_t, \lambda a_t)$.

By using the characterization of the first order dominance, we obtain the result. □

In the following Subsection, we recall the second order dominance relation on fuzzy variables introduced by Peng et al. [27]. We introduce the notions of crossing points of two fuzzy variables and characterize them. Then, we use this new notion to characterize $\succeq_2$.

4.2.2 The Second Order Dominance Relation: Definitions, Examples and Characterization

Definition and determination of crossing points

Let us recall the definition of the second order dominance relation.

Definition 4.2.2. *(Peng et al. [27], page 33, Definition 8) Let ξ_1 and ξ_2 be two fuzzy variables with Φ_1, Φ_2 their respective cumulative credibility distribution functions, ϕ_1 and ϕ_2 their respective density functions with $\phi_1 \neq \phi_2$.*

$$\xi_1 \succeq_2 \xi_2 \text{ if } \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0.$$

From the previous definition, we deduce the strict dominance of $\succeq_2$ by:

$$\xi_1 \succ_2 \xi_2 \text{ if } \begin{cases} \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0 \\ \exists t_0 \in \mathbb{R}, \int_{-\infty}^{t_0} [\Phi_2(r) - \Phi_1(r)]dr > 0 \end{cases}.$$

We note that $\int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr$ represents a balance of areas between Φ_1 and Φ_2, that means, the difference of areas resulting from integrating each function from $-\infty$ to t, with the following order: the area below Φ_2 minus the area below Φ_1.

In the following, we will characterize the second order dominance relation $\succeq_2$ by writing the fuzzy counterpart of the characterization of the second order dominance's characterization proposed recently by Osuna [25]. Therefore, we introduce, analogously as did Osuna [25] for random variables (see Definition 3.1 P 760), the two notions of interval of coincidence and crossing points for two fuzzy variables.

The intervals of coincidence of two fuzzy variables, denoted by IC, is the half open interval, open at the right, where the two curves of their distributions functions coincide. For example, in Figure 4.3, the two straight lines entitled I.C. are the two intervals of coincidence of Φ_1 and Φ_2. Formally, we have:

Definition 4.2.3. *The half-open interval* $[a, b)$*, with* $a < b$ *is an interval of coincidence (IC) for* Φ_1 *and* Φ_2 *if* $\Phi_1(t) = \Phi_2(t)$ *for all* $t \in [a, b)$*.*

From previous definition, we can deduce that any value t_0 belongs to an interval of coincidence if there exists some $\epsilon > 0$ such that the interval $[t_0, t_0 + \epsilon)$ is IC.

We now introduce two types of crossing points (CP) for fuzzy variables, namely, crossing points of type I and II. Analogously to Definition 3.2 of page 760 in Osuna [25], the crossing point of type II of ξ_1 and ξ_2 is the point where the two curves of their distribution functions intersect and the curve which strictly minimizes before that point strictly maximizes after it. The crossing point of type I of ξ_1 and ξ_2 is the upper bound of a given interval of coincidence (point where the two curves of their distribution functions coincide before it and are distinct after it). Formally, we have the following definition:

Definition 4.2.4. *1. Let* $[a, t_0)$ *be an IC.*

t_0 corresponds to a CP of type I if there exists some $\epsilon > 0$ such that for all $s \in (0, \epsilon)$,

$$\begin{cases} \Phi_1(a-s) \neq \Phi_2(a-s) \\ \Phi_1(t_0+s) \neq \Phi_2(t_0+s) \\ \begin{pmatrix} \Phi_1(a-s) - \Phi_2(a-s) < 0 \ \textit{and} \ \Phi_1(t_0+s) - \Phi_2(t_0+s) > 0 \\ \textit{or} \\ \Phi_1(a-s) - \Phi_2(a-s) > 0 \ \textit{and} \ \Phi_1(t_0+s) - \Phi_2(t_0+s) < 0 \end{pmatrix} \end{cases}.$$

2. *Any other value t_0 corresponds to a CP of type II if there exists some $\epsilon > 0$ such that for all $s \in (0, \epsilon)$, we have*

$$\begin{cases} \Phi_1(t_0-s) \neq \Phi_2(t_0-s) \\ \Phi_1(t_0+s) \neq \Phi_2(t_0+s) \\ \begin{pmatrix} \Phi_1(t_0-s) - \Phi_2(t_0-s) < 0 \ \textit{and} \ \Phi_1(t_0+s) - \Phi_2(t_0+s) > 0 \\ \textit{or} \\ \Phi_1(a-s) - \Phi_2(a-s) > 0 \ \textit{and} \ \Phi_1(t_0+s) - \Phi_2(t_0+s) < 0 \end{pmatrix} \end{cases}.$$

3. *Convention: (a) if t_0 belongs to an IC, it does not correspond to a CP;*

   (b) let $m_1 = \inf\{t/\Phi_1(t) > 0\}$ and $m_2 = \inf\{t/\Phi_2(t) > 0\}$, and let $t_1 = \min(m_1, m_2)$: the interval $(-\infty, t_1)$ is an IC and t_1 does not correspond to a CP.

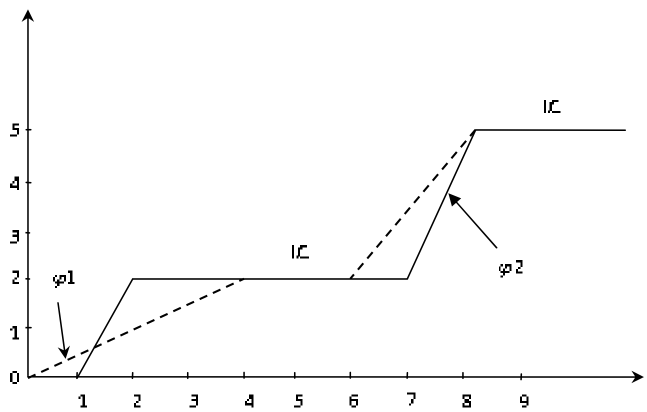

Figure 4.3: Intervals of coincidence (IC) of two curves.

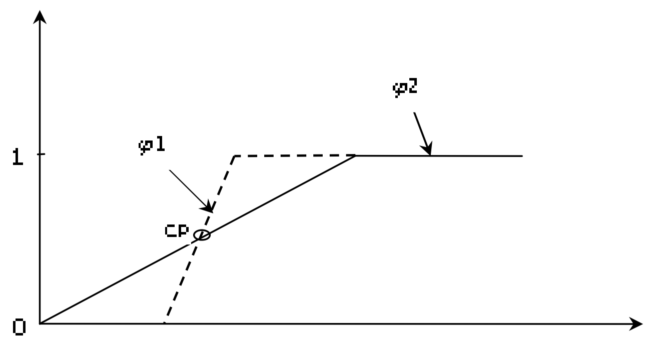

Figure 4.4: Crossing point(CP) of two distributions.

The following result establishes a characterization of the second order dominance relation. The proof is given in appendix.

Theorem 4.2.2. *Let ξ_1 and ξ_2 be two fuzzy variables, Φ_1 and Φ_2 their respective absolutely continuous credibility distributions. Let us suppose that there is a finite number of crossing points $\{t_{01},...,t_{0k}\}$ (ordered so increasing) such that $t_{01} > \min\{\inf\{t : \Phi_1(t) > 0\}, \inf\{t : \Phi_2(t) > 0\}\}$. Then*

$\xi_1 \succ_2 \xi_2$ if and only if

$$\left\{\begin{array}{l} \forall i \in \{1,2,...,k\}, \int_{-\infty}^{t_{0i}}[\Phi_2(r)-\Phi_1(r)]dr \geq 0 \\ \left(\begin{array}{c} \int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr = 0 \quad \textit{and } \exists t_{0h} \in \{t_{01},...,t_{0k}\}, \int_{-\infty}^{t_{0h}}[\Phi_2(r)-\Phi_1(r)]dr > 0 \\ \textit{or} \\ \int_{-\infty}^{\infty}[\Phi_2(r)-\Phi_1(r)]dr > 0 \end{array}\right) \end{array}\right. .$$

Remark 4.2.1. *When there is no crossing point, the distribution's curves do not intersect and we can use the first order dominance relation to compare two fuzzy variables.*

We end this Section by the characterization of crossing points.

Determination of Crossing Points of two fuzzy variables

The following result determines crossing points of two trapezoidal or triangular fuzzy variables in the following six cases: (i) the three first cases are illustrated by Figure 4.5 and (ii) the three last cases allow us to find crossing points when the kernel of one of at least one of the fuzzy variables is a peak.

For that, $\xi_i = (a_i, b_i, c_i, d_i)$ and $\xi_j = (a_j, b_j, c_j, d_j)$ are two fuzzy numbers. μ_i and μ_j are their respective membership functions, Φ_i and Φ_j are their respective credibility distribution functions. The proof of this proposition is given in appendix.

Proposition 4.2.2. *Let r_0 and ϵ be two reals numbers with $\epsilon > 0$. We have:*

1. $\forall s \in (0, \epsilon), \left\{\begin{array}{l} \mu_i(r_0-s) < \mu_j(r_0-s), \mu_i(r_0+s) > \mu_j(r_0+s) \\ r_0-s, r_0+s \in [a_i \vee a_j, b_i \wedge b_j] \end{array}\right. \Rightarrow$ *r_0 is a crossing point of type II.*

2. $\forall s \in (0, \epsilon), \begin{cases} \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s) \\ r_0 - s, r_0 + s \in [c_i \vee c_j, d_i \wedge d_j] \end{cases} \Rightarrow r_0$ *is a crossing point of type II.*

3. $([b_i, c_i] \subseteq [b_j, c_j]$ *and* $[a_i, d_i] \subseteq [a_j, d_j], b_i \neq c_i, b_j \neq c_j) \Rightarrow c_i$ *is a crossing point of type I and* $b_i = \min\{t/[t, c_i)$ *is I.C*$\}$.

4. $([a_i, d_i] \subseteq [a_j, d_j], b_i = c_i, b_j \neq c_j, b_i \in [b_j, c_j]) \Rightarrow c_i$ *is a crossing point of type II .*

5. $([a_i, d_i] \subseteq [a_j, d_j], b_i \neq c_i, b_j = c_j, b_j \in [b_i, c_i]) \Rightarrow c_j$ *is a crossing point of type II.*

6. $([a_i, d_i] \subseteq [a_j, d_j], b_i = c_i = b_j = c_j, a_i \neq a_j, d_i \neq d_j) \Rightarrow c_j$ *is a crossing point of type II.*

Remark 4.2.2. *The previous Proposition allows to obtain crossing points between two fuzzy variables directly by means of their membership functions.*

Remark 4.2.3. *We have an analogous result with* $r_0 \in \mathbb{R}$ *and* $\epsilon \in \mathbb{R}_*^+$ *in the following case:* $\forall s \in (0, \epsilon), \mu_i(r_0 - s) > \mu_j(r_0 - s)$ *and* $\mu_i(r_0 + s) < \mu_j(r_0 + s)$.

Remark 4.2.4.

- *We have an analogous result if* $\exists \epsilon > 0, \exists r_0, \forall s \in (0, \epsilon), \mu_i(r_0 - s) > \mu_j(r_0 - s)$ *and* $\mu_i(r_0 + s) < \mu_j(r_0 + s)$.

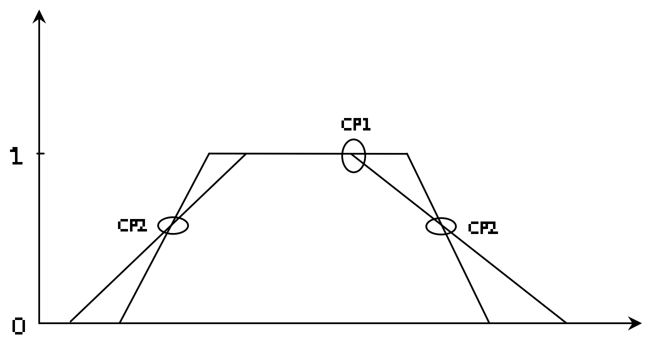

Figure 4.5: Crossings points of type I and type II of two fuzzy variables obtained by membership functions.

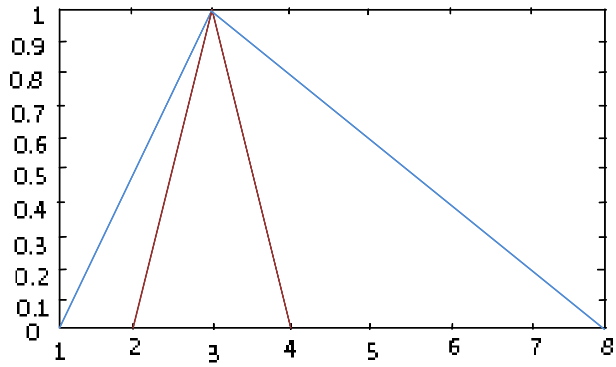

Figure 4.6: Incomparable fuzzy variables by means of $\succeq_2$.

- *The binary relation $\succeq_2$ on the set of fuzzy variables is not complete.*

  Let us prove that by considering the two following triangular fuzzy variables $\xi_1 = (1,3,8)$ and $\xi_2 = (2,3,4)$ drawn in Figure 4.6 and, Φ_1 and Φ_2 are their respective credibility distributions.

  By Proposition 4.2.2, we can prove that the only crossing point is obtained at $r_0 = 3$. Then, we have:

  $\int_{-\infty}^{3}[\Phi_1(r) - \Phi_2(r)]dr = \frac{1}{4} > 0, \int_{-\infty}^{+\infty}[\Phi_1(r) - \Phi_2(r)]dr = \frac{-1}{5} < 0$ *and by Theorem 4.2.2 we conclude that* $\xi_1 \not\succeq_2 \xi_2$ *and* $\xi_2 \not\succeq_2 \xi_1$.

The following example compares two trapezoidal fuzzy variables by means of the second order dominance $\succeq_2$.

Example 4.2.2. *Let $\eta_1 = (1,2,3,4)$ and $\eta_2 = (-1,0,1,2)$ be two trapezoidal fuzzy variables and Φ_1, Φ_2 their respective distribution functions.*

It is easy to check that there is no crossing point between Φ_1 and Φ_2. Therefore, we have: $\int_{-\infty}^{+\infty}[\Phi_2(x) - \Phi_1(x)]\,dx = 2 > 0$, *that is, $\eta_1 \succeq_2 \eta_2$ by Theorem 4.2.2.*

The following Section establishes the relationship between the three dominance relations and some common properties of such relations.

4.3 Other Properties of the three dominance relations

4.3.1 Relations between the three dominance relations

The following result justifies that $\succeq_1$ is stronger than $\succeq_{\alpha,\tau}$ and $\succeq_2$. Furthermore, $\succeq_2$ is stronger than $\succeq_{1,\tau}$.

Proposition 4.3.1. *Let ξ_1 and ξ_2 be two fuzzy variables with finite expected values. Then:*

1.

$$\xi_1 \succeq_1 \xi_2 \Rightarrow \begin{cases} \forall \alpha \in \mathbb{N}^*, \forall \tau \in \mathbb{R}, \xi_1 \succeq_{\alpha,\tau} \xi_2 \\ \xi_1 \succeq_2 \xi_2 \end{cases} .$$

2.

$$\xi_1 \succeq_2 \xi_2 \Rightarrow \forall \tau \in \mathbb{R}, \xi_1 \succeq_{1,\tau} \xi_2.$$

Proof: Let ξ_1 and ξ_2 be two fuzzy variables with uncertainty distributions Φ_1 and Φ_2 respectively, α and τ some given non null integer and real respectively.

1) We prove the first result.

a) Assume that $\xi_1 \succeq_1 \xi_2$ and we prove that $\xi_1 \succeq_{\alpha,\tau} \xi_2$.

$\xi_1 \succeq_1 \xi_2 \Rightarrow \forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$, that is,

$$\forall r \in \mathbb{R}, Cr\{\xi_1 \leq r\} \leq Cr\{\xi_2 \leq r\} \tag{4.7}$$

and

$$\forall r \in \mathbb{R}, Cr\{\xi_1 \geq r\} \geq Cr\{\xi_2 \geq r\} \tag{4.8}$$

According to the definition of $\succeq_1$.

On the other hand, we have: $E[\xi_i] = \int_0^{+\infty} Cr\{\xi_i \geq r\}\, dr \; - \; \int_{-\infty}^0 Cr\{\xi_i \leq r\}\, dr \;\forall i \in \{1;2\}$

According to (4.7) and (4.8), we conclude that $E[\xi_1] \geq E[\xi_2]$.

In the same manner, we have: $\text{FLPM}_{\alpha,\tau}[\xi_i] = \alpha \int_{-\infty}^{\tau} (\tau - x)^{\alpha-1} Cr\{\xi_i \leq x\} dx \ \forall i \in \{1; 2\}$

These last relations lead to $\text{FLPM}_{\alpha,\tau}[\xi_1] < \text{FLPM}_{\alpha,\tau}[\xi_2]$. Finally, we obtain $\xi_1 \succeq_{\alpha,\tau} \xi_2$.

b) Since $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$ then $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)] \ \ dr \geq 0$. We easily obtain the proof.

2) We prove the second result.

Let us assume that $\xi_1 \succeq_2 \xi_2$. The following equality

$E[\xi_i] = \int_0^{+\infty} (1 - \Phi_i(r)) \ \ dr - \int_{-\infty}^{0} \Phi_i(r) \ \ dr$, $\forall i \in \{1, 2\}$, leads to:

$E[\xi_1] - E[\xi_2] = \int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)] \ \ dr$. By using the characterization of $\succeq_2$ and by the fact that $\xi_1 \succeq_2 \xi_2$, we obtain $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)] \ \ dr \geq 0$, that means $E[\xi_1] \geq E[\xi_2]$.

On the other hand, by using relation (4.2), we get:

$\text{FLPM}_{1,\tau}[\xi_1] - \text{FLPM}_{1,\tau}[\xi_2] = \int_{-\infty}^{\tau} (\tau - u)^0 \ \ (\Phi_1 - \Phi_2)(u) \, du$ which implies that $\text{FLPM}_{1,\tau}[\xi_1] - \text{FLPM}_{1,\tau}[\xi_2] \leq 0$ by the fact that $\xi_1 \succeq_2 \xi_2$. Finally, by the fact that $E[\xi_1] \geq E[\xi_2]$ and $\text{FLPM}_{1,\tau}[\xi_1] \leq \text{FLPM}_{1,\tau}[\xi_2]$, we conclude that $\xi_1 \succeq_{1,\tau} \xi_2$. $\square$

The following example justifies that the converse of the two previous implications are false.

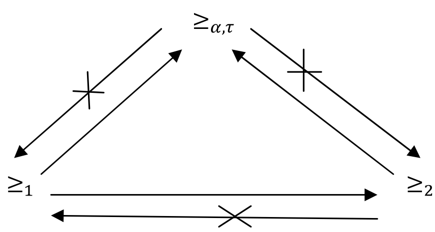

Figure 4.7: Links between the three dominance relations where only the link from $\succeq_2$ to $\succeq_{1,\tau}$ holds.

Example 4.3.1. *Let us consider the triangular fuzzy variables* $\xi_1 = (1, 3, 5)$ *and* $\xi_2 = (2, 3, 4)$.

- *By Proposition 4.2.2, the unique crossing point is* $r_0 = 3$. *Then, we have:*

  $\int_{-\infty}^{3}[\Phi_1(r) - \Phi_2(r)]dr = \frac{1}{4} > 0, \int_{-\infty}^{+\infty}[\Phi_1(r) - \Phi_2(r)]dr = 0$ *and by Theorem 5.2.1 we conclude that* $\xi_2 \succeq_2 \xi_1$. *But by Theorem 4.2.1,* $\xi_2 \not\succeq_1 \xi_1$.

- *By using the same fuzzy variables, we have:*

  $E[\xi_1] = E[\xi_2] = 3$, $FLPM_{2,3}[\xi_1] = \frac{2}{3}$ *and* $FLPM_{2,3}[\xi_2] = \frac{1}{6}$. *So* $FLPM_{2,3}[\xi_2] < FLPM_{2,3}[\xi_1]$. *Hence* $\xi_2 \succeq_{2,3} \xi_1$. *But by Theorem 4.2.1,* $\xi_2 \not\succeq_1 \xi_1$.

The following example specifies that the mean-risk dominance $\succeq_{\alpha,\tau}$ does not imply the second order dominance $\succeq_2$ (See Figure 4.7).

Example 4.3.2. *Let us consider the triangular fuzzy variables* $\xi_1 = (1.5, 4, 5)$ *and* $\xi_2 = (2, 3, 4)$ *with respective distribution functions* Φ_1 *and* Φ_2.

$r_0 = \frac{7}{3}$ *is the only crossing point between* Φ_1 *and* Φ_2. *We have:*

$\int_{-\infty}^{\frac{7}{3}}[\Phi_2(r) - \Phi_1(r)]\ \ dr \simeq -0.042$ *that is,* $\int_{-\infty}^{\frac{7}{3}}[\Phi_2(r) - \Phi_1(r)]\ \ dr < 0$ *and by Theorem 5.2.1, we conclude that* $\xi_1 \not\succeq_2 \xi_2$.

But, $E[\xi_1] = 3.625, E[\xi_2] = 3$, *i.e.,* $E[\xi_1] > E[\xi_2]$, $FLPM_{2,4}[\xi_1] \approx 1.042$ *and* $FLPM_{2,4}[\xi_2] = \frac{4}{3}$, *that means,* $FLPM_{2,4}[\xi_1] < FLPM_{2,4}[\xi_2]$. *Thus,* $\xi_1 \succeq_{2,4} \xi_2$.

In the following Subsection, we examine if each of the three dominance relations satisfies or violates some well-known properties of fuzzy variables.

4.3.2 Some properties of dominance relations

Let us recall six properties of dominance relations introduced by Wang and Kerre [39].

Let S be the set of independent trapezoidal fuzzy variables, $\mathcal{A}$ and $\mathcal{A}'$ two finite subsets of S

and $\succeq_M$ a method of comparison of two elements of S (dominance relation on S). We denote by $\sim_M$ and $\succeq_M$ its indifference and strict components. Let us introduce some well-known properties of $\succeq_M$.

Definition 4.3.1. *(Wang and Kerre [39])*

1. A_1) $\forall A \in \mathcal{A}, A \succeq_M A$.

2. A_2) $\forall (A,B) \in \mathcal{A}^2$, *If* $A \succeq_M B$ *and* $B \succeq_M A$, *then* $A \sim B$.

3. A_3) $\forall (A,B,C) \in \mathcal{A}^3, A \succeq_M B$ *and* $B \succeq_M C \Rightarrow A \succeq_M C$.

4. A_4) $\forall (A,B) \in \mathcal{A}^2, \inf supp(A) > \sup supp(B) \Rightarrow A \succeq_M B$.

   Stronger version: A'_4) $\forall (A,B) \in \mathcal{A}^2, \inf supp(A) > \sup supp(B) \Rightarrow A \succ_M B$.

5. A_5) *Let* $A, B \in \mathcal{A} \cap \mathcal{A}'$. $A \succeq_M B$ *on* $\mathcal{A} \Leftrightarrow A \succeq_M B$ *on* $\mathcal{A}'$.

6. A_6) *Let* $A, B \in \mathcal{A}$ *such that* $A + C, B + C$ *be elements of* $\mathcal{A}$.

   If $A \succeq_M B$, *then* $A + C \succeq_M B + C$.

   A'_6) *Let* $A, B \in \mathcal{A}$ *such that* $A + C, B + C$ *be elements of* $\mathcal{A}$ *with* $C \neq \emptyset$.

   If $A \succ_M B$, *then* $A + C \succ_M B + C$.

The following result consists on checking the properties given above when $\succeq_M \in \{\succeq_{\alpha,\tau}, \succeq_1, \succeq_2\}$.

Proposition 4.3.2. *1)* $\forall \alpha \in \mathbb{N}^*$, $\forall \tau \in \mathbb{R}$, $\succeq_{\alpha,\tau}$ *satisfies* A_1, A_2, A_3, A_4 *and* A_5 *and it violates* A_6 *and* A'_6.

2) $\succeq_1$ *satisfies* $A_1, A_2, A_3, A_4, A_5, A_6$ *and* A'_6.

3) $\succeq_2$ *satisfies* $A_1, A_2, A_3, A_4, A_5, A_6$ *and* A'_6.

Then, we summarize those results in the following table. Notice that given a line, Y (Yes) in a column indicates that the dominance relation in that line satisfies the property in the column and N (No) means that the fuzzy dominance violates it.

Dominances and properties	A_1	A_2	A_3	A_4	A'_4	A_5	A_6	A'_6
$\succeq_1$	Y	Y	Y	Y	Y	Y	Y	Y
$\succeq_2$	Y	Y	Y	Y	Y	Y	Y	Y
$\succeq_{\alpha,\tau}$	Y	Y	Y	Y	Y	Y	N	N

After introducing and analyzing the two approaches for comparing fuzzy variables and thereby portfolios of trapezoidal fuzzy variables, we apply the obtained results by solving the question of selection of the best portfolio of fuzzy variables in Finance.

Chapter Five

Application in Finance

In this Chapter, we apply theoretical results for each of the two approaches developed in the two preceded chapters to solve a portfolio optimization problem in Finance. We display numerical results and make some comparisons.

5.1 Main question

Let us consider an investor who likes to invest his capital in n securities in the proportion x_1, x_2,..., x_n such that $\forall i \in \{1, 2, ..., n\}$, $x_i \in [0, 1]$ and $\sum_{i=1}^{n} x_i = 1$. It is well-known that an investment of a part x_i of the capital in the i^{th} security generates a return denoted by $x_i \xi_i$ which is not currently known. As we raised earlier in the Introduction, we assume that the unknown future returns are fuzzy variables instead of random variables. In other words, making up such investment consists on constituting a portfolio $((x_i, \xi_i))_{1 \leq i \leq n}$ where the n fuzzy variables $x_1\xi_1, ..., x_n\xi_n$ are future returns of the n securities and the fuzzy variable $\xi = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$ is the total future return or the portfolio future return. In fact, ξ_i is given by $\frac{(p'_i + d_i - p_i)}{p_i}$ where p_i is the closing price of the ith security at present, p'_i is the estimated closing price in the next year and d_i is the estimated dividends during the coming year. It is clear that p'_i and d_i are unknown at present. If they are estimated as fuzzy variables, then ξ_i is a fuzzy variable.

The main question becomes the determination of best portfolios in the case where the future returns of securities are fuzzy variables in a credibility space. To study this question, first scholars (Huang [11], Li et al.[16]) proposed models based on parameters (three first moments and the first semi-moment) of fuzzy variables such as expected value (mean), variance, semi-variance and skewness.

In the following, we complement those models by proposing new ones that take into account the fourth moment or the second semi-moment. In addition, we propose a new approach based on first dominance relation on fuzzy variables inducing the core of the set of portfolios made up of non dominated portfolios.

5.2 Portfolio selection with fuzzy return: optimization models based on parameters of future return

5.2.1 New models and relationships with previous ones

Our models (the main one and its four variants) are based on expected return, variance, semi-variance, skewness, kurtosis and semi-kurtosis of a portfolio. Our main model has semi-kurtosis as objective function and expected return, variance and skewness as constraints. To define such constraints, we set the minimal expected return, the minimal skewness and the maximal risk (variance) denoted by α, γ and β respectively. We assune that investor has to select portfolio that maximizes its odd moments and minimizes its even moments or semi-moments. we deduce the following mean-variance-skewness-semi-kurtosis model.

$$\begin{cases} \text{minimize } K^S[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \geq \alpha \\ V[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \leq \beta \\ S[x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n] \geq \gamma \\ x_1 + x_2 + ... + x_n = 1 \\ x_i \geq 0, i = 1, 2, ..., n \end{cases} \tag{5.1}$$

where K^S, E, V and S designed the semi-kurtosis, the mean, the variance and the skewness operators respectively.

The first constraint of this model ensures that the expected return is no less than the given target value α, the second one assures that risk does not exceed the given level β the investor can bear, the third one assures that the skewness is no less than the given target value γ. The last two constraints stipulate that all the capital will be invested in n securities and short-selling is not allowed.

From model (5.1), Corollary 3.3.1 and Proposition 3.3.1, we obtain the following deterministic programm.

Theorem 5.2.1. *Let $(\xi_i = (a_i, b_i, c_i))_{i=1,2,...,n}$ be a family of n independent triangular fuzzy variables and f a function such that $f(x_1, x_2, ..., x_n) = \frac{1}{10\sum_{i=1}^{n} x_i(b_i - a_i)}[(\sum_{i=1}^{n} x_i(e_i - a_i))^5 + \frac{4}{\sum_{i=1}^{n} x_i(b_i - d_i)}(\sum_{i=1}^{n} x_i(b_i - e_i))^5 \min(0, \sum_{i=1}^{n} x_i(b_i - e_i))]$.*

Then model (5.1) becomes the following deterministic program:

$$\begin{cases} \min f(x_1, x_2, ..., x_n) \\ \text{subject to} \\ \sum_{i=1}^{n} x_i(a_i + 2b_i + c_i) \geq 4\alpha \\ 11(\sum_{i=1}^{i=n} x_i(c_i - a_i))^2 |\sum_{i=1}^{i=n} x_i(2b_i - a_i - c_i)| + \\ 2(8\sum_{i=1}^{i=n} x_i(c_i - a_i) + 3|\sum_{i=1}^{i=n} x_i(2b_i - a_i - c_i)|)((\sum_{i=1}^{i=n} x_i(c_i - b_i))^2 + \\ (\sum_{i=1}^{i=n} x_i(b_i - a_i))^2) \leq 192\beta(\sum_{i=1}^{i=n} x_i(c_i - a_i) + |\sum_{i=1}^{i=n} x_i(2b_i - a_i - c_i)|) \\ (\sum_{i=1}^{n} x_i(c_i - a_i))^2 \sum_{i=1}^{n} x_i(c_i - 2b_i + a_i) \geq 32\gamma \\ x_1 + x_2 + ... + x_n = 1 \\ x_i \geq 0, i = 1, 2, ..., n \end{cases} .$$

The other variants of model (5.1) can be deduced from the previous model by changing the objective function either by expected value, semi-variance, skewness or kurtosis. Therefore, we have the following four variants of the main model and deterministic program.

1. The first variant of model (5.1) minimizes risk (variance) when the expected return and the skewness are both no less than the given target values α and γ respectively and the semi-kurtosis is no more than the given target value θ. If one cancels the constraints on skewness and semi-kurtosis in that variant, then it degenerates to the mean-variance model proposed earlier by Huang ([11]).

2. The second variant of model (5.1) maximizes the expected return when the skewness is no less than the given target value γ and, the variance and the semi-kurtosis are no more than β and θ respectively.

3. The third variant of model (5.1) maximizes the skewness when the expected return is not less than α and, the variance and the semi-kurtosis are no more than the given target values β and θ respectively. If we cancel the second constraint on the semi-kurtosis in that variant, then it degenerates to the mean-variance-skewness model proposed by Li ([16]).

4. The fourth variant of model (5.1), introduced by Sadefo et al. ([29]), is the multi-objective nonlinear programming which minimizes the variance and the semi-kurtosis and maximizes the expected value and the skewness when the different target values are unknown.

In the following Subsection, we display numerical examples on the two new models, namely the mean-variance-skewness-kurtosis model and the mean-variance-skewness-semi-kurtosis model, and we compare obtained portfolios with those obtained by Huang ([11]) and Li et al. ([16]).

5.2.2 Numerical implementation of two new models and comparison of results

The data, we consider in this Section, are introduced and used by Huang ([11]) for the mean-semi-variance model and, used by Li et al. ([16]) for the mean-variance-skewness model. Those data are seven triangular security returns as presented in Table 1 below.

Security i	Fuzzy return	Security i	Fuzzy return
1	$\xi_1 = (-0.3, 1.8, 2.3)$	5	$\xi_5 = (-0.7, 2.4, 2.7)$
2	$\xi_2 = (-0.4, 2.0, 2.2)$	6	$\xi_6 = (-0.8, 2.5, 3.0)$
3	$\xi_3 = (-0.5, 1.9, 2.7)$	7	$\xi_7 = (-0.6, 1.8, 3.0)$
4	$\xi_4 = (-0.6, 2.2, 2.8)$		

Table 1: Fuzzy returns of 7 securities (units per stock).

For instance, the return of the first security is described by the fuzzy variable $\xi_1 = (-0.3, 1.8, 2.3)$ which represents about 1.8 units per stock.

To apply our the two new models, we use the following threshold values proposed by Li et al. ([16]): $\alpha = 1.6, \beta = 0.8$ and $\gamma = -0.6823$. In general, it is important to notice that γ must be at the most equal to -0.6823. Since the returns are asymmetric, the investor may employ either semi-variance or variance, either kurtosis or semi-kurtosis to determine an optimal portfolio. Thus, we consider the following four models:

1. the first one is the mean-semi-variance model from Huang ([11]):

$$\begin{cases} \quad \text{minimize } V^S[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \geq \alpha \\ x_1 + x_2 + \cdots + x_7 = 1 \\ x_i \geq 0, i = 1, 2, \ldots, 7 \end{cases} \tag{5.2}$$

2. the second one is the mean-variance-skewness model from Li ([16]):

$$\begin{cases} \text{maximize } S[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \geq \alpha \\ V[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \leq \beta \\ x_1 + x_2 + ... + x_7 = 1 \\ x_i \geq 0, i = 1, 2, ..., 7 \end{cases} \tag{5.3}$$

3. the two following models of Sadefo et al. ([29]): the mean-variance-skewness-kurtosis model and the mean-variance-skewness-semi-kurtosis model

$$\begin{cases} \text{minimize } K[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \geq \alpha \\ V[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \leq \beta \\ S[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \geq \gamma \\ x_1 + x_2 + ... + x_7 = 1 \\ x_i \geq 0, i = 1, 2, ..., 7 \end{cases} \tag{5.4}$$

and

$$\begin{cases} \text{minimize } K^S[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \geq \alpha \\ V[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \leq \beta \\ S[x_1\xi_1 + x_2\xi_2 + ... + x_7\xi_7] \geq \gamma \\ x_1 + x_2 + ... + x_7 = 1 \\ x_i \geq 0, i = 1, 2, ..., 7 \end{cases} \tag{5.5}$$

where K is the kurtosis operator.

