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On the Geometry of Cotangent Bundles of Lie Groups

By

Bakary Manga¹

Jury :

President - **Prof. Augustin Banyaga**, Pennsylvania State University (USA)

Reporters - **Prof. Augustin Banyaga**, Pennsylvania State University (USA)

- **Prof. Alberto Medina**, Université Montpellier II (France)

- **Prof. Joël Tossa**, Université d'Abomey-Calavi (Benin)

- **Prof. Léonard Todjihoundé**, Université d'Abomey-Calavi (Benin)

Examiners **Prof. Mahamane Bazanfaré**, Université Abdou Moumouni (Niger)

Prof. Issa Nourou, Université d'Abomey-Calavi (Benin)

Supervisors - **Prof. Alberto Medina**, Université Montpellier II (France)

- **Prof. Joël Tossa**, Université d'Abomey-Calavi (Benin)

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Dedication

To My Sister Binta Manga

Acknowledgements

As prelude to their works, authors often display a warning, a kind of umbrella, fortunately opened to protect themselves from possible backlash of their statement. As an author, if I was allowed to do the same, I will state the following: I acknowledge that It will be quite hard to find convenient words which can express my thanks to all those who helped me to achieve the dream of writing a PhD thesis. For those whose names are not written here, please be indulgent with me and accept my apologies.

I wish long life to the *Institut de Mathematiques et de Sciences Physiques (IMSP)*, the institute in which this thesis has been achieved.

The relations between *Professor Jean-Pierre Ezin*, director of the IMSP (1988-2008) and me have never been always easy. Nevertheless, Prof. Ezin has contributed to make me become a better person by allowing my personality, even overflowing at times, to be expressed. I am grateful to you for that and as well as for your strong and paramount battle related to support fundamental sciences in Africa. Thank you commissary ².

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²Mr. Ezin is nowadays commissary in charge of Human Resources, Sciences and New Technologies of the African Union.

him and never, he complained. Most of the results available in this thesis would have not been possible without his energetic support. Several of the results within this thesis are obtained in collaboration with him and some of them will be published in joint papers. Big brother, I am your servant!

Hermione Gandjidon and *Rekiath Yasso* have had to learn the L^AT_EX in order to become my "Secretaries". I thank both of you.

By being at the IMSP, I met a huge number of researchers, Physicists and Mathematicians. All interesting interactions and exchanges that we had had helped me to become a Mathematician. I thank you all.

At the campus of the "*Ecole Normale Supérieure Nadjo*" where live PhDs and Engineering students and where people call me *The Old*, I met guys funnier than me. I will miss the afternoon soccer's games of Tuesdays and Fridays through which I had opportunities to demonstrate to young people that *The Old* still has some youth in him.

I spent many years in this wonderful country and I would like to emphasize that the hospitality of the Beninese people has nothing to envy to the famous Senegalese "*Teranga*" ("*Teranga*" = hospitality). Here is an opportunity to express my gratitude to all the people of Benin.

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Résumé en Français

Cette thèse est une contribution à l'étude de la géométrie des fibrés cotangents des groupes de Lie et des espaces homogènes.

Nous caractérisons complètement les groupes des automorphismes des fibrés cotangents des groupes de Lie et montrons qu'ils sont des groupes de Lie super-symétriques (Theorem 2.3.2). Dans le cas particulier d'un groupe de Lie orthogonal, c'est-à-dire un groupe de Lie muni d'une métrique bi-invariante, nous utilisons la métrique pour réinterpréter les relations et retrouver des résultats connus de cohomologie.

Le fibré cotangent T^*G d'un groupe de Lie G peut être identifié au produit cartésien $G \times \mathcal{G}^*$, où \mathcal{G}^* est l'espace dual de l'algèbre de Lie \mathcal{G} de G . On peut alors munir T^*G de la structure de groupe de Lie obtenue par produit semi-direct de G et \mathcal{G}^* via la représentation coadjointe. Cette structure de groupe de Lie fait de T^*G un double de Drinfel'd.

Nous avons également étudié les préderivations des algèbres de Lie des fibrés cotangents des groupes de Lie. Nous montrons que l'algèbre des préderivations des algèbres de Lie des groupes de Lie fibrés cotangents des groupes de Lie sont réductives. Müller a montré que toutes les préderivations d'une algèbre de Lie semi-simple sont des dérivations (intérieures). Nous étendons ce résultat en montrant que si un groupe de Lie est semi-simple alors toutes les préderivations de l'algèbre de Lie de son fibré cotangent sont des dérivations quoi que le fibré cotangent soit non semi-simple (Theorem 3.4.1).

Un autre thème abordé dans cette thèse est l'étude des métriques bi-invariantes sur les fibrés cotangents des groupes de Lie. Nous caractérisons toutes les métriques bi-invariantes sur les fibrés cotangents des groupes de Lie et étudions le groupe de leurs isométries. L'algèbre de Lie de ce groupe d'isométries n'est rien d'autre que l'algèbre de toutes les préderivations de l'algèbre du fibré cotangent qui sont antisymétriques par rapport à la structure orthogonale induite sur l'algèbre de Lie du fibré cotangent.

Enfin, nous avons fait une introduction à la géométrie du groupe de Lie des transformations affines de la droite réelle. Nous donnons des expressions explicites d'une forme symplectique, d'une structure affine, des géodésiques de cette structure affine. La forme symplectique donnant lieu à une solution des équations Classiques de Yang-Baxter, nous avons également étudié le groupe de Lie double de Drinfel'd du groupe des transformations affines de la droite réelle.

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General Introduction

This thesis is a contribution to the large task of studying the geometry of cotangent bundles of Lie groups and the geometry of homogeneous spaces.

One of the greatest importance of the cotangent bundle T^*G of a Lie group G is that it is a symplectic manifold on which G acts by symplectomorphisms with a Lagrangian orbit. A symplectic Lie group is a pair (G, ω) consisting of a Lie group G and a closed non-degenerate 2-form ω which is invariant under left translations of G . If identified with the Cartesian product $G \times \mathfrak{G}^*$, the cotangent bundle T^*G possesses a Lie group structure obtained by semi-direct product $T^*G := G \ltimes \mathfrak{G}^*$ using the coadjoint action of G on the dual space \mathfrak{G}^* of the Lie algebra \mathfrak{G} of G . This Lie group structure together with the Liouville form is not a symplectic Lie group. But, if G carries a left-invariant affine structure, then its cotangent bundle carries a symplectic Lie group structure. This particular Lie group structure, sometimes called "*cotangent*", is obtained by taking the semi-direct product of G and the dual space \mathfrak{G}^* of its Lie algebra \mathfrak{G} by means of a natural action given by the affine structure on G . The corresponding Lie algebra is the semi-direct product via the left multiplications given by the left-symmetric product. The two Lie group structure on T^*G defined above are not isomorphic.

It is also well known, since the works of Drinfel'd ([32]), that T^*G (with the Lie group structure performed by semi-direct product $G \ltimes \mathfrak{G}^*$ via the coadjoint representation of G on \mathfrak{G}^*) is a particular case of the large class of the so-called Drinfel'd double Lie groups. As a Drinfel'd double Lie group, T^*G admits a metric which is invariant under left and right translations ([32]). This Lie group structure on T^*G does not admit a left-invariant symplectic form, except in the Abelian case.

In this thesis we deal with the Lie group structure which make T^*G a Drinfel'd double Lie group of G .

Studying the geometry of a given Lie group is to study invariant structures on it. The cotangent bundle T^*G of a Lie group G can exhibit very interesting and rich algebraic and geometric structures (affine, symplectic, pseudo-Riemannian, Kählerian,...[56], [47], [34], [32], [28], [5]).

Such structures can be better understood when one can exhibit the group of transformations which preserve them. This very often involves the automorphisms of T^*G , if in

particular, such structures are invariant under left or right multiplications by the elements of T^*G . This is one of the reason for which we deal with *automorphisms of cotangent bundles of Lie groups* in *Chapter II* of this dissertation. We study the connected component of the unit of the group of automorphisms of the Lie algebra $\mathcal{D} := T^*\mathcal{G}$ of T^*G . Such a connected component being spanned by exponentials of derivations of \mathcal{D} , we often work with those derivations. Let $\text{der}(\mathcal{G})$ stand for the Lie algebra of derivations of \mathcal{G} , while \mathcal{J} denotes that of linear maps $j : \mathcal{G} \rightarrow \mathcal{G}$ satisfying $j([x, y]) = [j(x), y]$, for every elements x, y of \mathcal{G} . We give a characterization of all derivations of $T^*\mathcal{G}$ (Theorem 2.3.1) and show that in particular, if G has a bi-invariant Riemannian or pseudo-Riemannian metric, then every derivation ϕ of \mathcal{D} can be expressed in terms of elements of $\text{der}(\mathcal{G})$ and \mathcal{J} alone. Furthermore, we give prominence to the fact that the Lie group $\text{Aut}(\mathcal{D})$ of automorphisms of \mathcal{D} is a super symmetric Lie group and its Lie algebra $\text{der}(\mathcal{D})$ possesses Lie subalgebras which are Lie superalgebras, *i.e.* they are $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebras with the Lie bracket satisfying $[x, y] = -(-1)^{\text{deg}(x)\text{deg}(y)}[y, x]$ (Theorem 2.3.2). We also consider particular cases (e.g. orthogonal Lie algebras, semi-simple Lie algebras, compact Lie algebras) and recover by simple methods interesting cohomological known results (Section 2.3.6).

In *Chapter III* we completely characterise the space of prederivations of \mathcal{D} , that is endomorphisms p of \mathcal{D} which satisfy

$$p([x, [y, z]]) = [p(x), [y, z]] + [x, [p(y), z]] + [x, [y, p(z)]],$$

for every elements x, y, z of \mathcal{D} . The Lie algebra $\text{der}(\mathcal{D})$ is a subalgebra of the Lie algebra $\text{Pder}(\mathcal{D})$ of prederivations of \mathcal{D} . Prederivations can be used to study bi-invariant metrics on Lie groups ([6], [9], [64]). One of the important results within this chapter is: *If G is a semi-simple Lie group with Lie algebra \mathcal{G} , then any prederivations of $T^*\mathcal{G}$ (not semi-simple) is a derivation.* This is an extension of the result of Müller ([64]) which states that any prederivation of a semi-simple Lie algebra is a derivation, hence an inner derivation. We also give a structure theorem for $\text{Pder}(\mathcal{D})$ which states that $\text{Pder}(\mathcal{D})$ decomposes into $\text{Pder}(T^*\mathcal{G}) = \mathcal{G}_0 \oplus \mathcal{G}_1$, where \mathcal{G}_0 is a reductive subalgebra of $\text{Pder}(\mathcal{D})$, that is $[\mathcal{G}_0, \mathcal{G}_0] \subset \mathcal{G}_0$ and $[\mathcal{G}_0, \mathcal{G}_1] \subset \mathcal{G}_1$. Semi-simple, compact and more generally orthogonal Lie algebras are also considered in this chapter.

We study bi-invariant metrics of cotangent bundles of Lie groups in *Chapter IV*. In this chapter we characterise all orthogonal structures on $T^*\mathcal{G}$ (Theorem 4.3.1) and their isometries. It is known that if (G, μ) is a connected and simply-connected orthogonal Lie group with Lie algebra \mathcal{G} , then the isotropy group of the neutral element of G in the group $I(G, \mu)$ of isometries of (G, μ) is isomorphic to the group of preautomorphisms of \mathcal{G} which preserve the non-degenerate bilinear form induced by μ on \mathcal{G} and whose Lie algebra is the whole set of skew-symmetric prederivations of \mathcal{G} ([64]). We characterise the isometries of bi-invariant metrics through the skew-symmetric prederivations with respect to the orthogonal structures induced on $T^*\mathcal{G}$ by the bi-invariant metrics (Proposition 4.3.2). In the case where \mathcal{G} possesses an orthogonal structure, we proved that any orthogonal structure on $T^*\mathcal{G}$ can be expressed in terms of the duality pairing and endomorphisms of \mathcal{G} which commute with all adjoint operators (Theorem 4.4.1). If \mathcal{G} is a semi-simple Lie algebra, we prove that any prederivation of $T^*\mathcal{G}$ which is skew-symmetric with respect to any orthogonal structure on

$T^*\mathcal{G}$ is an inner derivation (Proposition 4.5.1); that is the connected component of the unit of the Lie group of isometries of any bi-invariant metric on T^*G is spanned by exponentials of inner derivations of $T^*\mathcal{G}$. Examples of the affine Lie group of the real line, the special linear group $SL(2, \mathbb{R})$, the group $SO(3, \mathbb{R})$ of rotations and the 4-dimensional oscillator group are given.

The geometry of the Lie group of affine motions of the real line is explored in *Chapter V*. The geodesics of the left invariant affine structure induced by the symplectic structure are studied as well as integrale curves of left invariant vector fields. Since, the symplectic form considered on the affine Lie group corresponds to an invertible solution of the Classical Yang-Baxter equation, we have also studied the geometry of the corresponding double Lie group. The affine and complex structures on the double introduced by Diatta and Medina ([28]) are considered.

INVARIANT STRUCTURES ON LIE GROUPS

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This chapter is to make this thesis as self contained as possible. It defines basic notions and terminologies which might be useful throughout this dissertation.

1.1 Orthogonal Structures on Lie Groups

1.1.1 Definition and Examples

Let G be a Lie group, ϵ its identity element, \mathfrak{G} its Lie algebra and TG its tangent bundle. Let \mathfrak{G}^* stand for the dual space of \mathfrak{G} and let \mathbb{K} stand for the field \mathbb{R} of real numbers or the field \mathbb{C} of complex numbers.

Definition 1.1.1. *A bi-invariant pseudo-metric on G is a function $\mu : TG \rightarrow \mathbb{K}$ which is quadratic on each fiber, nondegenerate and invariant under both left and right translations of the group G .*

A bi-invariant pseudo-metric on the Lie group G corresponds to a non-degenerate quadratic form $q : \mathfrak{G} \rightarrow \mathbb{K}$ for which the adjoint operators are skew-symmetric; that is, if μ also denotes the corresponding symmetric bilinear form on \mathfrak{G} , we have

$$\mu([x, y], z) + \mu(y, [x, z]) = 0 \tag{1.1}$$

for every x, y, z in \mathfrak{G} .

Conversely, if \mathfrak{G} admits a non-degenerate symmetric bilinear form for which the adjoint operators are skew-symmetric, then there exists, on every connected Lie group with Lie algebra \mathfrak{G} , a bi-invariant pseudo-metric.

Throughout this work, a non-degenerate symmetric bilinear form on \mathfrak{G} for which the adjoint operators are skew-symmetric is called simply a bi-invariant scalar product or an orthogonal structure on \mathfrak{G} as well as a bi-invariant pseudo-metric on G is called a bi-invariant metric or an orthogonal structure on G .

Lie groups (resp. Lie algebras) with orthogonal structures are called orthogonal or quadratic (see e.g. [58], [60]). In [3] orthogonal Lie algebras are called *metrizable Lie algebras*.

In [58], Medina and Revoy have shown that every orthogonal Lie algebra is obtained by the so-called *double extension* procedure.

Consider the isomorphism of vector spaces $\theta : \mathfrak{G} \rightarrow \mathfrak{G}^*$ defined by

$$\langle \theta(x), y \rangle := \mu(x, y), \tag{1.2}$$

where \langle, \rangle on the left hand side, is the duality pairing $\langle x, f \rangle = f(x)$, between elements x of \mathfrak{G} and f of \mathfrak{G}^* . Then, θ is an isomorphism of \mathfrak{G} -modules in the sense that it is equivariant with respect to the adjoint and coadjoint actions of \mathfrak{G} on \mathfrak{G} and \mathfrak{G}^* respectively; *i.e.*

$$\theta \circ ad_x = ad_x^* \circ \theta, \tag{1.3}$$

for all x in \mathfrak{G} . Which is also

$$\theta^{-1} \circ ad_x^* = ad_x \circ \theta^{-1}, \tag{1.4}$$

for all x in \mathfrak{G} . The converse is also true. More precisely, a Lie group (resp. algebra) is orthogonal if and only if its adjoint and coadjoint representations are isomorphic. See Theorem 1.4. of [60].

Example 1.1.1. Any Abelian Lie algebra with any scalar product is an orthogonal Lie algebra.

Example 1.1.2. Semi-simple Lie algebras with the Killing forms are orthogonal Lie algebras.

Example 1.1.3 (Oscillator groups or diamond groups). Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, where $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are positive real numbers. On $\mathbb{R}^{2n+2} \equiv \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$, define the operation

$$(t, s; z_1, \dots, z_n) \cdot (t', s'; z'_1, \dots, z'_n) = \left(t + t', s + s' + \frac{1}{2} \sum_{j=1}^n Im(\bar{z}_j z'_j e^{i\lambda_j t}) \ ; \ z_1 + z'_1 e^{i\lambda_1 t}, \dots, z_n + z'_n e^{i\lambda_n t} \right), \tag{1.5}$$

where t, t', s, s' are real numbers while $z_i, z'_i, i = 1, 2, \dots, n$, are in \mathbb{C} . Endowed with the operation (1.5) and the adjacent manifold structure, \mathbb{R}^{2n+2} is a Lie group of dimension $2n + 2$. This Lie group is noted by G_λ and is called **Oscillator group** in [12], [29], [35],

[68] (for dimension $n = 1$), **twisted Heisenberg group** in [73] and **diamond group** in other literature ([73]).

The Lie algebra of G_λ , called **oscillator algebra** of dimension $2n + 2$, is the vector space spanned by $\{e_{-1}, e_0, e_1, e_2, \dots, e_n, \check{e}_1, \check{e}_2, \dots, \check{e}_n\}$ with the following non-zero brackets:

$$[e_{-1}, e_j] = \lambda_j \check{e}_j \quad ; \quad [e_{-1}, \check{e}_j] = -\lambda_j e_j \quad ; \quad [e_j, \check{e}_j] = e_0. \quad (1.6)$$

The oscillator Lie algebra is noted by \mathfrak{G}_λ . Every element x of \mathfrak{G}_λ can be written:

$$x = \alpha e_{-1} + \beta e_0 + \sum_{j=1}^n \alpha_j e_j + \sum_{j=1}^n \beta_j \check{e}_j. \quad (1.7)$$

where $\alpha, \beta, \alpha_j, \beta_j$ ($1 \leq j \leq n$) are real numbers. Then the following quadratic form defines an orthogonal structure on \mathfrak{G}_λ (see [12]):

$$k_\lambda(x, x) := 2\alpha\beta + \sum_{j=1}^n \frac{1}{\lambda_j} (\alpha_j^2 + \beta_j^2) \quad (1.8)$$

Oscillator groups are subject of a lot of studies ([29], [31], [35], [36], [50], [59], [68]). They appear in many branches of Physics and Mathematical Physics and give particular solutions of Einstein-Yang-Mills equations ([50]).

1.1.2 Hyperbolic Lie Algebras, Manin Lie Algebras

Definition 1.1.2. *A hyperbolic plan E is a 2-dimensional linear space endowed with a non-degenerate symmetric bilinear form B such that there exists a non-zero element v of E with $B(v, v) = 0$. A hyperbolic space is an orthogonal sum of hyperbolic plans.*

Remark 1.1.1. A hyperbolic space is of even dimension.

For more about hyperbolic spaces, see [48].

Definition 1.1.3. *A hyperbolic Lie algebra is an orthogonal Lie algebra which contains two totally isotropic subspaces in duality for the orthogonal structure.*

Definition 1.1.4. *A Manin Lie algebra is an orthogonal Lie algebra \mathfrak{G} such that:*

- \mathfrak{G} admits two totally isotropic subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 ;
- \mathfrak{h}_1 and \mathfrak{h}_2 are in duality one to the other for the orthogonal structure of \mathfrak{G} .

In this case, $(\mathfrak{G}, \mathfrak{h}_1, \mathfrak{h}_2)$ is called a Manin triple.

1.2 Poisson Structure on Lie Groups

1.2.1 Poisson Brackets, Poisson Tensors on a Manifold

Definition 1.2.1. A \mathcal{C}^∞ -smooth **Poisson structure (Poisson bracket)** on a \mathcal{C}^∞ -smooth finite-dimensional manifold M is an \mathbb{R} -bilinear skew-symmetric operation

$$\begin{aligned} \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) &\longrightarrow \mathcal{C}^\infty(M) \\ (f, g) &\longmapsto \{f, g\} \end{aligned} \quad (1.9)$$

on the space $\mathcal{C}^\infty(M)$ of real-valued \mathcal{C}^∞ -smooth functions on M , such that

- $(\mathcal{C}^\infty(M), \{, \})$ is a Lie algebra;
- $\{, \}$ is a derivation in each factor, that is it verifies the Liebniz identity

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad (1.10)$$

for every f, g, h in $\mathcal{C}^\infty(M)$.

A manifold equipped with such a bracket is called a **Poisson manifold**.

Example 1.2.1. Any manifold carries a trivial Poisson structure. One just has to put $\{f, g\} = 0$, for all smooth functions f and g on M .

Example 1.2.2. Let (x, y) denote coordinates on \mathbb{R}^2 and $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be an arbitrary smooth function. One defines a smooth Poisson structure on \mathbb{R}^2 by putting

$$\{f, g\} = \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) p, \quad (1.11)$$

for every f, g in $\mathcal{C}^\infty(\mathbb{R}^2)$.

Example 1.2.3. A symplectic manifold (M, ω) is a manifold M equipped with a non-degenerate closed differential 2-form ω , called a **symplectic form**. If $f : M \rightarrow \mathbb{R}$ is a function on (M, ω) , we define its **Hamiltonian vector field**, denoted by X_f , as follows:

$$i_{X_f} \omega := \omega(X_f, \cdot) = -Tf, \quad (1.12)$$

where Tf stands for the differential map of f . One defines on (M, ω) a natural bracket, called the **Poisson bracket of ω** , as follows:

$$\{f, g\} = \omega(X_f, X_g) = -\langle Tf, X_g \rangle = -X_g(f) = X_f(g), \quad (1.13)$$

for every $f, g \in \mathcal{C}^\infty(M)$. Thus, any symplectic manifold carries a Poisson structure.

Let $(M, \{, \})$ be a Poisson manifold. To every f in $\mathcal{C}^\infty(M)$ corresponds a unique vector field X_f on M such that

$$X_f(g) = \{f, g\}, \quad (1.14)$$

for every $g \in \mathcal{C}^\infty(M)$ (see e.g. [57]). This is an extension of the notion of Hamiltonian vector field from the symplectic to the Poisson context. Thus, X_f is called the **Hamiltonian vector field** associated to f as well as f is called the **Hamiltonian function** of X_f .

Definition 1.2.2. A map $\phi : (M_1, \{, \}_1) \rightarrow (M_2, \{, \}_2)$ between two Poisson manifolds is said to be a Poisson morphism if it is smooth and satisfies

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi, \tag{1.15}$$

for every f, g in $\mathcal{C}^\infty(M_2)$.

Example 1.2.4. If $(M_1, \{, \}_1)$ and $(M_2, \{, \}_2)$ are Poisson manifolds, then $M_1 \times M_2$ has a Poisson structure characterized by the following properties:

1. the projections $\pi_i : M_1 \times M_2 \rightarrow M_i, i = 1, 2$, are Poisson morphisms;
2. $\{f \circ \pi_1, g \circ \pi_2\}_1 = 0$, for any f in $\mathcal{C}^\infty(M_1)$ and g in $\mathcal{C}^\infty(M_2)$.

Example 1.2.5 (Kirilov-Kostant-Sauriau (KKS)). Let \mathcal{G} be a finite dimensional Lie algebra seen as the space of linear maps on its dual \mathcal{G}^* ; i.e. $\mathcal{G} \equiv (\mathcal{G}^*)^* \subset \mathcal{C}^\infty(\mathcal{G}^*)$. Let f, g be in $\mathcal{C}^\infty(\mathcal{G}^*)$ and α belongs to \mathcal{G}^* . If $T_\alpha f$ and $T_\alpha g$ denote the differentials of f and g at the point α (seen as elements of \mathcal{G}), one defines a linear Poisson structure on \mathcal{G}^* by putting:

$$\{f, g\}(\alpha) := \langle \alpha, [T_\alpha f, T_\alpha g]_{\mathcal{G}} \rangle, \tag{1.16}$$

where \langle, \rangle in the right hand side stands for the pairing between \mathcal{G} and its dual.

This structure plays an important role in many domains of mathematical physics, quantization, hyper-kählerian geometry, ...

Let M be a smooth manifold of dimension n ($n \in \mathbb{N}^*$) and p a positive integer. Denote by $\Lambda^p TM$ the space of tangent p -vectors of M . It is a vector bundle over M whose fiber over each point $x \in M$ is the space $\Lambda^p T_x M = \Lambda^p(T_x M)$, which is the exterior antisymmetric product of p copies of the tangent space $T_x M$. Of course $\Lambda^1 TM = TM$.

Let (x^1, \dots, x^n) be a local system of coordinates at $x \in M$. Then $\Lambda^p T_x M$ admits a linear basis consisting of the elements $\frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_p}}(x)$ with $i_1 < i_2 < \dots < i_p$.

Definition 1.2.3. A smooth p -vector field π on M is a smooth section of $\Lambda^p TM$, i.e. a map π from M to $\Lambda^p TM$, which associates to each point x of M a p -vector $\pi(x)$ of $\Lambda^p T_x M$, in a smooth way.

Given a 2-vector field π on a smooth manifold M , one defines a tensor field, also denoted by π , by the following formula:

$$\{f, h\} := \pi(f, h) := \langle Tf \wedge Th, \pi \rangle \tag{1.17}$$

Definition 1.2.4. A 2-vector field π , such that the bracket given by (1.17) is a Poisson bracket, is called a Poisson tensor, or also a Poisson structure.

Any Poisson tensor π arises from a 2-vector field which we will also denote by π .

Example 1.2.6. The Poisson tensor corresponding to the standard symplectic structure $\omega = \sum_{k=1}^n dx^k \wedge dy^k$ on \mathbb{R}^{2n} is $\sum_{k=1}^n \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial y^k}$.

1.2.2 Poisson-Lie Groups

Definition 1.2.5. Let G be a Lie group. A **Poisson-Lie structure** on G is a Poisson tensor π on the underlying manifold of G such that the multiplication map

$$\begin{aligned} G \times G &\rightarrow G \\ (g, g') &\mapsto gg' \end{aligned} \quad (1.18)$$

is a Poisson morphism (is grouped as said in [32]); $G \times G$ being equipped with the Poisson tensor product $\pi \times \pi$.

A Lie group G with a Poisson-Lie structure π is called a **Poisson-Lie group** and is denoted by (G, π) .

The Definition 1.2.5 is equivalent to the following:

$$\pi(gh) = T_h L_g \cdot \pi(h) + T_g R_h \cdot \pi(g), \quad (1.19)$$

for every g, h in G , where the differentials $T_g L_h$ and $T_g R_h$ of the left translation L_g and the right translation R_g are naturally extended to the linear space $\Lambda^2 T_g G$; i.e.

$$T_g L_h \cdot (X \wedge Y) := (T_g L_h \cdot X) \wedge (T_g L_h \cdot Y), \quad (1.20)$$

$$T_g R_h \cdot (X \wedge Y) := (T_g R_h \cdot X) \wedge (T_g R_h \cdot Y), \quad (1.21)$$

for any X, Y in $T_g^* G$.

Definition 1.2.6. Let G be a Lie group. A tensor field π on G is called *multiplicative* if it satisfies Relation (1.19).

Example 1.2.7. To every Lie algebra corresponds, at least, one Poisson-Lie group. Indeed, the dual space (seen as an Abelian Lie group) of any Lie algebra, endowed with its linear Poisson structure given in Example 1.2.5 is a Poisson-Lie group.

Every Lie group possesses, at least, one non-trivial Poisson tensor (see [24]).

1.2.3 Dual Lie Group, Drinfel'd Double of a Poisson-Lie Group

Let G be a Lie group with Lie algebra \mathfrak{G} , π a Poisson-Lie tensor on G with corresponding bracket $\{, \}$. One obtains a Lie algebra structure $[\cdot, \cdot]_*$ on the dual space \mathfrak{G}^* of \mathfrak{G} by setting

$$[\alpha, \beta]_* := T_\epsilon \{f, g\}, \quad (1.22)$$

where f and g are smooth functions on G such that α and β are equal, respectively, to the differentials of f and g at the identity element ϵ of G : $T_\epsilon f = \alpha$, $T_\epsilon g = \beta$.

Definition 1.2.7. 1. The bracket (1.22) is called the "linearized" bracket of π at ϵ and the pair $(\mathfrak{G}^*, [\cdot, \cdot]_*)$ is said to be the "linearized" or the **dual Lie algebra** of (G, π) or of (\mathfrak{G}, λ) , where $\lambda := T_\epsilon \pi : \mathfrak{G} \rightarrow \Lambda^2 \mathfrak{G}$ (λ is the transpose of $[\cdot, \cdot]_* : \Lambda^2 \mathfrak{G}^* \rightarrow \mathfrak{G}^*$).

2. Every Lie group, with Lie algebra $(\mathfrak{G}^*, [\cdot, \cdot]_*)$ is called a **dual Lie group** of (G, π) .

3. A subgroup H of a Poisson-Lie group (G, π) is said to be a Poisson-Lie subgroup of (G, π) if it is also a Poisson submanifold of (G, π) .

Definition 1.2.8. Let \mathfrak{G} be a Lie algebra.

1. A **Lie bi-algebra structure** on \mathfrak{G} is a 1-cocycle $\lambda : \mathfrak{G} \rightarrow \Lambda^2 \mathfrak{G}$ for the adjoint representation of \mathfrak{G} such that its transpose $\lambda^t : \Lambda^2 \mathfrak{G}^* \rightarrow \mathfrak{G}^*$ defines a Lie bracket on the vector space \mathfrak{G}^* .
2. A Lie algebra, with a Lie bi-algebra structure, is called a **Lie bi-algebra**.

A Lie bi-algebra will be denoted by (\mathfrak{G}, λ) or $(\mathfrak{G}, \mathfrak{G}^*)$.

Theorem 1.2.1. ([32]) Let G be a simply connected Lie group. A Poisson structure, with Poisson tensor π , on G bijectively corresponds to a Lie bi-algebra structure $\lambda = T_\epsilon \pi$ on the Lie algebra \mathfrak{G} of G .

Let $(\mathfrak{G}, [,])$ be a Lie algebra. Suppose $(\mathfrak{G}, \mathfrak{G}^*)$ is a Lie bi-algebra and let $[\cdot, \cdot]_*$ denote the induced Lie bracket on \mathfrak{G}^* . Then, the transpose of the Lie bracket of \mathfrak{G} is a 1-cocycle of the Lie algebra $(\mathfrak{G}^*, [\cdot, \cdot]_*)$. Hence, the Lie algebras $(\mathfrak{G}, [,])$ and $(\mathfrak{G}, [\cdot, \cdot]_*)$ act one to the other by their respective coadjoint actions. The space $\mathfrak{G} \times \mathfrak{G}^*$ can be equipped with the scalar product:

$$\langle (x, f), (y, g) \rangle := f(y) + g(x), \tag{1.23}$$

for every x, y in \mathfrak{G} and every f, g in \mathfrak{G}^* .

We have the following result due to Drinfel'd.

Theorem 1.2.2. ([32]) The following are equivalent:

1. $(\mathfrak{G}, \mathfrak{G}^*)$ is a Lie bi-algebra;
2. $\mathcal{D} := (\mathfrak{G} \times \mathfrak{G}^*, \langle, \rangle)$ is equipped with a unique structure $[\cdot, \cdot]_{\mathcal{D}}$ of orthogonal Lie algebra such that:

- (a) \mathfrak{G} and \mathfrak{G}^* are Lie subalgebras of \mathcal{D} ;
- (b) \mathfrak{G} and \mathfrak{G}^* are totally isotropic and in duality for the scalar product \langle, \rangle .

In this case, for every x in \mathfrak{G} and g in \mathfrak{G}^* ,

$$[x, g]_{\mathcal{D}} = ad_x^* g - ad_g^* x \tag{1.24}$$

Definition 1.2.9. The Lie algebra \mathcal{D} of the Theorem 1.2.2 is called the **Drinfel'd double Lie algebra** of (G, π) or of $(\mathfrak{G}, \lambda := T_\epsilon \pi)$. Every Lie group, with Lie algebra \mathcal{D} , is called a **Drinfel'd double Lie group** of (G, π) .

The Lie bracket on \mathcal{D} reads:

$$[(x, f), (y, g)]_{\mathcal{D}} = ([x, y] + ad_f^* y - ad_g^* x, [f, g]_* + ad_x^* g - ad_y^* f) \tag{1.25}$$

for every (x, f) and (y, g) in \mathcal{D} .

Since its introduction in 1983 ([32]), the notion of Drinfel'd double attracted many researchers ([2], [13], [18], [21],[22], [24],[28], [46]).

1.3 Yang-Baxter Equation

Let M be a smooth manifold. For any integer p , let $\Omega_p(M)$ stand for the space of smooth sections of $\Lambda^p TM$ and let $\Omega_*(M)$ be the direct sum of the spaces $\Omega_p(M)$, where

- $\Omega_p(M) = \{0\}$, if $p < 0$;
- $\Omega_0(M) = \mathcal{C}^\infty(M)$;
- $\Omega_1(M) = \mathfrak{X}(M)$ (smooth vector fields on M) ;
- $\Omega_p(M) = \{0\}$, if $p > \dim M$.

Theorem 1.3.1 (Schouten Bracket Theorem). *There exists a unique bilinear operation $[\cdot, \cdot] : \Omega_*(M) \times \Omega_*(M) \rightarrow \Omega_*(M)$ natural with respect to restriction to open sets, called the **Schouten Bracket**, that satisfies the following properties:*

1. $[\cdot, \cdot]$ is a biderivation of degree -1 , i.e. for all homogeneous elements A and B of $\Omega_*(M)$ and any C in $\Omega_*(M)$,
 - $\deg[A, B] = \deg A + \deg B - 1$; and
 - $[A, B \wedge C] = [A, B] \wedge C + (-1)^{(\deg A + 1) \deg B} B \wedge [A, C]$;
2. $[\cdot, \cdot]$ is defined on $\mathcal{C}^\infty(M)$ and on $\mathfrak{X}(M)$ by
 - (a) $[f, g] = 0$, for any f, g in $\mathcal{C}^\infty(M)$;
 - (b) $[X, f] = X \cdot f$, for all f in $\mathcal{C}^\infty(M)$ and any vector field X on M ;
 - (c) $[X, Y]$ is the usual Jacobi-Lie bracket of vector fields if X and Y are in $\mathfrak{X}(M)$;
3. $[A, B] = (-1)^{\deg A \deg B} [B, A]$.

In addition, we have the graded Jacobi identity

$$(-1)^{\deg A \deg C} [[A, B], C] + (-1)^{\deg B \deg A} [[B, C], A] + (-1)^{\deg C \deg B} [[C, A], B] = 0 \quad (1.26)$$

for all homogeneous elements A, B, C of $\Omega_*(M)$.

Let G be a Lie group with Lie algebra \mathfrak{G} . On the \mathbb{Z} -graded vector space $\Lambda \mathfrak{G} := \bigoplus_{p \in \mathbb{Z}} \Lambda^p \mathfrak{G}$ we consider the structure of graded Lie algebra obtained by the extension of the Lie bracket of \mathfrak{G} satisfying the properties of the definition of the Schouten's bracket.

Let $r \in \Lambda^2 \mathfrak{G}$ and note by η the 1-coboundary defined by

$$\eta(g) = Ad_g r - r, \quad (1.27)$$

for all g in G .

Let r^+ and r^- denote the left invariant and right invariant tensor fields associated to r respectively. One wonders whether the corresponding multiplicative tensor $\pi := r^+ - r^-$ is Poisson. The answer is given by the

Proposition 1.3.1. ([51]) Let Λ be a contravariant skew-symmetric 2-vector field on a manifold M .

(i) Λ is Poisson if and only if $[\Lambda, \Lambda] = 0$ (this is equivalent to the Jacobi identity);

(ii) If $[\Lambda, \Lambda] = 0$, then the map

$$\begin{aligned} \partial : \Omega_*(M) &\rightarrow \Omega_*(M) \\ P &\mapsto [\Lambda, P] \end{aligned} \quad (1.28)$$

is an operator of cohomology, i.e. $\partial_\Lambda \circ \partial_\Lambda = 0$.

Definition 1.3.1. The cohomology defined by (ii) of Proposition 1.3.1 is called the Poisson cohomology of the Poisson manifold (M, Λ) .

According to the Proposition 1.3.1, the tensor field π is Poisson if and only if the 3-tensor field

$$[\pi, \pi] = [r^+, r^+] + [r^-, r^-] = [r, r]^+ - [r, r]^- \quad (1.29)$$

vanishes identically or equivalently, for every $g \in G$, the 3-tensor

$$[\pi, \pi]_g := T_\epsilon(Ad_g[r, r] - [r, r]) \quad (1.30)$$

equals zero. Hence, π defines a Lie-Poisson tensor if and only if the 3-vector $[r, r]$ is Ad_G -invariant, i.e. for all $g \in G$,

$$Ad_g[r, r] = [r, r] \quad (1.31)$$

Definition 1.3.2. 1. Equation (1.31) below is called the **Generalized Yang-Baxter Equation (GYBE)** and its solutions are called **r -matrices**.

2. In the particular case where $[r, r] = 0$, one says that r is a solution of the **Classical Yang-Baxter Equation (CYBE)**.

1.4 Symmetric Spaces

Let G be a connected Lie group with Lie algebra \mathfrak{G} and identity element ϵ .

Definition 1.4.1. A **symmetric space** for G is a homogeneous space $M \equiv G/H$ such that the isotropy group H of any arbitrary point is an open subgroup of the fixed point set $G_\sigma := \{g \in G : \sigma(g) = g\}$ of an involution σ of G .

The involution σ is in fact an automorphism of G and fixes the identity element ϵ . Hence, the differential at ϵ of σ is an automorphism, also denote by σ , of the Lie algebra \mathfrak{G} with square equal to the identity mapping of \mathfrak{G} : $\sigma^2 = Id_{\mathfrak{G}}$. Then the eigenvalues of σ are $+1$ and -1 . The eigenspace associated to $+1$ is the Lie algebra \mathfrak{h} of H . We denote the -1 eigenspace by \mathfrak{m} . Since σ is a automorphism of \mathfrak{G} , we have the following decomposition

$$\mathfrak{G} = \mathfrak{h} \oplus \mathfrak{m} \quad (1.32)$$

with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad ; \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad ; \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (1.33)$$

Conversely, given any Lie algebra \mathfrak{G} with direct sum decomposition (1.32) satisfying (1.33), the linear map σ , equal to the identity on \mathfrak{h} and minus the identity on \mathfrak{m} , is an involutive automorphism of \mathfrak{G} .

1.5 Reductive Lie algebras

Let \mathfrak{G} be a finite dimensional Lie algebra over a field \mathbb{K} of characteristic zero.

Definition 1.5.1. *A subalgebra \mathfrak{h} is said to be reductive in \mathfrak{G} if \mathfrak{G} is a semisimple \mathfrak{h} -module; that is \mathfrak{G} is a sum of simple \mathfrak{h} -modules in the adjoint representation of \mathfrak{h} in \mathfrak{G} . The Lie algebra \mathfrak{G} is said to be reductive if it is a reductive subalgebra of itself.*

Let \mathfrak{G} be an arbitrary finite dimensional Lie algebra over \mathbb{K} , V be a semisimple \mathfrak{G} -module and I be the ideal $I = \{x \in \mathfrak{G} : x \cdot V = \{0\}\}$. Now set $\mathfrak{h} = \mathfrak{G}/I$. By a result (see [16]) due to E. Cartan and N. Jacobson, we have $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] + Z(\mathfrak{h})$, where $Z(\mathfrak{h})$ is the center of \mathfrak{h} , $[\mathfrak{h}, \mathfrak{h}]$ is semisimple, and V is semisimple as a $Z(\mathfrak{h})$ -module.

Now suppose that \mathfrak{G} is reductive. Since the center $Z(\mathfrak{G})$ of \mathfrak{G} is a \mathfrak{G} -submodule of \mathfrak{G} , there is an ideal J of \mathfrak{G} such that \mathfrak{G} is the direct sum $J \oplus Z(\mathfrak{G})$. By the result we have cited above, we have $J = [J, J] + Z(J)$, where $Z(J)$ is the center of J and $[J, J]$ is semisimple. But, of course, $[J, J] = [\mathfrak{G}, \mathfrak{G}]$ and $Z(J) = \{0\}$. Hence, $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}] + Z(\mathfrak{G})$ and $[\mathfrak{G}, \mathfrak{G}]$ is semisimple. Conversely, it is clear that if a finite dimensional Lie algebra \mathfrak{G} over \mathbb{K} satisfies these last conditions, then \mathfrak{G} is reductive.

1.6 Lie Superalgebras

Let \mathbb{K} be an Abelian field of characteristic zero.

1.6.1 Definition of a Lie Superalgebra

Definition 1.6.1. *A \mathbb{K} -linear space L is a \mathbb{K} -superalgebra, with superbracket $[\cdot, \cdot]$, if*

a) *it is $\mathbb{Z}/2\mathbb{Z}$ -graded, i.e.*

$$L = L_{\bar{0}} \oplus L_{\bar{1}} \quad ; \quad [L_{\bar{0}}, L_{\bar{0}}] \subset L_{\bar{0}} \quad ; \quad [L_{\bar{0}}, L_{\bar{1}}] \subset L_{\bar{1}} \quad ; \quad [L_{\bar{1}}, L_{\bar{1}}] \subset L_{\bar{0}} \quad (1.34)$$

b) *for every homogeneous elements a, b, c of L ,*

$$(-1)^{|a| \cdot |c|} [a, [b, c]] + (-1)^{|b| \cdot |a|} [b, [c, a]] + (-1)^{|c| \cdot |b|} [c, [a, b]] = 0. \quad (1.35)$$

where $|a|$ (respectively $|b|$ and $|c|$) stands for the degree of a (respectively b and c).

For a \mathbb{K} -superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$, $L_{\bar{0}}$ is an ordinary Lie algebra while $L_{\bar{1}}$ is a $L_{\bar{0}}$ -module. Each element z of L can be uniquely written:

$$z = z_{\bar{0}} + z_{\bar{1}} \quad (1.36)$$

where $z_{\bar{0}} \in L_{\bar{0}}$ and $z_{\bar{1}} \in L_{\bar{1}}$. One says that $z_{\bar{0}}$ is the component of degree $\bar{0}$ of z and $z_{\bar{1}}$ is the component of degree $\bar{1}$ of z .

1.6.2 Derivations of Lie Superalgebras

See [42] for more about Lie superalgebras.

Definition 1.6.2. *Let L be a Lie superalgebra over \mathbb{K} .*

1. *A derivation of degree $\bar{0}$ of L is an endomorphism $D : L \rightarrow L$ such that*

$$D(L_{\bar{0}}) \subset L_{\bar{0}} \quad ; \quad D(L_{\bar{1}}) \subset L_{\bar{1}} \quad ; \quad D[a, b] = [D(a), b] + [a, D(b)] \quad (1.37)$$

for every $a, b \in L$.

2. *A derivation of degree $\bar{1}$ of L is an endomorphism $D : L \rightarrow L$ such that*

$$D(L_{\bar{0}}) \subset L_{\bar{1}} \quad ; \quad D(L_{\bar{1}}) \subset L_{\bar{0}} \quad ; \quad D[a, b] = [D(a), b] + (-1)^{|a|}[a, D(b)] \quad (1.38)$$

for every homogeneous element a of L and every element $b \in L$.

3. *More generally, a derivation of degree r ($r \in \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$) of a Lie superalgebra L is an endomorphism $D \in \text{End}_r(L) := \{\varphi \in \text{End}(L) : \varphi(L_s) \subset L_{r+s}\}$.*

Note by $\text{der}_{\bar{0}}(L)$ the set of derivations of L of degree $\bar{0}$ and by $\text{der}_{\bar{1}}(L)$ the set of derivations of L of degree $\bar{1}$. Then,

$$\text{der}(L) = \text{der}_{\bar{0}}(L) \oplus \text{der}_{\bar{1}}(L) \quad (1.39)$$

Hence, to know the derivations of a Lie superalgebra L it suffices to know the derivations of degree $\bar{0}$ and the derivations of degree $\bar{1}$. If d is in $\text{der}(L)$, we write

$$d = d_{\bar{0}} + d_{\bar{1}} \quad (1.40)$$

with $d_{\bar{0}} \in \text{der}_{\bar{0}}(L)$ and $d_{\bar{1}} \in \text{der}_{\bar{1}}(L)$.

Proposition 1.6.1. *Let $d = d_{\bar{0}} + d_{\bar{1}} \in \text{der}(L)$. Then,*

1. *About $d_{\bar{0}}$:*

a) $d_{\bar{0}|L_{\bar{0}}} : L_{\bar{0}} \rightarrow L_{\bar{0}}$ *is a derivation of the Lie algebra $L_{\bar{0}}$.*

b) *If $z_{\bar{0}} \in L_{\bar{0}}$ and $z_{\bar{1}} \in L_{\bar{1}}$, then*

$$d_{\bar{0}}[z_{\bar{0}}, z_{\bar{1}}] = [d_{\bar{0}}(z_{\bar{0}}), z_{\bar{1}}] + [z_{\bar{0}}, d_{\bar{0}}(z_{\bar{1}})]; \quad (1.41)$$

that is the endomorphism $d_{\bar{0}|L_{\bar{1}}} : L_{\bar{1}} \rightarrow L_{\bar{1}}$ verifies

$$[d_{\bar{0}}, ad_{z_{\bar{0}}}]|_{L_{\bar{1}}} = ad_{(d_{\bar{0}}(z_{\bar{0}}))}|_{L_{\bar{1}}}. \quad (1.42)$$

c) *For every $z_{\bar{1}}, z'_{\bar{1}} \in L_{\bar{1}}$,*

$$d_{\bar{0}}[z_{\bar{1}}, z'_{\bar{1}}] = [d_{\bar{0}}(z_{\bar{1}}), z'_{\bar{1}}] + [z_{\bar{1}}, d_{\bar{0}}(z'_{\bar{1}})] \quad (1.43)$$

2. *About $d_{\bar{1}}$:*

a) *The morphism $d_{\bar{1}|L_{\bar{0}}} : L_{\bar{0}} \rightarrow L_{\bar{1}}$ verifies*

$$d_{\bar{1}}[z_{\bar{0}}, z'_{\bar{0}}] = [d_{\bar{1}}(z_{\bar{0}}), z'_{\bar{0}}] + [z_{\bar{0}}, d_{\bar{1}}(z'_{\bar{0}})], \quad (1.44)$$

for every $z_{\bar{0}}, z'_{\bar{0}} \in L_{\bar{0}}$; i.e. $d_{\bar{1}|L_{\bar{0}}} : L_{\bar{0}} \rightarrow L_{\bar{1}}$ is a 1-cocycle.

b) *The morphism $d_{\bar{1}|L_{\bar{1}}} : L_{\bar{1}} \rightarrow L_{\bar{0}}$ satisfies*

$$d_{\bar{1}}[z_{\bar{0}}, z_{\bar{1}}] = [d_{\bar{1}}(z_{\bar{0}}), z_{\bar{1}}] + [z_{\bar{0}}, d_{\bar{1}}(z_{\bar{1}})] \quad (1.45)$$

for every $z_{\bar{0}} \in L_{\bar{0}}$ and every $z_{\bar{1}} \in L_{\bar{1}}$

1.7 Cohomology of Lie Algebras

Let \mathcal{G} be a Lie algebra over a field \mathbb{K} of characteristic zero.

Definition 1.7.1. A \mathcal{G} -module is a linear space V of same dimension than \mathcal{G} with a bilinear map $\varphi : \mathcal{G} \times V \rightarrow V$ such that

$$\varphi([x, y], v) = \varphi(x, \varphi(y, v)) - \varphi(y, \varphi(x, v)), \quad (1.46)$$

for every elements x, y of \mathcal{G} and every vector v in V .

A \mathcal{G} -module corresponds to a representation of \mathcal{G} on the linear space V , that is a homomorphism $\Phi : \mathcal{G} \rightarrow gl(V)$ defined by:

$$\Phi(x)(v) = \varphi(x, v) := x \cdot v, \quad (1.47)$$

for every x in \mathcal{G} and every v in V .

Definition 1.7.2. Let V be a \mathcal{G} -module and p be a non-zero integer ($p \in \mathbb{N}^*$).

- A cochain of \mathcal{G} of degree p (or a p -cochain) with values in V is a skew-symmetric p times p -linear map from $\mathfrak{g}^p = \overbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}^p$ to V .
- A cochain of degree 0 (or a 0-cochain) of \mathcal{G} with values in V is a constant map from \mathcal{G} to V .

Note by $\mathcal{C}^p(\mathcal{G}, V)$ the space of p -cochains of \mathcal{G} with values in V . We have:

$$\mathcal{C}^p(\mathcal{G}, V) = \begin{cases} Hom(\Lambda^p \mathcal{G}, V), & \text{si } p \geq 1; \\ V & \text{si } p = 0; \\ \{0\} & \text{si } p < 0. \end{cases} \quad (1.48)$$

The space of cochains of \mathcal{G} with values in V is denoted by $\mathcal{C}^*(\mathcal{G}, V) := \bigoplus_p \mathcal{C}^p(\mathcal{G}, V)$. One endows $\mathcal{C}^p(\mathcal{G}, V)$ with a \mathcal{G} -module structure by setting :

$$(x \cdot \Phi)(x_1, \dots, x_p) = x \cdot \Phi(x_1, \dots, x_p) - \sum_{1 \leq i \leq p} \Phi(x_1, \dots, x_{i-1}, [x, x_i], x_{i+1}, \dots, x_p) \quad (1.49)$$

for all x, x_1, \dots, x_p in \mathcal{G} and all Φ in $\mathcal{C}^p(\mathcal{G}, V)$. This structure can be extended to the space $\mathcal{C}^*(\mathcal{G}, V)$.

Let us now define the endomorphism $d : \mathcal{C}^*(\mathcal{G}, V) \rightarrow \mathcal{C}^*(\mathcal{G}, V)$, called coboundary operator:

- If Φ is in $\mathcal{C}^0(\mathcal{G}, V) = V$ and x is in \mathcal{G} , then

$$(d\Phi)(x) = d\Phi(x) = x \cdot \Phi. \quad (1.50)$$

- For $p \geq 1$, x_1, \dots, x_{p+1} in \mathfrak{g} and Φ in $\mathcal{C}^p(\mathfrak{g}, V)$,

$$(d\Phi)(x_1, \dots, x_{p+1}) = \sum_{1 \leq s \leq p+1} (-1)^{s+1} x_s \cdot \Phi(x_1, \dots, \hat{x}_s, \dots, x_{p+1}) \quad (1.51)$$

$$+ \sum_{1 \leq s < t \leq p+1} (-1)^{s+t} \Phi([x_s, x_t], x_1, \dots, \hat{x}_s, \dots, \hat{x}_t, \dots, x_{p+1})$$

This endomorphism sends $\mathcal{C}^p(\mathcal{G}, V)$ on $\mathcal{C}^{p+1}(\mathcal{G}, V)$. One denotes by d_p the restriction of d to the space $\mathcal{C}^p(\mathcal{G}, V)$ of p -cochains. We have the

Proposition 1.7.1. $d^2 := d \circ d = 0$. More precisely $d_p \circ d_{p-1} = 0$, for every integer p .

The proof can be readed in [37].

Set $Z^p(\mathcal{G}, V) := \ker d_p$ and $B^p(\mathcal{G}, V) := \text{Im} d_{p-1}$.

Definition 1.7.3. An element of $Z^p(\mathcal{G}, V)$ is called a cocyle of degree p (or p -cocycle) of \mathcal{G} with values in V while an element of $B^p(\mathcal{G}, V)$ is said to be a coboundary of degree p (or p -coboundary) of \mathfrak{g} with values in V .

Since $d^2 = 0$, one has: $B^p(\mathcal{G}, V) \subset Z^p(\mathcal{G}, V)$. Then, we set, for $p \geq 1$,

$$H^p(\mathcal{G}, V) := Z^p(\mathcal{G}, V) / B^p(\mathcal{G}, V). \quad (1.52)$$

Definition 1.7.4. $H^p(\mathcal{G}, V)$ is the space of cohomology of \mathcal{G} of degree p (or p^{th} space of cohomology of \mathcal{G}) with values in V .

1.8 Affine Lie Groups

Definition 1.8.1. A n -dimensional affine manifold is smooth manifold M equipped with a smooth atlas $(U_i, \Phi_i)_i$ such that the transition functions $\Phi_i \circ \Phi_j^{-1}|_{\Phi_j(U_i \cap U_j)} : \Phi_j(U_i \cap U_j) \rightarrow \Phi_i(U_i \cap U_j)$, whenever they exist, are restrictions of affine transformations of \mathbb{R}^n .

The definition below is equivalent to the following: if (U_i, Φ_i) and (U_j, Φ_j) are two charts of the atlas satisfy $U_i \cap U_j$ then there exists an element θ_{ij} of the affine group $\text{Aff}(\mathbb{R}^n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ of \mathbb{R}^n such that

$$\Phi_i \circ \Phi_j^{-1}|_{\Phi_j(U_i \cap U_j)} = \theta_{ij}|_{\Phi_j(U_i \cap U_j)}. \quad (1.53)$$

Recall that $\text{Aff}(\mathbb{R}^n)$ is the group of diffeomorphisms of \mathbb{R}^n which preserve the standard connection ∇ of \mathbb{R}^n :

$$\nabla_X Y := \sum_{k=1}^n (X \cdot f_k) \frac{\partial}{\partial x_k}, \quad (1.54)$$

where $Y = \sum_{k=1}^n f_k \frac{\partial}{\partial x_k}$. Every open set U_i is then endowed with a connection ∇^i which is the reciprocal image by Φ_i of the connection induced on $\Phi_i(U_i)$ by ∇ . The connection ∇^i is torsion free and without curvature. Since the transition function preserve ∇ , the connections ∇^i can be perfectly glued into a zero-curvature and torsion free connection on the manifold M .

Definition 1.8.2. An application $f : M \rightarrow N$ between two affine manifold is called an affine transformation if

- it is smooth ;
- for all charts (U, Φ) of M and (V, Ψ) of N such that $f^{-1}(V) \cap U \neq \emptyset$, the function $\Psi \circ f \circ \Phi^{-1} : \Phi(f^{-1}(V) \cap U) \rightarrow \Psi(f^{-1}(V) \cap U)$ is the restriction of an element of $\text{Aff}(\mathbb{R}^n)$.

Definition 1.8.3. A Lie group G endowed with a left-invariant affine structure is called an affine Lie group. In other words, an affine Lie group is a Lie group equipped with an affine structure for which the left translations are affine transformations.

The existence of affine structures is a difficult and interesting problem ([39],[63], [54], [52]).

