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Laguerre-Hahn Orthogonal Polynomials with respect to the Hahn Operator : Fourth-Order Difference Equation for the rth Associated and the Laguerre-Freud Equations for the Recurrence coefficients.

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A mon épouse Adjara et ma fille Samihra.
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Contents

Chapter 1

Introduction

1.1 Historical introduction

1.1.1 The fourth-order differential and difference equation

Consider the family of monic polynomials $\{P_n\}_{n\in\mathbb{N}}$, orthogonal with respect to a linear functional $\mathcal{L}$ (see (2.5)). It satisfies a three-term recurrence relation (which we denote TTRR) [Chihara, 1978]

$$\begin{cases} P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), & n \geq 1, \\ P_0(x) = 1, P_1(x) = x - \beta_0, \end{cases}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0 \quad \forall n \in \mathbb{N}$.

The rth associated of $\{P_n\}_{n\in\mathbb{N}}$ is the family of monic polynomials $\{P_n^{(r)}\}_{n\in\mathbb{N}}$, defined by the previous relation in which β_n, γ_n and P_n are replaced by β_{n+r}, γ_{n+r} and $P_n^{(r)}$, respectively,

$$\begin{cases} P_{n+1}^{(r)}(x) = (x-\beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x), & n \geq 1, \\ P_0^{(r)}(x) = 1, P_1^{(r)}(x) = x - \beta_r. \end{cases}$$

The rth associated of the regular linear functional $\mathcal{L}$ is, by Favard Theorem [Favard, 1935], the unique linear functional $\mathcal{L}^{(r)}$ with respect to which $\{P_n^{(r)}\}_{n\in\mathbb{N}}$ is orthogonal and satisfices $\langle \mathcal{L}^{(r)}, 1\rangle = \gamma_r$.

Let $\{P_n\}_{n\in\mathbb{N}}$ be a family of polynomials, orthogonal with respect to the linear functional $\mathcal{L}$ and $S(\mathcal{L})$, the Stieltjes function of $\mathcal{L}$ given by

$$S(\mathcal{L})(x) = S(x) = -\sum_{n\geq 0} \frac{M_n}{x^{n+1}},$$

where M_n is the moment of order n of $\mathcal{L}$: $M_n = \langle \mathcal{L}, x^n\rangle$.

When S satisfies a Riccati differential equation

$$\phi(x)S(x)' = B(x)\,S(x)^2 + A(x)\,S(x) + D(x),$$

where ϕ, A, B and D are polynomials, then $\{P_n\}_{n\in\mathbb{N}}$ are called Laguerre-Hahn orthogonal polynomials [Magnus, 1984], [Dzoumba, 1985]. It is well-known [Magnus, 1984] that these polynomials satisfy a fourth-order linear differential equation.

Classical and semi-classical (continuous) orthogonal polynnomials are particular cases of Laguerre-Hahn orthogonal polynomials, and they satisfy a second-order linear differential equation.

The rth associated Laguerre-Hahn orthogonal polynomials are Laguerre-Hahn orthogonal polynomials, therefore they satisfy a fourth-order linear differential equation.

The search for these differential equations has been very intensive during the past few years. For $r = 1$, Grosjean (1985, 1986) found them for Legendre and Jacobi families, and Ronveaux (1988), has given a single equation valid for the first associated classical (continuous) orthogonal polynomials.

For an arbitrary r, computer algebra packages have been very useful due to the heavy computations involved. In this context we mention that Wimp (1987) has used the MACSYMA [ref] package to construct the fourth-order differential equations satisfied by the rth associated Jacobi polynomials (r in this case is integer or not). Belmehdi and Ronveaux (1989) devised a REDUCE program in order to obtain these differential equations for the associated classical orthogonal polynomials of integer (and fixed) order r.

Differential equations valid for the rth associated Laguerre-Hahn orthogonal polynomials and for any integer r were given by Belmehdi et al. (1991) using the properties of the Stieltjes function of the associated functional (see [Magnus, 1984], [Dzoumba, 1985]). Then, followed some papers giving, in a simple way, the single fourth-order differential equation for the associated classical orthogonal polynomials of any integer order r (see for instance [Ronveaux, 1991], [Zarzo et al., 1993], [Lewanowicz, 1995]).

As it was the case for the associated orthogonal polynomial of a continuous variable, many works have been done to give the fourth-order difference equation satisfied by the associated classical orthogonal polynomials of a discrete variable.

Atakishiyev et al. (1996) have derived the relation (already known for classical continuous orthogonal polynomials [Ronveaux, 1988]) giving the link between the first associated classical discrete orthogonal polynomials and the starting polynomials, and used this relation to prove that the first associated of the classical discrete orthogonal polynomials are solutions of a fourth-order linear difference equation which can be factored as product of two second-order linear difference equations.

Using the explicit representation of the associated Meixner polynomials (with the real association parameter $r \geq 0$) in terms of hypergeometric functions, Letessier et al.(1996) gave the fourth-order difference equation satisfied by the rth associated Meixner polynomials and deduced by an appropriate limit process the difference equation for the rth associated Charlier, Laguerre and Hermite polynomials.

This equation, thanks to the computer algebra system MATHEMATICA [Wolfram, 1993] and the relation proved in [Atakishiyev et al., 1996] is given explicitly for the first associated of Charlier, Meixner, Krawtchouk and Hahn polynomials [Ronveaux et al., 1998a].

The question one can ask is whether it is possible to give one fourth-order difference equation valid for the rth associated Laguerre-Hahn orthogonal polynomials including orthogonal polynomials of continuous, discrete variable and also q-polynomials? The answer is yes and the first part of this dissertation aimed at answering this question.

1.1.2 The non-linear difference equations

Here, we consider that the polynomials $\{P_n\}_{n\in\mathbb{N}}$, orthogonal with respect the semi-classical linear functional $\mathcal{L}$ is orthonormal ($\langle\mathcal{L}, P_nP_n\rangle = 1 \quad \forall n \in \mathbb{N}$), thus, satisfying

$$x\,P_n = a_{n+1}\,P_{n+1} + b_n\,P_n + a_nP_{n-1},\ n \geq 0,\ a_0\,P_{-1} = 0,$$

where a_n and b_n are complex numbers with $a_n \neq 0$.

The coefficients a_n and b_n can be given explicitly for classical (continuous) orthogonal polynomials in terms of the polynomials ϕ and ψ appearing in the Pearson differential equation, $\frac{d}{dx}(\phi\mathcal{L}) = \psi\mathcal{L}$, satisfied by the linear functional $\mathcal{L}$ with respect to which $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal (see for instance [Nikiforov et al., 1983] [Chihara, 1978], [Szegö, 1939], [Lesky, 1985], [Koepf et al., 1996]...).

These coefficients are also known for classical orthogonal polynomials of a discrete variable and for q-classical orthogonal polynomials ([Nikiforov et al., 1991], [Szegö, 1939], [Lesky, 1985], [Koepf et al., 1996], [Medem, 1996]...).

When the polynomials are semi-classical (instead of classical), except for some particular cases, it is difficult to give, in general situation, the coefficients a_n and b_n.

The properties of the coefficients a_n and b_n as well as those of the polynomials P_n have been investigated by many authors.

- Firstly, we cite for example Laguerre, who, in 1885, explored the properties of the orthogonal polynomials related to the weight function ρ satisfying

$$\frac{\rho'(x)}{\rho(x)} = R(x),$$

where $R(x)$ is a rational function of x. He also studied Padé approximations and continued fraction expansions of functions satisfying a differential equation of the form

$$W(x)f'(x) = 2V(x)f(x) + U(x),$$

where U, V and W are polynomials; and recovered orthogonal polynomials P_n as denominators of the approximants of f. He succeeded in showing that the orthogonal polynomials P_n satisfy the remarkable differential equation,

$$W\Theta_n y'' + [(2V + W')\Theta_n - W\Theta_n']y' + K_n y = 0,$$

where Θ_n and K_n are polynomials with bounded degrees, whose coefficients are solutions of certain (usually) *non-linear equations* which provide non-linear equations for a_n and b_n (see [Magnus, 1991] for more details about Laguerre equations).

- Secondly, we cite the works by Freud (see [Freud, 1976, 1977, 1986]) who investigated the asymptotic behaviour of the recurrence coefficients for special families of measures by a technique producing an infinite system of (usually non-linear) equations (called Freud equations) for these coefficients (see [Magnus, 1991] for more details about Freud equations). For example, if the polynomials P_n are related to the weight $\rho(x) = \exp(-x^4)$ on the whole real line, then the Freud equations are reduced to [Nevai, 1983]

$$\begin{cases} 4a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2) = n, \ n \geq 2, \ a_0 = 0, \ a_1^2 = \frac{\Gamma(3/4)}{\Gamma(1/4)}, \\ b_n = 0, \quad n \geq 0. \end{cases}$$

It should be noted that other people found similar non-linear equations and identities (see for instance [Laguerre, 1885], [Perron, 1929], [Shohat, 1939], see also [Nevai et al., 1986], [Magnus, 1991] for more details), but these authors did not study their solutions when no simple form could be found.

Using the Freud equations, Freud (1976) gave a conjecture about the asymptotic behaviour of recurrence coefficients when the polynomials P_n are related to the weight function $\rho(x) = |x|^\ell \exp(-|x|^\alpha)$ stating that :

Let a_n and b_n be the coefficients of the following recurrence relation

$$x P_n = a_{n+1} P_{n+1} + b_n P_n + a_n P_{n-1}, \ n \geq 0, \ a_0 P_{-1} = 0,$$

satisfied by the polynomials $\{P_n\}_{n\in\mathbb{N}}$, orthogonal with respect to the weight $\rho(x) = |x|^\ell \exp(-|x|^\alpha)$, $\ell > -1$, $\alpha > 0$, on the whole real line. Then a_n and b_n obey:

$$\lim_{n\to\infty} \frac{a_n}{[n/C(\alpha)]^{1/\alpha}} = 1, \ C(\alpha) = \frac{2\Gamma(\alpha)}{\Gamma(\alpha/2)^2}.$$

Important investigations have been devoted to the proof of Freud conjecture as well as to the study of the asymptotics for $\{P_n\}_{n\in\mathbb{N}}$, the distribution of zeros, the sharp estimates of the extreme zeros ... ([Chihara, 1978], [Freud, 1976, 1977, 1986], [Lubinsky, 1984, 1985a, 1985b], [Lubinsky et al. 1986, 1988] , [Magnus, 1984. 1985a, 1985b, 1986], [Bonan, 1984], [Máté et al., 1985], [Mhaskar et al., 1984a, 1984b], [Nevai, 1973, 1983, 1984a, 1984b, 1985, 1986], [Sheen, 1984] ..., for more details see [Magnus, 1984, 1985a, 1985b, 1986]).

Later, Belmehdi and Ronveaux (1994) gave a systematic way to obtain non-linear equations for the recurrence coefficients, valid for any semi-classical orthogonal polynomial of a continuous variable. In fact, given a semi-classical linear functional $\mathcal{L}$ satisfying $\frac{d}{dx}(\phi\mathcal{L}) = \psi\mathcal{L}$, where ϕ and ψ are polynomials, they were able to provide two non-linear equations for the coefficients a_n, b_n of the recurrence relation satisfied by the polynomials $\{P_n\}_{n\in\mathbb{N}}$ associated to $\mathcal{L}$, called Laguerre-Freud equations (denomination borrowed from Magnus [Magnus 1985b, 1986]).

In the second part of this dissertation, we give a generalisation of the previous results [Belmehdi et al., 1994] by giving the system of two non-linear difference equations satisfied by the recurrence coefficients: equations which are valid for semi-classical orthogonal polynomials of a continuous and discrete variable, and also for q-semi-classical orthogonal polynomials (both of class 1).

1.2 Summary of the main results

1.2.1 The fourth-order difference equation

1. Using the result in [Suslov, 1989], we prove the following:
   Consider $\mathcal{L}$ a regular linear functional satisfying $\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}$, where ψ is a first degree polynomial and ϕ a polynomial of degree at most two. $\mathcal{D}_q$ is the Hahn operator defined by

$$\mathcal{D}_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad q \neq 0, \quad q \neq 1, \mathcal{D}_q f(0) := f'(0).$$

   Then, if $\{P_n\}_{n\in\mathbb{N}}$ is the monic family of polynomials, orthogonal with respect to $\mathcal{L}$, then, the first associated $P_n^{(1)}$ of P_n satisfies the fourth-order difference equation

$$\mathcal{Q}^{**}_{2,n-1} \frac{\mathcal{Q}^{*}_{2,n-1}}{q^2\,(q-1)^2\,x^2}\left[P^{(1)}_{n-1}(x;q)\right] = 0.$$

   Operators $\mathcal{Q}^{**}_{2,n-1}$ and $\mathcal{Q}^{*}_{2,n-1}$ are given by:

$$\begin{aligned}
\mathcal{Q}^{*}_{2,n-1} &= \phi_{(2)}\,\mathcal{G}_q^2 - ((1+q)\phi_{(1)} + \psi_{(1)}\,t_1 - \lambda_{n,0}\,t_1^2)\mathcal{G}_q + q\,(\phi + \psi\,t)\,\mathcal{I}_d,\\
\mathcal{Q}^{**}_{2,n-1} &= (\phi_{(3)} + \psi_{(3)}\,t_3)[q^2\,A_1 + (1+q)\,\phi_{(2)} + \psi_{(2)}\,t_2]\mathcal{G}_q^2\\
&\quad -[q^3\,A_1\,(\phi_{(2)} + \psi_{(2)}\,t_2) + A_3\,(\phi_{(2)} + q\,A_1)]\,\mathcal{G}_q\\
&\quad +q\,\phi_{(1)}\,[q^2\,A_2 + (1+q)\,\phi_{(3)} + \psi_{(3)}\,t_3)]\,\mathcal{I}_d,
\end{aligned}$$

   with

$$\begin{aligned}
\lambda_{n,0} &= -[n]_q\{\psi' + [n-1]_{\frac{1}{q}}\frac{\phi''}{2q}\},\ [n]_q = \frac{q^n-1}{q-1},\ q \neq 1,\ n \geq 0,\ \mathcal{G}_qP(x) = P(qx) \quad \forall P \in \mathbb{P},\\
\phi_{(i)} &\equiv \phi(q^i x),\ \ \psi_{(i)} \equiv \psi(q^i x),\ \ t_i \equiv t(q^i x),\ \ t(x) = (q-1)x,\\
A_j &= (1+q)\phi_{(j)} + \psi_{(j)}\,t_j - \lambda_{n,0}\,t_j^2.
\end{aligned}$$

   This result [Foupouagnigni et al., 1998d], is used to deduce the factored form of the difference equations satisfied by the first associated classical orthogonal polynomials of a discrete variable [Ronveaux et al., 1998a] and also the factored form of the differential equation satisfied by the first associated classical continuous orthogonal polynomials [Ronveaux, 1988]. We have used, also, this result to prove that under certain conditions on the parameters, the first associated of little and big q-Jacobi polynomials are still classical. Moreover, we deduce that if $p_n(x;a,b\,|q)$ (respectively $P_n(x;a,b,c;q)$) denotes the monic little q-Jacobi polynomials (respectively monic big q-Jacobi polynomials), then they are related with their respective first associated by:

$$\begin{aligned}
p_n^{(1)}(x;a,\frac{1}{q\,a}\,|q) &= a^n\,q^n\,p_n(\frac{x}{a\,q};\frac{1}{a},a\,q\,|q),\\
P_n^{(1)}(x;a,\frac{1}{q\,a},c;q) &= a^n\,P_n(\frac{x}{a};\frac{1}{a},a\,q,c\,q;q).
\end{aligned}$$

2. We prove that the rth associated $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials satisfy the single fourth-order difference equation [Foupouagnigni et al., 1998e]

$$\sum_{j=0}^{4} I_j(n,r,q,x)\mathcal{D}_q^j\,P_n^{(r)} = 0,$$

   where $I_j(n,r,q,x)$ are polynomials in x.

We use suitable change of variable and limit processes to extend the above result to the rth associated Laguerre-Hahn orthogonal polynomials of a continuous and a discrete variable, respectively [Foupouagnigni et al 1998b].

We apply this result to compute explicitly the coefficients $I_j(n, r, q, x)$ for the rth associated classical orthogonal polynomials (including classical continuous, classical discrete and q-classical polynomials) [Foupouagnigni et. al., 1998b, 1998c, 1998e].

1.2.2 The non-linear recurrence equations

We prove the following theorem (see 8.1) which is the main result of the second part of this Dissertation.

Theorem

The coefficients β_n and γ_n of the three-term recurrence relation

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), n \geq 1, P_0(x) = 1, P_1(x) = x - \beta_0,$$

satisfied by the $\mathcal{D}_q$-semi-classical monic orthogonal polynomials of class at most one, $\{P_n\}_{n\in\mathbb{N}}$, can be computed recursively from the two non-linear equations

$$\begin{cases} (\psi_2 + [2n]_{\frac{1}{q}} \frac{\phi_3}{q})(\gamma_n + \gamma_{n+1}) = F_1(q; \beta_0, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n), \\ (\psi_2 + [2n+1]_{\frac{1}{q}} \frac{\phi_3}{q})\beta_{n+1}\gamma_{n+1} = F_2(q; \beta_0, \ldots, \beta_n; \gamma_1, \ldots, \gamma_{n+1}). \end{cases}$$

ϕ_j and ψ_j are the coefficients of the polynomials ϕ and ψ ($\phi(x) = \sum_{j=0}^{3} \phi_j x^j$, $\psi(x) = \sum_{j=0}^{2} \psi_j x^j$) appearing in the $\mathcal{D}_q$-Pearson equation, $\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}$, satisfied by the regular linear functional $\mathcal{L}$. F_1 is a polynomial of $2n+1$ variables and of degree 2; and F_2 a polynomial of $2n+2$ variables and of degree 3, with the initial conditions

$$\beta_0 = \frac{\langle \mathcal{L}, x\rangle}{\langle \mathcal{L}, 1\rangle}, \ \psi_2\gamma_1 = -\psi(\beta_0).$$

We use suitable change of variable and limit processes to extend the previous theorem to the $\mathcal{D}$ and Δ-semi-classical orthogonal polynomials of class at most one [Foupouagnigni et al., 1998a]. We then give the Laguerre-Freud equations for the generalised Charlier and generalised Meixner of class one and use these equations (numerical and symbolic computation with Maple V Release 4) to give a conjecture about the asymptotic behaviour of the coefficients β_n and γ_n of the generalised Charlier and generalised Meixner polynomials of class one:

Conjecture

The coefficients β_n and γ_n of the three-term recurrence relation satisfied by the monic generalised Meixner polynomials of class one obey:

$$\lim_{n\to\infty} \left(\beta_n - \frac{1+\mu}{1-\mu}n - \frac{\mu(\alpha_1+\alpha_2-1)}{1-\mu}\right) = 0, \ \lim_{n\to\infty}\left(\gamma_n - \frac{\mu(n+\alpha_1-1)(n+\alpha_2-1)}{(1-\mu)^2}\right) = 0,$$

and those of the three-term recurrence relation satisfied by the monic generalised Charlier polynomials of class one obey:

$$\lim_{n\to\infty}(\beta_n - n) = 0, \ \lim_{n\to\infty}(\gamma_n - \mu) = 0.$$

1.3 Outline of dissertation

In Chapter 2 we give some results and definitions on orthogonal and associated orthogonal polynomials. We also prove some characterisation theorems for classical orthogonal polynomials.

Chapter 3 gives some useful properties of the operators $A_{q,\omega}$ and $D_{q,\omega}$ and the proof of some characterisation theorems for $D_{q,\omega}$-classical and $D_{q,\omega}$-semi-classical orthogonal polynomials; characterisation theorem which are valid (by limit processes) for the operators $\frac{d}{dx}$, $\mathcal{D}_q$ and Δ.

Chapter 4 is devoted to the study of the $D_{q,\omega}$-Riccati difference equation satisfied by the Stieltjes function of the given associated linear functional. In particular, we prove that the affine $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials are the $D_{q,\omega}$-semi-classical orthogonal polynomials and conversely. In this chapter, it is also proved that the $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials can be obtained from the $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials by a change of variable.

In Chapter 5 we give the factored form of the fourth-order difference equation satisfied by the first associated $\mathcal{D}_q$-classical orthogonal polynomials and we deduce the difference equation for classical orthogonal polynomials of continuous and of discrete variable. We also consider the situations for which the first associated of the little and big q-Jacobi polynomials are still classical.

Chapter 6 describes the method used to obtain, for the general situation, the single fourth-order difference equation satisfied by the rth associated $\mathcal{D}$, $\mathcal{D}_q$ and Δ-Laguerre-Hahn orthogonal polynomials. The coefficients of the fourth-order difference equation for classical situations are also given explicitly.

Chapter 7 gives useful coefficients for classical orthogonal polynomials like β_n, γ_n, $T_{n,1}$ and $T_{n,2}$.

Chapter 8 presents the method used to obtain the two non-linear equations for the coefficients of the TTRR satisfied by the $\mathcal{D}_q$-semi-classical orthogonal polynomials of class at most one. We also show how these equations can be used to obtain the two non-linear equations for the coefficients of the TTRR satisfied by the $\mathcal{D}$ and Δ-semi-classical orthogonal polynomials of class at most one. The conjecture about the asymptotic behaviour of the coefficients of the TTRR satisfied by the generalised Charlier and the generalised Meixner polynomials of class one (conjecture obtained thanks to the two-non-linear equations) is also given.

The appendices I, II and III contain the data for classical orthogonal polynomials as well as the results on the fourth-order difference equations for classical situations.

It should be mentioned that:

- Chapter 2, devoted to the preliminaries, is based on [Chihara, 1978], [Guerfi, 1988], [Belmehdi, 1990a], [Salto, 1995] and [Medem, 1996].
- Chapters 3 and 4 generalise to the operator $D_{q,\omega}$ certain results given in the above mentioned references.
- The original results obtained in the framework of this thesis are presented in chapters 5, 6 and 8.

Chapter 2

Preliminaries

2.0.1 The notion of topology

We recall the notion of topology on polynomials and linear functional vector spaces. These notions have been defined in [Trèves, 1967], [Maroni, 1985, 1988], [Guerfi, 1988] and [Belmehdi, 1990a]. For these preliminaries, we shall exploit the works by Maroni [Maroni, 1988], Guerfi [Guerfi, 1988] and [Belmehdi, 1990a].

Let $\mathbb{P}$ be a vector space of polynomials in one real variable with complex coefficients, endowed with the strict inductive limit topology of the spaces $\mathbb{P}_n$. $\mathbb{P}_n \subset \mathbb{P}$ is the vector space of polynomials of degree at most n. It satisfies

$$\mathbb{P}_n \subset \mathbb{P}_{n+1},\ n \geq 0,\ \mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n,$$

and is endowed with its natural topology which makes it a Banach space.

Let $\mathbb{P}'$ be the dual of $\mathbb{P}$, equiped with its topology which is defined by the system of semi-norms:

$$||\mathcal{L}||_n = \sup_{k \leq n} |M_k|,$$

where M_k denotes the moments of the functional $\mathcal{L}$ with respect to the sequence $\{x^n\}_n$: $M_k = (\mathcal{L})_k = \langle \mathcal{L}, x^k \rangle$. $\mathbb{P}$ and $\mathbb{P}'$ are Fréchet spaces.

We consider $\mathbb{V}$ the vector space generated by the elements $\{\frac{(-1)^n}{n!}\mathcal{D}^n\delta\}_n$ ($\mathcal{D} = \frac{d}{dx}$) with its inductive limit topology. δ denotes the Dirac measure: $\langle \delta, f \rangle = f(0)$, $f \in C^\infty(\mathbb{R})$.

Let $\mathcal{F}$ be the linear application:

$$\begin{aligned} \mathcal{F} :\ \mathbb{V} &\longrightarrow \mathbb{P} \\ d = \sum_{j=0}^{n} d_j \frac{(-1)^j}{j!}\mathcal{D}^j\delta &\longrightarrow \mathcal{F}(d) = \sum_{j=0}^{n} d_j x^j. \end{aligned} \tag{2.1}$$

$\mathcal{F}$ verifies the following properties:

i) $\mathcal{F}$ is an isomorphism defined on $\mathbb{V}$ into $\mathbb{P}$.

ii) The transpose ${}^t\mathcal{F}$ of $\mathcal{F}$, is an isomorphism defined on $\mathbb{P}'$ into $\mathbb{V}'$.

iii) ${}^t\mathcal{F} = \mathcal{F}$ on $\mathbb{P}'$.

Thus,

$$\langle \mathcal{F}(\mathcal{L}), d \rangle = \langle \mathcal{L}, \mathcal{F}(d) \rangle, \forall \mathcal{L} \in \mathbb{P}',\ \forall d \in \mathbb{V}. \tag{2.2}$$

Since $\{\frac{(-1)^n}{n!}\mathcal{D}^n\delta\}_n$ forms a basis of $\mathbb{P}'$ [Maroni, 1988], that is, any element $\mathcal{L}$ of $\mathbb{P}'$ can be expressed as

$$\mathcal{L} = \sum_{n \geq 0} (\mathcal{L})_n \frac{(-1)^n}{n!}\mathcal{D}^n\delta, \tag{2.3}$$

it follows that

$$\mathcal{F}(\mathcal{L}) = \sum_{n\geq 0}(\mathcal{L})_n x^n. \tag{2.4}$$

$\mathbb{V}$' is therefore the vector space of formal series.

Remark 2.1 *Let $L(\mathbb{P},\mathbb{P})$ (respectively $L(\mathbb{P}',\mathbb{P}')$) be the vector space of continuous linear applications defined on $\mathbb{P}$ into $\mathbb{P}$ (respectively on $\mathbb{P}'$ into $\mathbb{P}'$). The transpose of any element of $L(\mathbb{P},\mathbb{P})$ is an element of $L(\mathbb{P}',\mathbb{P}')$. We shall use this process to define certain elements of $L(\mathbb{P}',\mathbb{P}')$ basically by transposing those of $L(\mathbb{P},\mathbb{P})$.*

2.0.2 Notations

We understand by linear functional any element $\mathcal{L}$ of $\mathbb{P}'$ and denote by $\langle \mathcal{L}, P\rangle$ the action of $\mathcal{L} \in \mathbb{P}'$ on $P \in \mathbb{P}$. We also denote by $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers and by $\mathbb{N}$ the set of integers. Henceforth, we will use interchangeably $\deg(\phi)$ and $\deg\ \phi$ to denote the degree of the polynomial ϕ. The operator $\mathcal{D}$ represents the usual derivative operator ($\mathcal{D} = \frac{d}{dx}$) while the Kronecker symbol $\delta_{n,j}$ is defined by

$$\delta_{n,j} = \begin{cases} 1 & \text{if} \quad n = j, \\ 0 & \text{if} \quad n \neq j \end{cases}.$$

2.1 Orthogonality and quasi-orthogonality

2.1.1 Orthogonal polynomials

Definition 2.1 *A set of polynomials $\{P_n\}_{n\in\mathbb{N}}$ is said to be an* orthogonal polynomial sequence *(OPS) associated to the linear functional $\mathcal{L} \in \mathbb{P}'$ if*

$$\begin{cases} \deg(P_n) = n, & \forall\ n \in \mathbb{N}, \\ \langle \mathcal{L}, P_n P_m\rangle = 0 & \forall\ m, n \in \mathbb{N},\ m \neq n, \\ \langle \mathcal{L}, P_n P_n\rangle \neq 0 & \forall\ n \in \mathbb{N}. \end{cases} \tag{2.5}$$

Definition 2.2 *A polynomial P is said to be monic if its leading coefficient is equal to one ($P = x^n + b_n x^{n-1} + \ldots$); and a monic polynomial family is a one in which any element is monic.*

Definition 2.3 *A linear functional $\mathcal{L} \in \mathbb{P}'$ is said to be* regular *if there exists an OPS associated to $\mathcal{L}$.*

Remark 2.2 *We state the following properties.*

1. *If $\mathcal{L}$ is a regular linear functional, then there exists a unique monic (OPS) associated to $\mathcal{L}$.*
2. *If $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}$, then $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathbb{P}$.*
3. *Any polynomial family $\{P_n\}_{n\in\mathbb{N}}$ with $\deg(P_n) = n \quad \forall n \in \mathbb{N}$ forms a basis of $\mathbb{P}$.*

Remark 2.3 *If $\{P_n\}_{n\in\mathbb{N}}$ is a set of polynomials with $\deg(P_n) = n \quad \forall n \in \mathbb{N}$ and $\mathcal{L}$ a given linear functional then the following properties are equivalent:*

$$\begin{array}{ll} i) & \langle \mathcal{L}, P_n P_m\rangle = 0 \quad \forall m, n \in \mathbb{N},\ n \neq m \text{ and } \langle \mathcal{L}, P_n P_n\rangle \neq 0 \quad \forall n \in \mathbb{N}. \\ ii) & \langle \mathcal{L}, x^m P_n\rangle = 0 \quad \forall m, n \in \mathbb{N},\ 0 \leq m < n \text{ and } \langle \mathcal{L}, x^n P_n\rangle \neq 0 \quad \forall n \in \mathbb{N}. \end{array}$$

The following theorem, proved in [Chihara, 1978], gives a necessary and sufficient condition for the regularity of a given linear functional.