We use Matlab to solve those four models and we obtain portfolios presented in Table 2.

Security i	1	2	3	4	5	6	7
Huang's model (5.2)	00.00%	47.06%	00.00%	35.28%	17.66%	00.00%	00.00%
Li et al.'s model (5.3)	20.00%	00.00%	00.00%	80.00%	00.00%	00.00%	00.00%
Sadefo et al.'s model (5.4)	20.04%	00.00%	00.00%	79.89%	00.00%	00.07%	00.00%
Sadefo et al.'s model (5.5)	20.00%	00.00%	00.00%	80.00%	00.00%	00.00%	00.00%

Table 2: Optimal selection from each model.

Let us explain the obtained investment's proportions by illustrating with line 4 of Table 2 which stipulates that: if one intends to invest 10000 units, he will invest 2004 units of the security 1, 7989 units of the security 4, 7 units of the security 6 and nothing elsewhere.

The computation of parameters of portfolios of the previous table are summarized in the following table.

	Mean	Variance	Semi-variance	Skewness	Kurtosis	Semi-kurtosis
Huang's model (5.2)	1.60	0.7235	0.6124	-0.7543	1.7972	1.7415
Li et al.'s model (5.3)	1.60	0.7019	0.6141	-0.6823	1.7291	1.6872
Sadefo et al.'s model (5.4)	1.60	0.7018	0.6140	-0.6823	1.7290	1.6873
Sadefo et al.'s model (5.5)	1.60	0.7019	0.6141	-0.6823	1.7291	1.6872

Table 3 : Comparison of the four first moments of different optimal portfolios.

We can make the following observations:

- When we consider semi-kurtosis (K^S) as an objective function, Li et al.'s model (5.3) and Sadefo et al's model (5.5) give the same optimal portfolio (according to lines 3 and 5 of Table 2).

  Therefore, the latter confirms and enhances results obtained by the first one. Those models allow to obtain a portfolio with the highest skewness (-0.6823) and the lowest semi-kurtosis (1.6872) (see lines 3 and 5 of Table 3) which are the optimal values of the objective functions of the two models respectively.

- When we consider kurtosis (K) as an objective function, Sadefo et al's model (5.4) provides the lowest variance (0.7018), the highest skewness (-0.623) and the lowest kurtosis (1.729) (see line 4 of Table 3).

  In that case, model (5.4) proposes an optimal portfolio different from the three other models (see Table 2).

- The histogram of Figure 5.1 illustrates parameters of the four total returns (combinations of the seven returns) obtained by different authors as described in Table 3.

Let us explain why Sadefo et al's model (5.5) with the semi-kurtosis and Li et al.'s model (5.3) coincide (as stipulated in the previous first observation).

Remark 5.2.1. *The main reason why the two models coincide (generate the same optimal portfolio) in our numerical examples with the seven fuzzy variables is: each of the seven variables* $\xi = (a_i, b_i, c_i)$ *have a large spread on their left, that is,* $\forall i \in \{1, 2, ..., 7\}, c_i - b_i < b_i - a_i$*, and thereby a small "good" part (right of the* b_i*). On one hand, the skewness assures the spread of the distribution on the left side (so that one can be able to say at what degree the distribution is concentrated on the left) and on the other hand, the semi-kurtosis allows to avoid penalizing the "good part" ("positive part") when applying the model. Therefore by adding the semi-kurtosis to Li et al.'s model (5.3) we obtain the same optimal portfolio from our seven variables.*

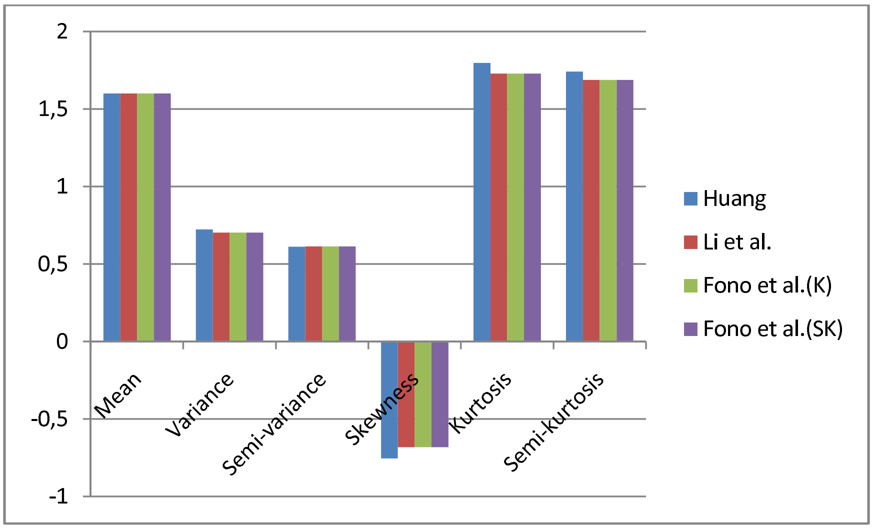

Figure 5.1: Comparison of different models.

1. *Now, if we replace the first fuzzy variable* $\xi_1 = (-0.3, 1.8, 2.3)$ *by the new fuzzy variable* $\xi_8 = (-0.1, 0.0, 2.0)$ *(its "positive part" is greater than the "negative part"), then with the seven fuzzy variables from* ξ_2 *to* ξ_8*, lines 3 and 5 of the two previous tables become respectively:*

Security i	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Li et al.'s model (5.3)	*00.00%*	*00.00%*	*33.00%*	*67.00%*	*00.00%*	*00.00%*	*00.00%*
Sadefo et al.'s model (5.5)	*00.00%*	*00.00%*	*36.00%*	*64.00%*	*00.00%*	*00.00%*	*00.00%*

Table 4: New optimal portfolios.

	Mean	*Variance*	*Semi-variance*	*Skewness*	*Semi-kurtosis*
Li et al.'s model (5.3)	*1.60*	*0.7213*	*0.6361*	*-0.6954*	*1.7931*
Sadefo et al.'s model (5.5)	*1.60*	*0.7164*	*0.6323*	*-0.6860*	*1.7702*

Table 5: Parameters of new optimal portfolios.

By comparing these new tables and the previous one, semi-kurtosis used in Sadefo et al.'s model (5.5) displays an optimal portfolio better than the one given by Li et al.'s model (5.3). In other words, by adding semi-kurtosis on Li et al.'s model, we improve the optimal portfolio with the same mean, the less variance, the less semi-variance, the greater skewness and the less semi-kurtosis.

In the following Section, we introduce the notion of core of the set of portfolios of a finite number of assets with respect to the first order dominance. We implement a part of the core on the example of seven assets of Table 1 to determine optimal portfolios as non dominated portfolios with respect to that dominance.

5.3 Core of portfolios: Definitions, First Properties and implementation

5.3.1 Core of a finite family of assets: Definition and non-emptiness

Let us consider the family $A = (\xi)_{1\leq i\leq n}$ of n assets where returns are described by trapezoidal fuzzy numbers. For example, for $n = 7$, we have the seven assets where returns are described by triangular fuzzy numbers of Table 1.

A portfolio return ξ associated with A is a convex linear combination of the n assets returns defined by $\xi = \sum_{i=1}^{n} x_i\xi_i$ where x_i represents the proportion of capital invested in asset i. The set of portfolios associated with A is $P = \{\xi = \sum_{i=1}^{n} x_i\xi_i, x_i \in [0, 1], \sum_{i=1}^{n} x_i = 1 \text{ and } \xi_i \in A\}$.

A main question is to determine non dominated portfolios by means of a dominance relation. Based on Game Theory terminology, we will determine the core $\mathcal{C}_R(P)$ of (P, R) where R is a dominance on P.

First, let us observe that according to Proposition 4.3.1 of Chapter 4, all portfolios which are not dominated through $\succeq_2$ or $\succeq_{\alpha,\tau}$ are not dominated through $\succeq_1$ too, that means, the set of non dominated portfolios through $\succeq_1$ contains the two other sets. More formally, we have:

$$\begin{cases} \mathcal{C}_{\succeq_2}(P) \subseteq \mathcal{C}_{\succeq_1}(P) \\ \mathcal{C}_{\succeq_{\alpha,\tau}}(P) \subseteq \mathcal{C}_{\succeq_1}(P) \end{cases}.$$

In the following, we will determine the core $\mathcal{C}_{\succeq_1}(P)$ defined by:

$$\mathcal{C}_{\succeq_1}(P) = \{\xi \in P, \forall\eta \in P \setminus \{\xi\}, \eta \not\succeq_1 \xi\}. \tag{5.6}$$

By using the fact that $\succeq_1$ is not a complete relation on P, we can express the core as a union of the following two sets:

$$\mathcal{C}_{\succeq_1}(P) = \{\xi \in P, \forall\eta \in P, \xi \neq \eta, \eta \not\succeq_1 \xi \text{ and } \xi \not\succeq_1 \eta\} \cup \{\xi \in P, \forall\eta \in P, \xi \succeq_1 \eta\} = \Lambda \cup \Gamma \tag{5.7}$$

where

$$\Gamma(P) = \{\xi \in P, \forall \eta \in P, \xi \succeq_1 \eta\} \tag{5.8}$$

is the set of best portfolios of P and $\Lambda(P) = \{\xi \in P, \forall \eta \in P, \xi \neq \eta, \eta \not\succeq_1 \xi$ and $\xi \not\succeq_1 \eta\}$ is the set of incomparable portfolios of P.

The following result establishes that the core of portfolios is non-empty.

Proposition 5.3.1. *Let us consider the family $A = (\xi)_{1 \leq i \leq n}$ of n trapezoidal fuzzy variables and $P = \{\xi = \sum_{i=1}^{n} x_i \xi_i, x_i \in [0,1], \sum_{i=1}^{n} x_i = 1$ and $\xi_i \in A\}$.*

$$\mathcal{C}_{\succeq_1}(P) \neq \emptyset. \tag{5.9}$$

Proof: Let us set: $\forall i \in \{1, ..., n\}, \xi_i = (a_i, b_i, c_i, d_i)$ and $a = \max_{1 \leq i \leq n} a_i$, $b = \max_{1 \leq i \leq n} b_i$, $c = \max_{1 \leq i \leq n} c_i$, $d = \max_{1 \leq i \leq n} d_i$.

There exists $j \in \{1, ..., n\}$ such that ξ_j has at least one value among a, b, c, d (without loss of generality, we assume that the only maximum value is a). It is obvious that ξ_j is generated by the vector $y = (0, 0, ..., 1, 0, ..., 0)$ where 1 is the j^{th} component and in that case we have: $\xi_j = \sum_{i=1}^{n} y_i \xi_i = (a, b_j, c_j, d_j)$. Therefore, for $x = (x_1, x_2, ..., x_j, ..., x_n)$ with $x_j \neq 1$, $\sum_{i=1}^{n} x_i a_i < a$ and $(\sum_{i=1}^{n} x_i a_i, \sum_{i=1}^{n} x_i b_i, \sum_{i=1}^{n} x_i c_i, \sum_{i=1}^{n} x_i d_i) \not\succeq_1 \xi_j$ and $\xi_j \in \mathcal{C}_{\succeq_1}(P)$.

The proof is obtained. $\square$.

Let us display necessary and sufficient conditions for the belonging of a portfolio to the core.

Remark 5.3.1. *Let us consider the family $x = (x_i)_{1 \leq i \leq n}$ such that $\sum_{i=1}^{n} x_i = 1$ and $\forall i \in \{1, ..., n\}, x_i \geq 0$, P is the set of portfolios of trapezoidal fuzzy assets $\xi_i = (a_i, b_i, c_i, d_i), \forall i \in$*

$\{1,...,n\}$. *Let us set:*

$$T_x = \{(y_i)_{1\leq i\leq n} / \sum_{i=1}^{n} y_i = 1, \forall i \in \{1,...,n\}, y_i \geq 0 \ \textit{and} \ \left\{ \begin{array}{l} \sum_{i=1}^{n} y_i a_i > \sum_{i=1}^{n} x_i a_i \\ \sum_{i=1}^{n} y_i b_i > \sum_{i=1}^{n} x_i b_i \\ \sum_{i=1}^{n} y_i c_i > \sum_{i=1}^{n} x_i c_i \\ \sum_{i=1}^{n} y_i d_i > \sum_{i=1}^{n} x_i d_i \end{array} \right. \}. \quad (5.10)$$

We have:

$$\sum_{i=1}^{n} x_i \xi_i \in \mathcal{C}_{\succeq 1}(P) \ \textit{if and only if} \ T_x = \emptyset. \quad (5.11)$$

The following result establishes that two comparable portfolios of two incomparable assets are equivalent.

Proposition 5.3.2. *Let* $\xi_1 = (a_1, b_1, c_1, d_1)$ *and* $\xi_2 = (a_2, b_2, c_2, d_2)$ *be two incomparable trapezoidal fuzzy variables by means of* $\succeq_1$, *and* G *the set of convex linear combinations of* ξ_1 *and* ξ_2.

If two portfolios of G *are comparable by means of* $\succeq_1$, *then they are equivalent.*

Proof: Let $x_1\xi_1 + x_2\xi_2$ and $y_1\xi_1 + y_2\xi_2$ be two comparable portfolios of G with positive reals numbers x_1, x_2, y_1, y_2 such that $x_1 + x_2 = 1$ and $y_1 + y_2 = 1$.

Without loss of generality, we assume that $a_1 \geq a_2$ and $b_1 \leq b_2$ (ξ_1 and ξ_2 are incomparable).

$$x_1\xi_1 + x_2\xi_2 \succeq_1 y_1\xi_1 + y_2\xi_2 \Leftrightarrow \left\{ \begin{array}{l} x_1a_1 + x_2a_2 \geq y_1a_1 + y_2a_2 \\ x_1b_1 + x_2b_2 \geq y_1b_1 + y_2b_2 \\ x_1c_1 + x_2c_2 \geq y_1c_1 + y_2c_2 \\ x_1d_1 + x_2d_2 \geq y_1d_1 + y_2d_2 \end{array} \right. .$$

We have:

$x_1a_1+x_2a_2 \geq y_1a_1+y_2a_2 \Leftrightarrow x_1a_1+(1-x_1)a_2 \geq y_1a_1+(1-y_1)a_2 \Leftrightarrow x_1(a_1-a_2) \geq y_1(a_1-a_2)$.

In the same way, we obtain: $x_1b_1+x_2b_2 \geq y_1b_1+y_2b_2 \Leftrightarrow x_1(b_1-b_2) \geq y_1(b_1-b_2)$. We consider two cases:

- If $a_1 \neq a_2$ and $b_1 \neq b_2$ (without loss of generality, we assume that $a_1 > a_2$ and $b_1 < b_2$), then $x_1(a_1 - a_2) \geq y_1(a_1 - a_2) \Rightarrow x_1 \geq y_1$ and $x_1(b_1 - b_2) \geq y_1(b_1 - b_2) \Rightarrow x_1 \leq y_1$. Thus, we obtain $x_1 = y_1$ which implies that $x_2 = y_2$. So, $x_1\xi_1 + x_2\xi_2 \sim_{\succeq_1} y_1\xi_1 + y_2\xi_2$.

- If $a_1 = a_2$ or $b_1 = b_2$, we use a similar way and the inequalities $x_1c_1 + x_2c_2 \geq y_1c_1 + y_2c_2$, $x_1d_1 + x_2d_2 \geq y_1d_1 + y_2d_2$ to get the result. □

In the following, we implement the set $\Gamma(P)$ of best portfolios contained in the core.

5.3.2 Numerical implementation of the set of best portfolios of finite family of assets

For the determination of $\Gamma(P)$, we introduce the following notations. For $(x_i)_{1\leq i\leq n}, (y_i)_{1\leq i\leq n}$ such that $x_i, y_i \in [0,1]$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 1$ and for all $i \in \{1,...,n\}$, $\xi_i = (a_i, b_i, c_i)$, we have:

$\xi = (f(x_1,...,x_n), g(x_1,...,x_n), h(x_1,...,x_n))$ and $\eta = (f(y_1,...,y_n), g(y_1,...,y_n), h(y_1,...,y_n))$ where $f(x_1,...,x_n) = \sum_{i=1}^n x_i a_i, g(x_1,...,x_n) = \sum_{i=1}^n x_i b_i$ and $h(x_1,...,x_n) = \sum_{i=1}^n x_i c_i$.

Based on characterization of $\succeq_1$ and those notations, (5.8) becomes:

$$\Gamma(P) = \{\sum_{i=1}^n x_i\xi_i, , \forall (y_i)_{1\leq i\leq n}, \begin{cases} f(x_1,...,x_n) \geq f(y_1,...,y_n) \\ g(x_1,...,x_n) \geq g(y_1,...,y_n) \\ h(x_1,...,x_n) \geq h(y_1,...,y_n) \\ \sum_{i=1}^n x_i = 1, \sum_{i=1}^n y_i = 1 \\ x_i \in [0,1], y_i \in [0,1], \forall i \in \{1,...,n\} \end{cases} \}. \tag{5.12}$$

Thereby, $\Gamma(P)$ is determined by the following optimization program:

$$\begin{cases} \max f(x_1,...,x_n) \\ \max g(x_1,...,x_n) \\ \max h(x_1,...,x_n) \\ \sum_{i=1}^n x_i = 1 \\ x_i \in [0,1] \forall i \in \{1,...,n\} \end{cases}. \tag{5.13}$$

In the following, we implement the previous program for the usual family $A = (\xi_i)_{1\leq i\leq 7}$ of seven assets with returns described in Table 1 of the previous Section.

In that case, the set of portfolios becomes $P = \{\xi = (-0.3x_1 - 0.4x_2 - 0.5x_3 - 0.6x_4 - 0.7x_5 - 0.8x_6 - 0.6x_7, 1.8x_1 + 2x_2 + 1.9x_3 + 2.2x_4 + 2.4x_5 + 2.5x_6 + 1.8x_7, 2.3x_1 + 2.2x_2 + 2.7x_3 + 2.8x_4 + 2.7x_5 + 3x_6 + 3x_7)$ where $\forall i \in \{1, ..., 7\}, x_i \in [0, 1]$ and $\sum_{i=1}^{7} x_i = 1\}$.

The optimization program which determines Γ becomes:

$$\begin{cases} \text{maximize } -0.3x_1 - 0.4x_2 - 0.5x_3 - 0.6x_4 - 0.7x_5 - 0.8x_6 - 0.6x_7 \\ \text{maximize } 1.8x_1 + 2x_2 + 1.9x_3 + 2.2x_4 + 2.4x_5 + 2.5x_6 + 1.8x_7 \\ \text{maximize } 2.3x_1 + 2.2x_2 + 2.7x_3 + 2.8x_4 + 2.7x_5 + 3x_6 + 3x_7 \\ \text{subject to} \\ x_1 + x_2 + ... + x_7 = 1 \\ x_i \geq 0, i = 1, ..., 7. \end{cases} \tag{5.14}$$

By solving (5.14) using Matlab, we obtain the following three elements of Γ: ξ, ξ', ξ'', chosen among many others which are presented in the three last lines of following table with optimal portfolios of Table 2.

Security i	1	2	3	4	5	6	7
Huang's model (5.2)	00.00%	47.06%	00.00%	35.28%	17.66%	00.00%	00.00%
Li et al.'s model (5.3)	20.00%	00.00%	00.00%	80.00%	00.00%	00.00%	00.00%
Sadefo et al.'s model (5.4)	20.04%	00.00%	00.00%	79.89%	00.00%	00.07%	00.00%
Sadefo et al.'s model (5.5)	20.00%	00.00%	00.00%	80.00%	00.00%	00.00%	00.00%
Best portfolio ξ	14.77%	35.01%	32.28%	21.7%	16.42%	20.17%	08.2%
Best portfolio ξ'	39.66%	09.2%	01.47%	01.28%	18.83%	25.82%	03.76%
Best portfolio ξ''	41.72%	05.55%	01.10%	00.86%	17.29%	31.47%	02.02%

Table 6: Optimal selection from models and best portfolios.

We can analyze the obtained best portfolios in the core in the way that a rational investor who intends to invest in the assets described by A must:

- diversify the capital on different assets (since values of x_i in each of the three portfolios of the core are non null)

- invest more on assets ξ_1, ξ_2, ξ_5 and ξ_6 (at least 17% of the capital) and less on assets ξ_3, ξ_4 and ξ_7 (at most 8% of the capital).

Finally, this investor can choose one of the three portfolios ξ, ξ', ξ'' as his shared capital.

Optimal portfolios obtained from optimization models and from the core can be viewed as triangular fuzzy variables in the following table.

Optimal portfolio	Triangular fuzzy variable
Huang's model (5.2)	$(-0.5235; 2.1412; 2.5)$
Li et al.'s model (5.3)	$(-0.54; 2.12; 2.7)$
Sadefo et al.'s model (5.4)	$(-0.54002; 2.12005; 2.6999)$
Sadefo et al.'s model (5.5)	$(-0.54; 2.12; 2.7)$
Best portfolio ξ	$(-0.5393; 2.1221; 2.5537)$
Best portfolio ξ'	$(-0.5317; 2.1191; 2.5859)$
Best portfolio ξ''	$(-0.5429; 2.1399; 2.607)$

Table 7 : Optimal portfolios from models and best portfolios viewed as triangular fuzzy variables.

We observe that portfolios of Table 7 obtained from different models are non dominated each other with respect to relation $\succeq_1$ according to Theorem 4.2.1. It is easy to check that, by implementing (5.11), the three portfolios belong to the core $\mathcal{C}_{\succeq_1}(P)$ contains all the portfolios of Table 7.

The following table presents some parameters (mean, variance, skewness, kurtosis, semi-variance and semi-kurtosis) of portfolios of Table 7 (the three best portfolios and those of Table 3).

Portfolio	Mean	Variance	Skewness	Kurtosis	Semi-variance	Semi-kurtosis
Huang's model [11]	1.6	0.7235	-0.7543	1.7972	0.6124	1.7415
Li et al.'s model [16]	1.6	0.7019	-0.6823	1.7291	0.6141	1.6872
Sadefo et al.'s model [29]	1.6	0.7018	-0.6823	1.7290	0.6140	1.6873
Sadefo et al.'s model [29]	1.6	0.7019	-0.6823	1.7291	0.6141	1.6872
Best portfolio ξ	1.5605	0.6973	-0.6666	1.7033	0.5832	1.5489
Best portfolio ξ'	1.5712	0.69	-0.6634	1.6668	0.5863	1.5585
Best portfolio ξ''	1.5849	0.7029	-0.687	1.7277	0.5994	1.6299

Table 8 : Comparison of four first moments of different optimal and best portfolios.

From Table 8, we can make the following analysis: except the mean, the two new best portfolios ξ and ξ' have better parameters (variance, skewness, kurtosis, semi-variance and semi-kurtosis) than those of portfolios obtained from quantitative approach whereas the third best portfolio ξ'' has better semi-variance and semi-kurtosis. The mean of the three best portfolios is less than those of the four optimal portfolios by the fact that the latter (models with parameters) were implemented with the target value of the mean equals to 1.6 (that was the minimal mean required by the investor). We notice that portfolios obtained from the set of best portfolios are suitable to get the shared capital of risk averse investors. Thereby, a risk averse investor who intends to invest on the seven assets can choose between the two best portfolios ξ and ξ' (see lines 6 and 7 of Table 2 or Table 1).

We can conclude that some main advantages of the approach based on the core (with best portfolios) are the following:

- it proposes more than one way of sharing a capital in different assets;

- the proposed results do not depend on targets values required to mean, variance, skewness and kurtosis. It means that, each investor can choose a portfolio according to his preference (maximum benefit, minimum risk);

- it contains all optimal portfolios obtained from models.

Finally, we illustrate these results by the following histogram:

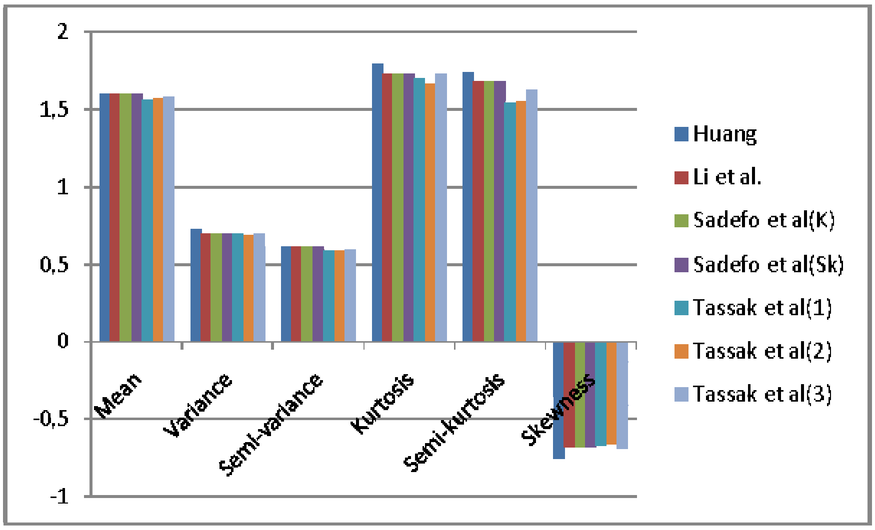

Figure 5.2: Comparison of characteristic values of optimal portfolios total returns.

CONCLUSION

In this thesis, we choose credibility measure to develop tools on fuzzy variables and apply some of them to improve portfolio optimization.

We define some new concepts as moments, semi-moments and partial moments of fuzzy variables and characterize them for trapezoidal fuzzy variables. Those concepts extend the first three moments of a fuzzy variable introduced and studied earlier by Liu [20], Huang[11], Li et al. [16]. We establish some of their useful properties and we analyze the particular case of kurtosis and semi-kurtosis (fourth moment and second semi-moment). Those results provide a new framework in statistics on fuzzy variables. Some applications of the obtained theoretical results enable us to describe the mean-variance-skewness-kurtosis and the mean-variance-skewness-semi-kurtosis portfolio optimization models. This quantitative approach for portfolio selection in fuzzy case improve the previous ones existing in the literature.

Furthermore, we introduce the mean risk dominance on fuzzy variables. That complements the two dominance relations existing in the literature, namely the first and second order dominance relations. We characterize these three dominance relations and establish that each of the three dominance relations satisfies many well-known properties of comparison tools of fuzzy variables. We justify that the first order dominance is stronger than the two others and the second order dominance is stronger than the mean risk dominance where the downside risk is the expected loss. The characterization of the second order dominance allows us to introduce and characterize two types of crossing points between two fuzzy variables. This result complements the literature on comparison of fuzzy variables. The first order dominance of that qualitative approach was applied in portfolio selection in order to introduce the set of non dominated portfolios, that is, the core of the set of portfolios. We establish the non-emptiness of the core. We implement a part of the core which is the set of best portfolios and we observe that the core contains optimal portfolios provided by deterministic models.

We implement, with Matlab, the proposed models for each of the two approaches in an example of the set of portfolios which components are seven assets introduced by Huang [11] used by Li et al.[16]. Numerical results justify that some portfolios proposed by dominance models are better than those proposed by quantitative models with targets value.

A next research topic is the theoretical determination of cores through respectively the second order dominance and the mean-risk dominance relations. These new cores are subsets of the core studied in this thesis. This open question leads us to the characterization of the minimal subset of the core of portfolios according to the first order dominance containing optimal portfolios obtained from deterministic models. Moreover, we intend to introduce a new poverty index based on fuzzy lower partial moment in order to evaluate an individual's poverty level in a population where individuals' incomes are unknown or imprecise.

Appendix

The Appendix is organized in two main parts:

1. The first part gives more details about the Fuzzy Lower Partial Moment.

2. The second part presents two scientific publications whose results are provided by this thesis.

Part I

Fuzzy Lower Partial moment and dominance relations

Some examples of Fuzzy Lower Partial Moment

Let us calculate the Fuzzy Lower Partial Moment (FLPM) of trapezoidal and triangular fuzzy variables.

Example 1:

1. The FLPM of trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is:

$$\mathrm{FLPM}_{\alpha,\tau}[\xi] = \begin{cases} 0, \text{ if } \tau < a \\ \frac{(\tau-a)^{\alpha+1}}{2(\alpha+1)(b-a)}, \text{ if } a \leq \tau < b \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)}, \text{ if } b \leq \tau < c \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{(\tau-c)^{\alpha+1}}{2(\alpha+1)(d-c)}, \text{ if } c \leq \tau < d \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{[(\tau-c)^{\alpha+1}-(\tau-d)^{\alpha+1}]}{2(\alpha+1)(d-c)}, \text{ if } \tau \geq d \end{cases}. \tag{5.15}$$

2. The FLPM of triangular fuzzy variable $\xi = (a, b, d)$ is:

$$\mathrm{FLPM}_{\alpha,\tau}[\xi] = \begin{cases} 0 \text{ if } \tau < a \\ \frac{(\tau-a)^{\alpha+1}}{2(\alpha+1)(b-a)}, \text{ if } a \leq \tau < b \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{(\tau-b)^{\alpha+1}}{2(\alpha+1)(d-b)}, \text{ if } b \leq \tau < d \\ \frac{[(\tau-a)^{\alpha+1}-(\tau-b)^{\alpha+1}]}{2(\alpha+1)(b-a)} + \frac{[(\tau-b)^{\alpha+1}-(\tau-d)^{\alpha+1}]}{2(\alpha+1)(d-b)}, \text{ if } \tau \geq d \end{cases}. \tag{5.16}$$

Remark 1:

From Example 1, we can deduce that:

1. Let $\xi_p = (\gamma_a(x), \gamma_b(x), \gamma_c(x), \gamma_d(x))$ where $\gamma_a(x) = \sum_{i=1}^n x_i a_i, \gamma_b(x) = \sum_{i=1}^n x_i b_i, \gamma_c(x) = \sum_{i=1}^n x_i c_i, \gamma_d(x) = \sum_{i=1}^n x_i d_i$, be a trapezoidal fuzzy return of a portfolio of n trapezoidal returns $(a_i, b_i, c_i, d_i)_{i=1,2,\ldots,n}$. Then the Fuzzy Lower Partial Moment of the portfolio return ξ_p is given by:

$$\mathrm{FLPM}_{\alpha,\tau}[\xi] = \begin{cases} 0 \text{ if } \tau < \gamma_a(x) \\ \frac{(\tau-\gamma_a(x))^{\alpha+1}}{2(n+1)(\gamma_b(x)-\gamma_a(x))}, \text{ if } \gamma_a(x) \leq \tau < \gamma_b(x) \\ \frac{[(\tau-\gamma_a(x))^{\alpha+1}-(\tau-\gamma_b(x))^{\alpha+1}]}{2(\alpha+1)(\gamma_b(x)-\gamma_a(x))}, \text{ if } \gamma_b(x) \leq \tau < \gamma_c(x) \\ \frac{[(\tau-\gamma_a(x))^{\alpha+1}-(\tau-\gamma_b(x))^{\alpha+1}]}{2(\alpha+1)(\gamma_b(x)-\gamma_a(x))} + \frac{(\tau-\gamma_c(x))^{\alpha+1}}{2(\alpha+1)(\gamma_d(x)-\gamma_c(x))}, \text{ if } \gamma_c(x) \leq \tau < \gamma_d(x) \\ \frac{[(\tau-\gamma_a(x))^{\alpha+1}-(\tau-\gamma_b(x))^{\alpha+1}]}{2(\alpha+1)(\gamma_b(x)-\gamma_a(x))} + \frac{[(\tau-\gamma_c(x))^{\alpha+1}-(\tau-\gamma_d(x))^{\alpha+1}]}{2(\alpha+1)(\gamma_d(x)-\gamma_c(x))}, \text{ if } \tau \geq \gamma_d(x) \end{cases} \tag{5.17}$$

where $\gamma_z(x) = \sum_{i=1}^n x_i z_i$ for $z = a, b, c, d \in \mathbb{R}$ with $a < b < c < d$.

2. When $\tau = E[\xi]$, we can deduce that: (i) $\mathrm{FLPM}_{2,\tau}[\xi]$ is the semi-variance of the portfolio return ξ and (ii) $\mathrm{FLPM}_{4,\tau}[\xi]$ is the semi-kurtosis of the portfolio return ξ.

Some results on Fuzzy Lower Partial Moment

The following result determines the credibility distribution function Φ of a fuzzy variable ξ by the derivatives of its FLPM when ξ has a lower bounded support. More precisely, it establishes that we can determine the credibility distribution $\Phi(\tau)$ only by the $\mathrm{FLPM}_{\alpha,\tau}$ with $\alpha \in \mathbb{N}^*$.