AUTOMORPHISMS OF COTANGENT BUNDLES OF LIE GROUPS

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2.1 Introduction

Let G be a Lie group whose Lie algebra \mathcal{G} is identified with its tangent space $T_\epsilon G$ at the unit ϵ . Throughout this work the cotangent bundle T^*G of G , is seen as a Lie group which is obtained by the semi-direct product $G \ltimes \mathcal{G}^*$ of G and the Abelian Lie group \mathcal{G}^* , where G acts on the dual space \mathcal{G}^* of \mathcal{G} via the coadjoint action. Here, using the trivialization by left translations, the manifold underlying T^*G has been identified with the trivial bundle $G \times \mathcal{G}^*$. We sometimes refer to the above Lie group structure on T^*G , as its natural Lie group structure. The Lie algebra $Lie(T^*G) = \mathcal{G} \ltimes \mathcal{G}^*$ of T^*G will be denoted by $T^*\mathcal{G}$ or simply by \mathcal{D} .

It is our aim in this work to fully study the connected component of the unit of the group of automorphisms of the Lie algebra $\mathcal{D} := T^*\mathcal{G}$. Such a connected component being spanned by exponentials of derivations of \mathcal{D} , we will work with those derivations and the first cohomology space $H^1(\mathcal{D}, \mathcal{D})$, where \mathcal{D} is seen as the \mathcal{D} -module for the adjoint representation.

Our motivation for this work comes from several interesting algebraic and geometric problems.

The cotangent bundle T^*G of a Lie group G can exhibit very interesting and rich algebraic and geometric structures ([5], [28], [32], [34], [47], [56]). Such structures can be better understood when one can compare, deform or classify them. This very often involves

the invertible homomorphisms (automorphisms) of T^*G , if in particular, such structures are invariant under left or right multiplications by the elements of T^*G . The derivatives at the unit of automorphisms of the Lie group T^*G are automorphisms of the Lie algebra \mathcal{D} . Conversely, if G is connected and simply connected, then so is T^*G and every automorphism of the Lie algebra \mathcal{D} integrates to an automorphism of the Lie group T^*G . A problem involving left or right invariant structures on a Lie group also usually transfers to one on its Lie algebra, with the Lie algebra automorphisms used as a means to compare or classify the corresponding induced structures.

In the purely algebraic point of view, finding and understanding the derivations of a given Lie algebra, is in itself an interesting problem ([23], [30], [40], [41], [49], [69]).

On the other hand, as a Lie group, the cotangent bundle T^*G is a common Drinfel'd double Lie group for all exact Poisson-Lie structures given by solutions of the Classical Yang-Baxter Equation in G . See e.g. [27]. Double Lie algebras (resp. groups) encode information on integrable Hamiltonian systems and Lax pairs ([4], [10], [32], [55]), Poisson homogeneous spaces of Poisson-Lie groups and the symplectic foliation of the corresponding Poisson structures ([27], [32], [55]). To that extend, the complete description of the group of automorphisms of the double Lie algebra of a Poisson-Lie structure would be a big contribution towards solving very interesting and hard problems. See Section 2.5 for wider discussions.

Interestingly, the space of derivations of \mathcal{D} encompasses interesting spaces of operators on \mathcal{G} , among which the derivations of \mathcal{G} , the second space of the left invariant de Rham cohomology $H_{inv}^2(G, \mathbb{R})$ of G , bi-invariant endomorphisms, in particular operators giving rise to complex group structure in G , when they exist.

Throughout this work, $\text{der}(\mathcal{G})$ will stand for the Lie algebra of derivations of \mathcal{G} , while \mathcal{J} will denote that of linear maps $j : \mathcal{G} \rightarrow \mathcal{G}$ satisfying $j([x, y]) = [j(x), y]$, for every elements x, y of \mathcal{G} . We consider Lie groups and Lie algebras over the field \mathbb{R} . However, most of the results within this chapter are valid for any field of characteristic zero.

We summarize some of our main results as follows.

Theorem A. *Let G be a Lie group, \mathcal{G} its Lie algebra, T^*G its cotangent bundle and $\mathcal{D} := \mathcal{G} \times \mathcal{G}^* = \text{Lie}(T^*G)$. A derivation of \mathcal{D} , has the form*

$$\phi(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + f \circ (j - \alpha) \right),$$

for all (x, f) in \mathcal{D} ; where

- $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is a derivation of the Lie algebra \mathcal{G} ;
- the linear map $j : \mathcal{G} \rightarrow \mathcal{G}$ is in \mathcal{J} ;
- $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ is a 1-cocycle of \mathcal{G} with values in \mathcal{G}^* for the coadjoint action of \mathcal{G} on \mathcal{G}^* ;
- $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ is a linear map satisfying the following conditions: for all x in \mathcal{G} and all f, g in \mathcal{G}^* ,

$$\psi \circ \text{ad}_x^* = \text{ad}_x \circ \psi \quad \text{and} \quad \text{ad}_{\psi(f)}^* g = \text{ad}_{\psi(g)}^* f.$$

If G has a bi-invariant Riemannian or pseudo-Riemannian metric, say μ , then every derivation ϕ of \mathcal{D} can be expressed in terms of elements of $\text{der}(\mathcal{G})$ and \mathcal{J} alone, as follows,

$$\phi(x, f) = \left(\alpha(x) + j \circ \theta^{-1}(f), \theta \circ \alpha'(x) + f \circ (j' - \alpha) \right),$$

for any element (x, f) of \mathcal{D} , where α, α' are derivations of \mathcal{G} ; the maps j, j' are in \mathcal{J} and $\theta : \mathcal{G} \rightarrow \mathcal{G}^*$ with $\theta(x)(y) := \mu(x, y)$, for every elements x, y of \mathcal{G} .

Theorem B. Let G be a Lie group and \mathcal{G} its Lie algebra. The group $\text{Aut}(\mathcal{D})$ of automorphisms of the Lie algebra \mathcal{D} of the cotangent bundle T^*G of G , is a super symmetric Lie group. More precisely, its Lie algebra $\text{der}(\mathcal{D})$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded symmetric (super-symmetric) Lie algebra which decomposes into a direct sum of vector spaces

$$\text{der}(\mathcal{D}) := \mathcal{G}_0 \oplus \mathcal{G}_1, \quad \text{with } [\mathcal{G}_i, \mathcal{G}_{i'}] \subset \mathcal{G}_{i+i'}, \quad i, i' \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$$

where \mathcal{G}_0 is the Lie algebra

$$\mathcal{G}_0 := \left\{ \phi : \mathcal{D} \rightarrow \mathcal{D}, \phi(x, f) = \left(\alpha(x), f \circ (\alpha - j) \right), \text{ with } \alpha \in \text{der}(\mathcal{G}) \text{ and } j \in \mathcal{J} \right\}$$

and \mathcal{G}_1 is the direct sum (as a vector space) of the space \mathcal{Q} of 1-cocycles $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ and the space Ψ of equivariant maps $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ with respect to the coadjoint and the adjoint representations, satisfying

$$\text{ad}_{\psi(f)}^* g = \text{ad}_{\psi(g)}^* f,$$

for every elements f, g of \mathcal{G}^* .

Moreover, $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}_1$ and $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}'_1$ are subalgebras of $\text{der}(\mathcal{D})$ which are Lie superalgebras, i.e. they are $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebras with the Lie bracket satisfying

$$[x, y] = -(-1)^{\text{deg}(x)\text{deg}(y)}[y, x],$$

where $\tilde{\mathcal{G}}_1 := \mathcal{Q}$ and $\tilde{\mathcal{G}}'_1 := \Psi$ are Abelian subalgebras of $\text{der}(\mathcal{D})$ and $\text{deg}(x) = i$, if $x \in \mathcal{G}_i$.

The Lie superalgebras $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}_1$ and $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}'_1$ respectively correspond to the subalgebras of all elements of $\text{der}(\mathcal{D})$ which preserve the subalgebra \mathcal{G} and the ideal \mathcal{G}^* of \mathcal{D} .

Theorem C. The first cohomology space $H^1(\mathcal{D}, \mathcal{D})$ of the (Chevalley-Eilenberg) cohomology associated with the adjoint action of \mathcal{D} on itself, satisfies

$$H^1(\mathcal{D}, \mathcal{D}) \cong H^1(\mathcal{G}, \mathcal{G}) \oplus \mathcal{J}^t \oplus H^1(\mathcal{G}, \mathcal{G}^*) \oplus \Psi,$$

where $H^1(\mathcal{G}, \mathcal{G})$ and $H^1(\mathcal{G}, \mathcal{G}^*)$ are the first cohomology spaces associated with the adjoint and coadjoint actions of \mathcal{G} , respectively; and $\mathcal{J}^t := \{j^t, j \in \mathcal{J}\}$ (space of transposes of elements of \mathcal{J}).

If \mathcal{G} is semi-simple, then $\Psi = \{0\}$ and thus $H^1(\mathcal{D}, \mathcal{D}) \cong \mathcal{J}^t$. Moreover, we have $\mathcal{J} \cong \mathbb{R}^p$, where p is the number of the simple ideals \mathfrak{s}_i of \mathcal{G} such that $\mathcal{G} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p$. Hence, of course, $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{R}^p$.

If \mathcal{G} is a compact Lie algebra, with centre $Z(\mathcal{G})$, we get

$$\begin{aligned} H^1(\mathcal{G}, \mathcal{G}) &\cong \text{End}(Z(\mathcal{G})), \\ \mathcal{J} &\cong \mathbb{R}^p \oplus \text{End}(Z(\mathcal{G})), \\ H^1(\mathcal{G}, \mathcal{G}^*) &\cong L(Z(\mathcal{G}), Z(\mathcal{G})^*), \\ \Psi &\cong L(Z(\mathcal{G})^*, Z(\mathcal{G})). \end{aligned}$$

Hence, we get

$$H^1(\mathcal{D}, \mathcal{D}) \cong (\text{End}(\mathbb{R}^k))^4 \oplus \mathbb{R}^p,$$

where k is the dimension of the center of \mathcal{G} , and p is the number of the simple components of the derived ideal $[\mathcal{G}, \mathcal{G}]$ of \mathcal{G} . Here, if E, F are vector spaces, $L(E, F)$ is the space of linear maps $E \rightarrow F$.

Traditionally, spectral sequences are used as a powerful tool for the study of the cohomology spaces of extensions of Lie groups or Lie algebras and more generally, of locally trivial fiber bundles (see e.g. [44], [65] for very interesting results and discussions). However, for the purpose of this investigation, we use a direct approach. Some parts of Theorem C can also be seen as a refinement, using our direct approach, of some already known results ([44]). Its proof is given by different lemmas and propositions, discussed in Sections 2.3.6 and 2.4.

The chapter is organized as follows. In Section 2.2, we explain some of the material and terminology needed to make the chapter more self contained. Sections 2.3 and 2.4 are the actual core of the work where the main calculations and proofs of theorems are carried out. In Section 2.5, we discuss some subjects related to this work, as well as some of the possible applications.

2.2 Preliminaries

Although not central to the main purpose of the work within this chapter, the following material might be useful, at least, as regards parts of the terminology used throughout this chapter.

2.2.1 The Cotangent Bundle of a Lie Group

Throughout this chapter, given a Lie group G , we will always let $G \times \mathcal{G}^*$ stand for the Lie group consisting of the Cartesian product $G \times \mathcal{G}^*$ as its underlying manifold, together with the group structure obtained by semi-direct product using the coadjoint action of G on \mathcal{G}^* . Recall that the trivialization by left translations, or simply the left trivialization of T^*G is given by the following isomorphism ζ of vector bundles

$$\zeta : T^*G \rightarrow G \times \mathcal{G}^*, \quad (\sigma, \nu_\sigma) \mapsto (\sigma, \nu_\sigma \circ T_\epsilon L_\sigma),$$

where L_σ is the left multiplication $L_\sigma : G \rightarrow G$, $\tau \mapsto L_\sigma(\tau) := \sigma\tau$ by σ in G and $T_\epsilon L_\sigma$ is the derivative of L_σ at the unit ϵ . In this chapter, T^*G will always be endowed with the

Lie group structure such that ζ is an isomorphism of Lie groups. The Lie algebra of T^*G is then the semi-direct product $\mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^*$. More precisely, the Lie bracket on \mathcal{D} reads

$$[(x, f), (y, g)] := ([x, y], ad_x^*g - ad_y^*f), \tag{2.1}$$

for any two elements (x, f) and (y, g) of \mathcal{D} .

In this work, we will refer to an object which is invariant under both left and right translations in a Lie group G , as a bi-invariant object. We discuss in this section, how T^*G is naturally endowed with a bi-invariant pseudo-Riemannian metric.

The cotangent bundle of any Lie group (with its natural Lie group structure, as above) and in general any element of the larger and interesting family of the so-called Drinfel'd doubles (see Section 2.2.2), are orthogonal Lie groups ([32]), as explained below.

As above, let $\mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^*$ be the Lie algebra of the cotangent bundle T^*G of G , seen as the semi-direct product of \mathcal{G} by \mathcal{G}^* via the coadjoint action of \mathcal{G} on \mathcal{G}^* , as in (2.1). Let μ_0 stand for the duality pairing $\langle \cdot, \cdot \rangle$, that is, for all $(x, f), (y, g)$ in \mathcal{D} ,

$$\mu_0\left((x, f), (y, g)\right) = f(y) + g(x). \tag{2.2}$$

Then, μ_0 satisfies the property (1.1) on \mathcal{D} and hence gives rise to a bi-invariant (pseudo-Riemannian) metric on T^*G .

2.2.2 Doubles of Poisson-Lie Groups and Yang-Baxter Equation

We explain in this section how cotangent bundles of Lie groups are part of the broader family of the so-called double Lie groups of Poisson-Lie groups.

A Poisson structure on a manifold M is given by a Lie bracket $\{, \}$ on the space $\mathcal{C}^\infty(M, \mathbb{R})$ of smooth real-valued functions on M , such that, for each f in $\mathcal{C}^\infty(M, \mathbb{R})$, the linear operator $X_f := \{f, \cdot\}$ on $\mathcal{C}^\infty(M, \mathbb{R})$, defined by $g \mapsto X_f \cdot g := \{f, g\}$, is a vector field on M . The bracket $\{, \}$ defines a 2-tensor, that is, a bivector field π which, seen as a bilinear skew-symmetric 'form' on the space of differential 1-forms on M , is given by $\pi(df, dg) := \{f, g\}$. The Jacobi identity for $\{, \}$ now reads $[\pi, \pi]_S = 0$, where $[\cdot, \cdot]_S$ is the so-called Schouten bracket, which is a natural extension to all multi-vector fields, of the natural Lie bracket of vector fields. Reciprocally, any bivector field π on M satisfying $[\pi, \pi]_S = 0$, is a Poisson tensor, *i.e.* defines a Poisson structure on M . See e.g. [55].

Recall that a Poisson-Lie structure on a Lie group G , is given by a Poisson tensor π on G , such that, when the Cartesian product $G \times G$ is equipped with the Poisson tensor $\pi \times \pi$, the multiplication $m : (\sigma, \tau) \mapsto \sigma\tau$ is a Poisson map between the Poisson manifolds $(G \times G, \pi \times \pi)$ and (G, π) . In other words, the derivative m_* of m satisfies $m_*(\pi \times \pi) = \pi$. If f, g are in \mathcal{G}^* and \bar{f}, \bar{g} are \mathcal{C}^∞ functions on G with respective derivatives $f = \bar{f}_{*,\epsilon}$, $g = \bar{g}_{*,\epsilon}$ at the unit ϵ of G , one defines another element $[f, g]_*$ of \mathcal{G}^* by setting $[f, g]_* := (\{\bar{f}, \bar{g}\})_{*,\epsilon}$. Then $[f, g]_*$ does not depend on the choice of \bar{f} and \bar{g} as above, and $(\mathcal{G}^*, [\cdot, \cdot]_*)$ is a Lie algebra. Now, there is a symmetric role played by the spaces \mathcal{G} and \mathcal{G}^* , dual to each other. Indeed, as well as acting on \mathcal{G}^* via the coadjoint action, \mathcal{G} is also acted on by \mathcal{G}^* using the

coadjoint action of $(\mathcal{G}^*, [,]_*)$. A lot of the most interesting properties and applications of π , are encoded in the new Lie algebra $(\mathcal{G} \oplus \mathcal{G}^*, [,]_\pi)$, where

$$[(x, f), (y, g)]_\pi := ([x, y] + ad_f^*y - ad_g^*x, ad_x^*g - ad_y^*f + [f, g]_*), \quad (2.3)$$

for every x, y in \mathcal{G} and every f, g in \mathcal{G}^* .

The Lie algebras $(\mathcal{G} \oplus \mathcal{G}^*, [,]_\pi)$ and $(\mathcal{G}^*, [,]_*)$ are respectively called the double and the dual Lie algebras of the Poisson-Lie group (G, π) . Endowed with the duality pairing defined in (2.2), the double Lie algebra of any Poisson-Lie group (G, π) , is an orthogonal Lie algebra, such that \mathcal{G} and \mathcal{G}^* are maximal totally isotropic (Lagrangian) subalgebras. The collection $(\mathcal{G} \oplus \mathcal{G}^*, \mathcal{G}, \mathcal{G}^*)$ is then called a Manin triple. More generally, $(\mathcal{G}_1 \oplus \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2)$ is called a Manin triple and $(\mathcal{G}_1, \mathcal{G}_2)$ a bi-algebra or a Manin pair, if $\mathcal{G}_1 \oplus \mathcal{G}_2$ is an orthogonal $2n$ -dimensional Lie algebra whose underlying adjoint-invariant pseudo-Riemannian metric is of index (n, n) and $\mathcal{G}_1, \mathcal{G}_2$ are two Lagrangian complementary subalgebras. See [27],[32], [55] for wider discussions.

Let r be an element of the wedge product $\wedge^2\mathcal{G}$. Denote by r^+ (resp. r^-) the left (resp. right) invariant bivector field on G with value $r = r_\epsilon^+$ (resp. $r = r_\epsilon^-$) at ϵ . If $\pi_r := r^+ - r^-$ is a Poisson tensor, then it is a Poisson-Lie tensor and r is called a solution of the Yang-Baxter Equation. If, in particular, r^+ is a (left invariant) Poisson tensor on G , then r is called a solution of the Classical Yang-Baxter Equation (CYBE) on G (or \mathcal{G}). In this latter case, the double Lie algebra $(\mathcal{G} \oplus \mathcal{G}^*, [,]_{\pi_r})$ is isomorphic to the Lie algebra \mathcal{D} of the cotangent bundle T^*G of G . See e.g. [28]. We may also consider the linear map $\tilde{r} : \mathcal{G}^* \rightarrow \mathcal{G}$, where $\tilde{r}(f) := r(f, \cdot)$. The linear map $\theta_r : (\mathcal{G} \oplus \mathcal{G}^*, [,]_{\pi_r}) \rightarrow \mathcal{D}$, $\theta_r(x, f) := (x + \tilde{r}(f), f)$, is an isomorphism of Lie algebras, between \mathcal{D} and the double Lie algebra of any Poisson-Lie group structure on G , given by a solution r of the CYBE.

2.3 Group of Automorphisms of $\mathcal{D} := T^*\mathcal{G}$

2.3.1 Derivations of $\mathcal{D} := T^*\mathcal{G}$

Consider a Lie group G of dimension n , with Lie algebra \mathcal{G} . Let us also denote by \mathcal{D} the vector space underlying the Lie algebra \mathcal{D} of the cotangent bundle T^*G , regarded as a \mathcal{D} -module under the adjoint action of \mathcal{D} . Consider the following complex with the coboundary operator ∂ , where $\partial \circ \partial = 0$:

$$0 \rightarrow \mathcal{D} \rightarrow Hom(\mathcal{D}, \mathcal{D}) \rightarrow Hom(\Lambda^2\mathcal{D}, \mathcal{D}) \rightarrow \dots \rightarrow Hom(\Lambda^{2n}\mathcal{D}, \mathcal{D}) \rightarrow 0. \quad (2.4)$$

We are interested in $Hom(\mathcal{D}, \mathcal{D}) := \{\phi : \mathcal{D} \rightarrow \mathcal{D}, \phi \text{ linear}\}$. The coboundary $\partial\phi$ of the element ϕ of $Hom(\mathcal{D}, \mathcal{D})$ is the element of $Hom(\Lambda^2\mathcal{D}, \mathcal{D})$ defined by

$$\partial\phi(u, v) := ad_u(\phi(v)) - ad_v(\phi(u)) - \phi([u, v]), \quad (2.5)$$

for any elements $u = (x, f)$ and $v = (y, g)$ in \mathcal{D} . An element ϕ of $Hom(\mathcal{D}, \mathcal{D})$ is a 1-cocycle if $\partial\phi = 0$, *i.e.*

$$\begin{aligned} \phi([u, v]) &= ad_u(\phi(v)) - ad_v(\phi(u)), \\ &= [u, \phi(v)] + [\phi(u), v]. \end{aligned} \quad (2.6)$$

In other words, 1-cocycles are the derivations of the Lie algebra \mathcal{D} . In Section 2.3.6, we will characterize the first cohomology space $H^1(\mathcal{D}, \mathcal{D}) := \ker(\partial^2)/\text{Im}(\partial^1)$ of the associated Chevalley-Eilenberg cohomology, where for clarity, we have denoted by ∂^1 and ∂^2 the following restrictions $\partial^1 : \mathcal{D} \rightarrow \text{Hom}(\mathcal{D}, \mathcal{D})$ and $\partial^2 : \text{Hom}(\mathcal{D}, \mathcal{D}) \rightarrow \text{Hom}(\wedge^2\mathcal{D}, \mathcal{D})$ of the coboundary operator ∂ .

Theorem 2.3.1. *Let G be a Lie group, \mathcal{G} its Lie algebra, T^*G its cotangent bundle and $\mathcal{D} := \mathcal{G} \times \mathcal{G}^*$ the Lie algebra of T^*G . A 1-cocycle (for the adjoint representation) hence a derivation of \mathcal{D} has the following form:*

$$\phi(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + \xi(f) \right), \quad (2.7)$$

for any (x, f) in \mathcal{D} ; where

- $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is a derivation of the Lie algebra \mathcal{G} ,
- $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ is a 1-cocycle of \mathcal{G} with values in \mathcal{G}^* for the coadjoint action of \mathcal{G} on \mathcal{G}^* ,
- $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ and $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ are linear maps satisfying the following conditions:

$$[\xi, \text{ad}_x^*] = \text{ad}_{\alpha(x)}^*, \quad \forall x \in \mathcal{G}, \quad (2.8)$$

$$\psi \circ \text{ad}_x^* = \text{ad}_x \circ \psi, \quad \forall x \in \mathcal{G}, \quad (2.9)$$

$$\text{ad}_{\psi(f)}^* g = \text{ad}_{\psi(g)}^* f, \quad \forall f, g \in \mathcal{G}^*. \quad (2.10)$$

The rest of this section is dedicated to the proof of Theorem 2.3.1.

Aiming to get a simpler expression for the derivations, let us write ϕ in terms of its components relative to the decomposition of \mathcal{D} into a direct sum $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$ of vector spaces as follows: for all (x, f) in \mathcal{D} ,

$$\phi(x, f) = \left(\phi_{11}(x) + \phi_{21}(f), \phi_{12}(x) + \phi_{22}(f) \right), \quad (2.11)$$

where $\phi_{11} : \mathcal{G} \rightarrow \mathcal{G}$, $\phi_{12} : \mathcal{G} \rightarrow \mathcal{G}^*$, $\phi_{21} : \mathcal{G}^* \rightarrow \mathcal{G}$ and $\phi_{22} : \mathcal{G}^* \rightarrow \mathcal{G}^*$ are all linear maps. In (2.11) we have made the identifications: $x = (x, 0)$, $f = (0, f)$ so that the element (x, f) can also be written $x + f$. Likewise, we can write

$$\phi(x) = (\phi_{11}(x), \phi_{12}(x)) \quad ; \quad \phi(f) = (\phi_{21}(f), \phi_{22}(f)), \quad (2.12)$$

for any x in \mathcal{G} and any f in \mathcal{G}^* ; or simply

$$\phi(x) = \phi_{11}(x) + \phi_{12}(x) \quad ; \quad \phi(f) = \phi_{21}(f) + \phi_{22}(f). \quad (2.13)$$

In order to find the ϕ_{ij} 's and hence all the derivations of \mathcal{D} , we are now going to use the cocycle condition (2.6).

For x, y in $\mathcal{G} \subset \mathcal{D}$ we have:

$$\phi([x, y]) = \phi_{11}([x, y]) + \phi_{12}([x, y]) \quad (2.14)$$

and

$$\begin{aligned} [\phi(x), y] + [x, \phi(y)] &= [\phi_{11}(x) + \phi_{12}(x), y] + [x, \phi_{11}(y) + \phi_{12}(y)] \\ &= [\phi_{11}(x), y] - ad_y^*(\phi_{12}(x)) + [x, \phi_{11}(y)] + ad_x^*(\phi_{12}(y)). \end{aligned} \quad (2.15)$$

Comparing (2.14) and (2.15), we first get

$$\phi_{11}([x, y]) = [\phi_{11}(x), y] + [x, \phi_{11}(y)], \quad (2.16)$$

for every x, y in \mathcal{G} . This means that ϕ_{11} is a derivation of the Lie algebra \mathcal{G} .

Secondly, for all elements x, y of \mathcal{G} , we have

$$\phi_{12}([x, y]) = ad_x^*(\phi_{12}(y)) - ad_y^*(\phi_{12}(x)). \quad (2.17)$$

Equation (2.17) means that $\phi_{12} : \mathcal{G} \rightarrow \mathcal{G}^*$ is a 1-cocycle of \mathcal{G} with values on \mathcal{G}^* for the coadjoint action of \mathcal{G} on \mathcal{G}^* .

Now we are going to examine the following case: for all x in \mathcal{G} and all f in \mathcal{G}^* ,

$$\begin{aligned} \phi([x, f]) &= \phi(ad_x^*f), \\ &= \phi_{21}(ad_x^*f) + \phi_{22}(ad_x^*f), \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} [\phi(x), f] + [x, \phi(f)] &= [\phi_{11}(x) + \phi_{12}(x), f] + [x, \phi_{21}(f) + \phi_{22}(f)], \\ &= ad_{\phi_{11}(x)}^*f + [x, \phi_{21}(f)] + ad_x^*(\phi_{22}(f)). \end{aligned} \quad (2.19)$$

Identifying (2.18) and (2.19) we obtain on the one hand

$$\begin{aligned} \phi_{21}(ad_x^*f) &= [x, \phi_{21}(f)], \\ &= ad_x(\phi_{21}(f)), \end{aligned}$$

for every x in \mathcal{G} , and every f in \mathcal{G}^* . We write the above as

$$\phi_{21} \circ ad_x^* = ad_x \circ \phi_{21},$$

for all x in \mathcal{G} . That is, $\phi_{21} : \mathcal{G}^* \rightarrow \mathcal{G}$ is equivariant (commutes) with respect to the adjoint and the coadjoint actions of \mathcal{G} on \mathcal{G} and \mathcal{G}^* respectively.

We have on the other hand

$$\phi_{22}(ad_x^*f) = ad_x^*(\phi_{22}(f)) + ad_{\phi_{11}(x)}^*f, \quad (2.20)$$

for all x in \mathcal{G} and all f in \mathcal{G}^* . Formula (2.20) can be rewritten as

$$\phi_{22} \circ ad_x^* - ad_x^* \circ \phi_{22} = ad_{\phi_{11}(x)}^*,$$

i.e. for any element x of \mathcal{G} ,

$$[\phi_{22}, ad_x^*] = ad_{\phi_{11}(x)}^*.$$

Last, for f and g in \mathcal{G}^* , we have

$$\phi([f, g]) = 0, \quad (2.21)$$

and

$$\begin{aligned} [\phi(f), g] + [f, \phi(g)] &= [\phi_{21}(f) + \phi_{22}(f), g] + [f, \phi_{21}(g) + \phi_{22}(g)], \\ &= ad_{\phi_{21}(f)}^* g - ad_{\phi_{21}(g)}^* f. \end{aligned} \quad (2.22)$$

From (2.21) and (2.22) it comes that for all elements f, g of \mathcal{G}^* ,

$$ad_{\phi_{21}(f)}^* g = ad_{\phi_{21}(g)}^* f.$$

Noting $\alpha := \phi_{11}$, $\beta := \phi_{12}$, $\psi := \phi_{21}$ and $\xi := \phi_{22}$, we get a proof of Theorem 2.3.1. \square

Remark 2.3.1. (*Notations*) From now on, if \mathcal{G} is a Lie algebra, then

1. \mathcal{E} will stand for the space of linear maps $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ satisfying Equation (2.8), for some derivation α of \mathcal{G} ;

2. set

$$\mathcal{G}_0 := \left\{ \phi : \mathcal{D} \rightarrow \mathcal{D}, \phi(x, f) = (\alpha(x), \xi(f)) : \alpha \in \text{der}(\mathcal{G}), \xi \in \mathcal{E}, [\xi, ad_x^*] = ad_{\alpha(x)}^*, \forall x \in \mathcal{G} \right\};$$

3. we may let \mathcal{Q} stand for the space of 1-cocycles $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ as in (2.17), whereas Ψ may be used for the space of equivariant linear $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ as in (2.9), which satisfy (2.10);

4. we will denote by \mathcal{G}_1 , the direct sum $\mathcal{G}_1 := \mathcal{Q} \oplus \Psi$ of the vector spaces \mathcal{Q} and Ψ .

Remark 2.3.2. The spaces $\text{der}(\mathcal{G})$ of derivations of \mathcal{G} , \mathcal{Q} and Ψ , as in Remark 2.3.1, are all subsets of $\text{der}(\mathcal{D})$, as follows. A derivation α of \mathcal{G} , an equivariant map ψ in Ψ , and a 1-cocycle β in \mathcal{Q} are respectively seen as the elements $\phi_\alpha, \phi_\psi, \phi_\beta$ of $\text{der}(\mathcal{D})$, with

$$\begin{aligned} \phi_\alpha(x, f) &:= (\alpha(x), -f \circ \alpha); \\ \phi_\psi(x, f) &:= (\psi(f), 0); \\ \phi_\beta(x, f) &:= (0, \beta(x)), \end{aligned}$$

for all (x, f) in \mathcal{D} .

Corollary 2.3.1. Every derivation of \mathcal{G} is the restriction to \mathcal{G} of a derivation of \mathcal{D} .

2.3.2 A Structure Theorem for the Group of Automorphisms of \mathcal{D}

Lemma 2.3.1. The space \mathcal{E} , as in Remark 2.3.1, is a Lie algebra.

Namely, if ξ_1, ξ_2 in \mathcal{E} satisfy

$$[\xi_1, ad_x^*] = ad_{\alpha_1(x)}^* \text{ and } [\xi_2, ad_x^*] = ad_{\alpha_2(x)}^*,$$

for all x in \mathcal{G} and some α_1, α_2 in $\text{der}(\mathcal{G})$, then their Lie bracket $[\xi_1, \xi_2]$ is in \mathcal{E} and satisfies

$$[[\xi_1, \xi_2], ad_x^*] = ad_{[\alpha_1, \alpha_2](x)}^*.$$

Proof. Using Jacobi identity in the Lie algebra $\mathfrak{gl}(\mathcal{G}^*)$ of endomorphisms of the vector space \mathcal{G}^* , we get: for any x in \mathcal{G}

$$\begin{aligned} [[\xi_1, \xi_2], ad_x^*] &= [[\xi_1, ad_x^*], \xi_2] + [\xi_1, [\xi_2, ad_x^*]] \\ &= [ad_{\alpha_1(x)}^*, \xi_2] + [\xi_1, ad_{\alpha_2(x)}^*] \\ &= -ad_{\alpha_2 \circ \alpha_1(x)}^* + ad_{\alpha_1 \circ \alpha_2(x)}^* \\ &= ad_{[\alpha_1, \alpha_2](x)}^*. \end{aligned}$$

□

Lemma 2.3.2. *The space \mathcal{G}_0 , as in Remark 2.3.1, is a Lie subalgebra of $\text{der}(\mathcal{D})$.*

Proof. This is a consequence of Lemma 2.3.1. If $\phi_1 := (\alpha_1, \xi_1)$, and $\phi_2 := (\alpha_2, \xi_2)$ are in \mathcal{G}_0 , then $[\phi_1, \phi_2] = ([\alpha_1, \alpha_2], [\xi_1, \xi_2])$, as can easily be seen, below. For every (x, f) in \mathcal{D} , we have

$$\begin{aligned} [\phi_1, \phi_2](x, f) &= \phi_1(\alpha_2(x), \xi_2(f)) - \phi_2(\alpha_1(x), \xi_1(f)) \\ &= (\alpha_1 \circ \alpha_2(x), \xi_1 \circ \xi_2(f)) - (\alpha_2 \circ \alpha_1(x), \xi_2 \circ \xi_1(f)) \\ &= ([\alpha_1, \alpha_2](x), [\xi_1, \xi_2](f)). \end{aligned}$$

□

Lemma 2.3.3. *Let β and ψ be in \mathcal{Q} and Ψ , respectively. Then $[\beta, \psi] = (-\psi \circ \beta, \beta \circ \psi)$ belongs to \mathcal{G}_0 , more precisely $\beta \circ \psi$ is in \mathcal{E} , $\psi \circ \beta$ is in $\text{der}(\mathcal{G})$ and $[\beta \circ \psi, ad_x^*] = -ad_{\psi \circ \beta(x)}^*$, for any x in \mathcal{G} .*

Proof. First, β being a 1-cocycle is equivalent to

$$\beta \circ ad_x(y) = ad_x^* \circ \beta(y) - ad_y^* \circ \beta(x), \quad (2.23)$$

for all x, y in \mathcal{G} . Now for every x in \mathcal{G} and every f in \mathcal{G}^* , we have

$$\begin{aligned} [\beta \circ \psi, ad_x^*](f) &= \beta \circ \psi \circ ad_x^*(f) - ad_x^* \circ \beta \circ \psi(f) \\ &= \beta \circ ad_x \circ \psi(f) - ad_x^* \circ \beta \circ \psi(f), \text{ now take } y = \psi(f) \text{ in (2.23)} \\ &= ad_x^* \circ \beta \circ \psi(f) - ad_{\psi(f)}^* \beta(x) - ad_x^* \circ \beta \circ \psi(f), \\ &= -ad_{\psi(f)}^* \beta(x), \text{ take } g = \beta(x) \text{ in (2.10)} \\ &= -ad_{\psi \circ \beta(x)}^* f = ad_{\alpha(x)}^* f, \text{ where } \alpha = -\psi \circ \beta. \end{aligned}$$

Next, the proof that $\psi \circ \beta$ is in $\text{der}(\mathcal{G})$, is straightforward. Indeed, for every elements x, y in \mathcal{G} , we have

$$\begin{aligned} \psi \circ \beta[x, y] &= \psi(ad_x^* \beta(y) - ad_y^* \beta(x)) \\ &= ad_x \circ \psi \circ \beta(y) - ad_y \circ \psi \circ \beta(x) \\ &= [x, \psi \circ \beta(y)] + [\psi \circ \beta(x), y] \end{aligned}$$

Hence $[\beta, \psi]$ belongs to \mathcal{G}_0 , for every β in \mathcal{Q} and every ψ in Ψ . □

Lemma 2.3.4. *Let $\phi := (\alpha, \xi)$ be in \mathcal{G}_0 , $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ and $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ be respectively in \mathcal{Q} and Ψ . Then both $[\phi, \beta]$ and $[\phi, \psi]$ are elements of \mathcal{G}_1 , more precisely $[\phi, \beta]$ is in \mathcal{Q} and $[\phi, \psi]$ is in Ψ . Moreover, we have $[\mathcal{Q}, \mathcal{Q}] = 0$ and $[\Psi, \Psi] = 0$.*

Proof. Let $\phi = (\alpha, \xi)$ be in \mathcal{G}_0 , $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ a 1-cocycle and $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ an equivariant linear map. Using ϕ_β and ϕ_ψ as in Remark 2.3.2, we obtain

$$\begin{aligned} [\phi, \phi_\beta](x, y) &= \phi(0, \beta(x)) - \phi_\beta(\alpha(x), \xi(f)) \\ &= (0, \xi \circ \beta(x)) - (0, \beta \circ \alpha(x)) \\ &= (0, (\xi \circ \beta - \beta \circ \alpha)(x)). \end{aligned}$$

Now, let us show that $\tilde{\beta} := \xi \circ \beta - \beta \circ \alpha : \mathcal{G} \rightarrow \mathcal{G}^*$ is a 1-cocycle. Indeed, on the one hand we have

$$\begin{aligned} \xi \circ \beta([x, y]) &= \xi(ad_x^* \beta(y) - ad_y^* \beta(x)) \\ &= ([\xi, ad_x^*] + ad_x^* \circ \xi)(\beta(y)) - ([\xi, ad_y^*] + ad_y^* \circ \xi)(\beta(x)) \\ &= ad_{\alpha(x)}^* \beta(y) + ad_x^*(\xi \circ \beta(y)) - ad_{\alpha(y)}^* \beta(x) - ad_y^*(\xi \circ \beta(x)) \\ &= ad_x^*(\xi \circ \beta(y)) - ad_y^*(\xi \circ \beta(x)) + ad_{\alpha(x)}^* \beta(y) - ad_{\alpha(y)}^* \beta(x) \end{aligned} \quad (2.24)$$

On the other hand, we also have

$$\begin{aligned} \beta \circ \alpha([x, y]) &= \beta([\alpha(x), y]) + \beta([x, \alpha(y)]) \\ &= ad_{\alpha(x)}^* \beta(y) - ad_y^*(\beta \circ \alpha(x)) + ad_x^*(\beta \circ \alpha(y)) - ad_{\alpha(y)}^* \beta(x) \end{aligned} \quad (2.25)$$

Subtracting (2.25) from (2.24), we see that $\tilde{\beta}[x, y]$ now reads

$$\begin{aligned} \tilde{\beta}[x, y] &= ad_x^*(\xi \circ \beta - \beta \circ \alpha)(y) - ad_y^*(\xi \circ \beta - \beta \circ \alpha)(x) \\ &= ad_x^* \tilde{\beta}(y) - ad_y^* \tilde{\beta}(x) \end{aligned}$$

Hence $\tilde{\beta}$ is an element of \mathcal{Q} .

In the same way, we also have

$$\begin{aligned} [\phi, \phi_\psi](x, y) &= \phi(\psi(f), 0) - \phi_\psi(\alpha(x), \xi(f)) \\ &= (\alpha \circ \psi(f), 0) - (\psi \circ \xi(f), 0) \\ &= ((\alpha \circ \psi - \psi \circ \xi)(f), 0) \end{aligned}$$

The linear map $\tilde{\psi} := \alpha \circ \psi - \psi \circ \xi : \mathcal{G}^* \rightarrow \mathcal{G}$ is equivariant, *i.e.* is an element of Ψ . As above, this is seen by first computing, for every elements x of \mathcal{G} and f of \mathcal{G}^* ,

$$\alpha \circ \psi(ad_x^* f) = \alpha([x, \psi(f)]) = [\alpha(x), \psi(f)] + [x, \alpha \circ \psi(f)] \quad (2.26)$$

and

$$\begin{aligned} \psi \circ \xi(ad_x^* f) &= \psi([\xi, ad_x^*] + ad_x^* \circ \xi)(f) \\ &= \psi(ad_{\alpha(x)}^* f) + \psi(ad_x^* \xi(f)) \\ &= [\alpha(x), \psi(f)] + [x, \psi \circ \xi(f)], \end{aligned} \quad (2.27)$$

then subtracting (2.26) and (2.27).

Now we have

$$[\phi_\beta, \phi_{\beta'}](x, f) = \phi_\beta(0, \beta'(x)) - \phi_{\beta'}(0, \beta(x)) = 0$$

and

$$[\phi_\psi, \phi_{\psi'}](x, f) = \phi_\psi(\psi'(f), 0) - \phi_{\psi'}(\psi(f), 0) = 0,$$

for all (x, f) in \mathcal{D} . In other words, $[\mathcal{Q}, \mathcal{Q}] = 0$ and $[\Psi, \Psi] = 0$. \square

We summarize all the above in the

Theorem 2.3.2. *Let G be a Lie group and \mathcal{G} its Lie algebra. The group $\text{Aut}(\mathcal{D})$ of automorphisms of the Lie algebra \mathcal{D} of the cotangent bundle T^*G of G , is a super symmetric Lie group. More precisely, its Lie algebra $\text{der}(\mathcal{D})$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded symmetric (supersymmetric) Lie algebra which decomposes into a direct sum of vector spaces*

$$\text{der}(\mathcal{D}) := \mathcal{G}_0 \oplus \mathcal{G}_1, \quad \text{with } [\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}, \quad i, j \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \quad (2.28)$$

where \mathcal{G}_0 is the Lie algebra of linear maps $\phi : \mathcal{D} \rightarrow \mathcal{D}$, $\phi(x, f) = (\alpha(x), \xi(f))$ with α a derivation of \mathcal{G} and the linear map $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ satisfies,

$$[\xi, \text{ad}_x^*] = \text{ad}_{\alpha(x)}^*, \quad (2.29)$$

for any x of \mathcal{G} ; and \mathcal{G}_1 is the direct sum (as a vector space) of the space \mathcal{Q} of 1-cocycles $\mathcal{G} \rightarrow \mathcal{G}^*$ and the space Ψ of linear maps $\mathcal{G}^* \rightarrow \mathcal{G}$ which are equivariant with respect to the coadjoint and the adjoint representations and satisfy (2.10).

Moreover, $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}_1$ and $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}'_1$ are subalgebras of $\text{der}(\mathcal{D})$ which are Lie superalgebras, i.e. they are $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebras with the Lie bracket satisfying

$$[x, y] = -(-1)^{\text{deg}(x)\text{deg}(y)}[y, x],$$

where $\tilde{\mathcal{G}}_1 := \mathcal{Q}$ and $\tilde{\mathcal{G}}'_1 := \Psi$ are Abelian subalgebras of $\text{der}(\mathcal{D})$ and $\text{deg}(x) = i$, if $x \in \mathcal{G}_i$.

Remark 2.3.3. (a) In Propositions 2.3.1 and 2.3.2, we will prove that every element ξ of \mathcal{E} is the transpose $\xi = (j - \alpha)^t$ of the sum of an adjoint-invariant endomorphism $j \in \mathcal{J}$ and a derivation $-\alpha$ of \mathcal{G} .

(b) The Lie superalgebras $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}_1$ and $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}'_1$ respectively correspond to the subalgebras of all elements of $\text{der}(\mathcal{D})$ which preserve the subalgebra \mathcal{G} and the ideal \mathcal{G}^* of \mathcal{D} . The Lie superalgebra $\mathcal{G}_0 \oplus \tilde{\mathcal{G}}'_1$ can be seen as part of the more general case of derivations of a semi-direct product Lie algebra $\mathcal{G} \ltimes \mathcal{N}$ which preserve the ideal \mathcal{N} and which are discussed in [65], among other interesting results therein.

Let us now have a closer look at maps ξ , ψ and β .

2.3.3 Maps ξ and Bi-invariant Tensors of Type (1,1)

Adjoint-invariant Endomorphisms

Linear operators acting on vector fields of a given Lie group G can be seen as fields of endomorphisms of its tangent spaces. Bi-invariant ones correspond to endomorphisms $j : \mathcal{G} \rightarrow \mathcal{G}$ of the Lie algebra \mathcal{G} of G , satisfying the condition $j[x, y] = [jx, y]$, for all x, y in \mathcal{G} . If we denote by ∇ the connection on G given on left invariant vector fields by

$$\nabla_x y := \frac{1}{2}[x, y],$$

then using the covariant derivative, we have $\nabla j = 0$, (see e.g. [70]). As above, let

$$\mathcal{J} := \{j : \mathcal{G} \rightarrow \mathcal{G}, \text{ linear and } j[x, y] = [jx, y], \forall x, y \in \mathcal{G}\}.$$

Endowed with the bracket

$$[j, j'] := j \circ j' - j' \circ j,$$

the space \mathcal{J} is a Lie algebra, and indeed a subalgebra of the Lie algebra $\mathcal{G}l(\mathcal{G})$ of all endomorphisms of \mathcal{G} .

In the case where the dimension of G is even and if in addition j satisfies $j^2 = -\text{identity}$, then (G, j) is a complex Lie group.

Maps $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$

Proposition 2.3.1. *Let \mathcal{G} be a Lie algebra and α a derivation of \mathcal{G} . A linear map $\xi' : \mathcal{G} \rightarrow \mathcal{G}$ satisfies $[\xi', ad_x] = ad_{\alpha(x)}$, for every element x of \mathcal{G} , if and only if there exists a linear map $j : \mathcal{G} \rightarrow \mathcal{G}$ satisfying*

$$j([x, y]) = [j(x), y] = [x, j(y)], \quad (2.30)$$

for all x, y in \mathcal{G} , such that $\xi' = j + \alpha$.

Proof. Let α be a derivation and ξ' an endomorphism of \mathcal{G} satisfying the hypothesis of Proposition 2.3.1, that is, $[\xi', ad_x] = ad_{\alpha(x)} = [\alpha, ad_x]$, for any x in \mathcal{G} . We then have,

$$[\xi' - \alpha, ad_x] = 0, \quad (2.31)$$

for any x of \mathcal{G} . So the endomorphism $j := \xi' - \alpha$ commutes with all adjoint operators.

Now a linear map $j : \mathcal{G} \rightarrow \mathcal{G}$ commuting with all adjoint operators, satisfies

$$0 = [j, ad_x](y) = j([x, y]) - [x, j(y)], \quad (2.32)$$

for all elements x, y of \mathcal{G} . We also have,

$$0 = [j, ad_y](x) = j([y, x]) - [y, j(x)], \quad (2.33)$$

for all x, y in \mathcal{G} . From (2.32) and (2.33), we have $j([x, y]) = [j(x), y] = [x, j(y)]$, for any x, y in \mathcal{G} .

Thus, (2.31) is equivalent to $\xi' = j + \alpha$, where j satisfies (2.30). \square

Remark 2.3.4. The above means that the space \mathcal{J} is the centralizer of the space $ad_{\mathcal{G}}$ of inner derivations of \mathcal{G} in $\mathcal{G}l(\mathcal{G}) := \{l : \mathcal{G} \rightarrow \mathcal{G} \text{ linear}\}$, i.e.

$$\mathcal{J} = Z_{\mathcal{G}l(\mathcal{G})}(ad_{\mathcal{G}}) := \{j : \mathcal{G} \rightarrow \mathcal{G} \text{ linear and } [j, ad_x] = 0, \forall x \in \mathcal{G}\}.$$

Proposition 2.3.2. Let \mathcal{G} be a nonabelian Lie algebra and \mathcal{S} the space of endomorphisms $\xi' : \mathcal{G} \rightarrow \mathcal{G}$ such that there exists a derivation α of \mathcal{G} and $[\xi', ad_x] = ad_{\alpha(x)}$ for all $x \in \mathcal{G}$. Then \mathcal{S} is a Lie algebra containing \mathcal{J} and $der(\mathcal{G})$ as subalgebras. In the case where \mathcal{G} has a trivial centre, then \mathcal{S} is the semi-direct product $\mathcal{S} = der(\mathcal{G}) \ltimes \mathcal{J}$ of \mathcal{J} and $der(\mathcal{G})$.

The following are equivalent

- (a) The linear map $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ is an element of \mathcal{E} with α as the corresponding derivation of \mathcal{G} , i.e. ξ satisfies (2.8) for the derivation α .
- (b) The transpose ξ^t of ξ is of the form $\xi^t = j - \alpha$, where j is in \mathcal{J} and α in $der(\mathcal{G})$.
- (c) ξ^t is an element of \mathcal{S} , with corresponding derivation $-\alpha$.

The transposition $\xi \mapsto \xi^t$ of linear maps is an anti-isomorphism between the Lie algebras \mathcal{E} and \mathcal{S} .

Proof. Using the same argument as in Lemma 2.3.1, if $[\xi'_1, ad_x] = ad_{\alpha_1(x)}$ and $[\xi'_2, ad_x] = ad_{\alpha_2(x)}$, for every x in \mathcal{G} , then $[[\xi'_1, \xi'_2], ad_x] = ad_{[\alpha_1, \alpha_2](x)}$, for any element x of \mathcal{G} . Thus \mathcal{S} is a Lie algebra. From Proposition 2.3.1, there exist j_i in \mathcal{J} such that $\xi'_i = \alpha_i + j_i$, $i = 1, 2$. Obviously, \mathcal{S} contains \mathcal{J} and $der(\mathcal{G})$. Thus, as a vector space, \mathcal{S} decomposes as $\mathcal{S} = der(\mathcal{G}) + \mathcal{J}$. Now, the Lie bracket in \mathcal{S} reads

$$[\xi'_1, \xi'_2] = [\alpha_1 + j_1, \alpha_2 + j_2] = [\alpha_1, \alpha_2] + [\alpha_1, j_2] + [j_1, \alpha_2] + [j_1, j_2] \quad (2.34)$$

Of course, $[\alpha_1, \alpha_2]$ is in $der(\mathcal{G})$. From Section 2.3.3, we know that \mathcal{J} is a Lie algebra, hence $[j_1, j_2]$ is in \mathcal{J} . It is easy to check that

$$[\alpha, j] \in \mathcal{J}, \quad (2.35)$$

for all α in $der(\mathcal{G})$ and for all j in \mathcal{J} . Indeed, the following holds

$$\begin{aligned} [\alpha, j]([x, y]) &= \alpha([j(x), y]) - j([\alpha(x), y] + [x, \alpha(y)]) \\ &= [\alpha \circ j(x), y] + [j(x), \alpha(y)] - [j \circ \alpha(x), y] - [j(x), \alpha(y)] \\ &= [[\alpha, j](x), y], \end{aligned} \quad (2.36)$$

for all x, y in \mathcal{G} . The intersection $der(\mathcal{G}) \cap \mathcal{J}$ is made of elements j of \mathcal{J} whose image $\text{Im}(j)$ is a subset of the centre $Z(\mathcal{G})$ of \mathcal{G} . Hence if $Z(\mathcal{G}) = 0$, then $\mathcal{S} = der(\mathcal{G}) \oplus \mathcal{J}$ and as a Lie algebra, $\mathcal{S} = der(\mathcal{G}) \ltimes \mathcal{J}$. Using this decomposition, we can also rewrite (2.34) as

$$[\xi'_1, \xi'_2] = [(\alpha_1, j_1), (\alpha_2, j_2)] = ([\alpha_1, \alpha_2], [j_1, j_2] + [\alpha_1, j_2] + [j_1, \alpha_2]) \quad (2.37)$$

The equivalence between (b) and (c) comes directly from Proposition 2.3.1.

Now let $\xi \in \mathcal{E}$, with $[\xi, ad_x^*] = ad_{\alpha(x)}^*$, $\alpha \in der(\mathcal{G})$, then

$$-ad_{\alpha(x)} = [\xi, ad_x^*]^t = -[\xi^t, (ad_x^*)^t] = [\xi^t, ad_x]$$

Hence $\xi^t \in \mathcal{S}$, with $ad_{\alpha'(x)} = [\xi^t, ad_x]$, for all $x \in \mathcal{G}$, where $\alpha' := -\alpha$. Thus, (a) implies (c). From Proposition 2.3.1, there exist $j \in \mathcal{J}$ such that $\xi^t = -\alpha + j$. Now it is straightforward that if (b) $\xi^t = -\alpha + j$ with α a derivation and j in \mathcal{J} , then ξ satisfies $[\xi, ad_x^*] = ad_{\alpha(x)}^*$, for all x in \mathcal{G} . Hence (c) implies (a).

Of course, we also know that $[\xi_1, \xi_2]^t = -[\xi_1^t, \xi_2^t]$, for every $\xi_1, \xi_2 \in \mathcal{E}$. \square

Lemma 2.3.5. *Let $\xi' : \mathcal{G} \rightarrow \mathcal{G}$ be a linear map such that there exists $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ linear and $[\xi', ad_x] = ad_{\alpha(x)}$, for all x in \mathcal{G} . Then ξ' preserves every ideal \mathcal{A} of \mathcal{G} satisfying $[\mathcal{A}, \mathcal{A}] = \mathcal{A}$. In particular, if \mathcal{G} is semi-simple and $\mathcal{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \dots \oplus \mathfrak{s}_p$ is a decomposition of \mathcal{G} into a sum of simple ideals $\mathfrak{s}_1, \dots, \mathfrak{s}_p$, then $\xi'(\mathfrak{s}_i) \subset \mathfrak{s}_i$, for $i = 1, \dots, p$.*

Proof. The proof is straightforward. Indeed, every element x of an ideal \mathcal{A} satisfying the hypothesis of Lemma 2.3.5, is a finite sum of the form $x = \sum_i [x_i, y_i]$ where x_i, y_i are all elements of \mathcal{A} . But as \mathcal{A} is an ideal,

$$\begin{aligned} \xi'([x_i, y_i]) &= \xi' \circ ad_{x_i}(y_i) = ([\xi', ad_{x_i}] + ad_{x_i} \circ \xi')(y_i) \\ &= (ad_{\alpha(x_i)} + ad_{x_i} \circ \xi')(y_i) = [\alpha(x_i), y_i] + [x_i, \xi'(y_i)] \end{aligned}$$

is again an element of \mathcal{A} . Hence we have $\xi'(x) = \sum_i ([\alpha(x_i), y_i] + [x_i, \xi'(y_i)])$ is in \mathcal{A} . \square

2.3.4 Equivariant Maps $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$

Let \mathcal{G} be a Lie algebra. In this section, we would like to explore properties of the space Ψ of linear maps $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ which are equivariant with respect to the adjoint and the coadjoint actions of \mathcal{G} on \mathcal{G} and \mathcal{G}^* respectively and satisfy: for all f, g in \mathcal{G}^* ,

$$ad_{\psi(f)}^* g = ad_{\psi(g)}^* f.$$

Lemma 2.3.6. *Let \mathcal{G} be a Lie algebra and ψ an element of Ψ . Then,*

- (a) *Im ψ is an Abelian ideal of \mathcal{G} and we have $\psi(ad_{\psi(g)}^* f) = 0$, for every f, g in \mathcal{G}^* ;*
- (b) *ψ sends closed forms on \mathcal{G} in the center of \mathcal{G} ;*
- (c) *$[Im\psi, \mathcal{G}] \subset \ker f$, for all f in $\ker \psi$;*
- (d) *the map ψ cannot be invertible if \mathcal{G} is not Abelian.*

Proof. (a) For every elements f of \mathcal{G}^* and x of \mathcal{G} , we have,

$$[\psi(f), x] = -(ad_x \circ \psi)(f) = -(\psi \circ ad_x^*)(f) \in Im\psi.$$

Hence $Im(\psi)$ is an ideal of \mathcal{G} .