Theorem 2.1 (Chihara, 1978) *Let $\mathcal{L}$ be a linear functional and M_n the moment of order n of $\mathcal{L}$ defined by $M_n = \langle \mathcal{L}, x^n \rangle$.*

A necessary and sufficient condition for the existence of an orthogonal polynomial sequence for $\mathcal{L}$ is

$$\Delta_n \neq 0 \quad \forall n \in \mathbb{N},$$

where the determinant Δ_n is defined by

$$\Delta_n = \det(M_{j+k})_{0 \le j,k \le n} = \begin{vmatrix} M_0 & M_1 & \dots & M_{n-1} & M_n \\ M_1 & M_2 & \dots & M_n & M_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{n-1} & M_n & \dots & M_{2n} & M_{2n-1} \\ M_n & M_{n+1} & \dots & M_n & M_{2n} \end{vmatrix}.$$

Definition 2.4 (Chihara, 1978) *A linear functional $\mathcal{L}$ is called positive-definite if $\langle \mathcal{L}, \pi(x) \rangle > 0$ for every polynomial π that is not identically zero and is non-negative for all real x.*

Theorem 2.2 (Chihara, 1978) *The linear functional $\mathcal{L}$ is positive-definite if and only if its moments are all real and $\Delta_n > 0 \quad \forall n \in \mathbb{N}$.*

The following theorem, taken from [Belmehdi, 1990a] gives in a more general situation some characterisations of a regular linear functional.

Theorem 2.3 (Maroni, 1987, Belmehdi, 1990a) *Let $\mathcal{L}$ be any linear functional; then the following properties are equivalent:*

i) The linear functional $\mathcal{L}$ is regular.

ii) There exists a polynomial sequence $\{P_n\}_{n \in \mathbb{N}}$ (with $\deg(P_n) = n \quad \forall n \in \mathbb{N}$) such that

$$\det(\langle \mathcal{L}, P_j\, P_k \rangle)_{0 \le j,k \le n} \neq 0 \quad \forall n \in \mathbb{N}.$$

iii) For any polynomial sequence $\{Q_n\}_{n \in \mathbb{N}}$ (with $\deg(Q_n) = n \quad \forall n \in \mathbb{N}$),

$$\det(\langle \mathcal{L}, Q_j\, Q_k \rangle)_{0 \le j,k \le n} \neq 0 \quad \forall n \in \mathbb{N}.$$

Theorem 2.4 (Szegö, 1939, Belmehdi, 1990a) *Given a regular linear functional $\mathcal{L}$, the monic orthogonal polynomials (O.P.) associated to $\mathcal{L}$ are given by*

$$P_n(x) = \frac{1}{\Delta^*_{n-1}} \begin{vmatrix} \langle \mathcal{L}, Q_0 Q_0 \rangle & \langle \mathcal{L}, Q_0 Q_1 \rangle & \dots & \langle \mathcal{L}, Q_0 Q_{n-1} \rangle & \langle \mathcal{L}, Q_0 Q_n \rangle \\ \langle \mathcal{L}, Q_1 Q_0 \rangle & \langle \mathcal{L}, Q_1 Q_1 \rangle & \dots & \langle \mathcal{L}, Q_1 Q_{n-1} \rangle & \langle \mathcal{L}, Q_1 Q_n \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathcal{L}, Q_{n-1} Q_0 \rangle & \langle \mathcal{L}, Q_{n-1} Q_1 \rangle & \dots & \langle \mathcal{L}, Q_{n-1} Q_{n-1} \rangle & \langle \mathcal{L}, Q_{n-1} Q_n \rangle \\ Q_0 & Q_1 & \dots & Q_{n-1} & Q_n \end{vmatrix} \tag{2.6}$$

where $\{Q_n\}_{n \in \mathbb{N}}$ is any monic polynomial family (with $\deg(Q_n) = n \quad \forall n \in \mathbb{N}$); and,

$$\Delta^*_n = \det(\langle \mathcal{L}, Q_j\, Q_k \rangle)_{0 \le j,k \le n},\ n \ge 0,$$

with the convention $\Delta^*_{-1} = 1$.

2.1.2 Quasi-orthogonal polynomials

The notion of quasi-orthogonal polynomials was introduced in [Riesz. 1923] and extended by Maroni and Van Rossum (for more information see [Belmehdi, 1990a]).

Definition 2.5 (Belmehdi, 1990a) *Let $\mathcal{L}$ be any linear functional and $\{P_n\}_{n\in\mathbb{N}}$ a polynomial family with* $\deg(P_n) = n \quad \forall n \in \mathbb{N}$. $\{P_n\}_{n\in\mathbb{N}}$ *is said to be quasi-orthogonal of order s with respect to $\mathcal{L}$ if*

$$\begin{cases} \langle \mathcal{L}, P_n P_m \rangle = 0, \ |n-m| > s, \\ \exists m \in \mathbb{N}, \ \langle \mathcal{L}, P_m P_{m+s} \rangle \neq 0. \end{cases} \tag{2.7}$$

$\{P_n\}_{n\in\mathbb{N}}$ *is said to be strictly quasi-orthogonal with respect to $\mathcal{L}$ if*

$$\begin{cases} \langle \mathcal{L}, P_n P_m \rangle = 0, \ |n-m| > s, \\ \langle \mathcal{L}, P_m P_{m+s} \rangle \neq 0 \quad \forall m \in \mathbb{N}. \end{cases} \tag{2.8}$$

Remark 2.4 (Belmehdi, 1990a) *1. Conditions (2.7) are equivalent to*

$$\begin{cases} \langle \mathcal{L}, x^m P_{m+t-s} \rangle = 0, \quad \forall m \in \mathbb{N} \quad \forall t \geq 1, \\ \exists m \in \mathbb{N}, \ \langle \mathcal{L}, x^m P_{m+s} \rangle \neq 0, \end{cases} \tag{2.9}$$

while (2.8) is equivalent to

$$\begin{cases} \langle \mathcal{L}, x^m P_{m+t+s} \rangle = 0, \quad \forall m \in \mathbb{N}, \quad \forall t \geq 1, \\ \langle \mathcal{L}, x^m P_{m+s} \rangle \neq 0, \quad \forall m \in \mathbb{N}. \end{cases} \tag{2.10}$$

2. *It follows from the definition 2.5 that if $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}$, then $\{P_n\}_{n\in\mathbb{N}}$ is strictly quasi-orthogonal of class $s = 0$ with respect to $\mathcal{L}$(see also [Shohat, 1937]).*

3. *Notice that quasi-orthogonality of class $s = 1$ was investigated in [Dickinson, 1961] and that the definition 2.5 was also given in [Chihara, 1957] and [Ronveaux, 1979] but without the second condition: $\exists m \in \mathbb{N}, \ \langle \mathcal{L}, P_m P_{m-s} \rangle \neq 0$.*

2.1.3 Other definitions

Definition 2.6 *Given a polynomial $f \in \mathbb{P}$ and a linear functional $\mathcal{L} \in \mathbb{P}'$, the product of f and $\mathcal{L}$, $f\mathcal{L}$, is defined as*

$$\begin{aligned} f\mathcal{L} \quad &: \quad \mathbb{P} \to \mathbb{C} \\ \langle f\mathcal{L}, P \rangle \quad &= \quad \langle \mathcal{L}, fP \rangle \qquad \forall P \in \mathbb{P}. \end{aligned}$$

Given f an element of $\mathbb{P}$, the application $\mathcal{L} \longrightarrow f\mathcal{L}$ belongs to $L(\mathbb{P}', \mathbb{P}')$ and is the transpose of the following element of $L(\mathbb{P}, \mathbb{P})$: $P \longrightarrow fP$.

Definition 2.7 *Given a polynomial $g \in \mathbb{P}$ and a linear functional $\mathcal{L} \in \mathbb{P}'$, the product of $\mathcal{L}$ and g, $\mathcal{L}g$, is a polynomial defined as*

$$\mathcal{L}g(x) = \sum_{j=0}^{n} \sum_{k=j}^{n} g_k \langle \mathcal{L}, x^{k-j} \rangle x^j. \tag{2.11}$$

where

$$g(x) = \sum_{j=0}^{n} g_j x^j.$$

Given a functional $\mathcal{L}$, the application $P \longrightarrow \mathcal{L}P$ belongs to $L(\mathbb{P}, \mathbb{P})$. By transposition, we define the product of two linear functionals $\mathcal{L}$ and $\mathcal{M}$ as:

Definition 2.8 *The product of two linear functionals $\mathcal{L}$ and $\mathcal{M}$ is defined by*

$$\langle \mathcal{L}\mathcal{M}, P\rangle = \langle \mathcal{L}, \mathcal{M}P\rangle, \ \forall P \in \mathbb{P}.$$

Definition 2.9 (Belmehdi, 1990a, Dini, 1988) *The operator θ_c is defined as*

$$\begin{aligned} \theta_c &: \mathbb{P} \to \mathbb{P} \\ (\theta_c P)(x) &= \begin{cases} \frac{P(x)-P(c)}{x-c}, & x \neq c \\ P'(c), & x = c \end{cases} \end{aligned} \tag{2.12}$$

where c is a complex number.

The application θ_c belongs to $L(\mathbb{P}, \mathbb{P})$.

Definition 2.10 *Consider the linear functional $\mathcal{L}$. From the above definition and by transposition (see remark 2.1), we define the linear functional $(x-c)^{-1}\mathcal{L}$, as*

$$\begin{aligned} (x-c)^{-1}\mathcal{L} &: \mathbb{P} \to \mathbb{C} \\ \langle (x-c)^{-1}\mathcal{L}, P\rangle &= \langle \mathcal{L}, \theta_c P\rangle \quad \forall P \in \mathbb{P}, \end{aligned} \tag{2.13}$$

where $c \in \mathbb{C}$.

Corollary 2.1 (Belmehdi, 1990a) *For any complex number c, and for any linear functional $\mathcal{L}$ the following holds:*

$$(x-c)\left[(x-c)^{-1}\mathcal{L}\right] = \mathcal{L}, \ (x-c)^{-1}\left[(x-c)\mathcal{L}\right] = \mathcal{L} - \langle \mathcal{L}, 1\rangle \delta_c, \tag{2.14}$$

where δ_c is the Dirac measure at the point c.

2.1.4 Dual basis

Definition 2.11 (Maroni, 1988) *Let $\{P_n\}_{n\in\mathbb{N}}$ be a monic polynomial family with* $\deg(P_n) = n \quad \forall n \in \mathbb{N}$*. Then $\{P_n\}_{n\in\mathbb{N}}$ forms a basis of $\mathbb{P}$ and therefore generates a unique basis of $\mathbb{P}'$, called* **dual basis** *associated to $\{P_n\}_{n\in\mathbb{N}}$, denoted by $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ and satisfying*

$$\langle \mathbf{P}_n, P_m\rangle = \delta_{n,m} \quad \forall m, n \in \mathbb{N}. \tag{2.15}$$

Any element $\mathcal{L}$ of $\mathbb{P}'$ can be expressed in this basis as (see [Roman et. al.. 1978], [Maroni, 1988]):

$$\mathcal{L} = \sum_{n\geq 0} \langle \mathcal{L}, P_n\rangle \mathbf{P}_n. \tag{2.16}$$

Proposition 2.1 *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family and $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ the dual basis associated to $\{P_n\}_{n\in\mathbb{N}}$. We have*

$$\mathbf{P}_n = \frac{P_n}{\langle \mathcal{L}, P_n P_n\rangle}\mathcal{L} \quad \forall n \in \mathbb{N}. \tag{2.17}$$

Proof: Let us write $P_n\mathcal{L} = \sum_j c_{n,j}\mathbf{P}_j$. We obtain

$$c_{n,j} = \langle \mathcal{L}, P_n P_j\rangle = \langle \mathcal{L}, P_n P_n\rangle \delta_{n,j}$$

by the fact that $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}$. Thus,

$$\mathbf{P}_n = \frac{P_n}{\langle \mathcal{L}, P_n P_n\rangle}\mathcal{L}.$$

□

2.2 Associated orthogonal polynomials

2.2.1 Three-term recurrence relation

We first give the following theorems which we shall use further to define associated orthogonal polynomials. The first is taken from [Chihara, 1978] and the second from [Favard,1935] (see also [Wintner,1929],[Stone,1932],[Sherman,1933],[Shohat,1938], [Peron,1957]).

Theorem 2.5 (Chihara,1978) *Let $\mathcal{L}$ be a regular linear functional and $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal polynomials. $\{P_n\}_{n\in\mathbb{N}}$ satisfy a three-term recurrence relation*

$$\begin{cases} P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), & n \geq 1, \\ P_0(x) = 1, P_1(x) = x - \beta_0, \end{cases} \tag{2.18}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0 \quad \forall n \in \mathbb{N}$.

Proof: Since $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}$, it forms a basis of $\mathbb{P}$ (see Remark 2.2). We therefore expand the polynomial xP_n on the basis $\{P_n\}_{n\in\mathbb{N}}$ and obtain

$$xP_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1} + \sum_{j=0}^{n-2} \eta_{n,j} P_j, \; n \geq 1, \tag{2.19}$$

where γ_n, β_n and $\eta_{n,j}$ are complex numbers.

To compute $\eta_{n,j}$, we apply the linear functional $\mathcal{L}$ to both sides of the equation obtained after multiplying the previous one by P_j, $j \leq n-2$ to get

$$\eta_{n,j}\, I_{0,j} = \langle \mathcal{L}, xP_nP_j \rangle = 0, \; j < n-1,$$

with $I_{0,n} = \langle \mathcal{L}, P_nP_n \rangle$.

Considering the fact that $I_{0,n} \neq 0 \quad \forall n \in \mathbb{N}$ (see (2.5)), it follows from the above equation that $\eta_{n,j} = 0$, $j < n-1$. Therefore equation (2.19) becomes

$$xP_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1}, \; n \geq 1.$$

Mimicking the approach used above to compute $\eta_{n,j}$, but with the previous equation, we express γ_n as

$$\gamma_n I_{0,n-1} = \langle \mathcal{L}, xP_{n-1}P_n \rangle = \langle \mathcal{L}, P_nP_n \rangle = I_{0,n} \neq 0, \; n \geq 1.$$

Hence $\gamma_n \neq 0$ $n \geq 1$.

By convention one takes $\gamma_0 = \langle \mathcal{L}, 1 \rangle$. □

The converse of the above theorem is due to Favard (1935) (see also [Chihara,1978]) .

Theorem 2.6 (Favard's Theorem) *Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ be two sequences of complex numbers and let $\{P_n\}_{n\in\mathbb{N}}$ be the family of polynomials defined by the recurrence formula*

$$\begin{cases} P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), & n \geq 1, \\ P_0(x) = 1, P_1(x) = x - \beta_0. \end{cases}$$

Then, there exists a unique linear functional $\mathcal{L}$ such that

$$\langle \mathcal{L}, 1 \rangle = \gamma_0 \text{ and } \langle \mathcal{L}, P_nP_m \rangle = 0 \quad \forall m,n \in \mathbb{N}, \; n \neq m.$$

$\mathcal{L}$ is regular and $\{P_n\}_{n\in\mathbb{N}}$ are the corresponding monic orthogonal polynomials if and only if

$$\gamma_n \neq 0 \quad \forall n \in \mathbb{N},$$

while $\mathcal{L}$ is positive-definite if and only if

$$\beta_n, \gamma_n \in \mathbb{R} \quad \forall n \in \mathbb{N}, \text{ and } \gamma_n > 0 \quad \forall n \in \mathbb{N}.$$

2.2.2 The first associated orthogonal polynomials

Definition 2.12 *Given a regular linear functional $\mathcal{L}$ and the corresponding monic orthogonal polynomials $\{P_n\}_{n\in\mathbb{N}}$, the first associated of the polynomial P_n is a monic polynomial of degree n, denoted by $P_n^{(1)}$ and defined by*

$$P_n^{(1)}(x) = \frac{1}{\gamma_0}\langle \mathcal{L}, \frac{P_{n+1}(x) - P_{n+1}(t)}{x-t}\rangle \quad \forall n \in \mathbb{N}, \tag{2.20}$$

with $\gamma_0 = \langle \mathcal{L}, 1\rangle$. It is understood that the linear functional $\mathcal{L}$ acts on the variable t.

Lemma 2.1 *The monic polynomial family $\{P_n^{(1)}\}_{n\in\mathbb{N}}$ satisfies the three-term recurrence relation*

$$\begin{cases} P_{n+1}^{(1)}(x) = (x - \beta_{n+1})P_n^{(1)}(x) - \gamma_{n+1}P_{n-1}^{(1)}(x), \; n \geq 1, \\ P_0^{(1)}(x) = 1, P_1^{(1)}(x) = x - \beta_1, \end{cases} \tag{2.21}$$

where β_n and γ_n are defined in (2.18).

Proof: Using the three-term recurrence relation satisfied by $\{P_n\}_{n\in\mathbb{N}}$ (see (2.18)) and (2.20) we obtain

$$\begin{aligned} P_{n+1}^{(1)}(x) &= \frac{1}{\gamma_0}\langle \mathcal{L}, \frac{P_{n+2}(x) - P_{n-2}(t)}{x-t}\rangle \\ &= \frac{1}{\gamma_0}\langle \mathcal{L}, \frac{(x-\beta_{n+1})P_{n-1}(x) - \gamma_{n+1}P_n(x)}{x-t} \\ &\quad - \frac{(t-\beta_{n+1})P_{n+1}(t) - \gamma_{n+1}P_n(t)}{x-t}\rangle \\ &= (x - \beta_{n-1})\frac{1}{\gamma_0}\langle \mathcal{L}, \frac{P_{n-1}(x) - P_{n+1}(t)}{x-t}\rangle \\ &\quad - \gamma_{n-1}\frac{1}{\gamma_0}\langle \mathcal{L}, \frac{P_n(x) - P_n(t)}{x-t}\rangle - \frac{1}{\gamma_0}\langle \mathcal{L}, P_{n+1}(t)\rangle \\ &= (x-\beta_{n-1})P_n^{(1)}(x) - \gamma_{n+1}P_{n-1}^{(1)}(x) \quad \forall n \in \mathbb{N}. \end{aligned}$$

□

We deduce from Theorem 2.6 and Lemma 2.1 that there exists a unique regular linear functional $\mathcal{L}^{(1)}$ with respect to which $\{P_n^{(1)}\}_{n\in\mathbb{N}}$ is orthogonal with $\langle \mathcal{L}^{(1)}, 1\rangle = \gamma_1$.

Iterating the above process, we define the general associated orthogonal polynomials.

2.2.3 The rth associated orthogonal polynomials

Definition 2.13 *Let $\mathcal{L}$ be a regular linear functional and $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal polynomials satisfying (2.18).*

The rth associated of the orthogonal polynomial P_n is a polynomial of degree n, denoted $P_n^{(r)}$ and defined by

$$\gamma_{r-1}P_n^{(r)}(x) = \langle \mathcal{L}^{(r-1)}, \frac{P_{n+1}^{(r-1)}(x) - P_{n+1}^{(r-1)}(t)}{x-t}\rangle, \quad n \geq 0, \quad r \geq 1, \tag{2.22}$$

with

$$\langle \mathcal{L}^{(r)}, 1\rangle = \gamma_r, \; r \geq 1,$$

assuming that $\gamma_0 \equiv \langle \mathcal{L}, 1\rangle$, $P_n^{(0)} \equiv P_n$, and $\mathcal{L}^{(0)} \equiv \mathcal{L}$; where $\mathcal{L}^{(r-1)}$ is the regular linear functional with respect to which $\{P_n^{(r-1)}\}_{n\in\mathbb{N}}$ is orthogonal; and it is understood that $\mathcal{L}^{(r-1)}$ acts on the variable t.

Lemma 2.2 *If $\mathcal{L}$ is a regular linear functional and $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal polynomials, then, the rth associated polynomials $\{P_n^{(r)}\}_{n\in\mathbb{N}}$ of $\{P_n\}_{n\in\mathbb{N}}$ satisfy the three-term recurrence relation*

$$\begin{cases} P_{n+1}^{(r)}(x) = (x-\beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r}P_{n-1}^{(r)}(x),\ n \geq 1, \\ P_0^{(r)}(x) = 1, P_1^{(r)}(x) = x-\beta_r, r \geq 0. \end{cases} \tag{2.23}$$

Proof: We shall prove the lemma by induction on r. For $r = 1$, (2.23) is satisfied thanks to Lemma 2.1. We suppose that (2.23) is satisfied up to a fixed integer r. Then using (2.22) we obtain

$$\begin{aligned} P_{n-1}^{(r+1)}(x) &= \frac{1}{\gamma_r}\langle \mathcal{L}^{(r)}, \frac{P_{n+2}^{(r)}(x) - P_{n+2}^{(r)}(t)}{x-t}\rangle \\ &= \frac{1}{\gamma_r}\langle \mathcal{L}^{(r)}, \frac{(x-\beta_{n+r+1})P_{n+1}^{(r)}(x) - \gamma_{n+r+1}P_n^{(r)}(x)}{x-t} \\ &\quad - \frac{(t-\beta_{n+r+1})P_{n+1}^{(r)}(t) - \gamma_{n+r+1}P_n^{(r)}(t)}{x-t}\rangle \\ &= (x-\beta_{n+r+1})\frac{1}{\gamma_r}\langle \mathcal{L}^{(r)}, \frac{P_{n+1}^{(r)}(x) - P_{n+1}^{(r)}(t)}{x-t}\rangle \\ &\quad -\gamma_{n+r+1}\frac{1}{\gamma_r}\langle \mathcal{L}^{(r)}, \frac{P_n^{(r)}(x) - P_n^{(r)}(t)}{x-t}\rangle - \frac{1}{\gamma_r}\langle \mathcal{L}^{(r)}, P_{n+1}^{(r)}(t)\rangle \\ &= (x-\beta_{n+r+1})P_n^{(r+1)}(x) - \gamma_{n+r+1}P_{n-1}^{(r+1)}(x) \quad \forall n \in \mathbb{N}. \end{aligned}$$

Thus $\{P_n^{(r)}\}_{n\in\mathbb{N}}$ satisfies (2.23) $\forall r \in \mathbb{N}$. □

As consequence of the previous lemma, we claim the following known result (see [Magnus, 1984], [Belmehdi, 1990b]).

Lemma 2.3 (**Magnus, 1984, Belmehdi, 1990b**) *The associated polynomials $P_n^{(r)}$ satisfy*

$$P_n^{(r)}P_n^{(r+1)} - P_{n+1}^{(r)}P_{n-1}^{(r+1)} = \prod_{k=1}^{n}\gamma_{r+k} \equiv \pi_{n,k} \quad \forall n \in \mathbb{N}, \quad \forall r \in \mathbb{N}. \tag{2.24}$$

Proof: In the first step we write (2.23) for $P_{n+1}^{(r)}$ and $P_n^{(r+1)}$

$$P_{n+1}^{(r)}(x) = (x-\beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r}P_{n-1}^{(r)}(x), \tag{2.25}$$

$$P_n^{(r+1)}(x) = (x-\beta_{n+r})P_{n-1}^{(r+1)}(x) - \gamma_{n+r}P_{n-2}^{(r+1)}(x). \tag{2.26}$$

In the second step we subtract the two equations obtained after multiplying (2.25) and (2.26) by $P_{n-1}^{(r+1)}$ and $P_n^{(r)}$, respectively,

$$P_n^{(r)}P_n^{(r+1)} - P_{n+1}^{(r)}P_{n-1}^{(r+1)} = \gamma_{n+r}(P_{n-1}^{(r)}P_{n-1}^{(r+1)} - P_n^{(r)}P_{n-2}^{(r+1)}).$$

Then relation (2.24) results by iterating the latter. □

2.3 Operators $\mathcal{D}$, $\mathcal{T}_\omega$, D_ω, $\mathcal{G}_q$ and $\mathcal{D}_q$

2.3.1 Operator $\mathcal{D}$

The application $P \longrightarrow \mathcal{D}P$ belongs to $L(\mathbb{P},\mathbb{P})$. By transposition, we define derivative of the linear functional as:

Definition 2.14 *Let $\mathcal{L}$ be a given linear functional, we define the $\mathcal{D}$-derivative of $\mathcal{L}$, $\mathcal{DL}$, as*

$$\begin{aligned} \mathcal{DL} &: \mathbb{P} \to \mathbb{C} \\ \langle \mathcal{DL}, P \rangle &= -\langle \mathcal{L}, \mathcal{D}P \rangle \qquad \forall P \in \mathbb{P}. \end{aligned} \tag{2.27}$$

Proposition 2.2 *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family and $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ the dual basis associated to $\{P_n\}_{n\in\mathbb{N}}$. If $\{\mathbf{Q}_{n,1}\}_{n\in\mathbb{N}}$ is the dual basis associated to the monic family $\{Q_{n,1}\}_{n\in\mathbb{N}}$ defined by*

$$Q_{n,1} = \frac{\mathcal{D}P_{n-1}}{n+1},$$

then we have

$$\mathcal{D}\mathbf{Q}_{n,1} = -(n+1)\mathbf{P}_{n+1}.$$

Proof: This follows from Proposition 3.5. □

Definition 2.15 *The regular linear functional $\mathcal{L}$ and the corresponding monic orthogonal polynomials are said to be $\mathcal{D}$-semi-classical (or semi-classical continuous) if there exist two polynomials ψ of degree at least one, and ϕ such that*

$$\mathcal{D}(\phi\mathcal{L}) = \psi\mathcal{L}. \tag{2.28}$$

Moreover, if ϕ is a polynomial of degree at most two and ψ a first-degree polynomial, then, the linear functional and the corresponding orthogonal polynomials are called $\mathcal{D}$-classical (classical continuous). For more details about $\mathcal{D}$-semi-classical orthogonal polynomials can be found in [Maroni, 1985, 1987], [Marcellàn, 1988], [Belmehdi, 1990a] and references therein.

2.3.2 Class of the $\mathcal{D}$-semi-classical linear functional

Let $\mathcal{L}$ be a $\mathcal{D}$-semi-classical linear functional satisfying

$$\mathcal{D}(\phi\mathcal{L}) = \psi\mathcal{L}, \tag{2.29}$$

where ϕ is any non-zero polynomial and ψ a polynomial of degree at least one. $\mathcal{L}$ satisfies $\mathcal{D}(f\phi\mathcal{L}) = (\phi\mathcal{D}f + \psi f)\mathcal{L}$, for any polynomial f.

Definition 2.16 *We define the class $\mathrm{cl}(\mathcal{L})$ of the $\mathcal{D}$-semi-classical linear functional $\mathcal{L}$ as*

$$\mathrm{cl}(\mathcal{L}) = \min_{(f,g)\in\mathcal{R}_1} \{\max(\deg(f) - 2, \deg(g) - 1)\},$$

where

$$\mathcal{R}_1 = \{(f,g) \in \mathbb{P}^2 \,/\, \deg(g) \geq 1 \text{ and } \mathcal{D}(f\mathcal{L}) = g\mathcal{L}\}.$$

Proposition 2.3 (Belmehdi, 1990a) *If $\mathcal{L}$ is a $\mathcal{D}$-semi-classical linear functional satisfying (2.29), then $\mathcal{L}$ is of class $s = \max(\deg(\phi) - 2, \deg(\psi) - 1)$ if and only if*

$$\prod_{c\in Z_\phi} (|r_c| + |\langle \mathcal{L}, \psi_c \rangle|) \neq 0, \tag{2.30}$$

where Z_ϕ is the set of zeros of ϕ. The complex number r_c and the polynomials ϕ_c, ψ_c are defined by

$$(x-c)\phi_c = \phi, \;\; \psi - \phi_c = (x-c)\psi_c + r_c. \tag{2.31}$$

Proof: For a proof see Proposition 3.4. □

Remark 2.5 *It follows from the definition of the class of the linear functional that the $\mathcal{D}$-classical linear functional is a $\mathcal{D}$-semi-classical linear functional of class $s = 0$.*

Lemma 2.4 *Let $\mathcal{L}$ be a regular linear functional.*

i) If there exist two polynomials $\psi \neq 0$, and ϕ such that

$$\mathcal{D}(\phi\mathcal{L}) = \psi\mathcal{L} \tag{2.32}$$

then ϕ is a non-zero polynomial.

ii) Conversely, if there exist two polynomials $\phi \neq 0$ and ψ such that (2.32) holds, then ψ is of degree at least one.

Proof: For a proof see Lemma 3.1. □

2.3.3 Characterisation of $\mathcal{D}$-classical orthogonal polynomials

The following theorem which is a corollary of the theorem 3.1 gives some characterisations of classical continuous orthogonal polynomials (see [Chihara, 1978],[Nikiforov et al., 1983], [Al-salam, 1990], [Marcellán et al., 1994], ...).

