



Existence and uniqueness of a hybrid system with variable coefficients



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Abstract

In this paper we consider a long flexible Euler-Bernoulli beam with boundary conditions imposed at the two ends, the resulting model being called hybrid system. The beam is hybrid in the sense that it holds both rigid and elastic motions. Our main result is to show the existence and uniqueness of the weak solution of close-loop system. The closed-loop system stability is shown through Lyapunov-based analysis.

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1. Introduction

Among various flexible members, the beam element has found its way into many applications such as flexible robots, civil engineering structures, aircraft wing, and space mechanisms. The flexible beam submitted to our study consists of a long flexible mast M or elastic beam joining two rigid bodies. One rigid body represents the space shuttle orbiter S , the other represents the antenna reflector A , each with prescribed inertial properties and each being controlled through ordinary differential equations (ODE), whereas the flexible mast satisfies a partial differential equation (PDE) with boundary conditions imposed at the two ends by the control forces and torques acting on S and A . The length of the beam is chosen to be l . Let us proceed to the statement of the problem. Consider the hyperbolic equation

$$\rho(x)u_{tt}(x, t) + (EI(x)u_{xx}(x, t))_{xx} = 0, x \in (0, l), t \geq 0,$$

with the initial conditions

$$u(0, t) = u_x(0, t) = 0, t \geq 0,$$

and the boundary conditions

$$\mu u_{tt}(l, t) - (EIu_{xx})_x(l, t) = 0, t \geq 0, \quad J u_{xtt}(l, t) + EI(l)u_{xx}(l, t) = z(t), t \geq 0,$$

where x stands for the position and t for the time. $EI(x)$ is the flexural rigidity of the beam, $\rho(x)$ is the

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mass density at x , m is the mass of the antenna A , and J the moment of inertia of A about its centroid, the point of attachment of the mast M . $z(t)$ is the control torque applied to A . In this paper, we shall always assume that:

$$\rho(x), EI(x) \in C^4[0, l]; \quad \rho, EI, m, J > 0.$$

Now, we suppose a direct-strain feedback control, as in [11] and [14], $z(t) = -ku_{xxt}(l, t)$, $k \geq 0$ (k is a real feedback gain). Then, the closed-loop system becomes:

$$\rho(x)u_{tt}(x, t) + (EI(x)u_{xx}(x, t))_{xx} = 0, \quad x \in (0, l), t \geq 0, \quad (1.1)$$

$$u(0, t) = u_x(0, t) = 0, \quad t \geq 0, \quad (1.2)$$

$$mu_{tt}(l, t) - (EIu_{xx})_x(l, t) = 0, \quad t \geq 0, \quad (1.3)$$

$$Ju_{xxt}(l, t) + EI(l)u_{xx}(l, t) + ku_{xxt}(l, t) = 0, \quad t \geq 0. \quad (1.4)$$

The energy associated to the above system $\mathcal{E}(t) = \mathcal{E}(u, v, \xi, \psi)$ is given by

$$\mathcal{E}(t) = \frac{1}{2} \int_0^l EIu_{xx}^2 dx + \frac{1}{2} \int_0^l \rho v^2 dx + \frac{1}{2m} \xi^2 + \frac{1}{2J} \psi^2.$$

Following [7], the energy identity is given by

$$\frac{d}{dt} \mathcal{E}(t) = -\frac{k}{J} EI(l)u_{xxt}^2(l) \leq 0. \quad (1.5)$$

The expression (1.5) shows that the energy $\mathcal{E}(t)$ is non-increasing and therefore defines a Lyapunov function.

In [9], the authors studied the uniform case where the system is modeled as a two-dimensional space-structure. The mathematical side of controllability and stability of the hybrid system have been proved. They also studied the strong stability in [10]. Using the concept of control by feedback boundary damping, this lead to an asymptotic stabilization. The uniform stability was studied in [14] using an energy multipliers method. The existence and uniqueness in the uniform case have been studied in [11], using properties of A -dependent operators.

We find in the literature several authors who have studied hybrid systems with variable coefficients, see [5, 12, 16]. In [7], concerning the closed-loop system (1.1)-(1.4) for $l = 1$, it was shown that the system under boundary feedback damping is a Riesz spectral system in the sense that the generalized eigenfunctions of the system form a Riesz basis on the suitable Hilbert space. The author also studied exponential stability under certain conditions. The spectrum-determined growth condition holds and an asymptotic expression of the spectrum is obtained. Moreover, the exact controllability and exact observability of the system are also presented.

The aim of this article is to show the existence, the uniqueness and higher regularity of the weak solution of system after having formulated it as an evolution problem. To prove existence, we use the intermediate spaces as defined in [8] and the Faedo-Galerkin method (see [12]).

Several authors have used this technique or method to show the existence and the uniqueness of the solution of the Euler Bernoulli beam (EBB) problems (see [1, 3, 4]). In [3, 4] the EBB problems are considered with constant coefficients, so the determination of spaces and the inner products for the proof of existence and uniqueness are not difficult to find. Concerning [1], the problem is with variable coefficients but the feedback conditions used are simple which makes the study a little more difficult than the previous ones. The particularity of our article lies in the fact of considering this time a higher-order feedback $z(t) = -ku_{xxt}(l, t)$. The question of whether the bilinear form $b_2(\cdot, \cdot)$ was well written has been a matter of much thought. This bilinear form appears in the definition of the weak solution and complicates the application of standard techniques discussed in the aforementioned articles above. So we are using new strategies.

The content of this article is as follows. In the Section 2 of this paper, we formulate the system (1.1)-(1.4) into an abstract Cauchy problem in Hilbert state space, and discuss some basic properties of system. The stability analysis of the closed-loop system through Lyapunov method is presented in "Stability analysis using Lyapunov" subsequent subsection. In Section 3, we use the properties of existence, uniqueness and higher regularity expressed in Lions [8], Yosida [17], and Temam [15], to show the existence of the weak solution. For the methodology, we approach in the same sense as the work already carried out in [4] and [12]. We end with a conclusion in Section 4.