Proposition 1:

The credibility distribution function Φ of a fuzzy variable ξ with lower bounded support satisfies the following relation:

$$\frac{d^\alpha}{d\tau^\alpha}\mathrm{FLPM}_{\alpha,\tau} = \alpha!\,\Phi(\tau), \text{ that is, } \Phi(\tau) = \frac{1}{\alpha!}\frac{d^\alpha}{d\tau^\alpha}\mathrm{FLPM}_{\alpha,\tau} \tag{5.18}$$

Proof: Let Φ be the credibility distribution function of fuzzy variable ξ with lower bounded support. We have:

$$\begin{aligned}\frac{d^\alpha}{d\tau^\alpha}\mathrm{FLPM}_{\alpha,\tau} &= \frac{d^\alpha}{d\tau^\alpha}[\int_{-\infty}^{\tau}(\tau-u)^\alpha d\Phi(u)]\\ &= \int_{-\infty}^{\tau}\frac{d^\alpha}{d\tau^\alpha}[(\tau-u)^\alpha d\Phi(u)].\end{aligned}$$

It is easy to check that $\forall\alpha \in \mathbb{N}^*$, $\frac{d^\alpha}{d\tau^\alpha}(\tau-u)^\alpha = \alpha!$ and finally, we have:

$$\begin{aligned}\frac{d^\alpha}{d\tau^\alpha}\mathrm{FLPM}_{\alpha,\tau} &= \int_{-\infty}^{\tau}\alpha! d\Phi(u) = \alpha!\int_{-\infty}^{\tau}d\Phi(u) = \alpha![\Phi(\tau) - \lim_{u\to-\infty}\Phi(u)]\\ &= \alpha![\Phi(\tau) - 0] = \alpha!\Phi(\tau).\end{aligned}$$

Hence the result. □

The following result determines necessary and sufficient conditions on a FLPM under which the density function ϕ of ξ satisfying a particular inequality, belongs to exponential family.

Proposition 2:

Let ϕ_γ be the credibility density function of a nonnegative fuzzy variable ξ satisfying the following condition:

$$\frac{d}{d\gamma}\phi_\gamma(u) \geq \phi_\gamma(u)(u + D'(\gamma)), \forall u \in (0, \infty) \tag{5.19}$$

where $D'(\gamma)$ is the derivative of $D(\cdot)$ with respect to γ.

ϕ_γ belongs to exponential family, that means,

$$\phi_\gamma(u) = e^{\gamma\, u + K(u) + D(\gamma)}, \qquad u \in (0, \infty), \quad \gamma > 0, \tag{5.20}$$

where $K(\cdot)$ is an arbitrary function, if and only if, its $\text{FLPM}_{\alpha,\tau}$ satisfy a recurrence relationship

$$\text{FLPM}_{\alpha+1,\tau} = (\tau + D^{'}(\gamma))\text{FLPM}_{\alpha,\tau} - \frac{d}{d\gamma}\text{FLPM}_{\alpha,\tau}, \tag{5.21}$$

.

Proof: ($\Rightarrow$) Assume that the credibility density function ϕ_γ is defined by: $\phi_\gamma(u) = e^{\gamma\, u + K(u) + D(\gamma)}$

where $\quad u \in (0, \infty), \quad \gamma > 0$, K and D two arbitrary functions.

By computing the derivative of ϕ_γ given by relation (5.20) with respect to γ, one can easily check that ϕ_γ satisfies relation (5.19).

Let us prove that $\text{FLPM}_{\alpha+1,\tau} = (\tau + D^{'}(\gamma))\text{FLPM}_{\alpha,\tau} - \frac{d}{d\gamma}\text{FLPM}_{\alpha,\tau}$.

We have:

$$\begin{aligned}
\frac{d}{d\gamma}\mathrm{FLPM}_{\alpha,\tau} &= \frac{d}{d\gamma}[\int_0^\tau (\tau-u)^\alpha e^{\gamma\, u+K(u)+D(\gamma)}du] \\
&= \int_0^\tau (\tau-u)^\alpha \frac{d}{d\gamma} e^{\gamma\, u+K(u)+D(\gamma)}du \\
&= \int_0^\tau (\tau-u)^\alpha (u+D'(\gamma)) e^{\gamma\, u+K(u)+D(\alpha)}du \\
&= \int_0^\tau u(\tau-u)^\alpha e^{\alpha\, u+K(u)+D(\gamma)}du + \int_0^\tau D'(\gamma)(\tau-u)^\alpha e^{\gamma\, u+K(u)+D(\gamma)}du \\
&= \int_0^\tau (u-\tau+\tau)(\tau-u)^\alpha e^{\gamma\, u+K(u)+D(\gamma)}du + D'(\gamma)\mathrm{FLPM}_{\alpha,\tau} \\
&= -\int_0^\tau (\tau-u)^{\alpha+1} e^{\gamma\, u+K(u)+D(\gamma)}du + \tau\int_0^\tau (\tau-u)^\alpha e^{\gamma\, u+K(u)+D(\gamma)}du + D'(\gamma)\mathrm{FLPM}_{\alpha,\tau} \\
&= -\mathrm{FLPM}_{\alpha+1,\tau} + (\tau+D'(\gamma))\mathrm{FLPM}_{\alpha,\tau}.
\end{aligned}$$

Hence the result.

($\Leftarrow$) Now we prove the sufficient condition.

By means of relation (4.2), relation (5.21) can be expressed as follows:

$\int_0^\tau (\tau-u)^{\alpha+1}\phi_\gamma(u)\ du = (\tau+D'(\gamma))\int_0^\tau (\tau-u)^\alpha\phi_\gamma(u)\ du - \frac{d}{d\gamma}\int_0^\tau (\tau-u)^{\alpha+1}\phi_\gamma(u)\ du$ which implies that:

$$\int_0^\tau [(\tau-u)^\alpha\phi(u)(-u-D'(\gamma)) + \frac{d}{d\gamma}((\tau-u)^\alpha\phi_\gamma(u))]\ du = 0 \tag{5.22}$$

By using the fact that ϕ_γ satisfies relation (5.19), relation (5.22) traduces the nullity of the integration of a positive function.

Thus, we obtain:

$$\frac{d}{d\gamma}((\tau-u)^\alpha\phi_\gamma(u)) = (\tau-u)^\alpha\phi_\gamma(u)(u+D'(\gamma)) \tag{5.23}$$

Finally, by integrating each side of relation (5.23) with respect to γ, we obtain:

$(\tau-u)^\alpha\phi_\gamma(u) = ke^{\gamma u+D(\gamma)}$, $k>0$, which leads to:

$\phi_\gamma(u) = e^{\gamma u + D(\gamma) + K(u)}$ with $K(u) = \ln(\frac{k}{((\tau - u)^\alpha)})$, $u \in]0;\tau[$.

It suffices to consider the function ϕ_γ defined as:

$\phi_\gamma(u) = e^{\gamma u + D(\gamma) + K(u)}$ with $K(u) = \ln|(\frac{k}{((\tau - u)^\alpha)}|$, $u \in (0;+\infty) \setminus \{\tau\}$ and $K(\tau) = 0$. $\square$

Some proofs on crossing points

To establish Proposition 4.2.2, we need the following Lemma.

Lemma 1:

Let r_0 and ϵ be two reals numbers with $\epsilon > 0$. We have:

1. $\forall s \in (0, \epsilon), \begin{cases} \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s) \\ r_0 - s, r_0 + s \in [a_i \vee a_j, b_i \wedge b_j] \end{cases} \Rightarrow \Phi_i(r_0 - s) < \Phi_j(r_0 - s), \Phi_i(r_0 + s) > \Phi_j(r_0 + s)$.

2. $\forall s \in (0, \epsilon), \begin{cases} \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s) \\ r_0 - s, r_0 + s \in [c_i \vee c_j, d_i \wedge d_j] \end{cases} \Rightarrow \Phi_i(r_0 - s) > \Phi_j(r_0 - s), \Phi_i(r_0 + s) < \Phi_j(r_0 + s)$.

3. $\forall r \in \mathbb{R}, (r \in [b_i \vee b_j, c_i \wedge c_j]) \Rightarrow \Phi_i(r) = \Phi_j(r)$.

Proof of Lemma 1: Let us recall that Φ is given by (2.5).

1) Let be a real number s such that $0 < s < \epsilon, \mu_i(r_0 - s) < \mu_j(r - s), \mu_i(r_0 + s) > \mu_j(r_0 + s)$ and $r_0 - s, r_0 + s \in [a_i \vee a_j, b_i \wedge b_j]$. We have:

$\mu_i(r_0 - s) < \mu_j(r_0 - s) \Rightarrow \Phi_i(r_0 - s) < \Phi_j(r_0 - s)$ (with $\Phi_i(r_0 - s) = \frac{1}{2}\mu_i(r_0 - s)$ and $\Phi_j(r_0 - s) = \frac{1}{2}\mu_j(r_0 - s)$) and

$\mu_i(r_0 + s) > \mu_j(r_0 + s) \Rightarrow \Phi_i(r_0 + s) > \Phi_j(r_0 + s)$ (with $\Phi_i(r_0 + s) = \frac{1}{2}\mu_i(r_0 + s)$ and $\Phi_j(r_0 + s) = \frac{1}{2}\mu_j(r_0 + s)$).

2) Let be a real number s such that $0 < s < \epsilon, \mu_i(r_0 - s) < \mu_j(r_0 - s), \mu_i(r_0 + s) > \mu_j(r_0 + s)$

and $r_0 - s, r_0 + s \in [c_i \vee c_j, d_i \wedge d_j]$. We have:

$\mu_i(r_0 - s) < \mu_j(r_0 - s) \Rightarrow \Phi_i(r_0 - s) > \Phi_j(r_0 - s)$ (with $\Phi_i(r_0 - s) = 1 - \frac{1}{2}\mu_i(r_0 - s)$ and $\Phi_j(r_0 - s) = 1 - \frac{1}{2}\mu_j(r_0 - s)$) and

$\mu_i(r_0 + s) > \mu_j(r_0 + s) \Rightarrow \Phi_i(r_0 + s) < \Phi_j(r_0 + s)$ (with $\Phi_i(r_0 + s) = 1 - \frac{1}{2}\mu_i(r_0 + s)$ and $\Phi_j(r_0 + s) = 1 - \frac{1}{2}\mu_j(r_0 + s)$).

3) If $r \in [b_i \vee b_j, c_i \wedge c_j]$, then $\Phi_i(r) = \Phi_j(r) = \frac{1}{2}$. $\square$

Proof of Proposition 4.2.2: 1) Let us consider $\epsilon \in \mathbb{R}_*^+, r_0 \in \mathbb{R}$ and s a real number such that $0 < s < \epsilon$ and $\mu_i(r_0 - s) < \mu_j(r_0 - s)$, $\mu_i(r_0 + s) > \mu_j(r_0 + s)$, with $r_0 - s, r_0 + s \in [a_i \vee a_j, b_i \wedge b_j]$. According to Lemma 1, we have $\Phi_i(r_0 - s) < \Phi_j(r_0 - s)$ and $\Phi_i(r_0 + s) > \Phi_j(r_0 + s)$ and by Definition 4.2.4, we can conclude that r_0 is a crossing point of type II.

We prove the converse case in the same manner.

2) We use the same method as in 1.

3) Let us prove that c_i is a crossing point of type I.

$[b_i, c_i] \subseteq [b_j, c_j] \Rightarrow [b_i \vee b_j, c_i \wedge c_j] = [b_i, c_i]$ and by Lemma 1 and Definition 4.2.3, we have:

$b_i = \min\{t/[t, c_i) is I.C\}$.

Now, let us find $\epsilon_0 > 0$ such that $\forall s : 0 < s < \epsilon_0, \Phi_i(b_i - s) < \Phi_j(b_i - s)$ and $\Phi_i(c_i + s) > \Phi_j(c_i + s)$.

i) If $b_i \neq b_j$ and $c_i \neq c_j$

Then we set $\epsilon_0 = (b_i - b_j) \wedge (c_j - c_i)$ and we easily check that $\epsilon_0 > 0$ according to the fact that $[b_i, c_i] \subset [b_j, c_j]$ and $b_i \neq b_j, c_i \neq c_j$. We have two cases:

<u>1st case</u>: $b_i - b_j < c_j - c_i$

We have $\epsilon_0 = b_i - b_j$, and $b_i - \epsilon_0 = b_j, c_i + \epsilon_0 = c_i + b_i - b_j$.

We obtain: $\Phi_i(b_i - s) < \Phi_i(b_i)$ (with $\Phi_i(b_i) = \frac{1}{2}$) because $b_i - s < b_i$ and Φ_i increases; on the other hand, by the fact that $0 < s < \epsilon_0$, and Φ_j increases, we have: $\Phi_j(b_i - s) > \Phi_j(b_i - \epsilon_0) = \Phi_j(b_j) = \frac{1}{2}$.

Furthermore, $\Phi_i(c_i + s) > \Phi_i(c_i)$ (with $\Phi_i(c_i) = \frac{1}{2}$) because Φ_i increases and

$\Phi_j(c_i + s) < \Phi_j(c_j) = \frac{1}{2}$ because $c_i + s < c_i + \epsilon_0 < c_i + c_j - c_i = c_j$ and Φ_j increases.

2nd case: $c_j - c_i < b_i - b_j$

We have $\epsilon_0 = c_j - c_i$, and $c_i + \epsilon_0 = c_j, b_i - \epsilon_0 = b_i - c_j + c_i$.

We obtain $\Phi_i(b_i - s) - \Phi_j(b_i - s) < 0$ because: $\Phi_i(b_i - s) < \Phi_i(b_i)$ (with $\Phi_i(b_i) = \frac{1}{2}$) and $b_i - \epsilon_0 - b_j = b_i - b_j - (c_j - c_i) > 0$, that is $b_i - \epsilon_0 > b_j$, so $\Phi_j(b_i - s) > \Phi_j(b_i - \epsilon_0) > \Phi_j(b_j)$ (with $\Phi_j(b_j) = \frac{1}{2}$) as Φ_j increases and $b_i - \epsilon_0 > b_j$.

Furthermore, $\Phi_i(c_i+s) - \Phi_j(c_i+s) > 0$; indeed, $c_i+s < c_i+\epsilon_0 = c_j$, so $\Phi_j(c_i+s) < \Phi_j(c_j) = \frac{1}{2}$.

On the other hand Φ_i increases and $\Phi_i(c_i + s) > \Phi_i(c_i)$ (with $\Phi_i(c_i) = \frac{1}{2}$).

ii) If $b_i = b_j$ and $c_i \neq c_j$

Then $\epsilon_0 = c_j - c_i$ and we easily conclude as in i).

iii) If $c_i = c_j$ and $b_i \neq b_j$

Then $\epsilon_0 = b_i - b_j$ and we easily conclude as in i).

iv) If $c_i = c_j$ and $b_i = b_j$.

Then we take $\epsilon_0 = (b_j - a_j) \wedge (d_j - c_j)$.

It is easy to check that for all s such that $0 < s < \epsilon_0$, we have: $b_j - s \in]a_j, b_i[$ and $c_j + s \in]c_i, d_j[$.

$(c_i = c_j, b_i = b_j) \Rightarrow [b_j, c_j] = [b_i, c_i]$; thus the support of ξ_i is included in the support of ξ_j and their kernels coincide that means μ_j and μ_i coincide only in $[b_j, c_j]$, and this justifies the fact that $\forall s \in]a_j, b_i[, \mu_j(s) > \mu_i(s)$ and $\forall s \in]c_i, d_j[, \mu_i(s) < \mu_j(s)$.

Furthermore, $\forall s \in [a_j, b_i[, \Phi_j(s) > \Phi_i(s)$ by the fact that $\mu_j(s) > \mu_i(s)$ and $\forall s \in [c_i, d_j[, \Phi_j(s) < \Phi_i(s)$ by the fact that $\mu_i(s) < \mu_j(s)$; these last inequalities lead us to $\Phi_i(c_j + s) > \Phi_j(c_j +$

$s), \Phi_i(b_j - s) < \Phi_j(b_j + s)$.

4) By taking $\epsilon_0 = \min(b_i - b_j, c_j - b_i)$, we can easily check that $\forall s$ such that: $0 < s < \epsilon_0$, $\Phi_i(c_i - s) < \Phi_j(c_i - s), \Phi_i(c_i + s) > \Phi_j(c_i + s)$.

5) By taking $\epsilon_0 = \min(b_j - b_i, c_i - b_j,)$ we can easily check that $\forall s$ such that: $0 < s < \epsilon_0$, $\Phi_j(c_j - s) < \Phi_i(c_j - s), \Phi_j(c_j + s) > \Phi_i(c_j + s)$.

6) By taking $\epsilon_0 = \min(a_i - a_j, d_j - d_i)$, we can easily check that $\forall s$ such that: $0 < s < \epsilon_0$, $\Phi_j(c_j - s) > \Phi_i(c_j - s), \Phi_j(c_j + s) < \Phi_i(c_j + s)$. $\square$

Proof of the characterization of the second order dominance relation

Without loss of generality, we assume that between ξ_1 and ξ_2, ξ_1 is the one that could dominates. In other words, the curve of Φ_1 starts below.

1) Necessity of Theorem 4.2.2.

According to the definition of $\succ_2$, we have: $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0$ and $\exists t_0 \in \mathbb{R}, \int_{-\infty}^{t_0} [\Phi_2(r) - \Phi_1(r)]dr > 0$.

(i) We have, according to the first assumption:

$(\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0) \Rightarrow \forall i \in \{1, 2, ..., k\}, \int_{-\infty}^{t_{0i}} [\Phi_2(r) - \Phi_1(r)]dr \geq 0$.

(ii) Furthermore, $(\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0) \Rightarrow \int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr \geq 0$, that means, $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr > 0$ or $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr = 0$.

- In the first case, where $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr > 0$, we immediately obtain the result.

- In the second case, where $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr = 0$, we justify that $\exists t_{0h} \in \{t_{01}, ..., t_{0k}\}, \int_{-\infty}^{t_{0h}} [\Phi_2(r) - \Phi_1(r)]dr > 0$.

Let us assume that $\forall s \in \{t_{01}, ..., t_{0k}\}, \int_{-\infty}^{s} [\Phi_2(r) - \Phi_1(r)]dr \leq 0$ and establish a contradiction.

According to the first assumption, we obtain in this second case: $\forall s \in \{t_{01}, ..., t_{0k}\}$, $\int_{-\infty}^{s}[\Phi_2(r) - \Phi_1(r)]dr = 0$.

As $\int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr = 0$, we have $\forall s \in \{t_{01}, ..., t_{0k}\}$, $\int_{-\infty}^{s}[\Phi_2(r)-\Phi_1(r)]dr+\int_{s}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr = 0$, that is,

$$\int_{s}^{+\infty}[\Phi_2(r) - \Phi_1(r)]dr = 0, \forall s \in \{t_{01}, ..., t_{0k}\} \tag{5.24}$$

Finally, to obtain a contradiction with respect to the second assumption, we prove that $\forall t \in \mathbb{R}, \int_{-\infty}^{t}[\Phi_2(r) - \Phi_1(r)]dr = 0$.

$*$) If $t < t_{01}$ where t_{01} is the first crossing point, then $\int_{-\infty}^{t}[\Phi_2(r) - \Phi_1(r)]dr = 0$ because $\int_{-\infty}^{t_{01}}[\Phi_2(r) - \Phi_1(r)]dr = 0$, and the sign of the quantity $\Phi_2(r) - \Phi_1(r)$ remains unchanged until the first crossing point t_{01}.

$*$) If $t_{0i} \leq t \leq t_{0j}$ where t_{0i} and t_{0i} are two consecutive crossing points such that $t_{0i} < t_{0j}$, then $\int_{-\infty}^{t}[\Phi_2(r) - \Phi_1(r)]dr = 0$ because $\int_{-\infty}^{t_{0j}}[\Phi_2(r) - \Phi_1(r)]dr = 0$, and the sign of the quantity $\Phi_2(r) - \Phi_1(r)$ remains unchanged between the two crossing points t_{0i} and t_{0j}.

$*$) If $t > t_{0k}$ where t_{0k} is the last crossing point, we have:

According to (5.24), $\int_{t_{0k}}^{+\infty}[\Phi_2(r) - \Phi_1(r)]dr = 0$, and consequently, $\int_{t_{0k}}^{t}[\Phi_2(r) - \Phi_1(r)]dr = 0$ by the fact that the sign of the quantity $\Phi_2(r) - \Phi_1(r)$ remains unchanged after the last crossing point. Thus, $\int_{-\infty}^{t_{0k}}[\Phi_2(r) - \Phi_1(r)]dr = 0$ and $\int_{t_{0k}}^{t}[\Phi_2(r) - \Phi_1(r)]dr = 0$, imply that $\int_{-\infty}^{t}[\Phi_2(r) - \Phi_1(r)]dr = 0$.

Hence, the contradiction is obtained by the fact that $\forall t \in \mathbb{R}, \int_{-\infty}^{t}[\Phi_2(r) - \Phi_1(r)]dr = 0$.

2) Sufficiency of Theorem 4.2.2.

From the assumptions of the the theorem, we have $\forall i \in \{1, 2, ..., k\}$, $\int_{-\infty}^{t_{0i}}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$, that means, the area balance sign condition for $\succ_2$ is fulfilled at all crossing points.

Since t_{01} corresponds to the first crossing point, we obtain $\int_{-\infty}^{t_{01}}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$ and $\forall t \in]-\infty, t_{01}], \int_{-\infty}^{t}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$, because t_{01} is the first crossing point and we have supposed that the curve of Φ_1 starts below. So the curve of Φ_1 should have stay below (or coincides in some intervals of coincidence) all the way from $-\infty$ to t_{01}.

Let us analyze the condition in the interval $]t_{01}, t_{02}]$. Since t_{02} corresponds to the second crossing point, it is clear from the definition of crossing point that, $\Phi_2(r) \geq \Phi_1(r), \forall r \in]-\infty, t_{01}]$ and $\Phi_2(r) \leq \Phi_1(r), \forall r \in]t_{01}, t_{01}]$. We can write: $\forall t \in]t_{01}, t_{02}], \int_{-\infty}^{t}[\Phi_2(r)-\Phi_1(r)]dr = \int_{-\infty}^{t_{02}}[\Phi_2(r)-\Phi_1(r)]dr - \int_{t_{01}}^{t_{02}}[\Phi_2(r)-\Phi_1(r)]dr$. Since $(\Phi_2(r) \leq \Phi_1(r)], \forall r \in]t_{01}, t_{02}] \Rightarrow \int_{t_{01}}^{t_{02}}[\Phi_2(r)-\Phi_1(r)]dr \leq 0)$ and $\int_{-\infty}^{t_{02}}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$ from the assumptions, we have: $\int_{-\infty}^{t}[\Phi_2(r)-\Phi_1(r)]dr \geq \int_{-\infty}^{t_{02}}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$.

We can say that the fulfillment of the condition at t_{01} and t_{02} involves the fulfillment at all t, $t \in]t_{01}, t_{02}]$. It follows inductively that for finite k, the fulfillment of the sign condition at all t_{0i}, $(t_{0i} \in \{t_{01}, t_{02}, ..., t_{0k}\})$ implies the fulfillment at all t, $t \in]-\infty, t_{0k}]$.

Now, we prove that the fulfillment of the sign condition at t_{0k} and $\int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$, will imply the fulfillment at all t, $t \in]t_{0k}, +\infty[$.

Let us assume (by considering k as an odd number) that $\Phi_2(r) \geq \Phi_1(r), \forall r \in]t_{0(k-1)}, t_{0k}]$ and $\Phi_2(r) \leq \Phi_1(r), \forall r \in]t_{0k}, +\infty[$ (where t_{0k} is the last crossing point). We have: $\forall t \in t \in]t_{0k}, +\infty[$,

$\int_{-\infty}^{t}[\Phi_2(r)-\Phi_1(r)]dr = \int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr - \int_{t}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr$. Since $(\Phi_2(r) \leq \Phi_1(r), \forall r \in]t_{0k}, +\infty[\Rightarrow \int_{t}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr \leq 0)$ and $\int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$ from the assumptions, we have: $\int_{-\infty}^{t}[\Phi_2(r)-\Phi_1(r)]dr \geq \int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$.

Therefore, if $\forall i \in \{1, 2, ..., k\}, \int_{-\infty}^{t_{0i}}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$ and $\int_{-\infty}^{+\infty}[\Phi_2(r)-\Phi_1(r)]dr \geq 0$, then $\int_{-\infty}^{t}[\Phi_2(r)-\Phi_1(r)]dr \geq 0, \forall t \in]-\infty, +\infty[$. This result and the condition of fulfillment as strict inequality at some $t_{0h} \in \{t_{01}, ..., t_{0k}\}$ or for the integral from $-\infty$ to $+\infty$ as stated in the theorem, implies $\succ_2$. The proof can be done in the same manner when k is even. This completes the proof for sufficiency and the proof of the theorem. □

Proofs of dominance relations properties

The proofs of some results of Proposition 4.3.2 require the following lemma:

Lemma 2:

Let ξ_1, ξ_2 and θ be three independent trapezoidal fuzzy variables. Φ_1 , Φ_2, Φ'_1 and Φ'_2 are respectively the credibility distributions functions of fuzzy variables ξ_1, ξ_2, $\xi_1 + \theta$ and $\xi_2 + \theta$. Then we have:

- $(\exists r_0 \in \mathbb{R}, \Phi_1(r_0) = \Phi_2(r_0)) \Leftrightarrow (\exists t_0 \in \mathbb{R}, \Phi'_1(t_0) = \Phi'_2(t_0))$.
- For all crossing point $v \in \mathbb{R}$ between Φ_1 and Φ_2, $\exists u_v \in \mathbb{R}$, crossing point between Φ'_1 and Φ'_2 such that: $\int_{-\infty}^{v} [\Phi_1(r) - \Phi_2(r)]\, dr = \int_{-\infty}^{u_v} [\Phi'_1(r) - \Phi'_2(r)]\, dr$.
- $\int_{-\infty}^{+\infty} [\Phi_1(r) - \Phi_2(r)]\, dr = \int_{-\infty}^{+\infty} [\Phi'_1(r) - \Phi'_2(r)]\, dr$.

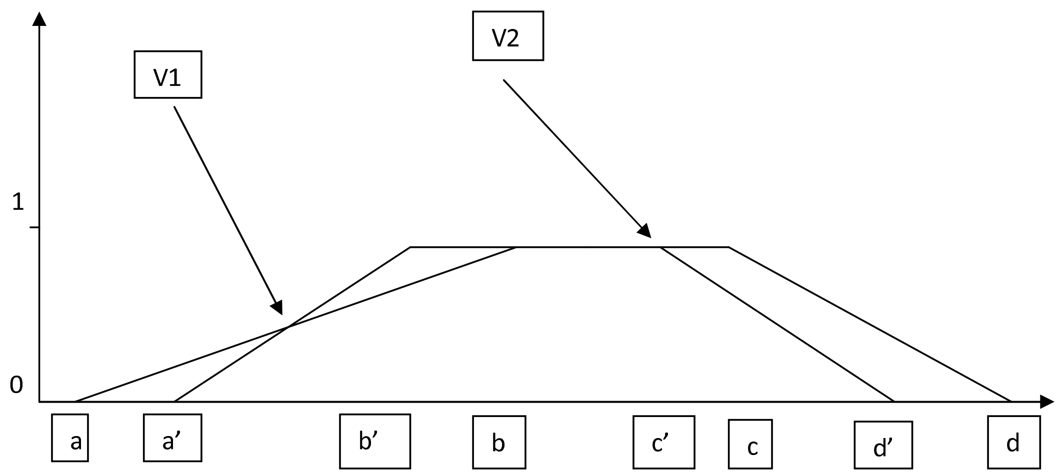

Figure 5.3: A particular position of two fuzzy variables.

Interpretation:

According to Lemma 2, there exists a crossing point between Φ_1 and Φ_2 if and only if there exists a crossing point between Φ'_1 and Φ'_2 and the area between two distributions functions keep unchanged by translating them.

Proof of Lemma 2: Let us consider the assumptions of the lemma.

We set: $\xi_1 = (a, b, c, d)$, $\xi_2 = (a', b', c', d')$ and $\theta = (a'', b'', c'', d'')$.

Without loss of generality, we suppose in all this proof that $a < a' < b' < b < c' < c < d' < d$ (see Figure 5.3). The other cases can be proved in the same way.

1) Let us recall that $\xi_1 + \theta = (a + a'', b + b'', c + c'', d + d'')$ and $\xi_2 + \theta = (a' + a'', b' + b'', c' + c'', d' + d'')$.

We have: $\Phi_1(r) = \Phi_2(r) \Leftrightarrow (\frac{r-a}{b-a} = \frac{r-a'}{b'-a'}$ or $r \in [b, c']) \Leftrightarrow (r = r_0 = \frac{a'(b-a)-a(b'-a')}{(b-a)-(b'-a')}$ or $r \in [b, c'])$ (see Figure 5.3 with $r_0 = v_1$). Furthermore, we have:

i) First, $\frac{r-(a+a'')}{(b+b'')-(a+a'')} = \frac{r-(a'+a'')}{(b'+b'')-(a'+a'')} \Leftrightarrow r = t_0 = \frac{(a'+a'')[(b+b'')-(a+a'')]-(a+a'')[(b'+b'')-(a'+a'']}{(b-a)-(b'-a')}$, that is $\Phi'_1(t_0) = \Phi'_2(t_0)$. t_0 exists if and only if r_0 exists and $t_0 \in [a' + a'', b' + b'']$ if and only if $r_0 \in [a', b']$.

ii) Secondly, we have: $r_0 \in [b, c'] \Leftrightarrow t_0 \in [b + b'', c' + c'']$, according to the expressions of $\xi_1 + \theta$ and $\xi_2 + \theta$.

2) Let $v \in \mathbb{R}$ be a crossing point between Φ_1 and Φ_2.

v satisfies $\Phi_1(v) = \Phi_2(v)$, thus, according to 1), $\exists u_v \in \mathbb{R}$ such that $\Phi'_1(u_v) = \Phi'_2(u_v)$. By the assumptions $a < a' < b' < b < c' < c < d' < d$, there exists two crossing points between Φ_1 and Φ_2: $v_1 \in [a', b']$ (type II) and $v_2 = c'$ (type I). (see Figure 5.3).

According to 1), we set:

i) $u_{v_1} = \frac{(a'+a'')[(b+b'')-(a+a'')]-(a+a'')[(b'+b'')-(a'+a'']}{[b-a]-[b'-a']}$ and we consider the real intervals $[a', b']$

and $[a'+a'', b'+b'']$ with $v_1 \in [a', b']$ and $u_{v_1} \in [a'+a'', b'+b'']$.

We have $\forall r \in [a', b']$: $\Phi_1(r) - \Phi_2(r) = \frac{r(a-b+b'-a') - a(b'-a') + a'(b-a)}{(b-a)(b'-a')}$ and,

$\forall r \in [a'+a'', b'+b'']$,

$\Phi_1'(r) - \Phi_2'(r) = \frac{r(a-b+b'-a') - (a+a'')[(b'+b'')-(a'+a'')] + (a'+a'')[(b+b'')-(a+a'')]}{[(b+b'')-(a+a'')][(b'+b'')-(a'+a'')]}$.

The sign of $\Phi_1(r) - \Phi_2(r)$ is the opposite sign of $A = \frac{a-b+b'-a'}{(b-a)(b'-a')}$ before v_1 and the same sign of A after v_1.

The sign of $\Phi_1'(r) - \Phi_2'(r)$ is the opposite sign of $B = \frac{a-b+b'-a'}{[(b+b'')-(a+a'')][(b'+b'')-(a'+a'')]}$ before u_{v_1} and the same sign of B after u_{v_1}.

By the fact that A and B have the same sign, we conclude that:

$(\forall r \leq v_1, \Phi_1(r) \leq \Phi_2(r)) \Leftrightarrow (\forall r \leq u_{v_1}, \Phi_1'(r) \leq \Phi_2'(r))$. That is, u_{v_1} is a crossing point of type II between Φ_1' and Φ_2'.

ii) $u_{v_2} = c' + c''$ and we check that u_{v_2} is a crossing point of type I between Φ_1' and Φ_2' by considering the interval of coincidence $[b+b'', c'+c'']$.

On the other hand, computations display that:

$\int_{-\infty}^{v_1} [\Phi_1(r) - \Phi_2(r)]\, dr = \int_{-\infty}^{u_{v_1}} [\Phi_1'(r) - \Phi_2'(r)]\, dr = \frac{(a'-a)^2}{4(b-a-b'+q')}$ and

$\int_{-\infty}^{v_2} [\Phi_1(r) - \Phi_2(r)]\, dr = \int_{-\infty}^{u_{v_2}} [\Phi_1'(r) - \Phi_2'(r)]\, dr = \frac{a'-a+b'-b}{4}$.

3) Finally, computations display that:

$\int_{-\infty}^{+\infty} [\Phi_1(r) - \Phi_2(r)]\, dr = \int_{-\infty}^{+\infty} [\Phi_1'(r) - \Phi_2'(r)]\, dr = \frac{a'-a+b'-b+c'-c+d'-d}{4}$.