Theorem 2.7 *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family and $Q_{n,m}$ the monic polynomial of degree n defined by*

$$B_{n,m}\, Q_{n,m} = \mathcal{D}^m P_{n+m},$$

with

$$B_{n,m} = \frac{(n+m)!}{n!}, \quad Q_{n,0} \equiv P_n.$$

The following properties are equivalent:

i) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that

$$\mathcal{D}(\phi\mathcal{L}) = \psi\mathcal{L}.$$

ii) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m,

$$\begin{aligned} &\mathcal{D}(\phi\mathcal{L}_m) = \psi_m\mathcal{L}, \\ &\langle \mathcal{L}_m, Q_{j,m}Q_{n,m}\rangle = k_n\delta_{j,n}, \quad \forall j,n \in \mathbb{N}, \ (k_n \neq 0\ \forall n \in \mathbb{N}), \end{aligned}$$

with the linear functional $\mathcal{L}_m$ and the polynomial ψ_m defined, recursively, by

$$\begin{aligned} \psi_{m+1} &= \mathcal{D}\phi + \psi_m, \quad \psi_0 \equiv \psi, \\ \mathcal{L}_{m+1} &= \phi\,\mathcal{L}_m, \quad \mathcal{L}_0 \equiv \mathcal{L} \end{aligned}$$

and given explicitly by

$$\psi_m(x) = m\phi'(x) + \psi(x), \tag{2.33}$$

$$\mathcal{L}_m = \phi^m\,\mathcal{L}. \tag{2.34}$$

iii) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m, the following second-order difference equation holds:

$$\phi\,\mathcal{D}^2 Q_{n,m} + \psi_m\,\mathcal{D}Q_{n,m} + \lambda^*_{n,m}\,Q_{n,m} = 0 \quad \forall n \in \mathbb{N},$$

with the polynomial ψ_m given by (2.33) and the constant $\lambda^*_{n,m}$ given by

$$\lambda^*_{n,m} = -n\,\{\psi'_m + (n-1)\frac{\phi''}{2}\} = -n\,\{\psi' + (2m+n-1)\frac{\phi''}{2}\}. \tag{2.35}$$

iv) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that, for any integer m, the following relation holds:

$$n\mathcal{D}\left[Q_{n-1,m+1}\mathcal{L}_{m+1}\right] = -\lambda^*_{n,m} Q_{n,m}\mathcal{L}_m \quad \forall n \in \mathbb{N}, \tag{2.36}$$

with the polynomial ϕ_m, the linear functional $\mathcal{L}_m$ and the constant $\lambda^*_{n,m}$ given, respectively, by (2.33), (2.34) and (2.35).

v) There exist a polynomial ϕ of degree at most two and three constants $c_{n,n+1}$, $c_{n,n}$, $c_{n,n-1}$ with $c_{n,n-1} \neq 0$ such that

$$\phi\mathcal{D}P_n = c_{n,n+1}P_{n+1} + c_{n,n}P_n + c_{n,n-1}P_{n-1}, \quad n \geq 1.$$

vi) For any non-zero integer m, there exist a sequence of complex numbers $\{u_{n,m}\}_{n\in\mathbb{N}}$ such that

$$Q_{n,m-1} = Q_{n,m} - u_{n-1,m}Q_{n-1,m} + v_{n-2,m}Q_{n-2,m}. \quad \forall n \in \mathbb{N} - \{0,1\}.$$

Remark 2.6 *Let us comment on the above properties.*

For all $m \in \mathbb{N}$, the derivative of order m, $\{Q_{n,m}\}_{n\in\mathbb{N}}$, of the family $\{P_{n+m}\}_{n\in\mathbb{N}}$ is classical and orthogonal with respect to the classical linear functional $\mathcal{L}_m$.

The functional version of the generalised Rodrigues formula [Nikiforov et al, 1983], [Belmehdi, 1990c], given below, is obtained by iterating the relation (2.36):

$$Q_{n,m}\phi^m\mathcal{L} = \prod_{j=0}^{n-1}\frac{1}{\psi' + (j+2m+n-1)\frac{\phi''}{2}}\mathcal{D}^n(\phi^{n+m}\mathcal{L}).$$

2.3.4 Operators $\mathcal{T}_\omega$ and D_ω

Definition 2.17 *The arithmetic shift operator $\mathcal{T}_\omega$ is defined by*

$$\begin{aligned} \mathcal{T}_\omega : \mathbb{P} &\longrightarrow \mathbb{P} \\ P &\longrightarrow \mathcal{T}_\omega P,\ \mathcal{T}_\omega P(x) = P(x+\omega),\ \omega \in \mathbb{R}. \end{aligned} \tag{2.37}$$

We denote $\mathcal{T}_1 = \mathcal{T}$.

Definition 2.18 *The difference operator D_ω is defined by*

$$\begin{aligned} D_\omega : \mathbb{P} &\longrightarrow \mathbb{P} \\ P &\longrightarrow D_\omega P,\ D_\omega P(x) = \frac{P(x+\omega) - P(x)}{\omega},\ \omega \in \mathbb{R},\ \omega \neq 0. \end{aligned} \tag{2.38}$$

We denote $D_1 = \Delta$ and $D_{-1} = \nabla$. Δ and ∇ denote the forward and the backward difference operators, respectively.

The applications $P \longrightarrow \mathcal{T}_\omega P$ and $P \longrightarrow D_\omega P$ belong to $L(\mathbb{P}, \mathbb{P})$. We, therefore, use their transposes to define the action of the operators $\mathcal{T}_\omega$ and D_ω on the linear functionals.

Definition 2.19 *The action of the arithmetic shift operator $\mathcal{T}_\omega$ on the functional $\mathcal{L}$ is defined by*

$$\langle \mathcal{T}_\omega\mathcal{L}, P\rangle = \langle \mathcal{L}, \mathcal{T}_{-\omega}P\rangle \quad \forall P \in \mathbb{P}. \tag{2.39}$$

Definition 2.20 *Given a linear functional $\mathcal{L}$, we define the D_ω derivative of $\mathcal{L}$, $D_\omega\mathcal{L}$, as*

$$\begin{aligned} D_\omega\mathcal{L} &: \mathbb{P} \to \mathbb{C} \\ \langle D_\omega\mathcal{L}, P\rangle &= -\langle \mathcal{L}, D_{-\omega}P\rangle \quad \forall P \in \mathbb{P}. \end{aligned} \tag{2.40}$$

Definition 2.21 *The regular linear functional $\mathcal{L}$ and the corresponding monic orthogonal polynomials are said to be D_ω-semi-classical if there exist two polynomials ψ of degree at least one, and ϕ such that*

$$D_\omega(\phi\mathcal{L}) = \psi\mathcal{L}. \tag{2.41}$$

Moreover, if ϕ is a polynomial of degree at most two and ψ a first-degree polynomial, then, the linear functional and the corresponding orthogonal polynomials are called classical discrete.

Using the above definitions, we obtain the following properties:

Proposition 2.4 (Salto, 1995)

$$\begin{aligned}
\mathcal{T}_{-\omega}D_\omega &= D_\omega\mathcal{T}_{-\omega} = D_{-\omega}, \quad D_\omega D_{-\omega} = D_{-\omega}D_\omega, & (2.42)\\
\mathcal{T}_\omega(fg) &= \mathcal{T}_\omega f\,\mathcal{T}_\omega g, \quad \mathcal{T}_\omega(f\mathcal{L}) = \mathcal{T}_\omega f\,\mathcal{T}_\omega\mathcal{L}, & (2.43)\\
D_\omega(fg) &= f\,D_\omega g + \mathcal{T}_\omega g D_\omega f = \mathcal{T}_\omega f D_\omega g + g\,D_\omega f, & (2.44)\\
D_\omega(f\mathcal{L}) &= f\,D_\omega\mathcal{L} + D_\omega f\,\mathcal{T}_\omega\mathcal{L} = \mathcal{T}_\omega f\,D_\omega\mathcal{L} - D_\omega f\,\mathcal{L}. & (2.45)\\
\omega D_\omega\mathcal{L} &= (\mathcal{T}_\omega - \mathcal{I}_d)\mathcal{L}. & (2.46)\\
D_\omega(fg\mathcal{L}) &= \mathcal{T}_\omega f\,D_\omega(g\mathcal{L}) - \mathcal{T}_\omega f\,D_\omega g\,\mathcal{L} + D_\omega(fg)\,\mathcal{L}, \ \forall f,g \in \mathbb{P}, \ \forall\mathcal{L} \in \mathbb{P}'. & (2.47)
\end{aligned}$$

Notice that equation (2.42) means that:

$$\begin{aligned}
\mathcal{T}_{-\omega}D_\omega\Phi &= D_\omega\mathcal{T}_{-\omega}\Phi = D_{-\omega}\Phi, \quad D_\omega D_{-\omega}\Phi = D_{-\omega}D_\omega\Phi, \ \forall\Phi \in \mathbb{P},\\
\mathcal{T}_{-\omega}D_\omega\bar{\Phi} &= D_\omega\mathcal{T}_{-\omega}\bar{\Phi} = D_{-\omega}\bar{\Phi}, \quad D_\omega D_{-\omega}\bar{\Phi} = D_{-\omega}D_\omega\bar{\Phi}, \ \forall\bar{\Phi} \in \mathbb{P}'.
\end{aligned}$$

Proof: This follows directly from Proposition 3.1. □

The following lemma proves that the arithmetic shift of the associated orthogonal polynomials (resp. regular linear functional) are the associated shifted orthogonal polynomials and shifted regular linear functional, respectively.

Lemma 2.5 *Given a regular linear functional $\mathcal{L}$ and $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal polynomials, the rth associated $P_n^{(r)}$ of P_n and $\mathcal{L}^{(r)}$ of $\mathcal{L}$ obey*

$$(\mathcal{T}_\omega P_n)^{(r)} = \mathcal{T}_\omega P_n^{(r)}, \quad (\mathcal{T}_\omega\mathcal{L})^{(r)} = \mathcal{T}_\omega\mathcal{L}^{(r)} \quad \forall r, n \in \mathbb{N}. \tag{2.48}$$

Proof: We shall give the proof by induction on r. It follows from Lemma 3.2 that $\{\mathcal{T}_\omega P_n\}_{n\in\mathbb{N}}$ are the monic orthogonal polynomials associated to $\mathcal{T}_\omega\mathcal{L}$.

For $r = 0$ $(\mathcal{T}_\omega P_n)^{(0)} = \mathcal{T}_\omega P_n^{(0)} = \mathcal{T}_\omega P_n$ and $(\mathcal{T}_\omega\mathcal{L})^{(0)} = \mathcal{T}_\omega\mathcal{L}^{(0)} = \mathcal{T}_\omega\mathcal{L}$.

Suppose that (2.48) is satisfied up to a fixed r. Then using (2.22) and the fact that $\mathcal{L}$ acts on the variable t, we get

$$\begin{aligned}
(\mathcal{T}_\omega P_n)^{(r+1)}(x) &= \frac{1}{\gamma_r}\langle(\mathcal{T}_\omega\mathcal{L})^{(r)}, \frac{(\mathcal{T}_\omega P_{n+1})^{(r)}(x) - (\mathcal{T}_\omega P_{n+1})^{(r)}(t)}{x - t}\rangle\\
&= \frac{1}{\gamma_r}\langle\mathcal{T}_\omega\mathcal{L}^{(r)}, \frac{\mathcal{T}_\omega P_{n+1}^{(r)}(x) - \mathcal{T}_\omega P_{n+1}^{(r)}(t)}{x - t}\rangle\\
&= \frac{1}{\gamma_r}\langle\mathcal{T}_\omega\mathcal{L}^{(r)}, \mathcal{T}_\omega\frac{\mathcal{T}_\omega P_{n+1}^{(r)}(x) - P_{n+1}^{(r)}(t)}{x - (t - \omega)}\rangle\\
&= \frac{1}{\gamma_r}\langle\mathcal{L}^{(r)}, \frac{P_{n+1}^{(r)}(x + \omega) - P_{n+1}^{(r)}(t)}{x + \omega - t}\rangle\\
&= \mathcal{T}_\omega P_n^{(r+1)}(x).
\end{aligned}$$

Then,

$$(\mathcal{T}_\omega P_n)^{(r)} = \mathcal{T}_\omega P_n^{(r)}, \quad \forall n \in \mathbb{N}, \ \forall r \in \mathbb{N}. \tag{2.49}$$

We use remark 2.3 to get

$$\langle (\mathcal{T}_\omega \mathcal{L})^{(r+1)}, (\mathcal{T}_\omega P_n)^{(r+1)} \rangle = 0 = \langle \mathcal{T}_\omega \mathcal{L}^{(r+1)}, \mathcal{T}_\omega P_n^{(r+1)} \rangle, \ n \geq 1, \ r \geq 0. \tag{2.50}$$

For $n = 0$ (see definition 2.13),

$$\langle (\mathcal{T}_\omega \mathcal{L})^{(r+1)}, 1 \rangle = \gamma_{r+1} = \langle \mathcal{T}_\omega \mathcal{L}^{(r+1)}, 1 \rangle. \tag{2.51}$$

We combine (2.49), (2.50) and (2.51) to get

$$\langle (\mathcal{T}_\omega \mathcal{L})^{(r+1)}, \mathcal{T}_\omega P_n^{(r+1)} \rangle = \langle \mathcal{T}_\omega \mathcal{L}^{(r+1)}, \mathcal{T}_\omega P_n^{(r+1)} \rangle \quad \forall n \geq 0.$$

Hence $(\mathcal{T}_\omega \mathcal{L})^{(r+1)} = \mathcal{T}_\omega \mathcal{L}^{(r+1)}$, thanks to the fact that $\{\mathcal{T}_\omega P_n^{(r+1)}\}_{n\in\mathbb{N}}$, which is orthogonal with respect to $\mathcal{T}_\omega \mathcal{L}^{(r+1)}$, forms a basis of $\mathbb{P}$. □

2.3.5 Class of the D_ω-semi-classical linear functional

Let $\mathcal{L}$ be a D_ω-semi-classical linear functional satisfying

$$D_\omega(\phi\mathcal{L}) = \psi\mathcal{L}, \tag{2.52}$$

where ϕ is any non-zero polynomial and ψ a polynomial of degree at least one. $\mathcal{L}$ satisfies $D_\omega(f\phi\mathcal{L}) = (\phi D_\omega f + \psi\mathcal{T}_\omega f)\mathcal{L}$, for any polynomial f.

Definition 2.22 *We define the class* $\mathrm{cl}(\mathcal{L})$ *of the* D_ω*-semi-classical linear functional* $\mathcal{L}$ *as*

$$\mathrm{cl}(\mathcal{L}) = \min_{(f,g)\in\mathcal{R}_2} \{\max(\deg(f) - 2. \deg(g) - 1)\},$$

where

$$\mathcal{R}_2 = \{(f,g) \in \mathbb{P}^2 \,/\, \deg(g) \geq 1 \text{ and } D_\omega(f\mathcal{L}) = g\mathcal{L}\}.$$

The following proposition give a characterisation of the class of a $\mathcal{D}_q$-semi-classical linear functional.

Proposition 2.5 (Salto, 1995) *If* $\mathcal{L}$ *is a* D_ω*-semi-classical linear functional satisfying (2.52), then* $\mathcal{L}$ *is of class* $s = \max(\deg(\phi) - 2, \deg(\psi) - 1)$ *if and only if*

$$\prod_{c\in Z_\phi} (|r_{c,\omega}| + |\langle \mathcal{L}, \psi_{c,\omega}\rangle|) \neq 0. \tag{2.53}$$

where Z_ϕ *is the set of zeros of* ϕ. *The complex number* $r_{c,\omega}$ *and the polynomials* ϕ_c, $\psi_{c,\omega}$ *are defined by*

$$(x - c)\phi_c = \phi, \ \psi - \phi_c = (x + \omega - c)\psi_{c,\omega} + r_{c,\omega}. \tag{2.54}$$

Proof: This follows from Proposition 3.4. □

More details about the class of a D_ω-semi-classical linear functional can be found in [Salto, 1996] and [Godoy et al., 1997b].

Remark 2.7 *From the definition of the class of the semi-classical linear functional. we deduce that the* D_ω*-classical linear functional is a* D_ω*-semi-classical linear functional of class* $s = 0$.

Lemma 2.6 *The linear functional* $\mathcal{L}$ *is regular if and only if* $\mathcal{T}_\omega\mathcal{L}$ *is regular.*

Proof: For a proof see Lemma 3.2. □

Lemma 2.7 *Let* $\mathcal{L}$ *be a regular linear functional.*

i) If there exist two polynomials $\psi \neq 0$ and ϕ such that

$$D_\omega(\phi\mathcal{L}) = \psi\mathcal{L}, \tag{2.55}$$

then ϕ is a non-zero polynomial.

ii) Conversely, if there exist two polynomials $\phi \neq 0$ and ψ such that (2.55) holds, then ψ is of degree at least one.

Proof: This follows from Lemma 3.1. □

Proposition 2.6 (Salto, 1995) *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family and $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ the dual basis associated to $\{P_n\}_{n\in\mathbb{N}}$. If $\{\tilde{\mathbf{Q}}_{n,1}\}_{n\in\mathbb{N}}$ is the dual basis associated to the monic family $\{\tilde{Q}_{n,1}\}_{n\in\mathbb{N}}$ defined by*

$$\tilde{Q}_{n,1} = \frac{D_\omega P_{n-1}}{n+1},$$

then we have

$$D_{-\omega}\tilde{\mathbf{Q}}_{n,1} = -(n+1)\mathbf{P}_{n+1}.$$

Proof: For a proof see Proposition 3.5. □

2.3.6 Characterisation of Δ-classical orthogonal polynomials

The following theorem which is a corollary of Theorem 3.1 gives a characterisation of the orthogonal polynomials of a discrete variable [Al-salam, 1990], [Nikiforov et al., 1991], [Garcia et al., 1995], [Salto, 1995].

Theorem 2.8 *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family and $Q_{n,m}$ the monic polynomial of degree n defined by*

$$B_{n,m}\, Q_{n,m} = \Delta^m P_{n+m}, \tag{2.56}$$

with

$$B_{n,m} = \frac{(n+m)!}{n!}, \quad Q_{n,0} \equiv P_n. \tag{2.57}$$

The following properties are equivalent:

i) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that

$$\Delta(\phi\mathcal{L}) = \psi\mathcal{L}.$$

ii) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m,

$$\Delta(\phi\mathcal{L}_m) = \psi_m\mathcal{L},$$
$$\langle\mathcal{L}_m, Q_{j,m}Q_{n,m}\rangle = k_n\delta_{j,n}, \quad (k_n \neq 0\ \forall n \in \mathbb{N}),$$

with the linear functional $\mathcal{L}_m$ and the polynomial ψ_m defined, recursively, by

$$\psi_{m-1} = \Delta\phi + \mathcal{T}\psi_m, \quad \psi_0 \equiv \psi,$$
$$\mathcal{L}_{m-1} = \mathcal{T}(\phi\,\mathcal{L}_m), \quad \mathcal{L}_0 \equiv \mathcal{L}$$

and given explicitly by

$$\psi_m(x) = \phi(x+m) - \phi(x) + \psi(x+m), \tag{2.58}$$

$$\mathcal{L}_m = \prod_{j=1}^{m} \phi(x+j)\,\mathcal{T}^m\mathcal{L}. \tag{2.59}$$

iii) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m, the following second-order difference equation holds:

$$\phi\, \Delta\nabla\, Q_{n,m} + \psi_m\, \Delta\, Q_{n,m} + \lambda^*_{n,m}\, Q_{n,m} = 0 \quad \forall n \in \mathbb{N},$$

with the polynomial ψ_m given by (2.58) and the constant $\lambda_{n,m}$ given by

$$\lambda^*_{n,m} = -n\,\{\psi'_m + (n-1)\frac{\phi''}{2}\} = -n\,\{\psi' + (2m+n-1)\frac{\phi''}{2}\}. \tag{2.60}$$

iv) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that, for any integer m, the following relation holds:

$$n\nabla\,[Q_{n-1,m+1}\mathcal{L}_{m-1}] = -\,\lambda^*_{n,m} Q_{n,m}\mathcal{L}_m \ \ \forall n \in \mathbb{N}, \tag{2.61}$$

with the polynomial ψ_m, the linear functional $\mathcal{L}_m$ and the constant $\lambda^*_{n,m}$ given, respectively, by (2.58), (2.59) and (2.60).

v) There exist a polynomial ϕ of degree at most two and three constants $c_{n,n+1}$, $c_{n,n}$, $c_{n,n-1}$ with $c_{n,n-1} \neq 0$ such that

$$\phi\nabla P_n = c_{n,n+1}P_{n-1} + c_{n,n}P_n + c_{n,n-1}P_{n-1},\ n > 1.$$

vi) For any non-zero integer m, there exist sequence of complex numbers $\{u_{n,m}\}_{n\in\mathbb{N}}$ such that

$$Q_{n,m-1} = Q_{n,m} + u_{n-1,m}Q_{n-1,m} + v_{n-2,m}Q_{n-2,m}, \qquad \forall n \in \mathbb{N} - \{0,1\}. \tag{2.62}$$

Remark 2.8 *1. For all $m \in \mathbb{N}$, the Δ-derivative of order m, $\{Q_{n,m}\}_{n\in\mathbb{N}}$, of the family $\{P_{n+m}\}_{n\in\mathbb{N}}$ is classical discrete and orthogonal with respect to the classical linear functional $\mathcal{L}_m$.*

2. The analogue of the functional version of the generalised Rodrigues formula [Nikiforov et al., 1991], [Salto, 1995] given below, is obtained by iterating the relation (2.61)

$$Q_{n,m}\prod_{j=1}^{m}\phi(x+j)\mathcal{T}^m\mathcal{L} = \prod_{j=0}^{n-1}\frac{1}{\psi' + (2m+j+n-1)\frac{\phi''}{2}}\nabla^n(\prod_{j=1}^{n+m}\phi(x+j)\mathcal{T}^{n+m}\mathcal{L}).$$

3. If the linear functional $\mathcal{L}$ is represented by the positive weight ρ on the interval $I =]a,b[$,

$$\langle\mathcal{L}, P\rangle = \sum_{x\in I}\rho(x)P(x) \quad \forall P \in \mathbb{P}, \tag{2.63}$$

with $x^n\,\phi(x)\,\rho(x)|_a^b = 0 \quad \forall n \in \mathbb{N}$, then we have the equivalence

$$\Delta(\phi\mathcal{L}) = \psi\mathcal{L} \Longleftrightarrow \Delta\,(\phi\rho) = \psi\rho. \tag{2.64}$$

2.3.7 Operators $\mathcal{G}_q$ and $\mathcal{D}_q$

Definition 2.23 *The geometric shift operator $\mathcal{G}_q$ is defined by*

$$\begin{aligned}\mathcal{G}_q : \mathbb{P} &\longrightarrow \mathbb{P}\\ P &\longrightarrow \mathcal{G}_qP,\ \mathcal{G}_qP(x) = P(qx),\ q \neq 0.\end{aligned} \tag{2.65}$$

Definition 2.24 (Hahn, 1948) *The q-difference operator $\mathcal{D}_q$, called* Hahn operator *is defined by*

$$\begin{aligned}\mathcal{D}_q : \mathbb{P} &\longrightarrow \mathbb{P}\\ P &\longrightarrow \mathcal{D}_qP,\ \mathcal{D}_qP(x) = \frac{P(qx) - P(x)}{(q-1)x},\ q \in \mathbb{C},\ q \neq 0,\ q \neq 1.\end{aligned} \tag{2.66}$$

The applications $P \longrightarrow \mathcal{G}_qP$ and $P \longrightarrow \mathcal{D}_qP$ belong to $L(\mathbb{P},\mathbb{P})$. We, therefore, use their transposes to define the action of the operators $\mathcal{G}_q$ and $\mathcal{D}_q$ on the linear functionals.

Definition 2.25 *The action of the geometric shift operator $\mathcal{G}_q$ on the functional $\mathcal{L}$ is defined by*

$$\langle \mathcal{G}_q \mathcal{L}, P \rangle = \frac{1}{q} \langle \mathcal{L}, \mathcal{G}_{\frac{1}{q}} P \rangle \quad \forall P \in \mathbb{P}. \tag{2.67}$$

Definition 2.26 *Given a linear functional $\mathcal{L}$, we define the $\mathcal{D}_q$-derivative of $\mathcal{L}$, $\mathcal{D}_q \mathcal{L}$, as*

$$\begin{aligned} \mathcal{D}_q \mathcal{L} &: \mathbb{P} \to \mathbb{C} \\ \langle \mathcal{D}_q \mathcal{L}, P \rangle &= -\frac{1}{q} \langle \mathcal{L}, \mathcal{D}_{\frac{1}{q}} P \rangle \quad \forall P \in \mathbb{P}. \end{aligned} \tag{2.68}$$

Definition 2.27 *Given a real number $q \neq 1$ and an integer n, we define the real number $[n]_q$ by*

$$[n]_q = \frac{q^n - 1}{q - 1}, \; q \neq 1, \; n \geq 0. \tag{2.69}$$

Definition 2.28 *The regular linear functional $\mathcal{L}$ and the corresponding monic orthogonal polynomials are said to be $\mathcal{D}_q$-semi-classical if there exist two polynomials ψ of degree at least one, and ϕ such that*

$$\mathcal{D}_q(\phi \mathcal{L}) = \psi \mathcal{L}. \tag{2.70}$$

Moreover, if ϕ is a polynomial of degree at most two and ψ a first-degree polynomial, then the linear functional and the corresponding orthogonal polynomials are called $\mathcal{D}_q$-classical or q-classical.

From the above definitions, we state the following corollary of Proposition 3.1 [Medem, 1996].

Proposition 2.7 (Medem, 1996)

$$\begin{aligned}
\mathcal{G}_q \mathcal{D}_{\frac{1}{q}} &= \mathcal{D}_q, \quad \mathcal{D}_q \mathcal{G}_t = t \mathcal{G}_t \mathcal{D}_q, \quad \mathcal{D}_{\frac{1}{q}} \mathcal{D}_q = q \mathcal{D}_q \mathcal{D}_{\frac{1}{q}}, & (2.71)\\
\mathcal{G}_q(f g) &= \mathcal{G}_q f \mathcal{G}_q g, \quad \mathcal{G}_q(f \mathcal{L}) = \mathcal{G}_q f \mathcal{G}_q \mathcal{L}, & (2.72)\\
\mathcal{D}_q(f g) &= f \mathcal{D}_q g + \mathcal{G}_q g \mathcal{D}_q f = \mathcal{G}_q f \mathcal{D}_q g + g \mathcal{D}_q f, & (2.73)\\
\mathcal{D}_q(f \mathcal{L}) &= f \mathcal{D}_q \mathcal{L} + \mathcal{D}_q f \mathcal{G}_q \mathcal{L} = \mathcal{G}_q f \mathcal{D}_q \mathcal{L} + \mathcal{D}_q f \mathcal{L}, & (2.74)\\
(q-1) \mathcal{D}_q \mathcal{L} &= x^{-1} (\mathcal{G}_q \mathcal{L} - \mathcal{L}), & (2.75)\\
\mathcal{D}_q(f g \mathcal{L}) &= \mathcal{G}_q f \mathcal{D}_q (g \mathcal{L}) - \mathcal{G}_q f \mathcal{D}_q g \mathcal{L} + \mathcal{D}_q(f g) \mathcal{L}, \; \forall f, g \in \mathbb{P}, \; \forall \mathcal{L} \in \mathbb{P}'. & (2.76)
\end{aligned}$$

Notice that the identities defined in (2.71) are valid when the operators $\mathcal{G}_q$ and $\mathcal{D}_q$ act on $\mathbb{P}$ and also on $\mathbb{P}'$.

2.3.8 Class of the $\mathcal{D}_q$-semi-classical linear functional

Let $\mathcal{L}$ be a $\mathcal{D}_q$-semi-classical linear functional satisfying

$$\mathcal{D}_q(\phi \mathcal{L}) = \psi \mathcal{L}, \tag{2.77}$$

where ϕ is any non-zero polynomial and ψ a polynomial of degree at least one. $\mathcal{L}$ satisfies $\mathcal{D}_q(f \phi \mathcal{L}) = (\phi \mathcal{D}_q f + \psi \mathcal{G}_q f) \mathcal{L}$, for any polynomial f. We, therefore, define the class of the $\mathcal{D}_q$-semi-classical linear functional $\mathcal{L}$ as:

Definition 2.29 *We define the class $\mathrm{cl}(\mathcal{L})$ of the $\mathcal{D}_q$-semi-classical linear functional $\mathcal{L}$ as*

$$\mathrm{cl}(\mathcal{L}) = \min_{(f, g) \in \mathcal{R}_3} \{\max(\deg(f) - 2, \deg(g) - 1)\}.$$

where

$$\mathcal{R}_3 = \{(f, g) \in \mathbb{P}^2 \, / \, \deg(g) \geq 1, \; \mathcal{D}_q(f \mathcal{L}) = g \mathcal{L}\}.$$

Proposition 2.8 (Medem, 1996) *If $\mathcal{L}$ is a $\mathcal{D}_q$-semi-classical linear functional satisfying (2.77), then $\mathcal{L}$ is of class $s = \max(\deg(\phi) - 2, \deg(\psi) - 1)$ if and only if*

$$\prod_{c \in Z_\phi} (|r_{c,q}| - |\langle \mathcal{L}, \psi_{c,q} \rangle|) \neq 0, \tag{2.78}$$

where Z_ϕ is the set of zeros of ϕ. The complex number $r_{c,q}$ and the polynomials ϕ_c, $\psi_{c,q}$ are defined by

$$(x - c)\phi_c = \phi, \quad \psi - \phi_c = (qx - c)\psi_{c,q} + r_{c,q}. \tag{2.79}$$

Proof: For a proof see Proposition 3.4. □

Remark 2.9 *It follows from the definition of the class of the linear functional that the $\mathcal{D}_q$-classical linear functional is a $\mathcal{D}_q$-semi-classical linear functional of class $s = 0$.*

Lemma 2.8 *The linear functional $\mathcal{L}$ is regular if and only if $\mathcal{G}_q\mathcal{L}$ (with $q \neq 0$) is regular.*

Proof: This follows from Lemma 3.2. □

Lemma 2.9 *Let $\mathcal{L}$ be a regular linear functional, we have:*

i) If there exist two polynomials $\psi \neq 0$, and ϕ such that

$$\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}, \tag{2.80}$$

then ϕ is a non-zero polynomial.

ii) Conversely, if there exist two polynomials $\phi \neq 0$ and ψ such that (2.80) holds, then ψ is of degree at least one.