2. Basic properties of the problem (1.1)-(1.4)

2.1. Formulation as an evolution problem

Let us introduce the following spaces:

$$H^m(0, l) = \{u : [0, l] \rightarrow \mathbb{C} \mid u, u^1 = \frac{\partial u}{\partial t}, \dots, u^m = \frac{\partial^m u}{\partial t^m} \in L^2(0, l)\},$$

we denote by $\|\cdot\|_m$ the associated norm of the space $H^m(0, l)$ and

$$L^2(0, l) = \{u : [0, l] \rightarrow \mathbb{C} \mid \int_0^l |u|^2 dx < \infty\}$$

with $\|\cdot\|$ the associated norm of the space. $D(0, l) :=$ the space of smooth functions with compact support, $D'(0, l) :=$ the space of continuous linear functions. Moreover, we introduce functional space:

$$V = \{u \in H^2(0, l), u(0) = u_x(0) = 0\}$$

and the following Hilbert space:

$$\mathcal{X} = \{u = (f, g, \xi, \psi)^T \mid f \in V, g \in L^2(0, l), \xi, \psi \in \mathbb{C}\} = V \times L^2(0, l) \times \mathbb{C}^2,$$

the superscript T stands for the transpose. The space \mathcal{X} is called energy space of the system (1.1)-(1.4). In the space \mathcal{X} , we define the inner-product:

$$(u, v)_{\mathcal{X}} = \int_0^l EI(x) f_1''(x) \overline{f_2''(x)} dx + \int_0^l \rho(x) g_1(x) \overline{g_2(x)} dx + \frac{1}{m} \xi_1 \overline{\xi_2} + \frac{1}{j} \psi_1 \overline{\psi_2},$$

where $u = (f_1, g_1, \xi_1, \psi_1)^T \in \mathcal{X}$, $v = u_t = (f_2, g_2, \xi_2, \psi_2)^T \in \mathcal{X}$ and $\overline{g_2(x)}$ denotes the complex conjugate of $g_2(x)$. We denote by $\|\cdot\|_{\mathcal{X}}$ the norm associated to the inner-product in the space \mathcal{X} .

Next, we define an unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \rightarrow \mathcal{X}$ as follows:

$$\mathcal{A} \begin{bmatrix} f \\ g \\ \xi \\ \psi \end{bmatrix} = \begin{bmatrix} g \\ -\frac{1}{\rho(x)} (EI(x) f''(x))'' \\ (EI f'')'(l) \\ -EI(l) f''(l) \end{bmatrix}$$

with the domain:

$$D(\mathcal{A}) = \{(f, g, \xi, \psi) \in (H^4(0, l) \cap V) \times V \times \mathbb{C}^2 \mid \xi = mg(l), \psi = Jg'(l) + kf''(l)\}.$$

The set of equations (1.1)-(1.4) can be formally written as a first order evolution problem

$$\begin{cases} \frac{d}{dt} w(t) = \mathcal{A}w(t), \\ w(0) = w_0 \in \mathcal{X}, \end{cases} \quad (2.1)$$

where

$$w(t) = (u(\cdot, t), u_t(\cdot, t), mu_t(l, t), Ju_{xt}(l, t) + ku_{xx}(l, t))^T, \quad w_0 = (u_0, u_1, mu_1(l), Ju'_1(l) + ku''_0(l))^T, \quad \forall t > 0$$

with

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad mu_t(l, 0) = mu_1(l), \quad Ju_{xt}(l, 0) + ku_{xx}(l, 0) = Ju'_1(l) + ku''_0(l).$$

It is known [13] that this result follows immediately from the theory of operator semigroups.

Theorem 2.1. *The operator \mathcal{A} defined as before generates a C_0 -semigroup of contractions on \mathcal{X} , denoted by $\{S(t)\}_{t \geq 0}$.*

Proof. We will firstly show that the operator \mathcal{A} is dissipative. Let $u = (f, g, \xi, \psi)^T \in D(\mathcal{A})$,

$$(\mathcal{A}u, u)_{\mathcal{X}} = \int_0^l [EI(x)g''(x)\overline{f''(x)} - (EI(x)f''(x))\overline{g''(x)}] dx + \frac{1}{m}(EI f'')'(l)\bar{\xi} - \frac{1}{j}EI(l)f''(l)\bar{\psi}.$$

Integrating twice by parts and using the boundary conditions (1.2)-(1.4) with $g(0) = 0$, after simplification, we get the following requirement:

$$\begin{aligned} (\mathcal{A}u, u)_{\mathcal{X}} &= \int_0^l EI(x)[g''(x)\overline{f''(x)} - f''(x)\overline{g''(x)}] dx - (EI(l)f''(l))\overline{g'(l)} + EI(l)f''(l)\overline{g'(l)} \\ &\quad + \frac{1}{m}(EI f'')'(l)\bar{\xi} - \frac{1}{j}EI(l)f''(l)\bar{\psi}, \\ (\mathcal{A}u, u)_{\mathcal{X}} &= \int_0^l EI(x)[g''(x)\overline{f''(x)} - f''(x)\overline{g''(x)}] dx - (EI(l)f''(l))\overline{g'(l)} + (EI f'')'(l)\overline{g'(l)} - \frac{k}{j}EI(l)|f''(l)|^2. \end{aligned}$$

Taking the real part of $(\mathcal{A}u, u)_{\mathcal{X}}$, we obtain $\text{Re}(\mathcal{A}u, u)_{\mathcal{X}} = -\frac{k}{j}EI(l)|f''(l)|^2 \leq 0$, with $g = f_t$ for all $u = (f, g, \xi, \psi)^T \in D(\mathcal{A})$. It follows that the operator \mathcal{A} is dissipative.