Now, for every f, g in \mathcal{G}^* , since $\psi(f)$ and $\psi(g)$ are elements of \mathcal{G} , we also have $\psi \circ ad_{\psi(f)}^* = ad_{\psi(f)} \circ \psi$ and $\psi \circ ad_{\psi(g)}^* = ad_{\psi(g)} \circ \psi$. On the one hand,

$$\begin{aligned} (\psi \circ ad_{\psi(f)}^*)(g) &= (ad_{\psi(f)} \circ \psi)(g) \\ \psi(ad_{\psi(f)}^* g) &= [\psi(f), \psi(g)] \end{aligned} \tag{2.38}$$

On the other hand,

$$\begin{aligned} (\psi \circ ad_{\psi(g)}^*)(f) &= (ad_{\psi(g)} \circ \psi)(f) \\ \psi(ad_{\psi(g)}^*f) &= [\psi(g), \psi(f)] \end{aligned} \quad (2.39)$$

Using (2.38) and (2.39) we get

$$[\psi(f), \psi(g)] = \psi(ad_{\psi(f)}^*g) = \psi(ad_{\psi(g)}^*f) = [\psi(g), \psi(f)] \quad (2.40)$$

Equation (2.40) implies the following

$$[\psi(f), \psi(g)] = \psi(ad_{\psi(g)}^*f) = 0, \quad (2.41)$$

for all elements f, g of \mathcal{G}^* . So we have proved (a).

(b) Let f be a closed form on \mathcal{G} , that is, f in \mathcal{G}^* and $ad_x^*f = 0$, for all x in \mathcal{G} . The relation (2.10) implies that $ad_{\psi(f)}^*g = 0$, for any g in \mathcal{G}^* . Thus, for any element y of \mathcal{G} and any element g of \mathcal{G}^* ,

$$g([\psi(f), y]) = 0,$$

and hence $[\psi(f), y] = 0$, for all y in \mathcal{G} . In other words $\psi(f)$ belongs to the center of \mathcal{G} .

(c) If $f \in \ker \psi$, then $ad_{\psi(g)}^*f = ad_{\psi(f)}^*g = 0$, for any g in \mathcal{G}^* , or equivalently, for every x in \mathcal{G} and g in \mathcal{G}^* , $f([\psi(g), x]) = 0$. It follows that $[Im\psi, \mathcal{G}] \subset \ker f$, for every f of $\ker \psi$.

(d) From (a), the map ψ satisfies $\psi(ad_{\psi(g)}^*f) = 0$, for any f, g in \mathcal{G}^* . There are two possibilities here:

- (i) either there exist f, g in \mathcal{G}^* such that $ad_{\psi(g)}^*f \neq 0$, in which case $ad_{\psi(g)}^*f$ belongs to $\ker \psi \neq 0$ and thus ψ is not invertible;
- (ii) or else, suppose $ad_{\psi(g)}^*f = 0$, for all f, g in \mathcal{G}^* . This implies that $\psi(g)$ belongs to the center of \mathcal{G} for every g in \mathcal{G}^* . In other words, the center of \mathcal{G} contains $Im(\psi)$. But since \mathcal{G} is not Abelian, the center of \mathcal{G} is different from \mathcal{G} , hence ψ is not invertible.

□

Lemma 2.3.7. *The space of equivariant maps $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ bijectively corresponds to that of \mathcal{G} -invariant bilinear forms on the \mathcal{G} -module \mathcal{G}^* for the coadjoint representation.*

Proof. Indeed, each such ψ defines a unique coadjoint-invariant bilinear form $\langle \cdot, \cdot \rangle_\psi$ on \mathcal{G}^* as follows:

$$\langle f, g \rangle_\psi := \langle \psi(f), g \rangle, \quad (2.42)$$

for all f, g in \mathcal{G}^* , where the right hand side is the duality pairing $\langle f, x \rangle = f(x)$, x in \mathcal{G} , f in \mathcal{G}^* , as above. The coadjoint-invariance reads

$$\langle ad_x^*f, g \rangle_\psi + \langle f, ad_x^*g \rangle_\psi = 0, \quad (2.43)$$

for all x in \mathcal{G} and all f, g in \mathcal{G}^* ; and is due to the simple equalities

$$\begin{aligned} \langle ad_x^*f, g \rangle_\psi &= \langle \psi(ad_x^*f), g \rangle = \langle ad_x \psi(f), g \rangle \\ &= -\langle \psi(f), ad_x^*g \rangle = -\langle f, ad_x^*g \rangle_\psi. \end{aligned}$$

Conversely, every \mathcal{G} -invariant bilinear form $\langle \cdot, \cdot \rangle_1$ on \mathcal{G}^* gives rise to a unique linear map $\psi_1 : \mathcal{G}^* \rightarrow \mathcal{G}$ which is equivariant with respect to the adjoint and coadjoint representations of \mathcal{G} , by the formula

$$\langle \psi_1(f), g \rangle := \langle f, g \rangle_1. \quad (2.44)$$

□

If ψ is symmetric or skew-symmetric, then so is $\langle \cdot, \cdot \rangle_\psi$ and vice versa. Otherwise, $\langle \cdot, \cdot \rangle_\psi$ can be decomposed into a symmetric and a skew-symmetric parts $\langle \cdot, \cdot \rangle_{\psi,s}$ and $\langle \cdot, \cdot \rangle_{\psi,a}$ respectively, defined by the following formulas:

$$\langle f, g \rangle_{\psi,s} := \frac{1}{2} [\langle f, g \rangle_\psi + \langle g, f \rangle_\psi], \quad (2.45)$$

$$\langle f, g \rangle_{\psi,a} := \frac{1}{2} [\langle f, g \rangle_\psi - \langle g, f \rangle_\psi]. \quad (2.46)$$

The symmetric and skew-symmetric parts $\langle \cdot, \cdot \rangle_{1,s}$ and $\langle \cdot, \cdot \rangle_{1,a}$ of a \mathcal{G} -invariant bilinear form $\langle \cdot, \cdot \rangle_1$, are also \mathcal{G} -invariant. From a remark in p. 2297 of [60], the radical $Rad \langle \cdot, \cdot \rangle_1 := \{f \in \mathcal{G}^*, \langle f, g \rangle_1 = 0, \forall g \in \mathcal{G}^*\}$ of a \mathcal{G} -invariant form $\langle \cdot, \cdot \rangle_1$, contains the coadjoint orbits of all its points.

2.3.5 Cocycles $\mathcal{G} \rightarrow \mathcal{G}^*$.

The 1-cocycles for the coadjoint representation of a Lie algebra \mathcal{G} are linear maps $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ satisfying the cocycle condition $\beta([x, y]) = ad_x^* \beta(y) - ad_y^* \beta(x)$, for every elements x, y of \mathcal{G} .

To any given 1-cocycle β , corresponds a bilinear form Ω_β on \mathcal{G} , by the formula

$$\Omega_\beta(x, y) := \langle \beta(x), y \rangle, \quad (2.47)$$

for all x, y in \mathcal{G} , where $\langle \cdot, \cdot \rangle$ is again the duality pairing between elements of \mathcal{G} and \mathcal{G}^* .

The bilinear form Ω_β is skew-symmetric (resp. symmetric, nondegenerate) if and only if β is skew-symmetric (resp. symmetric, invertible).

Skew-symmetric such cocycles β are in bijective correspondence with closed 2-forms in \mathcal{G} , via the formula (2.47). In this sense, the cohomology space $H^1(\mathcal{D}, \mathcal{D})$ contains the second cohomology space $H^2(\mathcal{G}, \mathbb{R})$ of \mathcal{G} with coefficients in \mathbb{R} for the trivial action of \mathcal{G} on \mathbb{R} . Hence, $H^1(\mathcal{D}, \mathcal{D})$ somehow contains the second space $H_{inv}^2(G, \mathbb{R})$ of left invariant de Rham cohomology $H_{inv}^*(G, \mathbb{R})$ of any Lie group G with Lie algebra \mathcal{G} .

Invertible skew-symmetric ones, when they exist, are those giving rise to symplectic forms or equivalently to invertible solutions of the Classical Yang-Baxter Equation. The study and classification of the solutions of the Classical Yang-Baxter Equation is a still open problem in Geometry, Theory of integrable systems. In Geometry, they give rise to very interesting structures in the framework of Symplectic Geometry, Affine Geometry, Theory of Homogeneous Kähler domains, (see e.g. [28] and references therein).

If \mathcal{G} is semi-simple, then every cocycle β is a coboundary, that is, there exists f_β in \mathcal{G}^* such that $\beta(x) = -ad_x^* f_\beta$, for any x in \mathcal{G} .

2.3.6 Cohomology Space $H^1(\mathcal{D}, \mathcal{D})$

Following Remarks 2.3.1 and 2.3.2, we can embed $\text{der}(\mathcal{G})$ as a subalgebra $\text{der}(\mathcal{G})_1$ of $\text{der}(\mathcal{D})$, using the linear map $\alpha \mapsto \phi_\alpha$, with $\phi_\alpha(x, f) = (\alpha(x), f \circ \alpha)$. In the same way, we have constructed \mathcal{Q} and Ψ as subspaces of $\text{der}(\mathcal{D})$. Likewise, $\mathcal{J}^t := \{j^t, \text{ where } j \in \mathcal{J}\}$ is seen as a subspace of $\text{der}(\mathcal{D})$, via the linear map $j^t \mapsto \phi_j$, with $\phi_j(x, f) = (0, f \circ j)$.

We give the following summary.

Theorem 2.3.3. *The first cohomology space $H^1(\mathcal{D}, \mathcal{D})$ of the (Chevalley-Eilenberg) cohomology associated with the adjoint action of \mathcal{D} on itself, satisfies*

$$H^1(\mathcal{D}, \mathcal{D}) \cong H^1(\mathcal{G}, \mathcal{G}) \oplus \mathcal{J}^t \oplus H^1(\mathcal{G}, \mathcal{G}^*) \oplus \Psi,$$

where $H^1(\mathcal{G}, \mathcal{G})$ and $H^1(\mathcal{G}, \mathcal{G}^*)$ are the first cohomology spaces associated with the adjoint and coadjoint actions of \mathcal{G} , respectively; and $\mathcal{J}^t := \{j^t, j \in \mathcal{J}\}$ (space of transposes of elements of \mathcal{J}).

If \mathcal{G} is semi-simple, then $\Psi = \{0\}$ and thus

$$H^1(\mathcal{D}, \mathcal{D}) \cong \mathcal{J}^t.$$

Moreover, we have $\mathcal{J} \cong \mathbb{R}^p$, where p is the number of the simple ideals \mathfrak{s}_i of \mathcal{G} such that $\mathcal{G} = \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_p$. Hence, of course, $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{R}^p$.

If \mathcal{G} is a compact Lie algebra, with centre $Z(\mathcal{G})$, we get

$$\begin{aligned} H^1(\mathcal{G}, \mathcal{G}) &\cong \text{End}(Z(\mathcal{G})), \\ \mathcal{J} &\cong \mathbb{R}^p \oplus \text{End}(Z(\mathcal{G})), \\ H^1(\mathcal{G}, \mathcal{G}^*) &\cong L(Z(\mathcal{G}), Z(\mathcal{G})^*), \\ \Psi &\cong L(Z(\mathcal{G})^*, Z(\mathcal{G})). \end{aligned}$$

Hence, we get

$$H^1(\mathcal{D}, \mathcal{D}) \cong (\text{End}(\mathbb{R}^k))^4 \oplus \mathbb{R}^p,$$

where k is the dimension of the centre of \mathcal{G} , and p is the number of the simple components of the derived ideal $[\mathcal{G}, \mathcal{G}]$ of \mathcal{G} . Here, if E, F are vector spaces, $L(E, F)$ is the space of linear maps $E \rightarrow F$.

The proof of Theorem 2.3.3 is given by Proposition 2.3.3 below and different lemmas and propositions, discussed in Section 2.4.

For the purpose of this investigation, we have favored a direct approach to exhibit detailed calculations of the first cohomology space, instead of the traditional powerful spectral sequences method commonly applied in the more general setting of locally trivial fiber bundles (see e.g. [44], [65]). Some parts of Theorem C can also be seen as a refinement, using our direct approach, of some already known results ([44]).

As a vector space, $\text{der}(\mathcal{D})$ is isomorphic to the direct sum $\text{der}(\mathcal{G}) \oplus \mathcal{J}^t \oplus \mathcal{Q} \oplus \Psi$ by

$$\Phi : \text{der}(\mathcal{G}) \oplus \mathcal{J}^t \oplus \mathcal{Q} \oplus \Psi \rightarrow \text{der}(\mathcal{D}); \quad (\alpha, j^t, \beta, \psi) \mapsto \phi_\alpha + \phi_j + \phi_\beta + \phi_\psi. \quad (2.48)$$

In this isomorphism, we have $\Phi(\text{der}(\mathcal{G}) \oplus \mathcal{J}^t) = \text{der}(\mathcal{G})_1 \oplus \mathcal{J}^t = \mathcal{G}_0$ and $\Phi(\mathcal{Q} \oplus \Psi) = \mathcal{G}_1$.

Now an exact derivation of \mathcal{D} , *i.e.* a 1-coboundary for the Chevalley-Eilenberg cohomology associated with the adjoint action of \mathcal{D} on \mathcal{D} , is of the form $\phi_0 = \partial v_0 = ad_{v_0}$ for some element $v_0 := (x_0, f_0)$ of the \mathcal{D} -module \mathcal{D} . That is,

$$\phi_0(x, f) = (\alpha_0(x), \beta_0(x) + \xi_0(f)),$$

where

$$\alpha_0(x) := [x_0, x], \quad \beta_0(x) = -ad_x^* f_0, \quad \xi_0(f) = ad_{x_0}^* f.$$

As we can see $\phi_0 = \phi_{\alpha_0} + \phi_{\beta_0} = \Phi(\alpha_0, 0, \beta_0, 0)$ and

Proposition 2.3.3. *The linear map Φ in (2.48) induces an isomorphism $\bar{\Phi}$ in cohomology, between the spaces $H^1(\mathcal{G}, \mathcal{G}) \oplus \mathcal{J}^t \oplus H^1(\mathcal{G}, \mathcal{G}^*) \oplus \Psi$ and $H^1(\mathcal{D}, \mathcal{D})$.*

Proof. The isomorphism in cohomology simply reads

$$\bar{\Phi}(\text{class}(\alpha), j^t, \text{class}(\beta), \psi) = \text{class}(\phi_\alpha + \phi_j + \phi_\beta + \phi_\psi).$$

□

2.4 Case of Orthogonal Lie Algebras

In this section, we prove that if a Lie algebra \mathcal{G} is orthogonal, then the Lie algebra $\text{der}(\mathcal{G})$ of its derivations and the Lie algebra \mathcal{J} of linear maps $j : \mathcal{G} \rightarrow \mathcal{G}$ satisfying $j[x, y] = [jx, y]$, for every x, y in \mathcal{G} , completely characterize the Lie algebra $\text{der}(\mathcal{D})$ of derivations of $\mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^*$, and hence the group of automorphisms of the cotangent bundle of any connected Lie group with Lie algebra \mathcal{G} . We also show that \mathcal{J} is isomorphic to the space of adjoint-invariant bilinear forms on \mathcal{G} .

Let (\mathcal{G}, μ) be an orthogonal Lie algebra and consider the isomorphism $\theta : \mathcal{G} \rightarrow \mathcal{G}^*$ of \mathcal{G} -modules, given by $\langle \theta(x), y \rangle := \mu(x, y)$, as in Section 2.2.1.

Of course, θ^{-1} is an equivariant map. But if \mathcal{G} is not Abelian, invertible equivariant linear maps do not contribute to the space of derivations of \mathcal{D} , as discussed in Lemma 2.3.6.

We pull coadjoint-invariant bilinear forms B' on \mathcal{G}^* back to adjoint-invariant bilinear forms on \mathcal{G} , as follows $B(x, y) := B'(\theta(x), \theta(y))$. Indeed, we have

$$\begin{aligned} B([x, y], z) &= B'(\theta([x, y]), \theta(z)) \\ &= B'(ad_x^* \theta(y), \theta(z)) \\ &= -B'(\theta(y), ad_x^* \theta(z)) \\ &= -B(y, [x, z]). \end{aligned}$$

Proposition 2.4.1. *If a Lie algebra \mathcal{G} is orthogonal, then there is an isomorphism between any two of the following vector spaces:*

- (a) the space \mathcal{J} of linear maps $j : \mathcal{G} \rightarrow \mathcal{G}$ satisfying $j[x, y] = [jx, y]$, for every x, y in \mathcal{G} ;

(b) the space of linear maps $\psi : \mathfrak{G}^* \rightarrow \mathfrak{G}$ which are equivariant with respect to the coadjoint and the adjoint representations of \mathfrak{G} ;

(c) the space of bilinear forms B on \mathfrak{G} which are adjoint-invariant, i.e.

$$B([x, y], z) + B(y, [x, z]) = 0, \quad (2.49)$$

for all x, y, z in \mathfrak{G} ;

(d) the space of bilinear forms B' on \mathfrak{G}^* which are coadjoint-invariant, i.e.

$$B'(ad_x^* f, g) + B'(f, ad_x^* g) = 0, \quad (2.50)$$

for all x in \mathfrak{G} , f, g in \mathfrak{G}^* .

Proof. • The linear map $\psi \mapsto \psi \circ \theta$ is an isomorphism between the space of equivariant linear maps $\psi : \mathfrak{G}^* \rightarrow \mathfrak{G}$ and the space \mathcal{J} . Indeed, if ψ is equivariant, we have

$$\psi \circ \theta([x, y]) = -\psi(ad_y^* \theta(x)) = -ad_y \psi(\theta(x)) = [\psi \circ \theta(x), y].$$

Hence $\psi \circ \theta$ is in \mathcal{J} . Conversely, if j is in \mathcal{J} , then $j \circ \theta^{-1}$ is equivariant, as it satisfies

$$j \circ \theta^{-1} \circ ad_x^* = j \circ ad_x \circ \theta^{-1} = ad_x \circ j \circ \theta^{-1}.$$

This correspondence is obviously linear and invertible. Hence we get the isomorphism between (a) and (b).

• The isomorphism between the space \mathcal{J} of adjoint-invariant endomorphisms and adjoint-invariant bilinear forms is given as follows

$$j \in \mathcal{J} \mapsto B_j, \text{ where } B_j(x, y) := \mu(j(x), y). \quad (2.51)$$

for any x, y in \mathfrak{G} . We have, for any x, y, z in \mathfrak{G}

$$B_j([x, y], z) := \mu(j([x, y]), z) = \mu([x, j(y)], z) = -\mu(j(y), [x, z]) = -B_j(y, [x, z]).$$

Conversely, if B is an adjoint-invariant bilinear form on \mathfrak{G} , then the endomorphism j , defined by

$$\mu(j(x), y) := B(x, y) \quad (2.52)$$

is an element of \mathcal{J} , as it satisfies

$$\mu(j([x, y]), z) := B([x, y], z) = B(x, [y, z]) = \mu(j(x), [y, z]) = \mu([j(x), y], z),$$

for all elements x, y, z of \mathfrak{G} .

• From Lemma 2.3.7, the space of equivariant linear maps ψ bijectively corresponds to that of coadjoint-invariant bilinear forms on \mathfrak{G}^* , via $\psi \mapsto \langle, \rangle_\psi$. \square

Now, suppose ψ is skew-symmetric. Let ω_ψ denote the corresponding skew-symmetric bilinear form on \mathcal{G} :

$$\omega_\psi(x, y) := \mu(\psi \circ \theta(x), y), \quad (2.53)$$

for all x, y in \mathcal{G} . Then, ω_ψ is adjoint-invariant. If we denote by ∂ the Chevalley-Eilenberg coboundary operator, that is,

$$(\partial\omega_\psi)(x, y, z) = -(\omega_\psi([x, y], z) + \omega_\psi([y, z], x) + \omega_\psi([z, x], y)),$$

the following formula holds true

$$(\partial\omega_\psi)(x, y, z) = -\omega_\psi([x, y], z).$$

for all x, y, z in \mathcal{G} .

Corollary 2.4.1. *The following are equivalent.*

- (a) ω_ψ is closed;
- (b) $\psi \circ \theta([x, y]) = 0$, for all x, y in \mathcal{G} ;
- (c) $Im(\psi)$ is in the centre of \mathcal{G} .

In particular, if $\dim[\mathcal{G}, \mathcal{G}] \geq \dim \mathcal{G} - 1$, then ω_ψ is closed if and only if $\psi = 0$.

Proof. The above equality also reads

$$\partial\omega_\psi(x, y, z) = -\omega_\psi([x, y], z) = -\mu(\psi \circ \theta([x, y]), z), \quad (2.54)$$

for all x, y, z in \mathcal{G} ; and gives the proof that (a) and (b) are equivalent. In particular, if $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$ then, obviously $\partial\omega_\psi = 0$ if and only if $\psi = 0$, as θ is invertible.

Now suppose $\dim[\mathcal{G}, \mathcal{G}] = \dim \mathcal{G} - 1$ and set $\mathcal{G} = \mathbb{R}x_0 \oplus [\mathcal{G}, \mathcal{G}]$, for some element x_0 of \mathcal{G} . If ω_ψ is closed, we already know that $\psi \circ \theta$ vanishes on $[\mathcal{G}, \mathcal{G}]$. Below, we show that, it also does on $\mathbb{R}x_0$. Indeed, the formula

$$0 = -\omega_\psi([x, y], x_0) = \omega_\psi(x_0, [x, y]) = \mu(\psi \circ \theta(x_0), [x, y]), \quad (2.55)$$

for all x, y in \mathcal{G} , obtained by taking $z = x_0$ in (2.54), coupled with the obvious equality $0 = \omega_\psi(x_0, x_0) = \mu(\psi \circ \theta(x_0), x_0)$, are equivalent to $\psi \circ \theta(x_0)$ satisfying $\mu(\psi \circ \theta(x_0), x) = 0$ for all x in \mathcal{G} . As μ is nondegenerate, this means that $\psi \circ \theta(x_0) = 0$. Hence $\psi \circ \theta = 0$, or equivalently $\psi = 0$.

Now, as every f in \mathcal{G}^* is of the form $f = \theta(y)$, for some y in \mathcal{G} , the formula

$$\psi \circ \theta([x, y]) = \psi \circ ad_x^* \theta(y) = ad_x \circ \psi \circ \theta(y) = [x, \psi \circ \theta(y)], \quad \forall x, y \in \mathcal{G}. \quad (2.56)$$

shows that $\psi \circ \theta([x, y]) = 0$, for all x, y on \mathcal{G} if and only if $Im(\psi)$ is a subset of the center of \mathcal{G} . Thus, (b) is equivalent to (c). \square

Now we pull every element ξ of \mathcal{E} back to an endomorphism ξ' of \mathcal{G} given by the formula

$$\xi' := \theta^{-1} \circ \xi \circ \theta.$$

Proposition 2.4.2. *Let (\mathcal{G}, μ) be an orthogonal Lie algebra and \mathcal{G}^* its dual space. Define $\theta : \mathcal{G} \rightarrow \mathcal{G}^*$ by $\langle \theta(x), y \rangle := \mu(x, y)$, as in Relation (1.2), and let \mathcal{E} and \mathcal{S} stand for the same Lie algebras as above. The linear map $Q : \xi \mapsto \xi' := \theta^{-1} \circ \xi \circ \theta$ is an isomorphism of Lie algebras between \mathcal{E} and \mathcal{S} .*

Proof. Let ξ be in \mathcal{E} , with $[\xi, ad_x^*] = ad_{\alpha(x)}^*$, for every x in \mathcal{G} . The image $Q(\xi) =: \xi'$ of ξ , satisfies, for any x in \mathcal{G} ,

$$\begin{aligned}
[\xi', ad_x] &:= \xi' \circ ad_x - ad_x \circ \xi' \\
&:= \theta^{-1} \circ \xi \circ \theta \circ ad_x - ad_x \circ \theta^{-1} \circ \xi \circ \theta \\
&= \theta^{-1} \circ \xi \circ ad_x^* \circ \theta - \theta^{-1} \circ ad_x^* \circ \xi \circ \theta \\
&= \theta^{-1} \circ (\xi \circ ad_x^* - ad_x^* \circ \xi) \circ \theta \\
&= \theta^{-1} \circ ad_{\alpha(x)}^* \circ \theta, \quad \text{since } [\xi, ad_x^*] = ad_{\alpha(x)}^*. \\
&= \theta^{-1} \circ \theta \circ ad_{\alpha(x)}, \quad \text{using (1.4).} \\
&= ad_{\alpha(x)}.
\end{aligned}$$

Now we have $[Q(\xi_1), Q(\xi_2)] = Q([\xi_1, \xi_2])$ for all ξ_1, ξ_2 in \mathcal{E} , as seen below.

$$\begin{aligned}
[Q(\xi_1), Q(\xi_2)] &:= Q(\xi_1)Q(\xi_2) - Q(\xi_2)Q(\xi_1) \\
&:= \theta^{-1} \circ \xi_1 \circ \theta \circ \theta^{-1} \circ \xi_2 \circ \theta - \theta^{-1} \circ \xi_2 \circ \theta \circ \theta^{-1} \circ \xi_1 \circ \theta \\
&= \theta^{-1} \circ [\xi_1, \xi_2] \circ \theta \\
&= Q([\xi_1, \xi_2]). \tag{2.57}
\end{aligned}$$

□

Proposition 2.4.3. *The linear map $P : \beta \mapsto D_\beta := \theta^{-1} \circ \beta$, is an isomorphism between the space of cocycles $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ and the space $der(\mathcal{G})$ of derivations of \mathcal{G} .*

Proof. The proof is straightforward. If $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ is a cocycle, then the linear map $D_\beta : \mathcal{G} \rightarrow \mathcal{G}$, $x \mapsto \theta^{-1}(\beta(x))$ is a derivation of \mathcal{G} , as we have

$$D_\beta[x, y] = \theta^{-1}(ad_x^* \beta(y) - ad_y^* \beta(x)) = [x, \theta^{-1}(\beta(y))] - [y, \theta^{-1}(\beta(x))].$$

Conversely, if D is a derivation of \mathcal{G} , then the linear map $\beta_D := P^{-1}(D) = \theta \circ D : \mathcal{G} \rightarrow \mathcal{G}^*$, is 1-cocycle. Indeed we have: for every x, y in \mathcal{G}

$$\beta_D[x, y] = \theta([Dx, y] + [x, Dy]) = -ad_y^*(\theta \circ D(x)) + ad_x^*(\theta \circ D(y)).$$

□

2.4.1 Case of Semi-simple Lie Algebras

Suppose now \mathcal{G} is semi-simple, then every derivation is inner. Thus in particular, the derivation ϕ_{11} obtained in (2.16), is of the form

$$\phi_{11} = ad_{x_0}, \tag{2.58}$$

for some x_0 in \mathfrak{G} . The semi-simplicity of \mathfrak{G} also implies that the 1-cocycle ϕ_{12} obtained in (2.17) is a coboundary. That is, there exists an element f_0 of \mathfrak{G}^* such that

$$\phi_{12}(x) = -ad_x^* f_0, \tag{2.59}$$

for all x in \mathfrak{G} . Here is a direct corollary of Lemma 2.3.6.

Proposition 2.4.4. *If \mathfrak{G} is a semi-simple Lie algebra, then every linear map $\psi : \mathfrak{G}^* \rightarrow \mathfrak{G}$ which is equivariant with respect to the adjoint and coadjoint actions of \mathfrak{G} and satisfies (2.10), is necessarily identically equal to zero.*

Proof. A Lie algebra is semi-simple if and only if it contains no nonzero proper Abelian ideal. But from Lemma 2.3.6, $Im(\psi)$ must be an Abelian ideal of \mathfrak{G} . So $Im(\psi) = \{0\}$ and hence $\psi = 0$. □

Remark 2.4.1. *From Propositions 2.3.3 and 2.4.4, the cohomology space $H^1(\mathcal{D}, \mathcal{D})$ is completely determine by the space \mathcal{J} of endomorphisms j with $j([x, y]) = [j(x), y]$, for all x, y in \mathfrak{G} , or equivalently, by the space of adjoint-invariant bilinear forms on \mathfrak{G} .*

Corollary 2.4.2. *If G is a semi-simple Lie group with Lie algebra \mathfrak{G} , then the space of bi-invariant bilinear forms on G is of dimension $dim H^1(\mathcal{D}, \mathcal{D})$.*

Proposition 2.4.5. *Suppose \mathfrak{G} is a simple Lie algebra. Then,*

- (a) every linear map $j : \mathfrak{G} \rightarrow \mathfrak{G}$ in \mathcal{J} , is of the form $j(x) = \lambda x$, for some λ in \mathbb{R} ;
- (b) every element ξ of \mathcal{E} is of the form

$$\xi = ad_{x_0}^* + \lambda Id_{\mathfrak{G}^*}, \tag{2.60}$$

for some x_0 in \mathfrak{G} and λ in \mathbb{R} .

Proof. The part (a) is obtained from relation (2.31) and the Schur's lemma. From Propositions 2.3.1 and 2.3.2, for every ξ in \mathcal{E} , there exist α in $der(\mathfrak{G})$ and j in \mathcal{J} such that $\xi^t = \alpha + j$. As \mathfrak{G} is simple and from part (a), there exist x_0 in \mathfrak{G} and λ in \mathbb{R} such that $\xi^t = ad_{x_0} + \lambda Id_{\mathfrak{G}}$. □

We also have the following.

Proposition 2.4.6. *Let G be a simple Lie group with Lie algebra \mathfrak{G} . Let $\mathcal{D} := \mathfrak{G} \times \mathfrak{G}^*$ be the Lie algebra of the cotangent bundle T^*G of G . Then, the first cohomology space of \mathcal{D} with coefficients in \mathcal{D} is $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{R}$.*

Proof. Indeed, a derivation $\phi : \mathcal{D} \rightarrow \mathcal{D}$ can be written: for every element (x, f) of \mathcal{D} ,

$$\phi(x, f) = ([x_0, x], ad_{x_0}^* f - ad_x^* f_0 + \lambda f), \tag{2.61}$$

where x_0 and f_0 are fixed elements in \mathfrak{G} and \mathfrak{G}^* respectively. The inner derivations are those with $\lambda = 0$. It follows that the first cohomology space of \mathcal{D} with values in \mathcal{D} is given by

$$\begin{aligned} H^1(\mathcal{D}, \mathcal{D}) &= \{ \phi : \mathcal{D} \rightarrow \mathcal{D} : \phi(x, f) = (0, \lambda f), \lambda \in \mathbb{R} \} \\ &= \{ \lambda(0, Id_{\mathfrak{G}^*}), \lambda \in \mathbb{R} \} \\ &= \mathbb{R} Id_{\mathfrak{G}^*} \end{aligned}$$

□

As a direct consequence, we get

Corollary 2.4.3. *If \mathcal{G} is a semi-simple Lie algebra over \mathbb{R} , then $\dim H^1(\mathcal{D}, \mathcal{D}) = p$, where p stands for the number of simple components of \mathcal{G} , in its decomposition into a direct sum $\mathcal{G} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p$ of simple ideals $\mathfrak{s}_1, \dots, \mathfrak{s}_p$.*

Consider a semi-simple Lie algebra \mathcal{G} and set $\mathcal{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, $p \in \mathbb{N}^*$, where \mathfrak{s}_i , $i = 1, \dots, p$ are simple Lie algebras. From Lemma 2.3.5, ξ' preserves each \mathfrak{s}_i . Thus from Proposition 2.4.5, the restriction ξ'_i of ξ' to each \mathfrak{s}_i , $i = 1, 2, \dots, p$ equals $\xi'_i = ad_{x_{0_i}} + \lambda_i Id_{\mathfrak{s}_i}$, for some x_{0_i} in \mathfrak{s}_i and a real number λ_i . Hence, $\xi' = ad_{x_0} \oplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i}$, where $x_0 = x_{0_1} + x_{0_2} + \cdots + x_{0_p} \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$ and $\oplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i}$ acts on $\mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$ as follows:

$$(\oplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i})(x_1 + x_2 + \cdots + x_p) = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p.$$

In particular, we have proved

Corollary 2.4.4. *Consider the decomposition of a semi-simple Lie algebra \mathcal{G} into a sum $\mathcal{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, of simple Lie algebras \mathfrak{s}_i , $i = 1, \dots, p \in \mathbb{N}^*$. If a linear map $j : \mathcal{G} \rightarrow \mathcal{G}$ satisfies $j[x, y] = [jx, y]$, then there exist real numbers $\lambda_1, \dots, \lambda_p$ such that*

$$j = \lambda_1 id_{\mathfrak{s}_1} \oplus \cdots \oplus \lambda_p id_{\mathfrak{s}_p}.$$

More precisely

$$j(x_1 + \cdots + x_p) = \lambda_1 x_1 + \cdots + \lambda_p x_p,$$

if x_i is in \mathfrak{s}_i , $i = 1, \dots, p$.

Now, we already know from Proposition 2.4.4, that each ψ vanishes identically. So a 1-cocycle of \mathcal{D} is given by:

$$\phi(x, f) = \left([x_0, x], ad_{x_0}^* f - ad_x^* f_0 + \sum_{i=1}^p \lambda_i f_i \right) \quad (2.62)$$

for every x in \mathcal{G} and every $f := f_1 + f_2 + \cdots + f_p$ in $\mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \cdots \oplus \mathfrak{s}_p^* = \mathcal{G}^*$, where x_0 is in \mathcal{G} , f_0 is in \mathcal{G}^* and λ_i , $i = 1, \dots, p$, are real numbers. We then have,

Proposition 2.4.7. *Let G be a semi-simple Lie group with Lie algebra \mathcal{G} over \mathbb{R} . Let $\mathcal{D} := \mathcal{G} \ltimes \mathcal{G}^*$ be the cotangent Lie algebra of G . Then, the first cohomology space of \mathcal{D} with coefficients in \mathcal{D} is given by $H^1(\mathcal{D}, \mathcal{D}) \cong \mathbb{R}^p$, where p is the number of the simple components of \mathcal{G} .*

2.4.2 Case of Compact Lie Algebras

It is well known that a compact Lie algebra \mathcal{G} decomposes as the direct sum $\mathcal{G} = [\mathcal{G}, \mathcal{G}] \oplus Z(\mathcal{G})$ of its derived ideal $[\mathcal{G}, \mathcal{G}]$ and its centre $Z(\mathcal{G})$, with $[\mathcal{G}, \mathcal{G}]$ semi-simple and compact. This yields a decomposition $\mathcal{G}^* = [\mathcal{G}, \mathcal{G}]^* \oplus Z(\mathcal{G})^*$ of \mathcal{G}^* into a direct sum of the dual spaces $[\mathcal{G}, \mathcal{G}]^*$, $Z(\mathcal{G})^*$ of $[\mathcal{G}, \mathcal{G}]$ and $Z(\mathcal{G})$ respectively, where $[\mathcal{G}, \mathcal{G}]^*$ (resp. $Z(\mathcal{G})^*$) is identified with the space of linear forms on \mathcal{G} which vanish on $Z(\mathcal{G})$ (resp. $[\mathcal{G}, \mathcal{G}]$). On the other hand, $[\mathcal{G}, \mathcal{G}]$ also decomposes into as a direct sum $[\mathcal{G}, \mathcal{G}] = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p$ of simple ideals \mathfrak{s}_i . From

Theorem 2.3.1, a derivation of the Lie algebra $\mathcal{D} := \mathcal{G} \times \mathcal{G}^*$ of the cotangent bundle of G has the following form $\phi(x, f) = (\alpha(x) + \psi(f), \beta(x) + \xi(f))$, with conditions listed in Theorem 2.3.1.

Let us look at the equivariant maps $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ satisfying $ad_{\psi(f)}^* g = ad_{\psi(g)}^* f$, for every f, g in \mathcal{G}^* . From Lemma 2.3.6, $Im(\psi)$ is an Abelian ideal of \mathcal{G} ; thus $Im(\psi) \subset Z(\mathcal{G})$. As consequence, we have $\psi(ad_x^* f) = [x, \psi(f)] = 0$, for every x of \mathcal{G} and every f of \mathcal{G}^* .

Lemma 2.4.1. *Let $(\tilde{\mathcal{G}}, \mu)$ be an orthogonal Lie algebra satisfying $\tilde{\mathcal{G}} = [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$. Then, every g in $\tilde{\mathcal{G}}^*$ is a finite sum of elements of the form $g_i = ad_{\bar{x}_i}^* \bar{g}_i$, for some \bar{x}_i in $\tilde{\mathcal{G}}$, \bar{g}_i in $\tilde{\mathcal{G}}^*$.*

Proof. Indeed, consider an isomorphism $\theta : \tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}^*$ of $\tilde{\mathcal{G}}$ -modules. For every g in $\tilde{\mathcal{G}}^*$, there exists x_g in $\tilde{\mathcal{G}}$ such that $g = \theta(x_g)$. But as $\tilde{\mathcal{G}} = [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}]$, we have $x_g = [x_1, y_1] + \dots + [x_s, y_s]$ for some x_i, y_i in $\tilde{\mathcal{G}}$. Thus

$$\begin{aligned} g &= \theta([x_1, y_1]) + \dots + \theta([x_s, y_s]) \\ &= ad_{x_1}^* \theta(y_1) + \dots + ad_{x_s}^* \theta(y_s) \\ &= ad_{\bar{x}_1}^* \bar{g}_1 + \dots + ad_{\bar{x}_s}^* \bar{g}_s, \end{aligned}$$

where $\bar{x}_i = x_i$ and $\bar{g}_i = \theta(y_i)$. □

A semi-simple Lie algebra being orthogonal (with, e.g. its Killing form as μ), from Lemma 2.4.1 and the equality $\psi(ad_x^* f) = 0$, for all x in \mathcal{G} , f in \mathcal{G}^* , each ψ in Ψ vanishes on $[\mathcal{G}, \mathcal{G}]^*$. Of course, the converse is true. Every linear map $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ with $Im(\psi) \subset Z(\mathcal{G})$ and $\psi([\mathcal{G}, \mathcal{G}]^*) = 0$, is in Ψ . Hence we can make the following identification.

Lemma 2.4.2. *Let \mathcal{G} be a compact Lie algebra, with centre $Z(\mathcal{G})$. Then Ψ is isomorphic to the space $L(Z(\mathcal{G})^*, Z(\mathcal{G}))$ of linear maps $Z(\mathcal{G})^* \rightarrow Z(\mathcal{G})$.*

The restriction of the cocycle β to the semi-simple ideal $[\mathcal{G}, \mathcal{G}]$ is a coboundary, that is, there exists an element f_0 in \mathcal{G}^* such that for every x_1 in $[\mathcal{G}, \mathcal{G}]$, $\beta(x_1) = -ad_{x_1}^* f_0$. Now for x_2 in $Z(\mathcal{G})$, one has $0 = \beta[x_2, y] = -ad_y^* \beta(x_2)$, for all y of \mathcal{G} , since x_2 is in $Z(\mathcal{G})$. In other words, $\beta(x_2)([y, z]) = 0$, for all y, z in \mathcal{G} . That is, $\beta(x_2)$ vanishes on $[\mathcal{G}, \mathcal{G}]$ for every $x_2 \in Z(\mathcal{G})$. Hence, we write

$$\beta(x) = -ad_{x_1}^* f_0 + \eta(x_2),$$

for all $x := x_1 + x_2$ in $[\mathcal{G}, \mathcal{G}] \oplus Z(\mathcal{G})$, where $\eta : Z(\mathcal{G}) \rightarrow Z(\mathcal{G})^*$ is a linear map. This simply means the following.

Lemma 2.4.3. *Let \mathcal{G} be a compact Lie algebra, with centre $Z(\mathcal{G})$. Then the first space $H^1(\mathcal{G}, \mathcal{G}^*)$ of the cohomology associated with the coadjoint action of \mathcal{G} , is isomorphic to the space $L(Z(\mathcal{G}), Z(\mathcal{G})^*)$.*

We have already seen that ξ is such that $\xi^t = \alpha + j$, where α is a derivation of \mathcal{G} and j is in \mathcal{J} . Both α and j preserve each of $[\mathcal{G}, \mathcal{G}]$ and $Z(\mathcal{G})$. Thus we can write $\alpha = ad_{x_{01}} \oplus \varphi$, for some $x_{01} \in [\mathcal{G}, \mathcal{G}]$, where φ is in $End(Z(\mathcal{G}))$. Here α acts on an element $x := x_1 + x_2$, where x_1 is in $[\mathcal{G}, \mathcal{G}]$, x_2 belongs to $Z(\mathcal{G})$, as follows:

$$\begin{aligned} \alpha(x) &= (ad_{x_{01}} \oplus \varphi)(x_1 + x_2) \\ &:= ad_{x_{01}} x_1 + \varphi(x_2). \end{aligned}$$

We summarize this as

Lemma 2.4.4. *If \mathcal{G} is a compact Lie algebra with centre $Z(\mathcal{G})$, then*

$$H^1(\mathcal{G}, \mathcal{G}) \cong \text{End}(Z(\mathcal{G})). \quad (2.63)$$

Now, suppose for the rest of this section, that \mathcal{G} is a compact Lie algebra. We write

$$j = \bigoplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i} \oplus \rho,$$

where $\rho : Z(\mathcal{G}) \rightarrow Z(\mathcal{G})$ is a linear map and j acts on an element $x := x_1 + x_2$ as follows:

$$j(x) = \left(\bigoplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i} \oplus \rho \right) (x_{11} + x_{12} + \cdots + x_{1p} + x_2) = \sum_{i=1}^p \lambda_i x_{1i} + \rho(x_2)$$

where $x_1 := x_{11} + x_{12} + \cdots + x_{1p}$ is in $[\mathcal{G}, \mathcal{G}]$, x_2 is in $Z(\mathcal{G})$ and x_{1i} belongs to \mathfrak{s}_i . Hence,

Lemma 2.4.5. *If \mathcal{G} is a compact Lie algebra with centre $Z(\mathcal{G})$, then $\mathcal{J} \cong \mathbb{R}^p \oplus \text{End}(Z(\mathcal{G}))$, where p is the number of simple components of $[\mathcal{G}, \mathcal{G}]$.*

So, the expression of ξ now reads

$$\xi = \left[-ad_{x_{01}}^* + \left(\bigoplus_{i=1}^p \lambda_i Id_{\mathfrak{s}_i^*} \right) \right] \oplus \varphi'',$$

with $(\varphi'')^t(x_2) = \rho(x_2) + \varphi(x_2)$, for all x_2 in $Z(\mathcal{G})$, where x_{01} is in $[\mathcal{G}, \mathcal{G}]$, λ_i is in \mathbb{R} , for all $i = 1, 2, \dots, p$.

By identifying $\text{End}(Z(\mathcal{G}))$, $L(Z(\mathcal{G})^*, Z(\mathcal{G}))$ and $L(Z(\mathcal{G}), Z(\mathcal{G})^*)$ to $\text{End}(\mathbb{R}^k)$, we get

$$H^1(\mathcal{D}, \mathcal{D}) = (\text{End}(\mathbb{R}^k))^4 \oplus \mathbb{R}^p. \quad (2.64)$$

2.5 Some Possible Applications and Open Problems

Given two left or right invariant structures of the same ‘nature’ (e.g. affine, symplectic, complex, Riemannian or pseudo-Riemannian, etc) on T^*G , one wonders whether they are equivalent, *i.e.* if there exists an automorphism of T^*G mapping one to the other. By taking the values of those structures at the unit of T^*G , the problem translates to finding an automorphism of Lie algebra mapping two structures of \mathcal{D} . The work within this chapter may also be seen as a useful tool for the study of such structures. Here are some examples of problems and framework for further extension and applications of this work. For more discussions about structures and problems on T^*G , one can have a look at [5], [28], [32], [34], [47], [56].

2.5.1 Some Examples

Here, we apply the above results to produce the following examples.

Example 2.5.1 (The Affine Lie Algebra of the Real Line). The 2-dimensional affine Lie algebra $\mathcal{G} = \text{aff}(\mathbb{R})$ is solvable nonnilpotent with Lie bracket

$$[e_1, e_2] = e_2,$$

in some basis (e_1, e_2) . The Lie algebra $\mathcal{D} = T^*\mathcal{G}$ of the cotangent bundle of any Lie group with Lie algebra \mathcal{G} , has a basis (e_1, e_2, e_3, e_4) with Lie bracket

$$[e_1, e_2] = e_2, \quad [e_1, e_4] = -e_4, \quad [e_2, e_4] = e_3,$$

where $e_3 := e_1^*$ and $e_4 := e_2^*$. This is the semi-direct product $\mathbb{R}e_1 \ltimes \mathcal{H}_3$ of the Heisenberg Lie algebra $\mathcal{H}_3 = \text{span}(e_2, e_3, e_4)$ with the line $\mathbb{R}e_1$, where e_1 acts on \mathcal{H}_3 by the restriction of the derivation of ad_{e_1} . The Lie algebra $\text{der}(\mathcal{D})$ has a basis $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$ where

$$\begin{aligned} \phi_1(e_1) &= e_2 & \phi_1(e_4) &= -e_3 & \phi_2(e_2) &= e_2, \\ \phi_2(e_4) &= -e_4 & \phi_3(e_1) &= e_3 & \phi_4(e_1) &= -e_4, \\ \phi_4(e_2) &= e_3 & \phi_5(e_3) &= e_3 & \phi_5(e_4) &= e_4, \end{aligned}$$

the remaining vectors $\phi_i(e_j)$ being zero, so that the Lie brackets are

$$[\phi_2, \phi_1] = \phi_1, \quad [\phi_2, \phi_3] = \phi_3, \quad [\phi_5, \phi_3] = \phi_3, \quad [\phi_5, \phi_4] = \phi_4.$$

This is the semi-direct product $\mathbb{R}^2 \ltimes \mathbb{R}^3$ of the Abelian Lie algebras $\mathbb{R}^3 = \text{span}_{\mathbb{R}}(\phi_1, \phi_3, \phi_4)$ and $\mathbb{R}^2 = \text{span}_{\mathbb{R}}(\phi_2, \phi_5)$. It has a contact structure (this is the Lie algebra number 18, for $p = q = 1$, in the list of Section 5.2).

Example 2.5.2 (The Lie Algebra of the Group $SO(3)$ of Rotations). Consider the Lie algebra $\mathfrak{so}(3) = \text{span}(e_1, e_2, e_3)$ with

$$[e_1, e_2] = -e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1.$$

The Lie algebra $\mathcal{D} = T^*\mathcal{G}$ has $(e_1, e_2, e_3, e_4, e_5, e_6)$ with Lie bracket

$$\begin{aligned} [e_1, e_5] &= -e_6 & [e_1, e_6] &= e_5 & [e_2, e_4] &= e_6, \\ [e_2, e_6] &= -e_4 & [e_3, e_4] &= -e_5 & [e_3, e_5] &= e_4, \end{aligned}$$

where $e_{3+i} = e_i^*$, $i = 1, 2, 3$. The Lie algebra $\text{der}(\mathcal{D})$ is spanned by the elements $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7$, where

$$\begin{aligned} \phi_1(e_1) &= -e_2 & \phi_1(e_2) &= e_1 & \phi_1(e_4) &= -e_5, \\ \phi_1(e_5) &= e_4 & \phi_2(e_1) &= -e_3 & \phi_2(e_3) &= e_1, \\ \phi_2(e_4) &= -e_6 & \phi_2(e_6) &= e_4 & \phi_3(e_2) &= -e_3, \\ \phi_3(e_3) &= e_2 & \phi_3(e_5) &= -e_6 & \phi_3(e_6) &= e_5, \\ \phi_4(e_1) &= -e_5 & \phi_4(e_2) &= e_4 & \phi_5(e_1) &= -e_6, \\ \phi_5(e_3) &= e_4 & \phi_6(e_4) &= e_4 & \phi_6(e_5) &= e_5, \\ \phi_6(e_6) &= e_6 & \phi_7(e_2) &= -e_6 & \phi_7(e_3) &= e_5, \end{aligned}$$

so that we have the following Lie brackets

$$\begin{aligned} [\phi_1, \phi_2] &= -\phi_3 & [\phi_1, \phi_3] &= \phi_2 & [\phi_1, \phi_5] &= -\phi_7 & [\phi_1, \phi_7] &= \phi_5, \\ [\phi_2, \phi_3] &= -\phi_1 & [\phi_2, \phi_4] &= \phi_7 & [\phi_2, \phi_7] &= -\phi_4 & [\phi_3, \phi_4] &= -\phi_5, \\ [\phi_3, \phi_5] &= \phi_4 & [\phi_4, \phi_6] &= -\phi_4 & [\phi_5, \phi_6] &= -\phi_5 & [\phi_6, \phi_7] &= \phi_7. \end{aligned} \quad (2.65)$$

This is the Lie algebra $\text{der}(\mathcal{D}) = \mathfrak{so}(3) \ltimes \mathcal{G}_{id}$, where $\mathfrak{so}(3) = \text{span}(\phi_1, \phi_2, \phi_3)$ and \mathcal{G}_{id} is the semi-direct product $\mathcal{G}_{id} = \mathbb{R}\phi_6 \ltimes \mathbb{R}^3$ of the abelian Lie algebras $\mathbb{R}^3 = \text{span}(\phi_4, \phi_5, \phi_7)$ and $\mathbb{R}\phi_6$ obtained by letting ϕ_6 acts as the identity map on \mathbb{R}^3 . Thus $\text{der}(\mathcal{D})$ is also a contact Lie algebra, as it is the Lie algebra number 4 of Section 5.3 in [26].

Example 2.5.3 (The Lie Algebra of the Group $SL(2, \mathbb{R})$ of Spacial Linear Group). The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ of $SL(2, \mathbb{R})$ has a basis (e_1, e_2, e_3) in which its Lie bracket reads

$$[e_1, e_2] = -2e_2 \quad , \quad [e_1, e_3] = 2e_3 \quad , \quad [e_2, e_3] = -e_1 \quad (2.66)$$

Set $e_1^* =: e_4, e_2^* =: e_5, e_3^* = e_6$, the Lie bracket of $\mathcal{D} := T^*\mathfrak{g}$ in the basis $(e_1, e_2, e_3, e_4, e_5, e_6)$ is given by

$$\begin{aligned} [e_1, e_2] &= -2e_2 \quad , \quad [e_1, e_3] = 2e_3 \quad , \quad [e_2, e_3] = -e_1, \\ [e_1, e_5] &= 2e_5 \quad , \quad [e_1, e_6] = -2e_6 \quad , \quad [e_2, e_4] = e_6, \\ [e_2, e_5] &= -2e_4 \quad , \quad [e_3, e_4] = -e_5 \quad , \quad [e_3, e_6] = 2e_4. \end{aligned} \quad (2.67)$$

The Lie algebra $\text{der}(\mathcal{D})$ is 7-dimensional. It has a basis $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7)$, where

$$\begin{aligned} \phi_1 &:= -e_{22} + e_{33} + e_{55} - e_{66} \quad , \quad \phi_2 := e_{44} + e_{55} + e_{66}, \\ \phi_3 &:= -e_{12} + 2e_{31} - 2e_{46} + e_{54} \quad , \quad \phi_4 := e_{13} - 2e_{21} + 2e_{45} - e_{64}, \\ \phi_5 &:= -e_{53} + e_{62} \quad , \quad \phi_6 := e_{42} - e_{51}, \\ \phi_7 &:= e_{43} - e_{61} \end{aligned} \quad (2.68)$$

Hence, the Lie bracket of $\text{der}(\mathcal{D})$ reads

$$\begin{aligned} [\phi_1, \phi_3] &= \phi_3 \quad , \quad [\phi_1, \phi_4] = -\phi_4 \quad , \quad [\phi_1, \phi_6] = \phi_6, \\ [\phi_1, \phi_7] &= -\phi_7 \quad , \quad [\phi_2, \phi_5] = \phi_5 \quad , \quad [\phi_2, \phi_6] = \phi_6, \\ [\phi_2, \phi_7] &= \phi_7 \quad , \quad [\phi_3, \phi_4] = 2\phi_1 \quad , \quad [\phi_3, \phi_5] = -2\phi_6, \\ [\phi_3, \phi_7] &= -\phi_5 \quad , \quad [\phi_4, \phi_5] = -2\phi_7 \quad , \quad [\phi_4, \phi_6] = -\phi_5. \end{aligned} \quad (2.69)$$

One realizes that $\text{der}(\mathcal{D}) = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{g}_{id}$, where $\mathfrak{sl}(2, \mathbb{R}) = \text{span}(\phi_1, \phi_3, \phi_4)$ and as above, \mathfrak{g}_{id} is the semi-direct product $\mathfrak{g}_{id} = \mathbb{R}\phi_2 \ltimes \mathbb{R}^3$ of the Abelian Lie algebras $\mathbb{R}^3 = \text{span}(\phi_5, \phi_6, \phi_7)$ and $\mathbb{R}\phi_2$ obtained by letting ϕ_2 act as the identity map on \mathbb{R}^3 . Again, $\text{der}(\mathcal{D})$ is also a contact Lie algebra, with $\eta := s\phi_1^* + t\phi_5^*$ as an example of a contact form, where $s, t \in \mathbb{R} - \{0\}$.

On the Lie algebra $\mathcal{D} = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{sl}(2, \mathbb{R})^*$, consider the following two forms

$$\varphi_1((x, f), (y, g)) = f(y) + g(y) \quad (2.70)$$

$$\varphi_2((x, f), (y, g)) = f(y) + g(y) + K(x, y) \quad (2.71)$$

where K stands for the Killing form of $\mathfrak{sl}(2, \mathbb{R})$. The matrix of φ_1 has two eigenvalues, -1 and $+1$, both of multiplicity 3. Therefore φ_1 is of signature $(3, 3)$. Now the matrix of φ_2 has four eigenvalues : $4 - \sqrt{17}$, $-2 - \sqrt{5}$, $2 - \sqrt{5}$, $2 + \sqrt{5}$, $-2 + \sqrt{5}$, $4 + \sqrt{17}$. The three first eigenvalues are less than zero and the three last ones are positive. Hence the signature of φ_2 is $(3, 3)$, too. It is now straightforward to check that

$$\text{der}(\mathcal{D}) = ad_{\mathcal{D}} \oplus \mathbb{R}\phi_2 \quad (2.72)$$

where $ad_{\mathcal{D}} := \text{span}(\phi_1, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7)$ is the space of inner d erivations of \mathcal{D} , ϕ_2 being, up to multiplication by a scalar, the unique exterior derivation of \mathcal{D} .

Let us look at the restrictions of φ_1 and φ_2 to the Levi subalgebra $\mathfrak{sl}(2, \mathbb{R})$ of \mathcal{D} . Since the restriction of φ_1 to $sl(2, \mathbb{R})$ is degenerate, so is the restriction of φ_1 to the

image $\exp(\phi_i)(\mathfrak{sl}(2, \mathbb{R}))$ of $\mathfrak{sl}(2, \mathbb{R})$ under the standard exponential map of any of the inner derivations ϕ_i , $i = 1, 3, 4, 5, 6, 7$, of \mathcal{D} . That is because special automorphisms $\exp(\phi_i)$, $i = 1, 3, 4, 5, 6, 7$, preserve the Levi subalgebra $\mathfrak{sl}(2, \mathbb{R})$ of \mathcal{D} . Suppose that there exists a special automorphism $\exp(\phi_i)$, $i = 1, 3, 4, 5, 6, 7$, mapping φ_1 to φ_2 , *i.e.*

$$\varphi_1\left(\exp(\phi_i)(x, f), \exp(\phi_i)(y, g)\right) = \varphi_2\left((x, f), (y, g)\right) \tag{2.73}$$

for all x, y in $\mathfrak{sl}(2, \mathbb{R})$ and all f, g in $\mathfrak{sl}(2, \mathbb{R})^*$. But the restriction of φ_1 to $\exp(\phi_i)(\mathfrak{sl}(2, \mathbb{R}))$ is degenerate while the restriction of φ_2 to $\mathfrak{sl}(2, \mathbb{R})$ is equal to the Killing form K of $\mathfrak{sl}(2, \mathbb{R})$, which is not degenerate. Hence, φ_1 and φ_2 are not homothetic via special automorphisms of \mathcal{D} .