By the previous results of items 1), 2) and 3), we obtain results of the lemma. □

Proof of Proposition 4.3.2:

We consider, $\xi = (a, b, c, d)$, $\eta = (a', b', c', d')$ and $\theta = (a'', b'', c'', d'')$ be three elements of $\mathcal{A}$ with respective credibility distributions functions Φ_1, Φ_2, Φ_3. Let us suppose that $\xi + \eta =$

$(a+a'', b+b'', c+c'', d+d'')$, $\eta+\theta=(a'+a'', b'+b'', c'+c'', d'+d'')$ are two elements of $\mathcal{A}$ and that Φ_1', Φ_2' are their respective credibility distributions functions.

1. Mean-risk dominance $\text{FLPM}_{\alpha,\tau}$, $\alpha \in \mathbb{N}^*, \tau \in \mathbb{R}$:

   A_1) We have: $\begin{cases} E[\xi] \geq E[\xi] \\ \text{FLPM}_{\alpha,\tau}[\xi] \leq \text{FLPM}_{\alpha,\tau}[\xi] \end{cases}$. Therefore, $\xi \succeq_{\alpha,\tau} \xi$.

   A_2) Let us assume that $\xi \succeq_{\alpha,\tau} \eta$ and $\eta \succeq_{\alpha,\tau} \xi$.

   We have: $\begin{cases} E[\xi] \geq E[\eta] \\ \text{FLPM}_{\alpha,\tau}[\xi] \leq \text{FLPM}_{\alpha,\tau}[\eta] \end{cases}$ and $\begin{cases} E[\eta] \geq E[\xi] \\ \text{FLPM}_{\alpha,\tau}[\eta] \leq \text{FLPM}_{\alpha,\tau}[\xi] \end{cases}$, that leads to $\begin{cases} E[\xi] = E[\eta] \\ \text{FLPM}_{\alpha,\tau}[\xi] = \text{FLPM}_{\alpha,\tau}[\eta] \end{cases}$. Thus, $\xi \sim_{\alpha,\tau} \eta$.

   A_3) Let us assume that $\xi \succeq_{\alpha,\tau} \eta$ and $\eta \succeq_{\alpha,\tau} \theta$.

   We have: $\begin{cases} E[\xi] \geq E[\eta] \\ \text{FLPM}_{\alpha,\tau}[\xi] \leq \text{FLPM}_{\alpha,\tau}[\eta] \end{cases}$ and $\begin{cases} E[\eta] \geq E[\theta] \\ \text{FLPM}_{\alpha,\tau}[\eta] \leq \text{FLPM}_{\alpha,\tau}[\theta] \end{cases}$; by the transitivity of inequalities, that leads to $\begin{cases} E[\xi] \geq E[\theta] \\ \text{FLPM}_{\alpha,\tau}[\xi] \leq \text{FLPM}_{\alpha,\tau}[\theta] \end{cases}$.

   Hence, $\xi \succeq_{\alpha,\tau} \theta$.

   A_4) Let us assume that $\inf supp(\xi) > \sup supp(\eta)$, that is, $a > d'$. Necessarily, we have: $a > a'$, $b > b'$, $c > c'$ and $d > d'$. By the fact that, $E[\xi] = \frac{a+b+c+d}{4}$ and $E[\eta] = \frac{a'+b'+c'+d'}{4}$, we have $E[\xi] \geq E[\theta]$.

   On the other hand, we have: $\text{FLPM}_{\alpha,\tau}[\xi] = \int_{-\infty}^{\tau}(\tau-u)^{\alpha-1}\Phi_1(u)du$ and $\text{FLPM}_{\alpha,\tau}[\eta] = \int_{-\infty}^{\tau}(\tau-u)^{\alpha-1}\Phi_2(u)du$. By the fact that $a > a'$, $b > b'$, $c > c'$ and $d > d'$, we have $\Phi_1(r) \leq \Phi_2(r)$, $\forall r \in \mathbb{R}$ (according to theorem 7). That leads to $\text{FLPM}_{\alpha,\tau}[\xi] \leq \text{FLPM}_{\alpha,\tau}[\eta]$. As $E[\xi] \geq E[\eta]$ and $\text{FLPM}_{\alpha,\tau}[\xi] \leq \text{FLPM}_{\alpha,\tau}[\eta]$, we conclude that $\xi \succeq_{\alpha,\tau} \eta$.

   A_4') The proof is similar to the one of A_4.

   A_5) The proof is justified by the fact that, the comparison between ξ and η following $\succeq_{\alpha,\tau}$ depends only on the parameters of these two fuzzy variables (mean, fuzzy lower partial moment) and not on the other variables.

A_6) Let us justify by a counterexample that this property is not satisfied by the dominance relation $\succeq_{\alpha,\tau} \eta$.

Let us consider the trapezoidal fuzzy variables $\xi_1 = (1, 2, 8, 9)$, $\xi_2 = (2, 3, 3.5, 4)$ and $\xi_0 = (9, 10, 11, 12)$.

We set: $\xi'_1 = \xi_1 + \xi_0 = (10, 12, 19, 21)$, $\xi'_2 = \xi_2 + \xi_0 = (11, 13, 14.5, 16)$, $\alpha = 2$ and $\tau = 10.5$.

If $\Phi_1, \Phi_2, \Phi'_1, \Phi'_2$ are respectively the credibility distribution functions of fuzzy variables $\xi_1, \xi_2, \xi'_1, \xi'_2$, then we have:

$$\Phi_1(x) = \begin{cases} 0 \text{ if } x < 1 \\ \dfrac{x-1}{2} \text{ if } 1 \leq x < 2 \\ \frac{1}{2} \text{ if } 2 \leq x < 8 \\ \dfrac{x-7}{2} \text{ if } 1 \leq x < 2 \\ 1 \text{ if } x \geq 9 \end{cases}, \quad \Phi_2(x) = \begin{cases} 0 \text{ if } x < 2 \\ \dfrac{x-2}{2} \text{ if } 2 \leq x < 3 \\ \frac{1}{2} \text{ if } 3 \leq x < 3.5 \\ x-3 \text{ if } 3.5 \leq x < 4 \\ 1 \text{ if } x \geq 4 \end{cases},$$

$$\Phi'_1(x) = \begin{cases} 0 \text{ if } x < 10 \\ \dfrac{x-10}{4} \text{ if } 10 \leq x < 12 \\ \frac{1}{2} \text{ if } 12 \leq x < 19 \\ \dfrac{x-17}{4} \text{ if } 19 \leq x < 21 \\ 1 \text{ if } x \geq 21 \end{cases}, \quad \Phi'_2(x) = \begin{cases} 0 \text{ if } x < 11 \\ \dfrac{x-11}{4} \text{ if } 11 \leq x < 13 \\ \frac{1}{2} \text{ if } 13 \leq x < 14.5 \\ \dfrac{x-13}{3} \text{ if } 14.5 \leq x < 16 \\ 1 \text{ if } x \geq 16 \end{cases}.$$

Computations display that: $E[\xi_1] = 5$, $E[\xi_2] = 3.125$, that is $E[\xi_1] \geq E[\xi_2]$ and $\int_{-\infty}^{10.5} [\Phi_1(x) - \Phi_2(x)](10.5 - x)\, dx \simeq -6.13$ that is $\mathrm{FLPM}_{\alpha,\tau}[\xi_1] \leq \mathrm{FLPM}_{\alpha,\tau}[\xi_2]$. Therefore, $\xi_1 \succeq_{2,10.5} \xi_2$.

We also obtain: $E[\xi'_1] = 15.5$, $E[\xi'_2] = 13.625$, that is $E[\xi'_1] \geq E[\xi'_2]$ and $\int_{-\infty}^{10.5} [\Phi'_1(x) - \Phi'_2(x)](10.5 - x)\, dx \simeq 0.005$ that is $\mathrm{FLPM}_{\alpha,\tau}[\xi'_1] \geq \mathrm{FLPM}_{\alpha,\tau}[\xi'_2]$. Therefore, $\xi'_1 \not\succeq_{2,10.5} \xi'_2$.

Finally, we have: $\xi_1 \succeq_{2,10.5} \xi_2$ and $\xi'_1 \not\succeq_{2,10.5} \xi'_2$, that means that the property A_6 is not satisfied by the dominance relation $\succeq_{\alpha,\tau} \eta$.

A'_6) The previous counterexample indicates that property A'_6 is not satisfied by the dominance relation $\succeq_{\alpha,\tau} \eta$.

2. First order dominance $\succeq_1$:

   A_1) We have: $a \geq a$, $b \geq b$, $c \geq c$ and $d \geq d$. So, $\xi \succeq \xi$.

   A_2) Let us assume that $\xi \succeq_1 \eta$ and $\eta \succeq 1_1\xi$.

   We have:

   $\xi \succeq_1 \eta \Rightarrow (a \geq a', b \geq b', c \geq c', d \geq d')$ and $\eta \succeq_1 \xi \Rightarrow (a' \geq a, b' \geq b, c' \geq c, d' \geq d)$. We obtain: $a = a'$, $b = b'$, $c = c'$ and $d = d'$. That is, $\xi \sim_1 \eta$.

   A_3) Let us assume that $\xi \succeq_1 \eta$ and $\eta \succeq_1 \theta$.

   We have:

   $\xi \succeq_1 \eta \Rightarrow (a \geq a', b \geq b', c \geq c', d \geq d')$ and $\eta \succeq_1 \theta \Rightarrow (a' \geq a'', b' \geq b'', c' \geq c'', d' \geq d'')$.
   By using the transitivity of inequalities, it follows that: $a \geq a''$, $b \geq b''$, $c \geq c''$ and $d \geq d''$. That is, $\xi \succeq_1 \theta$.

   A_4) Let us assume that $\inf supp(\xi) > \sup supp(\eta)$, that is, $a > d'$. Necessarily, we have: $a > a'$, $b > b'$, $c > c'$ and $d > d'$. That is, $\xi \succeq_1 \eta$. A_4') The proof is similar to the one of A_4.

   A_5) The proof is justified by the fact that, the comparison between ξ and η following $\succeq_1$ depends only on the parameters of these two fuzzy variables $(a, b, c, d, a', b', c', d')$ and not on the other variables.

   A_6) Let us assume that $\xi \succeq_1 \eta$.

   We have: $a \geq a'$, $b \geq b'$, $c \geq c'$, $d \geq d'$. Those inequalities imply that: $a + a'' \geq a' + a''$, $b + b'' \geq b' + b''$, $c + c'' \geq c' + c''$, $d + d'' \geq d' + d''$. Thus, $\xi + \theta \succeq_1 \eta + \theta$.

   A_6') The proof is similar to the one of A_6.

3. Second order dominance $\succeq_2$:

A_1) We have: $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_1(r) - \Phi_1(r)]\ dr \geq 0$. Therefore, $\xi \succeq_2 \xi$.

A_2) Let us assume that $\xi \succeq_2 \eta$ and $\eta \succeq_2 \xi$.

We have:

$\xi \succeq_2 \eta \Rightarrow \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]\ dr \geq 0$ and $\eta \succeq_2 \xi \Rightarrow \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_1(r) - \Phi_2(r)]\ dr \geq 0$.

We obtain by those two inequalities: $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_1(r) - \Phi_2(r)]\ dr = 0$. That is, $\xi \sim_2 \eta$.

A_3) Let us assume that $\xi \succeq_2 \eta$ and $\eta \succeq_2 \theta$.

We have:

$\xi \succeq_2 \eta \Rightarrow \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]\ dr \geq 0$ and $\eta \succeq_2 \theta \Rightarrow \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_3(r) - \Phi_2(r)]\ dr \geq 0$.

We obtain by the transitivity of inequalities: $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_3(r) - \Phi_1(r)]\ dr \geq 0$. That is, $\xi \succeq_2 \theta$.

A_4) Let us assume that $\inf supp(\xi) > \sup supp(\eta)$, that is, $a > d'$. Necessarily, we have: $a > a'$, $b > b'$, $c > c'$ and $d > d'$. Those inequalities imply that: $\Phi_1(r) \leq \Phi_2(r)$, $\forall r \in \mathbb{R}$ (according to Theorem 7). That leads to $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]\ dr \geq 0$. This means, $\xi \geq \eta$.

A_5) The proof is justified by the fact that, the comparison between ξ and η following $\succeq_2$ depends only on parameters of these two fuzzy variables (credibility distribution function) and not on parameters of other variables.

A_6) Let us assume that $\xi \succeq_2 \eta$.

By the characterization of $\succeq_2$, we have:

For all crossing point $v \in \mathbb{R}$, between Φ and Φ', $\int_{-\infty}^{v} [\Phi_2(r) - \Phi_1(r)]\ dr \geq 0$ and $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]\ dr \geq 0$.

According to Lemma 2: $\forall u_v \in \mathbb{R}$, crossing point between Φ_1' and Φ_2', $\int_{-\infty}^{v}[\Phi_2(r) - \Phi_1(r)]\ dr = \int_{-\infty}^{u_v}[\Phi_2'(r) - \Phi_1'(r)]\ dr \geq 0$ and $\int_{-\infty}^{+\infty}[\Phi_2(r) - \Phi_1(r)]\ dr = \int_{-\infty}^{+\infty}[\Phi_2'(r) - \Phi_1'(r)]\ dr \geq 0$. Thus, $\xi + \theta \succeq_2 \eta + \theta$.

A_6') The proof is similar to the one of A_6. $\square$

Part II

Our scientific publications

TITLE OF THE FIRST PAPER: MOMENTS AND SEMI-MOMENTS FOR FUZZY PORTFOLIO SELECTION.

PUBLISHED IN 2012 IN "INSURANCE: MATHEMATICS AND ECONOMICS".

Insurance: Mathematics and Economics 51 (2012) 517–530

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Moments and semi-moments for fuzzy portfolio selection

Jules Sadefo Kamdem [a,*], Christian Tassak Deffo [b], Louis Aimé Fono [c]

[a] LAMETA CNRS UMR 5474 et Faculté d'Economie, Université de Montpellier 1, France

[b] Laboratoire de Mathématiques Appliquées aux Sciences Sociales, Département de Mathématiques - Faculté des Sciences, University of Yaoundé I, Cameroon

[c] Département de Mathématiques et Informatique, Faculté des Sciences, Université de Douala, B.P. 24157 Douala, Cameroon

ARTICLE INFO

Article history:
Received January 2012
Received in revised form July 2012
Accepted 10 July 2012

ABSTRACT

The aim of this paper is to consider the moments and the semi-moments for credibilistic portfolio selection with fuzzy risk factors (for example trapezoidal risk factors). In order to measure the leptokurtocity of credibilistic portfolio return, notions of moments (for example Kurtosis) and semi-moments (for example Semi-kurtosis) for credibilistic portfolios are originally introduced in this paper, and their mathematical properties are studied. As an extension of the mean–variance–skewness model for credibilistic portfolio, the mean–variance–skewness–semi-kurtosis is presented and its four corresponding variants are also considered. We display numerical examples for our optimization models.

1. Introduction

An important area of finance research is portfolio selection which is to select a combination of assets under the constraints of the investor objectives. In classical portfolio theory security returns were assumed to be random variables, and the portfolio selection problem is handled by means of probabilistic theory (see Bachelier, 1900 and Markowitz, 1952). One core of portfolio selection is to find a quantitative risk definition of portfolio investment. Therefore, allocation capital in different risky assets to minimize risk and to maximize returns is the main concern of portfolio selection. Before the seminal work of Markowitz (1952), there were no measurable terms for risk. The mean–variance model, proposed by Markowitz (1952), opened the door for mathematical analysis of the portfolio selection problem by considering the trade-off between return and risk. As in Markowitz (1959), variance has been widely accepted as a risk measure by numerous portfolio selection models. However variance as a risk measure has some shortcomings and limitations (see Markowitz, 1959).

One important shortcoming is that analysis based on variance considers high returns as equally undesirable low returns (i.e. it does not take into account the asymmetry of the probability distribution). Then there is a controversy over the issue of whether higher moments should be considered in portfolio selection models. Some authors, such as Samuelson (1970), Kraus and Litzenberger (1976), Konno and Suzuki (1995), Konno et al. (1993), and Briec et al. (2007), have argued that it is important to take into account higher moments than the first and second ones. For instance Samuelson (1970) showed that investors would prefer a portfolio with a larger third order moment if the first and second moments are the same. The above literature assumed that the securities returns are random variables with fixed expected returns and variance values.

However randomness is not the only type of uncertainty in reality, especially when taking into account human factors. The security returns are sensitive to economic, environmental, political, social and people's psychological factors. Thus investors receive efficient or inefficient information from the real world and, ambiguous factors usually exist in it. As discussed in Hasuike et al. (2009), investors can make use of a fuzzy set to reflect the vagueness and ambiguity of securities (i.e. incompleteness of information due to the lack of data). In such situations, it may be convenient to express these uncertainties using various imprecise linguistic expressions: for example, investors could evaluate portfolio returns as follows: (i) the return is about b, but definitely not less than a and not greater than b, (ii) the return is most likely in the interval $[a, b]$ and similar. Therefore scholars have recognized that the Probability Theory could not be used in this context, but we can use Fuzzy Set Theory in Portfolio Selection Problems. And the security return is considered as a fuzzy variable.

In addition to the probabilistic portfolio approach, they are two approaches.

In one approach, some scholars have proposed the use of imprecise probability, possibility, etc., to deal with uncertainty in

* Corresponding author. Tel.: +33 4 34 43 25 11; fax: +33 4 34 43 24 61.
E-mail addresses: sadefo@yahoo.fr, sadefo@lameta.univ-montp1.fr (J. Sadefo Kamdem).

0167-6687/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2012.07.003

518 J. Sadefo Kamdem et al. / Insurance: Mathematics and Economics 51 (2012) 517–530

portfolio selection since the 1990's. For example, some authors such as Tanaka and Guo (1999) quantified mean and variance of a portfolio through fuzzy probability and possibility distributions, Carlsson et al. (2001, 2002) used their own definitions of mean and variance of fuzzy numbers.

Although, early researchers used possibility measure, it does not have the self-duality property which is very important and it is absolutely needed in application research. The self-duality property helps to make decision results consistent with the laws of contradiction and excluded middle. In fact, when the investors know the possibility level of a portfolio reaching a target return, they cannot know the possibility level of the opposite event, i.e., the event of this portfolio not being able to achieve the target return! This will confuse and worry the decision maker. Therefore, this approach will not be considered in this paper.

In the other approach we consider, Liu (2002) introduced a credibility measure in the new theory of uncertainty. It is a self-dual measure which has been applied in many application areas. Therefore, scholars such as Liu and Liu (2002), Li et al. (2010), Kar et al. (2011) and Huang (2010) used that measure to analyze portfolio selection with fuzzy returns. In that view, Li et al. (2010) and Huang (2008) quantified portfolio returns and risk by the expected value and variance based on credibility measure. More precisely, they introduced notions of mean, variance, semi-variance and skewness of a given fuzzy variable, determined some of their properties and applied those theoretical results in finance for a portfolio selection with fuzzy returns. They deduced the two following models and its variants: the mean–semi-variance model, then the mean–variance–skewness model.

Up to now, only the three first moments and/or semi-moments were examined and applied in fuzzy finance. Higher moments of fuzzy variables and their applications are not yet examined. Moreover, several empirical studies show that portfolio returns have fat tails. Generally, investors would prefer a portfolio return with smaller *kurtosis* which indicates the leptokurtosis (fat-tails or thin-tails) when the mean value, the variance and the asymmetry are the same.

In the continuation of the first research works, the goal of this paper is to contribute to a sound formal foundation of statistics and finance built upon the Fuzzy Set Theory. More precisely, (i) we introduce notions of k-moments (for example kurtosis for $k = 4$) and semi-moments (for example semi-kurtosis for $k = 4$), (ii) we determine their mathematical properties, (iii) we use them for portfolio selection with fuzzy risk factors and, (iv) we display some numerical examples for our optimization model on a family of independent triangular fuzzy returns.

The paper is organized as follows. In Section 2, we review some preliminary knowledge on fuzzy variables and credibility measure. We also recall the three first moments of a fuzzy variable namely mean, variance, and skewness of a fuzzy variable. In Section 3, we determine, for an integer $k > 1$, the k-moment of a symmetric trapezoidal fuzzy variable. We introduce kurtosis for fuzzy variables, study some of its properties and, we compute kurtosis of trapezoidal fuzzy numbers and triangular fuzzy numbers. In Section 4, we introduce the notion of semi-moment of order $n = 2p$ ($p \in \mathbb{N}^*$) of a fuzzy variable. We deduce that the particular cases of the semi-moment are the known notion of semi-variance and the new notion of semi-kurtosis for $p = 1$ and $p = 2$ respectively. We compute the semi-variance and the semi-kurtosis of a trapezoidal fuzzy variable. We establish some useful links between moments and semi-moments of fuzzy variables. Section 5 suggests some deterministic optimization programs with a family of independent triangular fuzzy numbers and, propose some numerical examples on mean–variance–skewness–semi-kurtosis program and mean–variance–skewness–kurtosis program with a family of seven triangular fuzzy returns introduced and used by Huang (2008) and, used by Li et al. (2010) in their models. Section 6 contains some concluding remarks and the proofs are in Section 7.

2. Preliminaries

2.1. On fuzzy variables and credibility measure

Let ξ be a fuzzy variable with membership function μ. For any $x \in \mathbb{R}$, $\mu(x)$ represents the possibility that ξ takes value x. For any set B, Liu and Liu (2002) defined the credibility measure as the average of possibility measure and necessity measure as follows:

$$\mathrm{Cr}(\{\xi \in B\}) = \frac{1}{2}\left(\sup_{x\in B}\mu(x) - \sup_{x\in B^c}\mu(x) + 1\right). \tag{1}$$

It is easy to show that the credibility measure is self-dual, that is, $\mathrm{Cr}(\{\xi \in B\}) + \mathrm{Cr}(\{\xi \in B^c\}) = 1$.

Remark 1. Note that for ξ taking values in B, Zadeh (1978) has defined the possibility measure of B by

$$\mathrm{Pos}(\{\xi \in B\}) = \sup_{x\in B}\mu(x)$$

and the necessity measure of ξ by

$$\mathrm{Nec}(\{\xi \in B\}) = 1 - \sup_{x\in B^c}\mu(x).$$

But neither, of these measures are self-dual. That reason also justified the introduction of the credibility measure by Liu (2002).

Example 1. 1. Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy number (with $a \le b \le c \le d$). For any $r \in \mathbb{R}$,

$$\mathrm{Cr}(\{\xi \le r\}) = \begin{cases} 0 & \text{if } r < a \\ \dfrac{1}{2}\left(\dfrac{r-a}{b-a}\right) & \text{if } a \le r < b \\ \dfrac{1}{2} & \text{if } b \le r < c \\ 1 - \dfrac{1}{2}\left(\dfrac{r-d}{c-d}\right) & \text{if } c \le r < d \\ 1 & \text{if } d \le r \end{cases}$$

and

$$\mathrm{Cr}(\{\xi \ge r\}) = \begin{cases} 1 & \text{if } r < a \\ 1 - \dfrac{1}{2}\left(\dfrac{r-a}{b-a}\right) & \text{if } a \le r < b \\ \dfrac{1}{2} & \text{if } b \le r < c \\ \dfrac{1}{2}\left(\dfrac{r-d}{c-d}\right) & \text{if } c \le r < d \\ 0 & \text{if } d \le r. \end{cases}$$

2. Let $\xi = (a, b, c)$ be a triangular fuzzy number (with $a \le b \le c$). For all $r \in \mathbb{R}$,

$$\mathrm{Cr}(\{\xi \le r\}) = \begin{cases} 0 & \text{if } r < a \\ \dfrac{1}{2}\left(\dfrac{r-a}{b-a}\right) & \text{if } a \le r < b \\ 1 - \dfrac{1}{2}\left(\dfrac{r-c}{b-c}\right) & \text{if } b \le r < c \\ 1 & \text{if } c \le r \end{cases}$$

and

$$\mathrm{Cr}(\{\xi \geq r\}) = \begin{cases} 1 & \text{if } r < a \\ 1 - \frac{1}{2}\left(\frac{r-a}{b-a}\right) & \text{if } a \leq r < b \\ \frac{1}{2}\left(\frac{r-c}{b-c}\right) & \text{if } b \leq r < c \\ 0 & \text{if } c \leq r. \end{cases}$$

Let us end this section by giving some notations useful throughout this paper.

- For a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ such that $a \neq b$ and $c \neq d$, $\mathrm{supp}(\xi) = [a, d]$ its support, $\mathrm{cor}(\xi) = [b, c]$ its core, l_s the length of $\mathrm{supp}(\xi)$ and l_c the length of $\mathrm{cor}(\xi)$. We set:

  $\alpha = b - a, \quad \beta = d - c, \quad l_s(\xi) = d - a$ and $l_c(\xi) = c - b.$
- For a triangular fuzzy variable $\xi = (a, b, c)$ such that $b \neq a$ and $c \neq a$, we set:

  $\alpha_1 = \max\{b - a, c - b\}$ and $\gamma = \min\{b - a, c - b\}.$
- $\xi = (a, b, c, d)$ is symmetric (that is $\exists t \in \mathbb{R}$, $\forall r \in \mathbb{R}$, $\mu(t-r) = \mu(t+r)$) if $\alpha = \beta$, and $\xi = (a, b, c)$ is symmetric if $\alpha_1 = \gamma$.

2.2. On the three first moments of fuzzy variables: expected value, variance and skewness

The definitions of the expected value, variance and skewness of fuzzy variables are obtained from Li et al. (2010).

Definition 1. Let ξ be a fuzzy variable. Then its expected value is defined as

$$E[\xi] = e = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\, dr - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\, dr \qquad (2)$$

provided that at least one of the above integrals is finite.

Remark 2. Note that, expected value is one of the most important concepts of a fuzzy variable, which gives the center of its distribution.

Example 2. The expected value of a trapezoidal fuzzy variable denoted $\xi = (a, b, c, d)$ is given by $E[\xi] = \frac{a+b+c+d}{4}$ and the expected value of a triangular fuzzy variable denoted $\xi = (a, b, c)$ is given by $E[\xi] = \frac{a+2b+c}{4}$.

Definition 2. Let ξ be a fuzzy variable with finite expected value e. Then its variance is defined as

$$V[\xi] = E[(\xi - e)^2]. \qquad (3)$$

Let us determine the variance of a trapezoidal fuzzy variable and that of a triangular fuzzy variable.

Example 3. 1. Let $\xi = (a, b, c, d)$ be a fuzzy trapezoidal variable with expected value $E[\xi] = \frac{a+b+c+d}{4} = e$. The variance $V[\xi]$ of ξ is given by:

$$V[\xi] = -\left[\frac{1}{4}(l_s(\xi) + l_c(\xi))\right]^3 \left(\frac{|\alpha - \beta|}{3\alpha\beta}\right) + \max\left(\frac{\left(\frac{|\alpha-\beta|}{4} - \frac{1}{2}l_c(\xi)\right)^3}{6\alpha \vee \beta}, 0\right) + \frac{|\alpha - \beta|}{2\alpha\beta}\left[\frac{1}{2}l_s(\xi) - \frac{(\alpha+\beta)}{4}\right]\left[\frac{1}{4}(l_s(\xi) + l_c(\xi))\right]^2 + \frac{\left(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_s(\xi)\right)^3}{6\alpha \vee \beta} - \frac{\left(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_c(\xi)\right)^3}{6\alpha \wedge \beta}.$$

2. We can easily check that if ξ is symmetric ($\alpha = \beta$), $V[\xi]$ simply becomes

$$V[\xi] = \frac{3[l_c(\xi) + \beta]^2 + \beta^2}{24}.$$

3. Let $\xi = (a, b, c)$ be a triangular fuzzy variable such that $E[\xi] = \frac{a+2b+c}{4} = e$.
The variance $V[\xi]$ of ξ can be deduced from the variance of a trapezoidal one by this way:

$$V[\xi] = \frac{33\alpha_1^3 + 21\alpha_1^2\gamma + 11\alpha_1\gamma^2 - \gamma^3}{384\alpha_1}.$$

4. More precisely:
   - The variances of the following three trapezoidal fuzzy variables are:
   $V[(-1, 2, 3, 4)] = \frac{41}{24}$, $\quad V[(1, 2, 3, 4)] = \frac{13}{24}$ and $V[(-1, 0, 1, 4)] = \frac{41}{24}$.
   - The variances of the following three triangular fuzzy variables are:
   $V[(-1, 0, 4)] = \frac{2491}{1536}$ and $V[(-1, 1, 2)] = V[(1, 2, 4)] = \frac{123}{256}$.

Remark 3. Note that, the variance of a fuzzy variable provides a degree of the spread of the distribution around its expected value. A small value of the variance indicates that the fuzzy variable is tightly concentrated around its expected value; and a large value of the variance indicates that the fuzzy variable has a wide spread around its expected value.

In finance, the investor use the variance to make a suitable choice among those of fuzzy variables that describe the same expected security's return. We can illustrate it in what follows: let $V_1 = (3, 4, 6, 7)$ and $V_2 = (2, 4, 6, 8)$ be two trapezoidal fuzzy variables with the same core [4, 6], the same mean 5, i.e., V_1 and V_2 describe the same return that we can name: "between 4 and 6" as depicted in Fig. 1.

Surely, a rational investor may have this question in his mind: "Which of the two variables best describe the previous return"?. An answer to that question is: V_1 is better than V_2 in describing the return "between 4 and 6".

The justification we can give is that the spread of V_1 (variance of $V_1 = 1.16$) is less than the one of V_2 (variance of $V_2 = 2.16$). The lower the variance is, the more it reduces uncertainty and allows the investor to make a good description of his portfolio's returns.

Let us end this subsection with some useful preliminaries on the skewness of a fuzzy variable.

Definition 3 (*See Li et al., 2010*). Let ξ be a fuzzy variable with finite expected value e. Then its skewness is defined as

$$Sk(\xi) = E[(\xi - e)^3]. \qquad (4)$$

Remark 4 (*See Li et al., 2010*). If ξ has a symmetric membership, then $Sk[\xi] = 0$.

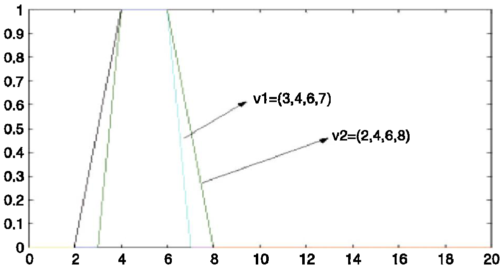

Fig. 1. Trapezoidal fuzzy numbers with the same core.

In the following example, we determine the skewness of a trapezoidal and triangular fuzzy variable respectively.

Example 4. 1. The skewness of a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is given by

$$Sk[\xi] = \frac{1}{8(b-a)}\left[\left(\frac{b-e}{4}\right)^4 - \left(\frac{a-e}{4}\right)^4\right] + \frac{1}{8(c-d)}\left[\left(\frac{c-e}{4}\right)^4 - \left(\frac{d-e}{4}\right)^4\right].$$

2. The skewness of a triangular fuzzy variable $\xi = (a, b, c)$ is given by

$$Sk[\xi] = \frac{1}{8(b-a)}\left[\left(\frac{b-e}{4}\right)^4 - \left(\frac{a-e}{4}\right)^4\right] + \frac{1}{8(b-c)}\left[\left(\frac{b-e}{4}\right)^4 - \left(\frac{c-e}{4}\right)^4\right],$$

that is,

$$Sk[\xi] = \frac{(c-a)^2}{32}(c + a - 2b).$$

Example 5. Let $\xi_1 = (2, \frac{23+\sqrt{73}}{4}, \frac{19+\sqrt{73}}{2})$ and $\xi_2 = (4, 5, \frac{13+\sqrt{73}}{2}, \frac{15+\sqrt{73}}{2})$ be two fuzzy variables.

We have $E(\xi_1) = E(\xi_2) = \frac{46+\sqrt{73}}{4}$, $V(\xi_1) = V(\xi_2) = \frac{298+30\sqrt{73}}{96}$ and $SK(\xi_1) = SK(\xi_2) = 0$.

The previous example justifies that if we have two portfolios where returns are ξ_1 and ξ_2, then the investor cannot discriminate since their three first moments are the same. Thereby, it is important to introduce and analyze k-moments and k-semi-moments for a fuzzy variable.

3. Moments of fuzzy variables

In the following subsection, we determine, for an integer $k > 1$, the k-moment of a symmetric trapezoidal fuzzy variable.

3.1. k-moment of a symmetric trapezoidal fuzzy variable

Proposition 1. *Let $\xi = (a, b, c, d)$ be a symmetric trapezoidal fuzzy variable with expected value $E[\xi] = e$. For an integer $k > 1$, the k-moment $m_k = E[(\xi - e)^k]$ is given by:*

$$m_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{\sum_{i=0}^{\frac{k}{2}} C_{k+1}^{2i+1}[(c-b)+\alpha]^{k-2i}}{2^{k+1}(k+1)} & \text{if } k \text{ is even.} \end{cases}$$

Corollary 1. *Let $\xi = (a, b, c)$ be a symmetric triangular fuzzy variable with expected value $E[\xi] = e$. For an integer $k \geq 1$, the k-moment $m_k = E[(\xi - e)^k]$ is given by:*

- *If $k = 2p$, then*

$$m_{2p} = m_k = 0. \tag{5}$$

- *If $k = 2p + 1$, then*

$$m_{2p+1} = \alpha^2 \frac{p}{p+1} m_{2p-1} \quad \text{that is,} \quad m_{2p+1} = \frac{\alpha^k}{2k+2}. \tag{6}$$

3.2. Kurtosis: Definitions, first properties and some particular cases

In this section, we introduce the kurtosis of a fuzzy variable. We study its properties and give some examples.