Let show that the operator is m -dissipative. To do this, we just prove that $(I - \mathcal{A})$ is surjective. Let $z = (f, g, \Lambda, \Xi)^T \in \mathcal{X}$, we need to find $w = (u, v, \xi, \psi)^T \in D(\mathcal{A})$ such as:

$$(I - \mathcal{A})w = z,$$

in other words, such that the following equations are satisfied:

$$v = u - f, \tag{2.2}$$

$$\rho v + (EIu_{xx})_{xx} = \rho g, \tag{2.3}$$

$$(EI(l)u_{xx})_x(l) - mu(l) = -\Lambda - mf(l), \tag{2.4}$$

$$EI(l)u_{xx}(l) + Ju_x(l) + ku_{xx}(l) = \Xi + Jf_x(l), \tag{2.5}$$

$$u(0, t) = u_x(0, t) = 0, \tag{2.6}$$

$$(EIu_{xx})_x(l, t) = mv(l) - mg(l), \tag{2.7}$$

$$-EI(l)u_{xx}(l, t) = Jv_x(l) + Jg_x(l) + kv_{xx}(l), \tag{2.8}$$

where $u \in H^2(0, l)$ et $f = u - v \in L^2(0, l)$.

Now we write the system of equations (2.3)-(2.8) in the weak form. Multiplying (2.3) by $\bar{\phi} \in V$ and integrating over $(0, l)$, we have:

$$\int_0^l \rho(x)v(x)\overline{\phi(x)} dx + \int_0^l (EI(x)u(x)_{xx})_{xx}\overline{\phi(x)} dx = \int_0^l \rho(x)g(x)\overline{\phi(x)} dx.$$

Using (2.2), we get

$$\int_0^l \rho(x)u(x)\overline{\phi(x)}dx + \int_0^l (EI(x)u(x)_{xx})_{xx}\overline{\phi(x)}dx = \int_0^l \rho(x)(f+g)(x)\overline{\phi(x)}dx.$$

Integrating twice by parts $\int_0^l (EI(x)u(x)_{xx})_{xx}\overline{\phi(x)}dx$ and taking into account the boundary conditions (2.6) to (2.8) yields:

$$\begin{aligned} & \int_0^l \rho u \overline{\phi} dx + \int_0^l EI u_{xx} \overline{\phi_{xx}} dx + (EI u_{xx})_x(l) \overline{\phi(l)} - EI(l) u_{xx}(l) \overline{\phi_x(l)} \\ &= \int_0^l \rho(x)(f+g)(x)\overline{\phi(x)}dx \int_0^l \rho u \overline{\phi} dx + \int_0^l EI u_{xx} \overline{\phi_{xx}} dx + m u(l) \overline{\phi(l)} + (J u_x(l) + k u_{xx}(l)) \overline{\phi_x(l)} \quad (2.9) \\ &= \int_0^l \rho(x)(f+g)(x)\overline{\phi(x)}dx + m(f+g)(l) \overline{\phi(l)} + (J f_x(l) + J g_x(l) + k f_{xx}(l)) \overline{\phi_x(l)}. \end{aligned}$$

Now, for all $(u, \phi)^T \in V \times V$, we set

$$a(u, \phi) = \int_0^l \rho u \phi dx + \int_0^l EI u_{xx} \phi_{xx} dx + m u(l) \phi(l) + (J u_x(l) + k u_{xx}(l)) \phi_x(l).$$

It is clear that $a(\cdot, \cdot)$ is symmetric bilinear form, bounded and coercive on $V \times V$. We define the continuous linear form L on V by

$$L(f) = \int_0^l \rho(x)(f+g)(x)\phi(x)dx + m(f+g)(l)\phi(l) + (J f_x(l) + J g_x(l) + k f_{xx}(l))\phi_x(l).$$

The Lax-Milgram theorem allows us to conclude the existence and uniqueness of the solution $u \in V$ such as $a(u, \phi) = L(\phi)$ for all $\phi \in V$.

2.2. Regularity of the solution

Let $\phi \in D(0, l)$, integrating twice by parts (2.9), yields

$$\int_0^l \rho u \phi dx + \int_0^l (EI u_{xx})_{xx} \phi dx = \int_0^l \ell \phi dx \text{ with } \ell = \rho(f+g)$$

or even

$$\int_0^l (\rho u + (EI u_{xx})_{xx} - \ell) \phi dx = 0,$$

this implies that

$$(\rho u + (EI u_{xx})_{xx} - \ell) \phi = 0, \quad \forall \phi \in D(0, l).$$

Thereby

$$\rho u + (EI u_{xx})_{xx} - \ell = 0 \text{ in } D'(0, l), \quad (2.10)$$

and also in $L^2(0, l)$, as:

$$(EI u_{xx})_{xx} = \ell - \rho u \text{ in } L^2(0, l)$$

and therefore, the equation (2.10) is in the sense of $L^2(0, l)$. Using in particular $\phi \in V$, we find boundary conditions (2.6)-(2.8). Hence there is a unique solution $u \in H^4(0, l) \cap V$ of problem (2.2)-(2.8). This shows that $(I - \mathcal{A})$ is surjective, then the operator \mathcal{A} is m -dissipative. So by the Lumer-Phillips Theorem (see, e.g., [13]), \mathcal{A} is infinitesimal generator of a C_0 -semigroup of contractions. \square

Remark 2.2. \mathcal{A} being m -dissipative on the Hilbert space \mathcal{X} , its domain $D(\mathcal{A})$ is dense in \mathcal{X} .

The following lemma is a direct consequence of the Remark 2.2 and the semigroups theory of linear operators.