Now, what about $\exp(\phi_2)$? One has $\exp(\phi_2) = e_{11} + e_{22} + e_{33} + e\phi_2$, where $e := \exp(1)$. For any elements (x, f) and (y, g) of \mathcal{D} , we have,

$$\begin{aligned} \varphi_2\left(\exp(\phi_2)(x, f), \exp(\phi_2)(y, g)\right) &= \varphi_2\left((x, ef), (y, eg)\right) \\ &= e\varphi_1\left((x, f), (y, g)\right) + K(x, y) \\ &\neq \varphi_1\left((x, f), (y, g)\right) \end{aligned} \tag{2.74}$$

Hence, the automorphism $\exp(\phi_2)$ too do not map φ_2 to φ_1 .

2.5.2 Invariant Riemannian or Pseudo-Riemannian Metrics

As discussed in Section 2.2.1, the Lie group T^*G possesses bi-invariant pseudo-Riemannian metrics.

Among others, one of the open problems in [60], is the question as to whether, given two bi-invariant pseudo-Riemannian metrics μ_1 and μ_2 on a Lie group \tilde{G} , the two pseudo-Riemannian manifolds (\tilde{G}, μ_1) and (\tilde{G}, μ_2) are homothetic via an automorphism of \tilde{G} . If this is the case, we say that μ_1 and μ_2 are isomorphic or equivalent.

When $\tilde{G} = T^*G$, the question as to how many non-isomorphic bi-invariant pseudo-Riemannian metrics can exist on T^*G is still open, in the general case. For example, suppose G itself has a bi-invariant Riemannian or pseudo-Riemannian metric μ and let μ stand again for the corresponding adjoint-invariant metric in the Lie algebra \mathfrak{G} of G . Then μ induces a new adjoint-invariant metric $\langle \cdot, \cdot \rangle_\mu$ on $\mathcal{D} = Lie(T^*G)$, with

$$\langle (x, f), (y, g) \rangle_\mu := \langle (x, f), (y, g) \rangle + \mu(x, y), \tag{2.75}$$

for all x, y in \mathfrak{G} and all f, g in \mathfrak{G}^* , where $\langle \cdot, \cdot \rangle$ on the right hand side is the duality pairing $\langle (x, f), (y, g) \rangle = f(y) + g(x)$. In some cases (see Section 2.5.3), the two metrics can even happen to have the same index, but are still not isomorphic via an automorphism of \tilde{G} . If $\tilde{\mu}$ is a bilinear symmetric form on \mathfrak{G}^* satisfying $\tilde{\mu}(ad_x^*f, g) = 0$, for every x in \mathfrak{G} and every f, g in \mathfrak{G}^* , then

$$\langle (x, f), (y, g) \rangle_{\tilde{\mu}} := \langle (x, f), (y, g) \rangle + \tilde{\mu}(f, g), \tag{2.76}$$

for all x, y in \mathfrak{G} , f, g in \mathfrak{G}^* , is also another adjoint-invariant metric on \mathcal{D} . Now in some cases, nonzero such bilinear forms $\tilde{\mu}$ may not exist, as is the case when one of the coadjoint orbits

of G , is an open subset of \mathcal{G}^* , or equivalently, when G has a left invariant exact symplectic structure.

More generally, these equivalence questions can also be simply extended to all left (resp. right) invariant Riemannian or pseudo-Riemannian structures on cotangent bundles of Lie groups.

2.5.3 Poisson-Lie Structures, Double Lie Algebras, Applications

The classification questions of double Lie algebras, Manin pairs, Lagrangian subalgebras, ... arising from Poisson-Lie groups, are still open problems [32], [55]. In [32], Lagrangian subalgebras of double Lie algebras are used as the main tool for classifying the so-called Poisson Homogeneous spaces of Poisson-Lie groups. A type of local action of those Lagrangian subalgebras is also used to describe symplectic foliations of Poisson Homogeneous spaces of Poisson-Lie groups in [32], [27].

It would be interesting to extend the results within this chapter to double Lie algebras of general Poisson-Lie groups. It is hard to get a common substantial description valid for the group of automorphism of the double Lie algebras of all possible Poisson-Lie structures in a given Lie group. This is due to the diversity of Poisson-Lie structures that can coexist in the same Lie group.

Among other things, the description of the group of automorphisms of the double Lie algebra of a Poisson-Lie structure is a big step forward towards solving very interesting and hard problems such as:

- the classification of Manin triples ([55]);
- the classification of Poisson homogeneous spaces of a Lie groups ([27], [55]);
- a full description and understanding of the foliations of Poisson homogeneous spaces of Poisson Lie groups. The leaves of such foliations trap the trajectories, under a Hamiltonian flow, passing through all its points. Hence, from their knowledge, one gets a great deal of information on Hamiltonian systems.
- etc.

2.5.4 Affine and Complex Structures on T^*G

In certain cases, T^*G possesses left invariant affine connections, that is, left invariant zero curvature and torsion free linear connections. Here, the classification problem involves $\text{Aut}(T^*G)$ as follows. The group $\text{Aut}(T^*G)$ acts on the space of left invariant affine connections on T^*G , the orbit of each connection being the set of equivalent (isomorphic) ones. Recall that among other results in [28], the authors proved that when G has an invertible solution of the Classical Yang-Baxter Equation (or equivalently a left invariant symplectic structure), then T^*G has a left invariant affine connection ∇ and a complex structure J such that $\nabla J = 0$.

PREDERIVATIONS OF LIE ALGEBRAS OF COTANGENT BUNDLES OF LIE GROUPS

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3.1 Introduction

In the sense of Felix Klein ([43]), studying the geometry of a "universe" is studying its invariant structures under the action of a suitable Lie group. In semi-Riemannian geometry, one of the suitable group used in this task is the group of isometries of pseudo-Riemannian metrics. So it seems reasonable to well known isometries of pseudo-Riemannian metrics. Among tools used, for instance in the case of bi-invariant (or orthogonal) Lie groups, there are prederivations of Lie algebras. Müller ([64]) gives an algebraic description of the group $I(G, \mu)$ of isometries of a connected orthogonal Lie group (G, μ) . He proves that if (G, μ) is a connected and simply-connected orthogonal Lie group with Lie algebra \mathcal{G} , then the stabilizer of the identity element of G in $I(G, \mu)$ is isomorphic to the group of preautomorphisms of \mathcal{G} which preserve the non-degenerate bilinear form induced by μ on \mathcal{G} and whose Lie algebra is the whole set of skew-symmetric prederivations of \mathcal{G} . In [9], Bajo studies the algebra of prederivations and skew-symmetric prederivations of a direct sum of Lie algebras and this study allows him to generalize some results in [6], [7] and in [72].

Prederivations also present an interest in the purely algebraic point of view. As well as Jacobson ([40]) proves that a Lie algebra admitting a non-singular derivation is necessarily nilpotent, the author quoted above establishes in [8] that a Lie algebra possessing a non-singular prederivation is necessarily nilpotent. Prederivations are also useful tools for construction of affine structures on Lie algebras (see Section 3.5.2).

In this chapter, we deal with prederivations of the Lie algebra of the cotangent bundle of a Lie group; which Lie algebra appears as the semi-direct sum $\mathcal{G} \ltimes \mathcal{A}$ of a Lie algebra \mathcal{G} and an Abelian Lie algebra \mathcal{A} . In the sequel, we will take \mathcal{G} to be the Lie algebra of a Lie group G , $\mathcal{A} = \mathcal{G}^*$, the dual space of \mathcal{G} and $T^*\mathcal{G} := \mathcal{G} \ltimes \mathcal{G}^*$ will be the semi-direct sum of the Lie algebra \mathcal{G} and the vector space \mathcal{G}^* via the coadjoint representation.

Our aim is to explore the structure of the Lie algebra $\text{Pder}(T^*\mathcal{G})$ of prederivations of $T^*\mathcal{G}$. We will have a particular attention to Lie algebras admitting quadratic or orthogonal structures.

The main results within this chapter are the following.

Theorem A. *A prederivation $p : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ is defined by :*

$$p(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + \xi(f) \right)$$

for any element (x, f) of $T^*\mathcal{G}$, where $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is a prederivation of \mathcal{G} and $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$, $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ and $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ are linear maps satisfying the following four relations :

$$\beta([x, [y, z]]) = -ad_{[y, z]}^*(\beta(x)) + ad_x^*(ad_y^*(\beta(z)) - ad_z^*(\beta(y)))$$

$$\psi \circ ad_{[x, y]}^* = ad_{[x, y]} \circ \psi$$

$$ad_x^*(ad_{\psi(f)}^*g - ad_{\psi(g)}^*f) = 0,$$

$$[\xi, ad_{[x, y]}^*] = ad_{([\alpha(x), y] + [x, \alpha(y)])}^*$$

for every elements x and y of \mathcal{G} and any elements f and g in \mathcal{G}^* .

About the structure of the Lie algebra of the Lie algebra $\text{Pder}(T^*\mathcal{G})$ of prederivations of $T^*\mathcal{G}$, we have the

Theorem B. *Let G be any finite-dimensional Lie group with Lie algebra \mathcal{G} . Then the Lie algebra $\text{Pder}(T^*\mathcal{G})$ of prederivations of the Lie algebra $T^*\mathcal{G}$ of the Lie group T^*G decomposes as follows : $\text{Pder}(T^*\mathcal{G}) = \mathcal{G}'_0 \oplus \mathcal{G}'_1$, where \mathcal{G}'_0 is a reductive subalgebra of $\text{Pder}(T^*\mathcal{G})$, that is $[\mathcal{G}'_0, \mathcal{G}'_0] \subset \mathcal{G}'_0$ and $[\mathcal{G}'_0, \mathcal{G}'_1] \subset \mathcal{G}'_1$.*

Recall that the Lie algebra of the cotangent bundle Lie group of a semi-simple Lie group is not semi-simple. Any way, as well as Müller proves that any prederivation of a semi-simple Lie algebra is a derivation we prove the following

Theorem C. *Let G be a semi-simple Lie group with Lie algebra \mathcal{G} . Then every prederivation of the Lie algebra $T^*\mathcal{G}$ of the cotangent bundle Lie group T^*G of G is a derivation.*

The chapter contains five (5) sections. In Section 3.2 is explained the link between prederivations and isometries of bi-invariant metrics on Lie groups. In Section 3.3 we study

the structure of the Lie algebra $\text{Pder}(T^*\mathcal{G})$ of prederivations of the Lie algebra $T^*\mathcal{G}$ of the cotangent bundle T^*G of a Lie group G with Lie \mathcal{G} . The particular case where the Lie group G possesses a bi-invariant metric is studied in Section 3.4. In Section 3.5 we give examples and some possible applications.

3.2 Preliminaries

3.2.1 Prederivations of a Lie algebra

Definition 3.2.1. *Let \mathcal{G} be a Lie algebra. A bijective endomorphism $P : \mathcal{G} \rightarrow \mathcal{G}$ such that*

$$P([x, [y, z]]) = [P(x), [P(y), P(z)]], \quad (3.1)$$

for all x, y, z in \mathcal{G} , is called a preautomorphism of \mathcal{G} .

The set of all preautomorphisms of \mathcal{G} forms a Lie group (see [64]) which we note by $\text{Paut}(T^*\mathcal{G})$. Its Lie algebra is the subset of the set $\mathcal{G}l(\mathcal{G})$ of endomorphisms of \mathcal{G} consisting of elements p which satisfy the following relation.

$$p([x, [y, z]]) = [p(x), [y, z]] + [x, [p(y), z]] + [x, [y, p(z)]] \quad (3.2)$$

for every elements x, y, z of \mathcal{G} .

One can easily convince himself that any derivation of a Lie algebra is a prederivation. Furthermore, there exists a class of Lie algebras that are such that any prederivation is a derivation. Semi-simple Lie algebras belong to that class of algebras (see [64]). We will prove in Section 3.4.2 that the Lie algebras (which are not semi-simple) of the cotangent Lie groups of semi-simple Lie groups are also members of this class.

As well as Jacobson ([40]) proves that a Lie algebra admitting a non-singular derivation is necessarily nilpotent, Bajo ([8]) shows that a Lie algebra which possesses a non-singular prederivation is necessarily nilpotent. The converses of the both results are false since there are nilpotent Lie algebra that admit only singular derivations and prederivations (see [8]).

3.2.2 Prederivations and Isometries of Orthogonal Structures

Müller ([64]) proves that if (G, μ) is a connected and simply-connected orthogonal Lie group with Lie algebra \mathcal{G} , then the isotropy group of the neutral element of G in the group $I(G, \mu)$ of isometries of (G, μ) is isomorphic to the subgroup of $GL(\mathcal{G})$ (group of endomorphisms of \mathcal{G}) consisting of preautomorphisms of \mathcal{G} which preserve the non-degenerate bilinear form induced by μ on \mathcal{G} and whose Lie algebra is the whole set of skew-symmetric prederivations of \mathcal{G} . See Section 4.2.3 for wider informations.

3.3 Preautomorphisms of $T^*\mathcal{G}$

3.3.1 Prederivations of $T^*\mathcal{G}$

Let \mathcal{G} be a Lie algebra and let $T^*\mathcal{G} := \mathcal{G} \ltimes \mathcal{G}^*$ stand for the semi-direct product of \mathcal{G} with its dual via the coadjoint representation. The Lie bracket of $T^*\mathcal{G}$ is given by

$$[(x, f), (y, g)] := ([x, y], ad_x^*g - ad_y^*f), \quad (3.3)$$

for any elements (x, f) and (y, g) of $T^*\mathcal{G}$.

Let $p : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ be a prederivation of $T^*\mathcal{G}$ and set

$$p(x, f) = (\alpha(x) + \psi(f), \beta(x) + \xi(f)), \quad (3.4)$$

where $\alpha : \mathcal{G} \rightarrow \mathcal{G}$, $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$, $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ and $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ are linear maps.

Let x, y, z be elements of \mathcal{G} . We have :

$$p([x, [y, z]]) = \underbrace{\alpha([x, [y, z]])}_{\in \mathcal{G}} + \underbrace{\beta([x, [y, z]])}_{\in \mathcal{G}^*} \quad (3.5)$$

On the other way, since p is a prederivation, then

$$\begin{aligned} p([x, [y, z]]) &= [p(x), [y, z]] + [x, [p(y), z]] + [x, [y, p(z)]] \\ &= [\alpha(x) + \beta(x), [y, z]] + [x, [\alpha(y) + \beta(y), z]] + [x, [y, \alpha(z) + \beta(z)]] \\ &= \left([\alpha(x), [y, z]] - ad_{[y, z]}^*(\beta(x)) \right) + [x, [\alpha(y), z] - ad_z^*(\beta(y))] \\ &\quad + [x, [y, \alpha(z)] + ad_y^*(\beta(z))] \\ &= \left([\alpha(x), [y, z]] - ad_{[y, z]}^*(\beta(x)) \right) + \left([x, [\alpha(y), z]] - ad_x^* \circ ad_z^*(\beta(y)) \right) \\ &\quad + \left([x, [y, \alpha(z)]] + ad_x^* \circ ad_y^*(\beta(z)) \right) \\ &= \underbrace{\left([\alpha(x), [y, z]] + [x, [\alpha(y), z]] + [x, [y, \alpha(z)]] \right)}_{\in \mathcal{G}} \\ &\quad + \underbrace{\left(-ad_{[y, z]}^*(\beta(x)) - ad_x^* \circ ad_z^*(\beta(y)) + ad_x^* \circ ad_y^*(\beta(z)) \right)}_{\in \mathcal{G}^*}. \end{aligned} \quad (3.6)$$

From relations (3.5) and (3.6) we have :

$$\alpha([x, [y, z]]) = [\alpha(x), [y, z]] + [x, [\alpha(y), z]] + [x, [y, \alpha(z)]]. \quad (3.7)$$

for any x, y, z in \mathcal{G} ; that is α is a prederivation of \mathcal{G} . Relations (3.5) and (3.6) also give

$$\beta([x, [y, z]]) = -ad_{[y, z]}^*(\beta(x)) - ad_x^* \circ ad_z^*(\beta(y)) + ad_x^* \circ ad_y^*(\beta(z)). \quad (3.8)$$

Now let x be an element of \mathcal{G} and f, g be in \mathcal{G}^* . We have

$$p([x, [f, g]]) = p(0) = 0. \quad (3.9)$$

We also have

$$\begin{aligned} p([x, [f, g]]) &= [p(x), [f, g]] + [x, [p(f), g]] + [x, [f, p(g)]] \\ &= 0 + [x, [\psi(f) + \xi(f), g]] + [x, [f, \psi(g) + \xi(g)]] \\ &= [x, ad_{\psi(f)}^*g] + [x, -ad_{\psi(g)}^*f] \\ &= [x, ad_{\psi(f)}^*g - ad_{\psi(g)}^*f] \\ &= ad_x^*(ad_{\psi(f)}^*g - ad_{\psi(g)}^*f). \end{aligned} \quad (3.10)$$

From (3.9) and (3.10) it comes that

$$ad_x^*(ad_{\psi(f)}^*g - ad_{\psi(g)}^*f) = 0, \quad (3.11)$$

for every x in \mathcal{G} and every f, g in \mathcal{G}^* . That is $ad_{\psi(f)}^*g - ad_{\psi(g)}^*f$ belongs the centralizer $Z_{T^*\mathcal{G}}(\mathcal{G})$ of \mathcal{G} in $T^*\mathcal{G}$, for any f and g in \mathcal{G}^* .

Let us now consider the following case. Let x and y be in \mathcal{G} and f be in \mathcal{G}^* .

$$p\left([x, [y, f]]\right) = \underbrace{\psi\left([x, [y, f]]\right)}_{\in \mathcal{G}} + \underbrace{\xi\left([x, [y, f]]\right)}_{\in \mathcal{G}^*} \quad (3.12)$$

Since p is a prederivation, one has

$$\begin{aligned} p\left([x, [y, f]]\right) &= [p(x), [y, f]] + [x, [p(y), f]] + [x, [y, p(f)]] \\ &= [\alpha(x) + \beta(x), [y, f]] + [x, [\alpha(y) + \beta(y), f]] + [x, [y, \psi(f) + \xi(f)]] \\ &= [\alpha(x) + \beta(x), ad_y^*f] + [x, ad_{\alpha(y)}^*f] + [x, [y, \psi(f)] + ad_y^*(\xi(f))] \\ &= ad_{\alpha(x)}^* \circ ad_y^*(f) + ad_x^* \circ ad_{\alpha(y)}^*(f) + [x, [y, \psi(f)]] + ad_x^* \circ ad_y^*(\xi(f)) \\ &= \underbrace{[x, [y, \psi(f)]]}_{\in \mathcal{G}} + \underbrace{\left(ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* + ad_x^* \circ ad_y^* \circ \xi\right)(f)}_{\in \mathcal{G}^*} \end{aligned} \quad (3.13)$$

On one hand, relations (3.12) and (3.13) imply

$$\begin{aligned} \psi\left([x, [y, f]]\right) &= [x, [y, \psi(f)]] \\ \psi\left(ad_x^* \circ ad_y^*f\right) &= ad_x \circ ad_y \circ \psi(f), \end{aligned}$$

that is

$$\psi \circ ad_x^* \circ ad_y^* = ad_x \circ ad_y \circ \psi, \quad (3.14)$$

for any x and y in \mathcal{G} . On the second hand relations (3.12) and (3.13) give

$$\xi\left([x, [y, f]]\right) = \left(ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* + ad_x^* \circ ad_y^* \circ \xi\right)(f)$$

$$\begin{aligned}\xi \circ ad_x^* \circ ad_y^*(f) &= \left(ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* + ad_x^* \circ ad_y^* \circ \xi \right)(f) \\ \left(\xi \circ ad_x^* \circ ad_y^* - ad_x^* \circ ad_y^* \circ \xi \right)(f) &= \left(ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* \right)(f)\end{aligned}$$

It comes that

$$[\xi, ad_x^* \circ ad_y^*] = ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* \quad (3.15)$$

Let us summarize in the

Theorem 3.3.1. *A prederivation $p : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ is defined by :*

$$p(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + \xi(f) \right) \quad (3.16)$$

for any element (x, f) of $T^*\mathcal{G}$, where

- $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is a prederivation of \mathcal{G} and
- $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$, $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ and $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ are linear maps satisfying the following four relations :

$$\beta([x, [y, z]]) = -ad_{[y, z]}^*(\beta(x)) + ad_x^*(ad_y^*(\beta(z)) - ad_z^*(\beta(y))) \quad (3.17)$$

$$\psi \circ ad_x^* \circ ad_y^* = ad_x \circ ad_y \circ \psi, \quad (3.18)$$

$$ad_x^*(ad_{\psi(f)}^*g - ad_{\psi(g)}^*f) = 0, \quad (3.19)$$

$$[\xi, ad_x^* \circ ad_y^*] = ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^*, \quad (3.20)$$

for every elements x and y of \mathcal{G} and any elements f and g in \mathcal{G}^* .

3.3.2 A Structure theorem for the Lie group $\text{Paut}(T^*\mathcal{G})$

Let us introduce the following notations :

1. $\text{Pder}(T^*\mathcal{G})$ stands for the space of prederivations of $T^*\mathcal{G}$;
2. $\text{Pder}(\mathcal{G})$ represents for the space of prederivations of \mathcal{G} ;
3. \mathcal{Q}' is the space of linear maps $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ satisfying relation (3.17) ;
4. \mathcal{E}' is the space of linear maps $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ such that

$$[\xi, ad_x^* \circ ad_y^*] = ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^*,$$

for some prederivation α of \mathcal{G} and any elements x and y of \mathcal{G} .

5. Ψ' stands for the space of linear maps $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ satisfying (3.18) and (3.19).
6. \mathcal{G}'_0 stands for the space of maps $\phi_{\alpha, \xi} : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$, $(x, f) \mapsto (\alpha(x), \xi(f))$, where α is in $\text{Pder}(\mathcal{G})$, ξ in \mathcal{E}' and $[\xi, ad_x^* \circ ad_y^*] = ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^*$, for all x, y in \mathcal{G} ;
7. $\mathcal{G}'_1 := \Psi' \oplus \mathcal{Q}'$ (direct sum of vector spaces).

Lemma 3.3.1. *The space \mathcal{E}' is a Lie algebra. Precisely, if ξ_1, ξ_2 in \mathcal{E}' satisfy*

$$\begin{aligned} [\xi_1, ad_x^* \circ ad_y^*] &= ad_{\alpha_1(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha_1(y)}^* \\ [\xi_2, ad_x^* \circ ad_y^*] &= ad_{\alpha_2(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha_2(y)}^* \end{aligned}$$

for all x, y in \mathcal{G} and some α_1, α_2 in $Pder(\mathcal{G})$, then $[\xi_1, \xi_2]$ belongs to \mathcal{E}' and satisfies

$$[[\xi_1, \xi_2], ad_x^* \circ ad_y^*] = ad_{[\alpha_1, \alpha_2](x)}^* \circ ad_y^* + ad_x^* \circ ad_{[\alpha_1, \alpha_2](y)}^*,$$

for all elements x, y of \mathcal{G} .

Proof. Consider ξ_1 and ξ_2 as in Lemma 3.3.1. We have,

$$\begin{aligned} [[\xi_1, \xi_2], ad_x^* \circ ad_y^*] &= [\xi_1 \circ \xi_2 - \xi_2 \circ \xi_1, ad_x^* \circ ad_y^*] \\ &= [\xi_1 \circ \xi_2, ad_x^* \circ ad_y^*] - [\xi_2 \circ \xi_1, ad_x^* \circ ad_y^*] \end{aligned} \quad (3.21)$$

$$\begin{aligned} [\xi_1 \circ \xi_2, ad_x^* \circ ad_y^*] &= (\xi_1 \circ \xi_2) \circ ad_x^* \circ ad_y^* - ad_x^* \circ ad_y^* \circ (\xi_1 \circ \xi_2) \\ &= \xi_1 \left(\xi_2 \circ ad_x^* \circ ad_y^* \right) - \left(ad_x^* \circ ad_y^* \circ \xi_1 \right) \circ \xi_2 \\ &= \left(\xi_1 \circ ad_{\alpha_2(x)}^* \circ ad_y^* + \xi_1 \circ ad_x^* \circ ad_{\alpha_2(y)}^* + \xi_1 \circ ad_x^* \circ ad_y^* \circ \xi_2 \right) \\ &\quad - \left(\xi_1 \circ ad_x^* \circ ad_y^* - ad_{\alpha_1(x)}^* \circ ad_y^* - ad_x^* \circ ad_{\alpha_1(y)}^* \right) \circ \xi_2 \end{aligned} \quad (3.22)$$

Hence,

$$\begin{aligned} [\xi_1 \circ \xi_2, ad_x^* \circ ad_y^*] &= \xi_1 \circ ad_{\alpha_2(x)}^* \circ ad_y^* + \xi_1 \circ ad_x^* \circ ad_{\alpha_2(y)}^* \\ &\quad + ad_{\alpha_1(x)}^* \circ ad_y^* \circ \xi_2 + ad_x^* \circ ad_{\alpha_1(y)}^* \circ \xi_2 \end{aligned} \quad (3.23)$$

We also have :

$$\begin{aligned} [\xi_2 \circ \xi_1, ad_x^* \circ ad_y^*] &= \xi_2 \circ ad_{\alpha_1(x)}^* \circ ad_y^* + \xi_2 \circ ad_x^* \circ ad_{\alpha_1(y)}^* \\ &\quad + ad_{\alpha_2(x)}^* \circ ad_y^* \circ \xi_1 + ad_x^* \circ ad_{\alpha_2(y)}^* \circ \xi_1 \end{aligned} \quad (3.24)$$

Now we have

$$\begin{aligned} [[\xi_1, \xi_2], ad_x^* \circ ad_y^*] &= \left(\xi_1 \circ ad_{\alpha_2(x)}^* \circ ad_y^* + \xi_1 \circ ad_x^* \circ ad_{\alpha_2(y)}^* \right. \\ &\quad \left. + ad_{\alpha_1(x)}^* \circ ad_y^* \circ \xi_2 + ad_x^* \circ ad_{\alpha_1(y)}^* \circ \xi_2 \right) \\ &\quad - \left(\xi_2 \circ ad_{\alpha_1(x)}^* \circ ad_y^* + \xi_2 \circ ad_x^* \circ ad_{\alpha_1(y)}^* \right. \\ &\quad \left. + ad_{\alpha_2(x)}^* \circ ad_y^* \circ \xi_1 + ad_x^* \circ ad_{\alpha_2(y)}^* \circ \xi_1 \right) \\ &= \left(\xi_1 \circ ad_{\alpha_2(x)}^* \circ ad_y^* - ad_{\alpha_2(x)}^* \circ ad_y^* \circ \xi_1 \right) \\ &\quad + \left(\xi_1 \circ ad_x^* \circ ad_{\alpha_2(y)}^* - ad_x^* \circ ad_{\alpha_2(y)}^* \circ \xi_1 \right) \\ &\quad + \left(ad_{\alpha_1(x)}^* \circ ad_y^* \circ \xi_2 - \xi_2 \circ ad_{\alpha_1(x)}^* \circ ad_y^* \right) \\ &\quad + \left(ad_x^* \circ ad_{\alpha_1(y)}^* \circ \xi_2 - \xi_2 \circ ad_x^* \circ ad_{\alpha_1(y)}^* \right) \end{aligned}$$

$$\begin{aligned}
 &= [\xi_1, ad_{\alpha_2(x)}^* \circ ad_y^*] + [\xi_1, ad_x^* \circ ad_{\alpha_2(y)}^*] \\
 &\quad - [\xi_2, ad_{\alpha_1(x)}^* \circ ad_y^*] - [\xi_2, ad_x^* \circ ad_{\alpha_1(y)}^*] \\
 &= \left(ad_{\alpha_1 \circ \alpha_2(x)}^* \circ ad_y^* + ad_{\alpha_2(x)}^* \circ ad_{\alpha_1(y)}^* \right) \\
 &\quad + \left(ad_{\alpha_1(x)}^* \circ ad_{\alpha_2(y)}^* + ad_x^* \circ ad_{\alpha_1 \circ \alpha_2(y)}^* \right) \\
 &\quad - \left(ad_{\alpha_2 \circ \alpha_1(x)}^* \circ ad_y^* + ad_{\alpha_1(x)}^* \circ ad_{\alpha_2(y)}^* \right) \\
 &\quad - \left(ad_{\alpha_2(x)}^* \circ ad_{\alpha_1(y)}^* + ad_x^* \circ ad_{\alpha_2 \circ \alpha_1(y)}^* \right).
 \end{aligned}$$

We then have

$$\left[[\xi_1, \xi_2], ad_x^* \circ ad_y^* \right] = ad_{[\alpha_1, \alpha_2](x)}^* \circ ad_y^* + ad_x^* \circ ad_{[\alpha_1, \alpha_2](y)}^*.$$

□

Lemma 3.3.2. *The space \mathcal{G}'_0 is a Lie subalgebra of the Lie algebra $Pder(T^*\mathcal{G})$.*

Proof. Let ϕ_{α_1, ξ_1} and ϕ_{α_2, ξ_2} be two elements of \mathcal{G}'_0 .

$$\begin{aligned}
 \left(\phi_{\alpha_1, \xi_1} \circ \phi_{\alpha_2, \xi_2} \right)(x, f) &= \phi_{\alpha_1, \xi_1} \left(\alpha_2(x), \xi_2(f) \right) \\
 &= \left(\alpha_1 \circ \alpha_2(x), \xi_1 \circ \xi_2(f) \right)
 \end{aligned}$$

We then have

$$\left(\phi_{\alpha_1, \xi_1} \circ \phi_{\alpha_2, \xi_2} \right) = \left(\alpha_1 \circ \alpha_2, \xi_1 \circ \xi_2 \right).$$

By the same way, we have

$$\left(\phi_{\alpha_2, \xi_2} \circ \phi_{\alpha_1, \xi_1} \right) = \left(\alpha_2 \circ \alpha_1, \xi_2 \circ \xi_1 \right).$$

Hence,

$$[\phi_{\alpha_1, \xi_1}, \phi_{\alpha_2, \xi_2}] = \left([\alpha_1, \alpha_2], [\xi_1, \xi_2] \right) \in \mathcal{G}'_0.$$

□

Lemma 3.3.3. *$[\mathcal{G}'_0, \Psi'] \subset \Psi'$ and $[\mathcal{G}'_0, \mathcal{Q}'] \subset \mathcal{Q}'$. Hence, $[\mathcal{G}'_0, \mathcal{G}'_1] \subset \mathcal{G}'_1$.*

Proof. Let $\phi_{\alpha, \xi}$ and $\phi_{\psi, 0}$ be elements of \mathcal{G}'_0 and Ψ' respectively. We have

$$\begin{aligned}
 \phi_{\alpha, \xi} \circ \phi_{\psi, 0}(x, f) &= \phi_{\alpha, \xi} \left(\psi(f), 0 \right) \\
 &= \left(\alpha \circ \psi(f), 0 \right).
 \end{aligned}$$

$$\begin{aligned}
 \phi_{\psi, 0} \circ \phi_{\alpha, \xi}(x, f) &= \phi_{\psi, 0} \left(\alpha(x), \xi(f) \right) \\
 &= \left(\psi \circ \xi(f), 0 \right).
 \end{aligned}$$

Then,

$$[\phi_{\alpha,\xi}, \phi_{\psi,\beta}](x, f) = \left((\alpha \circ \psi - \psi \circ \xi)(f), 0 \right).$$

Let us show that $(\alpha \circ \psi - \psi \circ \xi)$ belongs to Ψ' . For any elements x and y in \mathcal{G} , we have

$$\begin{aligned} (\alpha \circ \psi) \circ ad_x^* \circ ad_y^* &= \alpha \circ (\psi \circ ad_x^* \circ ad_y^*) \\ &= \alpha \circ (ad_x \circ ad_y \circ \psi) \\ &= (\alpha \circ ad_x \circ ad_y) \circ \psi. \end{aligned}$$

Let z be an element of \mathcal{G} . We have

$$\begin{aligned} (\alpha \circ ad_x \circ ad_y)(z) &= \alpha([x, [y, z]]) \\ &= [\alpha(x), [y, z]] + [x, [\alpha(y), z]] + [x, [y, \alpha(z)]] \\ &= \left(ad_{\alpha(x)} \circ ad_y + ad_x \circ ad_{\alpha(y)} + ad_x \circ ad_y \circ \alpha \right)(z). \end{aligned}$$

Then

$$\alpha \circ ad_x \circ ad_y = ad_{\alpha(x)} \circ ad_y + ad_x \circ ad_{\alpha(y)} + ad_x \circ ad_y \circ \alpha.$$

It comes that :

$$(\alpha \circ \psi) \circ ad_x^* \circ ad_y^* = ad_{\alpha(x)} \circ ad_y \circ \psi + ad_x \circ ad_{\alpha(y)} \circ \psi + ad_x \circ ad_y \circ \alpha \circ \psi. \quad (3.25)$$

Now, what about $(\psi \circ \xi) \circ ad_x^* \circ ad_y^*$?

$$\begin{aligned} (\psi \circ \xi) \circ ad_x^* \circ ad_y^* &= \psi \circ (\xi \circ ad_x^* \circ ad_y^*) \\ &= \psi \circ \left([\xi, ad_x^* \circ ad_y^*] + ad_x^* \circ ad_y^* \circ \xi \right) \\ &= \psi \circ \left(ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* + ad_x^* \circ ad_y^* \circ \xi \right). \end{aligned}$$

Hence,

$$(\psi \circ \xi) \circ ad_x^* \circ ad_y^* = \psi \circ ad_{\alpha(x)}^* \circ ad_y^* + \psi \circ ad_x^* \circ ad_{\alpha(y)}^* + \psi \circ ad_x^* \circ ad_y^* \circ \xi \quad (3.26)$$

From (3.25) and (3.26), we have :

$$\begin{aligned} (\alpha \circ \psi - \psi \circ \xi) \circ ad_x^* \circ ad_y^* &= (ad_{\alpha(x)} \circ ad_y \circ \psi + ad_x \circ ad_{\alpha(y)} \circ \psi + ad_x \circ ad_y \circ \alpha \circ \psi) \\ &\quad - (\psi \circ ad_{\alpha(x)}^* \circ ad_y^* + \psi \circ ad_x^* \circ ad_{\alpha(y)}^* + \psi \circ ad_x^* \circ ad_y^* \circ \xi). \end{aligned}$$

It comes that $(\alpha \circ \psi - \psi \circ \xi)$ satisfies (3.18) as it verifies

$$(\alpha \circ \psi - \psi \circ \xi) \circ ad_x^* \circ ad_y^* = ad_x \circ ad_y \circ (\alpha \circ \psi - \psi \circ \xi).$$

Let now f and g be elements of \mathcal{G}^* .

$$\begin{aligned} \Gamma &:= ad_{(\alpha \circ \psi - \psi \circ \xi)(f)}^* g - ad_{(\alpha \circ \psi - \psi \circ \xi)(g)}^* f \\ &= ad_{\alpha \circ \psi(f)}^* g - ad_{\psi \circ \xi(f)}^* g - ad_{\alpha \circ \psi(g)}^* f + ad_{\psi \circ \xi(g)}^* f \end{aligned}$$

For any x in \mathcal{G} , one has the following

$$\begin{aligned} ad_x^* \circ ad_{\alpha \circ \psi(f)}^* &= [\xi, ad_x^* \circ ad_{\psi(f)}^*] - ad_{\alpha(x)}^* \circ ad_{\psi(f)}^* \\ ad_x^* \circ ad_{\alpha \circ \psi(g)}^* &= [\xi, ad_x^* \circ ad_{\psi(g)}^*] - ad_{\alpha(x)}^* \circ ad_{\psi(g)}^* \end{aligned}$$

Then

$$\begin{aligned}
 ad_x^* \circ \Gamma &= ad_x^* \circ \left(ad_{(\alpha \circ \psi - \psi \circ \xi)(f)}^*(g) - ad_{(\alpha \circ \psi - \psi \circ \xi)(g)}^*(f) \right) \\
 &= \xi \circ ad_x^* \circ ad_{\psi(f)}^*(g) - ad_x^* \circ ad_{\psi(f)}^* \circ \xi(g) - ad_{\alpha(x)}^* \circ ad_{\psi(f)}^*(g) \\
 &\quad - \xi \circ ad_x^* \circ ad_{\psi(g)}^*(f) + ad_x^* \circ ad_{\psi(g)}^* \circ \xi(f) + ad_{\alpha(x)}^* \circ ad_{\psi(g)}^*(f) \\
 &\quad - ad_x^* \circ ad_{\psi \circ \xi(f)}^*(g) + ad_x^* \circ ad_{\psi \circ \xi(g)}^*(f) \\
 &= \underbrace{\xi \left(ad_x^* \circ ad_{\psi(f)}^*(g) - ad_x^* \circ ad_{\psi(g)}^*(f) \right)}_{=0, \text{ because of (3.19)}} + \underbrace{ad_{\alpha(x)}^* \left(ad_{\psi(g)}^*(f) - ad_{\psi(f)}^*(g) \right)}_{=0, \text{ because of (3.19)}} \\
 &\quad + \underbrace{ad_x^* \circ ad_{\psi \circ \xi(g)}^*(f) - ad_x^* \circ ad_{\psi(\xi(g))}^*(f)}_{=0} + \underbrace{ad_x^* \circ ad_{\psi(\xi(f))}^*(g) - ad_x^* \circ ad_{\psi \circ \xi(f)}^*(g)}_{=0}.
 \end{aligned}$$

Hence,

$$ad_x^* \left(ad_{(\alpha \circ \psi - \psi \circ \xi)(f)}^*(g) - ad_{(\alpha \circ \psi - \psi \circ \xi)(g)}^*(f) \right) = 0, \quad (3.27)$$

for any f, g in \mathcal{G}^* and any x in \mathcal{G} . We then have shown that $(\alpha \circ \psi - \psi \circ \xi)$ belongs to Ψ' . Hence, $[\mathcal{G}'_0, \Psi'] \subset \Psi'$.

Now we are going to show that $[\mathcal{G}'_0, \mathcal{Q}'] \subset \mathcal{Q}'$. For this goal, let $\phi_{\alpha, \xi}$ and $\phi_{0, \beta}$ be elements of \mathcal{G}'_0 and \mathcal{Q}' respectively.

$$\begin{aligned}
 \phi_{\alpha, \xi} \circ \phi_{0, \beta}(x, f) &= \phi_{\alpha, \xi}(0, \beta(x)) \\
 &= (0, \xi \circ \beta(x))
 \end{aligned} \quad (3.28)$$

$$\begin{aligned}
 \phi_{0, \beta} \circ \phi_{\alpha, \xi}(x, f) &= \phi_{0, \beta}(\alpha(x), \xi(f)) \\
 &= (0, \beta \circ \alpha(x))
 \end{aligned} \quad (3.29)$$

Then,

$$[\phi_{\alpha, \xi}, \phi_{0, \beta}](x, f) = (0, (\xi \circ \beta - \beta \circ \alpha)(x)) \quad (3.30)$$

Does $(\xi \circ \beta - \beta \circ \alpha)$ satisfies relation (3.17) ?

$$\begin{aligned}
 (\xi \circ \beta)[x, [y, z]] &= \xi \left(-ad_{[y, z]}^*(\beta(x)) - ad_x^* \circ ad_z^*(\beta(y)) + ad_x^* \circ ad_y^*(\beta(z)) \right) \\
 &= -\xi \circ ad_{[y, z]}^*(\beta(x)) - \xi \circ ad_x^* \circ ad_z^*(\beta(y)) + \xi \circ ad_x^* \circ ad_y^*(\beta(z)) \\
 &= -\xi \circ ad_y^* \circ ad_z^*(\beta(x)) + \xi \circ ad_z^* \circ ad_y^*(\beta(x)) \\
 &\quad - \xi \circ ad_x^* \circ ad_z^*(\beta(y)) + \xi \circ ad_x^* \circ ad_y^*(\beta(z)) \\
 &= -[\xi, ad_y^* \circ ad_z^*](\beta(x)) - ad_y^* \circ ad_z^* \circ \xi(\beta(x)) \\
 &\quad + [\xi, ad_z^* \circ ad_y^*](\beta(x)) + ad_z^* \circ ad_y^* \circ \xi(\beta(x)) \\
 &\quad - [\xi, ad_x^* \circ ad_z^*](\beta(y)) - ad_x^* \circ ad_z^* \circ \xi(\beta(y)) \\
 &\quad + [\xi, ad_x^* \circ ad_y^*](\beta(z)) + ad_x^* \circ ad_y^* \circ \xi(\beta(z)) \\
 &= -ad_{\alpha(y)}^* \circ ad_z^*(\beta(x)) - ad_y^* \circ ad_{\alpha(z)}^*(\beta(x)) - ad_y^* \circ ad_z^* \circ \xi(\beta(x)) \\
 &\quad + ad_{\alpha(z)}^* \circ ad_y^*(\beta(x)) + ad_z^* \circ ad_{\alpha(y)}^*(\beta(x)) + ad_z^* \circ ad_y^* \circ \xi(\beta(x)) \\
 &\quad - ad_{\alpha(x)}^* \circ ad_z^*(\beta(y)) - ad_x^* \circ ad_{\alpha(z)}^*(\beta(y)) - ad_x^* \circ ad_z^* \circ \xi(\beta(y)) \\
 &\quad + ad_{\alpha(x)}^* \circ ad_y^*(\beta(z)) + ad_x^* \circ ad_{\alpha(y)}^*(\beta(z)) + ad_x^* \circ ad_y^* \circ \xi(\beta(z)) \quad (3.31)
 \end{aligned}$$

$$\begin{aligned}
 (\beta \circ \alpha)[x, [y, z]] &= \beta\left([\alpha(x), [y, z]] + [x, [\alpha(y), z]] + [x, [y, \alpha(z)]]\right) \\
 &= -ad_{[y,z]}^*(\beta(\alpha(x))) - ad_{\alpha(x)}^* \circ ad_z^*(\beta(y)) + ad_{\alpha(x)}^* \circ ad_y^*(\beta(z)) \\
 &\quad - ad_{[\alpha(y),z]}^*(\beta(x)) - ad_x^* \circ ad_z^*(\beta(\alpha(y))) + ad_x^* \circ ad_{\alpha(y)}^*(\beta(z)) \\
 &\quad - ad_{[y,\alpha(z)]}^*(\beta(x)) - ad_x^* \circ ad_{\alpha(z)}^*(\beta(y)) + ad_x^* \circ ad_y^*(\beta(\alpha(z))) \quad (3.32)
 \end{aligned}$$

We then have

$$\begin{aligned}
 (\xi \circ \beta - \beta \circ \alpha)[x, [y, z]] &= -ad_{[y,z]}^*\left((\xi \circ \beta)(x)\right) - ad_{[\alpha(y),z]}^*(\beta(x)) - ad_{[y,\alpha(z)]}^*(\beta(x)) \\
 &\quad - ad_x^* \circ ad_z^*\left((\xi \circ \beta)(y)\right) + ad_x^* \circ ad_y^*\left((\xi \circ \beta)(z)\right) \\
 &\quad + ad_{[y,z]}^*\left((\beta \circ \alpha)(x)\right) + ad_{[\alpha(y),z]}^*(\beta(x)) + ad_x^* \circ ad_z^*\left((\beta \circ \alpha)(y)\right) \\
 &\quad + ad_{[y,\alpha(z)]}^*(\beta(x)) - ad_x^* \circ ad_y^*\left((\beta \circ \alpha)(z)\right) \\
 &= -ad_{[y,z]}^*\left((\xi \circ \beta - \beta \circ \alpha)(x)\right) - ad_x^* \circ ad_z^*\left((\xi \circ \beta - \beta \circ \alpha)(y)\right) \\
 &\quad + ad_x^* \circ ad_y^*\left((\xi \circ \beta - \beta \circ \alpha)(z)\right) \quad (3.33)
 \end{aligned}$$

That is $(\xi \circ \beta - \beta \circ \alpha)$ satisfies relation (3.17) and then $[\mathcal{G}'_0, \mathcal{Q}'] \subset \mathcal{Q}'$. It is now clear that $[\mathcal{G}'_0, \mathcal{G}'_1] \subset \mathcal{G}'_1$. □

We summarize the Lemmas above in the

Theorem 3.3.2. *Let \mathcal{G} be any finite-dimensional Lie algebra. Then the Lie algebra of prederivations of $T^*\mathcal{G}$ decomposes as follows : $Pder(T^*\mathcal{G}) = \mathcal{G}'_0 \oplus \mathcal{G}'_1$, where \mathcal{G}'_0 is a reductive subalgebra of $Pder(T^*\mathcal{G})$, that is*

$$[\mathcal{G}'_0, \mathcal{G}'_0] \subset \mathcal{G}'_0 \text{ and } [\mathcal{G}'_0, \mathcal{G}'_1] \subset \mathcal{G}'_1.$$

Remark 3.3.1. *$Pder(T^*\mathcal{G})$ is not a symmetric space as is $der(T^*\mathcal{G})$ (see Theorem 2.3.2) since $[\mathcal{G}'_1, \mathcal{G}'_1]$ is not a subset of \mathcal{G}'_0 . Precisely, let ϕ_{ψ_1, β_1} and ϕ_{ψ_2, β_2} be two elements of \mathcal{G}'_1 . Then,*

1. $[\phi_{\psi_1, \beta_1}, \phi_{\psi_2, \beta_2}] = \left((\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1), (\beta_1 \circ \psi_2 - \beta_2 \circ \psi_1) \right)$;
2. $[\phi_{\psi_1, \beta_1}, \phi_{\psi_2, \beta_2}]$ do not belong to \mathcal{G}'_0 even $(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)$ is a prederivation of \mathcal{G} .
3. $(\beta_1 \circ \psi_2 - \beta_2 \circ \psi_1)$ is not linked to $(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)$ by (3.20).

Indeed, let ϕ_{ψ_1, β_1} and ϕ_{ψ_2, β_2} be two elements of \mathcal{G}'_1 .

$$\begin{aligned}
 (\phi_{\psi_1, \beta_1} \circ \phi_{\psi_2, \beta_2})(x, f) &= \phi_{\psi_1, \beta_1}\left(\psi_2(f), \beta_2(x)\right) \\
 &= \left(\psi_1 \circ \beta_2(x), \beta_1 \circ \psi_2(f)\right) \quad (3.34)
 \end{aligned}$$

By the same way

$$(\phi_{\psi_2, \beta_2} \circ \phi_{\psi_1, \beta_1})(x, f) = \left(\psi_2 \circ \beta_1(x), \beta_2 \circ \psi_1(f)\right) \quad (3.35)$$

It comes that

$$[\phi_{\psi_1, \beta_1}, \phi_{\psi_2, \beta_2}](x, f) = \left((\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(x), (\beta_1 \circ \psi_2 - \beta_2 \circ \psi_1)(f) \right) \quad (3.36)$$

Let us see if $(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)$ is a prederivation of \mathcal{G} .

$$\begin{aligned} (\psi_1 \circ \beta_2)[x, [y, z]] &= \psi_1 \left(-ad_{[y, z]}^*(\beta_2(x)) - ad_x^* \circ ad_z^*(\beta_2(y)) + ad_x^* \circ ad_y^*(\beta_2(z)) \right) \\ &= -\psi_1 \circ ad_{[y, z]}^*(\beta_2(x)) - \psi_1 \circ ad_x^* \circ ad_z^*(\beta_2(y)) + \psi_1 \circ ad_x^* \circ ad_y^*(\beta_2(z)) \\ &= -ad_y \circ ad_z \circ \psi_1(\beta_2(x)) + ad_z \circ ad_y \circ \psi_1(\beta_2(x)) \\ &\quad - ad_x \circ ad_z \circ \psi_1(\beta_2(y)) + ad_x \circ ad_y \circ \psi_1(\beta_2(z)) \\ &= -[y, [z, \psi_1 \circ \beta_2(x)]] + [z, [y, \psi_1 \circ \beta_2(x)]] \\ &\quad - [x, [z, \psi_1 \circ \beta_2(y)]] + [x, [y, \psi_1 \circ \beta_2(z)]] \end{aligned} \quad (3.37)$$

We also have :

$$\begin{aligned} (\psi_2 \circ \beta_1)[x, [y, z]] &= -[y, [z, \psi_2 \circ \beta_1(x)]] + [z, [y, \psi_2 \circ \beta_1(x)]] \\ &\quad - [x, [z, \psi_2 \circ \beta_1(y)]] + [x, [y, \psi_2 \circ \beta_1(z)]] \end{aligned} \quad (3.38)$$

Now we have :

$$\begin{aligned} (\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)[x, [y, z]] &= -[y, [z, \psi_1 \circ \beta_2(x)]] + [z, [y, \psi_1 \circ \beta_2(x)]] \\ &\quad - [x, [z, \psi_1 \circ \beta_2(y)]] + [x, [y, \psi_1 \circ \beta_2(z)]] \\ &\quad + [y, [z, \psi_2 \circ \beta_1(x)]] - [z, [y, \psi_2 \circ \beta_1(x)]] \\ &\quad + [x, [z, \psi_2 \circ \beta_1(y)]] - [x, [y, \psi_2 \circ \beta_1(z)]] \\ &= -[y, [z, (\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(x)]] + [z, [y, (\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(x)]] \\ &\quad - [x, [z, (\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(y)]] + [x, [y, (\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(z)]] \\ &= [(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(x), [y, z]] + [x, [(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(y), z]] \\ &\quad + [x, [y, (\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(z)]] \end{aligned} \quad (3.39)$$

Then $(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)$ is a prederivation of \mathcal{G} .

We are now going to verify if $(\beta_1 \circ \psi_2 - \beta_2 \circ \psi_1)$ satisfies relation (3.20). Let x and y be two elements of \mathcal{G} and f be in \mathcal{G}^* .

$$\begin{aligned} [\beta_1 \circ \psi_2 - \beta_2 \circ \psi_1, ad_x^* \circ ad_y^*](f) &= [\beta_1 \circ \psi_2, ad_x^* \circ ad_y^*](f) - [\beta_2 \circ \psi_1, ad_x^* \circ ad_y^*](f) \\ &= \left(\beta_1 \circ \psi_2 \circ ad_x^* \circ ad_y^* - ad_x^* \circ ad_y^* \circ \beta_1 \circ \psi_2 \right)(f) \\ &\quad - \left(\beta_1 \circ \psi_2 \circ ad_x^* \circ ad_y^* - ad_x^* \circ ad_y^* \circ \beta_1 \circ \psi_2 \right)(f) \\ &= \left(\beta_1 \circ ad_x \circ ad_y \circ \psi_2 - ad_x^* \circ ad_y^* \circ \beta_1 \circ \psi_2 \right)(f) \\ &\quad - \left(\beta_2 \circ ad_x \circ ad_y \circ \psi_1 - ad_x^* \circ ad_y^* \circ \beta_2 \circ \psi_1 \right)(f) \end{aligned}$$

$$\begin{aligned}
 &= \beta_1([x, [y, \psi_2(f)]] - ad_x^* \circ ad_y^*(\beta_1(\psi_2(f))) \\
 &\quad - \beta_2([x, [y, \psi_1(f)]] + ad_x^* \circ ad_y^*(\beta_2(\psi_1(f)))) \\
 &= -ad_{[y, \psi_2(f)]}^*(\beta_1(x)) - ad_x^* \circ ad_{\psi_2(f)}^*(\beta_1(y)) \\
 &\quad + ad_x^* \circ ad_y^*(\beta_1(\psi_2(f))) - ad_x^* \circ ad_y^*(\beta_1(\psi_2(f))) \\
 &\quad + ad_{[y, \psi_1(f)]}^*(\beta_2(x)) + ad_x^* \circ ad_{\psi_1(f)}^*(\beta_2(y)) \\
 &\quad - ad_x^* \circ ad_y^*(\beta_2(\psi_1(f))) + ad_x^* \circ ad_y^*(\beta_2(\psi_1(f))) \\
 &= -ad_y^* \circ ad_{\psi_2(f)}^*(\beta_1(x)) + ad_{\psi_2(f)}^* \circ ad_y^*(\beta_1(x)) \\
 &\quad - ad_x^* \circ ad_{\psi_2(f)}^*(\beta_1(y)) + ad_y^* \circ ad_{\psi_1(f)}^*(\beta_2(x)) \\
 &\quad - ad_{\psi_1(f)}^* \circ ad_y^*(\beta_2(x)) + ad_x^* \circ ad_{\psi_1(f)}^*(\beta_2(y)) \\
 &= -ad_y^* \circ ad_{\psi_2(\beta_1(x))}^* f + ad_{\psi_2(f)}^* \circ ad_y^*(\beta_1(x)) \\
 &\quad - ad_x^* \circ ad_{\psi_2(\beta_1(y))}^* f + ad_y^* \circ ad_{\psi_1(\beta_2(x))}^* f \\
 &\quad - ad_{\psi_1(f)}^* \circ ad_y^*(\beta_2(x)) + ad_x^* \circ ad_{\psi_1(\beta_2(y))}^* f \quad (3.40)
 \end{aligned}$$

Hence,

$$\begin{aligned}
 [\beta_1 \circ \psi_2 - \beta_2 \circ \psi_1, ad_x^* \circ ad_y^*](f) &= ad_y^* \circ ad_{(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(x)}^* f + ad_x^* \circ ad_{(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)(y)}^* f \\
 &\quad + ad_{\psi_2(f)}^* \circ ad_y^*(\beta_1(x)) - ad_{\psi_1(f)}^* \circ ad_y^*(\beta_2(x)) \quad (3.41)
 \end{aligned}$$

The map $(\beta_1 \circ \psi_2 - \beta_2 \circ \psi_1)$ does not satisfy the relation (3.20) (with the prederivation $(\psi_1 \circ \beta_2 - \psi_2 \circ \beta_1)$) because of the term $ad_{\psi_2(f)}^* \circ ad_y^*(\beta_1(x)) - ad_{\psi_1(f)}^* \circ ad_y^*(\beta_2(x))$.

3.3.3 Maps $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$

Proposition 3.3.1. *Let \mathcal{G} be a Lie algebra and let α be a prederivation of \mathcal{G} . A linear map $\xi' : \mathcal{G} \rightarrow \mathcal{G}$ satisfies $[\xi', ad_x \circ ad_y] = -(ad_{\alpha(x)} \circ ad_y + ad_x \circ ad_{\alpha(y)})$, for any elements x and y of \mathcal{G} if and only if there exists a linear map $j : \mathcal{G} \rightarrow \mathcal{G}$ satisfying*

$$[j, ad_x \circ ad_y] = 0, \quad (3.42)$$

for any x, y and z in \mathcal{G} , such that $\xi' = j - \alpha$.