Proof: For a proof see Lemma 3.1. □

Proposition 2.9 (Medem, 1996) *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family and $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ the dual basis associated to $\{P_n\}_{n\in\mathbb{N}}$.*
If $\{\bar{\mathbf{Q}}_{n,1}\}_{n\in\mathbb{N}}$ is the dual basis associated to the monic family $\{\bar{Q}_{n,1}\}_{n\in\mathbb{N}}$ defined by

$$\bar{Q}_{n,1} = \frac{\mathcal{D}_q P_{n+1}}{[n+1]_q},$$

then we have

$$\mathcal{D}_{\frac{1}{q}} \bar{\mathbf{Q}}_{n,1} = -q\,[n+1]_q \mathbf{P}_{n+1}.$$

Proof: This follows from Proposition 3.5. □

2.3.9 Characterisation of $\mathcal{D}_q$-classical orthogonal polynomials

We give some characterisations for $\mathcal{D}_q$-classical orthogonal polynomials. The following theorem is a corollary of Theorem 3.1 [Medem, 1996].

Theorem 2.9 *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family, and $Q_{n,m}$ the monic polynomial of degree n defined by*

$$B_{n,m}(q)\, Q_{n,m} = \mathcal{D}_q^m P_{n+m},$$

with

$$B_{n,m}(q) = \prod_{j=0}^{m-1} [n+m-j]_q, \quad Q_{n,0} \equiv P_n.$$

The following properties are equivalent:

i) There exist two polynomials. ϕ of degree at most two and ψ of degree one, such that

$$\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}.$$

ii) There exist two polynomials. ϕ of degree at most two and ψ of degree one, such that for any integer m,

$$\mathcal{D}_q(\phi\mathcal{L}_m) = \psi_m\mathcal{L},$$
$$\langle\mathcal{L}_m, Q_{j,m}Q_{n,m}\rangle = k_n\delta_{j,n}, \quad \forall j,n \in \mathbb{N}, \quad (k_n \neq 0 \; \forall n \in \mathbb{N}),$$

with the linear functional $\mathcal{L}_m$ and the polynomial ψ_m defined. recursively, by

$$\psi_{m+1} = \mathcal{D}_q\phi + q\mathcal{G}_q\psi_m. \quad \psi_0 \equiv \psi.$$
$$\mathcal{L}_{m+1} = \mathcal{G}_q(\phi\,\mathcal{L}_m). \quad \mathcal{L}_0 \equiv \mathcal{L}$$

and given explicitly as

$$\psi_m(x) = \frac{\phi(q^m x) - \phi(x)}{(q-1)x} - q^m\psi(q^m x). \tag{2.81}$$

$$\mathcal{L}_m = \prod_{j=1}^{m}\phi(q^j x)\,\mathcal{G}_q^m\mathcal{L}. \tag{2.82}$$

iii) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m. the following second-order q-difference equation holds:

$$\phi\,\mathcal{D}_q\mathcal{D}_{\frac{1}{q}}\,Q_{n,m} + \psi_m\,\mathcal{D}_q\,Q_{n,m} - \lambda^{**}_{n,m}\,Q_{n,m} = 0 \quad \forall n \in \mathbb{N},$$

with the polynomial ψ_m given by (2.81) and the constant $\lambda^{**}_{n,m}$ given by

$$\lambda^{**}_{n,m} = -[n]_q\,\{\mathcal{D}_q\psi_m + [n-1]_{\frac{1}{q}}\,\frac{\phi''}{2\,q}\} = -[n]_q\,q^{2m}\,\{\psi' - [2m+n-1]_{\frac{1}{q}}\,\frac{\phi''}{2\,q}\}. \tag{2.83}$$

iv) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that. for any integer m. the following relation holds:

$$[n]_q\mathcal{D}_{\frac{1}{q}}\,[Q_{n-1,m+1}\mathcal{L}_{m-1}] = -q\,\lambda^{**}_{n,m}\;Q_{n,m}\,\mathcal{L}_m \quad \forall n \in \mathbb{N}. \tag{2.84}$$

with the polynomial ψ_m, the linear functional $\mathcal{L}_m$. and the constant $\lambda_{n,m}$ given respectively by (2.81). (2.82) and (2.83).

v) There exist a polynomial ϕ of degree at most two and three constants $c_{n,n+1}$, $c_{n,n}$, $c_{n,n-1}$ with $c_{n,n-1} \neq 0$ such that

$$\phi\mathcal{D}_{\frac{1}{q}}P_n = c_{n,n+1}P_{n+1} + c_{n,n}P_n + c_{n,n-1}P_{n-1}.$$

vi) For any non-zero integer m, there exist a sequence of complex numbers $\{u_{n,m}\}_{n\in\mathbb{N}}$ such that

$$Q_{n,m-1} = Q_{n,m} + u_{n-1,m}Q_{n-1,m} + u_{n-2,m}Q_{n-2,m}, \qquad \forall n \in \mathbb{N} - \{0,1\}.$$

Remark 2.10 *1. For all $m \in \mathbb{N}$, the $\mathcal{D}_q$-derivative of order m, $\{Q_{n,m}\}_{n\in\mathbb{N}}$, of the family $\{P_{n+m}\}_{n\in\mathbb{N}}$ is q-classical and orthogonal with respect to the q-classical linear functional $\mathcal{L}_m$.*

2. The q-analogue of the functional version of the generalised Rodrigues formula [Medem, 1996] given below, is obtained by iterating the relation (2.8_):

$$Q_{n,m}\mathcal{L}_m = (-1)^n q^{-\dots}\prod_{j=0}^{n-1}\frac{[n-j]_q}{\lambda^{**}_{n-j,m+j}}\,D^n_{\frac{1}{q}}\mathcal{L}_{n+m}.$$

2.4 The q-integration

In this section, exploiting the thesis of Medem [Medem, 1996], we recall the definition of the concept of the q-integration with the assumption $0 < q < 1$ and give some properties. More details can be found in [Jackson, 1919] and [Gasper et al., 1990] and [Medem, 1996].

2.4.1 The q-integration on the interval $[0, a]$, $a > 0$

Let f be a real function defined on the interval $[0, a]$ and $\mathcal{P}_q([0, a])$ the "q-partition" of the interval $[0, a]$ defined by

$$\mathcal{P}_q([0, a]) = \{\ldots aq^{n+1} < aq^n < \ldots < aq < a\}.$$

For any integer N, consider the "Riemann sum"

$$A_N(f) = \sum_{n=0}^{N}(aq^n - aq^{n+1})f(aq^n) = a(1-q)\sum_{n=0}^{N} q^n f(aq^n).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f on the interval $[0, a]$, denoted $\int_0^a f(s)d_qs$, is given by

$$\int_0^a f(s)d_qs = \lim_{N\to\infty} A_N(f) = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n). \tag{2.85}$$

2.4.2 The q-integration on the interval $[a, 0]$, $a < 0$

Let f be a real function defined on the interval $[0, a]$ and $\mathcal{P}_q([a, 0])$ the "q-partition" of the interval $[a, 0]$ defined by

$$\mathcal{P}_q([a, 0]) = \{a < aq < \ldots aq^n < aq^{n+1} < \ldots\} = \{aq^n, n \in \mathbb{N}\}.$$

For any integer N, consider the "Riemann sum"

$$A_N(f) = \sum_{n=0}^{N}(aq^{n+1} - aq^n)f(aq^n) = -a(1-q)\sum_{n=0}^{N} q^n f(aq^n).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f on the interval $[a, 0]$, denoted $\int_a^0 f(s)d_qs$, is given by

$$\int_a^0 f(s)d_qs = \lim_{N\to\infty} A_N(f) = -a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n). \tag{2.86}$$

2.4.3 The q-integration on the interval $[a, \infty[$, $a > 0$

Let f be a real function defined on the interval $[a, \infty[$ and $\mathcal{P}_q([a, \infty[)$ the "q-partition" of the interval $[a, \infty[$ defined by

$$\mathcal{P}_q([a, \infty[) = \{a < aq^{-1} < \ldots aq^{-n} < aq^{-n-1} < \ldots\} = \{aq^{-n}, n \in \mathbb{N}\}.$$

For any integer N, consider the "Riemann sum"

$$A_N(f) = \sum_{n=0}^{N}(aq^{-n-1} - aq^{-n})f(aq^{-n-1}) = a(\frac{1}{q} - 1)\sum_{n=0}^{N} q^{-n} f(aq^{-n-1}).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f on the interval $[a, \infty[$, denoted $\int_a^\infty f(s)d_qs$, is given by

$$\int_a^\infty f(s)d_qs = \lim_{N\to\infty} A_N(f) = a(\frac{1}{q} - 1)\sum_{n=0}^{\infty} q^{-n} f(aq^{-n-1}) \tag{2.87}$$

2.4.4 The q-integrationon the interval $]-\infty, a]$, $a < 0$

Let f be a real function defined on the interval $]-\infty, a]$ and $\mathcal{P}_q(]-\infty, a])$ the "q-partition" of the interval $]-\infty, a]$ defined by

$$\mathcal{P}_q(]-\infty, a]) = \{a > aq^{-1} > \ldots > aq^{-n-1} > \ldots\} = \{aq^{-n}, n \in \mathbb{N}\}.$$

For any integer N, consider the "Riemann sum"

$$A_N(f) = \sum_{n=0}^{N}(aq^{-n} - aq^{-n-1})f(aq^{-n-1}) = -a(\frac{1}{q} - 1)\sum_{n=0}^{N} q^{-n} f(aq^{-n-1}).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f on the interval $]-\infty, a]$, denoted $\int_{-\infty}^{a} f(s)d_qs$, is given by

$$\int_{-\infty}^{a} f(s)d_qs = \lim_{N\to\infty} A_N(f) = -a(\frac{1}{q} - 1)\sum_{n=0}^{\infty} q^{-n} f(aq^{-n-1}) \tag{2.88}$$

Remark 2.11 *The q-integration is extended to the whole real line by using relations (2.85)-(2.88) and the following rules*

$$\begin{aligned}
\int_a^b f(s)d_qs &= \int_a^0 f(s)d_qs + \int_0^b f(s)d_qs \quad \forall a, b \in \mathbb{R},\\
\int_a^{\infty} f(s)d_qs &= \int_a^b f(s)d_qs + \int_b^{\infty} f(s)d_qs \quad \forall a, b \in \mathbb{R},\ a < 0,\ b > 0\\
\int_{-\infty}^b f(s)d_qs &= \int_{-\infty}^a f(s)d_qs + \int_a^b f(s)d_qs \quad \forall a, b \in \mathbb{R},\ a < 0,\ b > 0\\
\int_{-\infty}^{\infty} f(s)d_qs &= \int_{-\infty}^a f(s)d_qs + \int_a^b f(s)d_qs + \int_b^{\infty} f(s)d_qs \quad \forall a, b \in \mathbb{R}.
\end{aligned} \tag{2.89}$$

As the usual integration, the q-integration enjoys some properties. Here, we give some, which are proved using the definition of the concept of the q-integration.

Lemma 2.10 *1. If f is a real function continuous at 0, then we have*

$$\int_0^a \mathcal{D}_q f(s)d_qs = f(a) - f(0).$$

2. For any function f integrable on $[0, a]$, we have

$$\mathcal{D}_q \int_0^a f(s)d_qs = f(a),$$

assuming that the operator $\mathcal{D}_q$ acts on the variable a.

3. If f is a real function continuous on the interval $[0, a]$, then f is q-integrable on $[0, a]$ and obeys

$$\lim_{q\to 1}\int_0^a f(s)d_qs = \int_0^a f(s)ds.$$

4. If f and g are two real functions, q-integrable on the interval $[0, a]$, then we have

$$\int_0^a f(s)\mathcal{D}_q g(s)d_qs = fg|_0^a - \int_0^a \mathcal{D}_q f(s)\, g(qs)d_qs = f(s/q)\, g(s)|_0^a - \frac{1}{q}\int_0^a g(s)\mathcal{D}_{\frac{1}{q}} f(s)d_qs.$$

with $fg|_0^a = f(a)g(a) - f(0)g(0)$.

Remark 2.12 *The previous lemma can be extended to the whole real line by using (2.89).*

Chapter 3

The $D_{q,\omega}$-semi-classical orthogonal polynomials

3.1 Introduction

We define the operators $A_{q,\omega}$ and $D_{q,\omega}$. The first generalises the operators $\mathcal{T}_\omega$ and $\mathcal{G}_q$ and the second generalises the operators $\mathcal{D}$, D_ω and $\mathcal{D}_q$. We give some definitions related to these operators and then give the characterisation theorems for $D_{q,\omega}$-semi-classical orthogonal polynomials: and deduce by limit processes the characterisation theorems for $\mathcal{D}$, D_ω and $\mathcal{D}_q$-semi-classical orthogonal polynomials.

3.1.1 Operators $A_{q,\omega}$ and $D_{q,\omega}$

Definition 3.1 *We combine the operators $\mathcal{T}_\omega$ and $\mathcal{G}_q$ to obtain a new operator denoted by $A_{q,\omega}$ and defined by*

$$\begin{aligned} A_{q,\omega} : \mathbb{P} &\longrightarrow \mathbb{P} \\ P &\longrightarrow A_{q,\omega}P,\ A_{q,\omega}P(x) = \mathcal{G}_q\mathcal{T}_\omega P(x) = P(qx + \omega),\ q \neq 0. \end{aligned} \tag{3.1}$$

We denote

$$A^*_{q,\omega} = A_{\frac{1}{q},-\frac{\omega}{q}}. \tag{3.2}$$

Definition 3.2 (Hahn, 1948) *The difference operator $D_{q,\omega}$ is defined by*

$$\begin{aligned} D_{q,\omega} : \mathbb{P} &\longrightarrow \mathbb{P} \\ P &\longrightarrow D_{q,\omega}P,\ D_{q,\omega}P(x) = \frac{P(qx + \omega) - P(x)}{(q-1)x + \omega},\ \omega \in \mathbb{R},\ q \in \mathbb{R},\ q \neq 0. \end{aligned} \tag{3.3}$$

We denote

$$D^*_{q,\omega} = D_{\frac{1}{q},-\frac{\omega}{q}} \tag{3.4}$$

The applications $P \longrightarrow A_{q,\omega}P$ and $P \longrightarrow D_{q,\omega}P$ belong to $L(\mathbb{P},\mathbb{P})$. We, therefore, use their transposes to define the action of the operators $A_{q,\omega}$ and $D_{q,\omega}$ on the linear functionals.

Definition 3.3 *We define the action of the operator $A_{q,\omega}$ on the functional $\mathcal{L}$ as*

$$\langle A_{q,\omega}\mathcal{L}, P\rangle = \frac{1}{q}\langle \mathcal{L}, A^*_{q,\omega}P\rangle \quad \forall P \in \mathbb{P},\ (q \neq 0). \tag{3.5}$$

Definition 3.4 *We define the $D_{q,\omega}$-derivative of a given linear functional $\mathcal{L}$, $D_{q,\omega}\mathcal{L}$, as*

$$\begin{aligned} D_{q,\omega}\mathcal{L} &: \mathbb{P} \to \mathbb{C} \\ \langle D_{q,\omega}\mathcal{L}, P\rangle &= -\frac{1}{q}\langle \mathcal{L}, D^*_{q,\omega}P\rangle \quad \forall P \in \mathbb{P}, (q \neq 0). \end{aligned} \tag{3.6}$$

Definition 3.5 *The regular linear functional $\mathcal{L}$ and the corresponding monic orthogonal polynomials are said to be $D_{q,\omega}$-semi-classical if there exist two polynomials ψ of degree at least one, and ϕ such that*

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}. \tag{3.7}$$

Moreover, if ϕ is a polynomial of degree at most two and ψ a first-degree polynomial, then the linear functional and the corresponding orthogonal polynomials are called $D_{q,\omega}$-classical.

Remark 3.1 *The operators D_ω and D_q generalise the operator $\frac{d}{dx}$ in the following way:*

$$\lim_{\omega\to 0} D_\omega = \frac{d}{dx}, \ \lim_{q\to 1} \mathcal{D}_q = \frac{d}{dx},$$

while the operators $\frac{d}{dx}$, D_ω and $\mathcal{D}_q$ can be obtained from the operator $D_{q,\omega}$ by the following limit processes:

$$\lim_{\omega\to 0} D_{q,\omega} = \mathcal{D}_q, \ \lim_{q\to 1} D_{q,\omega} = D_\omega, \ \lim_{q\to 1,\omega\to 0} D_{q,\omega} = \frac{d}{dx}.$$

Lemma 3.1 *Let $\mathcal{L}$ be a regular linear functional.*

i) If there exist two polynomials $\psi \neq 0$, and ϕ such that

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}, \tag{3.8}$$

then ϕ is a non-zero polynomial.

ii) Conversely, if there exist two polynomials $\phi \neq 0$ and ψ such that (3.8) holds, then ψ is of degree at least one.

Proof: We give the proof for the operator $D_{q,\omega}$ and extend it to the operators $\frac{d}{dx}$, D_ω and $\mathcal{D}_q$ by limit processes (see Remark 3.1).

i) Suppose that

$$\psi(x) = \sum_{j=0}^{r} \psi_j x^j,$$

with $\psi_r \neq 0$; and let $\{P_n\}_{n\in\mathbb{N}}$ be the monic polynomial family orthogonal with respect to $\mathcal{L}$. If $\phi = 0$, we apply both sides of (3.8) to the polynomial $\frac{1}{\psi_r}P_r$ and obtain

$$0 = \langle\psi\mathcal{L}, \frac{1}{\psi_r}P_r\rangle = \langle\mathcal{L}, P_rP_r\rangle.$$

Then $\langle\mathcal{L}, P_rP_r\rangle = 0$. This is a contradiction because $\{P_n\}_{n\in\mathbb{N}}$ is the monic (OPS) associated to $\mathcal{L}$ (see (2.5)). Thus ϕ is a non-zero polynomial.

ii) Suppose that ψ is a constant denoted ψ_0 and then apply both sides of (3.8) to the polynomial P_0P_0 ($P_0(x) = 1$) and get

$$\begin{aligned}\psi_0\langle\mathcal{L}, P_0P_0\rangle &= \langle D_{q,\omega}(\phi\mathcal{L}), P_0P_0\rangle \\ &= -\frac{1}{q}\langle\phi\mathcal{L}, D^*_{q,\omega}(P_0P_0)\rangle = 0.\end{aligned}$$

Since $\langle\mathcal{L}, P_0P_0\rangle \neq 0$, we deduce that $\psi_0 = 0$ and it results from (3.8) that

$$D_{q,\omega}(\phi\mathcal{L}) = 0.$$

The previous equation is equivalent to $\phi\mathcal{L} = 0$. In fact,

$$\begin{aligned} D_{q,\omega}(\phi\mathcal{L}) = 0 &\iff \langle D_{q,\omega}(\phi\mathcal{L}), P\rangle = 0 \quad \forall P \in \mathcal{P} \\ &\iff \\ \langle\phi\mathcal{L}, D^*_{q,\omega}P\rangle &= 0 \quad \forall P \in \mathcal{P} \\ &\iff \\ \phi\mathcal{L} &= 0.\end{aligned}$$

Since $\phi \neq 0$, we pose

$$\phi(x) = \sum_{j=0}^{t} \phi_j x^j,$$

with $\phi_t \neq 0$. Then applying both sides of (3.8) to the polynomial $\frac{1}{\phi_t} P_t$, we obtain

$$0 = \langle \phi\,\mathcal{L}, \frac{1}{\phi_t} P_t \rangle = \frac{1}{\phi_t} \langle \mathcal{L}, \phi\, P_t \rangle = \langle \mathcal{L}, P_t P_t \rangle.$$

The previous equation gives a contradiction since $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}$. We, therefore, conclude that the polynomial ψ is of degree at least one. □

Remark 3.2 *The operators D_ω, $\mathcal{D}_q$ or $D_{q,\omega}$ transform any polynomial P_n of degree n in a polynomial of degree $n-1$.*

Lemma 3.2 *If $\mathcal{L}$ is a linear functional and $\mathcal{Y}$ one of the difference operator $\{\mathcal{T}_\omega, \mathcal{G}_q, A_{q,\omega}\}$, $q \neq 0$, then the linear functional $\mathcal{Y}(\mathcal{L})$ is regular if and only if $\mathcal{L}$ is regular.*

Proof: We prove the lemma for the operator $A_{q,\omega}$ and extend it to the operators $\mathcal{T}_\omega$ and $\mathcal{G}_q$. If $\{P_n\}_{n\in\mathbb{N}}$ is the monic polynomial family orthogonal with respect to $\mathcal{L}$, then $\{q^{-n} A_{q,\omega} P_n\}_{n\in\mathbb{N}}$ is the monic polynomial family orthogonal with respect to $A_{q,\omega}\mathcal{L}$.

In fact,

$$\langle A_{q,\omega}\,\mathcal{L}, A_{q,\omega}\, P_n\, A_{q,\omega}\, P_m \rangle = \frac{1}{q}\,\delta_{n,m} \quad \forall n, m \in \mathbb{N}.$$

□

We prove the following proposition:

Proposition 3.1 *For all $q, \omega \in \mathbb{R}$, $q \neq 0$, $q \neq 1$; for all f, $g \in \mathbb{P}$; and for all $\mathcal{L} \in \mathbb{P}'$ the following properties hold:*

$$i) A_{q,\omega} A^*_{q,\omega} = A^*_{q,\omega} A_{q,\omega} = \mathcal{I}_d,\ D_{q,\omega} A_{q,\omega} = q A_{q,\omega} D_{q,\omega},\ D_{q,\omega} A^*_{q,\omega} = \frac{1}{q} A^*_{q,\omega} D_{q,\omega} = \frac{1}{q} D^*_{q,\omega}, \tag{3.9}$$

$$ii) D^*_{q,\omega} D_{q,\omega} = q D_{q,\omega} D^*_{q,\omega},\quad A_{q,\omega} A^*_{q,\omega} = A^*_{q,\omega} A_{q,\omega} = \mathcal{I}_d. \tag{3.10}$$

$$iii) A_{q,\omega}(f\,g) = A_{q,\omega} f A_{q,\omega} g,\quad A_{q,\omega}(f\,\mathcal{L}) = A_{q,\omega} f A_{q,\omega} \mathcal{L}. \tag{3.11}$$

$$iv) D_{q,\omega}(fg) = f D_{q,\omega} g + A_{q,\omega} g D_{q,\omega} f = A_{q,\omega} f D_{q,\omega} g + g D_{q,\omega} f, \tag{3.12}$$

$$v) D_{q,\omega}(f\mathcal{L}) = f D_{q,\omega}\mathcal{L} + D_{q,\omega} f A_{q,\omega}\mathcal{L} = A_{q,\omega} f D_{q,\omega}\mathcal{L} + D_{q,\omega} f \mathcal{L}, \tag{3.13}$$

$$vi) (q-1) D_{q,\omega}\mathcal{L} = (x - \frac{\omega}{1-q})^{-1} (A_{q,\omega}\,\mathcal{L} - \mathcal{L}). \tag{3.14}$$

$$D_{q,\omega}(f g \mathcal{L}) = A_{q,\omega}\, f\, D_{q,\omega}(g\mathcal{L}) - A_{q,\omega}\, f\, D_{q,\omega} g\, \mathcal{L} + D_{q,\omega}(fg)\, \mathcal{L},\ \forall f, g \in \mathbb{P},\ \forall \mathcal{L} \in \mathbb{P}'. \tag{3.15}$$

Proof: Properties i) and ii) are obtained directly from the definition.

It should be noted that the identities in relations (3.9) and (3.10) are valid when the operators $A_{q,\omega}$ and $D_{q,\omega}$ act on $\mathbb{P}$ and also on $\mathbb{P}'$.

For iii), use of (3.5) gives

$$\begin{aligned}
\langle A_{q,\omega}(f\mathcal{L}), P \rangle &= \frac{1}{q} \langle f\mathcal{L}, A^*_{q,\omega} P \rangle \\
&= \frac{1}{q} \langle \mathcal{L}, f A^*_{q,\omega} P \rangle \\
&= \frac{1}{q} \langle \mathcal{L}, A^*_{q,\omega} (A_{q,\omega} f P) \rangle \\
&= \langle A_{q,\omega}\mathcal{L}, A_{q,\omega} f P \rangle \\
&= \langle A_{q,\omega} f A_{q,\omega}\mathcal{L}, P \rangle,
\end{aligned}$$

thus

$$A_{q,\omega}(f\mathcal{L}) = A_{q,\omega}fA_{q,\omega}\mathcal{L}.$$

$$\begin{aligned}
D_{q,\omega}(fg) &= \frac{f(qx+\omega)g(qx+\omega)-f(x)g(x)}{(q-1)x+\omega} \\
&= f(x)\frac{g(qx+\omega)-g(x)}{(q-1)x+\omega} + g(qx+\omega)\frac{f(qx+\omega)-f(x)}{(q-1)x+\omega} \\
&= f(x)D_{q,\omega}g(x) + g(qx+\omega)D_{q,\omega}f(x).
\end{aligned}$$

then reversing the role of f and g, we deduce that

$$D_{q,\omega}(fg) = fD_{q,\omega}g + A_{q,\omega}gD_{q,\omega}f = A_{q,\omega}fD_{q,\omega}g + gD_{q,\omega}f.$$

We now use i), ii) and iii) to prove iv).

$$\begin{aligned}
\langle D_{q,\omega}(f\mathcal{L}), P\rangle &= -\frac{1}{q}\langle f\mathcal{L}, D^*_{q,\omega}P\rangle \\
&= -\frac{1}{q}\langle \mathcal{L}, fD^*_{q,\omega}P\rangle \\
&= -\frac{1}{q}\langle \mathcal{L}, D^*_{q,\omega}(fP) - A^*_{q,\omega}PD^*_{q,\omega}f \\
&= \langle D_{q,\omega}\mathcal{L}, fP\rangle + \frac{1}{q}\langle \mathcal{L}, D^*_{q,\omega}fA^*_{q,\omega}P\rangle \\
&= \langle fD_{q,\omega}\mathcal{L}, P\rangle + \frac{1}{q}\langle D^*_{q,\omega}f\mathcal{L}, A^*_{q,\omega}P\rangle \\
&= \langle fD_{q,\omega}\mathcal{L}, P\rangle + \langle A_{q,\omega}(D^*_{q,\omega}f\mathcal{L}), P\rangle \\
&= \langle fD_{q,\omega}\mathcal{L}, P\rangle + \langle D_{q,\omega}fA_{q,\omega}\mathcal{L}, P\rangle.
\end{aligned}$$

Then

$$D_{q,\omega}(f\mathcal{L}) = fD_{q,\omega}\mathcal{L} + D_{q,\omega}fA_{q,\omega}\mathcal{L} = A_{q,\omega}fD_{q,\omega}\mathcal{L} - D_{q,\omega}f\mathcal{L}.$$

For $q \neq 1$ and $q \neq 0$, we have

$$\begin{aligned}
\langle (x-\frac{\omega}{1-q})^{-1}(A_{q,\omega}\mathcal{L}-\mathcal{L}), P\rangle &= \langle A_{q,\omega}\mathcal{L}-\mathcal{L}, \theta_{\frac{\omega}{1-q}}P\rangle \\
&= \langle A_{q,\omega}\mathcal{L}-\mathcal{L}, \frac{P(x)-P(\frac{\omega}{1-q})}{x-\frac{\omega}{1-q}}\rangle \\
&= \frac{1}{q}\langle \mathcal{L}, \frac{P(\frac{x-\omega}{q})-P(\frac{\omega}{1-q})}{\frac{x-\omega}{q}-\frac{\omega}{1-q}}\rangle - \langle \mathcal{L}, \frac{P(x)-P(\frac{\omega}{1-q})}{x-\frac{\omega}{1-q}}\rangle \\
&= -(q-1)\frac{1}{q}\langle \mathcal{L}, \frac{P(\frac{x-\omega}{q})-P(x)}{(\frac{1}{q}-1)x-\frac{\omega}{q}}\rangle \\
&= -(q-1)\frac{1}{q}\langle \mathcal{L}, D^*_{q,\omega}P\rangle \\
&= (q-1)\langle D_{q,\omega}\mathcal{L}, P\rangle.
\end{aligned}$$

Then,

$$(q-1)D_{q,\omega}\mathcal{L} = (x-\frac{\omega}{1-q})^{-1}(A_{q,\omega}\mathcal{L}-\mathcal{L}).$$

The relation (3.15) follows straightforwardly from (3.12) and (3.13). □

Remark 3.3 *The proof of Proposition 2.4 (resp. Proposition 2.7) is obtained in the same way just by replacing q by one and ω by zero, respectively. In particular, to derive the relation (2.46) from (3.14), we first multiply both sides of (3.14) by $(x-\frac{\omega}{1-q})$, then use (2.14) to get*

$$((q-1)x+\omega)D_{q,\omega}\mathcal{L} = (A_{q,\omega} - \mathcal{I}_d)\mathcal{L}.$$

Therefore, (2.46) yields by taking $q=1$ in the previous relation.