Lemma 2.3. *For any initial condition $w_0 \in D(\mathcal{A})$, the equation (2.1) admits a unique strong solution $w \in C^1([0, \infty), \mathcal{X}) \cap C^0([0, \infty), D(\mathcal{A}))$. Hence $u \in C^2([0, \infty), L^2(0, l)) \cap C^1([0, \infty), V) \cap C^0([0, \infty), H^4(0, l) \cap V)$.*

The next result follows immediately from Theorem 2.1.

Theorem 2.4. *The equation (2.1) has a unique solution $w(t) = S(t)w_0 \in C([0, \infty), \mathcal{X}), \forall w_0 \in \mathcal{X}$.*

Proof. The application $S(t)$ defined by $D(\mathcal{A}) \rightarrow \mathcal{X} w_0 \mapsto w(t)$ extends in a contraction $S(t)$ on \mathcal{X} such that $(S(t))_{t \geq 0}$ is strongly continuous and for any initial condition $w_0 \in \mathcal{X}$, the weak solution of (2.1) is defined by $w(t) = S(t)w_0, \forall t \geq 0$, with $w(t) \in C^0([0, \infty), \mathcal{X})$. Thus, we have

$$u \in C^1([0, \infty[, L^2(0, l)) \cap C^0([0, \infty), V).$$

□

2.3. Stability analysis using Lyapunov

To analyze the stability of the closed-loop beam under the proposed boundary control laws, consider the following Lyapunov candidate $p : \mathcal{X} \rightarrow \mathbb{R}$ defined by:

$$p(w) = \frac{1}{2} \|w\|_{\mathcal{X}}^2 = \frac{1}{2} \left(\int_0^l [EI(x)|u_{xx}|^2 + \rho(x)|v|^2] dx + m^{-1}|\xi|^2 + J^{-1}|\psi|^2 \right). \quad (2.11)$$

Analogously as in (1.5), for all classical solutions w , it follows that the time derivative of the Lyapunov candidate (2.11) satisfies:

$$\frac{d}{dt} p(w) = \frac{d}{dt} \|w\|_{\mathcal{X}}^2 = -\frac{k}{J} EI(l) u_{xx}^2(l) \leq 0, \quad (2.12)$$

hence time evolution of the Lyapunov functional p along the classical solutions is non-increasing. We can say that the system (2.1) is stable in the sense of Lyapunov. Furthermore, from Theorem 2.1, the decay of energy along the classical solutions can be extended to mild solutions.

Theorem 2.5. *Assume that $w(t)$ is the mild solution of (2.1) for all $w_0 \in \mathcal{X}$. Then $w(t) \rightarrow 0$ in \mathcal{X} when $t \rightarrow \infty$.*

Now, the system of equations (1.1)-(1.4) is written in the weak form, and the existence and uniqueness of the weak solution are demonstrated.

3. Existence, uniqueness, and higher regularity of the weak solution

3.1. Weak formulation

Multiplying the equation $\rho(x)u_{tt}(x, t) + (EI(x)u_{xx}(x, t))_{xx} = 0$ by $\phi(x) \in V$ and integrating over $(0, l)$, we have:

$$\int_0^l \rho(x)u_{tt}\phi(x) dx + \int_0^l (EI(x)u_{xx})_{xx}\phi(x) dx = 0, \forall \phi \in V, t > 0.$$

Integrating twice by parts and taking into account the boundary conditions it follows:

$$\int_0^l \rho u_{tt} \phi dx + \int_0^l EI u_{xx} \phi_{xx} dx + (EI u_{xx})_x(l, t) \phi(l) - EI(l) u_{xx} \phi_x(l) = 0, \forall \phi \in V, t > 0.$$

$$\int_0^l \rho u_{tt} \phi \, dx + \int_0^l E I u_{xx} \phi_{xx} \, dx + m u_{tt}(l, t) \phi(l) + J u_{xtt}(l, t) \phi_x(l) + k u_{xxt}(l, t) \phi_x(l) = 0, \quad \forall \phi \in V, t > 0. \quad (3.1)$$

The first step in the definition of the weak formulation is the appropriate space setting. Following [2], we will define two Hilbert spaces which will allow us to reach the definition of a weak solution. The first Hilbert space is defined by $Y = \mathbb{R}^2 \times L^2(0, l)$ with the inner product:

$$\langle \eta, \xi \rangle_Y = J \eta_1 \xi_1 + m \eta_2 \xi_2 + \langle \rho \eta_3, \xi_3 \rangle_{L^2(0, l)}$$

for all $\eta = (\eta_1, \eta_2, \eta_3)$, $\xi = (\xi_1, \xi_2, \xi_3) \in Y$. The second Hilbert space is the following space $X = \mathbb{R}^2 \times V = \{\hat{u} = (u_x(l), u(l), u); u \in V\}$ with the inner product:

$$\langle \hat{u}_1, \hat{u}_2 \rangle_X = \langle (u_1)_{xx}, (u_2)_{xx} \rangle_{L^2(0, l)}.$$

It is natural to identify X with its dual X' . The Hilbert space X is reflexive; that is to say, $(X')' = X$. It is not difficult to show that X can be densely embedded in Y . Then the dual of Y is densely embedded in the dual of X . Choose Y to be a pivot space, $Y \equiv Y'$, then both X and Y are dense subspaces of X' and we have

$$X \subset Y \equiv Y' \subset X'.$$

Consider the bilinear forms:

$$b_1 : X \times X \longrightarrow \mathbb{R} \quad (\hat{u}_1, \hat{u}_2) \mapsto b_1(\hat{u}_1, \hat{u}_2) = \langle E I \hat{u}_1, \hat{u}_2 \rangle_X$$

and

$$b_2 : Y \times Y \longrightarrow \mathbb{R} \quad (\eta, \xi) \mapsto b_2(\eta, \xi) = k(\eta_1)_x \xi_1.$$

Let us give the definition of a weak solution.