Proof. Let ξ' and α be as in the Proposition 3.3.1. For any x, y, z in \mathcal{G} , one has

$$\begin{aligned}
 \xi' \circ ad_x \circ ad_y(z) - ad_x \circ ad_y \circ \xi'(z) &= -ad_{\alpha(x)} \circ ad_y(z) - ad_x \circ ad_{\alpha(y)}(z) \\
 &= -[\alpha(x), [y, z]] - [x, [\alpha(y), z]] \\
 &= -\alpha([x, [y, z]]) + [x, [y, \alpha(z)]] \\
 &= -\alpha \circ ad_x \circ ad_y(z) + ad_x \circ ad_y \circ \alpha(z).
 \end{aligned}$$

We then have

$$[(\xi' + \alpha), ad_x \circ ad_y] = 0, \quad (3.43)$$

for any elements x, y of \mathcal{G} . To finish the proof we just take $j = \xi' + \alpha$. □

Let us note by \mathcal{J}' the space of linear maps $j : \mathcal{G} \rightarrow \mathcal{G}$ which satisfy

$$[j, ad_x \circ ad_y] = 0, \tag{3.44}$$

for every x, y, z in \mathcal{G} .

Proposition 3.3.2. *Let \mathcal{G} be a non Abelian Lie algebra with dual space \mathcal{G}^* . A linear map $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ satisfies $[\xi, ad_x^* \circ ad_y^*] = ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^*$, for some prederivation α of \mathcal{G} and every x, y in \mathcal{G} if and only if its transpose $\xi^t : \mathcal{G} \rightarrow \mathcal{G}$ is of the form $\xi^t = j - \alpha$, where j is in \mathcal{J}' .*

Proof. Consider a linear map ξ satisfying the hypothesis of Proposition 3.3.2. That is, for every x, y in \mathcal{G} , $[\xi, ad_x^* \circ ad_y^*] = ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^*$, for some α in $Pder(\mathcal{G})$. Taking transposes of the two sides one has

$$\begin{aligned} [\xi, ad_x^* \circ ad_y^*]^t &= \left(ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* \right)^t \\ - \left[\xi^t, (ad_x^* \circ ad_y^*)^t \right] &= \left(ad_{\alpha(x)}^* \circ ad_y^* \right)^t + \left(ad_x^* \circ ad_{\alpha(y)}^* \right)^t \\ - \left[\xi^t, ad_y \circ ad_x \right] &= ad_y \circ ad_{\alpha(x)} + ad_{\alpha(y)} \circ ad_x \\ \left[\xi^t, ad_y \circ ad_x \right] &= - \left(ad_y \circ ad_{\alpha(x)} + ad_{\alpha(y)} \circ ad_x \right). \end{aligned} \tag{3.45}$$

From Proposition 3.3.1 we conclude that $\xi^t = j - \alpha$, where j is in \mathcal{J}' . □

As a consequence, we have the following corollary of Theorem 3.3.1.

Corollary 3.3.1. *A prederivation $p : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ is defined by :*

$$p(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + f \circ (j - \alpha) \right) \tag{3.46}$$

for any element (x, f) of $T^*\mathcal{G}$, where $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is a prederivation of \mathcal{G} , $j : \mathcal{G} \rightarrow \mathcal{G}$ is in \mathcal{J}' , $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ and $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ are linear maps satisfying relations (3.17), (3.18) and (3.19).

3.4 Orthogonal Lie algebras

3.4.1 Maps α, β, ψ, ξ in orthogonal Lie algebras

All over this section we consider an orthogonal Lie algebra (\mathcal{G}, μ) . Let $\theta : \mathcal{G} \rightarrow \mathcal{G}^*$ still stand for the isomorphism defined by $\langle \theta(x), y \rangle := \mu(x, y)$, for all x, y in \mathcal{G} .

Lemma 3.4.1. *The map $\beta \mapsto \alpha_\beta := \theta^{-1} \circ \beta$ is an isomorphism between the space \mathcal{Q}' of linear maps β satisfying relation (3.17) and the space $Pder(\mathcal{G})$ of prederivations of \mathcal{G} .*

Proof. Let β be an element of \mathcal{Q}' . For any x, y, z in \mathcal{G} , we have

$$\begin{aligned} \alpha_\beta([x, [y, z]]) &:= \theta^{-1} \circ \beta([x, [y, z]]) \\ &= \theta^{-1} \left(- ad_{[y, z]}^*(\beta(x)) + ad_x^* \circ ad_y^*(\beta(z)) - ad_x^* \circ ad_z^*(\beta(y)) \right) \\ &= - ad_{[y, z]} \circ \theta^{-1} \circ \beta(x) + ad_x \circ ad_y \circ \theta^{-1} \circ \beta(z) - ad_x \circ ad_z \circ \theta^{-1} \circ \beta(y) \end{aligned}$$

$$= -[[y, z], \alpha_\beta(x)] + [x, [y, \alpha_\beta(z)]] - [x, [z, \alpha_\beta(z)]] \quad (3.47)$$

Then α_β belongs to $\text{Pder}(\mathcal{G})$. Conversely, let α be a prederivation of \mathcal{G} and set $\beta_\alpha := \theta \circ \alpha$. For any x, y, z in \mathcal{G} one has

$$\begin{aligned} \beta_\alpha([x, [y, z]]) &:= \theta\left([\alpha(x), [y, z]] + [x, [\alpha(y), z]] + [x, [y, \alpha(z)]]\right) \\ &= -\theta \circ \text{ad}_{[x,y]}(\alpha(x)) + \theta \circ \text{ad}_x([\alpha(y), z]) + \theta \circ \text{ad}_x([y, \alpha(z)]) \\ &= -\text{ad}_{[y,z]}^*(\theta \circ \alpha)(x) + \text{ad}_x^* \circ \theta(-\text{ad}_z(\alpha(y))) + \text{ad}_x^* \circ \theta \circ \text{ad}_y(\alpha(z)) \\ &= -\text{ad}_{[y,z]}^* \beta_\alpha(x) - \text{ad}_x^* \circ \text{ad}_y^* \beta_\alpha(y) + \text{ad}_x^* \circ \text{ad}_y^* \beta_\alpha(z) \end{aligned}$$

Then β_α is an element of \mathcal{Q}' . This correspondence is obviously bijective. \square

Now we are going to look at maps ψ 's in the case where \mathcal{G} is an orthogonal Lie algebra.

Lemma 3.4.2. *The map $\psi \mapsto j_\psi := \psi \circ \theta$ is an isomorphism between the space of linear maps $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ which satisfy relations (3.18) and the space \mathcal{J}' .*

Proof. Take ψ as in the Lemma 3.4.2. Then for any elements x, y, z in \mathcal{G} , we have

$$\begin{aligned} [\psi \circ \theta, \text{ad}_x \circ \text{ad}_y] &= \psi \circ \theta \circ \text{ad}_x \circ \text{ad}_y - \text{ad}_x \circ \text{ad}_y \circ \psi \circ \theta \\ &= \psi \circ \text{ad}_x^* \circ \text{ad}_y^* \circ \theta - \psi \circ \text{ad}_x^* \circ \text{ad}_y^* \circ \theta \\ &= 0 \end{aligned}$$

Then $j_\psi := \psi \circ \theta$ is an element of \mathcal{J}' .

Now consider an element j of \mathcal{J}' and set $\psi_j := j \circ \theta^{-1}$. Taking two arbitrary elements x and y of \mathcal{G} , we have

$$\begin{aligned} \psi_j \circ \text{ad}_x^* \circ \text{ad}_y^* &= j \circ \theta^{-1} \circ \text{ad}_x^* \circ \text{ad}_y^* \\ &= j \circ \text{ad}_x \circ \text{ad}_y \circ \theta^{-1} \\ &= \text{ad}_x \circ \text{ad}_y \circ j \circ \psi \\ &= \text{ad}_x \circ \text{ad}_y \circ \psi_j \end{aligned} \quad (3.48)$$

Then ψ_j satisfies relation (3.18). This correspondence is linear and invertible. \square

Lemma 3.4.3. *The maps $\xi \mapsto \theta^{-1} \circ \xi \circ \theta$ is an isomorphism of Lie algebras between the Lie algebra \mathcal{E}' of linear maps $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ satisfying $[\xi, \text{ad}_x^* \circ \text{ad}_y^*] = \text{ad}_{\alpha(x)}^* \circ \text{ad}_y^* + \text{ad}_x^* \circ \text{ad}_{\alpha(y)}^*$, for some prederivation α of \mathcal{G} and any x and y in \mathcal{G} , and the Lie algebra \mathcal{S}' of linear maps $\xi' : \mathcal{G} \rightarrow \mathcal{G}$ such that $[\xi', \text{ad}_x \circ \text{ad}_y] = \text{ad}_{\alpha(x)} \circ \text{ad}_y + \text{ad}_x \circ \text{ad}_{\alpha(y)}$, for some prederivation α of \mathcal{G} and every elements x, y of \mathcal{G} .*

Proof. Consider an element ξ of \mathcal{E}' and set $\delta_\xi := \theta^{-1} \circ \xi \circ \theta$. For x, y, z in \mathcal{G} we have

$$\begin{aligned} [\delta_\xi, \text{ad}_x \circ \text{ad}_y] &= \delta_\xi \circ \text{ad}_x \circ \text{ad}_y - \text{ad}_x \circ \text{ad}_y \circ \delta_\xi \\ &= \theta^{-1} \circ \xi \circ \theta \circ \text{ad}_x \circ \text{ad}_y - \text{ad}_x \circ \text{ad}_y \circ \theta^{-1} \circ \xi \circ \theta \\ &= \theta^{-1} \circ \xi \circ \text{ad}_x^* \circ \text{ad}_y^* \circ \theta - \theta^{-1} \circ \text{ad}_x^* \circ \text{ad}_y^* \circ \xi \circ \theta \\ &= \theta^{-1} \circ (\xi \circ \text{ad}_x^* \circ \text{ad}_y^* - \text{ad}_x^* \circ \text{ad}_y^* \circ \xi) \circ \theta \\ &= \theta^{-1} \circ [\xi, \text{ad}_x^* \circ \text{ad}_y^*] \circ \theta \\ &= \theta^{-1} \circ \left(\text{ad}_{\alpha(x)}^* \circ \text{ad}_y^* + \text{ad}_x^* \circ \text{ad}_{\alpha(y)}^* \right) \circ \theta, \text{ for some } \alpha \in \text{Pder}(\mathcal{G}) \end{aligned}$$

$$\begin{aligned}
 &= \left(ad_{\alpha(x)} \circ ad_y + ad_x \circ ad_{\alpha(y)} \right) \circ \theta^{-1} \circ \theta \\
 &= ad_{\alpha(x)} \circ ad_y + ad_x \circ ad_{\alpha(y)}.
 \end{aligned} \tag{3.49}$$

Then δ_ξ is an element of \mathcal{S}' .

Now consider an element ξ' of \mathcal{S}' with α as corresponding prederivation of \mathcal{G} . Then $\xi := \theta \circ \xi' \circ \theta^{-1}$ is an element of \mathcal{E}' as one can see in the following. For any elements x, y in \mathcal{G} we have

$$\begin{aligned}
 [\xi, ad_x^* \circ ad_y^*] &= \xi \circ ad_x^* \circ ad_y^* - ad_x^* \circ ad_y^* \circ \xi \\
 &= \theta \circ \xi' \circ \theta^{-1} \circ ad_x^* \circ ad_y^* - ad_x^* \circ ad_y^* \circ \theta \circ \xi' \circ \theta^{-1} \\
 &= \theta \circ \xi' \circ ad_x \circ ad_y \circ \theta^{-1} - \theta \circ ad_x \circ ad_y \circ \xi' \circ \theta^{-1} \\
 &= \theta \circ (\xi' \circ ad_x \circ ad_y - ad_x \circ ad_y \circ \xi') \circ \theta^{-1} \\
 &= \theta \circ [\xi', ad_x \circ ad_y] \circ \theta^{-1} \\
 &= \theta \circ (ad_{\alpha(x)} \circ ad_y + ad_x \circ ad_{\alpha(y)}) \circ \theta^{-1} \\
 &= \left(ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^* \right) \circ \theta \circ \theta^{-1} \\
 &= ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^*.
 \end{aligned}$$

It comes that $\xi := \theta \circ \xi' \circ \theta^{-1}$ belongs to \mathcal{E}' .

Now note by $\delta : \mathcal{E}' \rightarrow \mathcal{S}'$, $\xi \mapsto \delta_\xi := \theta^{-1} \circ \xi \circ \theta$. For any ξ_1 and ξ_2 in \mathcal{E}' , we have :

$$\begin{aligned}
 [\delta_{\xi_1}, \delta_{\xi_2}] &= \delta_{\xi_1} \circ \delta_{\xi_2} - \delta_{\xi_2} \circ \delta_{\xi_1} \\
 &= \theta^{-1} \circ \xi_1 \circ \theta \circ \theta^{-1} \circ \xi_2 \circ \theta - \theta^{-1} \circ \xi_2 \circ \theta \circ \theta^{-1} \circ \xi_1 \circ \theta \\
 &= \theta^{-1} \circ \xi_1 \circ \xi_2 \circ \theta - \theta^{-1} \circ \xi_2 \circ \xi_1 \circ \theta \\
 &= \theta^{-1} \circ [\xi_1, \xi_2] \circ \theta \\
 &= \delta_{[\xi_1, \xi_2]}
 \end{aligned}$$

Thus, the map $\xi \mapsto \theta^{-1} \circ \xi \circ \theta$ is an isomorphism of Lie algebras. \square

Now we have the following result which states that $\text{Pder}(T^*\mathcal{G})$ is completely determined by $\text{Pder}(\mathcal{G})$ and \mathcal{J}' .

Proposition 3.4.1. *Let (\mathcal{G}, μ) be an orthogonal Lie algebra and \mathcal{G}^* its dual space. Consider the isomorphism $\theta : \mathcal{G} \rightarrow \mathcal{G}^*$ defined by $\langle \theta(x), y \rangle := \mu(x, y)$. Any prederivation ϕ of $T^*\mathcal{G}$ has the following form*

$$\phi(x, f) = \left(\alpha_1(x) + j_1 \circ \theta^{-1}(f), \theta \circ \alpha_2(x) + (j_2^t - \alpha_1^t)(f) \right), \tag{3.50}$$

for every (x, f) in $T^*\mathcal{G}$, where

- α_1, α_2 are prederivations of \mathcal{G} ,
- j_1, j_2 are in \mathcal{J}' , with $ad_x^*(ad_{j_1 \circ \theta^{-1}(f)}^* g - ad_{j_1 \circ \theta^{-1}(g)}^* f) = 0$, for all x in \mathcal{G} ; f, g in \mathcal{G}^* ,
- j_2^t and α_1^t are the transposes of j_2 and α_1 respectively.

Remark 3.4.1. 1. Recall relation (3.18) :

$$\psi \circ ad_x^* \circ ad_y^* = ad_x \circ ad_y \circ \psi,$$

for all x and y in \mathcal{G} . We can also write

$$\psi \circ ad_y^* \circ ad_x^* = ad_y \circ ad_x \circ \psi,$$

for all x and y in \mathcal{G} . Subtracting the above two relations we have, for all elements x, y of \mathcal{G} ,

$$\psi \circ [ad_x^*, ad_y^*] = [ad_x, ad_y] \circ \psi$$

which can be written

$$\psi \circ ad_{[x,y]}^* = ad_{[x,y]} \circ \psi.$$

Hence, if ψ satisfies relation (3.18) then

$$\psi \circ ad_{[x,y]}^* = ad_{[x,y]} \circ \psi, \tag{3.51}$$

for any x and y in \mathcal{G} .

2. By the same way, we recall relation (3.20) :

$$[\xi, ad_x^* \circ ad_y^*] = ad_{\alpha(x)}^* \circ ad_y^* + ad_x^* \circ ad_{\alpha(y)}^*$$

for any x and y in \mathcal{G} . Changing the roles played by x and y we obtain

$$[\xi, ad_y^* \circ ad_x^*] = ad_{\alpha(y)}^* \circ ad_x^* + ad_y^* \circ ad_{\alpha(x)}^*.$$

Subtracting again the two last relations above, we have

$$\begin{aligned} \left[\xi, [ad_x^*, ad_y^*] \right] &= [ad_{\alpha(x)}^*, ad_y^*] + [ad_x^*, ad_{\alpha(y)}^*] \\ \left[\xi, ad_{[x,y]}^* \right] &= ad_{[\alpha(x),y]}^* + ad_{[x,\alpha(y)]}^*. \end{aligned}$$

That is

$$\left[\xi, ad_{[x,y]}^* \right] = ad_{([\alpha(x),y]+[x,\alpha(y)])}^*, \tag{3.52}$$

for all x and y in \mathcal{G} .

3.4.2 Semi-simple Lie algebras

A semi-simple Lie algebra is an orthogonal Lie algebra. Moreover, if \mathcal{G} is semi-simple, then any prederivation is a derivation and hence an inner derivation ([64]). In this case, we can write relation (3.52) as follows

$$\left[\xi, ad_{[x,y]}^* \right] = ad_{\alpha([x,y])}^* \tag{3.53}$$

for all x, y in \mathcal{G} . Since $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, we can simply write

$$[\xi, ad_x^*] = ad_{\alpha(x)}^* \tag{3.54}$$

for any x of \mathcal{G} . The following lemma is an immediate consequence of Section 2.4.

Lemma 3.4.4. *If \mathcal{G} is a semi-simple Lie algebra, then \mathcal{E}' is nothing but the space of linear maps $\xi' : \mathcal{G}^* \rightarrow \mathcal{G}^*$ such that*

$$[\xi, ad_x^*] = ad_{\alpha(x)}^*,$$

for some derivation α of \mathcal{G} and any x in \mathcal{G} . Furthermore, if \mathcal{G} decomposes into simple ideal as follows $\mathcal{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \dots \oplus \mathfrak{s}_p$ ($p \in \mathbb{N}$), then

$$\xi = ad_{x_0}^* + \bigoplus_{i=1}^p \lambda_i id_{\mathfrak{s}_i^*}, \tag{3.55}$$

for some x_0 in \mathcal{G} and some real numbers $\lambda_1, \lambda_2, \dots, \lambda_p$.

Now we work on the maps β and the relation (3.17). The maps $\beta \mapsto \alpha_\beta := \theta^{-1} \circ \beta$ defined in Lemma 3.4.1 is then an isomorphism between the spaces $\text{der}(\mathcal{G})$ and \mathcal{Q}' , since $\text{Pder}(\mathcal{G}) = \text{der}(\mathcal{G})$. From Proposition 2.4.3 it comes that if α_β is a derivation then $\beta = \theta \circ \alpha_\beta$ is a 1-cocycle of \mathcal{G} with values in \mathcal{G}^* for the coadjoint representation of \mathcal{G} on \mathcal{G}^* . We then proved the following lemma.

Lemma 3.4.5. *If \mathcal{G} is a semi-simple Lie algebra, then any element β of \mathcal{Q}' is a 1-cocycle of \mathcal{G} with values in \mathcal{G}^* for the coadjoint representation of \mathcal{G} on \mathcal{G}^* .*

Lemma 3.4.6. *If \mathcal{G} is a semi-simple Lie algebra, then $\Psi' = \{0\}$.*

Proof. Relation (3.19) means that the linear form $ad_{\psi(f)}^*g - ad_{\psi(g)}^*f$ is closed, for any linear forms f and g on \mathcal{G} . Since \mathcal{G} is semi-simple, it is perfect ; that is \mathcal{G} is equal to its derived ideal ($[\mathcal{G}, \mathcal{G}] = \mathcal{G}$). It is known that any closed form on a perfect Lie algebra is zero. Then, the closed form $ad_{\psi(f)}^*g - ad_{\psi(g)}^*f$ is equal to zero for any f and g in \mathcal{G}^* , *i.e.*

$$ad_{\psi(f)}^*g - ad_{\psi(g)}^*f = 0, \tag{3.56}$$

for every f and g in \mathcal{G} . Again because $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, Relation (3.18) becomes

$$\psi \circ ad_x^* = ad_x \circ \psi \tag{3.57}$$

Now we conclude with Proposition 2.4.4. □

We summarize all the above by the following

Theorem 3.4.1. *Let G be a finite dimensional semi-simple Lie group with Lie algebra \mathcal{G} . Then every prederivation of the Lie algebra $T^*\mathcal{G}$ of the cotangent bundle Lie group T^*G of G is a derivation.*

Proof. This is direct consequence of Lemmas 3.4.4, 3.4.5, 3.4.6 and the fact that any prederivation of a semi-simple Lie algebra is a derivation. □

3.4.3 Compact Lie algebras

Lemma 3.4.7. *Let \mathfrak{G} be a compact Lie algebra. Then every prederivation α of \mathfrak{G} is of the form $\alpha = ad_{x_0} \oplus \varphi$, where x_0 belongs to the derived ideal of $[\mathfrak{G}, \mathfrak{G}]$ and φ is an endomorphism of $Z(\mathfrak{G})$. That is for any x_1 in $[\mathfrak{G}, \mathfrak{G}]$ and x_2 in $Z(\mathfrak{G})$,*

$$\alpha(x_1 + x_2) = [x_0, x_1] + \varphi(x_2), \tag{3.58}$$

where x_0 in $[\mathfrak{G}, \mathfrak{G}]$, where φ is an endomorphism of $Z(\mathfrak{G})$.

Proof. Recall that a compact Lie algebra is a Lie algebra \mathfrak{G} which decomposes into the sum $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}] \oplus Z(\mathfrak{G})$ of its derived ideal and its centre, with $[\mathfrak{G}, \mathfrak{G}]$ semi-simple and compact. A prederivation α of such a Lie algebra preserves each of the factor $[\mathfrak{G}, \mathfrak{G}]$ and $Z(\mathfrak{G})$. Then it can be written as a direct sum of a prederivation α_1 of $[\mathfrak{G}, \mathfrak{G}]$ and a prederivation α_2 of $Z(\mathfrak{G})$. Since $[\mathfrak{G}, \mathfrak{G}]$ is semi-simple, the prederivation α_1 is an inner derivation. Furthermore, the derivation α_2 of $Z(\mathfrak{G})$ is just an endomorphism of $Z(\mathfrak{G})$ because $Z(\mathfrak{G})$ is an Abelian ideal of \mathfrak{G} . Hence, we can write

$$\alpha = ad_{x_0} \oplus \varphi, \tag{3.59}$$

where x_0 is an element of the derived ideal of \mathfrak{G} and φ is an endomorphism of $Z(\mathfrak{G})$. The prederivation α acts on \mathfrak{G} as follows. If $x = x_1 + x_2$ with x_1 in $[\mathfrak{G}, \mathfrak{G}]$ and x_2 in $Z(\mathfrak{G})$,

$$\alpha(x) = \alpha(x_1 + x_2) = [x_0, x_1] + \varphi(x_2). \tag{3.60}$$

□

Lemma 3.4.8. *If \mathfrak{G} is a compact Lie algebra. Then the space Ψ' is isomorphic to the space $End(Z(\mathfrak{G}))$ of endomorphisms of the centre $Z(\mathfrak{G})$ of \mathfrak{G} .*

Proof. A compact Lie algebra admits an orthogonal structure, then from Lemma 3.4.2 the space of linear maps $\psi : \mathfrak{G}^* \rightarrow \mathfrak{G}$ which are equivariant with respect to the adjoint and coadjoint representation of $[\mathfrak{G}, \mathfrak{G}]$ on \mathfrak{G} and \mathfrak{G}^* respectively is isomorphic to the space \mathcal{J}' . The correspondence is given by $\psi = j \circ \theta^{-1}$, for some j in \mathcal{J}' . For any element x of $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}] \oplus Z(\mathfrak{G})$, we write $x = x_1 + x_2$, where x_1 is in the semi-simple ideal $[\mathfrak{G}, \mathfrak{G}]$ while x_2 belongs to the centre $Z(\mathfrak{G})$ of \mathfrak{G} . Set

$$j(x) = \underbrace{(j_{11}(x_1) + j_{21}(x_2))}_{\in [\mathfrak{G}, \mathfrak{G}]} + \underbrace{(j_{12}(x_1) + j_{22}(x_2))}_{\in Z(\mathfrak{G})}, \tag{3.61}$$

where $j_{11} : [\mathfrak{G}, \mathfrak{G}] \rightarrow [\mathfrak{G}, \mathfrak{G}]$, $j_{12} : [\mathfrak{G}, \mathfrak{G}] \rightarrow Z(\mathfrak{G})$, $j_{21} : Z(\mathfrak{G}) \rightarrow [\mathfrak{G}, \mathfrak{G}]$ and $j_{22} : Z(\mathfrak{G}) \rightarrow Z(\mathfrak{G})$ are linear maps. For any x, y, z in \mathfrak{G} , we have

$$\begin{aligned} j\left([[x, y], z]\right) &= [[x, y], j(z)] \\ j\left([[x_1, y_1], z_1]\right) &= [[x_1, y_1], j_{11}(z_1) + j_{21}(z_2)] \\ &= [[x_1, y_1], j_{11}(z_1)] + [[x_1, y_1], j_{21}(z_2)] \end{aligned} \tag{3.62}$$

If we take $z_2 = 0$, then $j\left([[x_1, y_1], z_1]\right) = [[x_1, y_1], j_{11}(z_1)]$ for all x_1, y_1, z_1 in $[\mathfrak{G}, \mathfrak{G}]$. Relation (3.62) gives $[[x_1, y_1], j_{21}(z_2)] = 0$ for all x_1, y_1 in $[\mathfrak{G}, \mathfrak{G}]$ and z_2 in $Z(\mathfrak{G})$; that is $j_{21}(z_2)$ belongs

to the centre of $[\mathfrak{G}, \mathfrak{G}]$ which is $\{0\}$ since $[\mathfrak{G}, \mathfrak{G}]$ is semi-simple. It comes that $j_{21} \equiv 0$. On the other way

$$\begin{aligned} j\left(\left[[x_1, y_1], z_1\right]\right) &= j_{11}\left(\left[[x_1, y_1], z_1\right]\right) + j_{12}\left(\left[[x_1, y_1], z_1\right]\right) \\ \underbrace{\left[[x_1, y_1], j_{11}(z_1)\right]}_{\in[\mathfrak{G}, \mathfrak{G}]} &= \underbrace{j_{11}\left(\left[[x_1, y_1], z_1\right]\right)}_{\in[\mathfrak{G}, \mathfrak{G}]} + \underbrace{j_{12}\left(\left[[x_1, y_1], z_1\right]\right)}_{\in Z(\mathfrak{G})} \end{aligned} \tag{3.63}$$

It comes that $j_{12}\left(\left[[x_1, y_1], z_1\right]\right) = 0$, for all x_1, y_1, z_1 in $[\mathfrak{G}, \mathfrak{G}]$. Then $J_{12} \equiv 0$ on the semi-simple ideal $[\mathfrak{G}, \mathfrak{G}]$. Now we have just

$$j(x) = j_{11}(x_1) + j_{22}(x_2), \tag{3.64}$$

where j_{11} is an endomorphism of $[\mathfrak{G}, \mathfrak{G}]$ satisfying

$$j_{11}\left(\left[[x_1, y_1], z_1\right]\right) = \left[[x_1, y_1], j_{11}(z_1)\right], \tag{3.65}$$

for all x_1, y_1, z_1 in $[\mathfrak{G}, \mathfrak{G}]$; and j_{22} is in $End(Z(\mathfrak{G}))$. Since $[\mathfrak{G}, \mathfrak{G}]$ is perfect then (3.65) can be written

$$j_{11}([x_1, y_1]) = [x_1, j_{11}(y_1)], \tag{3.66}$$

for all x_1, y_1 in $[\mathfrak{G}, \mathfrak{G}]$. It comes, from Corollary 2.4.4, that

$$j_{11}(x) = \sum_{i=1}^p \lambda_i x_{1i} + j_{22}(x_2), \tag{3.67}$$

where $x_1 = x_{11} + x_{12} + \dots + x_{1p}$ is the decomposition of x_1 into elements of the simple components of $[\mathfrak{G}, \mathfrak{G}]$. Now we have,

$$\begin{aligned} \psi(f) &= j \circ \theta^{-1}(f) \\ &= j\left(\sum_{k=1}^p \theta^{-1}(f_{1k}) + \theta^{-1}(f_2)\right) \\ &= \sum_{i=1}^p \lambda_i \theta^{-1}(f_{1i}) + j_{22} \circ \theta^{-1}(f_2) \end{aligned} \tag{3.68}$$

From Lemma 3.4.6 the restriction of ψ to the semi-simple ideal must be zero, then

$$\psi(f) = j_{22} \circ \theta^{-1}(f_2). \tag{3.69}$$

and we are done. □

Lemma 3.4.9. *If \mathfrak{G} is a compact Lie algebra, then the space \mathcal{Q}' is isomorphic to the space $ad_{[\mathfrak{G}, \mathfrak{G}]} \oplus End(Z(\mathfrak{G}))$, where $ad_{[\mathfrak{G}, \mathfrak{G}]}$ stands for the space of inner derivations of the derived ideal $[\mathfrak{G}, \mathfrak{G}]$ of \mathfrak{G} .*

Proof. The proof is straightforward. Lemma 3.4.1 asserts that \mathcal{Q}' is isomorphic to $Pder(\mathfrak{G})$ and Lemma 3.4.7 implies that $Pder(\mathfrak{G}) \cong ad_{[\mathfrak{G}, \mathfrak{G}]} \oplus End(Z(\mathfrak{G}))$. □

Lemma 3.4.10. *Let \mathcal{G} be a compact Lie algebra. Then any linear map $\xi : \mathcal{G}^* \rightarrow \mathcal{G}^*$ satisfying relation (3.20) can be written as*

$$\xi = \left(\bigoplus_{i=1}^p \lambda_i id_{\mathfrak{s}_i}^* + ad_{x_0}^* \right) \oplus \eta, \quad (3.70)$$

where η is an endomorphism of $Z(\mathcal{G}^*)$, x_0 is an element of $[\mathcal{G}, \mathcal{G}] = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \cdots \oplus \mathfrak{s}_p$; $\lambda_1, \lambda_2, \dots, \lambda_p$ are real numbers and p is the number of simple components of $[\mathcal{G}, \mathcal{G}]$. More precisely, if $f = f_1 + f_2$ is an element of \mathcal{G}^* with $f_1 = f_{11} + f_{12} + f_{13} + \cdots + f_{1p}$ in $[\mathcal{G}, \mathcal{G}]^* = \mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \cdots \oplus \mathfrak{s}_p^*$ and f_2 in $Z(\mathcal{G}^*)^*$, then

$$\xi(f) = \sum_{i=1}^p \lambda_i f_i + ad_{x_0}^* f_1 + \eta(f_2). \quad (3.71)$$

Proof. We have already seen that the transpose ξ^t of ξ has the form $\xi^t = j - \alpha$, where j is in \mathcal{J}' . From the proof of Lemma 3.4.8 (see Relation (3.67)) we have

$$j = \left(\bigoplus_{i=1}^p \lambda_i id_{\mathfrak{s}_i} \right) \oplus \rho_1,$$

where ρ_1 is an endomorphism of $Z(\mathcal{G})$. Hence,

$$\xi^t = \left(\bigoplus_{i=1}^p \lambda_i id_{\mathfrak{s}_i} \right) \oplus \rho_1 - (ad_{x_0} \oplus \varphi_1),$$

since a prederivation α of the compact Lie algebra \mathcal{G} is given by $\alpha = ad_{x_0} \oplus \varphi_1$, where x_0 is in $[\mathcal{G}, \mathcal{G}]$ and φ_1 is an endomorphism of $Z(\mathcal{G})$ (see Lemma 3.4.7). We simply write

$$\xi^t = \left(\bigoplus_{i=1}^p \lambda_i id_{\mathfrak{s}_i} - ad_{x_0} \right) \oplus \varphi_2,$$

where $\varphi_2 = \rho_1 + \varphi_1$ belongs to $End(Z(\mathcal{G}))$. It comes that

$$\xi = \left(\bigoplus_{i=1}^p \lambda_i id_{\mathfrak{s}_i}^* + ad_{x_0}^* \right) \oplus \eta,$$

where $\eta = \varphi_2^t$ is the transpose of φ_2 . □

The following Proposition holds and is a direct consequence of the Lemmas above.

Proposition 3.4.2. *Let \mathcal{G} be a compact Lie algebra. Let $\theta : \mathcal{G} \rightarrow \mathcal{G}^*$ be the isomorphism defined by $\langle \theta(x), y \rangle := \mu(x, y)$, where μ is an arbitrary orthogonal structure on \mathcal{G} . Then any prederivation of $T^*\mathcal{G}$ has the following form :*

$$\begin{aligned} \phi(x, f) &= \left([x_0, x_1] + \varphi_1(x_2) + \varphi_2 \circ \theta^{-1}(f_2), \theta([y_0, x_1]) + \theta \circ \varphi_3(x_2) \right. \\ &\quad \left. + \sum_{i=1}^p \lambda_i f_{1i} + ad_{x_0}^* f_1 + \eta(f_2) \right), \end{aligned} \quad (3.72)$$

where

- x_0, y_0 are in $[\mathfrak{G}, \mathfrak{G}]$;
- $\varphi_1, \varphi_2, \varphi_3$ are endomorphisms of $Z(\mathfrak{G})$;
- η is in $End(Z(\mathfrak{G})^*)$;
- $\nu_i, \lambda_i, i = 1, 2, \dots, p$ are real numbers;
- $x = x_1 + x_2 = x_{11} + x_{12} + \dots + x_{1p} + x_2 \in \mathfrak{G} = [\mathfrak{G}, \mathfrak{G}] \oplus Z(\mathfrak{G})$
- and $f = f_1 + f_2 = f_{11} + f_{12} + \dots + f_{1p} + f_2 \in \mathfrak{G}^* = [\mathfrak{G}, \mathfrak{G}]^* \oplus Z(\mathfrak{G})^*$;

p being the number of simple components of $[\mathfrak{G}, \mathfrak{G}] = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \dots \oplus \mathfrak{s}_p$.

Proof. Let $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}] \oplus Z(\mathfrak{G})$ be a compact Lie algebra. From Proposition 3.4.1 we have that any prederivation ϕ of \mathfrak{G} has the form

$$\phi(x, f) = \left(\alpha_1(x) + j_1 \circ \theta^{-1}(f), \theta \circ \alpha_2(x) + (j_2^t - \alpha_1^t)(f) \right),$$

for every (x, f) in $T^*\mathfrak{G}$, where α_1, α_2 are prederivations of \mathfrak{G} , j_1, j_2 are in \mathcal{J}' , j_2^t and α_1^t are the transposes of j_2 and α_1 respectively. Now Lemma 3.4.7 implies that $\alpha_1(x) = [x_0, x_1] + \varphi_1(x_2)$ and $\alpha_2(x) = [y_0, x_1] + \varphi_3(x_2)$, for every $x = x_1 + x_2 \in \mathfrak{G} = [\mathfrak{G}, \mathfrak{G}] \oplus Z(\mathfrak{G})$ where x_0 and y_0 are fix elements of the derived ideal $[\mathfrak{G}, \mathfrak{G}]$. From the proof of Lemma 3.4.8, we have

$$\begin{aligned} j_1(x) &= \sum_{i=1}^p \nu_i x_{1i} + \varphi_2(x_2), \\ j_2(x) &= \sum_{i=1}^p \lambda_i x_{1i} + \varphi_2'(x_2), \end{aligned}$$

where $x = x_1 + x_2 = x_{11} + x_{12} + \dots + x_{1p} + x_2 \in [\mathfrak{G}, \mathfrak{G}] \oplus Z(\mathfrak{G})$, $\lambda, \nu_i, i = 1, 2, \dots, p$ being real numbers. Then, for any $f = f_1 + f_2 = f_{11} + f_{12} + \dots + f_{1p} + f_2$ in $[\mathfrak{G}, \mathfrak{G}]^* \oplus Z(\mathfrak{G})^*$, we have

$$\begin{aligned} j_1 \circ \theta^{-1}(f) &= j_1(\theta^{-1}(f)) \\ &= \sum_{i=1}^p \nu_i \left(\theta^{-1}(f_{1i}) \right) + \varphi_2(\theta^{-1}(f_2)). \end{aligned} \tag{3.73}$$

The restriction of $\psi := j_1 \circ \theta^{-1}$ to the dual space of the semi-simple ideal $[\mathfrak{G}, \mathfrak{G}]$ must be zero (see Proposition 2.4.4). Then we have

$$j_1 \circ \theta^{-1}(f) = \varphi_2 \circ \theta^{-1}(f_2), \tag{3.74}$$

for all $f = f_1 + f_2 = f_{11} + f_{12} + \dots + f_{1p} + f_2$ in $[\mathfrak{G}, \mathfrak{G}]^* \oplus Z(\mathfrak{G})^*$.

Let us, now have a look at maps $\beta := \theta \circ \alpha_2$. We have

$$\beta(x) = \theta([y_0, x_1]) + \theta \circ \varphi_3(x_2) \tag{3.75}$$

for all $x = x_1 + x_2$ in $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}] \oplus Z(\mathfrak{G})$. □

We finish this chapter by giving some situations where prederivations are used.

3.5 Possible Applications and Examples

3.5.1 Examples

Example 3.5.1 (Affine Lie algebra of the real line). Let $\text{Aff}(\mathbb{R})$ stand for the affine Lie group of the real line and note by $\text{aff}(\mathbb{R})$ its Lie algebra. We recall (see Example 2.5.1) that $\mathcal{D}_1 := T^*\text{aff}(\mathbb{R}) = \text{span}(e_1, e_2, e_3)$ with the following brackets

$$[e_1, e_2] = e_2, \quad [e_1, e_4] = -e_4, \quad [e_2, e_4] = e_3,$$

One can readily verify that $\text{Pder}(\mathcal{D}_1) = \text{der}(\mathcal{D}_1) = \mathbb{R}^2 \ltimes \mathbb{R}^3$ (semi-direct product of the Abelian Lie algebras $\mathbb{R}^3 = \text{span}_{\mathbb{R}}(\phi_1, \phi_3, \phi_4)$ and $\mathbb{R}^2 = \text{span}_{\mathbb{R}}(\phi_2, \phi_5)$) with the following brackets (see Example 2.5.1) :

$$[\phi_2, \phi_1] = \phi_1, \quad [\phi_2, \phi_3] = \phi_3, \quad [\phi_5, \phi_3] = \phi_3, \quad [\phi_5, \phi_4] = \phi_4.$$

Example 3.5.2 (The Lie Algebra of the Group $SL(2, \mathbb{R})$ of Special Linear Group and the Lie Algebra of the Group $SO(3)$ of Rotations). The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is simple, then $\text{Pder}(T^*\mathfrak{sl}(2, \mathbb{R})) = \text{der}(T^*\mathfrak{sl}(2, \mathbb{R}))$ (See Example 2.5.3). By the same argument, $\text{Pder}(T^*\mathfrak{so}(3, \mathbb{R})) = \text{der}(T^*\mathfrak{so}(3, \mathbb{R}))$ (see Example 2.5.2).

Example 3.5.3 (The 4-dimensional Oscillator Group). In Example 1.1.3 we have define the Oscillator Lie group and its Lie algebra. The 4-dimensional oscillator algebra, is the space $\mathcal{G}_\lambda = \text{span}\{e_{-1}, e_0, e_1, \check{e}_1\}$ ($\lambda > 0$) with the following brackets :

$$[e_{-1}, e_1] = \lambda \check{e}_1 \quad ; \quad [e_{-1}, \check{e}_1] = -\lambda e_1 \quad ; \quad [e_1, \check{e}_1] = e_0. \quad (3.76)$$

Let $(e_{-1}^*, e_0^*, e_1^*, \check{e}_1^*)$ stand for the basis of \mathcal{G}_λ^* dual to $(e_{-1}, e_0, e_1, \check{e}_1)$. Then, the Lie algebra $T^*\mathcal{G}_\lambda = \text{span}(e_{-1}, e_0, e_1, \check{e}_1, e_{-1}^*, e_0^*, e_1^*, \check{e}_1^*)$ with the brackets

$$\begin{aligned} [e_{-1}, e_1] &= \lambda \check{e}_1 & ; & \quad [e_{-1}, \check{e}_1] = -\lambda e_1 & ; & \quad [e_1, \check{e}_1] = e_0 \\ [e_{-1}, e_1^*] &= \lambda \check{e}_1^* & ; & \quad [e_{-1}, \check{e}_1^*] = -\lambda e_1^* & ; & \quad [e_1, e_0^*] = -\check{e}_1^* \\ [\check{e}_1, e_1^*] &= -\lambda e_{-1}^* & ; & \quad [e_1, \check{e}_1^*] = \lambda e_{-1}^* & ; & \quad [\check{e}_1, e_0^*] = e_1^* \end{aligned} \quad (3.77)$$

Consider the form μ_λ defined on \mathcal{G}_λ by

$$\mu_\lambda(x, y) = x^{-1}y^0 + x^0y^{-1} + \frac{1}{\lambda}(x^1y^1 + \check{x}^1\check{y}^1) \quad (3.78)$$

for all $x = x^{-1}e_{-1} + x^0e_0 + x^1e_1 + \check{x}^1\check{e}_1$ and $y = y^{-1}e_{-1} + y^0e_0 + y^1e_1 + \check{y}^1\check{e}_1$. It is readily cheiked that the form (3.78) defines an orthogonal structure on \mathcal{G}_λ ([12]). The isomorphism $\theta : \mathcal{G}_\lambda \rightarrow \mathcal{G}_\lambda$ defined by $\langle \theta(x), y \rangle = \mu_\lambda(x, y)$, for all x, y in \mathcal{G}_λ is given by

$$\theta(e_{-1}) = e_0^* \quad , \quad \theta(e_0) = e_{-1}^* \quad , \quad \theta(e_1) = \frac{1}{\lambda}e_1^* \quad , \quad \theta(\check{e}_1) = \frac{1}{\lambda}\check{e}_1^*, \quad (3.79)$$

The inverse map of θ reads :

$$\theta^{-1}(e_{-1}^*) = e_0 \quad , \quad \theta^{-1}(e_0^*) = e_{-1} \quad , \quad \theta^{-1}(e_1^*) = \lambda e_1 \quad , \quad \theta^{-1}(\check{e}_1^*) = \lambda \check{e}_1. \quad (3.80)$$

Since \mathfrak{G}_λ is an orthogonal Lie algebra, any prederivation ϕ of $T^*\mathfrak{G}_\lambda$ can be written as follows.

$$\phi(x, f) = \left(\alpha_1(x) + j_1 \circ \theta^{-1}(f), \theta \circ \alpha_2(x) + (j_2^t - \alpha_1^t)(f) \right),$$

for every (x, f) in $T^*\mathfrak{G}$, where α_1, α_2 are in $\text{Pder}(\mathfrak{G}_\lambda)$ and j_1, j_2 are in \mathcal{J}' with conditions listed in Proposition 3.4.1. Now we have :

- A prederivation α of \mathfrak{G}_λ can be represented in the basis $(e_{-1}, e_0, e_1, \check{e}_1)$ by the following matrix.

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 2a_{33} & a_{23} & a_{24} \\ -\lambda a_{23} & 0 & a_{33} & a_{34} \\ -\lambda a_{24} & 0 & -a_{34} & a_{33} \end{pmatrix}, \quad (3.81)$$

where a_{ij} 's are reals numbers. We can put

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 2a_{33} & a_{23} & a_{24} \\ -\lambda a_{23} & 0 & a_{33} & a_{34} \\ -\lambda a_{24} & 0 & -a_{34} & a_{33} \end{pmatrix} ; \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{21} & 2b_{33} & b_{23} & b_{24} \\ -\lambda b_{23} & 0 & b_{33} & b_{34} \\ -\lambda b_{24} & 0 & -b_{34} & b_{33} \end{pmatrix}.$$

- A linear map $j : \mathfrak{G}_\lambda \rightarrow \mathfrak{G}_\lambda$ which satisfies (3.44) has the following form :

$$j = \begin{pmatrix} j_{11} & 0 & 0 & 0 \\ j_{21} & j_{11} & 0 & 0 \\ 0 & 0 & j_{11} & 0 \\ 0 & 0 & 0 & j_{11} \end{pmatrix}, \quad (3.82)$$

where j_{11} and j_{21} . We set

$$j_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{pmatrix} ; \quad j_2 = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1 \end{pmatrix}.$$

It follows that

$$j_1 \circ \theta^{-1} = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ 0 & 0 & \lambda a_1 & 0 \\ 0 & 0 & 0 & \lambda a_1 \end{pmatrix} ; \quad \theta \circ \alpha_2 = \begin{pmatrix} b_{21} & 2b_{33} & b_{23} & b_{24} \\ 0 & 0 & 0 & 0 \\ -b_{23} & 0 & \frac{1}{\lambda} b_{33} & \frac{1}{\lambda} b_{34} \\ -b_{24} & 0 & -\frac{1}{\lambda} b_{34} & \frac{1}{\lambda} b_{33} \end{pmatrix}.$$

If we consider the fact that $ad_x^*(ad_{j_1 \circ \theta^{-1}(f)}^* g - ad_{j_1 \circ \theta^{-1}(g)}^* f) = 0$, for all x in \mathfrak{G}_λ and any f, g in \mathfrak{G}_λ^* , we obtain

$$j_1 \circ \theta^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we write the matrix of a prederivation ϕ of $T^*\mathcal{G}_\lambda$.

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 2a_{33} & a_{23} & a_{24} & 0 & a_2 & 0 & 0 \\ -\lambda a_{23} & 0 & a_{33} & a_{34} & 0 & 0 & 0 & 0 \\ -\lambda a_{24} & 0 & -a_{34} & a_{33} & 0 & 0 & 0 & 0 \\ b_{21} & 2b_{33} & b_{23} & b_{24} & b_1 & b_2 - a_{21} & \lambda a_{23} & \lambda a_{24} \\ 0 & 0 & 0 & 0 & 0 & b_1 - 2a_{33} & 0 & 0 \\ -b_{23} & 0 & \frac{1}{\lambda}b_{33} & \frac{1}{\lambda}b_{34} & 0 & -a_{23} & b_1 - a_{33} & a_{34} \\ -b_{24} & 0 & -\frac{1}{\lambda}b_{34} & \frac{1}{\lambda}b_{33} & 0 & -a_{24} & -a_{34} & b_1 - a_{33} \end{pmatrix} \quad (3.83)$$

It comes that $\text{Pder}(T^*\mathcal{G}_\lambda) = \text{span}\{\phi_i, 1 \leq i \leq 13\}$, where

$\phi_1 = e_{21} - e_{56}$	$\phi_2 = 2e_{22} + e_{33} + e_{44} - 2e_{66} - e_{77} - e_{88}$
$\phi_3 = -e_{23} + \lambda e_{31} - \lambda e_{57} + e_{76}$	$\phi_4 = e_{24} - \lambda e_{41} + \lambda e_{58} - e_{86}$
$\phi_5 = e_{26}$	$\phi_6 = -\lambda e_{34} + \lambda e_{43} - \lambda e_{78} + \lambda e_{87}$
$\phi_7 = e_{51}$	$\phi_8 = 2e_{52} + \frac{1}{\lambda}e_{73} + \frac{1}{\lambda}e_{84}$
$\phi_9 = -\lambda e_{53} + \lambda e_{71}$	$\phi_{10} = \lambda e_{54} - \lambda e_{81}$
$\phi_{11} = e_{55} + e_{66} + e_{77} + e_{88}$	$\phi_{12} = e_{56}$
$\phi_{13} = -e_{74} + e_{83}$	

(3.84)

with the following brackets

$[\phi_1, \phi_2] = -\phi_1$	$[\phi_1, \phi_8] = -2\phi_7$	$[\phi_2, \phi_3] = \phi_3$
$[\phi_2, \phi_4] = \phi_4$	$[\phi_2, \phi_5] = 2\phi_5$	$[\phi_2, \phi_8] = -2\phi_8$
$[\phi_2, \phi_9] = -\phi_9$	$[\phi_2, \phi_{10}] = -\phi_{10}$	$[\phi_2, \phi_{12}] = 2\phi_{12}$
$[\phi_2, \phi_{13}] = -2\phi_{13}$	$[\phi_3, \phi_6] = \lambda\phi_4$	$[\phi_3, \phi_8] = -\frac{1}{\lambda}\phi_9$
$[\phi_3, \phi_{13}] = \phi_{10}$	$[\phi_4, \phi_6] = -\lambda\phi_3$	$[\phi_4, \phi_8] = -\frac{1}{\lambda}\phi_{10}$
$[\phi_4, \phi_{13}] = -\phi_9$	$[\phi_5, \phi_8] = -2\phi_{12}$	$[\phi_5, \phi_{11}] = \phi_5$
$[\phi_6, \phi_9] = -\lambda\phi_{10}$	$[\phi_6, \phi_{10}] = -\lambda\phi_{10}$	$[\phi_7, \phi_{11}] = -\phi_7$
$[\phi_8, \phi_{11}] = -\phi_8$	$[\phi_9, \phi_{11}] = -\phi_9$	$[\phi_{10}, \phi_{11}] = -\phi_{10}$
$[\phi_{11}, \phi_{12}] = \phi_{12}$	$[\phi_{11}, \phi_{13}] = \phi_{13}$	

(3.85)

One realizes that

$$\begin{aligned} \text{Pder}(T^*\mathcal{G}_\lambda) &= ad_{T^*\mathcal{G}_\lambda} \times \left[\mathbb{R}^2 \times (\mathbb{R}^2 \times \mathbb{R}^3) \right] \\ &= (\mathbb{R}^2 \times \mathbb{R}^4) \times \left[\mathbb{R}^2 \times (\mathbb{R}^2 \times \mathbb{R}^3) \right] \\ &= \left(\text{span}(\phi_6, \phi_{13}) \times \text{span}(\phi_3, \phi_4, \phi_9, \phi_{10}) \right) \\ &\quad \times \left[\text{span}(\phi_2, \phi_{11}) \times (\text{span}(\phi_1, \phi_5) \times \text{span}(\phi_7, \phi_8, \phi_{12})) \right] \end{aligned} \quad (3.86)$$

3.5.2 Possible Applications

The existence of affine structures is a difficult and interesting problem ([39],[63], [54], [52]).

Let \mathfrak{G} be a Lie algebra. If D is an invertible derivation of \mathfrak{G} , one defines an affine structure by the formula

$$\nabla_x y = D^{-1} \circ ad_x \circ D(y), \quad (3.87)$$

for every x and y in \mathfrak{G} . Unfortunately, there are Lie algebras that admits only singular derivations. Nilpotent such algebras are called characteristically nilpotent Lie algebras.

If the Lie algebra \mathfrak{G} admits no regular derivation, then one uses a regular prederivation whenever it exists. The idea of the construction is the following ([15]). Given an invertible prederivation P , set $\omega(x, y) = P([x, y]) - [x, P(y)] - [P(x), y]$, for all $x, y \in \mathfrak{G}$. Now we define

$$\nabla_x y = P^{-1} \circ ad_x \circ P(y) + \frac{1}{2} P^{-1} \circ \omega(x, y), \quad (3.88)$$

for every $x, y \in \mathfrak{G}$. In general the map $x \mapsto P^{-1} \circ ad_x \circ P + \frac{1}{2} P^{-1} \circ \omega(x, \cdot)$ might not be a representation. However, in many cases, (3.88) gives rise to an affine structure.

SKEW-SYMMETRIC PREDERIVATIONS AND BI-INVARIANT METRICS OF COTANGENT BUNDLES OF LIE GROUPS

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4.1 Introduction

A Lie group endowed with a pseudo-Riemannian metric which is invariant under both left and right translations, is called an orthogonal, a bi-invariant or a quadratic Lie group. The corresponding Lie algebra is called an orthogonal or quadratic Lie algebra.

Such Lie groups are important in Mathematics and in Physics. These objects are, for instance, useful in pseudo-Riemannian geometry, in the theory of Poisson-Lie groups, in relativity, in the theory of Hamiltonian systems,... ([4], [5], [10], [29], [32], [35], [50]).

According to the works of Medina and Revoy ([60],[58]), any orthogonal Lie algebra is obtained by the so-called double extension procedure. This is the case of orthogonal Lie algebras called hyperbolic and whose quadratic form is of signature (n, n) (where $2n$ is the dimension of the concerned Lie algebra). One particularly interesting case is the one where the Lie algebra admits two totally isotropic subalgebras which are in duality. These latter Lie algebras named Manin-Lie algebras describe simply connected Poisson-Lie groups.

It is known that the cotangent bundle T^*G of a Lie group G with Lie algebra \mathfrak{G} , considered with its Lie group structure obtained by semi-direct product of the Lie group

G and the Abelian Lie group \mathcal{G}^* (dual of \mathcal{G}) by means of the co-adjoint representation, possesses a hyperbolic metric : the duality pairing.

Here we seek to characterize, up to isometric automorphisms, all orthogonal structures on $T^*\mathcal{G}$ by means of the duality pairing and adjoint-invariant endomorphisms; hence, all bi-invariant metrics on T^*G ; and to determine the corresponding group of isometries. We will focus our study on orthogonal Lie algebras and on the more particular class of semi-simple Lie algebras.

We will take \mathcal{G} to be the Lie algebra of a Lie group G , \mathcal{G}^* will be the dual space of \mathcal{G} and $T^*\mathcal{G} := \mathcal{G} \ltimes \mathcal{G}^*$ will be the semi-direct sum of the Lie algebra \mathcal{G} and the vector space \mathcal{G}^* via the coadjoint representation. Recall that $\text{Pder}(\mathcal{G})$ stands for the Lie algebra of all prederivations of the Lie algebra \mathcal{G} while \mathcal{J}' is the set of all endomorphisms j of \mathcal{G} which satisfy $j([x, y], z) = [x, y], j(z)$, for any x, y, z in \mathcal{G} .

Among others, here are some of the important results contained in this chapter.

Theorem A

1. Let \mathcal{G} be a Lie algebra. Any orthogonal structure μ on $T^*\mathcal{G}$ is given by

$$\mu((x, f), (y, g)) = \langle g, j_{11}(x) \rangle + \langle f, j_{11}(y) \rangle + \langle g, j_{21}(f) \rangle + \langle j_{12}(x), y \rangle, \quad (4.1)$$

where $j_{11} : \mathcal{G} \rightarrow \mathcal{G}$, $j_{12} : \mathcal{G} \rightarrow \mathcal{G}^*$, $j_{21} : \mathcal{G}^* \rightarrow \mathcal{G}$ are as in Proposition 4.3.1.

2. If (\mathcal{G}, μ) is an orthogonal Lie algebra, then any orthogonal structure $\mu_{\mathcal{D}}$ on $T^*\mathcal{G}$ has the form

$$\mu_{\mathcal{D}}((x, f), (y, g)) = \langle g, j_{11}(x) \rangle + \langle f, j_{11}(y) \rangle + \langle g, j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x), y \rangle, \quad (4.2)$$

for all $(x, f), (y, g)$ in $T^*\mathcal{G}$, where θ is the isomorphism induced by the μ through the formula (1.2) and j_{11}, j_1, j_2 satisfy conditions listed in Lemma 4.4.1.

3. Let \mathcal{G} be a semi-simple Lie algebra. Any orthogonal structure μ on $T^*\mathcal{G}$ is given by

$$\mu((x, f), (y, g)) = \sum_{i=1}^p \lambda_i \langle (x_i, f_i), (y_i, g_i) \rangle_{\mathfrak{s}_i} + \sum_{k=1}^p \nu_k K_k(x_k, y_k), \quad (4.3)$$

for all $x, y \in \mathcal{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \dots \oplus \mathfrak{s}_p$ and $f, g \in \mathcal{G}^* = \mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \dots \oplus \mathfrak{s}_p^*$; where $\lambda_i \in \mathbb{R}^*$, $\nu_i \in \mathbb{R}$ and K_i stands for the Killing form on \mathfrak{s}_i for all $i = 1, 2, \dots, p$.