Proposition 3.2 *If $\mathcal{L}$ is a regular linear functional, and $q \in \mathbb{R} - \{0\}$, then we have*

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L} \Longleftrightarrow D^*_{q,\omega}(\tilde{\phi}\mathcal{L}) = \psi\mathcal{L}, \tag{3.16}$$

with

$$\tilde{\phi} = \frac{1}{q}(\phi + [(q-1)x + \omega]\psi). \tag{3.17}$$

Proof: Let ϕ and ψ be two polynomials, then using Proposition 1.3 we have

$$\begin{aligned}
& D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L} \Longleftrightarrow \langle D_{q,\omega}(\phi\mathcal{L}), A_{q,\omega}P\rangle = \langle \psi\mathcal{L}, A_{q,\omega}P\rangle \quad \forall P \in \mathbb{P} \\
& \Longleftrightarrow -\frac{1}{q}\langle \phi\mathcal{L}, D^*_{q,\omega}A_{q,\omega}P\rangle = \langle \psi\mathcal{L}, A_{q,\omega}P\rangle \quad \forall P \in \mathbb{P} \\
& \Longleftrightarrow -\langle \phi\mathcal{L}, A_{q,\omega}D^*_{q,\omega}P\rangle = \langle \psi\mathcal{L}, A_{q,\omega}P\rangle \quad \forall P \in \mathbb{P} \\
& \Longleftrightarrow -\langle \phi\mathcal{L}, D_{q,\omega}P\rangle = \langle \psi\mathcal{L}, [(q-1)x + \omega]D_{q,\omega}P + P\rangle \quad \forall P \in \mathbb{P} \\
& \Longleftrightarrow -\langle (\phi + [(q-1)x + \omega]\psi)\mathcal{L}, D_{q,\omega}P\rangle = \langle \psi\mathcal{L}, P\rangle \quad \forall P \in \mathbb{P} \\
& \Longleftrightarrow \frac{1}{q}\langle D^*_{q,\omega}\left[(\phi + [(q-1)x - \omega]\psi)\mathcal{L}\right], P\rangle = \langle \psi\mathcal{L}, P\rangle \quad \forall P \in \mathbb{P} \\
& \Longleftrightarrow D^*_{q,\omega}(\tilde{\phi}\mathcal{L}) = \psi\mathcal{L},
\end{aligned}$$

with $\tilde{\phi}$ given by (3.17). □

Corollary 3.1 (Salto, 1995, Medem, 1996) *From the above proposition, we deduce the following:*
i) $\mathcal{L}$ is $D_{q,\omega}$-semi-classical $\Longleftrightarrow$ $\mathcal{L}$ is $D^*_{q,\omega}$-semi-classical.
ii) $\mathcal{L}$ is $\mathcal{D}_q$-semi-classical $\Longleftrightarrow$ $\mathcal{L}$ is $\mathcal{D}_{\frac{1}{q}}$-semi-classical.
Indeed,

$$\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L} \Longleftrightarrow \mathcal{D}_{\frac{1}{q}}(\tilde{\phi}\mathcal{L}) = \psi\mathcal{L}.$$

with

$$\tilde{\phi} = \frac{1}{q}(\phi + (q-1)x\psi).$$

iii) $\mathcal{L}$ is D_ω-semi-classical $\Longleftrightarrow$ $\mathcal{L}$ is $D_{-\omega}$-semi-classical. Moreover,

$$D_\omega(\phi\mathcal{L}) = \psi\mathcal{L} \Longleftrightarrow D_{-\omega}(\tilde{\phi}\mathcal{L}) = \psi\mathcal{L},$$

where

$$\tilde{\phi} = \phi + \omega\psi.$$

Remark 3.4 *If $\mathcal{Y}$ represents one of the operators: $\mathcal{T}_\omega$, $\mathcal{G}_q$, $A_{q,\omega}$, $\frac{d}{dx}$, D_ω, $\mathcal{D}_q$, and $D_{q,\omega}$, we define the power of $\mathcal{Y}$, $\mathcal{Y}^m$ as*

$$\mathcal{Y}^m = \mathcal{Y}\mathcal{Y}^{m-1}, \ m \geq 1 \ \textit{with} \ \mathcal{Y}^0 \equiv \mathcal{I}_d,$$

where $\mathcal{I}_d$ is the identity operator.

Remark 3.5 *One proves easily that* $\forall P \in \mathbb{P}$ *and* $\forall n \in \mathbb{N}$

$$A^n_{q,\omega}P(x) = P(q^n x + \omega[n]_q), \ \mathcal{G}^n_q P(x) = P(q^n x), \ \mathcal{T}^n_\omega P(x) = P(x + n\omega). \tag{3.18}$$

3.1.2 Class of the $D_{q,\omega}$-semi-classical linear functional

Let $\mathcal{L}$ be a $D_{q,\omega}$-semi-classical linear functional satisfying

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}. \tag{3.19}$$

where ϕ is a non-zero polynomial and ψ a polynomial of degree at least one. $\mathcal{L}$ satisfies $D_{q,\omega}(f\phi\mathcal{L}) = (\phi D_{q,\omega}f + \psi A_{q,\omega}f)\mathcal{L}$, for any polynomial f.

Definition 3.6 *We define the class* $\mathrm{cl}(\mathcal{L})$ *of the* $D_{q,\omega}$*-semi-classical linear functional* $\mathcal{L}$ *as*

$$\mathrm{cl}(\mathcal{L}) = \min_{(f,g)\in\mathcal{R}}\{\max(\deg(f)-2, \deg(g)-1)\}, \tag{3.20}$$

where

$$\mathcal{R} = \{(f,g)\in\mathbb{P}^2 \,/\, \deg(g)\geq 1 \text{ and } D_{q,\omega}(f\mathcal{L}) = g\mathcal{L}\}. \tag{3.21}$$

We state the following lemmas and propostion which we shall use to prove the proposition characterising the class of the semi-classical linear functional.

Lemma 3.3 *Consider* $\mathcal{L}$ *a regular linear functional,* ψ *a non-zero polynomial and* ϕ *a polynomial of degree at least one. Then, for any zero,* c*, of* ϕ*, we have*

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L} \iff D_{q,\omega}(\phi_c\mathcal{L}) = \psi_{c,q,\omega}\mathcal{L} - \langle\mathcal{L},\psi_{c,q,\omega}\rangle\,\delta_{\frac{c-\omega}{q}} + \frac{1}{q}r_{c,q,\omega}\,(x-\frac{c-\omega}{q})^{-1}\mathcal{L}, \tag{3.22}$$

where

$$\phi = (x-c)\phi_c,\ \psi - \phi_c = (qx+\omega-c)\psi_{c,q,\omega} + r_{c,q,\omega}. \tag{3.23}$$

Proof: The proof is obtained straightforwardly by using (2.14), (3.13) and (3.23). □

Lemma 3.4 *Let* $\mathcal{L}$ *be a regular linear functional. If there exist four polynomials* ϕ, ψ, $\tilde{\phi}$ *and* $\tilde{\psi}$, *with* $\deg(\tilde{\phi}\geq 1)$, *such that*

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L},\ D_{q,\omega}(\phi\tilde{\phi}\mathcal{L}) = \tilde{\psi}\mathcal{L}. \tag{3.24}$$

then, for any zero, c*, of* $\tilde{\phi}$,

$$r_{c,q,\omega} = \langle\mathcal{L},\tilde{\psi}_{c,q,\omega}\rangle = 0. \tag{3.25}$$

where,

$$\tilde{\phi} = (x-c)\tilde{\phi}_c,\ \tilde{\psi} - \phi\tilde{\phi}_c = (qx+\omega-c)\tilde{\psi}_{c,q,\omega} + r_{c,q,\omega}. \tag{3.26}$$

Proof: The second relation of (3.24) thanks to Lemma 3.3 is equivalent to

$$D_{q,\omega}(\phi\tilde{\phi}_c\mathcal{L}) = \tilde{\psi}_{c,q,\omega}\mathcal{L} - \langle\mathcal{L},\tilde{\psi}_{c,q,\omega}\rangle\,\delta_{\frac{c-\omega}{q}} + \frac{1}{q}r_{c,q,\omega}\,(x-\frac{c-\omega}{q})^{-1}\mathcal{L}, \tag{3.27}$$

where $r_{c,q,\omega}$ and $\tilde{\psi}_{c,q,\omega}$ are defined by (3.26). The previous relation, used together with (3.13) and the first relation of (3.24) gives

$$(\psi A_{q,\omega}\tilde{\phi}_c + \phi D_{q,\omega}\tilde{\phi}_c - \tilde{\psi}_{c,q,\omega})\mathcal{L} = -\langle\mathcal{L},\tilde{\psi}_{c,q,\omega}\rangle\,\delta_{\frac{c-\omega}{q}} + \frac{1}{q}r_{c,q,\omega}\,(x-\frac{c-\omega}{q})^{-1}\mathcal{L}. \tag{3.28}$$

The multiplication of the latter equation by $(x-\frac{c-\omega}{q})$, use of (2.14) and the relation $(x-a)\delta_a = 0$, gives

$$(x-\frac{c-\omega}{q})(\psi A_{q,\omega}\tilde{\phi}_c + \phi D_{q,\omega}\tilde{\phi}_c - \tilde{\psi}_{c,q,\omega})\mathcal{L} = \frac{1}{q}r_{c,q,\omega}\,\mathcal{L}.$$

It follows from the previous equation and the fact that $\mathcal{L}$ is regular that,

$$(x-\frac{c-\omega}{q})(\psi A_{q,\omega}\tilde{\phi}_c + \phi D_{q,\omega}\tilde{\phi}_c - \tilde{\psi}_{c,q,\omega}) = \frac{1}{q}r_{c,q,\omega}.$$

Thus, $r_{c,q,\omega} = 0$ and $\psi A_{q,\omega}\tilde{\phi}_c + \phi D_{q,\omega}\tilde{\phi}_c - \tilde{\psi}_{c,q,\omega} = 0$. We, therefore, deduce that $\langle\mathcal{L},\tilde{\psi}_{c,q,\omega}\rangle = 0$. □

The following proposition, already known for the operator D_q (Salto, 1995), is also needed to characterise the class of the $D_{q,\omega}$-semi-classical linear functional.

Proposition 3.3 *Consider Φ_1, Φ_2, Ψ_1 and Ψ_2, four polynomials such that: $\Phi_1 \neq 0$, $\Phi_2 \neq 0$, $\deg(\Phi_1) \leq \deg(\Phi_2$, $\deg(\Psi_1) \geq 1$ and $\deg(\Psi_2) \geq 1$. Let $\mathcal{L}$ be a regular linear functional satisfying*

$$D_{q,\omega}(\Phi_1 \mathcal{L}) = \Psi_1 \mathcal{L}, \quad D_{q,\omega}(\Phi_2 \mathcal{L}) = \Psi_2 \mathcal{L}, \; q \neq 0. \tag{3.29}$$

If Φ denotes the highest common factor of Φ_1 and Φ_2: $\Phi = \mathrm{hcf}(\Phi_1, \Phi_2)$, then, there exists a polynomial Ψ such that,

$$D_{q,\omega}(\Phi \mathcal{L}) = \Psi \mathcal{L} \tag{3.30}$$

and

$$\max(\deg(\Phi) - 2, \deg(\Psi) - 1) \leq \max(\deg(\Phi_j) - 2, \deg(\Psi_j) - 1), \; j = 1, 2. \tag{3.31}$$

Moreover, If Φ_1 is not a divisor of Φ_2 ($\Phi_2 \neq f\,\Phi_1$, $\forall f \in \mathbb{P}$), then the previous relation becomes

$$\max(\deg(\Phi) - 2, \deg(\Psi) - 1) < \max(\deg(\Phi_j) - 2, \deg(\Psi_j) - 1), \; j = 1, 2. \tag{3.32}$$

Proof:
We shall give the proof mimicking the approach developed in [Salto, 1996] for the operator D_ω. Since $\Phi = \mathrm{hcf}(\Phi_1, \Phi_2)$, there exist two polynomials $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ satisfying

$$\Phi_1 = \Phi\,\tilde{\Phi}_1, \quad \Phi_2 = \Phi\,\tilde{\Phi}_2, \tag{3.33}$$

with $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ having no common zero.

In the first step, we combine (3.12), (3.15) and (3.29) to get

$$D_{q,\omega}(\tilde{\Phi}_2 \Phi_1 \mathcal{L}) = A_{q,\omega} \tilde{\Phi}_2 \Psi_1 \mathcal{L} - A_{q,\omega} \tilde{\Phi}_2 D_{q,\omega} \Phi_1 \mathcal{L} + D_{q,\omega}(\Phi_1 \tilde{\Phi}_2)\mathcal{L}, \tag{3.34}$$

$$D_{q,\omega}(\tilde{\Phi}_1 \Phi_2 \mathcal{L}) = A_{q,\omega} \tilde{\Phi}_1 \Psi_2 \mathcal{L} - A_{q,\omega} \tilde{\Phi}_1 D_{q,\omega} \Phi_2 \mathcal{L} + D_{q,\omega}(\Phi_2 \tilde{\Phi}_1)\mathcal{L}. \tag{3.35}$$

In the second step, we subtract the two previous equations taking care that $\tilde{\Phi}_1 \Phi_2 = \tilde{\Phi}_2 \Phi_1$, to get

$$\left[A_{q,\omega} \tilde{\Phi}_2 (\Psi_1 - \Phi D_{q,\omega} \tilde{\Phi}_1) - A_{q,\omega} \tilde{\Phi}_1 (\Psi_2 - \Phi D_{q,\omega} \tilde{\Phi}_2)\right] \mathcal{L} = 0.$$

Since $\mathcal{L}$ is regular, we deduce that

$$A_{q,\omega} \tilde{\Phi}_2 (\Psi_1 - \Phi D_{q,\omega} \tilde{\Phi}_1) = A_{q,\omega} \tilde{\Phi}_1 (\Psi_2 - \Phi D_{q,\omega} \tilde{\Phi}_2).$$

Using the previous relation and the fact that $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ have no common zero, it follows that there exists a polynomial Ψ verifying

$$A_{q,\omega} \tilde{\Phi}_1 \Psi = \Psi_1 - \Phi D_{q,\omega} \tilde{\Phi}_1, \quad A_{q,\omega} \tilde{\Phi}_2 \Psi = \Psi_2 - \Phi D_{q,\omega} \tilde{\Phi}_2. \tag{3.36}$$

Use of (3.33) and (3.36) transforms (3.29) in

$$A_{q,\omega} \tilde{\Phi}_1 D_{q,\omega}(\Phi \mathcal{L}) = A_{q,\omega} \tilde{\Phi}_1 \Psi \mathcal{L}, \tag{3.37}$$

$$A_{q,\omega} \tilde{\Phi}_2 D_{q,\omega}(\Phi \mathcal{L}) = A_{q,\omega} \tilde{\Phi}_2 \Psi \mathcal{L}. \tag{3.38}$$

Since $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ have no common zero, there exist two polynomials (Bezout identity) h_1 and h_2 such that $\tilde{\Phi}_1\, h_1 + \tilde{\Phi}_2\, h_2 = 1$. In the third step, we sum the two equations obtained by multiplying (3.37) and (3.38) by $A_{q,\omega} h_1$ and $A_{q,\omega} h_2$, respectively, and get

$$D_{q,\omega}(\Phi \mathcal{L}) = \Psi \mathcal{L}.$$

The latter equation, used together with Lemma 3.1 gives $\deg(\Psi) \geq 1$. In the fourth step, we use (3.33) and (3.36) to get

$$\deg(\Phi_j) = \deg(\Phi) + \deg(\tilde{\Phi}_j), \; \deg(\Psi) + \deg(\tilde{\Phi}_j) \leq \max(\deg(\Psi_j), \deg(\Phi_j) - 1), \; j = 1, 2. \tag{3.39}$$

We, therefore, deduce (3.31).

If we assume that polynomials Φ_1 and Φ_2 are such that $\Phi_1 \neq f\,\Phi_2$ $\forall f \in \mathbb{P}$, then, $\deg(\Phi) < \deg(\Phi_j)$, $j = 1, 2$. We finally use (3.39) to get (3.32). □

The following proposition gives a characterisation for the class of semi-classical linear functional.

Proposition 3.4 *If $\mathcal{L}$ is a $D_{q,\omega}$-semi-classical linear functional satisfying (3.19), then $\mathcal{L}$ is of class* $\mathrm{cl}(\mathcal{L}) = \max(\deg(\phi) - 2, \deg(\psi) - 1)$ *if and only if*

$$\prod_{c \in Z_\phi} (|r_{c,q,\omega}| + |\langle \mathcal{L}, \psi_{c,q,\omega}\rangle|) \neq 0. \tag{3.40}$$

where Z_ϕ is the set of zeros of ϕ. The complex number $r_{c,q,\omega}$ and the polynomials ϕ_c, $\psi_{c,q,\omega}$ are defined by

$$(x - c)\phi_c = \phi, \ \psi - \phi_c = (qx + \omega - c)\psi_{c,q,\omega} + r_{c,q,\omega}. \tag{3.41}$$

Proof: We first recall the definition of the class $\mathrm{cl}(\mathcal{L})$ of $\mathcal{L}$ (see (3.20) and (3.21)).

$$\mathrm{cl}(\mathcal{L}) = \min_{(f,g) \in \mathcal{R}} \{\max(\deg f) - 2, \deg(g) - 1 \},$$

where

$$\mathcal{R} = \{(f, g) \in \mathbb{P}^2 \,/\, \deg(g) \geq 1 \text{ and } D_{q,\omega}(f\mathcal{L}) = g\mathcal{L}\}.$$

Let $(\phi, \psi) \in \mathcal{R}$ such that there exists a zero, c, of ϕ verifying $r_{c,q,\omega} = \langle\mathcal{L}, \psi_{c,q,\omega}\rangle = 0$. We shall prove that,

$$\mathrm{cl}(\mathcal{L}) < \max(\deg(\phi) - 2, \deg(\psi) - 1).$$

Equation $D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}$, thanks to Lemma 3.3 is equivalent to

$$D_{q,\omega}(\phi_c\mathcal{L}) = \psi_{c,q,\omega}\mathcal{L},$$

therefore, $(\phi_c, \psi_{c,q,\omega})$ belongs to $\mathcal{R}$ (see Lemma 3.1). Moreover, the degree of ϕ, ψ, ϕ_c and $\psi_{c,q,\omega}$ obey

$$\max(\deg \phi_c) - 2, \deg(\psi_{c,q,\omega}) - 1) = \max(\deg(\phi) - 2, \deg(\psi) - 1) - 1.$$

Thus,

$$\mathrm{cl}(\mathcal{L}) \leq \max(\deg(\phi_c) - 2, \deg(\psi_{c,q,\omega}) - 1) < \max(\deg(\phi) - 2, \deg(\psi) - 1).$$

We conclude that for any $(\phi, \psi) \in \mathcal{R}$ such that $\mathrm{cl}(\mathcal{L}) = \max(\deg(\phi) - 2, \deg(\psi) - 1)$, then, for any zero, c, of ϕ,

$$|r_{c,q,\omega}| + |\langle \mathcal{L}, \psi_{c,q,\omega}\rangle| \neq 0. \tag{3.42}$$

Conversely, we shall prove that for any $(\phi, \psi) \in \mathcal{R}$ such that (3.42) holds for any zero, c, of ϕ, then $\mathrm{cl}(\mathcal{L}) = \max(\deg(\phi) - 2, \deg(\psi) - 1)$. Let $(\phi_m, \psi_m) \in \mathcal{R}$ such that $\mathrm{cl}(\mathcal{L}) = \max(\deg(\phi_m) - 2, \deg(\psi_m) - 1)$. We assume without loss of generality that $\deg(\phi_m) \leq \deg(\phi)$. We write

$$\phi = \phi_m f + R, \ R, f \in \mathbb{P}, \ \deg(R) < \deg(\phi_m).$$

- If $R \neq 0$, then, from Proposition 3.3, there exists $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{R}$, with $\tilde{\phi} = \mathrm{hcf}(\phi, \phi_m)$ such that

  $$\max(\deg(\tilde{\phi}) - 2, \deg(\tilde{\psi}) - 1) < \max(\deg(\phi_m) - 2, \deg(\psi_m) - 1) = \mathrm{cl}(\mathcal{L}).$$

  This is a contradiction because $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{R}$. Thus, $R = 0$.

- If $\deg(f) \geq 1$, then, it yields from Lemma 3.4, that for any zero, c, of f (then of ϕ),

  $$|r_{c,q,\omega}| + |\langle \mathcal{L}, \psi_{c,q,\omega}\rangle| = 0.$$

  The previous equation contradicts (3.42).

  Finally, f is a constant and we have $\phi = f\phi_m$, $\psi = f\psi_m$. Thus,

  $$\mathrm{cl}(\mathcal{L}) = \max(\deg(\phi) - 2, \deg(\psi) - 1) = \max(\deg(\phi_m) - 2, \deg(\psi_m) - 1).$$

□

The proof of the proposition is therefore complete. It should be noted that the proof of Propositions 2.3, 2.5 and 2.8 are deduced by limit processes (see Remark 3.1).

Remark 3.6 *It follows from the definition of the class of the linear functional that the $D_{q,\omega}$-classical linear functional is a $D_{q,\omega}$-semi-classical linear functional of class $s = 0$.*

Definition 3.7 *The Pearson-type difference equation (3.19) is said to be irreducible on $c \in Z_\phi$ if $|r_{c,q,\omega}| + |\langle \mathcal{L}, \psi_{c,q,\omega} \rangle| \neq 0$. Moreover, (3.19) is said to be irreducible if it is not reducible on any $c \in Z_\phi$.*

Proposition 3.5 *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family and $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ the dual basis associated to $\{P_n\}_{n\in\mathbb{N}}$. If $\{\mathbf{Q}_{n,1}\}_{n\in\mathbb{N}}$ is the dual basis associated to the monic family $\{Q_{n,1}\}_{n\in\mathbb{N}}$ defined by*

$$Q_{n,1} = \frac{D_{q,\omega} P_{n+1}}{[n+1]_q}, \tag{3.43}$$

then, we have

$$D^*_{q,\omega}\mathbf{Q}_{n,1} = -q\,[n+1]_q \mathbf{P}_{n+1}. \tag{3.44}$$

Proof:

$$\begin{aligned} \langle D^*_{q,\omega}\mathbf{Q}_{n,1}, P_{m+1}\rangle &= -q\langle \mathbf{Q}_{n,1}, D_{q,\omega}P_{m+1}\rangle \\ &= -q[m+1]_q\langle \mathbf{Q}_{n,1}, Q_{m,1}\rangle \\ &= -q[n+1]_q\delta_{n,m} \\ &= -q[n+1]_q\langle \mathbf{P}_{n+1}, P_{m+1}\rangle, \end{aligned}$$

then

$$D^*_{q,\omega}\mathbf{Q}_{n,1} = -q\,[n+1]_q \mathbf{P}_{n+1}.$$

□

3.2 Characterisation theorems for $D_{q,\omega}$-semi-classical orthogonal polynomials

3.2.1 $D_{q,\omega}$-classical orthogonal polynomials

Theorem 3.1 *Let $\mathcal{L}$ be a regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family, and $Q_{n,m}$ the monic polynomial of degree n defined by*

$$B_{n,m}(q)\, Q_{n,m} = D^m_{q,\omega} P_{n+m}, \tag{3.45}$$

with

$$B_{n,m}(q) = \prod_{j=0}^{m-1} [n+m-j]_q, \quad Q_{n,0} \equiv P_n \quad \forall n \in \mathbb{N}. \tag{3.46}$$

The following properties are equivalent:

i) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}. \tag{3.47}$$

ii) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m,

$$D_{q,\omega}(\phi\mathcal{L}_m) = \psi_m\mathcal{L}, \tag{3.48}$$

$$\langle \mathcal{L}_m, Q_{j,m}Q_{n,m}\rangle = k_n\delta_{j,n}, \quad \forall j, n \in \mathbb{N}, \ (k_n \neq 0\ \forall n \in \mathbb{N}), \tag{3.49}$$

with the linear functional $\mathcal{L}_m$ and the polynomial ψ_m defined, recursively, by

$$\psi_{m+1} = D_{q,\omega}\phi + qA_{q,\omega}\psi_m, \quad \psi_0 \equiv \psi, \tag{3.50}$$

$$\mathcal{L}_{m+1} = A_{q,\omega}(\phi\mathcal{L}_m), \quad \mathcal{L}_0 \equiv \mathcal{L} \tag{3.51}$$

and given explicitly by

$$\psi_m(x) = \frac{\phi(q^m x + \omega[m]_q) - \phi(x)}{(q-1)x + \omega} + q^m \psi(q^m x + \omega[m]_q). \tag{3.52}$$

$$\mathcal{L}_m = \prod_{j=1}^{m} \phi(q^j x + \omega[j]_q)\, A^m_{q,\omega}\mathcal{L}. \tag{3.53}$$

iii) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that for any integer m, the following second-order difference equation holds:

$$\phi\, D_{q,\omega}\mathcal{D}^*_{q,\omega}\, Q_{n,m} + \psi_m\, D_{q,\omega}\, Q_{n,m} + \lambda_{n,m}\, Q_{n,m} = 0 \quad \forall n \in \mathbb{N}, \tag{3.54}$$

with the polynomial ψ_m given by (3.52) and the constant $\lambda_{n,m}$ given by

$$\lambda_{n,m} = -[n]_q \{\mathcal{D}_q \psi_m + [n-1]_{\frac{1}{q}} \frac{\phi''}{2q}\}. \tag{3.55}$$

iv) There exist two polynomials, ϕ of degree at most two and ψ of degree one, such that, for any integer m, the following relation holds:

$$[n]_q D^*_{q,\omega}\, [Q_{n-1,m+1}\mathcal{L}_{m+1}] = -q\lambda_{n,m} Q_{n,m}\mathcal{L}_m \quad \forall n \in \mathbb{N}, \tag{3.56}$$

with the polynomial ψ_m, the linear functional $\mathcal{L}_m$ and the constant $\lambda_{n,m}$ given, respectively, by (3.52), (3.53) and (3.55).

v) There exist a polynomial ϕ of degree at most two and three constants $c_{n,n+1}$, $c_{n,n}$, $c_{n,n-1}$ with $c_{n,n-1} \neq 0$ such that

$$\phi D^*_{q,\omega} P_n = c_{n,n+1} P_{n+1} + c_{n,n} P_n + c_{n,n-1} P_{n-1}, \quad n \geq 1. \tag{3.57}$$

vi) For any non-zero integer m, there exist a sequence of complex numbers $\{u_{n,m}\}_{n\in\mathbb{N}}$ such that

$$Q_{n,m-1} = Q_{n,m} + u_{n-1,m} Q_{n-1,m} + v_{n-2,m} Q_{n-2,m}, \quad \forall n \in \mathbb{N} - \{0,1\}. \tag{3.58}$$

Proof: $i) \Longrightarrow ii)$. Suppose that the property i) is satisfied. We will show by induction on m that the relations (3.48) and (3.49) hold. From (3.47) and the orthogonality of the family $\{P_n\}_{n\in\mathbb{N}}$, it is obvious that the relations (3.48) and (3.49) are satisfied for $m = 0$. Suppose that relations (3.48) and (3.49) are satisfied up to a fixed integer m. Using Proposition 3.1, we have

$$\begin{aligned}
D_{q,\omega}(\phi\mathcal{L}_{m+1}) &= D_{q,\omega}(\phi A_{q,\omega}(\phi\mathcal{L}_m)) \\
&= D_{q,\omega}\phi A_{q,\omega}(\phi\mathcal{L}_m) + A_{q,\omega}\phi\, D_{q,\omega} A_{q,\omega}(\phi\mathcal{L}_m) \\
&= D_{q,\omega}\phi A_{q,\omega}(\phi\mathcal{L}_m) + A_{q,\omega}\phi\, q\, A_{q,\omega} D_{q,\omega}(\phi\mathcal{L}_m) \\
&= D_{q,\omega}\phi A_{q,\omega}(\phi\mathcal{L}_m) + q\, A_{q,\omega}\phi\, A_{q,\omega}(\psi_m\mathcal{L}_m) \\
&= D_{q,\omega}\phi A_{q,\omega}(\phi\mathcal{L}_m) + q\, A_{q,\omega}\psi_m\, A_{q,\omega}(\phi\mathcal{L}_m) \\
&= (D_{q,\omega}\phi + q\, A_{q,\omega}\psi_m)\, A_{q,\omega}(\phi\mathcal{L}_m) \\
&= \psi_{m+1}\mathcal{L}_{m+1}.
\end{aligned}$$

Thus, the relation (3.48) holds for all integers m.