Definition 3.1. Let $T > 0$ be fixed. We say that $\hat{u} = (u_x(l), u(l), u)$ is a weak solution of problem (1.1)-(1.4) on $(0, 1)$ if $\hat{u} \in L^2(0, T; X) \cap H^1(0, T; Y) \cap H^2(0, T; X')$ and satisfies

$$\langle \hat{u}_{tt}, \hat{\phi} \rangle_{X, X'} + b_1(\hat{u}, \hat{\phi}) + b_2(\hat{u}_t, \hat{\phi}) = 0 \quad (3.2)$$

for almost everywhere $t \in (0, T)$ and for all $\hat{\phi} \in X$ with the following initial conditions:

$$\hat{u}(0) = \hat{u}_0 = ((u_0)_x(l), u_0(l), u_0) \in X, \quad (3.3)$$

$$\hat{u}_t(0) = \hat{v}_0 = ((v_0)_x(l), v_0(l), v_0) \in Y. \quad (3.4)$$

In the Definition 3.1, the bilinear form $\langle \cdot, \cdot \rangle_{X, X'}$ is the duality pairing on $X' \times X$ (i.e., $\langle \cdot, \cdot \rangle_{X, X'}$ is a bilinear functional on the product space $X' \times X$). Moreover, the duality pairing on $X' \times X$ can be identified with the unique extension of the inner product in Y . Note that in the case where $u \in H^2(0, T; X)$, the formulation (3.2) is equivalent to equation (3.1).

3.2. Existence and uniqueness of the solution

In order to give a meaning to the initial conditions (3.3)-(3.4) we shall use the following lemma (special case of Theorem 3.1 in [8]).

Lemma 3.2. Let X and Y be two Hilbert spaces, such that X is dense and continuously embedded in Y . Assume that $u \in L^2(0, T; X)$ and $v = u_t \in L^2(0, T; Y)$, then $u \in C([0, T]; [X, Y]_{\frac{1}{2}})$ after, possibly, a modification on a set of measure zero.

Here the space $[X, Y]_{\frac{1}{2}}$ is called intermediate space and is defined as in [8]. It follows from [8] the duality theorem:

Lemma 3.3. Let X and Y be two Hilbert spaces, such that X is dense and continuous in Y . For all $\theta \in]0, 1[$,

$$[X, Y]'_\theta = [Y', X']_{1-\theta}$$

with equivalent norms.

We present the assertion of the Theorem 3.4 which will help us in the proof of Theorem 3.5.

Theorem 3.4. Let V be a subspace of $H^2(0, l)$. Then there exists a infinite sequence of functions $\{\phi_i\}_{i=1}^\infty$ such that: $\{\phi_i\}_{i=1}^\infty$ is an orthogonal basis of V and $\{\phi_i\}_{i=1}^\infty$ is an orthonormal basis of $L^2(0, l)$.

Proof. see [4, 12]. □

3.2.1. Existence of the weak solution

Theorem 3.5. The weak formulation (3.2)-(3.4) has a unique solution \hat{u} such that:

$$\hat{u} \in L^\infty(0, T; X), \hat{u}_t \in L^\infty(0, T; Y), \tag{3.5}$$

$$\hat{u} \in C([0, T]; [X, Y]_{\frac{1}{2}}), \tag{3.6}$$

$$\hat{u}_t \in C([0, T]; [X, Y]_{\frac{1}{2}}'). \tag{3.7}$$

The following proof is an adaption of the proof of Theorem 8.1 in [8], for the system studied here.

Proof. Let $\{\hat{\phi}_i\}_{i=1}^\infty$ be a sequence of functions that is an orthonormal basis for Y , and an orthogonal basis for X according to Theorem 3.4. We introduce the following finite dimensional spaces spanned by $\{\hat{\phi}_i\}_{i=1}^m$ given by:

$$\forall m \in \mathbb{N}, \hat{V}_m := \text{span}\{\hat{\phi}_1, \dots, \hat{\phi}_m\} = \left\{ \sum_{j=1}^m \alpha_j \hat{\phi}_j; \alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R} \right\}.$$

Step 1: Construction of approximate solutions.

We seek $\hat{u} = \hat{u}_m(t) \in \hat{V}_m$ the approximate solution of the problem in the form:

$$\hat{u}_m(t) = \sum_{i=1}^m g_{im}(t) \hat{\phi}_i,$$

where $g_{im}(t) \in \mathbb{R}$ ($0 \leq t \leq T, i = 1, \dots, m$) solves the formulation (3.2) on \hat{V}_m . For a fixed $m \in \mathbb{N}$, it follows

$$\langle (\hat{u}_m)_{tt}, \hat{\phi} \rangle_Y + b_1(\hat{u}_m, \hat{\phi}) + b_2((\hat{u}_m)_t, \hat{\phi}) = 0 \quad \forall \hat{\phi} \in \hat{V}_m. \tag{3.8}$$

And (3.8) is completed with the initial conditions:

$$\hat{u}_m(0) = \hat{u}_{m0}, \hat{u}_{m0} = \sum_{i=1}^m \alpha_{im} \hat{\phi}_i \longrightarrow \hat{u}_0 \text{ in } X \text{ when } m \longrightarrow \infty, \tag{3.9}$$

$$\hat{v}_m(0) = \hat{v}_{m0}, \hat{v}_{m0} = \sum_{i=1}^m \beta_{im} \hat{\phi}_i \longrightarrow \hat{v}_0 \text{ in } Y \text{ when } m \longrightarrow \infty, \tag{3.10}$$

with $\alpha_{im} = g_{im}(0)$ and $\beta_{im} = (g_{im})_t(0)$. The ordinary differential equation of the second order thus obtained admits a unique solution $\hat{u}_m \in C^2([0, T]; X)$ of (3.8)-(3.10) pour $0 \leq t \leq T$.