Definition 4. Let ξ be a fuzzy variable such that $E[\xi] = e < \infty$.

1. The kurtosis of ξ, denoted $K[\xi]$, is given by:
$K[\xi] = E[(\xi - e)^4]$.
2. The normalized kurtosis of ξ, denoted $K^1[\xi]$, is given by:
$$K_1[\xi] = \frac{E[(\xi - e)^4]}{(\sigma[\xi])^4}.$$

Let us rewrite $K[\xi]$ and $K_1[\xi]$ by means of a credibility measure. Let ξ be a fuzzy variable such that $E[\xi] = e < \infty$.

- The kurtosis $K[\xi]$ is given by:

$$K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\}\, dr. \tag{7}$$

- The normalized kurtosis $K_1[\xi]$ is given by:

$$K_1[\xi] = \frac{\int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\} dr}{\left[\int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^2 \geq r\} dr\right]^2}. \tag{8}$$

Example 6. Let ξ_1 and ξ_2 be the two fuzzy variables of Example 5.

We have $K[\xi_1] = \frac{1}{10}(\frac{15+\sqrt{73}}{4})^4 \simeq 120.027$ and $K[\xi_2] = \frac{5}{160}(\frac{5+\sqrt{73}}{2})^4 + \frac{1}{16}(\frac{5+\sqrt{73}}{2})^4 + \frac{1}{160} \simeq 68.6$.

The following result establishes some properties on the linearity of the kurtosis.

Proposition 2. *Let ξ be a fuzzy variable such that $E[\xi] = e$.*

1. *The kurtosis of ξ is defined by*

$$K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[4]{r}\} dr. \tag{9}$$

2. *The normalized kurtosis of ξ is defined by*

$$K^1[\xi] = \frac{\int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[4]{r}\} dr}{\left[\int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[2]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[2]{r}\} dr\right]^2}. \tag{10}$$

3. $\forall a, b \in \mathbb{R}$, $K[a\xi + b] = a^4 K[\xi]$.
4. $\forall a, b \in \mathbb{R}$, $K^1[a\xi + b] = K^1[\xi]$.

When ξ becomes a symmetric fuzzy variable, then the previous formulas become

Corollary 2. *If ξ is a symmetric fuzzy variable, then*

1. (9) *becomes*

$$K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\}dr. \tag{11}$$

2. (10) *becomes*

$$K^1[\xi] = \frac{\int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\}dr}{\left[\int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[2]{r}\}dr\right]^2}. \tag{12}$$

Let us end this section with the following proposition which determines the kurtosis of trapezoidal and triangular fuzzy variables.

Proposition 3. *Let $\xi = (a, b, c, d)$ be a fuzzy trapezoidal variable with expected value $E[\xi] = e$.*

1. *The kurtosis $K[\xi]$ of ξ is given by:*

$$\begin{aligned} K[\xi] = {} & -\left[\frac{1}{4}(l_s(\xi) + l_c(\xi))\right]^5 \left(\frac{|\alpha - \beta|}{5\alpha\beta}\right) \\ & + \max\left(\frac{\left(\frac{|\alpha-\beta|}{4} - \frac{1}{2}l_c(\xi)\right)^5}{10\alpha \vee \beta}, 0\right) \\ & + \frac{\left(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_s(\xi)\right)^5}{10\alpha \vee \beta} \frac{|\alpha - \beta|}{2\alpha\beta}\left[\frac{1}{2}l_s(\xi) - \frac{(\alpha + \beta)}{4}\right] \\ & \times \left[\frac{1}{4}(l_s(\xi) + l_c(\xi))\right]^4 - \frac{\left(\frac{|\alpha-\beta|}{4} + \frac{1}{2}l_c(\xi)\right)^5}{10\alpha \wedge \beta}. \end{aligned}$$

2. *If $\xi = (a, b, c, d)$ is symmetric, then*
   - *the previous expression of $K[\xi]$ becomes:*

   $$K[\xi] = \frac{5[l_c(\xi) + \beta]^4 + 10\beta^2[l_c(\xi) + \beta]^2 + \beta^4}{160} \tag{13}$$

   - *its normalized kurtosis $K_1[\xi]$ is*

   $$K_1[\xi] = \frac{5[l_c(\xi) + \beta]^4 + 10\beta^2[l_c(\xi) + \beta]^2 + \beta^4}{160\left[\frac{3[l_c(\xi)+\beta]^2+\beta^2}{24}\right]^2}.$$

3. *Let $\xi = (a, b, c)$ be a triangular fuzzy variable such that $E[\xi] = \frac{a+2b+c}{4} = e$.*
   The kurtosis $K[\xi]$ of ξ can be deduced from the kurtosis of a trapezoidal one by this way:

$$K[\xi] = \frac{253\alpha_1^5 + 395\alpha_1^4\gamma + 17\alpha_1\gamma^4 + 290\alpha_1^3\gamma^2 + 70\alpha_1^2\gamma^3 - \gamma^5}{10.240\alpha_1}.$$

We deduce from the previous formulae that the normalized kurtosis of some examples of trapezoidal fuzzy variables are:

$$K^1[(-1, 2, 3, 4)] = \frac{27\,414}{8405}, \qquad K^1[(1, 2, 3, 4)] = \frac{2178}{845},$$
$$K^1[(-2, -1, 3, 4)] = \frac{3798}{1805} \quad \text{and} \quad K^1[(1, 2, 2, 4)] = \frac{90\,928}{25\,215}.$$

We notice that: for $\xi = (a, b, c)$ a triangular fuzzy number, we have:

- if $b = a$, then $K[\xi] = \frac{253}{10.240}\gamma^4$ with $E[\xi] = \frac{3b+c}{4}$.
- if $b = c$, then $K[\xi] = \frac{253}{10.240}\alpha^4$ with $E[\xi] = \frac{a+3b}{4}$.

3.3. *First four moments of portfolio*

Example 7. Let $(\xi_i = (a_i, b_i, c_i, d_i))_{i=1,2,\ldots,n}$ be a family of n independent trapezoidal fuzzy variables and $x = (x_1, \ldots, x_n)$ a family of n positive reals. The expected value of

$$\xi(x) = \sum_{i=1}^n x_i\xi_i = \left(\sum_{i=1}^n x_i a_i, \sum_{i=1}^n x_i b_i, \sum_{i=1}^n x_i c_i, \sum_{i=1}^n x_i d_i\right) \tag{14}$$

is a fuzzy variable and its expectation is

$$e(x) = E[\xi(x)] = \frac{1}{4}\sum_{i=1}^n (a_i + b_i + c_i + d_i)\, x_i. \tag{15}$$

Proposition 4. *Let $\xi(x)$ be the portfolio return defined by (14).*

- *The variance of $\xi(x)$ is (see Box I).*
- *The skewness of $\xi(x)$ is*

$$\begin{aligned} Sk[\xi(x)] = {} & \frac{1}{8\sum_{k=1}^n x_k(b_k - a_k)} \left[\left(\frac{\sum_{k=1}^n x_k(b_k - e_k)}{4}\right)^4 \right. \\ & \left. - \left(\frac{\sum_{k=1}^n x_k(a_k - e_k)}{4}\right)^4\right] + \frac{1}{8\sum_{k=1}^n x_k(c_k - d_k)} \\ & \times \left[\left(\frac{\sum_{k=1}^n x_k(c_k - e_k)}{4}\right)^4 - \left(\frac{\sum_{k=1}^n x_k(d_k - e_k)}{4}\right)^4\right]. \end{aligned}$$

- *The kurtosis of $\xi(x)$ is (see Box II).*

Corollary 3. *Let $(\xi_i = (a_i, b_i, c_i))_{i=1,2,\ldots,n}$ be a family of n independent triangular fuzzy variables, $x = (x_1, \ldots, x_n)$ a family of n positive reals, and $\xi(x) = \sum_{i=1}^n x_i\xi_i$ be the portfolio return.*
Then

1. *The mean of $\xi(x)$ is:*

$$E[\xi(x)] = \frac{1}{4}\sum_{i=1}^n x_i(a_i + 2b_i + c_i).$$

2. *The variance of $\xi(x)$ is:*

$$\begin{aligned} V[\xi(x)] = {} & -\frac{1}{192\sum_{k=1}^n\sum_{l=1}^n x_k x_l \alpha_k \beta_l}\left[\sum_{k=1}^n x_k(l_s(\xi_k))\right]^3 \\ & \times \left|\sum_{k=1}^n x_k(\alpha_k - \beta_k)\right| + \left(\frac{1}{32\sum_{k=1}^n\sum_{l=1}^n x_k x_l \alpha_k \beta_l}\right. \\ & \times \left.\left[\sum_{k=1}^n x_k(l_s(\xi_k))\right]^2 \left|\sum_{k=1}^n x_k(\alpha_k - \beta_k)\right|\right) \\ & \times \left(\left[\frac{1}{4}\sum_{k=1}^n x_k(2l_s(\xi_k) - (\alpha_k + \beta_k))\right]\right) \end{aligned}$$

$$
\begin{aligned}
V[\xi] = &-\frac{1}{192\sum_{k=1}^{n}\sum_{l=1}^{n}x_kx_l\alpha_k\beta_l}\left[\sum_{k=1}^{n}x_k(l_s(\xi_k)+l_c(\xi_k))\right]^3\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right| \\
&+\left(\frac{1}{32\sum_{k=1}^{n}\sum_{l=1}^{n}x_kx_l\alpha_k\beta_l}\left[\sum_{k=1}^{n}x_k(l_s(\xi_k)+l_c(\xi_k))\right]^2\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|\right)\left(\left[\frac{1}{4}\sum_{k=1}^{n}x_k(2l_s(\xi_k)-(\alpha_k+\beta_k))\right]\right) \\
&+\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}+\frac{1}{2}\sum_{k=1}^{n}x_kl_s(\xi_k)\right)^3}{3\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}-\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}+\frac{1}{2}\sum_{k=1}^{n}x_kl_c(\xi_k)\right)^3}{3\sum_{k=1}^{n}x_k(\alpha_k+\beta_k-|\alpha_k-\beta_k|)} \\
&+\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}-\frac{1}{2}\sum_{k=1}^{n}x_kl_c(\xi_k)\right)^3+\left|\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}-\frac{1}{2}\sum_{k=1}^{n}x_kl_c(\xi_k)\right)^3\right|}{6\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}
\end{aligned}
$$

Box I.

$$
\begin{aligned}
K[\xi(x)] = &-\frac{1}{5120\sum_{k=1}^{n}\sum_{l=1}^{n}x_kx_l\alpha_k\beta_l}\left[\sum_{k=1}^{n}x_k(l_s(\xi_k)+l_c(\xi_k))\right]^5\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right| \\
&+\left(\frac{1}{512\sum_{k=1}^{n}\sum_{l=1}^{n}x_kx_l\alpha_k\beta_l}\left[\sum_{k=1}^{n}x_k(l_s(\xi_k)+l_c(\xi_k))\right]^4\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|\right)\left(\left[\frac{1}{4}\sum_{k=1}^{n}x_k(2l_s(\xi_k)-(\alpha_k+\beta_k))\right]\right) \\
&+\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}+\frac{1}{2}\sum_{k=1}^{n}x_kl_s(\xi_k)\right)^5}{5\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}-\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}+\frac{1}{2}\sum_{k=1}^{n}x_kl_c(\xi_k)\right)^5}{5\sum_{k=1}^{n}x_k(\alpha_k+\beta_k-|\alpha_k-\beta_k|)} \\
&+\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}-\frac{1}{2}\sum_{k=1}^{n}x_kl_c(\xi_k)\right)^5+\left|\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}-\frac{1}{2}\sum_{k=1}^{n}x_kl_c(\xi_k)\right)^5\right|}{10\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}
\end{aligned}
$$

Box II.

$$
+\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}+\frac{1}{2}\sum_{k=1}^{n}x_kl_s(\xi_k)\right)^3}{3\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}
-\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}\right)^3}{3\sum_{k=1}^{n}x_k(\alpha_k+\beta_k-|\alpha_k-\beta_k|)}
$$

$$
+\frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}\right)^3+\left|\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}\right|^3}{6\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}.
$$

3. *The skewness of $\xi(x)$ is:*

$$
SK[\xi(x)] = \frac{1}{32}\left(\sum_{i=1}^{n}x_i(c_i-a_i)\right)^2 . \sum_{i=1}^{n}x_i(c_i-2b_i+a_i).
$$

4. *The kurtosis of $\xi(x)$ is:*

$$
\begin{aligned}
K[\xi(x)] = &-\frac{1}{5120\sum_{k=1}^{n}\sum_{l=1}^{n}x_k x_l \alpha_k \beta_l}\left[\sum_{k=1}^{n}x_k(l_s(\xi_k))\right]^5 \\
&\times \left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right| + \Bigg(\frac{1}{512\sum_{k=1}^{n}\sum_{l=1}^{n}x_k x_l \alpha_k \beta_l} \\
&\times \left[\sum_{k=1}^{n}x_k(l_s(\xi_k))\right]^4 \left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|\Bigg) \\
&\times \left(\left[\frac{1}{4}\sum_{k=1}^{n}x_k(2l_s(\xi_k)-(\alpha_k+\beta_k))\right]\right) \\
&+ \frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}+\frac{1}{2}\sum_{k=1}^{n}x_k l_s(\xi_k)\right)^5}{5\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)} \\
&- \frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}\right)^5}{5\sum_{k=1}^{n}x_k(\alpha_k+\beta_k-|\alpha_k-\beta_k|)} \\
&+ \frac{\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}\right)^5+\left|\left(\frac{\left|\sum_{k=1}^{n}x_k(\alpha_k-\beta_k)\right|}{4}\right)^5\right|}{10\sum_{k=1}^{n}x_k(\alpha_k+\beta_k+|\alpha_k-\beta_k|)}.
\end{aligned}
$$

4. Semi-moment of fuzzy variables

Let ξ be a fuzzy variable with finite expected value e. We define the variable $(\xi - e)^-$ as follows:

$$(\xi-e)^- = \begin{cases} \xi - e & \text{if } \xi \le e \\ 0 & \text{if } \xi > e. \end{cases} \tag{16}$$

4.1. Definitions

Definition 5. Let $p \in \mathbb{N}^*$.

1. The semi-moment of order $n = 2p$ is
$$\begin{aligned} M^S_{2p}[\xi] &= M^S_n[\xi] = E[[(\xi-e)^-]^{2p}] \\ &= \int_0^{+\infty} \mathrm{Cr}\{[(\xi-e)^-]^{2p} \ge r\}dr. \end{aligned} \tag{17}$$
2. The normalized semi-moment of ξ is defined by:
$$M^{S,1}_{2p}[\xi] = \frac{M^S_{2p}[\xi]}{(M^S_2[\xi])^p}.$$

In the case where $p = 1$, we obtain the well-known semi-variance of ξ which is interpreted and described as follows.

Remark 5. The variance of ξ is used to measure the spread of its distribution about $e = E[\xi]$. Note that, variance concerns not only the part "ξ is less than e", but also the part "ξ is greater than e". If we are only interested with the first part, then we should use the concept of semi-variance.

Definition 6. Let ξ be a fuzzy variable with expected value e. The semi-variance of ξ is defined as

$$V^S[\xi] = E[[(\xi-e)^-]^2] = \int_0^{+\infty} \mathrm{Cr}\{[(\xi-e)^-]^2 \ge r\}dr. \tag{18}$$

For the example of semi-variance of triangular and trapezoidal fuzzy variables, we have the following example:

Example 8. 1. The semi-variance of a trapezoidal fuzzy number $\xi = (a, b, c, d)$ (where $a, b, c, d \in \mathbb{R}$ such that $a \ne b$ and $c \ne d$) with expected value $e = \frac{a+b+c+d}{4}$ is given by:
$$\begin{aligned} V^S[\xi] = &\frac{1}{6(b-a)}[(e-a)^3 + \min(0,(b-e)^3)] \\ &+ \frac{1}{6(d-c)}\max(0,(e-c)^3). \end{aligned}$$
2. The semi-variance of a triangular fuzzy number $\xi = (a, b, c)$ with expected value $e = \frac{a+2b+c}{4}$ is deduced from the semi-variance of a trapezoidal one by this way:
$$\begin{aligned} &V^S[\xi] \\ &= \frac{1}{6(b-a)}\left[(e-a)^3 + \frac{1}{(b-c)}(b-e)^3\min(0,(b-e))\right]. \end{aligned}$$

In the following subsection, we focus on the semi-kurtosis (i.e. $p = 2$ in (17)).

4.2. Semi-kurtosis: Definitions and examples

Definition 7. Let ξ be a fuzzy variable with finite expected value e. Then the semi-kurtosis of ξ is defined

$$K^S[\xi] = E[[(\xi-e)^-]^4] = \int_0^{+\infty} \mathrm{Cr}\{[(\xi-e)^-]^4 \ge r\}dr. \tag{19}$$

Let us give the semi-kurtosis of a trapezoidal fuzzy number and a triangular fuzzy number.

Example 9. 1. The semi-kurtosis of a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ with expected value $e = \frac{a+b+c+d}{4}$ is given by:
$$\begin{aligned} K^S[\xi] = &\frac{1}{10(b-a)}[(e-a)^5 + \min(0,(b-e)^5)] \\ &+ \frac{1}{10(d-c)}\max(0,(e-c)^5). \end{aligned}$$
2. The semi-kurtosis of a triangular fuzzy number $\xi = (a, b, c)$ with expected value $e = \frac{a+2b+c}{4}$ is deduced from the semi-kurtosis of a trapezoidal one by this way:
$$\begin{aligned} &K^S[\xi] \\ &= \frac{1}{10(b-a)}\left[(e-a)^5 + \frac{1}{(b-c)}(b-e)^5\min(0,(b-e))\right]. \end{aligned}$$

Definition 8. Let ξ a fuzzy variable with expected value e. The normalized semi-kurtosis of ξ is defined by:

$$K^S_1[\xi] = \frac{K^S[\xi]}{(V^S[\xi])^2}.$$

Example 10. 1. The normalized semi-kurtosis of a trapezoidal fuzzy variable $\xi = (a, b, c, d)$ with expected value e is defined as follows: $K^S_1[\xi]$ is given in Box III.

$$K_1^S[\xi] = \frac{\frac{1}{10(b-a)}[(e-a)^5 + \min(0,(b-e)^5)] + \frac{1}{10(d-c)}\max(0,(e-c)^5)}{\left[\frac{1}{6(b-a)}[(e-a)^3 + \min(0,(b-e)^3)] + \frac{1}{6(d-c)}\max(0,(e-c)^3)\right]^2}$$

Box III.

2. The normalized semi-kurtosis of a triangular fuzzy variable $\xi = (a, b, c)$ with expected value e is defined as follows:

$$K_1^S[\xi] = \frac{\frac{1}{10(b-a)}\left[(e-a)^5 + \frac{1}{(b-c)}(b-e)^5\min(0,(b-e))\right]}{\left[\frac{1}{6(b-a)}\left[(e-a)^3 + \frac{1}{(b-c)}(b-e)^3\min(0,(b-e))\right]\right]^2}.$$

Proposition 5. *Let $(\xi_k)_{k=1,\dots,n}$ be a family of independent trapezoidal fuzzy variables with finite expected values $(e_k)_{k=1,\dots,n}$, $(x_k)_{k=1,\dots,n}$ be a family of n positive reals and $\xi(x) = \sum_{k=1}^n x_k\xi_k$. Then*

1. *The semi-variance of $\xi(x)$ is*

$$V^S[\xi(x)] = \frac{1}{6\sum_{k=1}^n x_k(b_k-a_k)} \times \left[\left(\sum_{k=1}^n x_k(e_k-a_k)\right)^3 + \min\left(0, \left(\sum_{k=1}^n x_k(b_k-e_k)\right)^3\right)\right] + \frac{1}{6\sum_{k=1}^n x_k(d_k-c_k)}\max\left(0, \left(\sum_{k=1}^n x_k(e_k-c_k)\right)^3\right).$$

2. *The semi-kurtosis of $\xi(x)$ is*

$$K^S[\xi(x)] = \frac{1}{10\sum_{k=1}^n x_k(b_k-a_k)} \times \left[\left(\sum_{k=1}^n x_k(e_k-a_k)\right)^5 + \min\left(0, \left(\sum_{k=1}^n x_k(b_k-e_k)\right)^5\right)\right] + \frac{1}{10\sum_{k=1}^n x_k(d_k-c_k)}\max\left(0, \left(\sum_{k=1}^n x_k(e_k-c_k)\right)^5\right).$$

We end this section by establishing a link between moment and semi-moment of fuzzy variables.

4.3. Links between moments and semi-moments

Proposition 6. *Let ξ be a fuzzy variable with finite expected value e, $p \in \mathbb{N}$ and, $M_{2p}^S[\xi]$ and $M_{2p}[\xi]$ the semi-moment and moment of ξ respectively. Then*

$$0 \le M_{2p}^S[\xi] \le M_{2p}[\xi]. \quad (20)$$

Proposition 7. *Let ξ be a fuzzy variable with finite expected value e. Then*

$$M_{2p}[\xi] = 0 \quad \textit{if and only if} \quad \mathrm{Cr}\{\xi = e\} = 1. \quad (21)$$

Proposition 8. *Let ξ be a fuzzy variable with finite expected value e. Then*

$$M_{2p}^S[\xi] = 0 \quad \textit{if and only if} \quad \mathrm{Cr}\{\xi = e\} = 1, \quad \textit{i.e.}, M_{2p}[\xi] = 0. \quad (22)$$

Proposition 9. *Let ξ be a symmetric fuzzy variable with finite expected value e. Then*

$$M_{2p}^S[\xi] = M_{2p}[\xi]. \quad (23)$$

Remark 6. The previous results generalize those established by Huang (2008) when we consider moment and semi-moment as variance and semi-variance.

Furthermore, we can deduce the links between kurtosis and semi-kurtosis of a fuzzy variable.

Corollary 4. *Let ξ be a fuzzy variable with finite expected value e, $K^S[\xi]$ and $K[\xi]$ the semi-kurtosis and kurtosis of ξ respectively. Then*

1. $$0 \le K^S[\xi] \le K[\xi]. \quad (24)$$
2. $$K[\xi] = 0 \quad \textit{if and only if} \quad \mathrm{Cr}\{\xi = e\} = 1. \quad (25)$$
3. $$K^S[\xi] = 0 \quad \textit{if and only if} \quad \mathrm{Cr}\{\xi = e\} = 1, \quad \textit{i.e.}, K[\xi] = 0. \quad (26)$$
4. $$K^S[\xi] = K[\xi]. \quad (27)$$

5. An application in finance

5.1. Review, model, and a deterministic program with a family of triangular fuzzy numbers

Let ξ_i be a fuzzy variable representing the return of the ith security and let x_i be the proportion of the total capital invested in security i. In general, ξ_i is given as $\frac{(p_i'+d_i-p_i)}{p_i}$ where p_i is the closing price of the ith security at present, p_i' is the estimated closing price in the next year and d_i is the estimated dividends during the coming year.

It is clear that p_i' and d_i are unknown at present. If they are estimated as fuzzy variables, then ξ_i is also a fuzzy variable. Thereby, the returns $x_1\xi_1, \dots, x_n\xi_n$ of n securities and the total return $\xi = \xi_1x_1 + \xi_2x_2 + \cdots + \xi_nx_n$ are also fuzzy variables.

When minimal expected return, minimal skewness and maximal risk (variance) are given as α, γ and β respectively, the investors prefer a portfolio with small semi-kurtosis or kurtosis. Therefore, we deduce the following mean–variance–skewness–semi-kurtosis model:

$$\begin{cases} \text{minimize } K^S[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \ge \alpha \\ V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le \beta \\ S[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \ge \gamma \\ x_1 + x_2 + \cdots + x_n = 1 \\ x_i \ge 0, \quad i = 1, 2, \dots, n \end{cases} \quad (28)$$

where K^S, E, V and S designed the semi-kurtosis, the mean, the variance and the skewness operators respectively.

The first constraint of this model ensures that the expected return is no less than some target value α, the second one assures that risk does not exceed some given level β the investor can bear,

J. Sadefo Kamdem et al. / Insurance: Mathematics and Economics 51 (2012) 517–530 525

the third one assures that the skewness is no less than some target value γ. The last two constraints mean that all the capital will be invested in n securities and short-selling is not allowed.

From model (28) and, Propositions 4 and 5, we obtain the following deterministic program.

Theorem 1. *Let $(\xi_i = (a_i, b_i, c_i))_{i=1,2,\dots,n}$ be a family of n independent triangular fuzzy variables.*

Then model (28) becomes the following deterministic program:

$$\begin{cases}
\min \dfrac{1}{10\sum\limits_{i=1}^{n} x_i(b_i-a_i)}\left[\left(\sum\limits_{i=1}^{n} x_i(e_i-a_i)\right)^5 + \dfrac{1}{\sum\limits_{i=1}^{n} x_i(b_i-d_i)}\right. \\
\qquad \left. \times \left(\sum\limits_{i=1}^{n} x_i(b_i-e_i)\right)^5 \min\left(0, \sum\limits_{i=1}^{n} x_i(b_i-e_i)\right)\right] \\
\text{subject to} \\
\sum\limits_{i=1}^{n} x_i(a_i+2b_i+c_i) \geq 4\alpha \\
11\left(\sum\limits_{i=1}^{i=n} x_i(c_i-a_i)\right)^2 \left|\sum\limits_{i=1}^{i=n} x_i(2b_i-a_i-c_i)\right| \\
\qquad +2\left(8\sum\limits_{i=1}^{i=n} x_i(c_i-a_i) + 3\left|\sum\limits_{i=1}^{i=n} x_i(2b_i-a_i-c_i)\right|\right) \\
\qquad \times \left(\left(\sum\limits_{i=1}^{i=n} x_i(c_i-b_i)\right)^2 + \left(\sum\limits_{i=1}^{i=n} x_i(b_i-a_i)\right)^2\right) \\
\quad \leq 192\beta\left(\sum\limits_{i=1}^{i=n} x_i(c_i-a_i) + \left|\sum\limits_{i=1}^{i=n} x_i(2b_i-a_i-c_i)\right|\right) \\
\left(\sum\limits_{i=1}^{n} x_i(c_i-a_i)\right)^2 \sum\limits_{i=1}^{n} x_i(c_i-2b_i+a_i) \geq 32\gamma \\
x_1+x_2+\cdots+x_n = 1 \\
x_i \geq 0, \quad i=1,2,\dots,n.
\end{cases}$$

The other variants of model (28) can be deduced from the previous model by changing the objective function either by mean, variance, skewness or kurtosis.

Remark 7. It is important to notice that, similarly as above, one can write four variants of the previous model and deterministic program. These variants are described as follows.

1. The first variant of model (28) minimizes risk (variance) when the expected return and the skewness are both no less than some given target values α and γ respectively and the semi-kurtosis is no more than the given target value θ. If one cancels the constraints on skewness and semi-kurtosis in this variant, then this first variant degenerates to the mean-variance model proposed earlier by Huang (2008).
2. The second variant of model (28) maximizes the expected return when the skewness is no less than some given target values γ and, the variance and the semi-kurtosis are no more than β and θ respectively.
3. The third variant of model (28) maximizes the skewness when the expected return is not less than α and, the variance and the semi-kurtosis are no more than some given target values β and θ respectively. If we cancel the second constraint on the semi-kurtosis in this variant, then this third variant degenerates to the mean–variance–skewness model proposed by Li et al. (2010).
4. The fourth variant of model (28) is the multi-objective nonlinear programming which minimizes the risk and the semi-kurtosis and maximizes the expected value and the skewness when the different target values are unknown.

Let us explain why we minimize the fourth moment and semi-moment as in the well-known probability theory or Markowitz theory.[1]

Remark 8. In portfolio selection when return is a random variable and probability theory is used, it is well established that investors choose the portfolio that maximizes its even moments (e.g. mean and skewness) and that minimizes its odd moments or semi-moments (e.g. semi-variance, variance, semi-kurtosis and kurtosis). In portfolio selection when return is a fuzzy variable and credibilistic theory is used, the investor also has to select a portfolio that maximizes its even moments and minimizes its odd moments or semi-moments. The two reasons are:

1. In general, the moments in credibility theory and probability theory are defined "similarly" via the "Expectation" operator "E". For a given fuzzy variable ξ, $E[\xi]$ is given by (2).
   Baoding Liu and Liu (2002) (see Remark 1, p. 446) show that if the fuzzy variable ξ is replaced with a random variable (whose density function is ϕ) and Cr is replaced with the probability measure *Prob* (whose dual is itself), the representation of the expected value of the fuzzy variable is identical to that of the random variable. Technically speaking, $E[\xi]$ of (2) becomes $E[\xi] = \int_{-\infty}^{+\infty} x\phi(x)dx$ which is exactly the expected value of the random variable ξ.
   Let us recall the proof:
   $$E[\xi] = \int_0^{+\infty} Prob\{\xi \geq r\}dr - \int_{-\infty}^{0} Prob\{\xi \leq r\}dr \tag{29}$$
   $$= \int_0^{+\infty}\left[\int_r^{+\infty}\phi(x)dx\right]dr - \int_{-\infty}^{0}\left[\int_{-\infty}^{r}\phi(x)dx\right]dr$$
   $$= \int_0^{+\infty}\left[\int_0^{x}\phi(x)dr\right]dx - \int_{-\infty}^{0}\left[\int_x^{0}\phi(x)dr\right]dx$$
   $$= \int_0^{+\infty}\left[\phi(x)\int_0^{x}dr\right]dx + \int_{-\infty}^{0}\left[\phi(x)\int_0^{x}dr\right]dx$$
   $$= \int_0^{+\infty} x\phi(x)dx + \int_{-\infty}^{0} x\phi(x)dx$$
   $$= \int_{-\infty}^{+\infty} x\phi(x)dx. \tag{30}$$
   When the right-hand side of (29) is of form $\infty-\infty$, the expected value is not defined.
2. In particular, for a symmetric trapezoidal fuzzy variable $\xi = (a, b, c, d)$, the kurtosis is given by (13). That expression stipulates that the kurtosis is an increasing function of the length $l_c(\xi)$ of the core, that means, more the core increases, more the kurtosis increases. One of its meaningful interpretations is: a great length of the core which produces "fat tails" traduces a lack of information for the investors, and consequently there is more uncertainty when the core is large. So, according to the fact that, the investor needs more information for his portfolios fuzzy returns, it is better for him to minimize the kurtosis in order to have portfolio fuzzy returns with "thin tails" that means more information.

In the following subsection, we display numerical examples on our two models, namely the mean–variance–skweness–kurtosis model and the mean–variance–skewness–semi-kurtosis model, and those used by Huang (2008) and Li et al. (2010).

5.2. Numerical examples

The data, we consider in this section, are introduced and used by Huang (2008) for the mean-semi-variance model and, used by

[1] This explanation has been requested by an anonymous referee.

Table 1
Fuzzy returns of 7 securities (units per stock).

Security i	Fuzzy return
1	$\xi_1 = (-0.3, 1.8, 2.3)$
2	$\xi_2 = (-0.4, 2.0, 2.2)$
3	$\xi_3 = (-0.5, 1.9, 2.7)$
4	$\xi_4 = (-0.6, 2.2, 2.8)$
5	$\xi_5 = (-0.7, 2.4, 2.7)$
6	$\xi_6 = (-0.8, 2.5, 3.0)$
7	$\xi_7 = (-0.6, 1.8, 3.0)$

Li et al. (2010) for the mean–variance–skewness model. Those data are seven triangular security returns as illustrated in Table 1.

For instance, the return of the first security is described by the fuzzy variable $\xi_1 = (-0.3, 1.8, 2.3)$ which represents about 1.8 units per stock.

In order to use the previous proposed models to determine an optimal portfolio from these seven securities, the investor needs to set three parameters: the minimum expected return α, the bearable maximum risk β, and the minimum tolerable skewness γ. As in the previous case, we take $\alpha = 1.6$, $\beta = 0.8$ and $\gamma = -0.6823$ (it is important to notice that γ is at the most equal to -0.6823).

Since the returns are asymmetric, the investor may also employ either semi-variance or variance, either kurtosis or semi-kurtosis to create an optimal portfolio. We consider the following four models:

1. the first one is the mean–semi-variance model from Huang (2008):

$$\begin{cases} \text{minimize } V^S[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \geq \alpha \\ x_1 + x_2 + \cdots + x_7 = 1 \\ x_i \geq 0, \quad i = 1, 2, \ldots, 7 \end{cases} \tag{31}$$

2. the second one is the mean–variance–skewness model from Li et al. (2010)

$$\begin{cases} \text{maximize } S[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \geq \alpha \\ V[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \leq \beta \\ x_1 + x_2 + \cdots + x_7 = 1 \\ x_i \geq 0, \quad i = 1, 2, \ldots, 7 \end{cases} \tag{32}$$

3. the two following models of our paper: the mean–variance–skewness–kurtosis model and the mean–variance–skewness–semi-kurtosis model

$$\begin{cases} \text{minimize } K[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \geq \alpha \\ V[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \leq \beta \\ S[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \geq \gamma \\ x_1 + x_2 + \cdots + x_7 = 1 \\ x_i \geq 0, \quad i = 1, 2, \ldots, 7 \end{cases} \tag{33}$$

and

$$\begin{cases} \text{minimize } K^S[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \\ \text{subject to} \\ E[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \geq \alpha \\ V[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \leq \beta \\ S[x_1\xi_1 + x_2\xi_2 + \cdots + x_7\xi_7] \geq \gamma \\ x_1 + x_2 + \cdots + x_7 = 1 \\ x_i \geq 0, \quad i = 1, 2, \ldots, 7 \end{cases} \tag{34}$$

where K designed the kurtosis operator.