Theorem B

1. Let \mathcal{G} be a Lie algebra and let $\mu_{\mathcal{D}}$ be an orthogonal structure on $T^*\mathcal{G}$. A $\mu_{\mathcal{D}}$ -skew-symmetric prederivation ϕ of $T^*\mathcal{G}$ has the form

$$\phi(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + (j^t - \alpha^t)(f) \right), \quad (4.4)$$

where $j' : \mathcal{G} \rightarrow \mathcal{G}$ is in \mathcal{J}' , $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is in $\text{Pder}(\mathcal{G})$, $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ and $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ are as in Theorem 3.3.1, with the additional conditions listed in Proposition 4.3.2.

2. Let (\mathcal{G}, μ) be an orthogonal Lie algebra. Then any prederivation ϕ of $T^*\mathcal{G}$ which is skew-symmetric with respect to any orthogonal structure $\mu_{\mathcal{D}}$ on $T^*\mathcal{G}$ can be written as follows

$$\phi(x, f) = \left(\alpha_1(x) + j'_1 \circ \theta^{-1}(f), \theta \circ \alpha_2(x) + (j_2^t - \alpha_1^t)(f) \right), \quad (4.5)$$

where α_1, α_2 are in $Pder(\mathcal{G})$; j'_1, j'_2 are in \mathcal{J}' with the additional conditions listed in Proposition 4.4.1.

3. Let \mathcal{G} a semi-simple Lie algebra. Then any prederivation of $T^*\mathcal{G}$ which is skew-symmetric with respect to any orthogonal structure $\mu_{\mathcal{D}}$ on $T^*\mathcal{G}$ is an inner derivation of $T^*\mathcal{G}$.

In Section 4.2 is given some basic notions useful to make the chapter understandable. Section 4.3 is dedicated to the characterization of orthogonal structures on the Lie algebras of cotangent bundles of Lie groups and their groups of isometries. In Sections 4.4 and 4.5 are respectively studied the case of orthogonal Lie algebras and the case of semi-simple Lie algebras. The chapter finishes with some examples given in Section 4.6.

4.2 Preliminaries

4.2.1 Orthogonal Structures and Bi-invariant Endomorphisms

Let (G, μ) be an orthogonal Lie group with Lie algebra \mathcal{G} . The metric μ induces on \mathcal{G} an adjoint-invariant symmetric non-degenerate bilinear form \langle, \rangle , *i.e.* a symmetric and non-degenerate form \langle, \rangle such that

$$\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0, \quad (4.6)$$

for all x, y in \mathcal{G} . The form \langle, \rangle is called an orthogonal structure on \mathcal{G} . It is well known ([60]) that any other non-degenerate symmetric bilinear form B on \mathcal{G} can be written as

$$B(x, y) = \langle j(x), y \rangle, \quad (4.7)$$

for all x, y in \mathcal{G} , where j is a \langle, \rangle -symmetric automorphism of the vector space \mathcal{G} . Furthermore, the bilinear form B is adjoint-invariant, *i.e.* is an orthogonal structure, if and only if j commutes with all the adjoint operators ad_x , ($x \in \mathcal{G}$); that is

$$j([x, y]) = [j(x), y] = [x, j(y)], \quad (4.8)$$

for every x, y in \mathcal{G} .

An endomorphism j of \mathcal{G} which satisfies Relation (4.8) is called a bi-invariant or an adjoint-invariant endomorphism.

From what is said above it comes that if one orthogonal structure \langle, \rangle is known on \mathcal{G} then we will be able to characterize all the orthogonal structures on \mathcal{G} by characterizing all the \langle, \rangle -symmetric bi-invariant tensors on \mathcal{G} .

4.2.2 Isometries of Bi-invariant Metrics of Lie Groups

Let us begin by some reminders.

Definition 4.2.1. *Let M be a smooth manifold equipped with a pseudo-Riemannian metric μ . An isometry of the pseudo-Riemannian manifold (M, μ) is a diffeomorphism $f : M \rightarrow M$ such that $f^*\mu = \mu$, where f^* stands for the "pull-back" via f . More precisely, for all x in M and any vectors $X|_x, Y|_x$ in the tangent space $T_x M$,*

$$\mu_{f(x)}(T_x f \cdot X|_x, T_x f \cdot Y|_x) = \mu_x(X|_x, Y|_x). \quad (4.9)$$

We denote by $I(M, \mu)$ the set of all isometries of (M, μ) . Nomizu shows in [66] that the set $\text{Aff}(M, \nabla)$ of all affine transformations of the induced affine connection on M , with the compact-open topology, is a Lie transformation group. But $I(M, \mu)$ is a closed subgroup of $\text{Aff}(M, \nabla)$. Then, equipped with the compact-open topology, $I(M, \mu)$ is a Lie transformation group.

Let (G, μ) be an orthogonal Lie group with Lie algebra \mathfrak{G} . We note by $I(G, \mu) = I(G)$ the set of all isometries of (G, μ) , by $F(G)$ the set of those isometries which fix the identity element ϵ of G and by L_G the set of all left translations of G . We recall the following result due to Müller.

Lemma 4.2.1. ([64]) *Let G be a connected Lie group endowed with a bi-invariant metric.*

1. $F(G)$ is a closed Lie subgroup of $I(G)$;
2. L_G is a closed connected subgroup isomorphic to G ;
3. $I(G) = L_G F(G)$, with $L_G \cap F(G) = \{id\}$; $id : G \rightarrow G$ is the identity map of G ;
4. the manifolds $I(G)$ and $G \times F(G)$ are diffeomorphic.

4.2.3 Isometries of Bi-invariant Metrics and Preautomorphisms

Let (G, μ) be an orthogonal Lie group with unit element ϵ and let \mathfrak{G} be its Lie algebra. We will often note by $\langle, \rangle := \mu|_\epsilon$ the resulting non-degenerate adjoint-invariant bilinear form on \mathfrak{G} . The fact that G is orthogonal is equivalent to the one that the geodesics through ϵ are the one-parameter subgroups of G (see [38, Exercise 5, page 148]).

Let $\exp : \mathfrak{G} \rightarrow G$ denote the exponential map of the Lie group G . Consider an open neighborhood U of 0 in \mathfrak{G} such that the restriction $\exp|_U : U \rightarrow \exp(U)$ of \exp to U is a diffeomorphism and note by $\log : \exp(U) \rightarrow U$ the inverse of $\exp|_U$.

Definition 4.2.2. *A local isometry at ϵ is a diffeomorphism $\varphi : V_1 \rightarrow V_2$ between two open neighborhoods V_1 and V_2 of ϵ in G such that*

- φ fixes ϵ , i.e. $\varphi(\epsilon) = \epsilon$;
- $\varphi^*\mu = \mu$ on V_1 , where φ^* stands for the pull-back via φ .

Now, let φ be a local isometry at ϵ and x any arbitrary element of \mathfrak{G} . The local isometry φ maps any geodesic through ϵ onto a geodesic through ϵ on a neighborhood of ϵ . Then there exists an element y of \mathfrak{G} such that

$$\varphi(\exp(tx)) = \exp(ty), \tag{4.10}$$

for t small enough. Derivating relation (4.10) with respect to t at $t = 0$, one has

$$T_\epsilon\varphi \cdot x = y, \tag{4.11}$$

where $T_\epsilon\varphi$ is the tangent linear map of φ at ϵ . From relation (4.10) we have

$$y = \log \circ \varphi \circ \exp(x). \tag{4.12}$$

Relations (4.10) and (4.11) give the following nice formula

$$\varphi = \exp \circ T_\epsilon\varphi \circ \log \tag{4.13}$$

on a suitable neighborhood of ϵ in $\exp(U)$.

Thus if we identify two local isometries at ϵ that agree on a neighborhood of ϵ , it comes that a local isometry at ϵ is uniquely determine by its differential at ϵ . A local isometry at ϵ can be uniquely extended to an isometry on G . Indeed, it is well known that if M and N are two connected simply connected and geodesically complete pseudo-Riemannian manifolds, then every isometry between connected open subsets of M and N can be uniquely extended to an isometry between M and N ([45], [67], [71]).

Now we have the following theorem which establishes a link between local isometries at ϵ and the so-called preautomorphisms of the Lie algebra \mathfrak{G} of G .

Theorem 4.2.1. ([64]) *Let (G, μ) be a connected orthogonal Lie group with Lie algebra \mathfrak{G} . Let $\langle, \rangle := \mu|_\epsilon$ stands for the resulting orthogonal structure on \mathfrak{G} and let P be an endomorphism of \mathfrak{G} . Then there exists a local isometry φ of G at ϵ with $T_\epsilon\varphi = P$ if and only if P satisfies*

1. $\langle P(x), P(y) \rangle = \langle x, y \rangle$, for any x, y in \mathfrak{G} ;
2. $P([x, [y, z]]) = [P(x), [P(y), P(z)]]$, for all x, y, z in \mathfrak{G} .

Now, the author quoted above proves that once the Lie algebra of $F(G)$ is known one can readily construct the Lie algebra of $I(G)$. Here is his construction.

For g in G we define the following maps :

$$\begin{aligned} L_g : G &\rightarrow G & ; & & R_g : G &\rightarrow G & ; & & I_g : G &\rightarrow G \\ h &\mapsto gh & & & h &\mapsto hg & & & h &\mapsto ghg^{-1} \end{aligned} \tag{4.14}$$

This maps are respectively the left transformation, the right translation and the inner automorphism by the element g of G .

Let ξ be in \mathfrak{G} and $\exp : \mathfrak{G} \rightarrow G$ be the exponential map of G . Then, $L_{\exp(t\xi)}$, $R_{\exp(t\xi)}$, $I_{\exp(t\xi)}$ are one-parameter subgroups of $I(G)$. Let us note by $X^{\xi,L}$, $X^{\xi,R}$ and $X^{\xi,I}$ their respective infinitesimal generators. Now, we set

$$X^{\xi,s} = X^{\xi,R} + X^{\xi,L} \quad \text{and} \quad X^{\xi,a} = X^{\xi,R} - X^{\xi,L}. \tag{4.15}$$

If $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is a basis of the Lie algebra $\mathcal{F}(G)$ of $F(G)$ and $\{\xi_1, \xi_2, \dots, \xi_n\}$ a basis of \mathfrak{G} then $\{\alpha_1, \alpha_2, \dots, \alpha_m, X^{\xi_1,s}, X^{\xi_2,s}, \dots, X^{\xi_n,s}\}$ is a basis of the Lie algebra $\mathcal{J}(G)$.

It remains to compute the brackets on $\mathcal{J}(G)$. Let us first define some useful objects.

Let $\text{Paut}(\mathfrak{G})$ be the group of preautomorphisms of \mathfrak{G} and denote by $F(\mathfrak{G})$ the subgroup of $\text{Paut}(\mathfrak{G})$ consisting of those preautomorphisms which preserve the orthogonal structure on \mathfrak{G} . Now consider the map $T_\epsilon : F(G) \rightarrow F(\mathfrak{G})$ which associates to any element ϕ of $F(G)$ its differential $T_\epsilon\phi$ at the neutral element ϵ of G . The map T_ϵ induces a map $\partial_\epsilon : \mathcal{F}(G) \rightarrow \mathcal{F}(\mathfrak{G})$ between the Lie algebras $\mathcal{F}(G)$ and $\mathcal{F}(\mathfrak{G})$ of $F(G)$ and $F(\mathfrak{G})$ respectively :

$$\partial_\epsilon D = \left(\frac{d}{dt} T_\epsilon[\exp_{I(G)}(tD)] \right)_{|t=0} \tag{4.16}$$

Note that $\mathcal{F}(\mathfrak{G})$ consists of prederivations of \mathfrak{G} which are skew-symmetric with respect to the orthogonal structure on \mathfrak{G} .

The following theorem gives the brackets on $\mathcal{J}(G)$.

Theorem 4.2.2. ([64]) *Let ξ, η be in \mathfrak{G} and D be an element of $\mathcal{F}(G)$. Then,*

1. $[X^{\xi,s}, X^{\eta,s}] = [X^{\xi,a}, X^{\eta,a}] = -X^{[\xi,\eta],a}$;
2. $[D, X^{\xi,s}] = X^{\partial_\epsilon D(\xi),s}$;

Now we conclude that if we know the prederivations of \mathfrak{G} which are skew-symmetric with respect to the orthogonal structure on \mathfrak{G} we can calculate the Lie algebra $\mathcal{J}(G)$ of the group $I(G)$ of isometries of the orthogonal Lie group (G, μ) .

4.3 Bi-invariant Metrics on T^*G

Let G be a Lie group with Lie algebra \mathfrak{G} . We have already seen (Section 2.2.1) that the duality pairing \langle, \rangle given by

$$\langle(x, f), (y, g)\rangle = f(y) + g(x), \tag{4.17}$$

for all x, y in \mathfrak{G} , defines an orthogonal structure on the Lie algebra $T^*\mathfrak{G} = \mathfrak{G} \times \mathfrak{G}^*$ of the Lie group T^*G .

4.3.1 Bi-invariant Tensors on $T^*\mathcal{G}$

Let \langle, \rangle stand for the duality pairing on $T^*\mathcal{G}$, a bi-invariant metric on T^*G is given by an invertible linear map $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ satisfying

$$j([u, v]) = [u, j(v)]; \quad (4.18)$$

$$\langle j(u), v \rangle = \langle u, j(v) \rangle, \quad (4.19)$$

for all $u, v \in T^*\mathcal{G}$.

Lemma 4.3.1. *Any linear map $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ satisfying (4.18) can be written as*

$$j(x, f) = \left(j_{11}(x) + j_{21}(f), j_{12}(x) + j_{22}(f) \right), \quad (4.20)$$

for any $(x, f) \in T^*\mathcal{G}$; where $j_{11} : \mathcal{G} \rightarrow \mathcal{G}$, $j_{12} : \mathcal{G} \rightarrow \mathcal{G}^*$, $j_{21} : \mathcal{G}^* \rightarrow \mathcal{G}$ and $j_{22} : \mathcal{G}^* \rightarrow \mathcal{G}^*$ are linear maps such that for all x in \mathcal{G} , the following relations hold :

$$j_{11} \circ ad_x = ad_x \circ j_{11} \quad (4.21)$$

$$j_{12} \circ ad_x = ad_x^* \circ j_{12} \quad (4.22)$$

$$j_{21} \circ ad_x^* = ad_x \circ j_{21} = 0 \quad (4.23)$$

$$[j_{22}, ad_x^*] = 0 \quad (4.24)$$

$$ad_x^* \circ (j_{22} - j_{11}^t) = 0, \quad (4.25)$$

where j_{11}^t stands for the transpose of the map j_{11} .

Proof. If we write $u = (x, f)$ and $v = (y, g)$, then

$$\begin{aligned} j([u, v]) &= j([x, y], ad_x^*g - ad_y^*f) \\ &= \left(j_{11}([x, y]) + j_{21}(ad_x^*g - ad_y^*f), j_{12}([x, y]) + j_{22}(ad_x^*g - ad_y^*f) \right) \end{aligned} \quad (4.26)$$

and

$$\begin{aligned} [u, jv] &= [(x, f), (j_{11}(y) + j_{21}(g), j_{12}(y) + j_{22}(g))] \\ &= \left([x, j_{11}(y)] + [x, j_{21}(g)], ad_x^*(j_{12}(y) + j_{22}(g)) - ad_{j_{11}(y)+j_{21}(g)}^*f \right) \end{aligned} \quad (4.27)$$

Considering the case where $f = g = 0$ the equality between (4.27) and (4.28) gives

$$j_{11}([x, y]) = [x, j_{11}(y)] \text{ and } j_{12}([x, y]) = ad_x^*(j_{12}(y)),$$

for all $x, y \in \mathcal{G}$. The two relations above are equivalent to (4.21) and (4.22) respectively.

The equality between (4.26) and (4.27) now gives

$$\left(j_{21}(ad_x^*g - ad_y^*f), j_{22}(ad_x^*g - ad_y^*f) \right) = \left([x, j_{21}(g)], ad_x^*(j_{22}(g)) - ad_{j_{11}(y)+j_{21}(g)}^*f \right). \quad (4.28)$$

Taking $f = 0$, we get on one hand

$$j_{21}(ad_x^*g) = [x, j_{21}(g)],$$

for all $x \in \mathfrak{G}$ and all $g \in \mathfrak{G}^*$. Then Relation (4.23) is proved. On the second hand we have

$$j_{22}(ad_x^*g) = ad_x^*(j_{22}(g)),$$

for every g in \mathfrak{G}^* ; that is

$$j_{22} \circ ad_x^* = ad_x^* \circ j_{22}, \quad (4.29)$$

which is nothing but Relation (4.24). Now if we take $x = 0$, we get from Eq. (4.28)

$$\left(j_{21}(-ad_y^*f), j_{22}(-ad_y^*f) \right) = \left(0, -ad_{j_{11}(y)+j_{21}(g)}^*f \right). \quad (4.30)$$

The equality above implies the following two relations

$$\begin{aligned} j_{21}(-ad_y^*f) &= 0, \\ j_{22}(-ad_y^*f) &= -ad_{j_{11}(y)+j_{21}(g)}^*f, \end{aligned}$$

for all $y \in \mathfrak{G}$, $f \in \mathfrak{G}^*$. These equations are equivalent to

$$j_{21} \circ ad_y^* = 0, \quad (4.31)$$

$$j_{22} \circ ad_y^* = ad_{j_{11}(y)}^*, \quad (4.32)$$

$$ad_{j_{21}(g)}^* = 0, \quad (4.33)$$

for all $y \in \mathfrak{G}$ and $g \in \mathfrak{G}^*$. The relations

$$\begin{aligned} ad_y^* \circ j_{22} &= j_{22} \circ ad_y^* \\ &= ad_{j_{11}(y)}^* \\ &= ad_y^* \circ j_{11}^t, \end{aligned} \quad (4.34)$$

for all $y \in \mathfrak{G}$, coming from (4.29) and (4.32), give

$$ad_y^* \circ (j_{22} - j_{11}^t) = 0, \quad (4.35)$$

for all $y \in \mathfrak{G}$; which means that $(j_{22} - j_{11}^t)f$ is a closed 1-form, for every $f \in \mathfrak{G}^*$. \square

Remark 4.3.1. 1. The equality $ad_{j_{21}(g)}^* = 0$, for all g is equivalent to

$$Im(j_{21}) \subset Z(\mathfrak{G}), \quad (4.36)$$

where $Z(\mathfrak{G})$ is the centre of \mathfrak{G} . So if $Z(\mathfrak{G}) = \{0\}$ then $j_{21} = 0$.

2. Relation (4.34) also gives

$$\begin{aligned} -\langle j_{22}(f), [y, x] \rangle &= \langle j_{22}(ad_y^*(f)), x \rangle \\ &= -\langle f, [j_{11}(y), x] \rangle \\ &= -\langle f \circ j_{11}, [y, x] \rangle, \end{aligned} \quad (4.37)$$

for all f in \mathfrak{G}^* and any x, y in \mathfrak{G} .

Lemma 4.3.2. Any linear map $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ which is \langle, \rangle -symmetric can be written as

$$j(x, f) = \left(j_{11}(x) + j_{21}(f), j_{12}(x) + j_{11}^t(f) \right), \quad (4.38)$$

for any $(x, f) \in T^*\mathcal{G}$; where $j_{11} : \mathcal{G} \rightarrow \mathcal{G}$, $j_{12} : \mathcal{G} \rightarrow \mathcal{G}^*$, $j_{21} : \mathcal{G}^* \rightarrow \mathcal{G}$ are linear maps with the following relations valid for all x, y in \mathcal{G} and all f, g in \mathcal{G}^* .

$$\langle j_{12}(x), y \rangle = \langle x, j_{12}(y) \rangle \quad (4.39)$$

$$\langle j_{21}(f), g \rangle = \langle f, j_{21}(g) \rangle. \quad (4.40)$$

Proof. Let $u = (x, f)$ and $v = (y, g)$ be two elements of $T^*\mathcal{G}$.

$$\begin{aligned} \langle ju, v \rangle &= \left\langle \left(j_{11}(x) + j_{21}(f), j_{12}(x) + j_{22}(f) \right), (y, g) \right\rangle \\ &= \langle j_{11}(x), y \rangle + \langle j_{21}(f), y \rangle + \langle j_{12}(x), g \rangle + \langle j_{22}(f), g \rangle \end{aligned} \quad (4.41)$$

$$\begin{aligned} \langle u, jv \rangle &= \left\langle (x, f), \left(j_{11}(y) + j_{21}(g), j_{12}(y) + j_{22}(g) \right) \right\rangle \\ &= \langle f, j_{11}(y) \rangle + \langle f, j_{21}(g) \rangle + \langle x, j_{12}(y) \rangle + \langle x, j_{22}(g) \rangle \end{aligned} \quad (4.42)$$

For $f = g = 0$, the equality between (4.41) and (4.42) implies that for all x, y in \mathcal{G} ,

$$\langle j_{12}(x), y \rangle = \langle x, j_{12}(y) \rangle.$$

So (4.39) is established. By the same way, for $x = y = 0$, Relation (4.40) is obtained, as

$$\langle j_{21}(f), g \rangle = \langle f, j_{21}(g) \rangle.$$

Now the equality between (4.41) and (4.42) can be written

$$\langle j_{11}(x), y \rangle + \langle j_{22}(f), y \rangle = \langle f, j_{11}(y) \rangle + \langle x, j_{22}(g) \rangle.$$

We take $y = 0$ to obtain

$$\langle x, j_{11}^t(g) \rangle = \langle j_{11}(x), g \rangle = \langle x, j_{22}(g) \rangle$$

for all x in \mathcal{G} and g in \mathcal{G}^* ; or equivalently

$$j_{11}^t = j_{22}$$

□

Remark 4.3.2. Relations (4.39) and (4.40) mean that j_{12} and j_{21} are symmetric with respect to the duality.

Lemma 4.3.3. Let \mathcal{G} be a Lie algebra without centre, that is $Z(\mathcal{G}) = \{0\}$. A \langle, \rangle -symmetric bi-invariant endomorphism $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ defined by (4.38) is invertible if and only if j_{11} is invertible.

Proof. Let $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ be defined by (4.38) with conditions listed in Lemma 4.3.1. We have seen in Remark 4.3.1 that if $Z(\mathcal{G}) = \{0\}$ then $j_{21} = 0$. The determinant of j is

$$\det(j) = \begin{vmatrix} j_{11} & j_{21} \\ j_{12} & j_{11}^t \end{vmatrix} = \begin{vmatrix} j_{11} & 0 \\ j_{12} & j_{11}^t \end{vmatrix} = (\det(j_{11}))^2. \quad (4.43)$$

□

Let us summarize the above three Lemmas in the

Proposition 4.3.1. *Let \mathcal{G} be a Lie algebra. Any \langle, \rangle -symmetric and adjoint-invariant tensor $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ has the form*

$$j(x, f) = \left(j_{11}(x) + j_{21}(f), j_{12}(x) + j_{11}^t(f) \right), \quad (4.44)$$

for any $(x, f) \in T^*\mathcal{G}$; where $j_{11} : \mathcal{G} \rightarrow \mathcal{G}$, $j_{12} : \mathcal{G} \rightarrow \mathcal{G}^*$, $j_{21} : \mathcal{G}^* \rightarrow \mathcal{G}$ are linear maps with the following relations valid for all x, y in \mathcal{G} and all f, g in \mathcal{G}^* .

$$j_{11} \circ \text{ad}_x = \text{ad}_x \circ j_{11} \quad (4.45)$$

$$j_{12} \circ \text{ad}_x = \text{ad}_x^* \circ j_{12} \quad (4.46)$$

$$j_{21} \circ \text{ad}_x^* = \text{ad}_x \circ j_{21} = 0 \quad (4.47)$$

$$\langle j_{12}(x), y \rangle = \langle x, j_{12}(y) \rangle \quad (4.48)$$

$$\langle j_{21}(f), g \rangle = \langle f, j_{21}(g) \rangle. \quad (4.49)$$

If, in addition, the centre $Z(\mathcal{G})$ of the Lie algebra is $\{0\}$, then the endomorphism j is invertible if and only if j_{11} is invertible.

4.3.2 Orthogonal Structures on $T^*\mathcal{G}$

In the sequel we will use the duality pairing on $T^*\mathcal{G}$ and Relation (4.7) in order to determine all non-degenerate symmetric and adjoint-invariant form on $T^*\mathcal{G}$, hence all bi-invariant metrics on T^*G . Precisely, any other orthogonal structure μ on $T^*\mathcal{G}$ is linked to the duality pairing \langle, \rangle by

$$\mu((x, f), (y, g)) = \langle j(x, f), (y, g) \rangle, \quad (4.50)$$

for every $(x, f), (y, g)$ in $T^*\mathcal{G}$; where j is an endomorphism of $T^*\mathcal{G}$ such that :

- j is invertible;
- j is symmetric with respect to the duality, i.e. $\langle j(u), v \rangle = \langle u, j(v) \rangle$, for all $u, v \in T^*\mathcal{G}$;
- $j([u, v]) = [j(u), v] = [u, j(v)]$, for all $u, v \in T^*\mathcal{G}$.

We have the following characterization of all orthogonal structures on $T^*\mathcal{G}$.

Theorem 4.3.1. *Let \mathcal{G} be a Lie algebra. Any orthogonal structure μ on $T^*\mathcal{G}$ is given by*

$$\mu((x, f), (y, g)) = \langle g, j_{11}(x) \rangle + \langle f, j_{11}(y) \rangle + \langle g, j_{21}(f) \rangle + \langle j_{12}(x), y \rangle, \quad (4.51)$$

where $j_{11} : \mathcal{G} \rightarrow \mathcal{G}$, $j_{12} : \mathcal{G} \rightarrow \mathcal{G}^*$, $j_{21} : \mathcal{G}^* \rightarrow \mathcal{G}$ are as in Proposition 4.3.1.

Proof. Indeed, an orthogonal structure μ on $T^*\mathcal{G}$ is defined by

$$\mu((x, f), (y, g)) = \langle j(x, f), (y, g) \rangle,$$

where j is given by (4.44) and condition listed in Proposition 4.3.1. We then have

$$\begin{aligned} \mu((x, f), (y, g)) &= \left\langle \left(j_{11}(x) + j_{21}(f), j_{12}(x) + j_{11}^t(f) \right), (y, g) \right\rangle \\ &= \left\langle g, j_{11}(x) + j_{21}(f) \right\rangle + \left\langle j_{12}(x) + j_{11}^t(f), y \right\rangle \\ &= \left\langle g, j_{11}(x) \right\rangle + \left\langle g, j_{21}(f) \right\rangle + \left\langle j_{12}(x), y \right\rangle + \left\langle j_{11}^t(f), y \right\rangle \\ &= \left\langle g, j_{11}(x) \right\rangle + \left\langle g, j_{21}(f) \right\rangle + \left\langle j_{12}(x), y \right\rangle + \left\langle f, j_{11}(y) \right\rangle \end{aligned}$$

□

4.3.3 Skew-symmetric Prederivations on $T^*\mathcal{G}$

Let \mathcal{G} be a Lie algebra. On $T^*\mathcal{G}$ we consider the orthogonal structure μ defined by (4.51). We seek to characterise all prederivations of $T^*\mathcal{G}$ which are skew-symmetric with respect to the orthogonal structure μ .

Let us first recall the space $\mathcal{J}' = \{j' : \mathcal{G} \rightarrow \mathcal{G} \text{ linear} : [j', ad_x \circ ad_y] = 0, \forall x, y \in \mathcal{G}\}$.

Proposition 4.3.2. *A μ -skew-symmetric prederivation ϕ of $T^*\mathcal{G}$ has the form*

$$\phi(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + (j'^t - \alpha^t)(f) \right), \quad (4.52)$$

where $j' : \mathcal{G} \rightarrow \mathcal{G}$ is in \mathcal{J}' , $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is a prederivation of \mathcal{G} , $\beta : \mathcal{G} \rightarrow \mathcal{G}^*$ and $\psi : \mathcal{G}^* \rightarrow \mathcal{G}$ are as in Theorem 3.3.1, with the additional conditions

$$\langle \beta(x), j_{11}(y) \rangle + \langle \beta(y), j_{11}(x) \rangle + \langle j_{12} \circ \alpha(x), y \rangle + \langle j_{12} \circ \alpha(y), x \rangle = 0 \quad (4.53)$$

$$\begin{aligned} \langle f, j_{11} \circ \psi(g) \rangle + \langle g, j_{11} \circ \psi(f) \rangle + \langle f, j' \circ j_{21}(g) \rangle \\ + \langle g, j' \circ j_{21}(f) \rangle - \langle f, \alpha \circ j_{21}(g) \rangle - \langle g, \alpha \circ j_{21}(f) \rangle = 0 \end{aligned} \quad (4.54)$$

$$[j_{11}, \alpha] + j_{21} \circ \beta + j' \circ j_{11} + \psi^t \circ j_{12} = 0 \quad (4.55)$$

for every elements x and y of \mathcal{G} and any elements f and g in \mathcal{G}^* .

Proof. Recall that according to Corollary 3.3.1 a prederivation ϕ of $T^*\mathcal{G}$ can be written as

$$\phi(x, f) = \left(\alpha(x) + \psi(f), \beta(x) + (j'^t - \alpha^t)(f) \right),$$

where j' belongs to \mathcal{J}' and α, β, ψ are as in Theorem 3.3.1. The prederivation ϕ is μ -skew-symmetric if and only if

$$\mu(\phi(x, f), (y, g)) + \mu((x, f), \phi(y, g)) = 0, \quad (4.56)$$

for all (x, f) and (y, g) in $T^*\mathcal{G}$. On one hand, we have

$$\mu(\phi(x, f), (y, g)) = \mu\left((\alpha(x) + \psi(f), \beta(x) + (j'^t - \alpha^t)(f)), (y, g) \right)$$

$$\begin{aligned}
&= \langle g, j_{11}(\alpha(x) + \psi(f)) \rangle + \langle \beta(x) + (j^t - \alpha^t)(f), j_{11}(y) \rangle \\
&\quad \langle g, j_{21}(\beta(x) + (j^t - \alpha^t)(f)) \rangle + \langle j_{12}(\alpha(x) + \psi(f)), y \rangle \\
&= \langle g, j_{11} \circ \alpha(x) \rangle + \langle g, j_{11} \circ \psi(f) \rangle + \langle \beta(x), j_{11}(y) \rangle + \langle f, j' \circ j_{11}(y) \rangle \\
&\quad - \langle f, \alpha \circ j_{11}(y) \rangle + \langle g, j_{21} \circ \beta(x) \rangle + \langle g, j_{21} \circ j^t(f) \rangle \\
&\quad - \langle g, j_{21} \circ \alpha^t(f) \rangle + \langle j_{12} \circ \alpha(x), y \rangle + \langle j_{12} \circ \psi(f), y \rangle
\end{aligned} \tag{4.57}$$

On the other hand,

$$\begin{aligned}
\mu((x, f), \phi(y, g)) &= \mu((x, f), (\alpha(y) + \psi(g), \beta(y) + (j^t - \alpha^t)(g))) \\
&= \langle \beta(y) + (j^t - \alpha^t)(g), j_{11}(x) \rangle + \langle f, j_{11}(\alpha(y) + \psi(g)) \rangle \\
&\quad + \langle \beta(y) + (j^t - \alpha^t)(g), j_{21}(f) \rangle + \langle j_{12}(x), \alpha(y) + \psi(g) \rangle \\
&= \langle f, j_{11} \circ \alpha(y) \rangle + \langle f, j_{11} \circ \psi(g) \rangle + \langle \beta(y), j_{11}(x) \rangle \\
&\quad + \langle g, j' \circ j_{11}(x) \rangle - \langle g, \alpha \circ j_{11}(x) \rangle + \langle \beta(y), j_{21}(f) \rangle \\
&\quad + \langle g, j' \circ j_{21}(f) \rangle - \langle g, \alpha \circ j_{21}(f) \rangle + \langle j_{12} \circ \alpha(y), x \rangle \\
&\quad + \langle j_{12} \circ \psi(g), x \rangle, \text{ since } j_{21} \text{ is } \langle, \rangle\text{-symmetric}
\end{aligned} \tag{4.58}$$

Summing (4.57) and (4.58) and taking $f = g = 0$, Relation (4.56) becomes

$$\langle \beta(x), j_{11}(y) \rangle + \langle j_{12} \circ \alpha(x), y \rangle + \langle \beta(y), j_{11}(x) \rangle + \langle j_{12} \circ \alpha(y), x \rangle = 0$$

So (4.53) is proved.

Now we rewrite (4.56) and take $x = y = 0$ to obtain (4.54).

Last, (4.55) is obtained by summing (4.57) and (4.58), simplifying and taking $y = 0$. \square

4.4 Case of Orthogonal Lie Algebras

Let \mathcal{G} be a Lie algebra. We recall that an orthogonal structure μ on \mathcal{G} induces an isomorphism $\theta : \mathcal{G} \rightarrow \mathcal{G}^*$ by the formula $\langle \theta(x), y \rangle = \mu(x, y)$, for all x, y in \mathcal{G} . We also recall the set $\mathcal{J} = \{j : \mathcal{G} \rightarrow \mathcal{G} : j([x, y]) = [j(x), y], \forall x, y \in \mathcal{G}\}$.

4.4.1 Bi-invariant Metrics On Cotangent Bundles of Orthogonal Lie groups

Lemma 4.4.1. *Let (\mathcal{G}, μ) be an orthogonal Lie algebra. Any bi-invariant \langle, \rangle -symmetric invertible endomorphism $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ is given by*

$$j(x, f) = \left(j_{11}(x) + j_2 \circ \theta^{-1}(f), \theta \circ j_1(x) + j_{11}^t(f) \right), \tag{4.59}$$

for any (x, f) in $T^*\mathcal{G}$; where j_{11}, j_1, j_2 are elements of \mathcal{J} such that

- j_{11} is invertible,

- j_1 is μ -symmetric
- and j_2 satisfies the following two relations

$$j_2 \circ ad_x = 0, \quad (4.60)$$

$$\langle j_2 \circ \theta^{-1}(f), g \rangle = \langle f, j_2 \circ \theta^{-1}(g) \rangle \quad (4.61)$$

for all x in \mathcal{G} and all f, g in \mathcal{G}^* .

Proof. We have already seen that any bi-invariant \langle, \rangle -symmetric invertible endomorphism of $T^*\mathcal{G}$ is given by

$$j(x, f) = \left(j_{11}(x) + j_{21}(f), j_{12}(x) + j_{11}^t(f) \right),$$

with conditions listed in Proposition 4.3.1.

The linear maps $\theta^{-1} \circ j_{12} : \mathcal{G} \rightarrow \mathcal{G}$ and $j_{21} \circ \theta : \mathcal{G} \rightarrow \mathcal{G}$ commutes with all adjoint operators of \mathcal{G} ; that is there exists j_1, j_2 in \mathcal{J} such that $\theta^{-1} \circ j_{12} = j_1$ and $j_{21} \circ \theta = j_2$. It comes that $j_{12} = \theta \circ j_1$ and $j_{21} = j_2 \circ \theta^{-1}$.

Now, (4.47) is equivalent to $j_2 \circ \theta^{-1} \circ ad_x^* = 0$, for all x in \mathcal{G} . Since $\theta^{-1} \circ ad_x^* = ad_x \circ \theta^{-1}$, for all x in \mathcal{G} , we have $j_2 \circ ad_x \circ \theta^{-1} = 0$, for any x in \mathcal{G} . It comes that $j_2 \circ ad_x = 0$, for all x in \mathcal{G} , since θ^{-1} is an isomorphism.

Relation (4.48) becomes $\langle \theta \circ j_1(x), y \rangle = \langle x, \theta \circ j_1(y) \rangle$, for any x, y in \mathcal{G} . The latter can be written $\mu(j_1(x), y) = \mu(x, j_1(y))$, for any x, y in \mathcal{G} .

Last, Relation (4.49) reads $\langle j_2 \circ \theta^{-1}(f), g \rangle = \langle f, j_2 \circ \theta^{-1}(g) \rangle$, for all f, g in \mathcal{G}^* . \square

Remark 4.4.1. The relation $j_2 \circ ad_x = 0$, for all x in \mathcal{G} means that j_2 vanishes identically on the derived ideal $[\mathcal{G}, \mathcal{G}]$ of \mathcal{G} . So, if \mathcal{G} is perfect, then j_2 vanishes identically on all \mathcal{G} .

The following comes immediately from Lemma 4.4.1 and Remark 4.4.1

Corollary 4.4.1. Let (\mathcal{G}, μ) be an orthogonal and perfect Lie algebra. Then, any bi-invariant \langle, \rangle -symmetric invertible endomorphism $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ is given by

$$j(x, f) = \left(j_{11}(x), \theta \circ j_1(x) + j_{11}^t(f) \right), \quad (4.62)$$

for any (x, f) in $T^*\mathcal{G}$; where j_{11}, j_1 are bi-invariant tensors \mathcal{G} such that j_1 is μ -symmetric and j_{11} is invertible.

Theorem 4.4.1. Let (\mathcal{G}, μ) be an orthogonal Lie algebra. Any orthogonal structure $\mu_{\mathcal{D}}$ on $T^*\mathcal{G}$ is given by

$$\mu_{\mathcal{D}}\left((x, f), (y, g)\right) = \langle g, j_{11}(x) \rangle + \langle f, j_{11}(y) \rangle + \langle g, j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x), y \rangle, \quad (4.63)$$

for all $(x, f), (y, g)$ in $T^*\mathcal{G}$, where j_{11}, j_1 and j_2 satisfy conditions listed in Lemma 4.4.1.

Proof. Any orthogonal structure $\mu_{\mathcal{D}}$ on $T^*\mathcal{G}$ is linked to the duality pairing by means of a map $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ defined by (4.59) through the following formula valid for all $(x, f), (y, g)$ in $T^*\mathcal{G}$:

$$\begin{aligned} \mu_{\mathcal{D}}\left((x, f), (y, g)\right) &= \langle j(x, f), (y, g) \rangle. \\ &= \left\langle \left(j_{11}(x) + j_2 \circ \theta^{-1}(f), \theta \circ j_1(x) + j_{11}^t(f) \right), (y, g) \right\rangle \\ &= \langle g, j_{11}(x) + j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x) + j_{11}^t(f), y \rangle \\ &= \langle g, j_{11}(x) \rangle + \langle g, j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x), y \rangle + \langle j_{11}^t(f), y \rangle \\ &= \langle g, j_{11}(x) \rangle + \langle g, j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x), y \rangle + \langle f, j_{11}(y) \rangle \end{aligned}$$

□

We have the following consequence coming from Theorem 4.4.1 and Remark 4.4.1.

Corollary 4.4.2. *Let (\mathcal{G}, μ) be an orthogonal and perfect Lie algebra. Any orthogonal structure $\mu_{\mathcal{D}}$ on $T^*\mathcal{G}$ is given by*

$$\mu_{\mathcal{D}}\left((x, f), (y, g)\right) = \langle g, j_{11}(x) \rangle + \langle f, j_{11}(y) \rangle + \langle \theta \circ j_1(x), y \rangle, \quad (4.64)$$

for all $(x, f), (y, g)$ in $T^*\mathcal{G}$, where j_{11} and j_1 are bi-invariant tensors of \mathcal{G} such that j_{11} is invertible, j_1 is μ -symmetric.

Let us now characterise skew-symmetric prederivations of $T^*\mathcal{G}$ in the case where \mathcal{G} is an orthogonal Lie algebra.

4.4.2 Skew-symmetric Prederivations

Proposition 4.4.1. *Let (\mathcal{G}, μ) be an orthogonal Lie algebra. Then any prederivation ϕ of $T^*\mathcal{G}$ which is skew-symmetric with respect to the orthogonal structure $\mu_{\mathcal{D}}$ defined by (4.63) can be written as*

$$\phi(x, f) = \left(\alpha_1(x) + j'_1 \circ \theta^{-1}(f), \theta \circ \alpha_2(x) + (j_2'^t - \alpha_1^t)(f) \right), \quad (4.65)$$

where α_1, α_2 are in $Pder(\mathcal{G})$; j'_1, j'_2 are in \mathcal{J}' with the following additional conditions:

$$\theta \circ j_1 \circ \alpha_1 + \alpha_1^t \circ \theta \circ j_1 + \alpha_2^t \circ \theta \circ j_{11} = 0 \quad (4.66)$$

$$(j_{11} \circ j'_1 + j'_2 \circ j_2 - \alpha_1 \circ j_2) \circ \theta^{-1} + j_2 \circ \theta^{-1} \circ (j_2'^t - \alpha_1^t) + \theta^{-t} \circ j_1'^t \circ j_{11}^t = 0 \quad (4.67)$$

$$[j_{11}, \alpha_1] + j_2 \circ \alpha_2 + j'_2 \circ j_{11} + \theta^{-t} \circ j_1'^t \circ \theta \circ j_1 = 0, \quad (4.68)$$

where θ^{-t} is the transpose of θ^{-1} .

Proof. From Proposition 3.4.1, if \mathcal{G} is an orthogonal Lie algebra, then any prederivation of $T^*\mathcal{G}$ is given by

$$\phi(x, f) = \left(\alpha_1(x) + j'_1 \circ \theta^{-1}(f), \theta \circ \alpha_2(x) + (j_2'^t - \alpha_1^t)(f) \right),$$

for every (x, f) in $T^*\mathcal{G}$, where α_1, α_2 are prederivations of \mathcal{G} , j'_1, j'_2 are in \mathcal{J}' . The prederivation ϕ is μ_D -skew-symmetric means that

$$\mu_{\mathcal{D}}\left(\phi(x, f), (y, g)\right) + \mu_{\mathcal{D}}\left((x, f), \phi(y, g)\right) = 0, \quad (4.69)$$

for all $(x, f), (y, g)$ in $T^*\mathcal{G}$. Firstly, we compute $\mu_{\mathcal{D}}(\phi(x, f), (y, g))$.

$$\begin{aligned} \mu_{\mathcal{D}}\left(\phi(x, f), (y, g)\right) &= \mu_{\mathcal{D}}\left((\alpha_1(x) + j'_1 \circ \theta^{-1}(f), \theta \circ \alpha_2(x) + (j_2'^t - \alpha_1^t)(f)), (y, g)\right) \\ &= \left\langle g, j_{11}\left(\alpha_1(x) + j'_1 \circ \theta^{-1}(f)\right) \right\rangle + \langle f, j_{11}(y) \rangle \\ &\quad + \left\langle g, j_2 \circ \theta^{-1}\left(\theta \circ \alpha_2(x) + (j_2'^t - \alpha_1^t)(f)\right) \right\rangle \\ &\quad + \left\langle \theta \circ j_1\left(\alpha_1(x) + j'_1 \circ \theta^{-1}(f)\right), y \right\rangle \\ &= \left\langle g, j_{11} \circ \alpha_1(x) \right\rangle + \left\langle g, j_{11} \circ j'_1 \circ \theta^{-1}(f) \right\rangle + \langle f, j_{11}(y) \rangle \\ &\quad + \left\langle g, j_2 \circ \alpha_2(x) \right\rangle + \left\langle g, j_2 \circ \theta^{-1} \circ j_2'^t(f) \right\rangle \\ &\quad - \left\langle g, j_2 \circ \theta^{-1} \circ \alpha_1^t(f) \right\rangle + \left\langle \theta \circ j_1 \circ \alpha_1(x), y \right\rangle \\ &\quad + \left\langle \theta \circ j_1 \circ j'_1 \circ \theta^{-1}(f), y \right\rangle. \end{aligned} \quad (4.70)$$

Secondly,

$$\begin{aligned} \mu_{\mathcal{D}}\left((x, f), \phi(y, g)\right) &= \mu_{\mathcal{D}}\left((x, f), (\alpha_1(y) + j'_1 \circ \theta^{-1}(g), \theta \circ \alpha_2(y) + (j_2'^t - \alpha_1^t)(g))\right) \\ &= \left\langle \theta \circ \alpha_2(y) + (j_2'^t - \alpha_1^t)(g), j_{11}(x) \right\rangle + \left\langle f, j_{11}(\alpha_1(y) + j'_1 \circ \theta^{-1}(g)) \right\rangle \\ &\quad + \left\langle \theta \circ \alpha_2(y) + (j_2'^t - \alpha_1^t)(g), j_2 \circ \theta^{-1}(f) \right\rangle \\ &\quad + \left\langle \theta \circ j_1(x), \alpha_1(y) + j'_1 \circ \theta^{-1}(g) \right\rangle \\ &= \left\langle \theta \circ \alpha_2(y), j_{11}(x) \right\rangle + \left\langle j_2'^t(g), j_{11}(x) \right\rangle - \left\langle \alpha_1^t(g), j_{11}(x) \right\rangle \\ &\quad + \left\langle f, j_{11} \circ \alpha_1(y) \right\rangle + \left\langle f, j_{11} \circ j'_1 \circ \theta^{-1}(g) \right\rangle \\ &\quad + \left\langle \theta \circ \alpha_2(y), j_2 \circ \theta^{-1}(f) \right\rangle + \left\langle j_2'^t(g), j_2 \circ \theta^{-1}(f) \right\rangle \\ &\quad - \left\langle \alpha_1^t(g), j_2 \circ \theta^{-1}(f) \right\rangle + \left\langle \theta \circ j_1(x), \alpha_1(y) \right\rangle \\ &\quad + \left\langle \theta \circ j_1(x), j'_1 \circ \theta^{-1}(g) \right\rangle. \end{aligned} \quad (4.71)$$

Now, if we take $f = g = 0$, (4.69) gives

$$\left\langle \theta \circ j_1 \circ \alpha_1(x), y \right\rangle + \left\langle \theta \circ \alpha_2(y), j_{11}(x) \right\rangle + \left\langle \theta \circ j_1(x), \alpha_1(y) \right\rangle = 0$$

which can be written

$$\left\langle \theta \circ j_1 \circ \alpha_1(x), y \right\rangle + \left\langle y, \alpha_2^t \circ \theta \circ j_{11}(x) \right\rangle + \left\langle \alpha_1^t \circ \theta \circ j_1(x), y \right\rangle = 0$$

for all x, y in \mathcal{G} . Then,

$$\theta \circ j_1 \circ \alpha_1 + \alpha_1^t \circ \theta \circ j_1 + \alpha_2^t \circ \theta \circ j_{11} = 0.$$

We can rewrite (4.69) as follows

$$\begin{aligned}
& \langle g, j_{11} \circ \alpha_1(x) \rangle + \langle g, j_{11} \circ j'_1 \circ \theta^{-1}(f) \rangle + \langle f, j_{11}(y) \rangle \\
& + \langle g, j_2 \circ \alpha_2(x) \rangle + \langle g, j_2 \circ \theta^{-1} \circ j_2'^t(f) \rangle - \langle g, j_2 \circ \theta^{-1} \circ \alpha_1^t(f) \rangle \\
& + \langle \theta \circ j_1 \circ j'_1 \circ \theta^{-1}(f), y \rangle + \langle j_2'^t(g), j_{11}(x) \rangle - \langle \alpha_1^t(g), j_{11}(x) \rangle \\
& + \langle f, j_{11} \circ \alpha_1(y) \rangle + \langle f, j_{11} \circ j'_1 \circ \theta^{-1}(g) \rangle + \langle \theta \circ \alpha_2(y), j_2 \circ \theta^{-1}(f) \rangle \\
& + \langle j_2'^t(g), j_2 \circ \theta^{-1}(f) \rangle - \langle \alpha_1^t(g), j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x), j'_1 \circ \theta^{-1}(g) \rangle = 0.
\end{aligned} \tag{4.72}$$

If we take $x = y = 0$, (4.72) becomes

$$\begin{aligned}
& \langle g, j_{11} \circ j'_1 \circ \theta^{-1}(f) \rangle + \langle g, j_2 \circ \theta^{-1} \circ j_2'^t(f) \rangle - \langle g, j_2 \circ \theta^{-1} \circ \alpha_1^t(f) \rangle \\
& + \langle f, j_{11} \circ j'_1 \circ \theta^{-1}(g) \rangle + \langle j_2'^t(g), j_2 \circ \theta^{-1}(f) \rangle - \langle \alpha_1^t(g), j_2 \circ \theta^{-1}(f) \rangle = 0
\end{aligned} \tag{4.73}$$

which is

$$\begin{aligned}
& \langle g, j_{11} \circ j'_1 \circ \theta^{-1}(f) \rangle + \langle g, j_2 \circ \theta^{-1} \circ j_2'^t(f) \rangle - \langle g, j_2 \circ \theta^{-1} \circ \alpha_1^t(f) \rangle \\
& + \langle \theta^{-t} \circ j_1'^t \circ j_{11}^t(f), g \rangle + \langle g, j_2' \circ j_2 \circ \theta^{-1}(f) \rangle - \langle g, \alpha_1 \circ j_2 \circ \theta^{-1}(f) \rangle = 0,
\end{aligned}$$

for all f, g in \mathfrak{G}^* , where θ^{-t} is the transpose of θ^{-1} . This latter equality is equivalent to

$$(j_{11} \circ j'_1 + j_2' \circ j_2 - \alpha_1 \circ j_2) \circ \theta^{-1} + j_2 \circ \theta^{-1} \circ (j_2'^t - \alpha_1^t) + \theta^{-t} \circ j_1'^t \circ j_{11}^t = 0$$

Last, (4.69) have a more simple expression :

$$\begin{aligned}
& \langle g, j_{11} \circ \alpha_1(x) \rangle + \langle f, j_{11}(y) \rangle + \langle g, j_2 \circ \alpha_2(x) \rangle \\
& + \langle \theta \circ j_1 \circ j'_1 \circ \theta^{-1}(f), y \rangle + \langle j_2'^t(g), j_{11}(x) \rangle - \langle \alpha_1^t(g), j_{11}(x) \rangle \\
& + \langle f, j_{11} \circ \alpha_1(y) \rangle + \langle \theta \circ \alpha_2(y), j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x), j'_1 \circ \theta^{-1}(g) \rangle = 0.
\end{aligned} \tag{4.74}$$

We take $y = 0$ and obtain

$$\langle g, j_{11} \circ \alpha_1(x) \rangle + \langle g, j_2 \circ \alpha_2(x) \rangle + \langle g, j_2' \circ j_{11}(x) \rangle - \langle g, \alpha_1 \circ j_{11}(x) \rangle + \langle \theta \circ j_1(x), j'_1 \circ \theta^{-1}(g) \rangle = 0 \tag{4.75}$$

The latter equality is equivalent to

$$[j_{11}, \alpha_1] + j_2 \circ \alpha_2 + j_2' \circ j_{11} + \theta^{-t} \circ j_1'^t \circ \theta \circ j_1 = 0.$$

□

4.5 Case of Semi-simple Lie Algebras

Let $\mathfrak{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \dots \oplus \mathfrak{s}_p$ be a semi-simple Lie algebra, where \mathfrak{s}_i , $i = 1, 2, \dots, p$ ($p \in \mathbb{N}^*$) are simple ideals.

4.5.1 Bi-invariant metrics On Cotangent bundles of Semi-simple Lie groups

Lemma 4.5.1. Any \langle, \rangle -symmetric and invertible bi-invariant tensor $j : T^*\mathfrak{G} \rightarrow T^*\mathfrak{G}$ is defined by

$$j(x, f) = \left(\sum_{i=1}^p \lambda_i x_i, \sum_{k=1}^p \nu_k \theta(x_k) + \sum_{i=1}^p \lambda_i f_i \right), \quad (4.76)$$

where $x = x_1 + x_2 + \cdots + x_p \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, $f = f_1 + f_2 + \cdots + f_p \in \mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \cdots \oplus \mathfrak{s}_p^*$, λ_i , ($i = 1, 2, \dots, p$) are non-zero real numbers, ν_i ($i = 1, 2, \dots, p$) are any real numbers and $\theta : \mathfrak{G} \rightarrow \mathfrak{G}^*$ is the isomorphism induced by any orthogonal structure on \mathfrak{G} through the formula (1.2).

Proof. According to Corollary 4.4.1, any bi-invariant, \langle, \rangle -symmetric and invertible endomorphism $j : T^*\mathfrak{G} \rightarrow T^*\mathfrak{G}$ has the form

$$j(x, f) = \left(j_{11}(x), \theta \circ j_1(x) + j_{11}^t(f) \right), \quad (4.77)$$

for all (x, f) in $T^*\mathfrak{G}$, where j_{11} and j_1 are bi-invariant endomorphisms of \mathfrak{G} such that

- j_{11} is invertible,
- j_1 is symmetric with respect to any orthogonal structure on \mathfrak{G} .

Now we have seen in Section 2.4.1 that the map j_{11} must have the form

$$j_{11}(x) = \sum_{i=1}^p \lambda_i x_i, \quad (4.78)$$

if $x = x_1 + x_2 + \cdots + x_p \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, where $\lambda_i, i = 1, 2, \dots, p$ are real numbers. It comes that

$$j_{11}^t(f) = \sum_{i=1}^p \lambda_i f_i, \quad (4.79)$$

if $f = f_1 + f_2 + \cdots + f_p \in \mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \cdots \oplus \mathfrak{s}_p^*$.

We also have

$$j_1(x) = \sum_{k=1}^p \nu_k x_k, \quad (4.80)$$

where $\nu_k, k = 1, 2, \dots, p$, are real numbers. So we have

$$\begin{aligned} \theta \circ j_1(x) &= \theta \left(\sum_{k=1}^p \nu_k x_k \right) \\ &= \sum_{k=1}^p \nu_k \theta(x_k) \end{aligned} \quad (4.81)$$

We then can write j as follows

$$j(x, f) = \left(\sum_{i=1}^p \lambda_i x_i, \sum_{k=1}^p \nu_k \theta(x_k) + \sum_{i=1}^p \lambda_i f_i \right) \quad (4.82)$$

Now j is invertible if and only if $j_{11} = \sum_{i=1}^p \lambda_i Id_{\mathfrak{s}_i}$ is invertible.