Step 2: A-priori estimates on approximate solutions.

Let $\hat{E} : \mathbb{R} \times X \longrightarrow \mathbb{R}$ an energy functional, analogous to the Lyapunov functional in (2.11):

$$\begin{aligned} \hat{E}(t, \hat{u}) &= \frac{1}{2} \int_0^l EI \hat{u}_{xx}^2 dx + \frac{1}{2} \int_0^l \rho \hat{u}_t^2 dx + \frac{1}{2m} (m \hat{u}_t(l))^2 + \frac{1}{2J} [J \hat{u}_{xt}(l) + k \hat{u}_{xx}(l)]^2, \\ \hat{E}(t, \hat{u}) &= \|(\mathbf{u}, \mathbf{u}_t, m \mathbf{u}_t(l), J \mathbf{u}_{tx}(l) + k \mathbf{u}_{xx}(l))\|_X. \end{aligned}$$

For a solution $\hat{u}_m \in C^2([0, \tau]; \hat{V}_m)$ of (3.8) and taking $\hat{\phi} = (\hat{u}_m)_t$ in (3.8), a straightforward calculation

yields

$$\frac{d}{dt} \widehat{E}(t, \hat{u}_m) = -\frac{k}{J} EI(l) [((\hat{u}_m)_1)_x]_t(t)^2 \leq 0,$$

for all $t \in [0, \tau]$. Dissipation of the functional \widehat{E} corresponds to the decay in (2.12) for the classical solution. Hence,

$$\widehat{E}(t, \hat{u}_m) \leq \widehat{E}(0, \hat{u}_{m0}), \quad t \geq 0,$$

which implies:

$$\{\hat{u}_m\}_{m \in \mathbb{N}} \text{ is bounded in } C([0, T]; X), \tag{3.11}$$

$$\{(\hat{u}_m)_t\}_{m \in \mathbb{N}} \text{ is bounded in } C([0, T]; Y). \tag{3.12}$$

Considering the boundedness results in (3.11)-(3.12), for all $\hat{\phi} \in X$, we have:

$$|b_1(\hat{u}_m(t), \hat{\phi}) + b_2((\hat{u}_m)_t(t), \hat{\phi})| \leq M \|\hat{\phi}\|_X, \quad \forall t \in [0, T], \tag{3.13}$$

where M is a positive constant which does not depend on m . Let $m \in \mathbb{N}$ be fixed. Also, we consider $\hat{\phi} \in X$ and $\hat{\phi} = \hat{\phi}_1 + \hat{\phi}_2$ such that $\hat{\phi}_1 \in \hat{V}_m$ and $\hat{\phi}_2$ orthogonal to \hat{V}_m in Y . Then we obtain

$$\langle (\hat{u}_m)_{tt}, \hat{\phi} \rangle_Y = \langle (\hat{u}_m)_{tt}, \hat{\phi}_1 \rangle_Y.$$

From (3.8) and (3.13), we have:

$$\langle (\hat{u}_m)_{tt}, \hat{\phi} \rangle_Y = -b_1(\hat{u}_m(t), \hat{\phi}_1) - b_2((\hat{u}_m)_t(t), \hat{\phi}_1) \leq M \|\hat{\phi}_1\|_X \leq M \|\hat{\phi}\|_X.$$

This implies that:

$$\{(\hat{u}_m)_{tt}\}_{m \in \mathbb{N}} \text{ is bounded in } C([0, T]; X').$$

Step 3: Passage to the limit.

According to the Eberlein-Šmulian Theorem in [6, 17], we can extract weakly convergent subsequences $\{\hat{u}_{m_l}\}_{l \in \mathbb{N}}$, $\{(\hat{u}_{m_l})_t\}_{l \in \mathbb{N}}$ and $\{(\hat{u}_{m_l})_{tt}\}_{l \in \mathbb{N}}$ with $\hat{u} \in L^2(0, T; X)$, $\hat{u}_t \in L^2(0, T; Y)$ and $\hat{u}_{tt} \in L^2(0, T; X')$ such that:

$$\{\hat{u}_{m_l}\} \rightharpoonup \hat{u} \text{ in } L^2(0, T; X), \tag{3.14}$$

$$\{(\hat{u}_{m_l})_t\} \rightharpoonup \hat{u}_t \text{ in } L^2(0, T; Y), \tag{3.15}$$

$$\{(\hat{u}_{m_l})_{tt}\} \rightharpoonup \hat{u}_{tt} \text{ in } L^2(0, T; X'). \tag{3.16}$$

Moreover (3.15) yields

$$\{((\hat{u}_1)_{m_l})_t\} \rightharpoonup (\hat{u}_1)_t \text{ in } L^2(0, T; \mathbb{R})$$

for almost every $t \in [0, T]$. Let $m_0 \in \mathbb{N}$. For all functions $\hat{\phi} \in L^2(0, T; \hat{V}_{m_0})$ of the form

$$\hat{\phi}(t, x) = \sum_{j=1}^{m_0} \kappa_j(t) \phi_j(x), \tag{3.17}$$

where $\kappa_j \in L^2(0, T; \mathbb{R})$ and for all $m_l \geq m_0$, the formulation (3.8) becomes

$$\int_0^T \langle (\hat{u}_{m_l})_{tt}, \hat{\phi} \rangle_Y + b_1(\hat{u}_{m_l}, \hat{\phi}) + b_2((\hat{u}_{m_l})_t, \hat{\phi}) dt = 0. \tag{3.18}$$