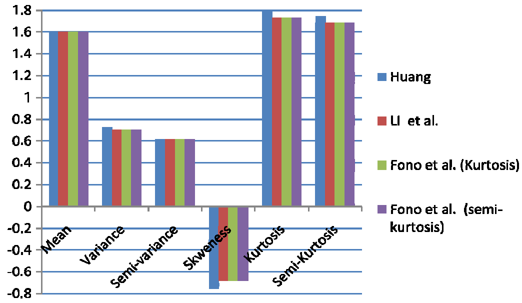

Fig. 2. Comparison of values of characteristics of different optimal total return of portfolio (combinations of the set of seven security returns) obtained by different authors.

We use MATLAB to solve the four models mentioned above and the computational results are shown in the following tables.

We can make the following observations:

- When we consider semi-kurtosis (K^S) as an objective function, Li et al.'s model (32) and Fono et al.'s model (34) give the same optimal portfolio (according to lines 2 and 4 of Table 2). Therefore, the latter confirms and enhances the results obtained by the first one. Those models allow us to obtain highest skewness (-0.6823) and lowest semi-kurtosis (1.6872) (see lines 2 and 3 of Table 3) which are the optimal values of the objective functions of the two models respectively.
- When we consider kurtosis (K) as an objective function, Fono et al.'s model (33) provides the lowest variance (0.7018), the highest skewness (-0.623) and the lowest kurtosis (1.729) (see line 3 of Table 3).
  In this case, model (33) proposes an optimal portfolio different from the three other models (see Table 2). More precisely, the investment's proportions obtained (according to line 3 of Table 2) just means that: if one wishes to invest 10 000 units, he will invest 2004 units of the security 1, 7989 units of the security 4, 7 units of the security 6 and nothing elsewhere.
- The histogram of Fig. 2 shows the different values of characteristics of the four total returns (combinations of the seven returns) obtained by these authors as described in Table 3.

Let us explain why Fono et al.'s model (34) with the semi-kurtosis and Li et al.'s model (32) coincide (as stipulated in the previous first observation).[2]

Remark 9. 1. The main reason why the two models coincide (generate the same optimal portfolio) in our numerical examples with the seven fuzzy variables is: each of the seven variables $\xi = (a_i, b_i, c_i)$ have a large spread on their left, that is, $\forall i \in \{1, 2, \ldots, 7\}$, $c_i - b_i < b_i - a_i$, and thereby a small "good" part (right of the b_i). On one hand, the skewness measures the spread of the distribution on the left side (so that one is able to say at what degree the distribution is concentrated on the left) and on the other hand, semi-kurtosis allows us to avoid penalizing the "good part" ("positive part") when applying the model. Therefore by adding the semi-kurtosis to Li et al.'s model (32) we obtain the same optimal portfolio from our seven variables.

2. Now, if we replace the first fuzzy variable $\xi_1 = (-0.3, 1.8, 2.3)$ by the new fuzzy variable $\xi_8 = (-0.1, 0.0, 2.0)$ (its "positive part" is greater than the "negative part"), then lines 2 and 4 of the two previous tables become respectively:

[2] This has been requested by an anonymous referee.

Table 2
Optimal selection from each model.

	Security i						
	1 (%)	2 (%)	3 (%)	4 (%)	5 (%)	6 (%)	7 (%)
Huang's model (31) (Huang, 2008)	00.00	47.06	00.00	35.28	17.66	00.00	00.00
Li et al.'s model (32) (Li et al., 2010)	20.00	00.00	00.00	80.00	00.00	00.00	00.00
Fono et al.'s model (33)	20.04	00.00	00.00	79.89	00.00	00.07	00.00
Fono et al.'s model (34)	20.00	00.00	00.00	80.00	00.00	00.00	00.00

Table 3
Comparison of the four first moments of the different optimal portfolios.

	Mean	Variance	Semi-variance	Skewness	Kurtosis	Semi-kurtosis
Huang's model (31) (Huang, 2008)	1.60	0.7235	0.6124	−0.7543	1.7972	1.7415
Li et al.'s model (32) (Li et al., 2010)	1.60	0.7019	0.6141	0.6823	1.7291	1.6872
Fono et al.'s model (33)	1.60	0.7018	0.6140	−0.6823	1.7290	1.6873
Fono et al.'s model (34)	1.60	0.7019	0.6141	−0.6823	1.7291	1.6872

With the seven last fuzzy variables (from ξ_2 from ξ_8), we have:

Security i	2 (%)	3 (%)	4 (%)	5 (%)	6 (%)	7 (%)	8 (%)
Li et al.'s model (32)	00.00	00.00	33.00	67.00	00.00	00.00	00.00
Fono et al.'s model (34)	00.00	00.00	36.00	64.00	00.00	00.00	00.00

	Mean	Variance	Semi-variance	Skewness	Semi-kurtosis
Li et al.'s model (32)	1.60	0.7213	0.6361	−0.6954	1.7931
Fono et al.'s model (34)	1.60	0.7164	0.6323	−0.6860	1.7702

By comparing these new tables and the previous one, semi-kurtosis used in Fono et al.'s model (34) displays an optimal portfolio better than the one given by Li et al.'s model (32). In other words, by adding semi-kurtosis to Li et al.'s model, we improve the optimal portfolio: the same mean, less variance, less semi-variance, greater skewness and less semi-kurtosis.

6. Concluding remarks

Different from Huang (2008) and Li et al. (2010), after recalling the definition of mean, variance, semi-variance and skewness, this paper introduces originally the k-moments (kurtosis for $k = 4$) and $2k$-semi-moments (semi-kurtosis for $k = 2$) for portfolio selection with fuzzy risk factors (i.e. returns) and, their mathematical properties are studied. Kurtosis and semi-kurtosis measure the leptokurtocity of credibilistic portfolio return. As an extension of the mean–variance–skewness model for credibilistic portfolio, the mean–variance–skewness–semi-kurtosis is presented and the corresponding variant (the mean–variance–skewness–kurtosis model) is also considered. We briefly give a numerical example for our optimization models and it appears that the application of those new theoretical notions enhances the allocation of a capital in the fuzzy environment.

7. Proof of the results

Throughout this section ξ is a fuzzy variable with $E[\xi] = e$.

Proof of Proposition 1. For a symmetric trapezoidal fuzzy variable $\xi = (a, b, c, d)$, we can easily show the following result:

$$\mathrm{Cr}\{(\xi - e)^k \geq r\} = \mathrm{Cr}\{\xi - e \geq \sqrt[k]{r}\} \vee \mathrm{Cr}\{\xi - e \geq \sqrt[k]{r}\}.$$

$$\mathrm{Cr}\{(\xi - e)^k \geq r\} = \begin{cases} \dfrac{1}{2}, & \text{if } 0 \leq r \leq \left(\dfrac{c-b}{2}\right)^k \\ -\dfrac{\sqrt[k]{r}}{2\beta} + \dfrac{c-b}{4\beta} + \dfrac{1}{2}, & \\ \quad \text{if } \left(\dfrac{c-b}{2}\right)^k \leq r \leq \left(\dfrac{c-b}{2} + \beta\right)^k & \\ 0, & \text{if } r \geq \left(\dfrac{c-b}{2} + \beta\right)^k \end{cases}$$

where $\alpha = d - c = b - a$.

So, we can conclude that:

$$\begin{aligned} m_k[\xi] &= \int_0^{\left(\frac{c-b}{2}+\beta\right)^k} \mathrm{Cr}\{(\xi - e)^k \geq r\} \\ &= \frac{\sum_{i=0}^{k}\sum_{j=0}^{k-i} C_{k\ \ i}^{j}(2\beta)^j (c-b)^{k-j}}{2^{k+1}(k+1)} \\ &= \frac{\sum_{j=0}^{k} C_{k+1}^{j+1}(2\beta)^j (c-b)^{k-j}}{2^{k+1}(k+1)} \\ &= \frac{\sum_{i=0}^{\frac{k}{2}} C_{k+1}^{2i+1}[(c-b)+\alpha]^{k-2i}}{2^{k+1}(k+1)}. \end{aligned}$$

The proof is complete. □

Proof of Corollary 1. We show that, for a symmetric fuzzy variable ξ, $m_k[\xi]$ is nil when k is an odd number.

By definition, we have:

$$\begin{aligned} m_k[\xi] &= E[(\xi - E[\xi])^k] \\ &= \int_0^{+\infty} \mathrm{Cr}\{(\xi - E[\xi])^k \geq r\}dr \\ &\quad - \int_{-\infty}^{0} \mathrm{Cr}\{(\xi - E[\xi])^k \leq r\}dr, \quad \forall k \in \mathbb{N}^*. \end{aligned}$$

In Li et al. (2010), Li has already proved that for a symmetric fuzzy variable ξ, $E[\xi] = e$ and $\mathrm{Cr}\{\xi - e \geq r\} = \mathrm{Cr}\{\xi - e \leq -r\}$, where e is a real number such that $\mu(e-r) = \mu(e+r)$, $\forall r \in \mathbb{R}$ and μ is the membership function of ξ.

Furthermore, we have:

$$\begin{aligned} m_k[\xi] &= \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^k \geq r\}dr - \int_{-\infty}^0 \mathrm{Cr}\{(\xi - e)^k \leq r\}dr \\ &= \int_0^{+\infty} kr^{k-1}\mathrm{Cr}\{\xi - e \geq r\}dr \\ &\quad - \int_{-\infty}^0 kr^{k-1}\mathrm{Cr}\{\xi - e \leq r\}dr \\ &= \int_0^{+\infty} kr^{k-1}\mathrm{Cr}\{\xi - e \leq -r\}dr \\ &\quad - \int_0^{+\infty} kr^{k-1}\mathrm{Cr}\{\xi - e \leq r\}dr = 0. \end{aligned}$$

Now, we assume that k is an even integer.

For a symmetric triangular fuzzy variable $\xi = (a, b, c)$, we can easily show the following result.

Since $\mathrm{Cr}\{(\xi - e)^k \geq r\} = \mathrm{Cr}\{\xi - e \geq \sqrt[k]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[k]{r}\}$, we have:

$$\mathrm{Cr}\{(\xi - e)^k \geq r\} = \begin{cases} \dfrac{\alpha - \sqrt[k]{r}}{2\alpha}, & \text{if } 0 \leq r \leq \alpha^k \\ 0, & \text{if } r \geq \alpha^k \end{cases}$$

where $\alpha = c - b = b - a$.

So, we can conclude that: $m_k[\xi] = \int_0^{\alpha^k} \frac{\alpha - \sqrt[k]{r}}{2\alpha}dr = \frac{1}{2k+2}\alpha^k$. □

Proof of Proposition 2. (1) It is easy to show that: $\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[4]{r}\}$. Hence we have the following equality:

$$\begin{aligned} K[\xi] &= \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\}dr \\ &= \int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[4]{r}\}dr. \end{aligned}$$

(2) We deduce the second result from the definition of $K^1[\xi]$ and by using the fact that:

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^2 \geq r\}dr \\ &= \int_0^{+\infty} \mathrm{Cr}\{\xi - e \geq \sqrt[2]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[2]{r}\}dr. \end{aligned}$$

(3) (i) Let $a, b \in \mathbb{R}$. We have $K[a\xi + b] = E[(a\xi + b - E[a\xi + b])^4]$. Since $E[a\xi + b] = aE[\xi] + b$, we deduce that $K[a\xi + b] = E[(a\xi + b - aE[\xi] - b)^4] = E[(a\xi - aE[\xi])^4] = a^4E[(\xi - E[\xi])^4] = a^4K[\xi]$.

(ii) Since $V[a\xi + b] = a^2V[\xi]$, we deduce $K^1[a\xi + b] = K^1[\xi]$. □

Proof of Corollary 2. When ξ is a symmetric fuzzy variable, we have: $\mathrm{Cr}\{(\xi - e)^4 \geq r\}dr = \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\}$ and $\mathrm{Cr}\{(\xi - e)^2 \geq r\}dr = \mathrm{Cr}\{\xi - e \geq \sqrt[2]{r}\}$ and the proof is complete. □

Proof of Proposition 3. (1) Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable such that $E[\xi] = e$, $\alpha = b - a$, $\beta = d - c$.

By using the fact that $\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[4]{r}\}$, we can easily obtain the following results:

(i) When $\alpha > \beta$, then $e < c$. We can so distinguish the two following cases as follows:

1st case: $e < b$

$$\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \begin{cases} 1 - \dfrac{\sqrt[4]{r} + e - a}{2\alpha}, & \text{if } 0 \leq r \leq (b - e)^4 \\ \dfrac{1}{2}, & \text{if } (b - e)^4 \leq r \leq (c - e)^4 \\ -\dfrac{\sqrt[4]{r} + e - d}{2\beta}, & \\ \quad \text{if } (c - e)^4 \leq r \leq \left(e - \dfrac{a+b}{2}\right)^4 & \\ \dfrac{-\sqrt[4]{r} + e - a}{2\alpha}, & \\ \quad \text{if } \left(e - \dfrac{a+b}{2}\right)^4 \leq r \leq (e - a)^4 & \\ 0, & \text{if } r \geq (e - a)^4 \end{cases}$$

and finally we get:

$$\begin{aligned} K[\xi] &= \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\}dr \\ &= \left(\frac{(e-a) + (e-b)}{2}\right)^5 \cdot \left(\frac{\beta - \alpha}{5\alpha\beta}\right) \\ &\quad + \left(\frac{(e-a) + (e-b)}{2}\right)^4 \cdot \left(\frac{\alpha(d-e) + \beta(e-a)}{2\alpha\beta}\right) \\ &\quad + \frac{(e-a)^5}{10\alpha} + \frac{(b-e)^5}{10\alpha} - \frac{(c-e)^5}{10\beta}. \end{aligned}$$

2nd case: $e > b$

$$\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \begin{cases} \dfrac{1}{2}, & \text{if } 0 \leq r \leq (c - e)^4 \\ -\dfrac{\sqrt[4]{r} + e - d}{2\beta}, & \\ \quad \text{if } (c - e)^4 \leq r \leq \left(e - \dfrac{a+b}{2}\right)^4 & \\ \dfrac{-\sqrt[4]{r} + e - a}{2\alpha}, & \\ \quad \text{if } \left(e - \dfrac{a+b}{2}\right)^4 \leq r \leq (e - a)^4 & \\ 0, & \text{if } r \geq (e - a)^4 \end{cases}$$

and finally we get:

$$\begin{aligned} K[\xi] &= \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\}dr \\ &= \left(\frac{(e-a) + (e-b)}{2}\right)^5 \cdot \left(\frac{\beta - \alpha}{5\alpha\beta}\right) \\ &\quad + \left(\frac{(e-a) + (e-b)}{2}\right)^4 \cdot \left(\frac{\alpha(d-e) + \beta(e-a)}{2\alpha\beta}\right) \\ &\quad + \frac{(e-a)^5}{10\alpha} - \frac{(c-e)^5}{10\beta}. \end{aligned}$$

(ii) When $\alpha < \beta$, we use a similar way to calculate $K[\xi]$.

(iii) When $\alpha = \beta$, we have:

$$\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq r \leq \left(\frac{c-b}{2}\right)^4 \\ -\frac{\sqrt[4]{r}}{2\beta} + \frac{c-b}{4\beta} + \frac{1}{2}, & \\ \quad \text{if } \left(\frac{c-b}{2}\right)^4 \leq r \leq \left(\frac{c-b}{2} + \beta\right)^4 & \\ 0, & \text{if } r \geq \left(\frac{c-b}{2} + \beta\right)^4 \end{cases}$$

$\alpha = d - c = b - a$ and this result implies that:

$$K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\} dr = \frac{5[(c-b)+\beta]^4 + 10\beta^2[(c-b)+\beta]^2 + \beta^4}{160}.$$

(2) Let $\xi = (a, b, c)$ be a triangular fuzzy variable such that $E[\xi] = e$, $\alpha = b - a$, $\beta = c - b$. By using the fact that $\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \mathrm{Cr}\{\xi - e \geq \sqrt[4]{r}\} \vee \mathrm{Cr}\{\xi - e \leq \sqrt[4]{r}\}$, we can easily obtain the following results:

(i) When $\alpha > \beta$, then $e < b$ and

$$\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \begin{cases} 1 - \frac{\sqrt[4]{r} + e - a}{2\alpha}, & \text{if } 0 \leq r \leq (b-e)^4 \\ -\frac{\sqrt[4]{r} + e - c}{2\beta}, & \\ \quad \text{if } (b-e)^4 \leq r \leq \left(\frac{\alpha+\beta}{4}\right)^4 & \\ \frac{-\sqrt[4]{r} + e - a}{2\alpha}, & \\ \quad \text{if } \left(\frac{\alpha+\beta}{4}\right)^4 \leq r \leq (e-a)^4 & \\ 0, & \text{if } r \geq (e-a)^4 \end{cases}$$

and finally we get:

$$K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\} dr = \frac{253\alpha^5 + 395\alpha^4\beta + 17\alpha\beta^4 + 290\alpha^3\beta^2 + 70\alpha^2\beta^3 - \beta^5}{10.240\alpha}.$$

(ii) When $\alpha < \beta$, we use a similar way to calculate $K[\xi]$.

(iii) When $\alpha = \beta$, we have:

$$\mathrm{Cr}\{(\xi - e)^4 \geq r\} = \begin{cases} \frac{\alpha - \sqrt[k]{4}}{2\alpha}, & \text{if } 0 \leq r \leq \alpha^k \\ 0, & \text{if } r \geq \alpha^4 \end{cases}$$

where $\alpha = c - b = b - a$ and this result implies that: $K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^4 \geq r\} dr = \frac{\alpha^4}{10}$. □

Proof of Corollary 3. We deduce these results from Proposition 3. □

Proof of Proposition 6. Let $\theta \in \Theta$ and $r \in \mathbb{R}$. With (16), we have: $[(\xi - e)^-]^{2p} = \begin{cases} (\xi - e)^{2p} & \text{si } \xi \leq e \\ 0 & \text{si } \xi > e \end{cases}$. Thus we distinguish two cases as follows:

(i) If $\xi(\theta) \leq e$, then $[(\xi(\theta) - e)^-]^{2p} = (\xi(\theta) - e)^{2p}$. And $[(\xi(\theta) - e)^-]^{2p} \geq r \Leftrightarrow (\xi(\theta) - e)^{2p} \geq r$.

(ii) If $\xi(\theta) > e$, then $[(\xi(\theta) - e)^-]^{2p} = 0$ and $(\xi(\theta) - e)^{2p} \geq [(\xi(\theta) - e)^-]^{2p}$. Thus the inequality $[(\xi(\theta) - e)^-]^{2p} \geq r$ implies $(\xi(\theta) - e)^{2p} \geq r$. We deduce that $\forall \theta, r$, $\{\theta / [(\xi(\theta) - e)^-]^{2p} \geq r\} \subseteq \{\theta / (\xi(\theta) - e)^{2p} \geq r\}$. Since Cr is monotone, we have: $\forall r$, $\mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\} \leq \mathrm{Cr}\{(\xi - e)^{2p} \geq r\}$. Hence $K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^{2p} \geq r\} dr \geq \int_0^{+\infty} \mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\} dr = K^S[\xi]$.

For $p = 2$, we show (24). □

Proof of Proposition 7. Assume that ξ is symmetric and let $p \in \mathbb{N}^*$.

($\Leftarrow$): Assume that $\mathrm{Cr}\{\xi = e\} = 1$. Thus we have: $\mathrm{Cr}\{\xi - e = 0\} = 1$ iff $\mathrm{Cr}\{(\xi - e)^{2p} = 0\} = 1$. With the self-duality of Cr, we have $\mathrm{Cr}\{(\xi - e)^{2p} \neq 0\} = 0$.

Let $r > 0$. We have: $\mathrm{Cr}\{(\xi - e)^{2p} \geq r\} \leq \mathrm{Cr}\{(\xi - e)^{2p} > 0\} \leq \mathrm{Cr}\{(\xi - e)^{2p} \neq 0\} = 0$. That means $\forall r > 0$, $\mathrm{Cr}\{(\xi - e)^{2p} \geq r\} = 0$. And we deduce $K[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^{2p} \geq r\} dr = 0$.

($\Rightarrow$:) Assume that $K[\xi] = 0$. Since Cr takes values in $[0; 1]$, this equality means $\mathrm{Cr}\{(\xi - e)^{2p} \geq r\} = 0, \forall r > 0$. Since Cr is self-dual, we have $\mathrm{Cr}\{(\xi - e)^{2p} = 0\} = 1$ and we deduce that $\mathrm{Cr}\{\xi - e = 0\} = 1$, that is, $\mathrm{Cr}\{\xi = e\} = 1$.

Assume that ξ is symmetric and replace $p = 2$ in the precede proof to obtain (25). □

Proof of Proposition 8. Let $p \in \mathbb{N}^*$. Assume that $M_{2p}[\xi] = 0$. With Proposition 6, we have $M_{2p}^S[\xi] = 0$.

Assume that $M_{2p}^S[\xi] = 0$. that is, $E[[(\xi - e)\ \ ]^{2p}] = 0$. Since $E[[(\xi - e)^-]^{2p}] = \int_0^{+\infty} \mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\} dr$, and the credibility measure Cr takes its value in $[0; 1]$, then $\mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\} = 0$, $\forall r > 0$. By the self-duality of Cr, we have $\mathrm{Cr}\{[(\xi - e)^-]^{2p} = 0\} = 1$ and, deduce that

$$\mathrm{Cr}\{(\xi - e)^- = 0\} = 1. \tag{35}$$

Since $\xi - e = (\xi - e)^- + (\xi - e)^+$, then (35) implies $\xi - e = (\xi - e)^+$. And $E[(\xi - e)] = E[(\xi - e)^+] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^+ \geq r\} dr = 0$. This equality implies that $\mathrm{Cr}\{(\xi - e)^+ \geq r\} = 0$, $\forall r > 0$. Since Cr is self-dual, we obtain $\mathrm{Cr}\{(\xi - e)^+ = 0\} = 1$.

With $\mathrm{Cr}\{(\xi - e)^- = 0\} = 1$ and $\mathrm{Cr}\{(\xi - e)^+ = 0\} = 1$, we deduce $\mathrm{Cr}\{(\xi - e) = 0\} = 1$, that is, $\mathrm{Cr}\{\xi = e\} = 1$. With Proposition 7, we have $M_{2p}[\xi] = 0$. When $p = 2$ and ξ is symmetric we obtain (26). □

Proof of Proposition 9. $p \in \mathbb{N}^*$. Assume that ξ is symmetric and let us show (27).

Since $M_{2p}[\xi] = \int_0^{+\infty} \mathrm{Cr}\{(\xi - e)^{2p} \geq r\} dr$ and $M_{2p}^S[\xi] = \int_0^{+\infty} \mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\} dr$, it suffices to show that: $\mathrm{Cr}\{(\xi - e)^{2p} \geq r\} = \mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\}$. For that we distinguish two cases:

- If $r < 0$, then we have $\mathrm{Cr}\{(\xi - e)^{2p} \geq r\} = \mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\} = \mathrm{Cr}\{\Theta\} = 1$.
- If $r \geq 0$, then (with $r = r'^{2p}$) and assume that $r' > 0$. We have $(\xi - e)^{2p} \geq r \Leftrightarrow (\xi - e) \in]-\infty; -r'] \cup [r'; +\infty[$, and $[(\xi - e)^-]^{2p} \geq r \Leftrightarrow (\xi - e)^- \in]-\infty; -r'] \cup [r'; +\infty[$. Therefore, we obtain $\mathrm{Cr}\{(\xi - e)^{2p} \geq r\} = 1 - \mathrm{Cr}\{-r' < \xi - e < r'\}$, $\mathrm{Cr}\{[(\xi - e)^-]^{2p} \geq r\} = 1 - \mathrm{Cr}\{-r' < (\xi - e)^- < r'\}$.

It rests to show that $\mathrm{Cr}\{-r' < \xi - e < r'\} = \mathrm{Cr}\{-r' < (\xi - e)^- < r'\}$.

Let μ be the membership function of $\xi - e$ and μ' be the membership function of $(\xi - e)^-$. Let us recall that $\mu' = \begin{cases} \mu & \text{if } \xi < e \\ 0 & \text{otherwise.} \end{cases}$

We have:

$$\mathrm{Cr}\{-r' < \xi - e < r'\} = \frac{1}{2}\left[1 + \sup_{x\in]-r';r'[} \mu(x) - \max\left(\sup_{x\in]-\infty;-r'[} \mu(x), \sup_{x\in]r';+\infty;[} \mu(x)\right)\right]$$

$$= \frac{1}{2}\left[1 + \sup_{x\in]-r';0[} - \sup_{x\in]-\infty;-r'[} \mu(x)\right].$$

We also have $\mathrm{Cr}\{-r' < (\xi - e)^- < r'\} = \mathrm{Cr}\{-r' < (\xi - e)^- \le 0\}$ since $(\xi - e)^- \le 0$. Therefore

$$\mathrm{Cr}\{-r' < (\xi - e)^- < r'\} = \frac{1}{2}\left[1 + \sup_{x\in]-r';0[} \mu'(x) - \max\left(\sup_{x\in]-\infty;-r'[} \mu'(x), \sup_{x\in]0;+\infty;[} \mu'(x)\right)\right]$$

$$= \frac{1}{2}\left[1 + \sup_{x\in]-r';0[} \mu'(x) - \sup_{x\in]-\infty;-r'[} \mu'(x)\right]. \quad (36)$$

Since $\mu'(x) = 0$, $\forall x \in]0; +\infty[$, hence

$\mathrm{Cr}\{-r' < \xi - e < r'\} = \mathrm{Cr}\{-r' < (\xi - e)^- < r'\}$.

Assume that ξ is symmetric. In the preceding proof, when $p = 2$ we obtain (27). □

Acknowledgments

The authors thank the referees for their valuable comments.

References

Bachelier, L., 1900. Théorie de la spéculation. Thesis. Annales Scientifiques de l'École Normale Supérieure, III 17, pp. 21–86.
Briec, W., Kerstens, K., Junkung, O., 2007. Mean–variance–skewness portfolio performance gauging: a general shortage function and dual approach. Management Science 53, 135–149.
Hasuike, T., Katagiri, H., Ishii, H., 2009. Portfolio selection problems with random fuzzy variable returns. Fuzzy Sets and Systems 160 (18), 2579–2596.
Huang, X., 2008. Mean-semivariance models for fuzzy portfolio selection. Journal of Computational and Applied Mathematics 217 (1), 1–8.
Huang, X., 2010. Portfolio Analysis: From Probabilistic to Credibilistic and Uncertain Approaches. In: Studies in Fuzziness and Soft Computing, Springer-Verlag, Berlin, Heidelberg.
Kar, S., Bhattacharyya, R., Majumbar, D.D., 2011. Fuzzy-mean-skewness portfolio selection models by interval analysis. Computers and Mathematics with Applications 61, 126–137.
Konno, H., Shirakawa, H., Yamazaki, H., 1993. A mean-absolute deviation-skewness portfolio optimization model. Annals of Operations Research 45, 205–220.
Konno, H., Suzuki, K., 1995. A mean–variance–skewness optimization model. Journal of the Operations Research Society of Japan 38, 137–187.
Kraus, A., Litzenberger, R., 1976. Skewness preference and the valuation of risky assets. Journal of Finance 21, 1085–1094.
Li, X., Qin, Z., Kar, S., 2010. Mean–variance–skewness model for portfolio selection with fuzzy returns. European Journal of Operational Research 202 (1), 239–247.
Liu, B., 2002. Theory and Practice of Uncertain Programming. Physica-Verlag, Heidelberg.
Liu, B., Liu, Y.K., 2002. Expected value of fuzzy variable and fuzzy expected value models. IEEE Transactions on Fuzzy Systems 10, 445–450.
Markowitz, H., 1952. Portfolio selection. Journal of Finance 7, 77–91.
Markowitz, H., 1959. Portfolio Selection: Efficient Diversification of Investments. Wiley, New York.
Samuelson, P., 1970. The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments. Review of Economic Studies 37, 537–542.
Tanaka, H., Guo, P., 1999. Portfolio selection based based on upper and lower exponential possibility distributions. European Journal of Operational Research 114, 115–126.
Zadeh, L.A., 1978. Fuzzy set as a basis for a theory of possibility. Fuzzy Sets and Systems 1, 3–28.

TITLE OF THE SECOND PAPER: CHARACTERIZATION OF ORDER DOMINANCE ON FUZZY VARIABLES FOR PORTFOLIO SELECTION WITH FUZZY RETURNS.

PUBLISHED IN 2016 IN "JOURNAL OF OPERATIONAL RESEARCH SOCIETY".

Journal of the Operational Research Society (2017)

www.palgrave.com/journals

Characterization of order dominances on fuzzy variables for portfolio selection with fuzzy returns

Christian Deffo Tassak[1], Jules Sadefo Kamdem[2,3]*, Louis Aimé Fono[4] and Nicolas Gabriel Andjiga[5]
[1]Laboratoire de Mathématiques et Applications Fondamentales, UFDR MIBA - CRFD STG, Université de Yaoundé I, B.P. 812, Yaoundé, Cameroon; [2]LAMETA CNRS UMR 5474 (Montpellier), Montpellier, France; [3]DFR SJE, Campus de Troubiran, Université de Guyane, B.P. 792, 97337 Cayenne Cedex, France; [4]Laboratoire de Mathématiques et Faculté des Sciences, Université de Douala, B.P. 24157, Douala, Cameroun; and [5]Laboratoire de Mathématiques et Applications Fondamentales, UFDR MIBA - CRFD STG et ENS Yaoundé, Université de Yaoundé I, B.P. 47, Yaoundé, Cameroon

Peng *et al* (Int J Uncertain Fuzziness Knowl Based Syst 15:29–41, 2007) introduced, by means of the credibility measure, two dominance relations on fuzzy variables, namely the first- and the second-order dominances. In this paper, we characterize each of these dominance relations, and we justify that they satisfy six well-known properties of comparison methods. We propose a Game Theory approach for the determination of optimal portfolios when returns are fuzzy by introducing the set of best portfolios with respect to the first- and the second-order dominances. Based on the characterization of the first-order dominance, we numerically display some of the best portfolios of the classical set of portfolios of seven independent assets described by triangular fuzzy numbers.
Journal of the Operational Research Society (2017). doi:10.1057/s41274-016-0164-5

Keywords: credibility measure; fuzzy variable; first-order dominance; second-order dominance; set of best portfolios

1. Introduction

A part of the literature on portfolio selection deals with the fact that asset future returns are represented by random variables. Thus, from seminal works of Markowitz (1952) and Tobin (1965), many scholars (Brogan and Stidham 2008; Dentcheva and Ruszczynski 2004; Grauer and Best 1991; Konno and Suzuki 1995; Kraus and Litzenberger 1976; Samuelson 1970; Sengupta 1989; Sharpe 1971) develop tools on random variables in order to determine the best portfolios. We do not consider this literature in this paper.

However, we sometimes faced up to situations where random variables values are not completely known. In some cases, an investor seeks expert's advice to get an idea on the investment's future returns and the expert's opinion can be a vague concept such as "around 20MU[1]," "about 20MU," "between 18MU and 22MU." There are much other information and knowledge that cannot generally be well described by random variables because of database with incomplete or wrong information and sometimes the lack of sample data. For instance, investors in energy sector would like to estimate the coal reserves in some area, but even so after exploration, analysis drawn by appraisers will always be "about billions of tons." The precede estimation "billions of tons" is a value expert's estimation rather than observation because the coal reserve has an exact true value that we do not know but estimate. In such situations, future returns expressed as vague concepts are represented by fuzzy variables. Therefore, scholars (Liu and Liu 2002; Huang 2008; Li *et al*, 2010; Sadefo Kamdem *et al*, 2012) studied, by means of the credibility measure, moments and semi-moments of fuzzy variables and they used the obtained theoretical results to propose portfolio optimization models when returns are fuzzy. In that approach based on parameters of fuzzy variables (namely quantitative approach), we have the mean-variance model proposed by Huang (2008), the mean–variance–skewness model proposed by Li *et al*, (2010) and the mean–variance–skewness–semi-kurtosis model proposed by Sadefo Kamdem *et al* (2012). More recently, Chen and Tsaur (2016) proposed a weighted fuzzy portfolio model based on a weighted function of possibility mean and variance in order to approach portfolio selection differently in response to the varying investment return. Saborido *et al* (2016) proposed the mean–downside risk skewness model for portfolio selection which takes into account the multidimensional nature of the portfolio selection problem. Bilbao-Terol *et al* (2016) proposed a sequential goal programming model with fuzzy hierarchies for solving portfolio selection problem. In fuzzy portfolio theory, some proposed models are sometimes implemented by efficient algorithms such as mutation, crossover and reparation operators proposed by Saborido

**Correspondence: Jules Sadefo Kamdem, DFR SJE, Campus de Troubiran, Université de Guyane, B.P. 792, 97337 Cayenne Cedex, France.*
E-mail: jsadefo@gmail.com; jules.sadefo@univ-guyane.fr; sadefo@lameta.univ-montp1.fr

[1]MU: Monetary Unit

et al (2016) and the fuzzy goal programming and linear physical programming applied by Kucukbay and Araz (2016).