Let j and n be two integers such that $j < n$. Using Proposition 3.1 and the fact that (3.48) and (3.49) hold up to a fixed integer m, we have

$$\begin{aligned}
&[j+1]_q [n+1]_q \langle \mathcal{L}_{m+1}, Q_{j,m+1} Q_{n,m+1}\rangle \\
&= \langle A_{q,\omega}(\phi\mathcal{L}_m), D_{q,\omega} Q_{j+1,m} D_{q,\omega} Q_{n+1,m}\rangle \\
&= \langle A_{q,\omega}(\phi\mathcal{L}_m), D_{q,\omega}[Q_{n+1,m} D_{q,\omega} Q_{j+1,m}] - A_{q,\omega} Q_{n+1,m} D^2_{q,\omega} Q_{j+1,m}\rangle \\
&= -\frac{1}{q}\langle D^*_{q,\omega} A_{q,\omega}(\phi\mathcal{L}_m), Q_{n+1,m} D_{q,\omega} Q_{j+1,m}\rangle - \frac{1}{q}\langle \phi\mathcal{L}_m, Q_{n+1,m} A^*_{q,\omega} D^2_{q,\omega} Q_{j+1,m}\rangle
\end{aligned}$$

$$= -\langle D_{q,\omega}(\phi\mathcal{L}_m), Q_{n+1,m} D_{q,\omega} Q_{j+1,m}\rangle - \frac{1}{q}\langle \mathcal{L}_m, Q_{n+1,m}\phi A^*_{q,\omega} D^2_{q,\omega} Q_{j+1,m}\rangle$$
$$= -\langle \mathcal{L}_m, \psi_m Q_{n+1,m} D_{q,\omega} Q_{j+1,m}\rangle - \frac{1}{q}\langle \mathcal{L}_m, Q_{n+1,m}\phi A^*_{q,\omega} D^2_{q,\omega} Q_{j+1,m}\rangle$$
$$= 0.$$

because $\deg(\phi A^*_{q,\omega} D^2_{q,\omega} Q_{j-1,m}) \leq \deg(\psi_m D_{q,\omega} Q_{j+1,m}) = j+1 < n+1$ (see Lemma 3.5).

Repeated use of the following relations, proved in Lemma 3.5,

$$[n+1]_q[n+1]_q \langle \mathcal{L}_{m+1}, Q_{n,m+1}Q_{n,m+1}\rangle = \lambda_{n+1,m}\langle \mathcal{L}_m, Q_{n+1,m}Q_{n+1,m}\rangle, \tag{3.59}$$

$$\lambda_{n+1,m} = \frac{[n+1]_q q^{2m}}{[n+1+2m]_q}\lambda_{n+1+2m,0} \neq 0 \quad \forall n, m \in \mathbb{N} \tag{3.60}$$

gives

$$\langle \mathcal{L}_m, Q_{n,m}Q_{n,m}\rangle = \prod_{j=0}^{m-1} \frac{\lambda_{n+1+j,m-1-j}}{[n+j+1]_q^2}\langle \mathcal{L}, P_{n+m}P_{n+m}\rangle. \tag{3.61}$$

Thus,

$$\langle \mathcal{L}_m, Q_{n,m}Q_{n,m}\rangle \neq 0 \quad \forall n, m \in \mathbb{N}.$$

Iteration of relations (3.50) and taking into account (3.9) lead to

$$\begin{aligned}
\psi_m &= \sum_{j=0}^{m-1} q^j A^j_{q,\omega} D_{q,\omega}\phi(x) + q^m A^m_{q,\omega}\psi(x) \\
&= \sum_{j=0}^{m-1} q^j q^{-j} D_{q,\omega} A^j_{q,\omega}\phi(x) + q^m A^m_{q,\omega}\psi(x) \\
&= \sum_{j=0}^{m-1} \frac{A^{j+1}_{q,\omega}\phi(x) - A^j_{q,\omega}\phi(x)}{(q-1)x+\omega} + q^m A^m_{q,\omega}\psi(x).
\end{aligned}$$

Thus,

$$\psi_m(x) = \frac{\phi(q^m x + \omega[m]_q) - \phi(x)}{(q-1)x+\omega} + q^m\psi(q^m x + \omega[m]_q).$$

Taking into account Remark 3.5, relation (3.53) follows directly from the iteration of (3.51).

ii) $\Longrightarrow$ *iii)*. Assuming that the property ii) holds, it follows that for any integer m, the monic polynomial family $\{Q_{n,m}\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}_m$; thus, $\{Q_{n,m}\}_{n\in\mathbb{N}}$ forms a basis of $\mathcal{P}$.

From the following expansion

$$\phi D_{q,\omega} D^*_{q,\omega} Q_{n,m} + \psi_m D_{q,\omega} Q_{n,m} = -\sum_{j=0}^{n} \lambda_{j,m} Q_{j,m}, \tag{3.62}$$

we obtain

$$\begin{aligned}
\lambda_{j,m}\langle \mathcal{L}_m, Q_{j,m}Q_{j,m}\rangle &= -\langle \mathcal{L}_m, \phi Q_{j,m} D_{q,\omega} D^*_{q,\omega} Q_{n,m} + \psi_m Q_{j,m} D_{q,\omega} Q_{n,m}\rangle \\
&= -\langle \mathcal{L}_m, \phi Q_{j,m} D_{q,\omega} D^*_{q,\omega} Q_{n,m}\rangle - \langle \psi_m \mathcal{L}_m, Q_{j,m} D_{q,\omega} Q_{n,m}\rangle \\
&= -\langle \mathcal{L}_m, \phi Q_{j,m} D_{q,\omega} D^*_{q,\omega} Q_{n,m}\rangle - \langle D_{q,\omega}(\phi\mathcal{L}_m), Q_{j,m} D_{q,\omega} Q_{n,m}\rangle \\
&= -\langle \mathcal{L}_m, \phi Q_{j,m} D_{q,\omega} D^*_{q,\omega} Q_{n,m}\rangle + \frac{1}{q}\langle \phi\mathcal{L}_m, D^*_{q,\omega}(Q_{j,m} D_{q,\omega} Q_{n,m})\rangle \\
&= -\langle \mathcal{L}_m, \phi Q_{j,m} D_{q,\omega} D^*_{q,\omega} Q_{n,m}\rangle + \frac{1}{q}\langle \phi\mathcal{L}_m, Q_{j,m} D^*_{q,\omega} D_{q,\omega} Q_{n,m}\rangle \\
&\quad + \frac{1}{q}\langle \phi\mathcal{L}_m, D^*_{q,\omega} Q_{j,m} A^*_{q,\omega} D_{q,\omega} Q_{n,m}\rangle \\
&= -\langle \mathcal{L}_m, \phi Q_{j,m} D_{q,\omega} D^*_{q,\omega} Q_{n,m}\rangle - \langle \phi\mathcal{L}_m, Q_{j,m} D_{q,\omega} D^*_{q,\omega} Q_{n,m}\rangle
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{q}\langle \phi\,\mathcal{L}_m, D^*_{q,\omega}Q_{j,m}A^*_{q,\omega}D_{q,\omega}Q_{n,m}\rangle \\
= & +\frac{1}{q}\langle \phi\,\mathcal{L}_m, A^*_{q,\omega}D_{q,\omega}Q_{j,m}A^*_{q,\omega}D_{q,\omega}\,Q_{n,m}\rangle \\
= & +\frac{1}{q}\langle \phi\,\mathcal{L}_m, A^*_{q,\omega}(D_{q,\omega}Q_{j,m}D_{q,\omega}\,Q_{n,m})\rangle \\
= & \ \langle A_{q,\omega}(\phi\,\mathcal{L}_m), D_{q,\omega}Q_{j,m}D_{q,\omega}Q_{n,m}\rangle \\
= & \ [j]_q[n]_q\langle \mathcal{L}_{m-1}, Q_{j-1,m+1}\,Q_{n-1,m+1}\rangle \\
= & \ 0 \text{ for } j<n,
\end{aligned}
$$

by orthogonality of $\{Q_{n,m+1}\}_{n\in\mathbb{N}}$ with respect to $\mathcal{L}_{m+1}$. Thus,

$$\phi\,D_{q,\omega}D^*_{q,\omega}\,Q_{n,m} + \psi_m\,D_{q,\omega}\,Q_{n,m} + \lambda_{n,m}\,Q_{n,m} = 0 \qquad \forall n\in\mathbb{N}.$$

Identification of the coefficients of x^n in the previous equation gives

$$\frac{\phi''}{2}[n-1]_q[n]_{\frac{1}{q}} + \psi'_m[n]_q - \lambda_{n,m} = 0.$$

Then using the following relation

$$[n]_{\frac{1}{q}} = q^{1-n}[n]_q, \quad \forall n\in\mathbb{N}, \tag{3.63}$$

we obtain

$$\lambda_{n,m} = -[n]_q(\psi'_m + [n-1]_{\frac{1}{q}}\frac{\phi''}{2q}).$$

iii) $\Longrightarrow$ *i)*. Assuming that the property iii) holds, elementary computations using (3.12) and (3.54) for $m=1$ give

$$
\begin{aligned}
[n+1]_q\langle D_{q,\omega}(\phi\mathcal{L}) - \psi\mathcal{L}, Q_{n,1}\rangle &= -\frac{1}{q}\langle \phi\mathcal{L}, D^*_{q,\omega}D_{q,\omega}\,P_{n+1}\rangle - \langle \psi\mathcal{L}, D_{q,\omega}\,P_{n+1}\rangle \\
&= -\langle \mathcal{L}, \phi\,D_{q,\omega}D^*_{q,\omega}\,P_{n+1} + \psi\,D_{q,\omega}\,P_{n+1}\rangle \\
&= \langle \mathcal{L}, \lambda_{n+1,0}\,P_{n+1}\rangle \\
&= 0 \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{3.64}
$$

Since the family $\{Q_{n,1}\}_{n\in\mathbb{N}}$ forms a basis of $\mathcal{P}'$, it is clear that $D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}$.

iii) $\Longleftrightarrow$ *vi)*. Computations using Proposition 3.1 show straightforwardly that given an integer m, (3.56) is equivalent to

$$(\phi\,D_{q,\omega}D^*_{q,\omega}\,Q_{n,m} + \psi_m\,D_{q,\omega}\,Q_{n,m} - \lambda_{n,m}\,Q_{n,m})\mathcal{L}_m = 0 \qquad \forall n\in\mathbb{N}.$$

Since $\mathcal{L}_m$ is regular (see property ii)) it is obvious that properties iii) and iv) are equivalent.

i) $\Longrightarrow$ *v)*. Expanding the polynomial $\phi(x)D^*_{q,\omega}P_n$ in the basis $\{P_n\}_{n\in\mathbb{N}}$ of $\mathcal{P}$,

$$\phi(x)D^*_{q,\omega}P_n = \sum_{j=0}^{n+1} c_{n,j}P_j,$$

we obtain, using (3.10),

$$
\begin{aligned}
c_{n,j}\langle \mathcal{L}, P_jP_j\rangle &= \langle \phi\mathcal{L}, P_jD^*_{q,\omega}P_n\rangle \\
&= \langle \phi\mathcal{L}, D^*_{q,\omega}(A_{q,\omega}P_jP_n) - P_nD^*_{q,\omega}A_{q,\omega}P_j\rangle \\
&= -q\langle D_{q,\omega}(\phi\mathcal{L}), A_{q,\omega}P_jP_n\rangle - q\langle \phi\mathcal{L}, P_nD_{q,\omega}P_j\rangle \\
&= -q\langle \mathcal{L}, (\psi A_{q,\omega}P_j - \phi D_{q,\omega}P_j)P_n\rangle \\
&= 0 \quad \text{for } j<n-1.
\end{aligned}
$$

Thus,

$$\phi(x)D^*_{q,\omega}P_n = c_{n,n+1}P_{n+1} + c_{n,n}P_n + c_{n,n-1}P_{n-1}.$$

When we set $j = n-1$ in the above equations we obtain, taking into account (3.63),

$$\begin{aligned} c_{n,n-1}\langle\mathcal{L}, P_{n-1}P_{n-1}\rangle \\ &= -q\langle\mathcal{L}, (\psi A_{q,\omega}P_{n-1} + \phi D_{q,\omega}P_{n-1})P_n\rangle \\ &= -q(q^{n-1}\psi' + [n-1]_q\frac{\phi''}{2})\langle\mathcal{L}, x^nP_n\rangle \\ &= -q^n(\psi' + [n-1]_{\frac{1}{q}}\frac{\phi''}{2q})\langle\mathcal{L}, P_nP_n\rangle \\ &= q^n\frac{\lambda_{n,0}}{[n]_q}\langle\mathcal{L}, P_nP_n\rangle \\ &\neq 0 \text{ for } n \geq 1 \end{aligned}$$

by relation (3.74).

$v) \Longrightarrow i)$. Expanding the linear functional $D_{q,\omega}(\phi\mathcal{L})$ in the dual basis $\{\mathbf{P}_n\}_{n\in n}$ of the orthogonal family $\{P_n\}_{n\in\mathbb{N}}$,

$$D_{q,\omega}(\phi\mathcal{L}) = \sum_{n\geq 0} h_n\mathbf{P}_n, \tag{3.65}$$

we obtain, using (3.57),

$$\begin{aligned} h_n &= \langle D_{q,\omega}(\phi\mathcal{L}), P_n\rangle \\ &= -\frac{1}{q}\langle\mathcal{L}, \phi D^*_{q,\omega}P_n\rangle \\ &= -\frac{1}{q}\langle\mathcal{L}, c_{n,n+1}P_{n+1} + c_{n,n}P_n + c_{n,n-1}P_{n-1}\rangle \\ &= 0 \text{ for } n \geq 2, \end{aligned}$$

then

$$h_n = 0 \text{ for } n \geq 2. \tag{3.66}$$

On the other hand,

$$h_0 = \langle D_{q,\omega}(\phi\mathcal{L}), 1\rangle = -\frac{1}{q}\langle\phi\mathcal{L}, D^*_{q,\omega}1\rangle = 0, \tag{3.67}$$

$$h_1 = -\frac{c_{1,0}}{q}\langle\mathcal{L}, P_0P_0\rangle \neq 0 \tag{3.68}$$

because by hypothesis, $c_{n,n-1} \neq 0$ for $n \geq 1$.

Use of (3.65)-(3.68) and the fact that $\mathbf{P}_1 = \frac{P_1}{\langle\mathcal{L}, P_1P_1\rangle}\mathcal{L}$ (see (2.17)) give

$$D_{q,\omega}(\phi\mathcal{L}) = h_1\mathbf{P}_1 = -\frac{c_{1,0}\langle\mathcal{L}, P_0P_0\rangle P_1}{q\langle\mathcal{L}, P_1P_1\rangle}\mathcal{L}. \tag{3.69}$$

Then the regular linear functional $\mathcal{L}$ satisfies $D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}$, where

$$\psi(x) = -\frac{c_{1,0}\langle\mathcal{L}, P_0P_0\rangle}{q\langle\mathcal{L}, P_1P_1\rangle}P_1(x),$$

with $\deg(\phi) \leq 2$ and $\deg(\psi) = \deg(P_1) = 1$. Thus $\mathcal{L}$ is $D_{q,\omega}$-classical, and therefore properties $i)$ and $v)$ are equivalent.

$i) \Longrightarrow vi)$. Since properties i) and ii) are equivalent, assuming that the property i) is satisfied, it yields that for any integer m the monic polynomial family $\{Q_{n,m}\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}_m$. Let m be a non-zero integer. We expand the polynomial $Q_{n,m-1}$ in the basis $\{Q_{n,m}\}_{n\in\mathbb{N}}$ of $\mathbb{P}$,

$$Q_{n,m-1} = Q_{n,m} + \sum_{j=0}^{n-1} u_{j,m}Q_{j,m}$$

and obtain

$$\begin{aligned}
u_{j,m}[j+1]_q\langle\mathcal{L}_m, Q_{j,m}Q_{j,m}\rangle &= [j+1]_q\langle\mathcal{L}_m, Q_{n,m-1}Q_{j,m}\rangle \\
&= \langle\mathcal{L}_m, D_{q,\omega}Q_{j+1,m-1}Q_{n,m-1}\rangle \\
&= \langle A_{q,\omega}(\phi\mathcal{L}_{m-1}), D_{q,\omega}(Q_{j+1,m-1}Q_{n,m-1})\rangle \\
&\quad -\langle A_{q,\omega}(\phi\mathcal{L}_{m-1}), A_{q,\omega}Q_{j+1,m-1}D_{q,\omega}Q_{n,m-1}\rangle \\
&= -\frac{1}{q}\langle D^*_{q,\omega}A_{q,\omega}(\phi\mathcal{L}_{m-1}), Q_{j+1,m-1}Q_{n,m-1}\rangle \\
&\quad -[n]_q\langle\mathcal{L}_m, A_{q,\omega}Q_{j+1,m-1}Q_{n-1,m}\rangle \\
&= -\langle D_{q,\omega}(\phi\mathcal{L}_{m-1}), Q_{j+1,m-1}Q_{n,m-1}\rangle \\
&\quad -[n]_q\langle\mathcal{L}_m, A_{q,\omega}Q_{j+1,m-1}Q_{n-1,m}\rangle \\
&= -\langle\mathcal{L}_{m-1}, \psi_{m-1}Q_{j+1,m-1}Q_{n,m-1}\rangle \\
&\quad -[n]_q\langle\mathcal{L}_m, A_{q,\omega}Q_{j+1,m-1}Q_{n-1,m}\rangle \\
&= 0 \text{ for } j < n-2,
\end{aligned}$$

by the orthogonality of $\{Q_{n,m}\}_{n\in\mathbb{N}}$ and $\{Q_{n,m-1}\}_{n\in\mathbb{N}}$ with respect to $\mathcal{L}_m$ and $\mathcal{L}_{m-1}$, respectively. Therefore,

$$Q_{n,m-1} = Q_{n,m} - u_{n-1,m}Q_{n-1,m} + u_{n-2,m}Q_{n-2,m} \quad \forall n\in\mathbb{N}-\{0,1\} \quad \forall m\in\mathbb{N}-\{0\}.$$

$iv) \Longrightarrow i)$. Let $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{Q}_{n,m}\}_{n\in\mathbb{N}}$ be the dual basis associated to the monic families $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_{n,m}\}_{n\in\mathbb{N}}$, respectively.

In the first step we expand $\mathbf{Q}_{0,1}$ in the dual basis $\{\mathbf{Q}_{n,0}\}_{n\in\mathbb{N}}$,

$$\mathbf{Q}_{0,1} = \sum_{j\geq 0}\alpha_j\mathbf{Q}_{j,0}$$

and obtain, using, (3.58)

$$\begin{aligned}
\alpha_j &= \langle\mathbf{Q}_{0,1}, Q_{j,0}\rangle \\
&= \langle\mathbf{Q}_{0,1}, Q_{j,1} - u_{j-1,1}Q_{j-1,1} + u_{j-2,1}Q_{j-2,1}\rangle \\
&= 0 \text{ for } j \geq 3.
\end{aligned}$$

Using (2.17), we, therefore, obtain,

$$\mathbf{Q}_{0,1} = \sum_{j=0}^{2}\alpha_j\mathbf{Q}_{j,0} = \sum_{j=0}^{2}\alpha_j\mathbf{P}_j = \phi\mathcal{L}, \tag{3.70}$$

where

$$\phi(x) = \sum_{j=0}^{2}\frac{\alpha_j P_j(x)}{\langle\mathcal{L}, P_jP_j\rangle}. \tag{3.71}$$

In the second step, we compute $D^*_{q,\omega}\mathbf{Q}_{0,1}$ using (2.17), (3.44) and obtain

$$D^*_{q,\omega}(\mathbf{Q}_{0,1}) = -q\,\mathbf{P}_1 = \psi\mathcal{L}, \tag{3.72}$$

where

$$\psi(x) = \frac{-qP_1(x)}{\langle\mathcal{L}, P_1P_1\rangle}. \tag{3.73}$$

Use of (3.70)-(3.72) permit us to conclude that

$$D^*_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}.$$

The previous equation, thanks to Proposition 3.2, is equivalent to

$$D_{q,\omega}(\tilde{\phi}\mathcal{L}) = \psi\mathcal{L},$$

with

$$\tilde{\phi} = q\phi - [(q-1)x + \omega]\psi.$$

We complete the proof of vi) $\Longrightarrow$ *i*), by remarking that $\deg(\phi) \leq \deg(P_2) = 2$ and $\deg(\psi) = \deg(P_1) = 1$ (see (3.71) and (3.73)).

Summing up, we have proved that *i*) $\Longrightarrow$ *ii*) $\Longrightarrow$ *iii*) $\Longrightarrow$ *i*), *iii* $\Longleftrightarrow$ *iv*), *i*) $\Longleftrightarrow$ *v*) and *i*) $\Longleftrightarrow$ *vi*); thus, the proof of the theorem is complete. □

Lemma 3.5 *Let $\mathcal{L}$ be a regular linear functional satisfying $D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}$, where ϕ is a polynomial of degree at most two and ψ a first-degree polynomial. The following properties hold:*

$$i) \qquad \lambda_{n+1,0} \neq 0 \quad \forall n \in \mathbb{N}, \tag{3.74}$$

$$ii) \qquad \lambda_{n+1,m} = \frac{[n+1]_q\, q^{2m}}{[n+1+2m]_q}\lambda_{n+1+2m,0}; \tag{3.75}$$

$$iii) \qquad D_{q,\omega}\psi_m \neq 0 \quad \forall m \in \mathbb{N}. \tag{3.76}$$

$$iv) \qquad [n+1]_q^2\langle \mathcal{L}_{m+1}, Q_{n,m+1}Q_{n,m+1}\rangle = \lambda_{n+1,m}\langle \mathcal{L}_m, Q_{n+1,m}Q_{n+1,m}\rangle. \tag{3.77}$$

with $\mathcal{L}_m$, $\lambda_{n,m}$ and $Q_{n,m}$ defined in Theorem 3.1.

Proof: i) From the relation

$$D^{*}_{q,\omega}x^n = [n]_{\frac{1}{q}}\, x^{n-1} + \sum_{j=2}^{n} a^{*}_{n,j}(q,\omega)\, x^{n-j},$$

where $a^{*}_{n,j}(q,\omega)$ are complex numbers given by

$$a^{*}_{n,j}(q,\omega) = \left(\frac{\omega}{q}\right)^{j-1}\sum_{k=0}^{n-1} q^{k}\binom{k}{j-1}, \tag{3.78}$$

we obtain,

$$\begin{aligned}
D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L} &\Longleftrightarrow \langle D_{q,\omega}(\phi\mathcal{L}), x^n\rangle = \langle\psi\mathcal{L}, x^n\rangle \qquad \forall n \in \mathbb{N} \\
&\Longleftrightarrow -\frac{1}{q}\langle \phi\mathcal{L}, D^{*}_{q,\omega}x^n\rangle = \langle\psi\mathcal{L}, x^n\rangle \qquad \forall n \in \mathbb{N} \\
&\Longleftrightarrow (\psi' + [n]_{\frac{1}{q}}\frac{\phi''}{2q})M_{n+1} = \sum_{j=0}^{n}\tilde{f}_j M_j \qquad \forall n \in \mathbb{N} \\
&\Longleftrightarrow -\frac{\lambda_{n+1,0}}{[n+1]_q}M_{n+1} = \sum_{j=0}^{n}\tilde{f}_j M_j \qquad \forall n \in \mathbb{N},
\end{aligned} \tag{3.79}$$

where $M_j \equiv \langle\mathcal{L}, x^j\rangle$ is the moment of order j of the linear functional $\mathcal{L}$ and $\tilde{f}_j$ are complex numbers easily computed as function of coefficients $a^{*}_{n,j}$ and those of the polynomials ϕ and ψ. Since $\mathcal{L}$ is regular, to have all its moments given in the unique way by the previous ones, it is necessary to have

$$\lambda_{n+1,0} \neq 0 \quad \forall n \in \mathbb{N}. \tag{3.80}$$

ii) The $D_{q,\omega}$-derivative of (3.52), taking into account (3.63), gives

$$\begin{aligned}
D_{q,\omega}\psi_m = \psi'_m &= [2m]_q\frac{\phi''}{2} + q^{2m}\psi' \\
&= q^{2m}(\psi' + q^{1-2m}[2m]_q\frac{\phi''}{2q}) \\
&= q^{2m}(\psi' + [2m]_{\frac{1}{q}}\frac{\phi''}{2}).
\end{aligned}$$

then

$$D_{q,\omega}\psi_m = q^{2m}(\psi' + [2m]_{\frac{1}{q}}\frac{\phi''}{2q}) = \frac{-q^{2m}}{[2m+1]_q}\lambda_{2m+1,0} \quad \forall m \in \mathbb{N}. \tag{3.81}$$

We, therefore, conclude using (3.80) that for any integer m, $\psi'_m \neq 0$ and ψ_m is a first-degree polynomial.

iii) use of (3.55), (3.63) and (3.81) give

$$\begin{aligned}
\lambda_{n,m} &= -[n]_q\{\psi'_m + [n-1]_{\frac{1}{q}}\frac{\phi''}{2q}\} \\
&= -[n]_q\{q^{2m}(\psi' + [2m]_{\frac{1}{q}}\frac{\phi''}{2q}) + [n-1]_{\frac{1}{q}}\frac{\phi''}{2q}\} \\
&= -[n]_q q^{2m}\{\psi' + ([2m]_{\frac{1}{q}} + q^{-2m}[n-1]_{\frac{1}{q}})\frac{\phi''}{2q}\} \\
&= -[n]_q\, q^{2m}\{\psi' + [2m+n-1]_{\frac{1}{q}}\frac{\phi''}{2q}\} \\
&= \frac{[n]_q\, q^{2m}}{[n+2m]_q}\lambda_{n+2m,0}.
\end{aligned}$$

We derive the relation iv) using Proposition 3.1, the second property of Theorem 3.1 and the orthogonality of the family $\{Q_{n,m}\}_{n\in\mathbb{N}}$ with respect to $\mathcal{L}_m$. In fact,

$$\begin{aligned}
& [n+1]_q[n+1]_q\langle\mathcal{L}_{m+1}, Q_{n,m+1}Q_{n,m-1}\rangle \\
= \;& \langle A_{q,\omega}(\phi\mathcal{L}_m), D_{q,\omega}Q_{n+1,m}D_{q,\omega}Q_{n+1,m}\rangle \\
= \;& \langle A_{q,\omega}(\phi\mathcal{L}_m), D_{q,\omega}(Q_{n-1,m}D_{q,\omega}Q_{n+1,m}) - A_{q,\omega}Q_{n+1,m}D^2_{q,\omega}Q_{n+1,m}\rangle \\
= \;& -\frac{1}{q}\langle D^*_{q,\omega}A_{q,\omega}(\phi\mathcal{L}_m), Q_{n+1,m}D_{q,\omega}Q_{n+1,m}\rangle \\
& -\frac{1}{q}\langle\phi\mathcal{L}_m - Q_{n+1,m}A^*_{q,\omega}D^2_{q,\omega}Q_{n+1,m}\rangle \\
= \;& -\langle\mathcal{L}_m, \psi_m Q_{n-1,m}D_{q,\omega}Q_{n-1,m}\rangle - \frac{1}{q}\langle\mathcal{L}_m, \phi Q_{n+1,m}D^*_{q,\omega}D_{q,\omega}Q_{n+1,m}\rangle \\
= \;& -[n-1]_q\psi'_m\langle\mathcal{L}_m, Q_{n+1,m}Q_{n+1,m}\rangle - [n+1]_q[n]_{\frac{1}{q}}\frac{\phi''}{2q}\langle\mathcal{L}_m, Q_{n+1,m}Q_{n+1,m}\rangle \\
= \;& -[n-1]_q\{\psi'_m + [n]_{\frac{1}{q}}\frac{\phi''}{2q}\}\langle\mathcal{L}_m, Q_{n+1,m}Q_{n+1,m}\rangle \\
= \;& \lambda_{n+1,m}\langle\mathcal{L}_m, Q_{n+1,m}Q_{n+1,m}\rangle.
\end{aligned}$$

□

3.2.2 $D_{q,\omega}$-semi-classical orthogonal polynomials

Let $\mathcal{L}$ be a regular linear functional and $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family. When the linear functional $\mathcal{L}$ is $D_{q,\omega}$-semi-classical of class $s > 0$ satisfying (3.47), the characterisation theorem (see Theorem 3.1) is not valid anymore. In particular, the derivative $D_{q,\omega}P_n$ of P_n is not orthogonal with respect to $A_{q,\omega}(\phi\mathcal{L})$ but quasi-orthogonal of class s with respect to $A_{q,\omega}(\phi\mathcal{L})$. The following theorem, which generalise some results in [Salto, 1995] and [Medem, 1996], gives some characterisations for $D_{q,\omega}$-semi-classical orthogonal polynomials.