Therefore, passing to the limit in (3.18) for $m = m_l$, when $l \rightarrow \infty$ and using the convergence results (3.14)-(3.16), we obtain

$$\int_0^T \langle \hat{u}_{tt}, \hat{\phi} \rangle_{X, X'} + b_1(\hat{u}, \hat{\phi}) + b_2(\hat{u}_t, \hat{\phi}) dt = 0, \tag{3.19}$$

consequently $\langle \hat{u}_{tt}, \hat{\phi} \rangle_{X, X'} + b_1(\hat{u}, \hat{\phi}) + b_2(\hat{u}_t, \hat{\phi}) = 0$ on $[0, T]$ for all $\hat{\phi} \in L^2(0, T; X)$. The functions $\hat{\phi}$ of (3.17) are dense in $L^2(0, T; X)$ and therefore (3.19) is well defined for any $\hat{\phi} \in L^2(0, T; X)$. This implies that the expression of the weak formulation (3.2) is satisfied almost everywhere on $[0, T]$. Hence \hat{u} is the solution of the weak formulation.

Concerning additional regularities, by definition of weak solution and (3.11)-(3.12), \hat{u} satisfies (3.5). As for (3.6), it immediately arises from the Lemma 3.2, after, possibly a modification on a set of measure zero, and finally, the regularity (3.7) is deduced from the Lemma 3.2 and from Lemma 3.3. \square

The proof of the existence of the weak solution is now complete. Before showing the uniqueness of the solution, we prove that the solution \hat{u} satisfies initial conditions (3.3)-(3.4). Let $\hat{\phi} \in C^2([0, T]; Y)$ such that $\hat{\phi}(T) = 0$ and $\hat{\phi}_t(T) = 0$. Integrating (3.2) on $[0, T]$, we have:

$$\int_0^T [\langle \hat{u}_{tt}, \hat{\phi} \rangle_{X, X'} + b_1(\hat{u}, \hat{\phi}) + b_2(\hat{u}_t, \hat{\phi})] d\tau = 0.$$

Integrating twice by parts on $[0, T]$ under the duality pairing, we have:

$$\int_0^T [\langle \hat{u}, \hat{\phi}_{tt} \rangle_Y + b_1(\hat{u}, \hat{\phi}) + b_2(\hat{u}_t, \hat{\phi})] d\tau = \langle \hat{u}_t(0), \hat{\phi}(0) \rangle_{X, X'} - \langle \hat{u}(0), \hat{\phi}_t(0) \rangle_Y. \quad (3.20)$$

For a fixed m , similarly from (3.8), it follows:

$$\int_0^T [\langle \hat{u}_m, \hat{\phi}_{tt} \rangle_Y + b_1(\hat{u}_m, \hat{\phi}) + b_2((\hat{u}_m)_t, \hat{\phi})] d\tau = \langle \hat{v}_{m0}, \hat{\phi}(0) \rangle_Y - \langle \hat{u}_{m0}, \hat{\phi}_t(0) \rangle_Y. \quad (3.21)$$

Using (3.9)-(3.10) and (3.14)-(3.16), passing to the limit in (3.21) along the convergent subsequence, we obtain:

$$\int_0^T [\langle \hat{u}, \hat{\phi}_{tt} \rangle_Y + b_1(\hat{u}, \hat{\phi}) + b_2(\hat{u}_t, \hat{\phi})] d\tau = \langle \hat{v}_0, \hat{\phi}(0) \rangle_Y - \langle \hat{u}_0, \hat{\phi}_t(0) \rangle_Y. \quad (3.22)$$

Comparing (3.20) with (3.22), we deduce that $\hat{u}(0) = \hat{u}_0$ and $\hat{u}_t(0) = \hat{v}_0$ so initial conditions (3.3) and (3.4) are verified.

3.2.2. Uniqueness of the Weak Solution

Theorem 3.6. *The solution \hat{u} of weak formulation (3.2) with the initial conditions (3.3)-(3.4) is unique.*

Proof. Now, we show the uniqueness of the weak solution of (3.2). For this, let fix $0 \leq s \leq T$ and let introduce this auxiliary function: $\hat{\psi} :]0, T[\rightarrow \mathbb{R}$,

$$\hat{\psi}(t) := \begin{cases} \int_t^s \hat{u}(\tau) d\tau, & 0 < t < s, \\ 0, & t \geq s. \end{cases}$$

Integrating (3.2) on $[0, T]$, using one integration by part and taking $\hat{\psi}(t) = \hat{\phi}(t)$ in (3.2), we have

$$\int_0^s [\langle \hat{u}_t(\tau), \hat{u}(\tau) \rangle_Y - b_1(\hat{\psi}_t(\tau), \hat{\psi}(\tau)) + b_2(\hat{u}(\tau), \hat{u}(\tau))] d\tau = 0. \quad (3.23)$$

We deduce from (3.23),

$$\int_0^s \frac{d}{dt} \left[\frac{1}{2} \|\hat{u}(\tau)\|_Y^2 - \frac{1}{2} b_1(\hat{\psi}(\tau), \hat{\psi}(\tau)) \right] d\tau = - \int_0^s b_2(\hat{u}(\tau), \hat{u}(\tau)) d\tau.$$

This is equivalent to

$$\left[\frac{1}{2} \|\hat{u}(\tau)\|_X^2 - \frac{1}{2} b_1(\hat{\psi}(\tau), \hat{\psi}(\tau)) \right]_0^s = - \int_0^s b_2(\hat{u}(\tau), \hat{u}(\tau)) d\tau.$$

Therefore

$$\frac{1}{2} \|\hat{u}(s)\|_X^2 + \frac{1}{2} b_1(\hat{\psi}(0), \hat{\psi}(0)) \leq 0.$$

Since the bilinear form $b_1(\cdot, \cdot)$ is coercive, $\hat{u}(s) \equiv 0$ and $\hat{\psi}(0) = 0$. As $s \in]0, T[$ was arbitrary, then $\hat{u} \equiv 0$. \square

Due to the spaces defined above under [2], we were able to give the definition of a weak solution, to show the existence and the uniqueness. Now, we show the regularity of the solution.