For all $i = 1, 2, \dots, p$, note by n_i the dimension of the simple ideal \mathfrak{s}_i ($n_i := \dim \mathfrak{s}_i$). The matrix of j_{11} , in some basis $\{e_{11}, e_{12}, \dots, e_{1n_1}; e_{21}, e_{22}, \dots, e_{2n_2}; \dots; e_{p1}, e_{p2}, \dots, e_{pn_p}\}$, is the following

$$\begin{pmatrix} A_1 & O & \dots & O \\ O & A_2 & \dots & O \\ \vdots & \vdots & \ddots & O \\ O & O & \dots & A_p \end{pmatrix} \tag{4.83}$$

where A_i is the $n_i \times n_i$ -matrix defined as follows

$$A_i = \text{diag}(\underbrace{\lambda_i, \lambda_i, \dots, \lambda_i}_{n_i \text{ times}}) = \begin{pmatrix} \lambda_i & O & \dots & O \\ O & \lambda_i & \dots & O \\ \vdots & \vdots & \ddots & O \\ O & O & \dots & \lambda_i \end{pmatrix} \tag{4.84}$$

The determinant of j_{11} is then

$$\det(j_{11}) = \prod_{k=1}^p \lambda_k^{n_k}. \tag{4.85}$$

Then, j_{11} is invertible if and only if each of λ_k is non-zero. □

Remark 4.5.1. The matrix of j in the basis $\{e_{11}, e_{12}, \dots, e_{1n_1}; e_{21}, e_{22}, \dots, e_{2n_2}; \dots; e_{p1}, e_{p2}, \dots, e_{pn_p}; e_{11}^*, e_{12}^*, \dots, e_{1n_1}^*; e_{21}^*, e_{22}^*, \dots, e_{2n_2}^*; \dots; e_{p1}^*, e_{p2}^*, \dots, e_{pn_p}^*\}$ is given by

$$\begin{pmatrix} A_1 & O & \dots & O & O & O & \dots & O \\ O & A_2 & \dots & O & O & O & \dots & O \\ \vdots & \vdots & \ddots & O & \vdots & \vdots & \vdots & \vdots \\ O & O & \dots & A_p & O & O & \dots & O \\ & & & & A_1 & O & \dots & O \\ & & & & O & A_2 & \dots & O \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & O & O & \dots & A_p \end{pmatrix} \tag{4.86}$$

and the determinant of j is

$$\det(j) = \prod_{k=1}^p \lambda_k^{2n_k}. \tag{4.87}$$

Note by $\langle \cdot, \cdot \rangle_{\mathfrak{s}_i}$ the duality pairing between \mathfrak{s}_i and its dual space \mathfrak{s}_i^* . The following theorem characterizes all orthogonal structures on $T^*\mathcal{G}$.

Theorem 4.5.1. Let \mathcal{G} be a semi-simple Lie algebra. Any orthogonal structure μ on $T^*\mathcal{G}$ is given by

$$\mu((x, f), (y, g)) = \sum_{i=1}^p \lambda_i \langle (x_i, f_i), (y_i, g_i) \rangle_{\mathfrak{s}_i} + \sum_{k=1}^p \nu_k \langle \theta(x_k), y_k \rangle_{\mathfrak{s}_k}, \tag{4.88}$$

for all x, y in $\mathfrak{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$ and f, g in $\mathfrak{G}^* = \mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \cdots \oplus \mathfrak{s}_p^*$; where for all $i = 1, 2, \dots, p$, (λ_i, ν_i) belongs to $\mathbb{R}^* \times \mathbb{R}$.

In particular, if \mathfrak{G} is simple, then any orthogonal structure can be written

$$\mu((x, f), (y, g)) = \lambda \langle (x, f), (y, g) \rangle + \nu \langle \theta(x), y \rangle, \quad (4.89)$$

for all $(x, f), (y, g)$ in $T^*\mathfrak{G}$, where (λ, ν) is in $\mathbb{R}^* \times \mathbb{R}$.

Proof. Any orthogonal structure on $T^*\mathfrak{G}$ is given by an adjoint-invariant invertible and \langle, \rangle -symmetric endomorphism j through the formula

$$\mu_j((x, f), (y, g)) = \langle j(x, f), (y, g) \rangle, \quad (4.90)$$

We have seen in Lemma 4.5.1 that

$$j(x, f) = \left(\sum_{i=1}^p \lambda_i x_i, \sum_{k=1}^p \nu_k \theta(x_k) + \sum_{i=1}^p \lambda_i f_i \right),$$

where $x = x_1 + x_2 + \cdots + x_p \in \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$, $f = f_1 + f_2 + \cdots + f_p \in \mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \cdots \oplus \mathfrak{s}_p^*$, λ_i, ν_i , ($i = 1, 2, \dots, p$) are real numbers, $\lambda_i \neq 0$, for all $i = 1, 2, \dots, p$. We can write

$$\begin{aligned} \mu_j((x, f), (y, g)) &= \sum_{i=1}^p \lambda_i g(x_i) + \sum_{k=1}^p \nu_k \theta(x_k)(y) + \sum_{i=1}^p \lambda_i f_i(y) \\ &= \sum_{i=1}^p \lambda_i (g(x_i) + f_i(y)) + \sum_{k=1}^p \nu_k \theta(x_k)(y) \\ &= \sum_{i=1}^p \lambda_i (g_i(x_i) + f_i(y_i)) + \sum_{k=1}^p \nu_k \theta(x_k)(y_k) \\ &= \sum_{i=1}^p \lambda_i \langle (x_i, f_i), (y_i, g_i) \rangle_{\mathfrak{s}_i} + \sum_{k=1}^p \nu_k \langle \theta(x_k), y_k \rangle_{\mathfrak{s}_k} \end{aligned} \quad (4.91)$$

So, the first part of the theorem is proved.

Now, if \mathfrak{G} is simple, then $\mathfrak{G} = \mathfrak{s}_1$, $x = x_1$, $f = f_1$ and Relation (4.91) becomes

$$\mu_j((x, f), (y, g)) = \lambda \langle (x, f), (y, g) \rangle + \nu \langle \theta(x), y \rangle \quad (4.92)$$

where $\lambda \in \mathbb{R}^*$ and $\nu \in \mathbb{R}$. □

Remark 4.5.2. If $\mathfrak{G} = \mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_p$ is a semi-simple Lie algebra, where $\mathfrak{s}_i, i = 1, 2, \dots, p$ are simple ideals, then on any of the ideals \mathfrak{s}_i the Killing form $K_{\mathfrak{s}_i}$ defines an orthogonal structure. Thus, Relation (4.88) can be written

$$\mu((x, f), (y, g)) = \sum_{i=1}^p \lambda_i \langle (x_i, f_i), (y_i, g_i) \rangle_{\mathfrak{s}_i} + \sum_{k=1}^p \nu_k K_{\mathfrak{s}_k}(x_k, y_k) \quad (4.93)$$

and Relation (4.89) can be expressed as

$$\mu_j((x, f), (y, g)) = \lambda \langle (x, f), (y, g) \rangle + \nu K(x, y), \quad (4.94)$$

where K stands for the Killing form on \mathfrak{G} .

Let us now study the group of isometries of bi-invariant metrics on T^*G , when G is a semi-simple Lie group.

4.5.2 Skew-symmetric Prederivations

Proposition 4.5.1. *Let \mathcal{G} be a semi-simple Lie. Then any prederivation of $T^*\mathcal{G}$ which is skew-symmetric with respect to any orthogonal structure μ on $T^*\mathcal{G}$ is an inner derivation of $T^*\mathcal{G}$; that is if ϕ is a μ -skew-symmetric prederivation of $T^*\mathcal{G}$, then there exist x_0 in \mathcal{G} and f_0 in \mathcal{G}^* such that*

$$\phi(x, f) = ([x_0, x], ad_{x_0}^* f - ad_x^* f_0), \quad (4.95)$$

for every x and f in \mathcal{G} and \mathcal{G}^* respectively.

Proof. We have shown in Theorem 3.4.1 that if \mathcal{G} is a semi-simple Lie algebra, then every prederivation of $T^*\mathcal{G}$ is a derivation. From Relation (2.62) we have that, if \mathcal{G} is semi-simple, any derivation of $T^*\mathcal{G}$ has the form

$$\phi(x, f) = ([x_0, x], ad_{x_0}^* f - ad_x^* f_0 + \sum_{i=1}^p \gamma_i f_i) \quad (4.96)$$

for every x in \mathcal{G} and every $f := f_1 + f_2 + \dots + f_p$ in $\mathfrak{s}_1^* \oplus \mathfrak{s}_2^* \oplus \dots \oplus \mathfrak{s}_p^* = \mathcal{G}^*$, where x_0 is in \mathcal{G} , f_0 is in \mathcal{G}^* and $\gamma_i, i = 1, \dots, p$, are real numbers. To prove the Proposition it suffices to prove that ϕ is μ -skew-symmetric if and only if $\gamma_i = 0$, for all $i = 1, 2, \dots, p$.

$$\begin{aligned} \mu(\phi(x, f), (y, g)) &= \mu\left([x_0, x], ad_{x_0}^* f - ad_x^* f_0 + \sum_{i=1}^p \gamma_i f_i, (y, g)\right) \\ &= \sum_{i=1}^p \lambda_i \left\langle \left([x_0, x]_i, (ad_{x_0}^* f - ad_x^* f_0 + \sum_{k=1}^p \gamma_k f_k)_i\right), (y_i, g_i) \right\rangle_{\mathfrak{s}_i} \\ &\quad + \sum_{k=1}^p \nu_k \left\langle \theta(y_k), [x_0, x]_k \right\rangle_{\mathfrak{s}_k} \end{aligned}$$

That is

$$\begin{aligned} \mu(\phi(x, f), (y, g)) &= \sum_{i=1}^p \lambda_i \left[g_i([x_0, x]_i) + (ad_{x_0}^* f)_i(y_i) - (ad_x^* f_0)_i(y_i) + \left(\sum_{k=1}^p \gamma_k f_k\right)_i(y_i) \right] \\ &\quad + \sum_{k=1}^p \nu_k \langle \theta(y_k), [x_0, x]_k \rangle \end{aligned} \quad (4.97)$$

By the same way we obtain

$$\begin{aligned} \mu((x, f), \phi(y, g)) &= \sum_{i=1}^p \lambda_i \left[f_i([x_0, y]_i) + (ad_{x_0}^* g)_i(x_i) - (ad_y^* f_0)_i(x_i) + \left(\sum_{k=1}^p \gamma_k g_k\right)_i(x_i) \right] \\ &\quad + \sum_{k=1}^p \nu_k \langle \theta([x_0, y]_k), x_k \rangle. \end{aligned} \quad (4.98)$$

Now ϕ is μ -symmetric means that

$$0 = \mu(\phi(x, f), (y, g)) + \mu((x, f), \phi(y, g))$$

$$\begin{aligned}
&= - \sum_{i=1}^p \lambda_i \left[(ad_x^* f_0)_i(y_i) + (ad_y^* f_0)(x_i) \right] + \sum_{k=1}^p \nu_k \left[\langle \theta(y_k), [x_0, x]_k \rangle + \langle \theta([x_0, y]_k), x_k \rangle \right] \\
&\quad + \sum_{i=1}^p \lambda_i g_i([x_0, x]_i) + \sum_{i=1}^p \lambda_i (ad_{x_0}^* f)_i(y_i) + \sum_{i=1}^p \lambda_i \left(\sum_{k=1}^p \gamma_k f_k \right)_i (y_i) \\
&\quad + \sum_{i=1}^p \lambda_i f_i([x_0, y]_i) + \sum_{i=1}^p \lambda_i (ad_{x_0}^* g)_i(x_i) + \sum_{i=1}^p \lambda_i \left(\sum_{k=1}^p \gamma_k g_k \right)_i (x_i) \\
&= \sum_{i=1}^p \lambda_i f_0 \left(\underbrace{[x_i, y_i] + [y_i, x_i]}_{=0} \right) + \sum_{k=1}^p \nu_k \left[\underbrace{\langle \theta(y_k), [x_0, x]_k \rangle - \langle \theta(y_k), [x_0, x]_k \rangle}_{=0} \right] \\
&\quad + \sum_{i=1}^p \lambda_i \left[\underbrace{g_i([x_0, x]_i) + (ad_{x_0}^* g)_i(x_i)}_{=0} \right] + \sum_{i=1}^p \left[\underbrace{f([x_0, y]_i) + (ad_{x_0}^* f)_i(y_i)}_{=0} \right] \\
&\quad + \sum_{i=1}^p \lambda_i \left[\left(\sum_{k=1}^p \gamma_k f_k \right)_i (y_i) + \left(\sum_{k=1}^p \gamma_k g_k \right)_i (x_i) \right]. \tag{4.99}
\end{aligned}$$

We then have

$$\begin{aligned}
\sum_{i=1}^p \lambda_i \left[\left(\sum_{k=1}^p \gamma_k f_k \right)_i (y_i) + \left(\sum_{k=1}^p \gamma_k g_k \right)_i (x_i) \right] &= 0 \\
\sum_{i=1}^p \lambda_i \left[\gamma_i f_i(y_i) + \gamma_i g_i(x_i) \right] &= 0 \tag{4.100}
\end{aligned}$$

for all x, y in \mathcal{G} and all f, g in \mathcal{G}^* . Now if we take $y = 0$, we obtain

$$\sum_{i=1}^p \lambda_i \left(\sum_{k=1}^p \gamma_k g_k \right)_i (x_i) = 0, \tag{4.101}$$

for all x, y in \mathcal{G} and all f, g in \mathcal{G}^* . That is

$$\sum_{i=1}^p \lambda_i \gamma_i g_i(x_i) = 0, \tag{4.102}$$

for all $x = x_1 + x_2 + \dots + x_p \in \mathcal{G}$ and for all $g = g_1 + g_2 + \dots + g_p$ in \mathcal{G}^* . Since, $\lambda_i \neq 0$, for any $i = 1, 2, \dots, p$, then we have $\gamma_i = 0$, for any $i = 1, 2, \dots, p$. \square

4.6 Examples

4.6.1 The Affine Lie Group of the Real Line \mathbb{R}

Let $G := \text{Aff}(\mathbb{R})$ be the Lie group of affine motions of \mathbb{R} and let $\mathcal{G} := \text{aff}(\mathbb{R})$ its Lie algebra. We note by $T^*\mathcal{G} := T^*\text{aff}(\mathbb{R}) = \mathcal{G} \ltimes \mathcal{G}^*$ the Lie algebra of the cotangent bundle of the Lie group G . If $\mathcal{G} := \text{span}\{e_1, e_2\}$ and $\{e_3, e_4\}$ denotes the dual basis of $\{e_1, e_2\}$, we have the following brackets:

$$[e_1, e_2] = e_2, \quad [e_1, e_4] = -e_4, \quad [e_2, e_4] = e_3 \tag{4.103}$$

Proposition 4.6.1. *Let \langle, \rangle denote the duality pairing between \mathcal{G} and \mathcal{G}^* .*

1. *Any orthogonal structure μ on $T^*\mathcal{G}$ is given by the following expression:*

$$\mu\left((x, f), (y, g)\right) = a\langle(x, f), (y, g)\rangle + bx_1y_1, \quad (4.104)$$

for all $(x, f) = x_1e_1 + x_2e_2 + f_3e_3 + f_4e_4$ and $(y, g) = y_1e_1 + y_2e_2 + g_3e_3 + g_4e_4$ in $T^*\mathcal{G}$, where $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$.

2. *Any orthogonal structure μ on $T^*\mathcal{G}$ is of signature $(2, 2)$.*

Proof. On the basis (e_1, e_2, e_3, e_4) of $T^*\mathcal{G}$ an invertible bi-invariant tensor $j : T^*\mathcal{G} \rightarrow T^*\mathcal{G}$ has the following matrix

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ b & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad a \in \mathbb{R}^*, \quad b \in \mathbb{R}. \quad (4.105)$$

We note $j_{a,b} := j$ if j is defined by (4.105). One can verify that $j_{a,b}$ is \langle, \rangle -symmetric.

Now any adjoint-invariant scalar product on $T^*\mathcal{G}$ is given by an adjoint-invariant invertible and \langle, \rangle -symmetric endomorphism $j_{a,b}$. We note it by $\mu_{j_{a,b}}$ or simply by $\mu_{a,b}$. We have

$$\mu_{a,b}\left((x, f), (y, g)\right) = \langle j_{a,b}(x, f), (y, g)\rangle, \quad (4.106)$$

for all (x, f) and (y, g) in $T^*\mathcal{G}$. Now if we set $(x, f) = x_1e_1 + x_2e_2 + f_3e_3 + f_4e_4$ and $(y, g) = y_1e_1 + y_2e_2 + g_3e_3 + g_4e_4$, then we have

$$\begin{aligned} \mu_{a,b}\left((x, f), (y, g)\right) &= \langle ax_1e_1 + ax_2e_2 + (bx_1 + af_3)e_3 + af_4, y_1e_1 + y_2e_2 + g_3e_3 + g_4e_4 \rangle \\ &= ax_1g_3 + ax_2g_4 + (bx_1 + af_3)y_1 + ay_2f_4 \\ &= a(x_1g_3 + x_2g_4 + y_1f_3 + y_2f_4) + bx_1y_1. \end{aligned}$$

That is

$$\mu_{a,b}\left((x, f), (y, g)\right) = a\langle(x, f), (y, g)\rangle + bx_1y_1,$$

for all (x, f) and (y, g) in $T^*\mathcal{G}$, where $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$.

Let us study the signature of the scalar product $\mu_{a,b}$. The matrix of $\mu_{a,b}$ on the basis (e_1, e_2, e_3, e_4) of $T^*\mathcal{G}$ is given by

$$M_{a,b} = \begin{pmatrix} b & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}. \quad (4.107)$$

The characteristic polynomial of $M_{a,b}$ is

$$P_{a,b}(t) = (t - a)(t + a)(t^2 - bt - a^2), \quad (4.108)$$

for any $t \in \mathbb{R}$. It is now a little matter to check that $M_{a,b}$ has four eigenvalues :

$$-a \quad , \quad a \quad , \quad \frac{b - \sqrt{b^2 + 4a^2}}{2} \quad , \quad \frac{b + \sqrt{b^2 + 4a^2}}{2}.$$

Two of these eigenvalues are less than zero and two are greater than zero. Hence, $\mu_{a,b}$ is of signature $(2, 2)$ for all $(a, b) \in \mathbb{R}^* \times \mathbb{R}$. \square

Let G_1 stand for the identity connected component of G . We also note by $\mu_{a,b}$ the bi-invariant metric induced on the Lie group T^*G_1 by the orthogonal structure $\mu_{a,b}$ given by (4.104).

Proposition 4.6.2. *Any $\mu_{a,b}$ -skew-symmetric prederivation ϕ of $T^*\mathcal{G}$ is an inner derivation of $T^*\mathcal{G}$.*

Proof. A prederivation α of $T^*\mathcal{G}$ has the following matrix on the basis (e_1, e_2, e_3, e_4) of $T^*\mathcal{G}$ (see Examples 2.5.1 and 3.5.1):

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & -\alpha_{21} \\ -\alpha_{32} & 0 & 0 & \alpha_{33} - \alpha_{22} \end{pmatrix} \quad (4.109)$$

A $\mu_{a,b}$ -skew-symmetric prederivation α of $T^*\mathcal{G}$ is characterised by

$$\mu_{a,b}(\alpha(x, f), (y, g)) = -\mu_{a,b}((x, f), \alpha(y, g)), \quad (4.110)$$

for all (x, f) and (y, g) in $T^*\mathcal{G}$. The left hand side of the equality above is

$$\begin{aligned} \mu_{a,b}(\alpha(x, f), (y, g)) &= \mu_{a,b}\left((\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{24}f_4)e_2 + (\alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}f_3 - \alpha_{21}f_4)e_3 \right. \\ &\quad \left. + (-\alpha_{32}x_1 + (\alpha_{33} - \alpha_{22})f_4)e_4 \quad , \quad y_1e_1 + y_2e_2 + g_3e_3 + g_4e_4\right) \\ &= a \left\langle (\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{24}f_4)e_2 + (\alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}f_3 - \alpha_{21}f_4)e_3 \right. \\ &\quad \left. + (-\alpha_{32}x_1 + (\alpha_{33} - \alpha_{22})f_4)e_4 \quad , \quad y_1e_1 + y_2e_2 + g_3e_3 + g_4e_4 \right\rangle \\ &= a \left[(\alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{24}f_4)g_4 + (\alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}f_3 - \alpha_{21}f_4)y_1 \right. \\ &\quad \left. + (-\alpha_{32}x_1 + (\alpha_{33} - \alpha_{22})f_4)y_2 \right] \\ &= a \left[\alpha_{21}(x_1g_4 - y_1f_4) + \alpha_{22}x_2g_4 + \alpha_{24}f_4g_4 + \alpha_{31}x_1y_1 \right. \\ &\quad \left. + \alpha_{32}(x_2y_1 - x_1y_2) + \alpha_{33}y_1f_3 + \alpha_{42}x_2y_2 + \alpha_{44}y_2f_4 \right] \end{aligned} \quad (4.111)$$

The right hand side gives

$$\begin{aligned} -\mu_{a,b}((x, f), \alpha(y, g)) &= -\mu_{a,b}\left(x_1e_1 + x_2e_2 + f_3e_3 + f_4e_4 \quad , \quad (\alpha_{21}y_1 + \alpha_{22}y_2 + \alpha_{24}g_4)e_2 \right. \\ &\quad \left. + (\alpha_{31}y_1 + \alpha_{32}y_2 + \alpha_{33}g_3 - \alpha_{21}g_4)e_3 + (-\alpha_{32}y_1 + \alpha_{42}y_2 + \alpha_{44}g_4)e_4\right) \end{aligned}$$

$$\begin{aligned}
&= -a \left[(\alpha_{21}y_1 + \alpha_{22}y_2 + \alpha_{24}g_4)f_4 + x_1(\alpha_{31}y_1 + \alpha_{32}y_2 + \alpha_{33}g_3 - \alpha_{21}g_4) \right. \\
&\quad \left. + x_2(-\alpha_{32}y_1 + \alpha_{42}y_2 + \alpha_{44}g_4) \right] \\
&= -a \left[\alpha_{21}(-x_1g_4 + y_1f_4) + \alpha_{22}y_2f_4 + \alpha_{24}f_4g_4 + \alpha_{31}x_1y_1 \right. \\
&\quad \left. + \alpha_{32}(-x_2y_1 + x_1y_2) + \alpha_{33}x_1g_3 + \alpha_{42}x_2y_2 + \alpha_{44}x_2g_4 \right] \quad (4.112)
\end{aligned}$$

The equality (4.110) then implies that $\alpha_{24} = \alpha_{31} = \alpha_{33} = 0$. In this case α can be written $\alpha = \alpha_{21}\phi_1 + \alpha_{22}\phi_2 + \alpha_{32}\phi_4$. One can readily check that $\phi_1 = -ad_{e_2}$, $\phi_2 = ad_{e_1}$, $\phi_4 = -ad_{e_4}$. \square

Let us now focus our attention on isometries of bi-invariant metrics on T^*G_1 . Recall first the following materials.

- The operation law of T^*G_1 is given by (see Proposition 5.3.1) :

$$x \cdot y = (x_1 + y_1, x_2 + y_2e^{x_1}, x_3 + y_3 + x_2y_4e^{-x_1}, x_4 + y_4e^{-x_1}). \quad (4.113)$$

The unit element is $\epsilon = (0, 0, 0, 0)$ and the inverse of an element $x = (x_1, x_2, x_3, x_4)$ is the element $x^{-1} = (-x_1, -x_2e^{-x_1}, -x_3 + x_2x_4, -x_4e^{x_1})$.

- The exponential map of T^*G_1 is defined as follows (see Chapter 5, Corollary 5.3.1) : let $\xi = \xi_1e_1 + \xi_2e_2 + \xi_3e_3 + \xi_4e_4$ be in $T^*\mathcal{G}$,

$$\exp(\xi) = \begin{cases} \left(0, \xi_2, \frac{1}{2}\xi_2\xi_4 + \xi_3, \xi_4 \right), & \text{if } \xi_1 = 0 \\ \left(\xi_1, \frac{\xi_2}{\xi_1}[\exp(\xi_1)-1], \xi_3 + \frac{\xi_2\xi_4}{\xi_1} + \frac{\xi_2\xi_4}{\xi_1^2}[\exp(-\xi_1)-1], \frac{\xi_4}{\xi_1}[1-\exp(-\xi_1)] \right) & \text{if } \xi_1 \neq 0. \end{cases}$$

For $i = 1, 2, 3, 4$, note by X^{iL} , X^{iR} the infinitesimal generators of the one-parameter subgroups $L_{\exp(te_i)}$ and $R_{\exp(te_i)}$ respectively. It is readily checked that for all $g = (x_1, x_2, x_3, x_4)$ in T^*G_1 , we have :

$$\begin{aligned}
X_{|g}^{1,L} &= (1, x_2, 0, -x_4) \quad , \quad X_{|g}^{1,R} = (1, 0, 0, 0) \\
X_{|g}^{2,L} &= (0, 1, x_4, 0) \quad , \quad X_{|g}^{2,R} = (0, e^{x_1}, 0, 0) \\
X_{|g}^{3,L} &= (0, 0, 1, 0) \quad , \quad X_{|g}^{3,R} = (0, 0, 1, 0) \\
X_{|g}^{4,L} &= (0, 0, 0, 1) \quad , \quad X_{|g}^{4,R} = (0, 0, x_2e^{-x_1}, 0).
\end{aligned} \quad (4.114)$$

It is now easy to obtain

$$\begin{aligned}
X_{|g}^{1,s} &= (2, x_2, 0, -x_4) & , & \quad X_{|g}^{1,a} = (0, -x_2, 0, x_4) \\
X_{|g}^{2,s} &= (0, 1 + e^{x_1}, x_4, 0) & , & \quad X_{|g}^{2,a} = (0, e^{x_1} - 1, -x_4, 0) \\
X_{|g}^{3,s} &= (0, 0, 2, 0) & , & \quad X_{|g}^{3,a} = (0, 0, 0, 0) \\
X_{|g}^{4,s} &= (0, 0, x_2e^{-x_1}, 1 + e^{-x_1}) & , & \quad X_{|g}^{4,a} = (0, 0, x_2e^{-x_1}, e^{-x_1} - 1)
\end{aligned} \tag{4.115}$$

Now $\{\phi_1, \phi_2, \phi_4, X^{1,s}, X^{2,s}, X^{3,s}, X^{4,s}\}$ is a basis of the Lie algebra $\mathcal{J}(G_1)$ of the Lie group $I(G_1)$ of isometries of any bi-invariant metric on T^*G_1 . Now we just have to compute the brackets on $\mathcal{J}(G_1)$ by Theorem 4.2.2. The non-vanishing brackets are the following :

$$\begin{aligned}
[\phi_1, \phi_2] &= -\phi_1 & , & \quad [\phi_1, X^{1,s}] = X^{2,s} & , & \quad [\phi_1, X^{4,s}] = -X^{1,s} \\
[\phi_2, X^{2,s}] &= X^{2,s} & , & \quad [\phi_2, X^{4,s}] = -X^{4,s} & , & \quad [\phi_4, X^{1,s}] = X^{4,s} \\
[\phi_4, X^{2,s}] &= -X^{3,s} & , & \quad [X^{1,s}, X^{2,s}] = -X^{2,s} & , & \quad [X^{1,s}, X^{4,s}] = X^{4,s} \\
[X^{2,s}, X^{4,s}] &= -X^{3,s}.
\end{aligned} \tag{4.116}$$

4.6.2 The Special Linear Group $SL(2, \mathbb{R})$

Let G_2 denote the special linear group $SL(2, \mathbb{R})$. Set $\mathcal{G}_2 := \mathfrak{sl}(2, \mathbb{R}) = \text{span}\{e_1, e_2, e_3\}$ and $T^*\mathcal{G}_2 := \mathcal{G}_2 \times \mathcal{G}_2^* = \text{span}\{e_1, e_2, e_3, e_4, e_5, e_6\}$. We have the following brackets (see Example 2.5.3):

$$\begin{aligned}
[e_1, e_2] &= -2e_2 & [e_1, e_3] &= 2e_3 & [e_1, e_5] &= 2e_5 \\
[e_1, e_6] &= -2e_6 & [e_2, e_3] &= -e_1 & [e_2, e_4] &= e_6 \\
[e_2, e_5] &= -2e_4 & [e_3, e_4] &= -e_5 & [e_3, e_6] &= 2e_4
\end{aligned} \tag{4.117}$$

An element (x, f) of $T^*\mathcal{G}_2$ can be written $(x, f) = x_1e_1 + x_2e_2 + x_3e_3 + f_4e_4 + f_5e_5 + f_6e_6$.

Proposition 4.6.3. *Let \langle, \rangle stand for the duality pairing between \mathcal{G}_2 and \mathcal{G}_2^* and μ be any orthogonal structure on $T^*\mathcal{G}_2$. Then,*

1. *there exist (a, b) in $\mathbb{R}^* \times \mathbb{R}$ such that*

$$\mu((x, f), (y, g)) = a\langle(x, f), (y, g)\rangle + 4b(2x_1y_1 + x_2y_3 + x_3y_2), \tag{4.118}$$

for all (x, f) and (y, g) in $T^\mathcal{G}_2$.*

2. *The orthogonal structure μ on $T^*\mathcal{G}_2$ is of signature $(3, 3)$.*

Proof. 1. Since \mathcal{G}_2 is a simple Lie algebra, Theorem 4.5.1 asserts that any orthogonal structure $\mu_{a,b}$ on $T^*\mathcal{G}_2$ is given by

$$\mu_{a,b}((x, f), (y, g)) = a\langle(x, f), (y, g)\rangle + b\langle\theta(x), y\rangle, \tag{4.119}$$

for all $(x, f), (y, g)$ in $T^*\mathcal{G}_2$, where $(a, b) \in \mathbb{R}^* \times \mathbb{R}$. Because of the simplicity of \mathcal{G}_2 every orthogonal structure on \mathcal{G}_2 is a multiple of the Killing form. So, we let θ be induced by the Killing form K of \mathcal{G}_2 , *i.e.* $\langle\theta(x), y\rangle = K(x, y)$, for all x, y in \mathcal{G}_2 . Then,

$$\mu((x, f), (y, g)) = a\langle(x, f), (y, g)\rangle + bK(x, y),$$

for all $(x, f), (y, g)$ in $T^*\mathcal{G}_2$.

By definition, the Killing form is given by

$$K(x, y) = \text{trace}(ad_x \circ ad_y), \quad (4.120)$$

for all x, y in \mathcal{G}_2 . On the basis of (e_1, e_2, e_3) of \mathcal{G}_2 , we have the following matrix of K

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

Now it is a little matter to checked that

$$\mu_{a,b}((x, f), (y, g)) = a\langle (x, f), (y, g) \rangle + b(8x_1y_1 + 4x_2y_3 + 4x_3y_2)$$

2. The matrix of $\mu_{a,b}$ on the basis of $T^*\mathcal{G}_2$ is

$$M_{a,b} = \begin{pmatrix} 8b & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 4b & 0 & a & 0 \\ 0 & 4b & 0 & 0 & 0 & a \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \end{pmatrix}. \quad (4.121)$$

The characteristic polynomial of $M_{a,b}$ is

$$P_{a,b}(t) = (t^2 - 8bt - a^2)[t^4 - 2(a^2 + 8b^2)t^2 + a^4], \quad (4.122)$$

for all $t \in \mathbb{R}$. The roots of $P_{a,b}$ are

$$\begin{aligned} t_1 &= 4b - \sqrt{a^2 + 16b^2} & ; & \quad t_2 = 4b + \sqrt{a^2 + 16b^2} \\ t_3 &= -\sqrt{a^2 + 8b^2 - 4|b|\sqrt{a^2 + 4b^2}} & ; & \quad t_4 = +\sqrt{a^2 + 8b^2 - 4|b|\sqrt{a^2 + 4b^2}} \\ t_5 &= -\sqrt{a^2 + 8b^2 + 4|b|\sqrt{a^2 + 4b^2}} & ; & \quad t_6 = +\sqrt{a^2 + 8b^2 + 4|b|\sqrt{a^2 + 4b^2}} \end{aligned}$$

The roots t_1, t_3 and t_5 are negative while t_2, t_4 and t_6 are positive. Then, the signature of the bilinear form $\mu_{a,b}$ is $(3, 3)$, for all $(a, b) \in \mathbb{R}^* \times \mathbb{R}$. □

From Theorem 3.4.1, any prederivation of $T^*\mathcal{G}_2$ is a derivation. Then the space of prederivations of $T^*\mathcal{G}_2$ is $\text{Pder}(T^*\mathcal{G}_2) = \text{der}(T^*\mathcal{G}_2) = \text{span}(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7)$, where $\phi_i, i = 1, 2, 3, 4, 5, 6, 7$ are defined in Example 2.5.3. If ϕ is a prederivation of $T^*\mathcal{G}_2$,

$$\phi = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4 + \alpha_5\phi_5 + \alpha_6\phi_6 + \alpha_7\phi_7, \quad (4.123)$$

where $\alpha_i, i = 1, 2, 3, 4, 5, 6, 7,$ are real numbers. Now, since \mathfrak{G}_2 is simple then ϕ is skew-symmetric with respect to the form (4.118) if and only if it is an inner derivation (Proposition 4.5.1). Hence, $\alpha_2 = 0$ and then any $\mu_{a,b}$ -skew-symmetric prederivation of $T^*\mathfrak{G}_2$ has the form

$$\phi = \alpha_1\phi_1 + \alpha_3\phi_3 + \alpha_4\phi_4 + \alpha_5\phi_5 + \alpha_6\phi_6 + \alpha_7\phi_7, \tag{4.124}$$

where $\alpha_i, i = 1, 3, 4, 5, 6, 7,$ are real numbers. In the basis of $T^*\mathfrak{G}_2$, such prederivation has the following matrix

$$\begin{pmatrix} 0 & -\alpha_3 & \alpha_4 & 0 & 0 & 0 \\ -2\alpha_4 & -\alpha_1 & 0 & 0 & 0 & 0 \\ 2\alpha_3 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_6 & \alpha_7 & 0 & 2\alpha_4 & -2\alpha_3 \\ -\alpha_6 & 0 & -\alpha_5 & \alpha_3 & \alpha_1 & 0 \\ -\alpha_7 & \alpha_5 & 0 & -\alpha_4 & 0 & -\alpha_1 \end{pmatrix} \tag{4.125}$$

4.6.3 The 4-dimensional Oscillator Lie Group

In Example 3.5.3 we have defined the 4-dimensional oscillator Lie group G_λ and its Lie algebra \mathfrak{G}_λ .

Proposition 4.6.4. *Any bi-invariant metric on the 4-dimensional oscillator group is Lorentzian and its induced orthogonal structure \langle, \rangle_λ on the Lie algebra \mathfrak{G}_λ has the following form :*

$$\langle x, y \rangle_\lambda = k\mu_\lambda(x, y) + mx^{-1}y^{-1}, \tag{4.126}$$

for any $x = x^{-1}e_{-1} + x^0e_0 + x^1e_1 + \check{x}^1\check{e}_1$ and $y = y^{-1}e_{-1} + y^0e_0 + y^1e_1 + \check{y}^1\check{e}_1$ in \mathfrak{G}_λ , where μ_λ is the orthogonal structure on \mathfrak{G}_λ given by (3.78) and (k, m) is in $\mathbb{R}^* \times \mathbb{R}$.

Proof. We have already seen that the form μ_λ given by (3.78) is an orthogonal structure on \mathfrak{G}_λ . Then, any other orthogonal structure \langle, \rangle_λ on \mathfrak{G}_λ is given by $\langle x, y \rangle_\lambda = \mu_\lambda(j(x), y)$, for all x, y in \mathfrak{G}_λ , where $j : \mathfrak{G}_\lambda \rightarrow \mathfrak{G}_\lambda$ is an μ_λ -symmetric and invertible bi-invariant tensor. Now, one can check that such map j is given by $j(x) = kx^{-1}e_{-1} + (mx^{-1} + kx^0)e_0 + kx^1e_1 + k\check{x}^1\check{e}_1$, where (k, m) is in $\mathbb{R}^* \times \mathbb{R}$. So, we have

$$\begin{aligned} \langle x, y \rangle_\lambda &= \mu_\lambda\left(kx^{-1}e_{-1} + (mx^{-1} + kx^0)e_0 + kx^1e_1 + k\check{x}^1\check{e}_1, y^{-1}e_{-1} + y^0e_0 + y^1e_1 + \check{y}^1\check{e}_1\right) \\ &= kx^{-1}y^0 + (mx^{-1} + kx^0)y^{-1} + \frac{1}{\lambda}(kx^1y^1 + k\check{x}^1\check{y}^1) \\ &= kx^{-1}y^0 + mx^{-1}y^{-1} + kx^0y^{-1} + \frac{1}{\lambda}(kx^1y^1 + k\check{x}^1\check{y}^1) \\ &= k\left[x^{-1}y^0 + x^0y^{-1} + \frac{1}{\lambda}(x^1y^1 + \check{x}^1\check{y}^1)\right] + mx^{-1}y^{-1} \\ &= k\mu_\lambda(x, y) + mx^{-1}y^{-1}. \end{aligned}$$

Now the matrix of \langle, \rangle_λ on the basis $(e_{-1}, e_0, e_1, \check{e}_1)$ of \mathfrak{G}_λ is given by

$$M_\lambda = \begin{pmatrix} m & k & 0 & 0 \\ k & 0 & 0 & 0 \\ 0 & 0 & \frac{k}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{k}{\lambda} \end{pmatrix} \tag{4.127}$$

The characteristic polynomial of M_λ is $P_\lambda(t) = (t - \frac{k}{\lambda})^2 (t^2 - mt - k^2)$ and its roots are

$$t_1 = \frac{k}{\lambda} \quad ; \quad t_2 = \frac{1}{2}(m + \sqrt{m^2 + 4k^2}) \quad ; \quad t_3 = \frac{1}{2}(m - \sqrt{m^2 + 4k^2}) \quad (4.128)$$

t_2 and t_3 are simple roots while t_1 is of multiplicity 2. It is clear that $t_2 \geq 0$, $t_3 \leq 0$ and t_1 has the same sign as k . □

Proposition 4.6.5. *Any orthogonal structure μ_λ^* on $T^*\mathcal{G}_\lambda$ can be written as*

$$\mu_\lambda^* \left((x, f), (y, g) \right) = A \langle (x, f), (y, g) \rangle + B \mu_\lambda(x, y) + C(x^{-1}g^0 + y^{-1}f^0) + D x^{-1}y^{-1} + E f^0 g^0, \quad (4.129)$$

for all elements $(x, f) = x^{-1}e_{-1} + x^0e_0 + x^1e_1 + \check{x}^1\check{e}_1 + f^{-1}e_{-1}^* + f^0e_0^* + f^1e_1^* + \check{f}^1\check{e}_1^*$ and $(y, g) = y^{-1}e_{-1} + y^0e_0 + y^1e_1 + \check{y}^1\check{e}_1 + g^{-1}e_{-1}^* + g^0e_0^* + g^1e_1^* + \check{g}^1\check{e}_1^*$ in $T^*\mathcal{G}_\lambda$, where $(A; B, C, D, E)$ is in $\mathbb{R}^* \times \mathbb{R}^4$ and \langle, \rangle stands for the duality pairing between \mathcal{G}_λ and \mathcal{G}_λ^* .

Proof. Since \mathcal{G}_λ is an orthogonal Lie algebra, then (see Theorem 4.4.1) any orthogonal structure μ_λ^* on $T^*\mathcal{G}_\lambda$ is given by

$$\mu_\lambda^* \left((x, f), (y, g) \right) = \langle g, j_{11}(x) \rangle + \langle f, j_{11}(y) \rangle + \langle g, j_2 \circ \theta^{-1}(f) \rangle + \langle \theta \circ j_1(x), y \rangle,$$

for all $(x, f), (y, g)$ in $T^*\mathcal{G}_\lambda$, where $\theta : \mathcal{G}_\lambda \rightarrow \mathcal{G}_\lambda^*$, $\langle \theta(x), y \rangle = \mu_\lambda(x, y)$ for all x, y in \mathcal{G}_λ ; j_{11} , j_1 and j_2 commute with all adjoint operators of \mathcal{G} and satisfy the following conditions :

1. j_{11} is invertible ;
2. j_1 is μ_λ -symmetric (\langle, \rangle_λ being defined by (4.126));
3. $j_2 \circ ad_x = 0$ and $\langle j_2 \circ \theta^{-1}(f), g \rangle = \langle f, j_2 \circ \theta^{-1}(g) \rangle$, for all x in \mathcal{G} and all f, g in \mathcal{G}^* .

One can easily establish that the endomorphisms j_{11} , j_1 and j_2 have the following matrices on the basis $(e_{-1}, e_0, e_1, \check{e}_1)$ of \mathcal{G}_λ :

$$j_{11} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix} ; j_1 = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{11} & 0 & 0 \\ 0 & 0 & b_{11} & 0 \\ 0 & 0 & 0 & b_{11} \end{pmatrix} ; j_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $a_{11} \neq 0$, $a_{21}, b_{11}, b_{21}, c_{21}$ are real numbers. It is also readily checked that, with respect to the basis $(e_{-1}, e_0, e_1, \check{e}_1)$ and $(e_{-1}, e_0, e_1, \check{e}_1)$ of \mathcal{G}_λ and \mathcal{G}_λ^* respectively, θ and θ^{-1} have the following matrices :

$$\theta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix} ; \theta^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \quad (4.130)$$

It is now a little matter to establish that

$$\begin{aligned} \mu_\lambda^* \left((x, f), (y, g) \right) &= a_{11} \langle (x, f), (y, g) \rangle + b_{11} \mu_\lambda(x, y) \\ &\quad + a_{21}(x^{-1}g^0 + y^{-1}f^0) + b_{21}x^{-1}y^{-1} + c_{21}f^0g^0. \end{aligned} \quad (4.131)$$

This latter equality is nothing but (4.129); one just has to put $A = a_{11}$, $B = b_{11}$, $C = a_{21}$, $D = b_{21}$, $E = c_{21}$. □

KÄHLERIAN STRUCTURE ON THE LIE GROUP OF AFFINE MOTIONS OF THE REAL LINE \mathbb{R}

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5.1 Introduction

An almost complex structure on a Lie algebra \mathfrak{G} , when it exists, is defined by a linear endomorphism $J : \mathfrak{G} \rightarrow \mathfrak{G}$, $J^2 = -Id$ (Id is the identity map of \mathfrak{G}). If, in addition, J satisfies the condition

$$J[x, y] - [Jx, y] - [x, Jy] - J[Jx, Jy] = 0,$$

for all x and y in \mathfrak{G} , we will say that J is integrable. Now, an integrable almost complex structure is called a complex structure. In this case the pair (\mathfrak{G}, J) is called a complex algebra.

A metric μ on a complex algebra (\mathfrak{G}, J) is called Kählerian if it is hermitian, that is,

$$\mu(Jx, Jy) = \mu(x, y),$$

for all vectors x and y in \mathfrak{G} , and if J is a parallel tensor with respect to the connection arising from μ . Likewise, given a Lie algebra with metric μ , we shall say that a complex structure J on \mathfrak{G} is Kählerian if μ is Kählerian with respect to J in the above sense. Such a pair (J, μ) defines a Kählerian structure on \mathfrak{G} .

A Poisson-Lie structure on a Lie group G , is given by a Poisson tensor π on G , such that, when the Cartesian product $G \times G$ is equipped with the Poisson tensor $\pi \times \pi$, the multiplication $m : (\sigma, \tau) \mapsto \sigma\tau$ is a Poisson map between the Poisson manifolds

$(G \times G, \pi \times \pi)$ and (G, π) . If f, g are in \mathcal{G}^* and \bar{f}, \bar{g} are \mathcal{C}^∞ functions on G with respective derivatives $f = \bar{f}_{*,\epsilon}$, $g = \bar{g}_{*,\epsilon}$ at the unit ϵ of G , one defines another element $[f, g]_*$ of \mathcal{G}^* by setting

$$[f, g]_* := (\{\bar{f}, \bar{g}\})_{*,\epsilon}.$$

Then $[f, g]_*$ does not depend on the choice of \bar{f} and \bar{g} as above, and $(\mathcal{G}^*, [,]_*)$ is a Lie algebra. Now, there is a symmetric role played by the spaces \mathcal{G} and \mathcal{G}^* , dual to each other. Indeed, as well as acting on \mathcal{G}^* via the coadjoint action, \mathcal{G} is also acted on by \mathcal{G}^* using the coadjoint action of $(\mathcal{G}^*, [,]_*)$. A lot of the most interesting properties and applications of π , are encoded in the new Lie algebra $(\mathcal{G} \oplus \mathcal{G}^*, [,]_\pi)$, where

$$[(x, f), (y, g)]_\pi := ([x, y] + ad_f^* y - ad_g^* x, ad_x^* g - ad_y^* f + [f, g]_*), \quad (5.1)$$

for every x, y in \mathcal{G} and every f, g in \mathcal{G}^* .

The Lie algebras $(\mathcal{G} \oplus \mathcal{G}^*, [,]_\pi)$ and $(\mathcal{G}^*, [,]_*)$ are respectively called the double and the dual Lie algebras of the Poisson-Lie group (G, π) . Endowed with the duality pairing defined in (2.2), the double Lie algebra of any Poisson-Lie group (G, π) , is an orthogonal Lie algebra, such that \mathcal{G} and \mathcal{G}^* are maximal totally isotropic (Lagrangian) subalgebras.

Let r be an element of the wedge product $\wedge^2 \mathcal{G}$. Denote by r^+ (resp. r^-) the left (resp. right) invariant bivector field on G with value $r = r_\epsilon^+$ (resp. $r = r_\epsilon^-$) at ϵ . If $\pi_r := r^+ - r^-$ is a Poisson tensor, then it is a Poisson-Lie tensor and r is called a solution of the Yang-Baxter Equation. If, in particular, r^+ is a (left invariant) Poisson tensor on G , then r is called a solution of the Classical Yang-Baxter Equation (CYBE) on G (or \mathcal{G}). In this latter case, the double Lie algebra $(\mathcal{G} \oplus \mathcal{G}^*, [,]_{\pi_r})$ is isomorphic to the Lie algebra \mathcal{D} of the cotangent bundle T^*G of G . See e.g. [28]. We may also consider the linear map $\tilde{r} : \mathcal{G}^* \rightarrow \mathcal{G}$, where $\tilde{r}(f) := r(f, \cdot)$. The linear map $\theta_r : (\mathcal{G} \oplus \mathcal{G}^*, [,]_{\pi_r}) \rightarrow \mathcal{D}$,

$$\theta_r(x, f) := (x + \tilde{r}(f), f),$$

is an isomorphism of Lie algebras, between \mathcal{D} and the double Lie algebra of any Poisson-Lie group structure on G , given by a solution r of the CYBE.

Let $G = \text{Aff}(\mathbb{R})$ denote the group of affine motions of the real line \mathbb{R} . It possesses a lot of interesting structures : symplectic ([1],[5]), complex, affine ([11]), Kählerian ([53]). Note by $\mathcal{G} = \text{aff}(\mathbb{R})$ its Lie algebra. We wish to study the geometry of G as a Kählerian Lie group. This supposes to describe the symplectic structures, the complex structures, the affine transformations and transformations which preserve those structures. Furthermore, the symplectic structure corresponds to a solution of the Classical Yang-Baxter equation r (see [28]). So we will also study the double Lie group $\mathcal{D}(G, r)$ of G associated to r .

In Section 5.2 we show how can man construct a symplectic structure on G , determine the induced left-invariant affine structure, study the geodesics of this affine structure and compare these geodesics with the integral curves of left-invariant vector fields on G . In Section 5.3 we deal with the geometry of a double Lie group of G associated to a solution of the Classical Yang-Baxter equation. As explained below, the Lie algebra $\mathcal{D}(\mathcal{G}, r) := \mathcal{G} \times \mathcal{G}^*(r)$ of any double Lie group $\mathcal{D}(G, r)$ of G is isomorphic to the Lie algebra $T^*\mathcal{G}$ of the cotangent bundle $T^*G := G \times \mathcal{G}^*$ ([28]). The Lie group structure on T^*G considered here is

the one obtained by semi-direct product via the coadjoint action of G on the dual space \mathfrak{G}^* of \mathfrak{G} , considered as an Abelian Lie group. We construct a double Lie group $\mathcal{D}(G, r)$ with Lie algebra $\mathcal{D}(\mathfrak{G}, r) \simeq T^*\mathfrak{G}$. The double Lie group $\mathcal{D}(G, r)$ admits an affine structure and a complex structure ([28]). We study the both structures.

5.2 The Affine Lie Group of the Real Line

5.2.1 The Affine Lie Group of \mathbb{R} and its Lie algebra

An affine transformation of \mathbb{R} is a function $\mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto ax + b$, where a and b are real numbers ($a \neq 0$). Let $\text{Aff}(\mathbb{R})$ denote the set of all such transformations. We identify the sets $\text{Aff}(\mathbb{R})$ and $\mathbb{R}^* \times \mathbb{R}$ by putting $f = (a, b)$ if $f : x \mapsto ax + b$. Now consider the operation

$$(a_1, b_1)(a_2, b_2) := (a_1a_2, a_1b_2 + b_1), \tag{5.2}$$

for all $(a_1, b_1), (a_2, b_2)$ in $\text{Aff}(\mathbb{R})$. It is readily checked that, endowed with the composition rule (5.2) $\text{Aff}(\mathbb{R})$ is a group. The identity element is $\epsilon := (1, 0)$ and the inverse of the element $(a, b) \in G$ is given by $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$. Considering the underlying manifold $\mathbb{R}^* \times \mathbb{R}$, G is a Lie group. We will note it by G .

Remark 5.2.1. *The operation law (5.2) is nothing but the composition law of maps.*

The Lie algebra of G is $\mathfrak{G} = \mathbb{R}^2$ (as a set). The bracket on \mathfrak{G} is given by

$$[(u', v'), (u, v)] = (0, -uv' + vu'). \tag{5.3}$$

In some basis (e_1, e_2) of \mathfrak{G} we have the following :

$$[e_1, e_2] = e_2 \tag{5.4}$$

5.2.2 Symplectic and Affine Structures on the Affine Lie group

We first recall the affine Lie group of \mathbb{R}^n ($n \in \mathbb{N}$) and how can man construct a symplectic form on it (see [1]). The affine group of \mathbb{R}^n is the even dimensional Lie group

$$\text{Aff}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}, A \in GL(n, \mathbb{R}), v \in \mathbb{R}^n \right\} \tag{5.5}$$

Its Lie algebra is

$$\text{aff}(\mathbb{R}^n) := \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix}, A \in \mathfrak{gl}(n, \mathbb{R}), v \in \mathbb{R}^n \right\} \tag{5.6}$$

Let e_{ij} be the matrix such that the $(i - j)$ -entry equals 1 and the other entries are zero. $(e_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n+1}}$ forms a basis of $\text{aff}(n, \mathbb{R})$. Denote by $(e_{ij}^*)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n+1}}$ its dual basis and set

$$\alpha_0 := \sum_{k=1}^n e_{k,k+1}^* \quad \text{and} \quad \omega_0 := -d\alpha_0. \tag{5.7}$$

Hence, if x, y belong to $\text{aff}(n, \mathbb{R})$,

$$\begin{aligned}\omega_0(x, y) &= -d\alpha_0(x, y), \\ &= \alpha_0([x, y])\end{aligned}$$

One can check that the 2-form ω_0 is non-degenerate and gives a left invariant symplectic structure ω on $\text{Aff}(n, \mathbb{R})$ by the following formula :

$$\omega_g(X_{|g}, Y_{|g}) := \omega_0(T_g L_{g^{-1}} \cdot X_{|g}, T_g L_{g^{-1}} \cdot Y_{|g}), \tag{5.8}$$

for all g in $\text{Aff}(n, \mathbb{R})$ and all vectors $X_{|g}, Y_{|g}$ in $T_g \text{Aff}(n, \mathbb{R})$, where $L_g : \text{Aff}(n, \mathbb{R}) \rightarrow \text{Aff}(n, \mathbb{R})$, $h \mapsto g \cdot h$ stands for the left translation by g in G .

From now on we focus our attention on $G = \text{Aff}(\mathbb{R})$. The Lie algebra of G can be written

$$\mathfrak{G} := \text{aff}(\mathbb{R}) = \left\{ \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}; u, v \in \mathbb{R} \right\} \tag{5.9}$$

Put

$$e_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\mathfrak{G} = \text{vect}\{e_{11}, e_{12}\}$. We rename the vectors as follow : $e_{11} = e_1, e_{12} = e_2$. We have

$$[e_1, e_2] = e_2. \tag{5.10}$$

Now we put

$$\alpha_0 = e_{12}^* = e_2^* \text{ and } \omega_0 = -d\alpha_0 = -de_2^*,$$

that is

$$\omega_0(x, y) = e_2^*([x, y]), \tag{5.11}$$

for all x, y in \mathfrak{G} . We then have

$$\omega_0(e_1, e_1) = 0 \quad ; \quad \omega_0(e_1, e_2) = 1 \quad ; \quad \omega_0(e_2, e_2) = 0. \tag{5.12}$$

ω_0 is a non-degenerate 2-form on \mathfrak{G} . It induces a symplectic form ω on G by relation (5.8).

For any ξ in \mathfrak{G} , let X^ξ denote the associated left invariant vector field. That is

$$X_{|g}^\xi := T_\epsilon L_g \cdot \xi, \tag{5.13}$$

for any g in G , where ϵ is the identity element of G . One defines an affine connection ∇ on G by the following formula (see [17]) : $\forall \xi, \eta, \sigma \in \mathfrak{G}$,

$$\omega(\nabla_{X^\xi} X^\eta, X^\sigma) = -\omega(X^\eta, [X^\xi, X^\sigma]). \tag{5.14}$$

We obtain

$$\nabla_{e_1} e_1 = -e_1 \quad ; \quad \nabla_{e_1} e_2 = 0 \quad ; \quad \nabla_{e_2} e_1 = -e_2 \quad ; \quad \nabla_{e_2} e_2 = 0. \tag{5.15}$$

Note by Γ_{ij}^k the symbols of Christoffel, that is $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ (Einstein's summation). We then have the following symbols of Christoffel

$$\begin{aligned}\Gamma_{11}^1 &= -1 \quad ; \quad \Gamma_{11}^2 = 0 \quad ; \quad \Gamma_{12}^1 = 0 \quad ; \quad \Gamma_{12}^2 = 0 \\ \Gamma_{21}^1 &= 0 \quad ; \quad \Gamma_{21}^2 = -1 \quad ; \quad \Gamma_{22}^1 = 0 \quad ; \quad \Gamma_{22}^2 = 0\end{aligned} \tag{5.16}$$

In the sequel, we will consider the connected component of the unit ϵ of G . Let us note it by G_0 . That is $G_0 = \mathbb{R}_+^* \times \mathbb{R} = \{(x, y) \in \mathbb{R} \times \mathbb{R}, x > 0\}$. We endow G_0 with the connection (also denote by ∇) induced on G_0 by the connection ∇ defined by relation (5.15).

5.2.3 Geodesics of (G_0, ∇) at the unit

Let (ξ_0, η_0) be an element of \mathcal{G} and let $\gamma = (\gamma_1, \gamma_2)$ be an auto-parallel ($\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$, for all t) curve such that

$$\gamma(0) = \epsilon = (1, 0) \quad ; \quad \dot{\gamma}(0) = (\xi_0, \eta_0). \quad (5.17)$$

The curve γ satisfies the equations :

$$\ddot{\gamma}_1 - \dot{\gamma}_1^2 = 0. \quad (5.18)$$

$$\ddot{\gamma}_2 - \dot{\gamma}_1\dot{\gamma}_2 = 0. \quad (5.19)$$

Now we consider the following two cases.

Case 1. $\gamma(0) = (1, 0)$ and $\dot{\gamma}(0) = (0, \eta_0)$.