Theorem 3.2 *Let $\mathcal{L}$ be a regular linear functional and $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal family. The following properties are equivalent:*

i) There exist two polynomials: ψ of degree at least one and ϕ such that

$$D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}. \tag{3.82}$$

ii) There exists a polynomials $\phi \neq 0$ and an integer s with $\deg\phi \leq s+2$ such that

$$\begin{cases} \langle A_{q,\omega}(\phi\mathcal{L}), Q_{m,1}Q_{n,1}\rangle = 0, & |n-m| > s \\ \langle A_{q,\omega}(\phi\mathcal{L}), Q_{m,1}Q_{m+s,1}\rangle \neq 0, & \forall m \geq 1, \end{cases} \tag{3.83}$$

where polynomials $Q_{n,1}$ are defined in theorem 3.1.

iii) There exists a polynomial $\phi \neq 0$ and an integer s with $t = \deg\phi \leq s+2$ such that

$$\phi D^*_{q,\omega} P_n = \sum_{j=n-s-1}^{n+t-1} \xi_{n,j} P_j \quad n > s+1, \tag{3.84}$$

with

$$\xi_{n,n-s-1} \neq 0, \ n > s+1. \tag{3.85}$$

Proof: $i) \Longrightarrow ii)$. Suppose that (3.82) is satisfied. Then $\phi \neq 0$ by Lemma 3.1 and (3.82).

Let m and n be two integers such that $n > m+s$ and pose $s = \max\{\deg(\phi)-2, \deg(\psi)-1\}$. Using (3.82) and Proposition 1.3, we get

$$\begin{aligned}
&[m+1]_q [n+1]_q \langle A_{q,\omega}(\phi\mathcal{L}), Q_{n,1}Q_{m,1}\rangle \\
&= \langle A_{q,\omega}(\phi\mathcal{L}), D_{q,\omega} P_{m+1} D_{q,\omega} P_{n+1}\rangle \\
&= \langle A_{q,\omega}(\phi\mathcal{L}), D_{q,\omega} [P_{n+1} D_{q,\omega} P_{m+1}] - A_{q,\omega} P_{n+1} D^2_{q,\omega} P_{m+1}\rangle \\
&= -\frac{1}{q}\langle D^*_{q,\omega} A_{q,\omega}(\phi\mathcal{L}), P_{n+1} D_{q,\omega} P_{m-1}\rangle - \frac{1}{q}\langle \phi\mathcal{L}, P_{n+1} A^*_{q,\omega} D^2_{q,\omega} P_{m+1}\rangle \\
&= -\langle \mathcal{L}, P_{n+1}\psi D_{q,\omega} P_{m+1}\rangle - \frac{1}{q}\langle \mathcal{L}, P_{n+1}\phi A^*_{q,\omega} D^2_{q,\omega} P_{m+1}\rangle \\
&= 0,
\end{aligned}$$

because $\deg(\frac{1}{q}\phi A^*_{q,\omega} D^2_{q,\omega} P_{m+1} + \psi D_{q,\omega} P_{m+1}) \leq m+s+1 < n+1$.

Given non-zero integer m, we have, from the previous computations,

$$\begin{aligned}
&\langle A_{q,\omega}(\phi\mathcal{L}), D_{q,\omega} P_{m+1} D_{q,\omega} P_{m-s+1}\rangle \\
&= -\langle \mathcal{L}, P_{m+s+1}\psi D_{q,\omega} P_{m+1}\rangle - \frac{1}{q}\langle \mathcal{L}, P_{m+s+1}\phi A^*_{q,\omega} D^2_{q,\omega} P_{m+1} \\
&= -\{[m+1]_q \psi_p \delta_{p,s+1} + [m]_q [m+1]_q q^{1-m} \frac{\phi_t}{q}\delta_{t,s+2}\} I_{0,m-s+1} \\
&= -[m+1]_q \{\psi_p \delta_{p,s+1} + \frac{\phi_t}{q}[m]_{\frac{1}{q}} \delta_{t,s+2}\} I_{0,m+s+1},
\end{aligned} \tag{3.86}$$

where $I_{0,m}$ is defined by

$$I_{0,m} = \langle \mathcal{L}, P_m P_m\rangle, \ m \geq 0, \tag{3.87}$$

and the polynomials ϕ and ψ are given by

$$\phi(x) = \sum_{j=0}^{t} \phi_j x^j, \ \psi(x) = \sum_{j=0}^{p} \psi_j x^j, \tag{3.88}$$

with $|\phi_t||\psi_p| \neq 0$.

It results from (3.86) that $\langle A_{q,\omega}(\phi\mathcal{L}), D_{q,\omega} P_{m+1} D_{q,\omega} P_{m+s+1}\rangle (I_{0,n+s+1})^{-1} = U(m,s)$, takes one of the three values:

i): $t < s+2$, $p = s+1$.

$$U(m,s) = -[m+1]_q \psi_{s-1} \neq 0, \ m \geq 0,$$

ii): $t = s+2$, $p < s+1$.

$$U(m,s) = -[m]_{\frac{1}{q}} [m+1]_q \frac{\phi_{s+2}}{q} \neq 0, \ m \geq 1,$$

iii): $t = s+2,\ p = s+1,$

$$U(m,s) = -[m+1]_q\,\{\psi_{s+1} + \frac{\phi_{s+2}}{q}[m]_{\frac{1}{q}}\}.$$

$U(m,s)$ for the case iii) is not zero by the regularity of the linear functional $\mathcal{L}$.

In fact, mimicking the approach used in (3.79), we conclude that if $\mathcal{L}$ is regular and satisfies (3.82), with ϕ and ψ given by (3.88) and $t = p+1$, then we have

$$\psi_p + \frac{\phi_{p+1}}{q}[m]_{\frac{1}{q}} \neq 0 \quad \forall m \in \mathbb{N}. \tag{3.89}$$

We deduce that $\langle A_{q,\omega}(\phi\mathcal{L}), Q_{m,1}Q_{m+s,1}\rangle \neq 0 \quad \forall m \geq 1$ and therefore that the property ii) is fulfilled.

ii) $\Longrightarrow$ *iii)*. We assume that *ii)* holds and expand $\phi D^{*}_{q,\omega}P_n$ on the basis $\{P_n\}_{n\in\mathbb{N}}$

$$\phi D^{*}_{q,\omega}P_n \sum_{j=0}^{n+t} \xi_{n,j}\,P_j,$$

where $t = \deg(\phi)$, and get

$$\begin{aligned}
\xi_{n,j}I_{0,j} &= \langle \phi\mathcal{L}, P_j\,D^{*}_{q,\omega}P_n\rangle \\
&= \langle \phi\mathcal{L}, A^{*}_{q,\omega}(A_{q,\omega}P_j\,D_{q,\omega}P_n)\rangle \\
&= q[n]_q\langle A_{q,\omega}(\phi\mathcal{L}).\, A_{q,\omega}P_j\,Q_{n-1,1}\rangle \\
&= 0 \text{ for } n > j+s+1,
\end{aligned}$$

by (3.83).

Moreover, for $n > s+1$,

$$\xi_{n,n-s-1}I_{0,n-s-1} = q[n]_q\langle A_{q,\omega}(\phi\mathcal{L}), A_{q,\omega}P_{n-s-1}\,Q_{n-1,1}\rangle \neq 0$$

also by (3.83).

iii) $\Longrightarrow$ *i)*.

Let $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$ be the dual basis associated to the monic family $\{P_n\}_{n\in\mathbb{N}}$ and t the degree of ϕ.

We expand the linear functional $D_{q,\omega}(\phi\mathcal{L})$ in the basis $\{\mathbf{P}_n\}_{n\in\mathbb{N}}$

$$D_{q,\omega}(\phi\mathcal{L}) = \sum_{n\geq 0}\alpha_n\,\mathbf{P}_n$$

and get

$$\begin{aligned}
\alpha_n &= \langle D_{q,\omega}(\phi\mathcal{L}), P_n\rangle \\
&= -\frac{1}{q}\,\langle \mathcal{L}, \phi D^{*}_{q,\omega}\,P_n\rangle \\
&= -\frac{1}{q}\langle \mathcal{L}, \sum_{j=n-s-1}^{s+t-1} \xi_{n,j}P_j\rangle \\
&= 0 \text{ for } n > s+1.
\end{aligned}$$

Then

$$\begin{aligned}
D_{q,\omega}(\phi\mathcal{L}) &= \sum_{j=0}^{s+1}\alpha_j\,\mathbf{P}_j \\
&= \sum_{j=0}^{s+1}\alpha_j\,\frac{P_j}{\langle \mathcal{L}, P_jP_j\rangle}\mathcal{L} \\
&= \psi\mathcal{L}
\end{aligned}$$

thanks to Proposition 2.1. We deduce from the previous equations, Lemma 3.1 and the fact that $\phi \neq 0$, that ψ is of degree at least one. The linear functional $\mathcal{L}$ is, therefore, $D_{q,\omega}$-semi-classical. □

Chapter 4

The formal Stieltjes function

4.1 The Stieltjes function and the Riccati difference equation

4.1.1 Some definitions

Definition 4.1 *The formal Stieltjes function $S(\mathcal{L})$ of a given linear functional $\mathcal{L} \in \mathbb{P}'$ is defined by*

$$S(\mathcal{L})(x) = -\sum_{k\geq 0} \frac{(\mathcal{L})_k}{x^{k+1}}, \tag{4.1}$$

where $(\mathcal{L})_k = \langle \mathcal{L}, x^k \rangle$, represents the moment of order k of the linear functional $\mathcal{L}$ with respect to the sequence $\{x^n\}_{n\geq 0}$.

We define the action of the operators $\mathcal{D}$, $\mathcal{T}_\omega$, D_ω, $\mathcal{G}_q$, $\mathcal{D}_q$, $\mathcal{D}$, $A_{q,\omega}$ and $D_{q,\omega}$ on the Stieltjes function $S(\mathcal{L})$ as is done in [Medem, 1996] (for more information see [Medem, 1996, p. 357]).

Definition 4.2 (Medem, 1996) *The operators $\mathcal{T}_\omega$, D_ω, $\mathcal{G}_q$, $\mathcal{D}_q$, $\mathcal{D}$, $A_{q,\omega}$ and $D_{q,\omega}$ act on the Stieltjes function $S(\mathcal{L})$ of the linear functional $\mathcal{L}$ in the following ways:*

$$\begin{aligned}
\mathcal{T}_\omega S(\mathcal{L})(x) &= S(\mathcal{L})(x+\omega) = -\sum_{n\geq 0} \frac{(\mathcal{L})_n}{(x+\omega)^{n+1}}, \\
\mathcal{G}_q S(\mathcal{L})(x) &= S(\mathcal{L})(q\,x) = -\sum_{n\geq 0} \frac{(\mathcal{L})_n}{q^{n+1}\,x^{n+1}}, \; q \neq 0. \\
A_{q,\omega} S(\mathcal{L})(x) &= S(\mathcal{L})(q\,x+\omega) - \sum_{n\geq 0} \frac{(\mathcal{L})_n}{(qx+\omega)^{n+1}}, \\
\mathcal{D} S(\mathcal{L})(x) &= \sum_{n\geq 0} (n+1) \frac{(\mathcal{L})_n}{x^{n+2}}, \\
\mathcal{D}_q S(\mathcal{L})(x) &= \frac{S(\mathcal{L})(qx) - S(\mathcal{L})(x)}{(q-1)x} = \sum_{n\geq 0} \frac{[n+1]_q\,(\mathcal{L})_n}{q^{n+1}\,x^{n+2}}. \\
D_\omega S(\mathcal{L})(x) &= \frac{S(\mathcal{L})(x+\omega) - S(\mathcal{L})(x)}{\omega} = -\sum_{n\geq 0} (\mathcal{L})_n \frac{1}{\omega} \left(\frac{1}{(x+\omega)^{n+1}} - \frac{1}{x^{n+1}} \right), \\
D_{q,\omega} S(\mathcal{L})(x) &= \frac{S(\mathcal{L})(q\,x+\omega) - S(\mathcal{L})(x)}{(q-1)\,x+\omega} = -\sum_{n\geq 0} (\mathcal{L})_n \frac{1}{(q-1)\,x+\omega} \left(\frac{1}{(q\,x+\omega)^{n+1}} - \frac{1}{x^{n+1}} \right).
\end{aligned}$$

Definition 4.3 *The formal Stieltjes function $S(\mathcal{L}) = S$ (see (4.1)) of the regular linear functional $\mathcal{L}$ satisfies a Riccati differential equation if S satisfies an equation of type [Magnus, 1984], [Dzoumba, 1985]*

$$\phi(x)\mathcal{D}S(x) = A(x)S(x)^2 + B(x)S(x) + C(x), \tag{4.2}$$

where ϕ is a non-zero polynomial and A, B and C are polynomials.

When $A = 0$, the Riccati differential equation is called the affine Riccati differential equation.

Definition 4.4 *The formal Stieltjes function, $S(\mathcal{L}) = S$, of the regular linear functional $\mathcal{L}$ satisfies a D_ω-Riccati difference equation if S satisfies an equation of type*

$$\begin{aligned}\phi(x)D_\omega S(x) &= G(x;\omega)S(x)\mathcal{T}_\omega S(x) + E(x;\omega)S(x) \\ &\quad + F(x;\omega)\mathcal{T}_\omega S(x) + H(x;\omega),\end{aligned} \tag{4.3}$$

where ϕ is a non-zero polynomial and E, F, G and H are polynomials in the variable x and depending on ω.

When $G = 0$, the D_ω-Riccati difference equation is called the affine D_ω-Riccati difference equation.

Definition 4.5 *The formal Stieltjes function $S(\mathcal{L}) = S$ of the regular linear functional $\mathcal{L}$ satisfies a $\mathcal{D}_q$-Riccati difference equation if S satisfies an equation of type*

$$\begin{aligned}\phi(x)\mathcal{D}_q S(x) &= G(x;q)S(x)\mathcal{G}_q S(x) + E(x;q)S(x) \\ &\quad + F(x;q)\mathcal{G}_q S(x) + H(x:q)\end{aligned} \tag{4.4}$$

where ϕ is a non-zero polynomial and E, F, G and H are polynomials in the variable x and depending on q.

When $G = 0$, the $\mathcal{D}_q$-Riccati difference equation is called the affine $\mathcal{D}_q$-Riccati difference equation.

Definition 4.6 *The formal Stieltjes function, $S(\mathcal{L}) \equiv S$, of the regular linear functional $\mathcal{L}$ satisfies a $D_{q,\omega}$-Riccati difference equation if S satisfies an equation of type*

$$\begin{aligned}\phi(qx+\omega)D_{q,\omega}S(x) &= G(x;q,\omega)S(x)A_{q,\omega}S(x) + E(x;,q,\omega)S(x) \\ &\quad + F(x;,q,\omega)A_{q,\omega}S(x) + H(x;q,\omega),\end{aligned} \tag{4.5}$$

where ϕ is a non-zero polynomial and E, F, G and H are polynomials in the variable x and depending on q and ω.

When $G = 0$, the $D_{q,\omega}$-Riccati difference equation is called the affine $D_{q,\omega}$-Riccati difference equation.

Definition 4.7 *Let $\mathcal{Y}$ be any one of the four operators $\{\frac{d}{dx}, \mathcal{D}, \mathcal{D}_q, D_{q,\omega}\}$. Then, the regular linear functional $\mathcal{L}$ and the corresponding monic orthogonal polynomials belong to the $\mathcal{Y}$-Laguerre-Hahn class (resp. affine $\mathcal{Y}$-Laguerre-Hahn class) if the Stieltjes function of $\mathcal{L}$ satisfies a $\mathcal{Y}$-Riccati difference equation (an affine $\mathcal{Y}$-Riccati difference equation). The regular linear functional and the corresponding orthogonal polynomials belonging to the $\mathcal{Y}$-Laguerre-Hahn class are called $\mathcal{Y}$-Laguerre-Hahn linear functional and $\mathcal{Y}$-Laguerre-Hahn orthogonal polynomials, respectively (see [Magnus, 1984], [Dzoumba, 1985], [Guerfi. 1988], [Medem, 1995], [Salto, 1996], [Marcellàn et al.. 1998]).*

4.1.2 Some properties

Proposition 4.1 *The formal Stieltjes function, $S(\mathcal{L})$, of a given linear functional $\mathcal{L} \in \mathbb{P}'$ obeys the relations*

$$i)\ S(\alpha\mathcal{L} + \beta\mathcal{M}) = \alpha S(\mathcal{L}) + \beta S(\mathcal{M}) \quad \forall\alpha,\beta \in \mathbb{C}, \quad \forall\mathcal{M} \in \mathbb{P}' \tag{4.6}$$

$$ii)\ S(f\mathcal{L}) = fS(\mathcal{L}) + \mathcal{L}\theta_0 f, \quad \forall f \in \mathbb{P}. \tag{4.7}$$

$$iii)\ A_{q,\omega}S(\mathcal{L}) = S(A_{q,\omega}\mathcal{L}), \tag{4.8}$$

$$iv)\ S(D_{q,\omega}\mathcal{L}) = D_{q,\omega}S(\mathcal{L}). \tag{4.9}$$

Proof: i) Let α, β be two complex numbers and $\mathcal{L}$, $\mathcal{M}$ two linear functionals. Then,

$$\begin{aligned} S(\alpha\mathcal{L}+\beta\mathcal{M}) &= = -\sum_{k\geq 0} \frac{\langle \alpha\mathcal{L}+\beta\mathcal{M}, x^k\rangle}{x^{k+1}} \\ &= -\alpha\sum_{k\geq 0}\frac{\langle \mathcal{L}, x^k\rangle}{x^{k+1}} - \beta\sum_{k\geq 0}\frac{\langle \mathcal{M}, x^k\rangle}{x^{k+1}} \\ &= \alpha S(\mathcal{L}) + \beta S(\mathcal{M}). \end{aligned}$$

ii) Let k be an integer; we shall prove that

$$S(x^k\mathcal{L}) = x^k S(\mathcal{L}) + \mathcal{L}\theta_0 x^k \quad \forall k \in \mathbb{N}$$

and use property i) to deduce that (4.8) holds for any $f \in \mathbb{F}$.

In fact,

$$\begin{aligned} S(x^k\mathcal{L})(x) &= -\sum_{n=0}^{\infty}\frac{\langle x^k\mathcal{L}, x^n\rangle}{x^{n+1}} \\ &= -\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, x^{n+k}\rangle}{x^{n+1}} \\ &= -\sum_{m=k}^{\infty}\frac{\langle \mathcal{L}, x^m\rangle}{x^{m-k+1}} \text{ (taking } k+n=m) \\ &= -x^k\Big(\sum_{m=0}^{\infty}\frac{\langle \mathcal{L}, x^m\rangle}{x^{m+1}} - \sum_{m=0}^{k-1}\frac{\langle \mathcal{L}, x^m\rangle}{x^{m+1}}\Big) \\ &= x^k S(\mathcal{L})(x) + \sum_{m=0}^{k-1}\langle \mathcal{L}, x^m\rangle x^{k-1-m} \\ &= x^k S(\mathcal{L})(x) + \sum_{j=1}^{k}\langle \mathcal{L}, x^{k-j}\rangle x^{j-1} \text{ (taking } k-m=j) \\ &= x^k S(\mathcal{L})(x) + \mathcal{L}\theta_0 x^k. \end{aligned}$$

For property iii),

$$\begin{aligned} S(A_{q,\omega}\mathcal{L})(x) &= -\sum_{n=0}^{\infty}\frac{\langle A_{q,\omega}\mathcal{L}, x^n\rangle}{x^{n+1}} \\ &= -\frac{1}{q}\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, A^*_{q,\omega}x^n\rangle}{x^{n+1}} \\ &= -\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, (x-\omega)^n\rangle}{(qx)^{n+1}} \\ &= -\sum_{n=0}^{\infty}\frac{1}{(qx)^{n+1}}\langle \mathcal{L}, \sum_{j=0}^{n} x^j\binom{n}{j}(-\omega)^{n-j}\rangle \\ &= -\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-\omega)^{n-j}\frac{\langle \mathcal{L}, x^j\rangle}{(qx)^{n+1}}, \end{aligned}$$

where

$$\binom{n}{j} = \begin{cases} \frac{n!}{j!(n-j)!} & \text{if } j \leq n, \\ 0 & \text{if } j > n. \end{cases} \tag{4.10}$$

Then,

$$S(A_{q,\omega}\mathcal{L})(x) = -\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(-\omega)^{n-j}\frac{\langle\mathcal{L},x^j\rangle}{(qx)^{n+1}}. \tag{4.11}$$

On the other hand, using the series expansion of $\frac{1}{(qx+\omega)^{n+1}}$,

$$\frac{1}{(qx+\omega)^{n+1}} = \sum_{p=0}^{\infty}\frac{\binom{n+p}{n}(-\omega)^p}{(qx)^{n+1+p}}, \tag{4.12}$$

we obtain

$$\begin{aligned} A_{q,\omega}S(\mathcal{L})(x) &= -\sum_{n=0}^{\infty}\frac{\langle\mathcal{L},x^n\rangle}{(qx+\omega)^{n+1}} \\ &= -\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\binom{n+p}{n}(-\omega)^p\langle\mathcal{L},x^n\rangle}{(qx)^{n+p+1}}. \end{aligned}$$

Then changing the variable $n+p=j$, the previous equation gives

$$\begin{aligned} A_{q,\omega}S(\mathcal{L})(x) &= -\sum_{j=0}^{\infty}\sum_{n=0}^{j}\binom{j}{n}(-\omega)^{j-n}\frac{\langle\mathcal{L},x^n\rangle}{(qx)^{j+1}} \\ &= S(A_{q,\omega}\mathcal{L})(x), \end{aligned}$$

by (4.11) after reversing the role of n and j.

To derive relation iv), we compute both sides of (4.9) and remark that they are the same. In fact,

$$\begin{aligned} S(D_{q,\omega}\mathcal{L})(x) &= -\sum_{n=0}^{\infty}\frac{\langle D_{q,\omega}\mathcal{L},x^n\rangle}{x^{n+1}} \\ &= \frac{1}{q}\sum_{n=1}^{\infty}\frac{\langle\mathcal{L},D^*_{q,\omega}x^n\rangle}{x^{n+1}} \\ &= \frac{1}{q}\sum_{n=1}^{\infty}\frac{1}{x^{n+1}}\langle\mathcal{L},\frac{(\frac{x-\omega}{q})^n - x^n}{\frac{x-\omega}{q}-x}\rangle. \end{aligned}$$

Then, using the relations (derived by induction on n),

$$(a+b)^n = \sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j} \quad \forall a,b\in\mathbb{C} \quad \forall n\in\mathbb{N}, \tag{4.13}$$

$$a^n - b^n = (a-b)\sum_{j=0}^{n-1}a^j b^{n-1-j} \quad \forall a,b\in\mathbb{C} \quad \forall n\in\mathbb{N}, \tag{4.14}$$

we obtain

$$S(D_{q,\omega}\mathcal{L})(x) = \frac{1}{q}\sum_{n=1}^{\infty}\frac{1}{x^{n+1}}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{k}{j}(-\omega)^{k-j}q^{-k}\langle\mathcal{L},x^{n+j-1-k}\rangle,$$

On replacing n by $n+1$, we obtain,

$$S(D_{q,\omega}\mathcal{L})(x) = \sum_{n=0}^{\infty}\frac{1}{x^{n+2}}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-\omega)^{k-j}q^{-k-1}\langle\mathcal{L},x^{n+j-k}\rangle. \tag{4.15}$$

Let us compute $D_{q,\omega}S(\mathcal{L})(x)$.

$$\begin{aligned} D_{q,\omega}S(\mathcal{L})(x) &= -\sum_{n=0}^{\infty}\langle\mathcal{L},x^n\rangle D_{q,\omega}\frac{1}{x^{n+1}} \\ &= \sum_{n=0}^{\infty}\langle\mathcal{L},x^n\rangle\frac{D_{q,\omega}x^{n+1}}{x^{n+1}\,(qx+\omega)^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{(qx+\omega)^{n+1}-x^{n+1}}{qx+\omega-x}\,\frac{\langle\mathcal{L},x^n\rangle}{x^{n+1}\,(qx+\omega)^{n+1}}. \end{aligned}$$

Use of (4.14) transforms the previous relation into

$$\begin{aligned} D_{q,\omega}S(\mathcal{L})(x) &= \sum_{n=0}^{\infty}\frac{\langle\mathcal{L},x^n\rangle}{(qx+\omega)^{n+1}\,x^{n+1}}\sum_{j=0}^{n}(qx+\omega)^j\,x^{n-j} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{\langle\mathcal{L},x^n\rangle}{(qx+\omega)^{n-j+1}\,x^{j+1}}. \end{aligned}$$

Rewriting this equation, taking into account the series expansion of $\frac{1}{(qx+\omega)^{n-j+1}}$ (see (4.12)), leads to

$$\begin{aligned} D_{q,\omega}S(\mathcal{L})(x) &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{\langle\mathcal{L},x^n\rangle}{x^{j+1}}\sum_{k=0}^{\infty}\frac{(n+k-j)!\,(-\omega)^k}{k!\,(n-j)!\,(qx)^{n+k-j+1}} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{\infty}\frac{(n+k-j)!\,(-\omega)^k\langle\mathcal{L},x^n\rangle}{k!\,(n-j)!\,x^{n+k+2}\,q^{n+k-j+1}}. \end{aligned}$$

The change of variable, $n+k=t$, transforms the latter equation into

$$D_{q,\omega}S(\mathcal{L})(x) = \sum_{t=0}^{\infty}\sum_{k=0}^{t}\sum_{j=0}^{t-k}\frac{(t-j)!\,(-\omega)^k\langle\mathcal{L},x^{t-k}\rangle}{k!\,(t-k-j)!\,x^{t+2}\,q^{t-j+1}}.$$

We obtain after replacing in the previous equation $t-j$ by m that

$$\begin{aligned} D_{q,\omega}S(\mathcal{L})(x) &= \sum_{t=0}^{\infty}\sum_{k=0}^{t}\sum_{m=k}^{t}\frac{m!\,(-\omega)^k\langle\mathcal{L},x^{t-k}\rangle}{k!\,(m-k)!\,x^{t+2}\,q^{m+1}} \\ &= \sum_{t=0}^{\infty}\sum_{k=0}^{t}\sum_{m=k}^{t}\frac{\binom{m}{k}(-\omega)^k\langle\mathcal{L},x^{t-k}\rangle}{x^{t+2}\,q^{m-1}} \\ &= \sum_{t=0}^{\infty}\sum_{k=0}^{t}\sum_{m=0}^{t}\frac{\binom{m}{k}(-\omega)^k\langle\mathcal{L},x^{t-k}\rangle}{x^{t+2}\,q^{m-1}} \\ &= \sum_{t=0}^{\infty}\sum_{m=0}^{t}\sum_{k=0}^{m}\frac{\binom{m}{k}(-\omega)^k\langle\mathcal{L},x^{t-k}\rangle}{x^{t+2}\,q^{m-1}}. \end{aligned}$$

Again, replacing $m-k$ by l in the above equation, we obtain

$$\begin{aligned} D_{q,\omega}S(\mathcal{L})(x) &= \sum_{t=0}^{\infty}\sum_{m=0}^{t}\sum_{l=0}^{m}\frac{\binom{m}{m-l}(-\omega)^{m-l}\langle\mathcal{L},x^{t-l-m}\rangle}{x^{t+2}\,q^{m-1}} \\ &= \sum_{t=0}^{\infty}\sum_{m=0}^{t}\sum_{l=0}^{m}\frac{\binom{m}{l}(-\omega)^{m-l}\langle\mathcal{L},x^{t+l-m}\rangle}{x^{t+2}\,q^{m-1}} \\ &= S(D_{q,\omega}\mathcal{L})(x), \end{aligned}$$

by (4.15); hence the proof of Proposition 4.1 is complete. □We give some consequences of the previous proposition, already given in [Guerfi, 1988], [Salto, 1995], [Medem, 1996].

Corollary 4.1 *The formal Stieltjes function $S(\mathcal{L})$ of a given linear functional $\mathcal{L} \in \mathbb{P}'$ obeys the relations:*

$$\begin{aligned} \mathcal{T}_\omega S(\mathcal{L}) &= S(\mathcal{T}_\omega \mathcal{L}),\ S(D_\omega \mathcal{L}) = D_\omega S(\mathcal{L}), \\ \mathcal{G}_q S(\mathcal{L}) &= S(\mathcal{G}_q \mathcal{L}),\ S(\mathcal{D}_q \mathcal{L}) = \mathcal{D}_q S(\mathcal{L}). \end{aligned}$$

We announce another corollary of Proposition 4.1. This result has been given for the operators $\mathcal{D}$, $\mathcal{D}_q$ and D_ω (see [Dzoumba, 1985], [Guerfi, 1988], [Medem. 1996]).