3.3. Higher regularity results

Here, we recall the Lemma 8.1 of [8] which will be used in the next theorem. Before, let's give the following definition.

Definition 3.7. Let Y be a Banach space. Then

$$C_u([0, T]; Y) = \{u \in L^\infty(0, T; Y) : t \mapsto \langle f, u(t) \rangle \text{ is continuous on } [0, T], \forall f \in Y'\}$$

denotes the space of weakly continuous functions with values in Y .

Lemma 3.8. Let X and Y be two Banach spaces, $X \subset Y$ with continuous injection, and X being reflexive. Then

$$L^\infty(0, T; X) \cap C_u(0, T; Y) = C_u(0, T; X).$$

Theorem 3.9. The weak solution \hat{u} of (3.2)-(3.4) satisfies

$$\hat{u} \in C([0, T]; X), \quad \hat{u}_t \in C([0, T]; Y),$$

after, possibly, a modification on a set of measure zero.

Proof. This proof is an adaption of standard strategies to the situation at hand (cf. Section 8.4 of [8] and the section 2.4 of [15]). Using Lemma 3.8, it follows from (3.5)-(3.7) that $\hat{u} \in C_u([0, T]; X)$. In addition, (3.5) and (3.7) imply that $\hat{u}_t \in C_u([0, T]; Y)$. We set $\xi \in C^\infty(\mathbb{R})$ a fixed scalar function such as $\xi(x) = 1$ if $x \in J \subset \subset [0, T]$ and $\xi(x) = 0$ else. The function $\xi\hat{u}$ is compactly supported. Let η^ε be a standard mollifier in time. For example, the function may be given by

$$\eta^\varepsilon(t) := \varepsilon^{-1} \eta\left(\frac{t}{\varepsilon}\right),$$

where

$$\eta(t) := \begin{cases} C \exp[1/(1 - |t|^2)], & |t| < 1, \\ 0, & |t| \geq 1, \end{cases}$$

belongs to $C_c^\infty(\mathbb{R})$ for any constant C . We choose C such that $\int_{\mathbb{R}} \eta dx = 1$. We set $\hat{u}^\varepsilon := \eta^\varepsilon * \xi\hat{u} \in C_c^\infty(\mathbb{R}, X)$. \hat{u}^ε converges to \hat{u} in X and $(\hat{u}^\varepsilon)_t$ converges to \hat{u}_t a.e. in Y for all element on J . Hence, $\hat{E}(t, \hat{u}^\varepsilon)$ converges to $\hat{E}(t, \hat{u})$ a.e. on J . Since \hat{u}^ε is smooth, a straightforward calculation on J gives:

$$\frac{d}{dt} \hat{E}(t, \hat{u}^\varepsilon) = -\frac{k}{J} EI(l) [((\hat{u}^\varepsilon)_t)_x(t)]^2.$$

Passing to the limit when $\varepsilon \rightarrow 0$,

$$\frac{d}{dt} \hat{E}(t, \hat{u}) = -\frac{k}{J} EI(l) [((\hat{u}_t)_x(t))]^2 \quad (3.24)$$

holds in the sense of distributions on J . Since J was arbitrary, (3.24) holds on all compact subintervals of $[0, T]$. Let $t \in [0, \infty[$ be fixed and $\lim_{n \rightarrow \infty} t_n = t$. Taking the sequence $(\sigma_n)_{n \in \mathbb{N}}$ defined by

$$\begin{aligned} \sigma_n = & \frac{1}{2} \|\sqrt{EI}(\hat{u}(t) - \hat{u}(t_n))\|_X^2 + \frac{1}{2} \|\hat{u}_t(t) - \hat{u}_t(t_n)\|_Y^2 \\ & + k [(\hat{u}_1)_t(t)(\hat{u}_1)_x(t) - (\hat{u}_1)_t(t_n)(\hat{u}_1)_x(t_n)] + \frac{1}{2J} (k(\hat{u}_1)_x(t) - k(\hat{u}_1)_x(t_n))^2, \end{aligned}$$

we have for all $n \in \mathbb{N}$,

$$\sigma_n = \widehat{E}(t, \hat{u}) + \widehat{E}(t_n, \hat{u}) - \langle EI\hat{u}(t), \hat{u}(t_n) \rangle_X - \langle \hat{u}_t(t), \hat{u}_t(t_n) \rangle_Y - \frac{k^2}{J} (\hat{u}_1)_x(t)(\hat{u}_1)_x(t_n). \quad (3.25)$$

Since \hat{u}, \hat{u}_t are weakly continuous and \widehat{E} is continuous in t , passing to the limit in (3.25), it follows:

$$\sigma_n \longrightarrow 0, \text{ when } n \longrightarrow \infty.$$

Therefore, this implies that

$$\|\hat{u}(t) - \hat{u}(t_n)\|_X^2 \longrightarrow 0 \text{ when } n \longrightarrow \infty, \quad \|\hat{u}_t(t) - \hat{u}_t(t_n)\|_Y^2 \longrightarrow 0 \text{ when } n \longrightarrow \infty.$$

Thus, we get $\hat{u} \in C([0, T]; X)$ and $\hat{u}_t \in C([0, T]; Y)$. □

4. Conclusion

In this paper, a long flexible Euler-Bernoulli beam with boundary conditions imposed at the two ends is developed. The definition of good spaces due to [2], the use of intermediate spaces and the Faedo-Galerkin method play a very large role in demonstration of the existence and uniqueness of the solution in the Euler-Bernoulli beam equation with variable coefficient. In addition, our proposed scheme can be also extended to other Euler-Bernoulli beam problems.

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