It is important to notice that for most of the previously cited models, constraints are defined by means of target values, that is, all selected (best) portfolios depend on given target values. For instance, risky investors would like to maximize their benefits as possible with a maximum risk level to avoid, whereas risk-averse investors intend first to reduce the risk of investment with a minimum benefit to obtain. This method is not flexible in the sense that it proposes optimal portfolios by taking into account investor's preferences given by different target values, which can vary from one to another and consequently does not solve this problem in a general way. On the other hand, it is burdensome for an investor to evaluate risk in all situations. However, Georgescu and Kinnunen (2013) proved that the risk evaluation depends on the fact that the agent or investor is more or less risk averse and they propose the credibility index of riskiness of Aumann–Serrano type to evaluate and to compare two risks described by fuzzy variables. Our new approach focuses on the determination of optimal portfolios in the case where there is no information relative to the investor's preferences.

This approach is based on ranking of several alternatives (variables). Some scholars (Cheng *et al,* 2012; Chu and Tsao 2002; Detyniecki and Yager 2001; Peng *et al,* 2007; Saeidifar 2011 and, Wang and Kerre 2001) proposed several approaches and properties for ranking fuzzy variables with respect to possibility or credibility measures. In this paper, we focus on two dominances, namely the first- and the second-order dominances, introduced by Peng *et al* (2007) as a fuzzy extension of stochastic dominance of random variables to fuzzy variables. In order to improve first results obtained by Peng *et al* (2007), we propose characterizations of these two dominances, we determine some properties for those dominances, and we bring a contribution to portfolio selection problem by introducing an approach, inspired from Game Theory and based on dominances, for obtaining optimal portfolios.

The paper is planned as follows: Section 2 reviews some useful notions on credibility measure and credibility distribution of a fuzzy variable introduced by Liu and Liu (2002). We end by recalling the first- and the second-order dominance relations on fuzzy variables introduced by Peng *et al* (2007). In Section 3, we characterize the first-order dominance for trapezoidal fuzzy numbers and we justify that it is not a complete binary relation (it does not compare some couples of trapezoidal fuzzy numbers). We introduce the two notions of interval of coincidence and crossing point of two fuzzy variables. We use crossing points to characterize the second-order dominance relation and we justify that it is not a complete relation. We establish that the first-order dominance is stronger that the second one and we prove that the two dominances satisfy six well-known properties of comparison methods of fuzzy variables. We apply the two proposed dominances in portfolio selection by introducing the set of best portfolios of a finite number of assets with respect to the first- and the second-order dominance. Based on characterization of the first-order dominance, we implement new notion to numerically display some of the best portfolios of the usual example of the set of portfolios of seven assets introduced by Huang (2008). Comparisons of the three best portfolios with those obtained by the quantitative approach are presented. Section 4 gives some concluding remarks, and Section 1 is Appendix containing characterization of crossing points of fuzzy variables, proofs of some results and some parameters of a triangular fuzzy number.

2. Preliminaries

Let ξ be a fuzzy variable described on $\mathbb{R}$ by its membership function μ interpreted as: for any $x \in \mathbb{R}$, $\mu(x)$ represents the degree that ξ takes value x. The core of ξ is a crisp subset of $\mathbb{R}$ defined by $Cor(\xi) = \{x \in \mathbb{R}, \mu(x) = 1\}$.

A fuzzy number ξ which is a fuzzy variable satisfying: $\exists a, b, c, d \in \mathbb{R}$ with $a \le b \le c \le d$ such that (1) μ is upper semi-continuous, (2) $\forall r \notin [a, d], \mu(r) = 0$, (3) μ is increasing on $[a, b]$ and decreasing on $[c, d]$ and (4) $\forall r \in [b, c], \mu(r) = 1$. Thus, we denote it by $\xi = (a, b, c, d)$. In the particular case where μ is a straight line on $[a, b]$ and $[c, d]$, $\xi = (a, b, c, d)$ is the usual and well-known trapezoidal fuzzy number. If $b = c$, then $\xi = (a, b, d)$ is a triangular fuzzy number.

Liu (2004, 2014) and Liu and Liu (2002) introduced the credibility measure defined as follows: for any set B,

$$Cr(\{\xi \in B\}) = \frac{1}{2}\left(\sup_{x \in B} \mu(x) - \sup_{x \in B^c} \mu(x) + 1\right). \tag{1}$$

Notice that it is an average of the possibility and necessity measures introduced earlier by Zadeh (1978) as follows: for any set B, $Pos(\{\xi \in B\}) = \sup_{x \in B} \mu(x)$ and $Nec(\{\xi \in B\}) = 1 - \sup_{x \in B^c} \mu(x)$.

Liu (2004) defined the cumulative credibility distribution function (for short distribution function) $\Phi : \mathbb{R} \to [0, 1]$ of a fuzzy variable ξ as follows:

$$\forall t \in \mathbb{R}, \Phi(t) = Cr\{\xi \le t\} = \frac{1}{2}\left[1 + \sup_{x \in]-\infty; t]} \mu(x) - \sup_{x \in]t; +\infty[} \mu(x)\right]. \tag{2}$$

Liu (2004) proved that Φ is a continuous function.

The distribution function Φ of a fuzzy number $\xi = (a, b, c, d)$ is defined by:

$$\forall r \in \mathbb{R}, \Phi(r) = \begin{cases} 0 & \text{if} \quad r < a \\ \frac{1}{2}\mu(r) & \text{if} \quad a \le r < b \\ \frac{1}{2} & \text{if} \quad b < r < c \\ 1 - \frac{1}{2}\mu(r) & \text{if} \quad c \le r < d \\ 1 & \text{if} \quad d \le r \end{cases}. \tag{3}$$

Φ is an increasing function, that is, $\forall x \in [a, b], \forall y \in [b, c], \forall z \in [c, d]$, $\Phi(x) \leq \Phi(y) \leq \Phi(z)$. With (3), it is easy to check that Φ is a continuous function using the fact that the membership function μ is upper semi-continuous.

Throughout this paper: Φ_1, Φ_2 are distribution functions of the fuzzy variables ξ_1 and ξ_2, respectively.

Let us end this section by recalling the first- and second-order dominance relations on the set of fuzzy variables introduced by Peng *et al* (2005).

Definition 1 (See Peng *et al*, 2007, pages 32 and 33, Definitions 7 and 8)

1. The first-order dominance is the binary relation on fuzzy variables denoted $\succeq_1$ and defined by:

$$\xi_1 \succeq_1 \xi_2 \quad \text{if} \quad \forall \mathrm{r} \in \mathbb{R}, \Phi_1(\mathrm{r}) \leq \Phi_2(\mathrm{r}). \tag{4}$$

2. The second-order dominance is the binary relation on fuzzy variables denoted $\succeq_2$ and defined by:

$$\xi_1 \succeq_2 \xi_2 \quad \text{if} \quad \forall \mathrm{t} \in \mathbb{R}, \int_{-\infty}^{\mathrm{t}} [\Phi_2(\mathrm{r}) - \Phi_1(\mathrm{r})]\mathrm{dr} \geq 0. \tag{5}$$

In the definition of $\succeq_2$, we note that $\int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr$ represents a balance of areas between the curves of Φ_1 and Φ_2, that is, the difference of areas resulting from integrating each function from $-\infty$ to t, with the following order: the area below the curve of Φ_2 minus the area below the curve of Φ_1.

From the previous definitions, we deduce strict dominance relations by: $\xi_1 \succ_1 \xi_2$ iff $(\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$ and $\exists r_0 \in \mathbb{R}, \Phi_1(r_0) < \Phi_2(r_0))$ and $\xi_1 \succ_2 \xi_2$ iff $\begin{cases} \forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]dr \geq 0 \\ \exists t_0 \in \mathbb{R}, \int_{-\infty}^{t_0} [\Phi_2(r) - \Phi_1(r)]dr > 0 \end{cases}$. In addition, the indifference of $\succeq_1$ is given by: $\xi_1 \sim_1 \xi_2$ **if** $\forall r \in \mathbb{R},\ \Phi_1(r) = \Phi_2(r)$.

In the next section, we study the two-order dominance relations on fuzzy variables.

3. First- and second-order dominances: characterization, properties and application

In the following subsection, we characterize the first-order dominance on trapezoidal fuzzy numbers and we justify that it is not a complete binary relation on fuzzy variables.

3.1. First-order dominance: characterization and not completeness

Our first main result characterizes $\succeq_1$ for two trapezoidal fuzzy variables.

Theorem 1 *Let $\xi_1 = (a, b, c, d)$ and $\xi_2 = (a', b', c', d')$ be two trapezoidal fuzzy variables.*

1.

$$\xi_1 \succeq_1 \xi_2 \Leftrightarrow \begin{cases} a \geq a' \\ b \geq b' \\ c \geq c' \\ d \geq d' \end{cases}. \tag{6}$$

2. $\xi_1 \sim_1 \xi_2$ ifandonlyif $\xi_1 = \xi_2$.

In other words, $\xi_1 \not\succeq_1 \xi_2$ if and only if $(a < a'$ or $b < b'$ or $c < c'$ or $d < d')$.

By means of $\succeq_1$, Figure 1 illustrates that on left, trapezoidal fuzzy number $\xi_2 = (a', b', c', d')$ dominates $\xi_1 = (a, b, c, d)$ while on right, neither dominates another.

Proof 1) ($\Rightarrow$) It is obvious to prove the necessary condition. ($\Leftarrow$) Assume that that $a \geq a'$, $b \geq b'$, $c \geq c'$ and $d \geq d'$. Let us prove that $\xi_1 \succeq_1 \xi_2$, that is, $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$.

We consider the 8 following cases : $r \in]-\infty; a']$, $r \in [a'; a]$, $r \in [b'; a]$, $r \in [\max(a, b'); b]$, $r \in [b; c']$, $r \in [c'; \min(c, d')]$, $r \in [\min(c, d'); d]$, $r \in [d; +\infty[$, and the results are easily obtained according to relation (3). $\square$

The following example compares two trapezoidal fuzzy numbers by means of the previous characterization, and it justifies that $\succeq_1$ is not a complete binary relation.

Example 1 1. Let us consider the three trapezoidal fuzzy numbers: $\rho_1 = (-2, -1, 4, 9)$, $\rho_2 = (1, 2, 3, 7)$ and $\rho_3 = (2, 3, 4, 8)$. We have the three following comparisons: $\rho_3 \succeq_1 \rho_2$, $\rho_3 \not\succeq_1 \rho_1$ since $8 < 9$ and $\rho_2 \not\succeq_1 \rho_1$ since $3 < 4$ and $7 < 9$.

2. ρ_1 and ρ_2 are incomparable by means of the first-order dominance.

In the following subsection, we characterize the second-order dominance relation $\succeq_2$. For that, we proceed as follows: we introduce the notion of interval of coincidence of two fuzzy variables, and we use it to introduce the notion of crossing points of two fuzzy variables. We then characterize the dominance by means of crossing points. We characterize crossing points for two trapezoidal fuzzy variables in Appendix.

3.2. Characterization of the second-order dominance

3.2.1. Interval of coincidence and crossing points for fuzzy variables The intervals of coincidence of two fuzzy variables are the half-open interval, open at the right, where the two curves of their distributions functions coincide. For example, in Figure 2, the two straight lines entitled I.C. are the two intervals of coincidence of two curves. Formally, we have:

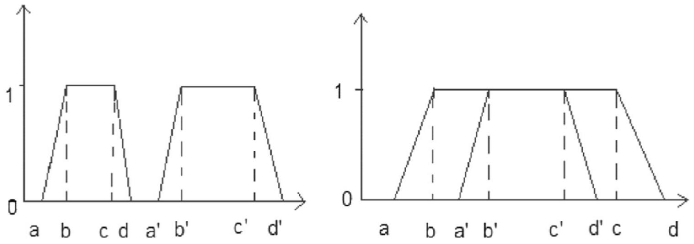

Figure 1 On *left*,(a', b', c', d') dominates (a, b, c, d), and on *right*, they are incomparable, with respect to the first-order dominance.

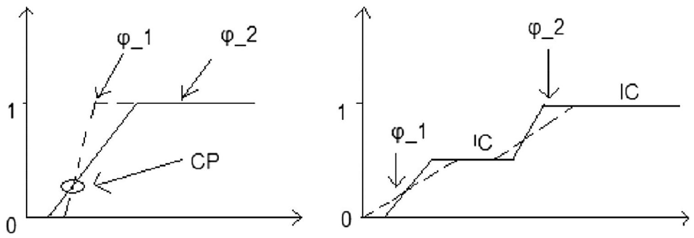

Figure 2 On *left*, crossing a point of two distribution function and on *right* intervals of coincidence of two distribution functions.

Definition 2 The half-open interval $[a, b)$, with $a<b$ is an interval of coincidence (IC) for Φ_1 and Φ_2 if $\Phi_1(t) = \Phi_2(t)$ for all $t \in [a, b)$.

From this definition, we can deduce that any value t_0 belongs to an interval of coincidence if there exists some $\epsilon > 0$ such that the interval $[t_0, t_0 + \epsilon)$ is IC.

We now introduce two types of crossing points for fuzzy variables, namely crossing points of types I and II. The crossing point of type II of ξ_1 and ξ_2 is the point where the two curves of their distribution functions intersect and the curve which strictly minimizes before that point strictly maximizes after it. The crossing point of type I of ξ_1 and ξ_2 is the upper bound of a given interval of coincidence (point where the two curves of the distribution functions coincide before it and are distinct after it). Formally, we have the following definition. Characterizations of crossing points for trapezoidal fuzzy numbers are given in Appendix.

Definition 3 Crossing points (CP)

1. If t_0 does not belong to an IC, but $[a, t_0)$ is an IC and $a = \inf t$ such that $[a, t_0)$ *is an IC*, t_0 corresponds to a CP of type I if there exists some $\epsilon > 0$ such that for all $s \in (0, \epsilon)$, we have

$$\begin{cases} \Phi_1(a-s) \neq \Phi_2(a-s) \\ \Phi_1(t_0+s) \neq \Phi_2(t_0+s) \\ \begin{pmatrix} \Phi_1(a-s)-\Phi_2(a-s)<0 \text{ and } \Phi_1(t_0+s)-\Phi_2(t_0-s)>0 \\ \text{or} \\ \Phi_1(a-s)-\Phi_2(a-s)>0 \text{ and } \Phi_1(t_0+s)-\Phi_2(t_0+s)<0 \end{pmatrix} \end{cases}.$$

2. Any other value t_0 corresponds to a CP of type II if there exists some $\epsilon > 0$ such that for all $s \in (0, \epsilon)$, we have

$$\begin{cases} \Phi_1(t_0-s) \neq \Phi_2(t_0-s) \\ \Phi_1(t_0+s) \neq \Phi_2(t_0+s) \\ \begin{pmatrix} \Phi_1(t_0-s)-\Phi_2(t_0-s)<0 \text{ and } \Phi_1(t_0-s)-\Phi_2(t_0+s)>0 \\ \text{or} \\ \Phi_1(a-s)-\Phi_2(a-s)>0 \text{ and } \Phi_1(t_0-s)-\Phi_2(t_0+s)<0 \end{pmatrix} \end{cases}.$$

3. Convention: (a) if t_0 belongs to an IC, it does not correspond to a CP; (b) let $m_1 = \inf\{t/\Phi_1(t) > 0\}$ and $m_2 = \inf\{t/\Phi_2(t) > 0\}$, and let $t_1 = \min(m_1, m_2)$: the interval $(-\infty, t_1)$ is an IC and t_1 does not correspond to a CP.

We notice that Osuna (2012) (Definition 3.2 of page 760) introduced interval of coincidence and crossing points for random variables. Therefore, the previous definitions are fuzzy counterparts of such notions.

In the following, we characterize by means of crossing points the second-order dominance of two fuzzy variables. We display one example of comparison of two trapezoidal fuzzy variables by means of such characterization, and we justify that this dominance does not compare some couples of trapezoidal fuzzy variables.

3.2.2. Second-order dominance: characterization and non-completeness Our second main result establishes a characterization of the second-order dominance relation. The definition of this dominance stipulates that we compare two fuzzy variables by checking the positivity of the balance area between the two curves of their distribution functions from the left $(-\infty)$ to each real number t. Since we have infinite real numbers, we have to check infinite areas in order to compare two fuzzy variables by means of the second-order dominance. In addition to this definition, the following theorem stipulates that we have to check a finite number of areas from the left to each crossing point between the two variables (since crossing points are finite). Its proof is in Appendix.

Theorem 2 *Let ξ_1 and ξ_2 be two fuzzy variables with a finite number of crossing points $\{t_{01}, \ldots, t_{0k}\}$ (ordered so increasing) such that $t_{01} > \min\{\inf\{t : \Phi_1(t) > 0\}, \inf\{t : \Phi_2(t) > 0\}\}$. Let Φ_1 and Φ_2 their respective absolutely continuous credibility distributions. Then,*
$\xi_1 \succ_2 \xi_2$ if and only if

$$\begin{cases} \forall i \in \{1,2,\ldots,k\}, \int_{-\infty}^{t_{0i}} [\Phi_2(r) - \Phi_1(r)]dr \geq 0 \\ \begin{pmatrix} \int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]dr = 0 \quad \text{and} \\ \exists t_{0h} \in \{t_{01}, \ldots, t_{0k}\}, \int_{-\infty}^{t_{0h}} [\Phi_2(r) - \Phi_1(r)]dr > 0 \\ \text{or} \\ \int_{-\infty}^{\infty} [\Phi_2(r) - \Phi_1(r)]dr > 0 \end{pmatrix} \end{cases}. \tag{7}$$

Remark 1 1. We have an analogous result if $\exists \epsilon > 0$, $\exists r_0, \forall s \in (0, \epsilon), \mu_i(r_0 - s) > \mu_j(r_0 - s)$ and $\mu_i(r_0 + s) < \mu_j(r_0 + s)$.

2. When there is no crossing point, the distribution's curves do not intersect and we can use the first-order dominance to compare two fuzzy variables.

3. In other words, the second-order dominance relation $\succeq_2$ is useful for comparing two fuzzy variables when they cannot be compared with respect to the first-order dominance relation $\succeq_1$.

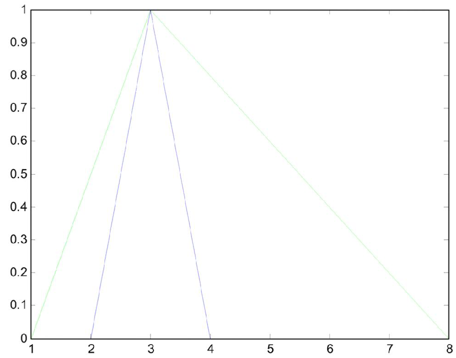

Figure 3 Incomparable fuzzy variables by means of the second-order dominance.

The following example compares two trapezoidal fuzzy variables by means of the second-order dominance relation. It also justifies that this dominance is not a complete binary relation on fuzzy variables (see Figure 2).

Example 2 1. Let $\xi_1 = (1,2,3,4)$ and $\xi_2 = (-1,0,1,2)$ be two trapezoidal fuzzy variables. It is easy to check that there is no crossing point between Φ_1 and Φ_2. We have: $\int_{-\infty}^{+\infty} [\Phi_2(x) - \Phi_1(x)]\, dx = 2 > 0$, that is, $\xi_1 \succ_2 \xi_2$ by (7).

2. The binary relation $\succeq_2$ on the set of fuzzy variables is not complete. Let $\xi_1 = (1,3,8)$ and $\xi_2 = (2,3,4)$ be the two triangular fuzzy variables in Figure 3. The only crossing point is $r_0 = 3$ (by using Proposition 3 in Appendix). Then, we have: $\int_{-\infty}^{3} [\Phi_1(r) - \Phi_2(r)]dr = \frac{1}{4} > 0, \int_{-\infty}^{+\infty} [\Phi_1(r) - \Phi_2(r)]dr = \frac{-1}{5} < 0$ and by Theorem 2, we conclude that $\xi_1 \not\succeq_2 \xi_2$ and $\xi_2 \not\succeq_2 \xi_1$.

In the following subsection, we establish some properties of the dominances. For that, we establish relationships between the two dominances and we show that these dominances satisfy some well-known properties of comparisons methods of fuzzy variables.

3.3. Comparison and some properties of dominances

The following result stipulates that $\succeq_1$ is stronger than $\succeq_2$.

Proposition 1 *Let ξ_1 and ξ_2 be two fuzzy variables. Then,*

$$\xi_1 \succeq_1 \xi_2 \Rightarrow \xi_1 \succeq_2 \xi_2.$$

Proof Let us assume that $\xi_1 \succeq_1 \xi_2$ and we prove that $\xi_1 \succeq_2 \xi_2$.

Since $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$ $\forall t \in \mathbb{R}, \int_{-\infty}^{t}[\Phi_2(r) - \Phi_1(r)]\ dr \geq 0.$ □

The following example justifies that the converse of the previous implication is not true.

Example 3 Let us consider the triangular fuzzy variables $\xi_1 = (1, 3, 5)$ and $\xi_2 = (2, 3, 4)$. The only crossing point is $r_0 = 3$ (by using Proposition 3 established in Appendix). Then, we have:

$\int_{-\infty}^{3}[\Phi_1(r) - \Phi_2(r)]dr = \frac{1}{4} > 0,\quad \int_{-\infty}^{+\infty}[\Phi_1(r) - \Phi_2(r)]\ dr = 0$, and by Theorem 2 , we conclude that $\xi_2 \succeq_2 \xi_1$. But by Theorem 1, $\xi_2 \not\succeq_1 \xi_1$.

Remark 2 Let us notice that, according to Proposition 1, if ξ_1 does not dominate ξ_2 with respect to $\succeq_2$, then ξ_1 does not dominate ξ_2 with respect to $\succeq_1$.

Let us recall six reasonable properties for ordering fuzzy quantities introduced by Wang and Kerre (2001).

Let S be the set of independent trapezoidal fuzzy variables, $\mathcal{A}$ and $\mathcal{A}'$ two finite subsets of S and $\succeq_M$ a comparison method of two elements of S (dominance relation on S). We denote by $\sim_M$ and $\succeq_M$ its indifference and strict components. Let us introduce some well-known properties of $\succeq_M$.

Definition 4 (Wang and Kerre 2001, page 380, Section 3)

1. A_1) $\forall A \in \mathcal{A}, A \succeq_M A$.
2. A_2) $\forall (A, B) \in \mathcal{A}^2, A \succeq_M B$ and $B \succeq_M A$, then $A \sim_M B$.
3. A_3) $\forall (A, B, C) \in \mathcal{A}^3, A \succeq_M B$ and $B \succeq_M C \Rightarrow A \succeq_M C$.
4. A_4) $\forall (A,B) \in \mathcal{A}^2, \inf supp(A) > \sup supp(B) \Rightarrow A \succeq_M B$. Stronger version: A_4') $\forall (A,B) \in \mathcal{A}^2, \inf supp(A) > \sup supp(B) \Rightarrow A \succ_M B$.
5. m A_5) *Let* $A, B \in \mathcal{A} \cap \mathcal{A}'$. $A \succeq_M B$ on $\mathcal{A} \Leftrightarrow A \succeq_M B$ on $\mathcal{A}'$.
6. A_6) Let $A, B \in \mathcal{A}$ such that $A + C, B + C$ be elements of $\mathcal{A}$. If $A \succeq_M B$, then $A + C \succeq_M B + C$. A_6') Let $A, B \in \mathcal{A}$ such that $A + C, B + C$ be elements of $\mathcal{A}$ with $C \neq \emptyset$. If $A \succ_M B$, then $A + C \succ_M B + C$.

Our third main result establishes that first- and second-order dominances satisfy previous properties.

Proposition 2 1) $\succeq_1$ *satisfies* $A_1, A_2, A_3, A_4, A_5, A_6$ *and* A_6'.

2) $\succeq_2$ *satisfies* $A_1, A_2, A_3, A_4, A_5, A_6$ *and* A_6'.

Proofs of some results of Proposition 2 require the following lemma which stipulates that translating two trapezoidal fuzzy variables by adding a third trapezoidal fuzzy variable preserves the crossing points. The proofs of the proposition and the lemma are in Appendix.

Lemma 1 *Let* ξ_1, ξ_2 *and* θ *be three independent trapezoidal fuzzy variables.* Φ_1, Φ_2, Φ_1' *and* Φ_2' *are, respectively, the credibility distributions functions of fuzzy variables* ξ_1, ξ_2, $\xi_1 + \theta$ *and* $\xi_2 + \theta$. *Then, we have*:

- $(\exists r_0 \in \mathbb{R}, \Phi_1(r_0) = \Phi_2(r_0)) \Leftrightarrow (\exists t_0 \in \mathbb{R}, \Phi_1'(t_0) = \Phi_2'(t_0))$.
- *For all crossing points* $v \in \mathbb{R}$ *between* Φ_1 *and* Φ_2 , $\exists u_v \in \mathbb{R}$, *crossing point between* Φ_1' *and* Φ_2' *such that:* $\int_{-\infty}^{v}[\Phi_1(r) - \Phi_2(r)]\ dr = \int_{-\infty}^{u_v}[\Phi_1'(r) - \Phi_2'(r)]\ dr.$
- $\int_{-\infty}^{+\infty}[\Phi_1(r) - \Phi_2(r)]\ dr = \int_{-\infty}^{+\infty}[\Phi_1'(r) - \Phi_2'(r)]\ dr.$

In the following subsection, we apply the characterization of the first-order dominance for Portfolio Selection in Finance. For that, we introduce the set of best portfolios as a subset of portfolios which dominate in each pairwise comparison. Based on the characterization of the first dominance, we determine the optimization model that defines that set. We implement, by means of MATLAB, the obtained model to determine some of the best portfolios in the set of portfolios of seven assets where future returns are triangular fuzzy variables introduced by Huang (2008). We compare parameters of these best portfolios with the ones of selected portfolios obtained with quantitative models.

3.4. Application in Finance: set of best portfolios of a finite number of assets

Let us consider a family $A = (\xi_i)_{1 \leq i \leq n}$ of n assets where returns are described by triangular fuzzy variables. A portfolio return ξ associated with A is a linear combination of the n asset returns defined by $\xi = \sum_{i=1}^{n} x_i \xi_i$ where x_i represents the proportion of capital invested in asset i. We have $x_i \in [0, 1]$ and we assume that and $\sum_{i=1}^{n} x_i = 1$, that is, all the capital is invested (shared) in the n assets. Hence, we have the set of portfolios associated with A is

$$\mathcal{P}_A = \left\{ \xi = \sum_{i=1}^{n} x_i \xi_i, x_i \in [0, 1], \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad \xi_i \in \mathrm{A} \right\}. \tag{8}$$

The main question is to determine the best portfolios of $\mathcal{P}_A$ with respect to one of the two dominance relations. Notice that the subset of best elements of a set according to a dominance relation is extremely studied in Game Theory and Social Choice. Using the well-known Game Theory terminology, we can say that the best portfolios are Condorcet winners in $\mathcal{P}_A$, that is, portfolios which dominate other ones in each pairwise comparison. More formally, the set of best portfolios of $\mathcal{P}_A$ with respect to $\succeq_1$ or with respect to $\succeq_2$, respectively, denoted by $\mathcal{B}_{\succeq_1}(\mathcal{P}_A)$ and $\mathcal{B}_{\succeq_2}(\mathcal{P}_A)$, are, respectively, defined by:

$$\mathcal{B}_{\succ_1}(\mathcal{P}_A) = \{\xi \in \mathcal{P}_A, \forall \eta \in \mathcal{P}_A, \xi \succeq_1 \eta\}. \tag{9}$$

and

$$\mathcal{B}_{\succ_2}(\mathcal{P}_A) = \{\xi \in \mathcal{P}_A, \forall \eta \in \mathcal{P}_A, \xi \succeq_2 \eta\}. \tag{10}$$

According to Remark 2, non-dominated portfolios with respect to $\succeq_2$ belong to the family of non-dominated portfolios with respect to $\succeq_1$. In other words, we have: $\mathcal{B}_{\succeq_2}(\mathcal{P}_A) \subset \mathcal{B}_{\succeq_1}(\mathcal{P}_A)$. In the rest of this section, we will implement the larger set of best portfolios $\mathcal{B}_{\succeq_1}(\mathcal{P}_A)$ defined by (9).

For that, we introduce the following notations. For $(x_i)_{1\leq i\leq n}$, $(y_i)_{1\leq i\leq n}$ such that $x_i, y_i \in [0,1]$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 1$ and for all $i \in \{1,\ldots,n\}$, $\xi_i = (a_i, b_i, c_i)$, we have: $\xi = (f(x_1,\ldots,x_n), g(x_1,\ldots,x_n), h(x_1,\ldots,x_n))$ and $\eta = (f(y_1,\ldots,y_n), g(y_1,\ldots,y_n), h(y_1,\ldots,y_n))$ where $f(x_1,\ldots,x_n) = \sum_{i=1}^n x_i a_i$, $g(x_1,\ldots,x_n) = \sum_{i=1}^n x_i b_i$ and $h(x_1,\ldots,x_n) = \sum_{i=1}^n x_i c_i$.

Based on characterization of $\succeq_1$ (see Theorem 1) and those notations, (9) becomes:

$$\mathcal{B}_{\succeq_1}(\mathcal{P}_A) = \left\{ \sum_{i=1}^n x_i \xi_i, , \forall (y_i)_{1\leq i\leq n}, \left\{ \begin{array}{l} f(x_1,\ldots,x_n) \geq f(y_1,\ldots,y_n) \\ g(x_1,\ldots,x_n) \geq g(y_1,\ldots,y_n) \\ h(x_1,\ldots,x_n) \geq h(y_1,\ldots,y_n) \\ \sum_{i=1}^n x_i = 1, \sum_{i=1}^n y_i = 1 \\ x_i \in [0,1], y_i \in [0,1], \forall i \in \{1,\ldots,n\} \end{array} \right\} \right. . \tag{11}$$

$\mathcal{B}_{\succeq_1}(\mathcal{P}_A)$ defined by (11) is determined by the following multiobjective optimization program:

$$\left\{ \begin{array}{l} \max f(x_1,\ldots,x_n) \\ \max g(x_1,\ldots,x_n) \\ \max h(x_1,\ldots,x_n) \\ \sum_{i=1}^n x_i = 1 \\ x_i \in [0,1] \forall i \in \{1,\ldots,n\} \end{array} \right. . \tag{12}$$

In the following, we implement the previous multiobjective program for the usual family $A = (\xi_i)_{1\leq i\leq 7}$ of seven assets with returns described by the following triangular fuzzy variables (see Huang 2008; Li *et al,* 2010; Sadefo Kamdem *et al,* 2012):

$\xi_1 = (-0.3, 1.8, 2.3)$, $\xi_2 = (-0.4, 2.0, 2.2)$, $\xi_3 = (-0.5, 1.9, 2.7)$, $\xi_4 = (-0.6, 2.2, 2.8)$, $\xi_5 = (-0.7, 2.4, 2.7)$, $\xi_6 = (-0.8, 2.5, 3.0)$ and $\xi_7 = (-0.6, 1.8, 3.0)$.

In that case, the set of portfolios defined by (8) becomes $\mathcal{P}_A = \{\xi = (-0.3x_1 - 0.4x_2 - 0.5x_3 - 0.6x_4 - 0.7x_5 - 0.8x_6 - 0.6x_7, 1.8x_1 + 2x_2 + 1.9x_3 + 2.2x_4 + 2.4x_5 + 2.5x_6 + 1.8\ x_7, 2.3x_1 + 2.2x_2 + 2.7x_3 + 2.8x_4 + 2.7x_5 + 3x_6 + 3x_7)$ where $\forall i \in \{1,\ldots,7\}, x_i \in [0,1]$ and $\sum_{i=1}^7 x_i = 1\}$.

The multiobjective optimization program which determines $\mathcal{B}_{\succeq_1}(\mathcal{P}_A)$, defined by (12), becomes:

$$\left\{ \begin{array}{ll} \text{maximize} & -0.3x_1 - 0.4x_2 - 0.5x_3 - 0.6x_4 - 0.7x_5 - 0.8x_6 - 0.6x_7 \\ \text{maximize} & 1.8x_1 + 2x_2 + 1.9x_3 + 2.2x_4 + 2.4x_5 + 2.5x_6 + 1.8x_7 \\ \text{maximize} & 2.3x_1 + 2.2x_2 + 2.7x_3 + 2.8x_4 + 2.7x_5 + 3x_6 + 3x_7 \\ \text{subject to} & \\ x_1 + x_2 + \cdots + x_7 = 1 & \\ x_i \geq 0, i = 1,\ldots,7. & \end{array} \right. \tag{13}$$

The implementation is done with the MATLAB's command "gamultiobj" used for multiobjective optimization problems[2].

The three last lines of the following table show the three best portfolios (elements of $\mathcal{B}_{\succeq_1}(\mathcal{P}_A)$) obtained by solving (13)). In addition, lines 2 to 5 in Table 1 recall the best portfolios obtained previously with quantitative approach (by solving models based on parameters) and given in Table 2 of page 527 of Sadefo Kamdem *et al* (2012). Notice that each line indicates how the portfolio shares the capital in the seven assets.