If γ_1 is not constant, then for any $t \in \mathbb{R}$ such that $\gamma_1(t) \neq 0$, we can solve equation (5.18) as follows :

$$\begin{aligned} \ddot{\gamma}_1 - \dot{\gamma}_1^2 = 0 &\Leftrightarrow \frac{\ddot{\gamma}_1}{\dot{\gamma}_1^2} = 1 \\ &\Leftrightarrow -\frac{1}{\dot{\gamma}_1} = t + c, \quad c \in \mathbb{R} \\ &\Leftrightarrow \dot{\gamma}_1(t) = -\frac{1}{t + c}. \end{aligned} \quad (5.20)$$

From the condition $\dot{\gamma}_1(0) = 0$, we obtain : $-\frac{1}{c} = 0$, which is not possible, then $\dot{\gamma}_1(t) = 0$, for all $t \in \mathbb{R}$. We then have :

- $\gamma_1(t) = \text{constante} = a$, for all real number t ;
- Equation (5.18) gives $\gamma_2(t) = bt + d$, for all t in \mathbb{R} , where b and d belong to \mathbb{R} .

From conditions $\gamma(0) = (1, 0)$ and $\dot{\gamma}(0) = (0, \eta_0)$, we have :

$$a = 1 \quad ; \quad d = 0 \quad ; \quad b = \eta_0. \quad (5.21)$$

So, for any $(0, \eta_0)$ in \mathcal{G} the geodesic through ϵ with velocity $(0, \eta_0)$ is given by

$$\gamma(t) = (1, \eta_0 t), \quad (5.22)$$

for all $t \in \mathbb{R}$.

Case 2. Now we consider an element (ξ_0, η_0) of $\text{aff}(\mathbb{R})$, with $\xi_0 \neq 0$. Equation (5.18) can be solve as above and

$$\dot{\gamma}_1(t) = -\frac{1}{t + c}. \quad (5.23)$$

From the condition $\dot{\gamma}_1(0) = \xi_0$ we obtain : $-\frac{1}{c} = \xi_0$, *i.e.* $c = -\frac{1}{\xi_0}$. Now we have :

$$\begin{aligned} \dot{\gamma}_1(t) &= -\frac{1}{t - \frac{1}{\xi_0}} = -\frac{\xi_0}{\xi_0 t - 1}, \quad t \neq \frac{1}{\xi_0} \\ \gamma_1(t) &= -\ln |\xi_0 t - 1| + \text{cste}. \end{aligned} \quad (5.24)$$

Since $\gamma_1(0) = 1$, we have $cste = 1$. It comes that :

$$\gamma_1(t) = 1 - \ln |\xi_0 t - 1|, \tag{5.25}$$

for any $t \neq \frac{1}{\xi_0}$ and $\frac{1-e}{\xi_0} < t < \frac{1+e}{\xi_0}$, since $\gamma_1(t) > 0$, if it is defined. Equation (5.19) now becomes

$$\ddot{\gamma}_2 + \frac{\xi_0}{\xi_0 t - 1} \dot{\gamma}_2 = 0. \tag{5.26}$$

So we have

$$\dot{\gamma}_2(t) = \frac{C_1}{|\xi_0 t - 1|}, \tag{5.27}$$

where C_1 is a real number. Since $\dot{\gamma}_2(0) = \eta_0$, we have $C_1 = \eta_0$ and

$$\dot{\gamma}_2(t) = \frac{\eta_0}{|\xi_0 t - 1|}. \tag{5.28}$$

Integrating the latter we have :

$$\gamma_2(t) = \frac{\eta_0}{\xi_0} \ln(k|\xi_0 t - 1|), \tag{5.29}$$

where $k > 0$, $t \neq \frac{1}{\xi_0}$ and $\frac{1-e}{\xi_0} < t < \frac{1+e}{\xi_0}$. Since $\gamma_2(0) = 0$, we have $k = 1$ and

$$\gamma_2(t) = \frac{\eta_0}{\xi_0} \ln |\xi_0 t - 1|, \tag{5.30}$$

for any $t \neq \frac{1}{\xi_0}$ and $\frac{1-e}{\xi_0} < t < \frac{1+e}{\xi_0}$. Hence, for any element (ξ_0, η_0) , with $\xi_0 \neq 0$, the geodesic through ϵ with velocity (ξ_0, η_0) is given by :

$$\gamma(t) = \left(1 - \ln |\xi_0 t - 1|, \frac{\eta_0}{\xi_0} \ln |\xi_0 t - 1| \right), \tag{5.31}$$

for all $t \neq \frac{1}{\xi_0}$ and $\frac{1-e}{\xi_0} < t < \frac{1+e}{\xi_0}$.

Now we can summarize.

Proposition 5.2.1. *A geodesic through the unit element of (G_0, ∇) with velocity (ξ, η) in \mathcal{G} is given by*

$$\gamma(t) = \begin{cases} (1, \eta t) & \text{if } \xi = 0, \\ (1 - \ln |\xi t - 1|, \frac{\eta}{\xi} \ln |\xi t - 1|) & \text{if } \xi \neq 0. \end{cases} \tag{5.32}$$

for all $t \neq \frac{1}{\xi}$ and $\frac{1-e}{\xi} < t < \frac{1+e}{\xi}$.

In order to represent them, let us write Cartesian equations of the geodesics. Set

$$\gamma(t) = (y(t), x(t)). \tag{5.33}$$

- If $\xi = 0$, the Cartesian equation of the geodesic through ϵ with velocity $(0, \eta)$ reads

$$y = 1. \tag{5.34}$$

This is the unique complete geodesic and is just a horizontal line through ϵ .

- If $\xi \neq 0$ and $\eta = 0$, the Cartesian equation of the geodesic through ϵ with velocity $(\xi, 0)$ is given by

$$\begin{cases} x = 0 \\ y > 0. \end{cases} \tag{5.35}$$

This geodesic is not complete and is the vertical line through ϵ contained in the half-plan $\{(y, x) \in \mathbb{R}^2 : y > 0\}$.

- If $\xi \neq 0$ and $\eta \neq 0$, the Cartesian equation of the geodesic through ϵ with velocity (ξ, η) is $y = -\frac{\xi}{\eta}x + 1$ and $y > 0$. We can simply write

$$\begin{cases} y = ax + 1 \\ y > 0. \end{cases} \quad a \neq \mathbb{R}. \tag{5.36}$$

These geodesics are oblique lines through ϵ . They are not complete.

The Figure 5.1 represents the geodesics of (G_0, ∇) through ϵ .

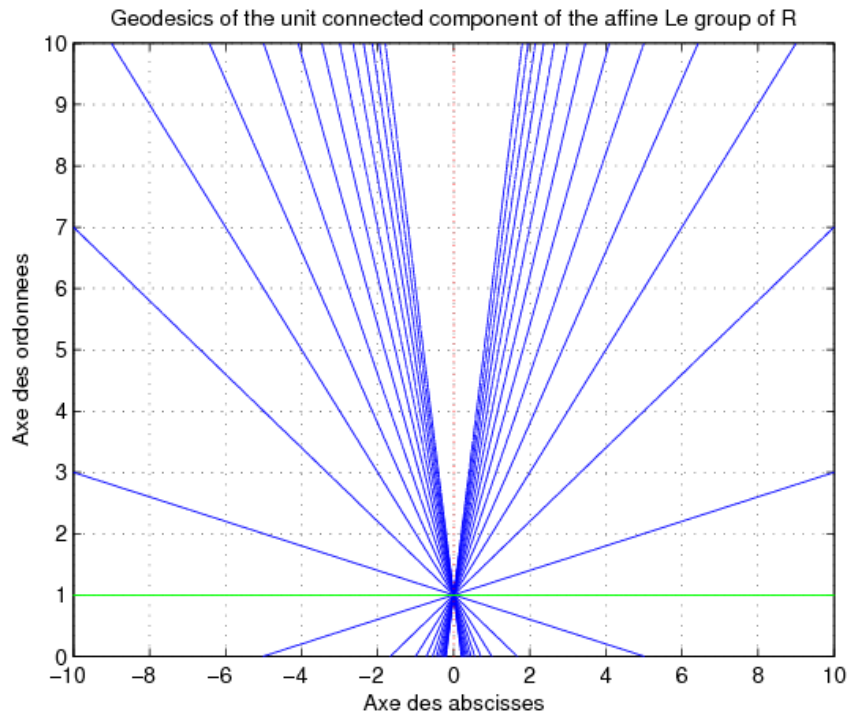


Figure 5.1: Geodesics of G_0 through ϵ

We just take the geodesics at $t = 1$, whenever it is possible, to get the

Corollary 5.2.1. *The exponential map of the affine structure ∇ on G_0 is defined for any (ξ, η) in \mathcal{G} such that $\xi \neq 1$ and $1 - e < \xi < 1 + e$, by*

$$Exp_e(\xi, \eta) = \begin{cases} (1, \eta) & \text{if } \xi = 0, \\ (1 - \ln |\xi - 1|, \frac{\eta}{\xi} \ln |\xi - 1|) & \text{if } \xi \neq 0. \end{cases} \quad (5.37)$$

Let us now look at the inverse of Exp_e , whenever it exists. Let (y, x) be an element of G_e . We wish to find (ξ, η) in \mathcal{G} such that $Exp_e(\xi, \eta) = (y, x)$.

1. If $y = 1$, then the unique solution is $(\xi, \eta) = (0, x)$.
2. Suppose now $y \neq 1$, then the equation $Exp_e(\xi, \eta) = (y, x)$ gives the following two equations :

$$1 - \ln |\xi - 1| = y \quad (5.38)$$

$$\frac{\eta}{\xi} \ln |\xi - 1| = x \quad (5.39)$$

Equation (5.38) is equivalent to

$$\begin{aligned} \ln |\xi - 1| &= 1 - y \\ |\xi - 1| &= \exp(1 - y) \\ \xi - 1 &= \begin{cases} \exp(1 - y) & \text{if } \xi > 1 \\ -\exp(1 - y) & \text{if } \xi < 1 \end{cases} \\ \xi &= \begin{cases} 1 + \exp(1 - y) & \text{if } \xi > 1 \\ 1 - \exp(1 - y) & \text{if } \xi < 1 \end{cases} \end{aligned} \quad (5.40)$$

Relation (5.39) becomes

$$\begin{aligned} \frac{\eta}{\xi}(1 - y) &= x \\ \eta &= \frac{\xi}{1 - y}x \\ \eta &= \begin{cases} \frac{1 + \exp(1 - y)}{1 - y}x & \text{if } \xi > 1 \\ \frac{1 - \exp(1 - y)}{1 - y}x & \text{if } \xi < 1 \end{cases} \end{aligned} \quad (5.41)$$

It comes that the logarithm map of (G_0, ∇) which is the inverse of Exp_e , is defined only on the subset $\{(1, x), x \in \mathbb{R}\}$ of G_0 .

Proposition 5.2.2. *The Logarithm map of (G_0, ∇) is defined on $\{(1, x), x \in \mathbb{R}\}$ and is given by*

$$Log_e(1, x) = (0, x). \quad (5.42)$$

5.2.4 Integral curves of left invariant vector fields on G

If $g = (y, x)$ et $h = (y', x')$ are two elements of G_0 , the left translation L_g by g acts on h as follows :

$$L_g h = (yy', yx' + x). \quad (5.43)$$

It comes that the differential map at ϵ of L_g on the basis (e_1, e_2) has the following matrix :

$$T_\epsilon L_g = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

Let $\xi = \xi_1 e_1 + \xi_2 e_2$ be an element of \mathcal{G} . We wish to compute the exponential $\exp_G(\xi)$ of ξ . Let X^ξ stand for the left invariant vector field associated to ξ . We have :

$$X_{|g}^\xi := T_\epsilon L_g \cdot \xi = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = y\xi_1 \frac{\partial}{\partial y} + y\xi_2 \frac{\partial}{\partial x}. \quad (5.44)$$

Now let $\gamma = (\gamma_1, \gamma_2)$ be the curve such that

$$\gamma(0) = \epsilon = (1, 0) \text{ and } \dot{\gamma}(t) = X_{|\gamma(t)}^\xi. \quad (5.45)$$

We then have the following equations :

$$\begin{cases} \dot{\gamma}_1(t) = \gamma_1(t)\xi_1; \\ \dot{\gamma}_2(t) = \gamma_1(t)\xi_2. \end{cases} \quad (5.46)$$

The first relation of the system (5.46) has the following solution :

$$\gamma_1(t) = k \exp(\xi_1 t), \quad t \in \mathbb{R}, \quad (5.47)$$

where $k > 0$. Since $\gamma_1(0) = 1$, we have $k = 1$ and

$$\gamma_1(t) = \exp(\xi_1 t), \quad t \in \mathbb{R}. \quad (5.48)$$

Now the second equation of the system (5.46) becomes :

$$\dot{\gamma}_2(t) = \xi_2 \exp(\xi_1 t). \quad (5.49)$$

We consider two cases :

(i) If $\xi_1 = 0$, then we have

$$\begin{cases} \dot{\gamma}_2(t) = \xi_2 \\ \gamma_2(t) = \xi_2 t + q, \end{cases} \quad (5.50)$$

where q is a real number. But $\gamma_2(0) = 0$, then $q = 0$ and

$$\gamma_2(t) = \xi_2 t, \quad t \in \mathbb{R}. \quad (5.51)$$

(ii) If $\xi_1 \neq 0$, then the equation (5.49) integrates to

$$\gamma_2(t) = \frac{\xi_2}{\xi_1} \exp(\xi_1 t) + r, \tag{5.52}$$

for some real number r . Since, $\gamma_2(0) = 0$, then $\frac{\xi_2}{\xi_1} + r = 0$, i.e. $r = -\frac{\xi_2}{\xi_1}$. Hence, we have :

$$\gamma_2(t) = \frac{\xi_2}{\xi_1} \left[\exp(\xi_1 t) - 1 \right], \tag{5.53}$$

for all t in \mathbb{R} .

We then summarize in the

Proposition 5.2.3. *For any element $\xi = \xi_1 e_1 + \xi_2 e_2$ in \mathcal{G} , the integral curve γ_ξ of the left-invariant vector field associated to ξ is defined by*

$$\gamma(t) = \begin{cases} (1, \xi_2 t) & \text{if } \xi_1 = 0; \\ \left(\exp(\xi_1 t), \frac{\xi_2}{\xi_1} [\exp(\xi_1 t) - 1] \right) & \text{if } \xi_1 \neq 0. \end{cases} \tag{5.54}$$

for all t in \mathbb{R} .

It is readily checked that the Cartesian equation of these integral curves are given as follows. Set $\gamma_\xi(t) = (y, x)$.

- If $\xi_1 = 0$, then the Cartesian equation of γ_ξ is $y = 1$. These curve is complete.
- If $\xi_1 \neq 0$ and $\xi_2 = 0$, then $x = 0$ and $y > 0$. This is a non-complete integral curve.
- If $\xi_1 \neq 0$ and $\xi_2 \neq 0$, we have the equation : $y = \frac{\xi_1}{\xi_2} x + 1$ and $y > 0$ or simply $y = ax + 1$ and $y > 0$, where a is a non-zero real number.

Remark 5.2.2. *The integral curves of the left-invariant vector fields associated to the elements of \mathcal{G} globally coincides with the geodesics through ϵ of G_0 obtained in (5.32).*

Corollary 5.2.2. *The exponential map of the group G_0 is defined by*

$$\exp_G(\xi) = \begin{cases} (1, \xi_2) & \text{if } \xi_1 = 0; \\ \left(\exp(\xi_1), \frac{\xi_2}{\xi_1} [\exp(\xi_1) - 1] \right) & \text{if } \xi_1 \neq 0. \end{cases} \tag{5.55}$$

for all $\xi = \xi_1 e_1 + \xi_2 e_2$ in \mathcal{G} .

Let us now define the Logarithm map of G_0 . Let $(y, x) \in G = \mathbb{R}_+^* \times \mathbb{R}$. We want to find $\xi \in \mathcal{G}$ such that $\exp_G(\xi) = (y, x)$.

1. If $y = 1$, then the unique solution is $\xi = (0, x)$.

2. Suppose $y \neq 1$, then

$$\left(\exp(\xi_1), \frac{\xi_2}{\xi_1}[\exp(\xi_1) - 1] \right) = (y, x). \tag{5.56}$$

That is

$$\begin{cases} \exp(\xi_1) & = & y \\ \frac{\xi_2}{\xi_1}[\exp(\xi_1) - 1] & = & x \end{cases} \tag{5.57}$$

and then

$$\begin{cases} \xi_1 & = & \ln y \\ \xi_2 & = & \frac{x}{y-1} \ln y. \end{cases} \tag{5.58}$$

We get,

Proposition 5.2.4. *The map \exp_G is invertible and its inverse is the map $\text{Log}_G : G \rightarrow \mathcal{G}$ defined by*

$$\text{Log}_G(y, x) = \begin{cases} (0, x) & \text{if } y = 1 \\ \left(\ln y, \frac{x}{y-1} \ln y \right) & \text{if } y \neq 1 \end{cases} \tag{5.59}$$

5.3 Double Lie groups of the affine Lie group of \mathbb{R}

5.3.1 Double Lie group of the affine Lie group of \mathbb{R}

Let $G := \text{Aff}(\mathbb{R})$ be the affine Lie group of \mathbb{R} and let $\mathcal{G} := \text{aff}(\mathbb{R})$ stand for the affine Lie algebra. It is shown (see [28]) that the double Lie algebra $\mathcal{D}(\mathcal{G}, r)$ (where $r : \mathcal{G}^* \rightarrow \mathcal{G}$ is a solution of the Classical Yang-Baxter Equation), of a double Lie group $\mathcal{D}(G, r)$ of G , is isomorphic to the Lie algebra $T^*\mathcal{G}$ of the cotangent bundle $T^*G = G \times \mathcal{G}^*$ endowed with the Lie group structure obtained by semi-direct product via the coadjoint action of the Lie group G and the Abelian Lie group \mathcal{G}^* . The Lie bracket of $T^*\mathcal{G}$, on some basis (e_1, e_2, e_3, e_4) , reads :

$$[e_1, e_2] = e_2 \quad , \quad [e_1, e_4] = -e_4 \quad , \quad [e_2, e_4] = e_3. \tag{5.60}$$

We write $T^*\mathcal{G} = \mathbb{R}e_1 \times \mathcal{H}_3$ (see Example 2.5.1), where e_1 acts on the Heisenberg Lie algebra $\mathcal{H}_3 = \text{span}(e_2, e_3, e_4)$ by the restriction of ad_{e_1} . Set $D := ad_{e_1} = \text{diag}(1, 0, -1)$. One Lie group of $T^*\mathcal{G}$ is then the group $\mathbb{R} \times \mathbb{H}_3$, where \mathbb{H}_3 is the 3-dimensional Heisenberg group ($\text{Lie}(\mathbb{H}_3) = \mathcal{H}_3$) and \mathbb{R} acts on \mathbb{H}_3 via the standard exponential of the endomorphism D by $\rho : \mathbb{R} \rightarrow \text{Aut}(\mathbb{H}_3)$. Precisely,

$$\rho_t : y \mapsto \exp_{\mathbb{H}_3} \left(\text{Exp}(tD)Y \right), \tag{5.61}$$

where $\exp_{\mathbb{H}_3}$ stands for the exponential map of the Lie group \mathbb{H}_3 , Exp is the standard exponential of endomorphisms and Y is an element of \mathcal{G} such that $\exp_{\mathbb{H}_3}(Y) = y$. The product on $\mathbb{R} \times \mathbb{H}_3$ reads :

$$(t, x)(s, y) = (t + s, x \cdot \rho_t(y)), \tag{5.62}$$

where $x \cdot \rho_t(y)$ is the product in the Heisenberg group of the elements x and $\rho_t(y)$. We then need to know the exponential and the logarithm map of the Heisenberg group.

Lemma 5.3.1. *Let $\exp_{\mathbb{H}_3}$ and $\ln_{\mathbb{H}_3}$ be the exponential and the logarithm maps of the Heisenberg group \mathbb{H}_3 . Then,*

1. *the exponential map is defined by : for all $\xi = (\xi_2, \xi_3, \xi_4)$ in \mathcal{H}_3 ,*

$$\exp_{\mathbb{H}_3}(\xi) = (\xi_2, \xi_3 + \frac{1}{2}\xi_2\xi_4, \xi_4) \tag{5.63}$$

2. *while the logarithm is given by*

$$\ln_{\mathbb{H}_3}(y) = (y_2, y_3 - \frac{1}{2}y_2y_4, y_4), \tag{5.64}$$

for any (y_2, y_3, y_4) in \mathbb{H}_3 .

Proof. Recall that the multiplication of \mathbb{H}_3 is given by :

$$x \cdot y = (x_2, x_3, x_4) \cdot (y_2, y_3, y_4) = (x_2 + y_2, x_3 + y_3 + x_2y_4, x_4 + y_4) \tag{5.65}$$

and the unit element of \mathbb{H}_3 is $\epsilon_{\mathbb{H}_3} = (0, 0, 0)$. The differential map at $\epsilon_{\mathbb{H}_3}$ of the left translation by an element x has the following matrix on the basis (e_2, e_3, e_4) :

$$T_{\epsilon_{\mathbb{H}_3}}L_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.66}$$

Let us compute the exponential map $\exp_{\mathbb{H}_3}$. Consider an element $\xi = \xi_2e_2 + \xi_3e_3 + \xi_4e_4$ of \mathcal{H}_3 and let X^ξ be the associated left invariant vector field. We note by γ_ξ the integral curve of X^ξ with initial conditions $\gamma_\xi(0) = \epsilon_{\mathbb{H}_3}$ and $\dot{\gamma}_\xi(0) = \xi$. We have :

$$X^\xi_{|(x_2, x_3, x_4)} = T_{\epsilon}L_{(x_2, x_3, x_4)} \cdot \xi = (\xi_2, \xi_3 + x_2\xi_4, \xi_4) \tag{5.67}$$

and

$$\dot{\gamma}_\xi(l) = X^\xi_{|\gamma_\xi(l)}. \tag{5.68}$$

If we set $\gamma_\xi = (\gamma_\xi^2, \gamma_\xi^3, \gamma_\xi^4)$, we obtain :

$$\dot{\gamma}_\xi^2(l) = \xi_2 \tag{5.69}$$

$$\dot{\gamma}_\xi^3(l) = \xi_3 + \gamma_\xi^2(l)\xi_4 \tag{5.70}$$

$$\dot{\gamma}_\xi^4(l) = \xi_4. \tag{5.71}$$

With the initial conditions we obtain the following solution :

$$\gamma_\xi^2(l) = \xi_2l \tag{5.72}$$

$$\gamma_\xi^3(l) = \xi_3l + \frac{1}{2}\xi_2\xi_4l^2 \tag{5.73}$$

$$\gamma_\xi^4(l) = \xi_4l. \tag{5.74}$$

It comes that

$$\exp_{\mathbb{H}_3}(\xi) = (\xi_2, \xi_3 + \frac{1}{2}\xi_2\xi_4, \xi_4). \tag{5.75}$$

Let now $Y = (Y_2, Y_3, Y_4)$ be an element of \mathcal{H}_3 such that $\exp_{\mathbb{H}_3}(Y) = y$. Then

$$(Y_2, Y_3 + \frac{1}{2}Y_2Y_4, Y_4) = (y_2, y_3, y_4).$$

We obtain that

$$Y_2 = y_2 \quad ; \quad Y_3 = y_3 - \frac{1}{2}y_2y_4 \quad ; \quad Y_4 = y_4. \tag{5.76}$$

That is

$$\ln_{\mathbb{H}_3}(y) = (y_2, y_3 - \frac{1}{2}y_2y_4, y_4), \tag{5.77}$$

where $\ln_{\mathbb{H}_3}$ is the inverse map of $\exp_{\mathbb{H}_3}$. □

Now we have :

$$Exp(tD) = \text{diag}(e^t, 1, e^{-t}) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}. \tag{5.78}$$

Hence,

$$\begin{aligned} \rho_t(y) &= \exp_{\mathbb{H}_3}(Exp(tD)Y) \\ &= \exp_{\mathbb{H}_3}\left(y_2e^t, y_3 - \frac{1}{2}y_2y_4, y_4e^{-t}\right) \\ &= \left(y_2e^t, y_3 - \frac{1}{2}y_2y_4 + \frac{1}{2}y_2e^t \times y_4e^{-t}, y_4e^{-t}\right) \\ &= \left(y_2e^t, y_3, y_4e^{-t}\right) \end{aligned} \tag{5.79}$$

Now we can write the multiplication on $\mathbb{R} \times \mathbb{H}_3$ as follows :

$$\begin{aligned} (t, x) \cdot (s, y) &= (t, x_2, x_3, x_4) \cdot (s, y_2, y_3, y_4) \\ &= \left(t + s, (x_2, x_3, x_4) \cdot \rho_t(y_2, y_3, y_4)\right) \\ &= \left(t + s, (x_2, x_3, x_4) \cdot (y_2e^t, y_3, y_4e^{-t})\right) \\ &= \left(t + s, x_2 + y_2e^t, x_3 + y_3 + x_2y_4e^{-t}, x_4 + y_4e^{-t}\right). \end{aligned} \tag{5.80}$$

Hence, the product of two elements $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$ of $\mathbb{R} \times \mathbb{H}_3$ is given by

$$x \cdot y = \left(x_1 + y_1, x_2 + y_2e^{x_1}, x_3 + y_3 + x_2y_4e^{-x_1}, x_4 + y_4e^{-x_1}\right). \tag{5.81}$$

The unit element is $\epsilon = (0, 0, 0, 0)$ and the inverse of an element $x = (x_1, x_2, x_3, x_4)$ is the element $x^{-1} = (-x_1, -x_2e^{-x_1}, -x_3 + x_2x_4, -x_4e^{x_1})$.

We have prove the following

Proposition 5.3.1. *Endowed with the product (5.81), $\mathbb{R} \times \mathbb{H}_3$ is a double Lie group of the affine Lie group of \mathbb{R} .*

Let $\mathcal{D}(G, r)$ denote the double Lie group of G defined in Proposition 5.3.1.

5.3.2 Connection on the double of the affine Lie group

Among a lot of possibilities, we are interested in the left invariant affine connection on $\mathcal{D}(G, r)$, introduced on any double Lie group of a symplectic Lie group by Diatta and Medina in [28] :

$$\nabla_{(x,\alpha)^+}(y, \beta)^+ = (x \cdot y + ad_\alpha y, ad_{r(\alpha)}^* \beta + ad_x^* \beta)^+, \tag{5.82}$$

where $x \cdot y$ is the product induced by the symplectic structure on \mathcal{G} through the formula

$$\omega(x \cdot y, z) = -\omega(y, [x, z]), \tag{5.83}$$

for all $x, y, z \in \mathcal{G}$.

Proposition 5.3.2. *On the basis of $\mathcal{D}(\mathcal{G}, r)$, the connection is given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_1 & ; & \nabla_{e_1} e_2 = 0 & ; & \nabla_{e_1} e_3 = e_2 & ; & \nabla_{e_1} e_4 = -e_1 - e_4 \\ \nabla_{e_2} e_1 &= -e_2 & ; & \nabla_{e_2} e_2 = 0 & ; & \nabla_{e_2} e_3 = 0 & ; & \nabla_{e_2} e_4 = e_3 \\ \nabla_{e_3} e_1 &= e_2 & ; & \nabla_{e_3} e_2 = 0 & ; & \nabla_{e_3} e_3 = 0 & ; & \nabla_{e_3} e_4 = -e_3 \\ \nabla_{e_4} e_1 &= -e_1 - e_4 & ; & \nabla_{e_4} e_2 = e_3 & ; & \nabla_{e_4} e_3 = 0 & ; & \nabla_{e_4} e_4 = -e_4 \end{aligned}$$

Proof. We have (see Relation (5.12))

$$\omega(e_1, e_2) = 1 \tag{5.84}$$

and (see Relation (5.15))

$$e_1 \cdot e_1 = -e_1 \quad ; \quad e_2 \cdot e_1 = -e_2. \tag{5.85}$$

Let us compute the bracket $[e_3, e_4]_*$ on $\mathcal{G}^*(r)$.

$$[e_3, e_4]_* = ad_{r(e_3)}^* e_4 - ad_{r(e_4)}^* e_3, \tag{5.86}$$

where $r := q^{-1} : \mathcal{G}^* \mapsto \mathcal{G}$, with $\langle q(x), y \rangle = \omega(x, y)$.

$$\langle q(e_1), e_1 \rangle = \omega(e_1, e_1) = 0. \tag{5.87}$$

$$\langle q(e_1), e_2 \rangle = \omega(e_1, e_2) = 1. \tag{5.88}$$

Then, $q(e_1) = e_2^* = e_4$.

$$\langle q(e_2), e_1 \rangle = \omega(e_2, e_1) = -1. \tag{5.89}$$

$$\langle q(e_2), e_2 \rangle = \omega(e_2, e_2) = 0. \tag{5.90}$$

Hence, $q(e_2) = -e_1^* = -e_3$. We then have the matrix of q on the basis (e_1, e_2) and (e_3, e_4) of \mathcal{G} and \mathcal{G}^* respectively :

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.91}$$

The matrix M est invertible and its inverse reads :

$$M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.92}$$

It comes that $r(e_3) = -e_2$ et $r(e_4) = e_1$. Hence,

$$\begin{aligned} [e_3, e_4]_* &= -ad_{e_2}^* e_4 - ad_{e_1}^* e_3. \\ &= +e_4 \circ ad_{e_2} + e_3 \circ ad_{e_1}. \end{aligned} \tag{5.93}$$

$$e_4 \circ ad_{e_2}(e_i) = \begin{cases} -1 & \text{si } i = 1 \\ 0 & \text{si } i = 2 \end{cases} \Rightarrow e_4 \circ ad_{e_2} = -e_3. \tag{5.94}$$

$$e_3 \circ ad_{e_1}(e_i) = \begin{cases} 0 & \text{si } i = 1 \\ 0 & \text{si } i = 2 \end{cases} \Rightarrow e_3 \circ ad_{e_1} = 0. \tag{5.95}$$

We then have $[e_3, e_4]_* = -e_3$.

We can now compute the connection on the basis of $\mathcal{D}(\mathcal{G}, r)$.

$$\nabla_{e_1} e_1 = e_1 \cdot e_1 = -e_1. \tag{5.96}$$

$$\nabla_{e_1} e_2 = e_1 \cdot e_2 = 0. \tag{5.97}$$

$$\nabla_{e_1} e_3 = ad_{e_3}^* e_1 + ad_{e_1}^* e_3 = ad_{e_3}^* e_1 = -e_1 \circ ad_{e_3|_{\mathcal{G}^*(r)}} = e_2 \tag{5.98}$$

$$\nabla_{e_1} e_4 = ad_{e_4}^* e_1 + ad_{e_1}^* e_4 = -e_1 \circ ad_{e_4|_{\mathcal{G}^*(r)}} + -e_4 \circ ad_{e_1|_{\mathcal{G}}} = -e_1 - e_4. \tag{5.99}$$

$$\nabla_{e_2} e_1 = e_2 \cdot e_1 = -e_2. \tag{5.100}$$

$$\nabla_{e_2} e_2 = e_2 \cdot e_2 = 0. \tag{5.101}$$

$$\nabla_{e_2} e_3 = ad_{e_3}^* e_2 + ad_{e_2}^* e_3 = 0. \tag{5.102}$$

$$\nabla_{e_2} e_4 = ad_{e_4}^* e_2 + ad_{e_2}^* e_4 = e_3. \tag{5.103}$$

$$\nabla_{e_3} e_1 = \nabla_{e_1} e_3 = e_2. \tag{5.104}$$

$$\nabla_{e_3} e_2 = \nabla_{e_2} e_3 = 0. \tag{5.105}$$

$$\nabla_{e_3} e_3 = ad_{r(e_3)}^* e_3 = -ad_{e_2}^* e_3 = 0. \tag{5.106}$$

$$\nabla_{e_3} e_4 = ad_{r(e_3)}^* e_4 = -ad_{e_2}^* e_4 = -e_3. \tag{5.107}$$

$$\nabla_{e_4} e_1 = \nabla_{e_1} e_4 = -e_1 - e_4. \tag{5.108}$$

$$\nabla_{e_4} e_2 = \nabla_{e_2} e_4 = e_3. \tag{5.109}$$

$$\nabla_{e_4} e_3 = ad_{r(e_4)}^* e_3 = ad_{e_1}^* e_3 = 0. \tag{5.110}$$

$$\nabla_{e_4} e_4 = ad_{r(e_4)}^* e_4 = ad_{e_1}^* e_4 = -e_4. \tag{5.110}$$

Thus, if we set $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ (Einstein's summation), the Christoffel's symbols Γ_{ij}^k for

this connection are :

$$\begin{aligned}
 \Gamma_{11}^1 &= -1 & ; & \Gamma_{11}^2 = 0 & ; & \Gamma_{11}^3 = 0 & ; & \Gamma_{11}^4 = 0 \\
 \Gamma_{12}^1 &= 0 & ; & \Gamma_{12}^2 = 0 & ; & \Gamma_{12}^3 = 0 & ; & \Gamma_{12}^4 = 0 \\
 \Gamma_{13}^1 &= 0 & ; & \Gamma_{13}^2 = 1 & ; & \Gamma_{13}^3 = 0 & ; & \Gamma_{13}^4 = 0 \\
 \Gamma_{14}^1 &= -1 & ; & \Gamma_{14}^2 = 0 & ; & \Gamma_{14}^3 = 0 & ; & \Gamma_{14}^4 = -1 \\
 \Gamma_{21}^1 &= 0 & ; & \Gamma_{21}^2 = -1 & ; & \Gamma_{21}^3 = 0 & ; & \Gamma_{21}^4 = 0 \\
 \Gamma_{22}^1 &= 0 & ; & \Gamma_{22}^2 = 0 & ; & \Gamma_{22}^3 = 0 & ; & \Gamma_{22}^4 = 0 \\
 \Gamma_{23}^1 &= 0 & ; & \Gamma_{23}^2 = 0 & ; & \Gamma_{23}^3 = 0 & ; & \Gamma_{23}^4 = 0 \\
 \Gamma_{24}^1 &= 0 & ; & \Gamma_{24}^2 = 0 & ; & \Gamma_{24}^3 = 1 & ; & \Gamma_{24}^4 = 0 \\
 \Gamma_{31}^1 &= 0 & ; & \Gamma_{31}^2 = 1 & ; & \Gamma_{31}^3 = 0 & ; & \Gamma_{31}^4 = 0 \\
 \Gamma_{32}^1 &= 0 & ; & \Gamma_{32}^2 = 0 & ; & \Gamma_{32}^3 = 0 & ; & \Gamma_{32}^4 = 0 \\
 \Gamma_{33}^1 &= 0 & ; & \Gamma_{33}^2 = 0 & ; & \Gamma_{33}^3 = 0 & ; & \Gamma_{33}^4 = 0 \\
 \Gamma_{34}^1 &= 0 & ; & \Gamma_{34}^2 = 0 & ; & \Gamma_{34}^3 = -1 & ; & \Gamma_{34}^4 = 0 \\
 \Gamma_{41}^1 &= -1 & ; & \Gamma_{41}^2 = 0 & ; & \Gamma_{41}^3 = 0 & ; & \Gamma_{41}^4 = -1 \\
 \Gamma_{42}^1 &= 0 & ; & \Gamma_{42}^2 = 0 & ; & \Gamma_{42}^3 = 1 & ; & \Gamma_{42}^4 = 0 \\
 \Gamma_{43}^1 &= 0 & ; & \Gamma_{43}^2 = 0 & ; & \Gamma_{43}^3 = 0 & ; & \Gamma_{43}^4 = 0 \\
 \Gamma_{44}^1 &= 0 & ; & \Gamma_{44}^2 = 0 & ; & \Gamma_{44}^3 = 0 & ; & \Gamma_{44}^4 = -1
 \end{aligned}$$

The only non vanishing symbols are :

$$\begin{aligned}
 \Gamma_{11}^1 &= -1 & ; & \Gamma_{13}^2 = 1 & ; & \Gamma_{14}^1 = -1 & ; & \Gamma_{14}^4 = -1 \\
 \Gamma_{21}^2 &= -1 & ; & \Gamma_{24}^3 = 1 & ; & \Gamma_{31}^2 = 1 & ; & \Gamma_{34}^3 = -1 \\
 \Gamma_{41}^1 &= -1 & ; & \Gamma_{41}^4 = -1 & ; & \Gamma_{42}^3 = 1 & ; & \Gamma_{44}^4 = -1
 \end{aligned}$$

□

5.3.3 Geodesics of $(\mathcal{D}(G, r), \nabla)$

Now let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ be a geodesic such that $\gamma(0) = (0, 0, 0, 0)$ in $T^*G \simeq \mathcal{D}(G, r)$ and $\dot{\gamma}(0) = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{D}(\mathcal{G}, r)$. We have the following non-linear differential equations which are the equations of geodesics of $\mathcal{D}(G, r)$.

$$\begin{aligned}
 \ddot{\gamma}_1 - \dot{\gamma}_1^2 - \dot{\gamma}_1\dot{\gamma}_4 - \dot{\gamma}_1\dot{\gamma}_4 &= 0, \\
 \ddot{\gamma}_2 + \dot{\gamma}_1\dot{\gamma}_3 - \dot{\gamma}_1\dot{\gamma}_2 + \dot{\gamma}_1\dot{\gamma}_3 &= 0, \\
 \ddot{\gamma}_3 + \dot{\gamma}_2\dot{\gamma}_4 - \dot{\gamma}_3\dot{\gamma}_4 + \dot{\gamma}_2\dot{\gamma}_4 &= 0, \\
 \ddot{\gamma}_4 - \dot{\gamma}_4^2 - \dot{\gamma}_1\dot{\gamma}_4 - \dot{\gamma}_1\dot{\gamma}_4 &= 0.
 \end{aligned}$$

We rearrange the latter equations as follows :

$$\ddot{\gamma}_1 - \dot{\gamma}_1^2 - 2\dot{\gamma}_1\dot{\gamma}_4 = 0, \tag{5.111}$$

$$\ddot{\gamma}_2 - \dot{\gamma}_1\dot{\gamma}_2 + 2\dot{\gamma}_1\dot{\gamma}_3 = 0, \tag{5.112}$$

$$\ddot{\gamma}_3 - \dot{\gamma}_3\dot{\gamma}_4 + 2\dot{\gamma}_2\dot{\gamma}_4 = 0, \tag{5.113}$$

$$\ddot{\gamma}_4 - \dot{\gamma}_4^2 - 2\dot{\gamma}_1\dot{\gamma}_4 = 0. \tag{5.114}$$

Unfortunately we do not yet have a solution for this system.

5.3.4 Integral curves of left-invariant vector fields on the double Lie group of the affine Lie group of \mathbb{R}

Let $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$ be in $\mathcal{D}(\mathcal{G}, r)$. The left invariant vector field X^ξ associated to ξ is given by

$$\begin{aligned} X^\xi_{|(x_1, x_2, x_3, x_4)} &= T_\epsilon L_{(x_1, x_2, x_3, x_4)} \cdot \xi \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{x_1} & 0 & 0 \\ 0 & 0 & 1 & x_2 e^{-x_1} \\ 0 & 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \\ &= (\xi_1, \xi_2 e^{x_1}, \xi_3 + x_2 \xi_4 e^{-x_1}, \xi_4 e^{-x_1}). \end{aligned} \tag{5.115}$$

Let γ_ξ the unique curve such that $\gamma_\xi(0) = \epsilon, \dot{\gamma}_\xi(0) = \xi$ and

$$\dot{\gamma}_\xi(t) = X^\xi_{|\gamma_\xi(t)}. \tag{5.116}$$

If $\gamma_\xi(t) = (\gamma_\xi^1(t), \gamma_\xi^2(t), \gamma_\xi^3(t), \gamma_\xi^4(t))$, we have the following equations coming from (5.116) :

$$\dot{\gamma}_\xi^1(t) = \xi_1 \tag{5.117}$$

$$\dot{\gamma}_\xi^2(t) = \xi_2 \exp[\gamma_\xi^1(t)] \tag{5.118}$$

$$\dot{\gamma}_\xi^3(t) = \xi_3 + \xi_4 \gamma_\xi^2(t) \exp[-\gamma_\xi^1(t)] \tag{5.119}$$

$$\dot{\gamma}_\xi^4(t) = \xi_4 \exp[-\gamma_\xi^1(t)]. \tag{5.120}$$

Resolving relation (5.117) and taking care with the initial conditions we have :

$$\gamma_\xi^1(t) = \xi_1 t. \tag{5.121}$$

Relation (5.118) is resolved as follows.

- If $\xi_1 = 0$, then

$$\begin{aligned} \gamma_\xi^2(t) &= \xi_2 t + b, \quad b \in \mathbb{R} \\ &= \xi_2 t \quad \text{since } \gamma_\xi^2(0) = 0. \end{aligned} \tag{5.122}$$

- If $\xi_1 \neq 0$, then we have

$$\begin{aligned} \gamma_\xi^2(t) &= \frac{\xi_2}{\xi_1} \exp(\xi_1 t) + b_1, \quad b_1 \in \mathbb{R}; \\ &= \frac{\xi_2}{\xi_1} [\exp(\xi_1 t) - 1], \quad \text{since } \gamma_\xi^2(0) = 0. \end{aligned} \tag{5.123}$$

Now we come to the relation (5.119) and again consider two cases.

- If $\xi_1 = 0$, the equation (5.119) can be written as

$$\dot{\gamma}_\xi^3(t) = \xi_3 + \xi_4 \xi_2 t. \tag{5.124}$$

The latter equation is solved as follows :

$$\begin{aligned} \gamma_\xi^3(t) &= \xi_3 t + \frac{1}{2} \xi_4 \xi_2 t^2 + c_1, \quad c_1 \in \mathbb{R}; \\ &= \xi_3 t + \frac{1}{2} \xi_4 \xi_2 t^2, \quad \text{as } \gamma_\xi^3(0) = 0. \end{aligned} \tag{5.125}$$

- If $\xi_1 \neq 0$, the relation (5.119) can be written as

$$\begin{aligned} \dot{\gamma}_\xi^3(t) &= \xi_3 + \xi_4 \frac{\xi_2}{\xi_1} [\exp(\xi_1 t) - 1] \exp(-\xi_1 t) \\ &= \xi_3 + \frac{\xi_2 \xi_4}{\xi_1} [1 - \exp(-\xi_1 t)]. \end{aligned} \tag{5.126}$$

and is integrated as

$$\begin{aligned} \gamma_\xi^3(t) &= \xi_3 t + \frac{\xi_2 \xi_4}{\xi_1} \left[t + \frac{1}{\xi_1} \exp(-\xi_1 t) \right] + c_2 \\ &= \xi_3 t + \frac{\xi_2 \xi_4}{\xi_1} \left[t + \frac{1}{\xi_1} \exp(-\xi_1 t) \right] - \frac{\xi_2 \xi_4}{\xi_1^2}, \quad \text{since } \gamma_\xi^3(0) = 0. \\ &= \left(\xi_3 + \frac{\xi_2 \xi_4}{\xi_1} \right) t + \frac{\xi_2 \xi_4}{\xi_1^2} [\exp(-\xi_1 t) - 1]. \end{aligned} \tag{5.127}$$

Let us now solve equation (5.120).

- If $\xi_1 = 0$, then (with the condition $\gamma_\xi^4(0) = 0$)

$$\gamma_\xi^4(t) = \xi_4 t. \tag{5.128}$$

- If $\xi_1 \neq 0$, then

$$\begin{aligned} \dot{\gamma}_\xi^4(t) &= -\frac{\xi_4}{\xi_1} \exp(-\xi_1 t) + d, \quad d \in \mathbb{R}; \\ &= \frac{\xi_4}{\xi_1} [1 - \exp(-\xi_1 t)], \quad \text{since } \gamma_\xi^4(0) = 0. \end{aligned} \tag{5.129}$$

Let us summarize all the above.

Proposition 5.3.3. *The integral curve of the left invariant vector field associated to any element $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$ of $\mathcal{D}(\mathcal{G}, r)$ is defined by*

$$\gamma_\xi(t) = \left(0, \xi_2 t, \frac{1}{2} \xi_2 \xi_4 t^2 + \xi_3 t, \xi_4 t \right), \tag{5.130}$$

if $\xi_1 = 0$; and by

$$\gamma_\xi(t) = \left(\xi_1 t, \frac{\xi_2}{\xi_1} [\exp(\xi_1 t) - 1], \left(\xi_3 + \frac{\xi_2 \xi_4}{\xi_1} \right) t + \frac{\xi_2 \xi_4}{\xi_1^2} [\exp(-\xi_1 t) - 1], \frac{\xi_4}{\xi_1} [1 - \exp(-\xi_1 t)] \right) \tag{5.131}$$

if $\xi_1 \neq 0$.

As an immediate consequence, we have the

Corollary 5.3.1. *The exponential map of the double Lie group $\mathcal{D}(G, r)$ of the affine Lie group of \mathbb{R} is defined as follows. For any $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4$,*

$$\exp_{\mathcal{D}(G,r)}(\xi) = \begin{cases} \left(0, \xi_2, \frac{1}{2}\xi_2\xi_4 + \xi_3, \xi_4 \right), & \text{if } \xi_1 = 0 \\ \left(\xi_1, \frac{\xi_2}{\xi_1}[\exp(\xi_1)-1], \xi_3 + \frac{\xi_2\xi_4}{\xi_1} + \frac{\xi_2\xi_4}{\xi_1^2}[\exp(-\xi_1)-1], \frac{\xi_4}{\xi_1}[1-\exp(-\xi_1)] \right) & \text{if } \xi_1 \neq 0. \end{cases}$$

We are now going to deal with the invertibility of the exponential map above. Let (x, y, z, t) be an arbitrary element of $\mathcal{D}(G, r)$. Our goal is to find ξ in $T^*\mathcal{G}$ such that $\exp_{\mathcal{D}(G,r)}(\xi) = (x, y, z, t)$.

1. If $x = 0$, then

$$\begin{aligned} & \left(0, \xi_2, \frac{1}{2}\xi_2\xi_4 + \xi_3, \xi_4 \right) = (x, y, z, t) \\ & \begin{cases} 0 & = & x \\ \xi_2 & = & y \\ \frac{1}{2}\xi_2\xi_4 + \xi_3 & = & z \\ \xi_4 & = & t \end{cases} \iff \begin{cases} 0 & = & x \\ \xi_2 & = & y \\ \xi_3 & = & z - \frac{1}{2}yt \\ \xi_4 & = & t \end{cases} \end{aligned} \tag{5.132}$$

We then have that

Lemma 5.3.2. *The restriction $\exp_{\{0\} \times \mathbb{R}^3}$ of the exponential map of $\mathcal{D}(G, r)$ to the subset $\{0\} \times \mathbb{R}^3$ of $\mathcal{D}(G, r)$ is invertible and its inverse is given by*

$$(0, y, z, t) \mapsto \left(0, y, z - \frac{1}{2}yt, t \right) \tag{5.133}$$

2. Suppose $x \neq 0$, then we have

$$\left(\xi_1, \frac{\xi_2}{\xi_1}[e^{\xi_1} - 1], \xi_3 + \frac{\xi_2\xi_4}{\xi_1} + \frac{\xi_2\xi_4}{\xi_1^2}[e^{-\xi_1} - 1], \frac{\xi_4}{\xi_1}[1 - e^{-\xi_1}] \right) = (x, y, z, t) \tag{5.134}$$

$$\begin{cases} \xi_1 & = & x \\ \frac{\xi_2}{\xi_1}[e^{\xi_1} - 1] & = & y \\ \xi_3 + \frac{\xi_2\xi_4}{\xi_1} + \frac{\xi_2\xi_4}{\xi_1^2}[e^{-\xi_1} - 1] & = & z \\ \frac{\xi_4}{\xi_1}[1 - e^{-\xi_1}] & = & t \end{cases} \iff \begin{cases} \xi_1 & = & x \\ \xi_2 & = & \frac{xy}{e^x - 1} \\ \xi_3 & = & z + \frac{xyt}{(e^x - 1)(e^{-x} - 1)} + \frac{yt}{e^x - 1} \\ \xi_4 & = & \frac{xt}{1 - e^{-x}} \end{cases}$$

Hence,

Lemma 5.3.3. *The restriction $\exp_{\mathbb{R}^* \times \mathbb{R}^3}$ of the exponential map of the Lie group $\mathcal{D}(G, r)$ to the subset $\mathbb{R}^* \times \mathbb{R}^3$ is invertible and its inverse is the map defined as :*

$$(x, y, z, t) \mapsto \left(x, \frac{xy}{e^x - 1}, z + \frac{yt}{e^x - 1} \left[\frac{x}{e^{-x} - 1} + 1 \right], \frac{xt}{1 - e^{-x}} \right) \quad (5.135)$$

The precedent two Lemmas imply

Proposition 5.3.4. *The exponential map of the Lie group $\mathcal{D}(G, r)$ is invertible and its inverse is the map $\text{Log}_{\mathcal{D}(G, r)} : \mathcal{D}(G, r) \rightarrow \mathcal{D}(\mathcal{G}, r)$ defined as follows :*

$$\text{Log}_{\mathcal{D}(G, r)}(x, y, z, t) = \begin{cases} (0, y, z - \frac{1}{2}yt, t) & \text{if } x = 0 \\ \left(x, \frac{xy}{e^x - 1}, z + \frac{yt}{e^x - 1} \left[\frac{x}{e^{-x} - 1} + 1 \right], \frac{xt}{1 - e^{-x}} \right) & \text{if } x \neq 0 \end{cases} \quad (5.136)$$

5.3.5 A Left Invariant Complex Structure On The Double of The Affine Lie group of \mathbb{R}

From [28], the following formula defines a left-invariant complex structure J on any Lie group with Lie algebra $\mathcal{D}(\mathcal{G}, r)$:

$$J((x, \alpha)^+) := (-r(\alpha), q(x))^+, \quad (5.137)$$

for any (x, α) in $\mathcal{D}(\mathcal{G}, r)$. We have

$$Je_1^+ = (q(e_1))^+ = e_4^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{x_1} & 0 & 0 \\ 0 & 0 & 1 & x_2 e^{-x_1} \\ 0 & 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_2 e^{-x_1} \\ e^{-x_1} \end{pmatrix} \quad (5.138)$$

$$Je_2^+ = (q(e_2))^+ = -e_3^+ = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{x_1} & 0 & 0 \\ 0 & 0 & 1 & x_2 e^{-x_1} \\ 0 & 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (5.139)$$

$$Je_3^+ = (-r(e_3))^+ = e_2^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{x_1} & 0 & 0 \\ 0 & 0 & 1 & x_2 e^{-x_1} \\ 0 & 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{x_1} \\ 0 \\ 0 \end{pmatrix} \quad (5.140)$$

$$Je_4^+ = (-r(e_4))^+ = -e_1^+ = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{x_1} & 0 & 0 \\ 0 & 0 & 1 & x_2 e^{-x_1} \\ 0 & 0 & 0 & e^{-x_1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (5.141)$$

Let us summarize

$$Je_1^+ = x_2 e^{-x_1} \frac{\partial}{\partial x_3} + e^{-x_1} \frac{\partial}{\partial x_4} \quad (5.142)$$

$$Je_2^+ = -\frac{\partial}{\partial x_3} \quad (5.143)$$

$$Je_3^+ = e^{x_1} \frac{\partial}{\partial x_2} \quad (5.144)$$

$$Je_4^+ = -\frac{\partial}{\partial x_1} \quad (5.145)$$

This tensor is not bi-invariant, that is it does not commute with the adjoint operators of \mathcal{G} . Indeed, if $j := J|_{\mathfrak{e}}$, we have

- $j[e_1, e_2] = je_2 = -e_3$;
- $[e_1, je_2] = [e_1, -e_3] = 0$,

then $j[e_1, e_2] \neq [e_1, je_2]$.

General Conclusion

In this thesis we study some aspect of the geometry of cotangent bundles of Lie groups as Drinfel'd double Lie groups. Automorphisms of cotangent bundles of Lie groups are completely characterized and interesting results are obtained. We give prominence to the fact that the Lie groups of automorphisms of cotangent bundles of Lie groups are supersymmetric Lie groups (Theorem 2.3.2). In the cases of semi-simple Lie algebras, compact Lie algebras and more generally orthogonal Lie algebras, we recover by simple methods interesting co-homological known results (Section 2.3.6).

Another theme in this thesis is the study of prederivations of cotangent bundles of Lie groups. The Lie algebra of prederivations encompasses the one of derivations as a subalgebra. We find out that Lie algebras of cotangent Lie groups (which are not semi-simple) of semi-simple Lie groups have the property that all their prederivations are derivations. This result is an extension of a well known result due to Müller ([64]). The structure of the Lie algebra of prederivations of Lie algebras of cotangent bundles of Lie groups is explored and we have shown that the Lie algebra of prederivations of Lie algebras of cotangent bundle of Lie groups are reductive Lie algebras.

Prederivations are useful tools for classifying objects like pseudo-Riemannian metrics ([9], [64]). We have studied bi-invariant metrics on cotangent bundles of Lie groups and their isometries. The Lie algebra of the Lie group of isometries of a bi-invariant metric on a Lie group is entirely determined by prederivations of the Lie algebra which are skew-symmetric with respect to the induced orthogonal structure on the Lie algebra. We have shown that the Lie group of isometries of any bi-invariant metric on the cotangent bundle of any semi-simple Lie groups is given by inner derivations of the Lie algebra of the cotangent bundle.

Last, we have dealt with an introduction to the geometry of the Lie group of affine motions of the real line \mathbb{R} , which is a Kählerian Lie group (see [53]). We describe, through explicit expressions, a symplectic structure, a complex structure, geodesics. Since the symplectic structure corresponds to a solution r of the Classical Yang-Baxter equation (see [28]), we also study the double Lie group associated to r .

Admittedly, questions remain. Can it be otherwise ?

Let \mathcal{G} be a Lie algebra. We have said that the Lie algebra $\text{Pder}(T^*\mathcal{G})$ of prederivations of $T^*\mathcal{G}$ contains the one $\text{der}(T^*\mathcal{G})$ of derivations as a Lie subalgebra. It would be interesting

to know :

- if $\text{Pder}(T^*\mathcal{G})$ can be decomposed into a semi-direct sum of $\text{der}(T^*\mathcal{G})$ and a subspace \mathfrak{h} of $\text{Pder}(T^*\mathcal{G})$: $\text{Pder}(T^*\mathcal{G}) = \text{der}(T^*\mathcal{G}) \ltimes \mathfrak{h}$;
- if $\text{der}(T^*\mathcal{G})$ is an ideal of $\text{Pder}(T^*\mathcal{G})$.

We have seen that if \mathcal{G} is semi-simple, then $\text{Pder}(T^*\mathcal{G}) = \text{der}(T^*\mathcal{G})$. It would be a great step to find necessary and sufficient conditions for which the latter equality holds.

Another open question is the following. As the cotangent bundle of any Lie group is a Drinfel'd double Lie group, it seems to be a good idea to extend the results within this thesis to the class of Lie groups which are double Lie groups of Poisson-Lie groups.

One other step is to list all orthogonal Lie groups of low-dimension using the double extension procedure of Medina and Revoy. Studying the isomorphic classes, up to isometric automorphisms, of bi-invariant metrics on cotangent bundles of Lie groups is also an interesting subject. It seems to be reasonable to begin by simple Lie algebras.

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