Contrary to the four first portfolios obtained by means of parameters which indicated that an investor who intends to invest in the assets described by A must share its capital only on some assets (in each four first lines, at least one percentage x_i is null), the three best portfolios ξ, ξ' and ξ'' indicate that he must (1) diversify the capital on different assets (since values of x_i in each of the three portfolios are non-null), and (2) invest more on the assets ξ_1, ξ_2, ξ_5 and ξ_6 (at least 17%) and less on assets ξ_3, ξ_4 and ξ_7 (at most 8%). Portfolios in Table 1 can be viewed as triangular fuzzy variables in the following table.

Lines 2 to 5 of the following table recall parameters (mean, variance, skewness, kurtosis, semi-variance and semi-kurtosis) of the four first portfolios in Table 2 (given in Table 3 of page 527 in Sadefo Kamdem *et al,* 2012), and its three last lines present the ones of the three best portfolios that we have computed with formulas recalled in Appendix.

[2]Characteristics of the computer used for this implementation are: Pentium (R)4, CPU (1.80 GHZ), RAM (512 MO)

Table 1 Optimal selection from models based on parameters and from the set of best portfolios of $\succeq_1$.

Security i	*1 (%)*	*2 (%)*	*3 (%)*	*4 (%)*	*5 (%)*	*6 (%)*	*7 (%)*
Huang (2008)	00.00	47.06	00.00	35.28	17.66	00.00	00.00
Li *et al* (2010)	20.00	00.00	00.00	80.00	00.00	00.00	00.00
Sadefo Kamdem *et al* (2012)	20.04	00.00	00.00	79.89	00.00	00.07	00.00
Sadefo Kamdem *et al* (2012)	20.00	00.00	00.00	80.00	00.00	00.00	00.00
Tassak *et al* (Best portfolio ξ)	14.77	35.01	32.28	21.7	16.42	20.17	08.2
Tassak *et al* (Best portfolio ξ')	39.66	09.2	01.47	01.28	18.83	25.82	03.76
Tassak *et al* (Best portfolio ξ'')	41.72	05.55	01.10	00.86	17.29	31.47	02.02

Table 2 Optimal portfolios from models and the best portfolios viewed as triangular fuzzy variables

Optimal portfolio	*Triangular fuzzy variable*
Huang (2008)	$(-0.5; 2.1; 2.5)$
Li *et al* (2010)	$(-0.5; 2.1; 2.7)$
Sadefo Kamdem *et al* (2012)	$(-0.5; 2.1; 2.7)$
Sadefo Kamdem *et al* (2012)	$(-0.5; 2.12; 2.7)$
Tassak *et al* (Best portfolio ξ)	$(-0.5; 2.1; 2.6)$
Tassak *et al* (Best portfolio ξ')	$(-0.5; 2.1; 2.6)$
Tassak *et al* (Best portfolio ξ'')	$(-0.5; 2.1; 2.6)$

Analysis in Table 3 shows that, except the mean, the two new best portfolios ξ and ξ' have better parameters (variance, skewness, kurtosis, semi-variance and semi-kurtosis) than those of portfolios obtained from quantitative approach, whereas the third best portfolio ξ'' has better semi-variance and semi-kurtosis. The mean of the three best portfolios is less than those of the four optimal portfolios by the fact that the latter (models with parameters) were implemented with the target value of the mean equals to 1.6 (that was the minimal mean required by the investor). Therefore, the investor who intends to invest on the seven assets can choose between the two best portfolios ξ and ξ' (see lines 6 and 7 in Table 2 or Table 1).

We can illustrate these different results by the following histogram illustrated in Figure 4 (where BP means best portfolio):

4. Concluding remarks

In this paper, we characterize two tools to compare fuzzy variables, namely the first- and the second-order dominance relations. We justify that the first-order dominance is stronger than the second one and these two binary relations on fuzzy variables are not complete. We prove that they satisfy six properties. These results complement the literature on dominance relations on fuzzy variables. The characterization of the second-order dominance is based on new notions of crossing points between two fuzzy variables' distribution functions' curves . In addition, crossing points of two fuzzy variables have been characterized by means of their membership functions and thus complement the literature on characteristics of fuzzy variables.

We introduce the set of best portfolios as a subset of portfolios which dominate in each pairwise comparison other ones through one of the two dominances. The characterization of the first-order dominance allows us to write an optimization model describing elements of its set of the best portfolios. The numerical implementation of the model with the example of seven triangular assets, introduced by Huang (2008) and used by Li *et al* (2010) and Sadefo Kamdem *et al* (2012), displays the best portfolios which have better parameters than those obtained by Huang (2008), Li *et al* (2010) and Sadefo Kamdem *et al* (2012). This new approach (qualitative approach) for portfolio selection with fuzzy returns, based on dominances of fuzzy variables, is flexible (it is not restricted to target values) and it improves previous approach (quantitative approach) by providing the best portfolios with better parameters.

One can examine some open questions such as: (1) comparisons between the first- and second-order dominance relations and those existing in the literature (for instance, comparison methods introduced by Wang and Kerre (2001), (2) determination of the set of best portfolios with respect to the second-order dominance by using its characterization and

Table 3 Comparison of the four first moments of different optimal and best portfolios

Portfolio	*Mean*	*Variance*	*Skewness*	*Kurtosis*	*Semi-variance*	*Semi-kurtosis*
Huang (2008)	1.6	0.7235	−0.7543	1.7972	0.6124	1.7415
Li *et al* (2010)	1.6	0.7019	−0.6823	1.7291	0.6141	1.6872
Sadefo Kamdem *et al* (2012)	1.6	0.7018	−0.6823	1.7290	0.6140	1.6873
Sadefo Kamdem *et al* (2012)	1.6	0.7019	−0.6823	1.7291	0.6141	1.6872
Tassak *et al* (Best portfolio ξ)	1.5605	0.6973	−0.6666	1.7033	0.5832	1.5489
Tassak *et al* (Best portfolio ξ')	1.5712	0.69	−0.6634	1.6668	0.5863	1.5585
Tassak *et al* (Best portfolio ξ'')	1.5849	0.7029	−0.687	1.7277	0.5994	1.6299

[!t]

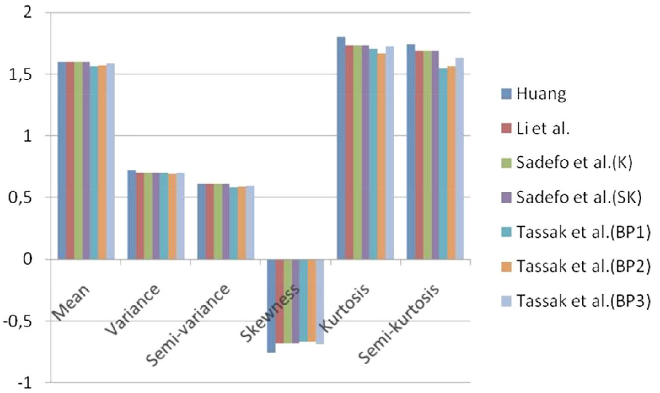

Figure 4 Comparison of characteristic values of optimal portfolio total returns.

(3) determination of necessary and sufficient conditions of the non-emptiness of the set of best portfolios with respect to an order dominance relation for a finite number of assets represented by trapezoidal fuzzy numbers.

References

Bilbao-Terol A., Arenas-Parra M., Cañal-Fernández V. and Jiménez M. (2016): A sequential goal programming model with fuzzy hierarchies to sustainable and responsible portfolio selection problem. *Journal of the Operational Research Society*, **67**(10): 1259–1273.

Brogan J.A. and Stidham S.Jr (2008): Non-separation in the mean-lower-partial-moment portfolio optimization problem, *European Journal of Operational Research*, **184**(2):701–710.

Chen I-F. and Tsaur R-C. (2016): Fuzzy portfolio selection using a weighted function of possibilistic mean and variance in business cycles, *International Journal of Fuzzy Systems*, **18**(2):151–159.

Cheng S-M, Munif A., Cheng G-S, Liu H-C. and Kuo B-C. (2012): Fuzzy risk analysis based on ranking generalized fuzzy numbers with different left heights and right heights, *Expert Systems With Applications*, **39**(7):6320–6334.

Chu T-C and Tsao C-T. (2002): Ranking fuzzy numbers with an area between the centroid point and original point, *Computers and Mathematics with Applications*, **43**(1–2):111–117.

Dentcheva D. and Ruszczynski A. (2004): Semi-infinite probabilistic optimization: First-order stochastic dominance constraint, *Optimization*, **53**(5–6):583–601.

Detyniecki M. and Yager R. (2001): Ranking fuzzy numbers using α-weighted valuations, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems,* **8**(5):573.

Georgescu I. and Kinnunen J. (2013): A Risk Approach by Credibility Theory, *Springer-Verlag Berlin Heidelberg and Fuzzy Information and Engineering Branch of the Operation Research Society of China*, **5**(4):399–416.

Grauer R. and Best M. (1991): Sensitivity analysis for mean-variance portfolio problems, *Management Science*, **37**(8):980–989.

Huang X. (2008): Mean-semivariance models for fuzzy portfolio selection, *Journal of Computational and Applied Mathematics*, **217**(1):1–8.

Konno H. and Suzuki K. (1995): A mean-variance-skewness optimization model, *Journal of the Operations Research Society of Japan*, **38**:137–187.

Kraus A. and Litzenberger R. (1976): Skewness preference and the valuation of risky assets, *Journal of Finance*, **31**(4):1085–1100.

Kucukbay F. and Araz C. (2016): Portfolio selection problem: a comparison of fuzzy goal programming and linear physical programming, *An International Journal of Optimization and Control: Theories and Applications (IJOCTA)*, **6**(2):121–128.

Li X., Qin Z. and Kar S. (2010): Mean-variance-skewness model for portfolio selection with fuzzy returns, *European Journal of Operational Research*, **202**(1):239–247.

Liu B. (2004): Uncertainty theory: An introduction to its axiomatics foundations. *Springer-Verlag, Berlin*.

Liu B. (2014): Uncertainty theory, 4th ed., *Springer-Verlag, Berlin*.

Liu B. and Liu Y.K. (2002): Expected value of fuzzy variable and fuzzy expected value models. *IEEE Transactions on Fuzzy Systems*, **10**(4):445–450.

Markowitz H. (1952): Portfolio selection,*Journal of Finance*, **7**(1):77–91.

Osuna E. (2012): Crossing points of distribution and a theorem that relates them to second order stochastic dominance. *Statistics and Probability Letters*, **82**(4):758–764.

Peng J., Mok H. and Tse W. (2005): Fuzzy dominance based on credibility distributions. *Springer-Verlag, Berlin*, 295–303.

Peng J., Jiang Q. and Rao C. (2007): Fuzzy dominance: a new approach for ranking fuzzy variables via credibility measure. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **15**(supp02):29–41.

Saborido R., Ruiz A., Bermúdez J., Vercher E. and Luque M. (2016): Evolutionary multi-objective optimization algorithms for fuzzy portfolio selection. *Applied Soft Computing*, **39**(c): 48–63.

Sadefo Kamdem J, Tassak DC, Fono LA (2012) Moments and semi-moments for fuzzy portfolio selection. *Insurance Mathematics and Economics*, **51**(3):517–530

Saeidifar A. (2011): Application of weighting functions to the ranking of fuzzy numbers. *Computers and Mathematics With Applications*, **62**(5):2246–2258.

Samuelson P. (1970): The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments, *Reviews of Economics Studies*, **37**(4):537–542.

Sengupta J. (1989): Portfolio decision as games, *International Journal of Systems Science*, **20**(8):1323–1334.

Sharpe W. (1971): A linear programming approximation for the general portfolio analysis problem, *Journal of Financial and Quantitative Analysis*, **6**(5):1263–1275.

Tobin J. (1965): The theory of portfolio selection, in F.H. Hahn and F.P.R. Brechling, eds., *The Theory of Interest Rates*, London

Wang X. and Kerre E. (2001): Reasonable properties for the ordering of fuzzy quantities, *Fuzzy Sets and Systems*, **118**(3):375–385.

Zadeh L.A. (1978): Fuzzy set as a basis of theory of possibility, *Fuzzy Sets and Systems*, **1**(1):3–28.

Appendix

Determination of crossing points of two trapezoidal fuzzy numbers

Definition 3 introduces crossing points of two fuzzy variables by means of their distributions functions. Since characterization of the second dominance is based on those points, it is important to give a simple way to determine those points for two trapezoidal fuzzy numbers. Therefore, the three first cases of the following result stipulate that a crossing point of two fuzzy numbers is either the intersection point of the increasing parts of the curves of their membership functions, either the intersection point of the decreasing parts of such curves, either the upper bound of the interval of coincidence of the constant parts of the curves. As illustrated in Figure 5, CP1 is the crossing point of type I and CP2 is the crossing point of type II. In addition, the three last cases of the result allow us to find crossing points when the core of at least one of the fuzzy numbers is a single point.

Proposition 3 *Let $\xi_i = (a_i, b_i, c_i, d_i)$ and $\xi_j = (a_j, b_j, c_j, d_j)$ be two fuzzy variables with μ_i and μ_j are their respective membership functions, Φ_i and Φ_j are their respective distribution functions. Let r_0 and ϵ be two reals numbers with $\epsilon > 0$. We have:*

1. $\forall s \in (0,\epsilon), \begin{cases} \mu_i(r_0-s) < \mu_j(r_0-s), \mu_i(r_0+s) > \mu_j(r_0+s) \\ r_0-s, r_0+s \in [a_i \vee a_j, b_i \wedge b_j] \end{cases} \Rightarrow r_0$ *is a crossing point of type II.*
2. $\forall s \in (0,\epsilon), \begin{cases} \mu_i(r_0-s) < \mu_j(r_0-s), \mu_i(r_0+s) > \mu_j(r_0+s) \\ r_0-s, r_0+s \in [c_i \vee c_j, d_i \wedge d_j] \end{cases} \Rightarrow r_0$ *is a crossing point of type II.*
3. $([b_i, c_i] \subseteq [b_j, c_j]$ and $[a_i, d_i] \subseteq [a_j, d_j], b_i \neq c_i, b_j \neq c_j) \Rightarrow c_i$ *is a crossing point of type I and* $b_i = \min\{t/[t, c_i)$ *is I.C*$\}$.
4. $([a_i, d_i] \subseteq [a_j, d_j], b_i = c_i, b_j \neq c_j, b_i \in [b_j, c_j]) \Rightarrow c_i$ *is a crossing point of type II .*
5. $([a_i, d_i] \subseteq [a_j, d_j], b_i \neq c_i, b_j = c_j, b_j \in [b_i, c_i]) \Rightarrow c_j$ *is a crossing point of type II.*

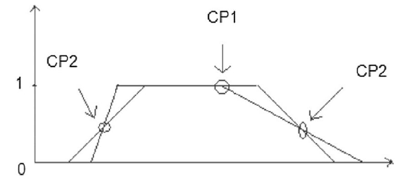

Figure 5 Crossing points: type I (CP1) and type II (CP2) of two trapezoidal fuzzy variables based on their membership functions.

6. $([a_i, d_i] \subseteq [a_j, d_j], b_i = c_i = b_j = c_j, a_i \neq a_j, d_i \neq d_j)$ $\Rightarrow c_j$ *is a crossing point of type II.*

To establish our result, we need the following lemma.

Lemma 2 *Let r_0 and ϵ be two reals numbers with $\epsilon > 0$. We have:*

1. $\forall s \in (0,\epsilon), \begin{cases} \mu_i(r_0-s) < \mu_j(r_0-s), \mu_i(r_0+s) > \mu_j(r_0+s) \\ r_0-s, r_0+s \in [a_i \vee a_j, b_i \wedge b_j] \end{cases} \Rightarrow \Phi_i(r_0-s) < \Phi_j(r_0-s), \Phi_i(r_0+s) > \Phi_j(r_0+s).$
2. $\forall s \in (0,\epsilon), \begin{cases} \mu_i(r_0-s) < \mu_j(r_0-s), \mu_i(r_0+s) > \mu_j(r_0+s) \\ r_0-s, r_0+s \in [c_i \vee c_j, d_i \wedge d_j] \end{cases}$ $\Rightarrow \Phi_i(r_0-s) > \Phi_j(r_0-s), \Phi_i(r_0+s) < \Phi_j(r_0+s).$
3. $\forall r \in \mathbb{R}, (r \in [b_i \vee b_j, c_i \wedge c_j]) \Rightarrow \Phi_i(r) = \Phi_j(r).$

The proof of this lemma is obvious.

Remark 3 We have an analogous result with $r_0 \in \mathbb{R}$ and $\epsilon > 0$ in the following case: $\forall s \in (0,\epsilon), \mu_i(r_0-s) > \mu_j(r_0-s)$ and $\mu_i(r_0+s) < \mu_j(r_0+s)$.

We now give the proof of Proposition 3.

Proof of Proposition 3

1. Let us consider $\epsilon > 0, r_0 \in \mathbb{R}$ and s a real number such that $0 < s < \epsilon$ and $\mu_i(r_0-s) < \mu_j(r_0-s)$, $\mu_i(r_0+s) > \mu_j(r_0+s)$, with $r_0-s, r_0+s \in [a_i \vee a_j, b_i \wedge b_j]$. According to Lemma 1, we have $\Phi_i(r_0-s) < \Phi_j(r_0-s)$ and $\Phi_i(r_0+s) > \Phi_j(r_0+s)$ and by Definition 3, we can conclude that r_0 is a crossing point of type II. We prove the converse case by the same manner.
2. We use the same method as in (1).
3. Let us prove that c_i is a crossing point of type I. $[b_i, c_i] \subseteq [b_j, c_j] \Rightarrow [b_i \vee b_j, c_i \wedge c_j] = [b_i, c_i]$ and by Lemma 2 and Definition 2, we have: $b_i = \min\{t/[t, c_i) is I.C\}$. Now, let us find $\epsilon_0 > 0$ such that $\forall s : 0 < s < \epsilon_0, \Phi_i(b_i - s) < \Phi_j(b_i - s)$ and $\Phi_i(c_i + s) > \Phi_j(c_i + s)$.

   i) If $b_i \neq b_j$ and $c_i \neq c_j$ Then, we set $\epsilon_0 = (b_i - b_j) \wedge (c_j - c_i)$ and we easily check that $\epsilon_0 > 0$ according to the fact that $[b_i, c_i] \subset [b_j, c_j]$ and $b_i \neq b_j, c_i \neq c_j$. We have two cases: first case: $b_i - b_j < c_j - c_i$ We have $\epsilon_0 = b_i - b_j$, and $b_i - \epsilon_0 = b_j, c_i + \epsilon_0 = c_i + b_i - b_j$. We obtain: $\Phi_i(b_i - s) < \Phi_i(b_i) = \frac{1}{2}$ because $b_i - s < b_i$ and Φ_i increases; on the other hand, by the fact that $0 < s < \epsilon_0$, and Φ_j increases, we have: $\Phi_j(b_i - s) > \Phi_j(b_i - \epsilon_0) = \Phi_j(b_j) = \frac{1}{2}$. Furthermore, $\Phi_i(c_i + s) > \Phi_i(c_i) = \frac{1}{2}$ because Φ_i increases and $\Phi_j(c_i + s) < \Phi_j(c_j) = \frac{1}{2}$ because $c_i + s < c_i + \epsilon_0 < c_i + c_j - c_i = c_j$ and Φ_j increases. <u>Second case</u>: $c_j - c_i < b_i - b_j$ We have $\epsilon_0 = c_j - c_i$, and $c_i + \epsilon_0 = c_j, b_i - \epsilon_0 = b_i - c_j + c_i$. We obtain $\Phi_i(b_i - s) - \Phi_j(b_i - s) < 0$ because: $\Phi_i(b_i - s) < \Phi_i(b_i) = \frac{1}{2}$ and $b_i - \epsilon_0 - b_j = b_i - b_j - (c_j - c_i) > 0$, that is, $b_i - \epsilon_0 > b_j$, so $\Phi_j(b_i - s) > \Phi_j(b_i - \epsilon_0) > \Phi_j(b_j) = \frac{1}{2}$ as Φ_j increases and $b_i - \epsilon_0 > b_j$. Furthermore, $\Phi_i(c_i + s) - \Phi_j(c_i + s) > 0$; indeed, $c_i + s < c_i + \epsilon_0 = c_j$, so $\Phi_j(c_i + s) < \Phi_j(c_j) = \frac{1}{2}$. On the other hand Φ_i increases and $\Phi_i(c_i + s) > \Phi_i(c_i) = \frac{1}{2}$.

ii) If $b_i = b_j$ and $c_i \neq c_j$ Then, $\epsilon_0 = c_j - c_i$ and we easily conclude as in i).

iii) If $c_i = c_j$ and $b_i \neq b_j$ Then, $\epsilon_0 = b_i - b_j$ and we easily conclude as in i).

iv) If $c_i = c_j$ and $b_i = b_j$. Then, we take $\epsilon_0 = (b_j - a_j) \wedge (d_j - c_j)$. It is easy to check that for all s such that $0 < s < \epsilon_0$, we have: $b_j - s \in (a_j, b_i)$ and $c_j + s \in (c_i, d_j)$. $(c_i = c_j, b_i = b_j) \Rightarrow [b_j, c_j] = [b_i, c_i]$; thus, the support of ξ_i is included in the support of ξ_j and their cores coincide that means μ_j and μ_i coincide only in $[b_j, c_j]$, and this justifies the fact that $\forall s \in (a_j, b_i), \mu_j(s) > \mu_i(s)$ and $\forall s \in (c_i, d_j), \mu_i(s) < \mu_j(s)$. Furthermore, $\forall s \in [a_j, b_i), \Phi_j(s) > \Phi_i(s)$ by the fact that $\mu_j(s) > \mu_i(s)$ and $\forall s \in [c_i, d_j), \Phi_j(s) < \Phi_i(s)$ by the fact that $\mu_i(s) < \mu_j(s)$; these last inequalities lead us to $\Phi_i(c_j + s) > \Phi_j(c_j + s), \Phi_i(b_j - s) < \Phi_j(b_j + s)$.

4. By taking $\epsilon_0 = \min(b_i - b_j, c_j - b_i)$, we can easily check that $\forall s$ such that: $0 < s < \epsilon$, $\Phi_i(c_i - s) < \Phi_j(c_i - s), \Phi_i(c_i + s) > \Phi_j(c_i + s)$.
5. By taking $\epsilon_0 = \min(b_j - b_i, c_i - b_j,)$ we can easily check that $\forall s$ such that: $0 < s < \epsilon$, $\Phi_j(c_j - s) < \Phi_i(c_j - s)$, $\Phi_j(c_j + s) > \Phi_i(c_j + s)$.
6. By taking $\epsilon_0 = \min(a_i - a_j, d_j - d_i)$, we can easily check that $\forall s$ such that: $0 < s < \epsilon_0$, $\Phi_j(c_j - s) > \Phi_i(c_j - s), \Phi_j(c_j + s) < \Phi_i(c_j + s)$. □

Proof of Theorem 2 The proof is similar to the one proposed by Osuna (2012) for random variables.

In the following, we establish the proofs of the following results: Lemma 1 and Proposition 2. □

Proofs of some results

We establish the proof of Lemma 1.

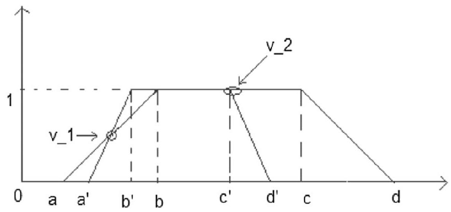

Figure 6 A particular position of two trapezoidal fuzzy variables.

Proof of Lemma 1 Let us consider the assumptions of the lemma. We set: $\xi_1 = (a, b, c, d)$, $\xi_2 = (a', b', c', d')$ and $\theta = (a'', b'', c'', d'')$. Without loss of generality, we suppose in all these proofs that $a < a' < b' < b < c' < c < d' < d$ (see Figure 6 for illustration). The other cases can be proved in the same way. By writing $\xi_1 + \theta = (a + a'', b + b'', c + c'', d + d'')$ and $\xi_2 + \theta = (a' + a'', b' + b'', c' + c'', d' + d'')$ and by considering the crossing points $v_1 = \frac{a'(b-a)-a(b'-a'}{(b-a)-(b'-a')}$ (type II) and $v_2 = c'$ (type I)(see Figure 6), we easily obtain the proof.

We now establish the proof of Proposition 2. □

Proof of Proposition 2 We consider, $\xi = (a, b, c, d)$, $\eta = (a', b', c', d')$ and $\theta = (a'', b'', c'', d'')$ be three elements of $\mathcal{A}$ with respective credibility distributions functions Φ_1, Φ_2, Φ_3. Let us suppose that $\xi + \eta = (a + a'', b + b'', c + c'', d + d'')$, $\eta + \theta = (a' + a'', b' + b'', c' + c'', d' + d'')$ are two elements of $\mathcal{A}$ and that Φ_1', Φ_2' are their respective credibility distributions functions.

1. Those properties can be easily proved for the first-order dominance relation $\succeq_1$.
2. Second-order dominance $\succeq_2$: Properties A_1,A_2, A_3, A_4 and A_5 can be checked easily. A_6) Let us assume that $\xi \succeq_2 \eta$. By the characterization of $\succeq_2$, we have: For all crossing points $v \in \mathbb{R}$, between Φ and Φ' , $\int_{-\infty}^{v} [\Phi_2(r) - \Phi_1(r)]\, dr \geq 0$ and $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]\, dr \geq 0$. According to Lemma 2, $\forall u_v \in \mathbb{R}$, crossing point between Φ_1' and Φ_2', $\int_{-\infty}^{v} [\Phi_2(r) - \Phi_1(r)]\, dr = \int_{-\infty}^{u_v} [\Phi_2'(r) - \Phi_1'(r)]\, dr \geq 0$. and $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]\, dr = \int_{-\infty}^{+\infty} [\Phi_2'(r) - \Phi_1'(r)]\, dr \geq 0$. Thus, $\xi + \theta \succeq_2 \eta + \theta$. A_6') The proof is similar to the one of A_6.

The proof is ended. □

We recall formulas of different characteristics of a triangular fuzzy variable $\xi = (a, b, c)$ with finite expected value e (see Sadefo Kamdem *et al*, 2012). For that, we set: $\alpha_1 = \max(b - a, c - b)$ and $\gamma = \min(b - a, c - b)$.

1. Expected value and variance:

$$e = E[\xi] = \frac{a + 2b + c}{4} \quad \text{and}$$

$$V[\xi] = \frac{33\alpha_1^3 + 21\alpha_1^2\beta_1 + 11\alpha_1\beta_1^2 - \beta_1^3}{384\alpha_1}.$$

2. Skewness:

$$S[\xi] = \frac{(c - a)^2}{32}(c + a - 2b).$$

3. Kurtosis:

$$K[\xi] = \frac{253\alpha_1^5 + 395\alpha_1^4\gamma + 17\alpha_1\gamma^4 + 290\alpha_1^3\gamma^2 + 70\alpha_1^2\gamma^3 - \gamma^5}{10.240\alpha_1}.$$

4. Semi-variance:

$$SV[\xi] = \frac{1}{6(b-a)}\left[(e-a)^3 + \frac{4}{(b-c)}(b-e)^3 \min(0, (b-e)^3)\right].$$

5. Semi-kurtosis:

$$K^S[\xi] = \frac{1}{10(b-a)}\left[(e-a)^5 + \frac{4}{(b-c)}(b-e)^5 \min(0, (b-e))\right].$$

Received 8 June 2016;
accepted 28 November 2016

Bibliography

[1] J. Adamo (1980): Fuzzy decisions trees, *Fuzzy Sets and Systems,* 4, 207-219.

[2] B. Asady and M. Zendehnam (2007): Ranking of fuzzy numbers by minimize distance, *Appl. Math. Model,* 31, 2589-2598.

[3] B. Bouchon-Meunier, L. Foulloy and M. Ramdani (1986): Logique floue, *Cepadues-Editions.*

[4] W. Briec, K. Kerstens and O. Junkung (2007): Mean-variance-skewness portfolio performance gauging: A general shortage function and dual approach, *Management Science,* 53, 135-149.

[5] J. Brogan and S. Jr Stidham (2008): Non-separation in the mean-lower-partial-moment portfolio optimization problem, *European Journal of Operational Research*, 184, 2, 701-710.

[6] C. Carlsson, R. Fullér and P. Majlender (2002): A possibilistic approach to selecting portfolios with highest utility score, *Fuzzy Sets and Systems,* 131, 13-21.

[7] R. Ezzati, T. Allahviranloo, S. khezerloo and M. khezerloo (2012): An approach for ranking of fuzzy numbers, *Experts Systems with Applications,* 39, 690-695.

[8] Fishburn P.C. (1977) Mean-Risk Analysis with Risk associated with Below-Target Returns. *The American Economic Review,* 67, **(2)**, 116-126.

[9] R. Grauer and M. Best (1991): Sensitivity analysis for mean-variance portfolio problems, *Management Science*, 37, 981-989.

[10] X. Huang (2007): Portfolio selection with fuzzy returns, *Journal of Intelligent and Fuzzy Systems*, 18, 383-390.

[11] X. Huang (2008): Mean-Semivariance models for fuzzy portfolio selection, *Journal of Computational and Applied Mathematics*, 217, 1-8.

[12] H. Levy, (1982): Stochastic dominance Rules for Truncated Normal Distributions: A Note *Journal of Finance,* 37, 1299-1303.

[13] H. Konno and K. Suzuki (1995): A mean-variance-skewness optimization model, *Journal of the Operations Research Society of Japan*, 38, 137-187.

[14] A. Kraus and R. Litzenberger (1976): Skewness preference and the valuation of risky assets, *Journal of Finance*, 21, 1085-1094.

[15] S. Li and J. Peng (2012): A new approach to risk comparison via uncertain measure. *Industrial Engineering and Management Systems*, 11, 176-182.

[16] X. Li, Z. Qin and S. Kar (2010): Mean-variance-skewness model for portfolio selection with fuzzy returns, *European Journal of Operational Research,* 202, 239-247.

[17] B. Liu (2002): Theory and Practice of Uncertain Programming, Physica-Verlag, Heidelberg.

[18] B. Liu (2004): Uncertainty Theory: An Introduction to its Axiomatics Foundations. Springer-Verlag, Berlin.

[19] B. Liu (2014): Uncertainty Theory, 4th ed., Springer-Verlag, Berlin.

[20] B. Liu and Y.K. Liu, (2002): Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems,* 10, 445-450.

[21] B. Liu and Y.K. Liu, (2003): Expected value operator of random fuzzy variable and random fuzzy expected models, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 11, 195-215

[22] H. Markowitz (1952): Portfolio selection, *Journal of Finance* 7, 77-91.

[23] H. Markowitz (1959): Portfolio selection: Efficient diversification of investments, Wiley, New York.

[24] W. Ogryczak and A. Ruszczynski (1999): From stochastic dominance to mean-risk models: Semideviations as risk measures. *European Journal of Operational Research,* 116, 33-50.

[25] E.Osuna (2012): Crossing points of distribution and a theorem that relates them to second order stochastic dominance. *Statistics and Probability Letters*, 82, 758-764.

[26] J. Peng, H. Mok and W. Tse (2005): *Fuzzy dominance based on credibility distributions* Springer-Verlag, Berlin, 295-303.

[27] Peng J., Jiang Q. and Rao C. (2007): Fuzzy dominance: a new approach for ranking fuzzy variables via credibility measure. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 15, 29-41.

[28] B. Rom and K. Ferguson (1994): Post-modern portfolio theory comes of age, *Journal of Investing*, 3, 11-17.

[29] J. Sadefo Kamdem, C. D. Tassak and L. A. Fono (2012): Moments and semi-moments for fuzzy portfolio selection. *Insurance: Mathematics and Economics,* 51, 517-530.

[30] A. Saeidifar and E. Pasha (2009): The possibilistic moments of fuzzy numbers and their applications. *Journal of Computation and Applied Mathematics,* 223, 1028-1042.

[31] P. Samuelson (1970): The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments, *Reviews of Economics Studies*, 37, 537-542.

[32] J. Sengupta (1989): Portfolio decision as games. *International Journal of Systems Science,* 20, 1323-1334.

[33] W. Sharpe (1971): A linear programming approximation for the general portfolio analysis problem. *Journal of Financial and Quantitative Analysis,* 6, 1263-1275.

[34] K. Smimou, C.R. Bector and G. Jacoby (2007): A subjective assessment of approximate probabilities with a portfolio application. *Research in International Business and Finance,* 21, 134-160.

[35] B. Stone (1973): A linear programming formulation of the general portfolio selection problem. *Journal of Financial and Quantitative Analysis,* 8, 621-636.

[36] S. Sunoj and S. Maya, (2008): The role of lower partial moments in stochastic modeling, *METRON-International Journal of Statistics,* 2, 223-242.

[37] H. Tanaka, P. Guo, B. Türksen (2000): Portfolio selection based on fuzzy probabilities and possibility distributions, *Fuzzy Sets and Systems*, 111, 387-397.

[38] Tassak C.D., Sadefo K.J., Fono L.A. and Andjiga N.G. (2017) Characterization of order dominances on fuzzy variables for portfolio selection with fuzzy returns. *Journal of the Operational Research society.* On line since 09 February 2017.

[39] X. Wang and E. Kerre (2001): Reasonable properties for the ordering of fuzzy quantities. *Fuzzy Sets and Systems,* 118, 375-385.

[40] L.A. Zadeh (1978): Fuzzy Set as a basis of theory of possibility. *Fuzzy Sets and Systems,* 1, 3-28.