Theorem 4.1 *Let $\mathcal{L}$ be a regular linear functional. $\mathcal{L}$ belongs to the affine $D_{q,\omega}$-Laguerre-Hahn class if and only if $\mathcal{L}$ is $D_{q,\omega}$-semi-classical.*

Proof: Suppose that $\mathcal{L}$ is $D_{q,\omega}$-semi-classical and satisfies $D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}$, where ϕ is a non-zero polynomial and ψ a polynomial of degree at least one. We first use Propositions 3.1 and 4.1 to compute $S(D_{q,\omega}(\phi\mathcal{L}))$ and obtain

$$\begin{aligned} S(D_{q,\omega}(\phi\mathcal{L})) &= S(A_{q,\omega}\phi D_{q,\omega}\mathcal{L} + D_{q,\omega}\phi\mathcal{L}) \\ &= A_{q,\omega}\phi S(D_{q,\omega}\mathcal{L}) + (D_{q,\omega}\mathcal{L})\theta_0 A_{q,\omega}\phi + D_{q,\omega}\phi S(\mathcal{L}) + \mathcal{L}\theta_0 D_{q,\omega}\phi \\ &= A_{q,\omega}\phi D_{q,\omega} S(\mathcal{L}) + (D_{q,\omega}\mathcal{L})\theta_0 A_{q,\omega}\phi + D_{q,\omega}\phi S(\mathcal{L}) + \mathcal{L}\theta_0 D_{q,\omega}\phi. \end{aligned}$$

Secondly, we use again Proposition 4.1 to compute $S(\psi\mathcal{L})$ and we obtain

$$S(\psi\mathcal{L}) = \psi S(\mathcal{L}) + \mathcal{L}\theta_0\psi.$$

Since $D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}$ and ϕ is a non-zero polynomial, we deduce from the above computations that $S(\mathcal{L})$ satisfies the affine $D_{q,\omega}$-Riccati difference equation

$$\begin{aligned} \phi(qx+\omega)D_{q,\omega}S(\mathcal{L})(x) &= (\psi(x) - D_{q,\omega}\phi(x))S(\mathcal{L})(x) + \mathcal{L}\theta_0\psi(x) \\ &\quad -(D_{q,\omega}\mathcal{L})\theta_0\phi(qx+\omega) - \mathcal{L}\theta_0 D_{q,\omega}\phi(x). \end{aligned}$$

Thus, $\mathcal{L}$ belongs to the affine $D_{q,\omega}$-Laguerre-Hahn class.

Conversely, assume that the Stieltjes function $S(\mathcal{L})$ of the regular linear functional $\mathcal{L}$ satisfies an affine $D_{q,\omega}$-Riccati difference equation

$$A(x)D_{q,\omega}S(\mathcal{L})(x) = B(x)S(\mathcal{L})(x) + C(x),$$

where B and C are any polynomials and A is a non-zero polynomial. Using Propositions 3.1 and 4.1 we obtain

$$\begin{aligned} A(x)D_{q,\omega}S(\mathcal{L})(x) &= B(x)S(\mathcal{L})(x) + C(x) \\ &\Longleftrightarrow \\ S(A(x)D_{q,\omega}\mathcal{L})(x) - (D_{q,\omega}\mathcal{L})\theta_0 A(x) &= S(B(x)\mathcal{L})(x) - \mathcal{L}\theta_0 B(x) + C(x) \\ &\Longleftrightarrow \\ S(A(x)D_{q,\omega}\mathcal{L} - B(x)\mathcal{L})(x) &= (D_{q,\omega}\mathcal{L})\theta_0 A(x) - \mathcal{L}\theta_0 B(x) + C(x). \end{aligned}$$

The right hand-side of the previous equation is a polynomial while the left hand-side is, by definition of the Stieltjes function of a given linear functional, an infinite (unless it vanishes) linear combination of $\{\frac{1}{x^{n+1}}, n \in \mathbb{N}\}$. Therefore, both sides of the previous equation vanish and we obtain

$$A(x)D_{q,\omega}\mathcal{L} - B(x)\mathcal{L} = 0 \tag{4.16}$$

and

$$(D_{q,\omega}\mathcal{L})\theta_0 A(x) - \mathcal{L}\theta_0 B(x) + C(x) = 0 \quad \forall x \in \mathbb{R}. \tag{4.17}$$

Again, we use Proposition 3.1 to deduce that (4.16) is equivalent to

$$D_{q,\omega}(A^*_{q,\omega}A\mathcal{L}) = (B - \frac{1}{q}D^*_{q,\omega}A)\mathcal{L}. \tag{4.18}$$

The previous equation, used together with Lemma 3.1 taking into account the fact that $A \neq 0$, allows us to conclude that the degree of $B + \frac{1}{q}D^*_{q,\omega}A$ is at least one. Then the regular linear functional $\mathcal{L}$ is $D_{q,\omega}$-semi-classical. □

4.2 $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials as $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials

In this section we prove that the $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials can be deduced from $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials by a change of variable and then we give some consequences.

Theorem 4.2 *Let $\mathcal{L}$ be any regular linear functional, then we have:*

i) $\mathcal{L}$ is a $D_{q,\omega}$-Laguerre-Hahn linear functional if and only if $A_{a,\frac{\omega}{1-q}}\mathcal{L}$ is a $\mathcal{D}_q$-Laguerre-Hahn linear functional. This means that the Stieltjes function $S(\mathcal{L})$ of $\mathcal{L}$ satisfies

$$\begin{aligned}\phi(qx+\omega)D_{q,\omega}S(\mathcal{L})(x) &= G(x;q,\omega)S(\mathcal{L})(x)A_{q,\omega}S(\mathcal{L})(x)+E(x;q,\omega)S(\mathcal{L})(x)\\ &\quad +F(x;q,\omega)A_{q,\omega}S(\mathcal{L})(x)+H(x;q,\omega),\end{aligned} \tag{4.19}$$

where a is any non-zero real number, ϕ is a non-zero polynomial, E, F, G and H are polynomials in the variable x, if and only if the Stieltjes function $S(A_{a,\frac{\omega}{1-q}}\mathcal{L})$ of $A_{a,\frac{\omega}{1-q}}\mathcal{L}$ satisfies

$$\begin{aligned}\tilde{\phi}(qx)\mathcal{D}_qS(\tilde{\mathcal{L}})(x) &= \tilde{G}(x;q,\omega)S(\tilde{\mathcal{L}})(x)\mathcal{G}_q\,S(\tilde{\mathcal{L}})(x)+\tilde{E}(x;q,\omega)S(\tilde{\mathcal{L}})(x)\\ &\quad +\tilde{F}(x;q,\omega)\mathcal{G}_qS(\tilde{\mathcal{L}})(x)+\tilde{H}(x;q,\omega),\end{aligned} \tag{4.20}$$

where $\tilde{\phi}=a^{-1}A_{a,\frac{\omega}{1-q}}\phi$, $\tilde{\Phi}=A_{a,\frac{\omega}{1-q}}\Phi$ for $\Phi\in\{\mathcal{L},\ E,\ F,\ G,\ H\}$.

ii) Let $\mathcal{L}$ be a $D_{q,\omega}$-Laguerre-Hahn linear functional satisfying (4.19). If $\{P_n\}_{n\in\mathbb{N}}$ and $\{\tilde{P}_n\}_{n\in\mathbb{N}}$ represent the monic orthogonal families associated to $\mathcal{L}$ and $A_{a,\frac{\omega}{1-q}}\mathcal{L}$, respectively, then we have the following results:

$$\tilde{P}_n(x)=a^{-n}\,P_n(ax+\frac{\omega}{1-q})\quad \forall x\in\mathbb{R}, \tag{4.21}$$

$$\beta_n(q,\omega,\phi,E,F,G,H)=a\,\tilde{\beta}_n(q,0,\tilde{\phi},\tilde{E},\tilde{F},\tilde{G},\tilde{H})+\frac{\omega}{1-q},$$

$$\gamma_n(q,\omega,\phi,E,F,G,H)=a^2\,\tilde{\gamma}_n(q,0,\tilde{\phi},\tilde{E},\tilde{F},\tilde{G},\tilde{H}),$$

where β_n, γ_n, $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ are coefficients of the three-term recurrence relation satisfied by $\{P_n\}_{n\in\mathbb{N}}$ and $\{\tilde{P}_n\}_{n\in\mathbb{N}}$:

$$\left\{\begin{array}{l} P_{n+1}(x)=(x-\beta_n(q,\omega,\phi,E,F,G,H))P_n(x)-\gamma_n(q,\omega,\phi,E,F,G,H)P_{n-1}(x),\ n\geq 0,\\ \tilde{P}_{n+1}(x)=(x-\tilde{\beta}_n(q,0,\tilde{\phi},\tilde{E},\tilde{F},\tilde{G},\tilde{H}))\tilde{P}_n(x)-\tilde{\gamma}_n(q,0,\tilde{\phi},\tilde{E},\tilde{F},\tilde{G},\tilde{H})\tilde{P}_{n-1}(x),\ n\geq 0,\\ P_{-1}(x)=0,\ P_0(x)=1,\ \tilde{P}_{-1}(x)=0,\tilde{P}_0(x)=1.\end{array}\right. \tag{4.22}$$

Proof: i) We use the relation [Guerfi, 1988], [Medem, 1996]

$$A_{a,\frac{\omega}{1-q}}\,D_{q,\omega}=a^{-1}\,\mathcal{D}_q\,A_{a,\frac{\omega}{1-q}},\ A_{a,\frac{\omega}{1-q}}\,A_{q,\omega}=\mathcal{G}_q\,A_{a,\frac{\omega}{1-q}},\ q\neq 1,\ a\neq 0 \tag{4.23}$$

and get

$$\phi(qx+\omega)D_{q,\omega}S(\mathcal{L})(x)=G(x;q,\omega)S(\mathcal{L})(x)A_{q,\omega}S(\mathcal{L})(x)+E(x;q,\omega)S(\mathcal{L})(x)$$
$$+F(x;q,\omega)A_{q,\omega}S(\mathcal{L})(x)+H(x;q,\omega)$$
$$\Longleftrightarrow$$
$$A_{a,\frac{\omega}{1-q}}A_{q,\omega}\phi(x)A_{a,\frac{\omega}{1-q}}D_{q,\omega}S(\mathcal{L})(x)=\tilde{G}(x;q,\omega)A_{a,\frac{\omega}{1-q}}S(\mathcal{L})(x)A_{a,\frac{\omega}{1-q}}A_{q,\omega}S(\mathcal{L})(x)$$
$$+\tilde{E}(x;q,\omega)A_{a,\frac{\omega}{1-q}}S(\mathcal{L})(x)+\tilde{F}(x;q,\omega)A_{a,\frac{\omega}{1-q}}A_{q,\omega}S(\mathcal{L})(x)+\tilde{H}(x;q,\omega)$$
$$\Longleftrightarrow$$
$$a^{-1}\mathcal{G}_q\,A_{a,\frac{\omega}{1-q}}\phi(x)\mathcal{D}_q\,A_{a,\frac{\omega}{1-q}}S(\mathcal{L})(x)=\tilde{G}(x;q,\omega)A_{a,\frac{\omega}{1-q}}S(\mathcal{L})(x)\,\mathcal{G}_q\,A_{a,\frac{\omega}{1-q}}S(\mathcal{L})(x)$$
$$+\tilde{E}(x;q,\omega)A_{a,\frac{\omega}{1-q}}S(\mathcal{L})(x)+\tilde{F}(x;q,\omega)\,\mathcal{G}_q\,A_{a,\frac{\omega}{1-q}}S(\mathcal{L})(x)+\tilde{H}(x;q,\omega)$$
$$\Longleftrightarrow$$
$$\tilde{\phi}(qx)\mathcal{D}_qS(\tilde{\mathcal{L}})(x)=\tilde{G}(x;q,\omega)S(\tilde{\mathcal{L}})(x)\mathcal{G}_q\,S(\tilde{\mathcal{L}})(x)+\tilde{E}(x;q,\omega)S(\tilde{\mathcal{L}})(x)$$
$$+\tilde{F}(x;q,\omega)\mathcal{G}_qS(\tilde{\mathcal{L}})(x)+\tilde{H}(x;q,\omega),$$

by the relation (4.8): $S(A_{q,\omega}\,\mathcal{L}) = A_{q,\omega}\,S(\mathcal{L})$.

ii) Since the family $\{A_{a,\frac{\omega}{1-q}} P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect $A_{a,\frac{\omega}{1-q}}\,\mathcal{L}$ (see Lemma 3.2), we deduce that $\tilde{P}_n = a^{-n}\,A_{a,\frac{\omega}{1-q}} P_n$, thanks to the uniqueness of the monic orthogonal polynomial family associated to a given regular linear functional.

Since $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to the linear functional $\mathcal{L}$, it satisfies

$$P_{n+1}(x) = (x - \beta_n(q,\omega,\phi,E,F,G,H))P_n(x) - \gamma_n(q,\omega,\phi,E,F,G,H)P_{n-1}(x),$$

where $\beta_n(q,\omega,\phi,E,F,G,H)$, $\gamma_n(q,\omega,\phi,E,F,G,H)$ are complex numbers depending on q, ω, ϕ, E, F, G and H.

After applying the operator $A_{a,\frac{\omega}{1-q}}$ to both sides of the previous equation, we obtain that

$$A_{a,\frac{\omega}{1-q}}\,P_{n+1}(x) = (a\,x + \frac{\omega}{1-q} - \beta_n(q,\omega,\phi,\psi))A_{a,\frac{\omega}{1-q}}\,P_n(x) - \gamma_n(q,\omega,\phi,\psi)A_{a,\frac{\omega}{1-q}}\,P_{n-1}(x).$$

This latter equation, used together with (4.21), gives

$$\tilde{P}_{n+1}(x) = (x + \frac{\omega}{(1-q)a} - \frac{1}{a}\beta_n(q,\omega,\phi,\psi))\tilde{P}_n(x) - \frac{1}{a^2}\gamma_n(q,\omega,\phi,\psi)\tilde{P}_{n-1}(x).$$

We complete the proof of the theorem by identifying the coefficients of the previous equation with the ones of the three-term recurrence relation satisfied by family $\{\tilde{P}_n\}_{n\in\mathbb{N}}$, orthogonal with respect to the $\mathcal{D}_q$-semi-classical linear functional $A_{a,\frac{\omega}{1-q}}\,\mathcal{L}$

$$\tilde{P}_{n+1}(x) = (x - \tilde{\beta}_n(q,0,\tilde{\phi},\tilde{\psi}))\tilde{P}_n(x) - \tilde{\gamma}_n(q,0,\tilde{\phi},\tilde{\psi})\tilde{P}_{n-1}(x),\ n \geq 0,$$

with $\tilde{\phi}$ and $\tilde{\psi}$ defined by (4.24). □

Remark 4.1 *Since the results stated in Theorem 4.2 are valid for any real number $a \neq 0$, without loss of generality, we choose $a = 1$. In this case $A_{1,\frac{\omega}{1-q}} = \mathcal{T}_{\frac{\omega}{1-q}}$ and we, therefore, get the following consequences:*

Corollary 4.2 *Let $\mathcal{L}$ be any regular linear functional, $\{P_n\}_{n\in\mathbb{N}}$ and $\{\tilde{P}_n\}_{n\in\mathbb{N}}$ represent the orthogonal families associated to $\mathcal{L}$ and $\mathcal{T}_{\frac{\omega}{1-q}}\mathcal{L}$, respectively. Then, we have the following results:*

1. *$\mathcal{L}$ is $D_{q,\omega}$-semi-classical if and only if $\mathcal{T}_{\frac{\omega}{1-q}}\mathcal{L}$ is $\mathcal{D}_q$-semi-classical, i.e.,*

$$D_{q,\omega}(\phi\,\mathcal{L}) = \psi \iff \mathcal{D}_q(\tilde{\phi}\,\tilde{\mathcal{L}}) = \tilde{\psi}\,\tilde{\mathcal{L}},$$

where ϕ is any polynomial and ψ a polynomial of degree at least one, with

$$\tilde{\mathcal{L}} = \mathcal{T}_{\frac{\omega}{1-q}}\mathcal{L},\ \tilde{\phi}(x) = \phi(x + \frac{\omega}{1-q}),\ \tilde{\psi}(x) = \psi(x + \frac{\omega}{1-q}). \tag{4.24}$$

2. *The coefficients of the TTRR satisfied by $\{P_n\}_{n\in\mathbb{N}}$ and $\{\tilde{P}_n\}_{n\in\mathbb{N}}$ are related by*

$$\beta_n(q,\omega,\phi,\psi) = \tilde{\beta}_n(q,0,\tilde{\phi},\tilde{\psi}) - \frac{\omega}{1-q},$$
$$\gamma_n(q,\omega,\phi,\psi) = \tilde{\gamma}_n(q,0,\tilde{\phi},\tilde{\psi}),$$

where β_n, γ_n, $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ are coefficients of the three-term recurrence relation

$$\begin{cases} P_{n+1}(x) = (x - \beta_n(q,\omega,\phi,\psi))P_n(x) - \gamma_n(q,\omega,\phi,\psi)P_{n-1}(x),\ n \geq 0, \\ \tilde{P}_{n+1}(x) = (x - \tilde{\beta}_n(q,0,\tilde{\phi},\tilde{\psi}))\tilde{P}_n(x) - \tilde{\gamma}_n(q,0,\tilde{\phi},\tilde{\psi})\tilde{P}_{n-1}(x),\ n \geq 0, \\ P_{-1}(x) = 0, P_0(x) = 1,\ \tilde{P}_{-1}(x) = 0, \tilde{P}_0(x) = 1 \end{cases}$$

and $\tilde{\phi}$, $\tilde{\psi}$ given by (4.24).

Proof: The proof is similar to the one given for $D_{q,\omega}$-Laguerre-Hahn case. In particular, we have,

$$\begin{aligned} D_{q,\omega}(\phi\,\mathcal{L}) = \psi \quad &\iff \quad \mathcal{T}_{\frac{\omega}{1-q}}\,D_{q,\omega}(\phi\,\mathcal{L}) = \mathcal{T}_{\frac{\omega}{1-q}}\,\psi\,\mathcal{T}_{\frac{\omega}{1-q}}\,\mathcal{L} \\ &\iff \quad \mathcal{D}_q\,\mathcal{T}_{\frac{\omega}{1-q}}\,(\phi\,\mathcal{L}) = \mathcal{T}_{\frac{\omega}{1-q}}\,\psi\,\mathcal{T}_{\frac{\omega}{1-q}}\,\mathcal{L} \\ &\iff \quad \mathcal{D}_q(\mathcal{T}_{\frac{\omega}{1-q}}\,\phi\,\mathcal{T}_{\frac{\omega}{1-q}}\,\mathcal{L}) = \mathcal{T}_{\frac{\omega}{1-q}}\,\psi\,\mathcal{T}_{\frac{\omega}{1-q}}\,\mathcal{L}. \end{aligned}$$

□

Chapter 5

Difference equations for the first associated classical orthogonal polynomials

5.1 Introduction

In this chapter we derive the single fourth order difference equation satisfied by the first associated of the q-classical orthogonal polynomials. We give this equation in the factored and simple form, we then use Theorem 4.2 to deduce the single fourth order difference (resp. differential) equation satisfied by the first associated of the classical orthogonal polynomials of a discrete variable and continuous variable, respectively.
Although the main result of this section is contained in the general theory given in the next chapter, this method is worth to be communicated because it uses the properties of the functions of a discrete variable of the second kind [Suslov, 1989] rather than the properties of the Stieltjes function which are used in the next chapter. It also allows us to have a factored and simple form for the fourth-order difference equation and to confirm the results obtained by the general theory.

5.2 q-classical weight

Let $\rho(x)$ be a positive weight function defined on the interval $I =]a, b[$ and let $\mathcal{L}$ be a linear functional defined by

$$\langle \mathcal{L}, P \rangle = \int_I P(s)\, \rho(s) d_q s. \tag{5.1}$$

The orthogonality weight ρ (defined in the interval I) is said to be q-classical if ρ satisfies:

i) There exists a monic polynomial family $\{P_n\}_{n\in\mathbb{N}}$, orthogonal with respect to ρ, i.e.,

$$\int_I P_n(s)\, P_m(s)\, \rho(s) d_q s = k_n\, \delta_{n,m} \quad \forall n, m \in \mathbb{N}. \quad (k_n \neq 0 \quad \forall n \in \mathbb{N}). \tag{5.2}$$

ii) There exist two polynomials ϕ of degree at most two and ψ of degree one such that

$$\mathcal{D}_q\,(\phi\, \rho) = \psi\, \rho, \tag{5.3}$$

with

$$x^n\, \phi(x)\, \rho(x)|_a^b = 0 \quad \forall n \in \mathbb{N}. \tag{5.4}$$

Lemma 5.1 *The linear functional $\mathcal{L}$ represented by the q-classical weight ρ (see (5.1)) is $\mathcal{D}_q$-classical and satisfies*

$$\mathcal{D}_q(\phi\,\mathcal{L}) = \psi\,\mathcal{L}. \tag{5.5}$$

Proof: If P is any element of $\mathbb{P}$, we use (5.1)-(5.4) and get,

$$\begin{aligned}
\langle \mathcal{D}_q(\phi\,\mathcal{L}), P\rangle &= -\frac{1}{q}\langle \phi\mathcal{L}, \mathcal{D}_{\frac{1}{q}}P\rangle \\
&= -\frac{1}{q}\int_I \phi(s)\mathcal{D}_{\frac{1}{q}}P(s)\,\rho(s)d_qs \\
&= -\frac{1}{q}\int_I (\mathcal{D}_{\frac{1}{q}}(\phi(qs)\,\rho(qs)\,P(s)) - \mathcal{D}_{\frac{1}{q}}(\phi(qs)\rho(qs))\,P(s))d_qs \\
&= -\int_I (\mathcal{D}_q(\phi(s)\,\rho(s)\,P(s/q)) + \mathcal{D}_q(\phi(s)\rho(s))\,P(s))d_qs \\
&= -\phi(s)\,\rho(s)\,P(s/q)|_a^b + \int_I \psi(s)\rho(s)\,P(s)d_qs \\
&= \int_I \psi(s)\rho(s)\,P(s)d_qs \\
&= \langle \psi\mathcal{L}, P\rangle.
\end{aligned}$$

Hence, $\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}$. We complete the proof by remarking that $\{P_n\}_{n\in\mathbb{N}}$ is orthogonal with respect to $\mathcal{L}$ (see (5.2)). □

The monic polynomials $\{P_n\}_{n\in\mathbb{N}}$, orthogonal with respect to $\mathcal{L}$, satisfy the second order q-difference equation (see Theorem 3.1),

$$\mathcal{Q}_{2,n}[y(x)] \equiv [\phi\mathcal{D}_q\,\mathcal{D}_{\frac{1}{q}} + \psi\,\mathcal{D}_q + \lambda_{n,0}\mathcal{I}_d]\,y(x) = 0, \tag{5.6}$$

an equation which can be written in the q-shifted form,

$$[(\phi_{(1)} + \psi_{(1)}t_1)\mathcal{G}_q^2 - ((1+q)\,\phi_{(1)} + \psi_{(1)}t_1 - \lambda_{n,0}t_1^2)\mathcal{G}_q + q\phi_{(1)}\,\mathcal{I}_d]y(x) = 0. \tag{5.7}$$

with

$$\begin{aligned}
\lambda_{n,0} &= -[n]_q\{\psi' + [n-1]_{\frac{1}{q}}\frac{\phi''}{2q}\}, \\
\phi_{(i)} &\equiv \phi(q^ix),\quad \psi_{(i)} \equiv \psi(q^ix),\quad t_i \equiv t(q^ix),\quad t(x) = (q-1)x.
\end{aligned} \tag{5.8}$$

5.3 Fourth-order q-difference equation for $P_{n-1}^{(1)}(x;q)$

The first associated $P_{n-1}^{(1)}(x;q)$ of $P_{n-1}(x;q)$ is a monic polynomial of degree $n-1$ defined by

$$P_{n-1}^{(1)}(x;q) = \frac{1}{\gamma_0}\langle\mathcal{L}, \frac{P_n(s;q) - P_n(x;q)}{s-x}\rangle = \frac{1}{\gamma_0}\int_I \frac{P_n(s;q) - P_n(x;q)}{s-x}\rho(s)d_qs, \tag{5.9}$$

where γ_0 is given by $\gamma_0 = \langle\mathcal{L}, 1\rangle = \int_I \rho(s)d_qs$.

Relation (5.9) can be rewritten as

$$P_{n-1}^{(1)}(x;q) = \rho(x)\,Q_n(x;q) - P_n(x;q)\rho(x)\,Q_0(x;q), \tag{5.10}$$

where

$$Q_n(x;q) = \frac{1}{\gamma_0\,\rho(x)}\int_I \frac{P_n(s;q)}{s-x}\rho(s)d_qs.$$

It is well-known [Suslov, 1989] that $Q_n(x;q)$ also satisfies Equation (5.6); hence, by (5.10)

$$\mathcal{Q}_{2,n}\left[\frac{P_{n-1}^{(1)}(x;q)}{\rho(x)}+P_n(x;q)Q_0(x;q)\right]=0. \tag{5.11}$$

In a first step, we eliminate $\rho(x)$ and $Q_0(x;q)$ in Equation (5.11) using Equation (5.3) and Equation (5.6) for $P_n(x;q)$. This can be easily carried out using a computer algebra system—we used Maple V Release 4 [Char et al., 1991]—and gives the relation

$$(\phi_{(1)}+\psi_{(1)}\,t_1)\,\mathcal{Q}^*_{2,n-1}\left[P_{n-1}^{(1)}(x;q)\right]=\left[e\,\mathcal{G}_q+f\,\mathcal{I}_d\right]P_n(x;q), \tag{5.12}$$

with

$$\begin{aligned}
\mathcal{Q}^*_{2,n-1} &= \phi_{(2)}\,\mathcal{G}_q^2-((1+q)\phi_{(1)}+\psi_{(1)}\,t_1-\lambda_{n,0}\,t_1^2)\mathcal{G}_q+q\,(\phi+\psi\,t)\,\mathcal{I}_d. \\
e &= (\frac{\phi''}{2}-\psi')\,((1+q)\phi_{(1)}+\psi_{(1)}\,t_1-\lambda_{n,0}\,t_1^2)\,t_1, \\
f &= -(\frac{\phi''}{2}-\psi')\,((q+1)\phi_{(1)}+\psi_{(1)}\,t_1)\,t_1.
\end{aligned} \tag{5.13}$$

In a second step, we use Equations (5.12), (5.13) and the fact that the polynomials $P_n(x;q)$ satisfy Equation (5.6), again. This gives—after some computations with Maple V.4—the operator $\mathcal{Q}^{**}_{2,n-1}$ annihilating the right-hand side of Equation (5.12),

$$\begin{aligned}
\mathcal{Q}^{**}_{2,n-1} = {} & (\phi_{(3)}+\psi_{(3)}\,t_3)[q^2\,A_1+(1+q)\,\phi_{(2)}+\psi_{(2)}\,t_2]\mathcal{G}_q^2 \\
& -[q^3\,A_1\,(\phi_{(2)}+\psi_{(2)}\,t_2)+A_3\,(\phi_{(2)}+q\,A_1)]\,\mathcal{G}_q \\
& +q\,\phi_{(1)}\,[q^2\,A_2+(1+q)\,\phi_{(3)}+\psi_{(3)}\,t_3)]\,\mathcal{I}_d,
\end{aligned} \tag{5.14}$$

where $A_j=(1+q)\phi_{(j)}+\psi_{(j)}\,t_j-\lambda_{n,0}\,t_j^2$.

We, therefore, obtain the factored form of the fourth-order q-difference equation satisfied by each $P_{n-1}^{(1)}(x;q)$,

$$\mathcal{Q}^{**}_{2,n-1}\,\frac{\mathcal{Q}^*_{2,n-1}}{q^2\,(q-1)^2\,x^2}\left[P_{n-1}^{(1)}(x;q)\right]=0. \tag{5.15}$$

5.4 Applications

5.4.1 The first associated Little and Big q-Jacobi polynomials

For the Little q-Jacobi polynomials, $p_n(x;a,b|q)$ [Area et al., 1998a],[Koekoek et. al, 1996]

$$\phi(x)=\frac{x\,(x-1)}{q},\ \psi(x)=\frac{1-a\,q+(abq^2-1)\,x}{q(q-1)}.$$

and for the Big q-Jacobi polynomials, $P_n(x;a,b,c;q)$ [Area et al., 1998a],[Koekoek et. al 1996]

$$\phi(x)=acq-(a+c)x+\frac{x^2}{q},\ \psi(x)=\frac{cq+aq(1-(b+c)q)+(abq^2-1)x}{q(q-1)},$$

the constant $\phi''-2\psi'$ is equal to $\frac{2\,(1-a\,b\,q)}{q-1}$. Therefore, the first associated of the Little q-Jacobi polynomials (resp. Big q-Jacobi polynomials) is still in the Little q-Jacobi (resp. Big q-Jacobi) family when $a\,b\,q=1$.

Let $\beta_n(a,b|q)$ and $\gamma_n(a,b|q)$ (resp. $\beta_n(a,b,c;q)$ and $\gamma_n(a,b,c;q)$) be the coefficients of the three-term relation (see (2.18)) satisfied by the Little q-Jacobi polynomials $p_n(x;a,b|q)$ and the Big q-Jacobi polynomials $P_n(x;a,b,c;q)$, respectively.

It follows immediately from Lemma 7.1 that they obey:

$$\beta_{n-1}(a,\frac{1}{q\,a}|q)=q\,a\,\beta_n(\frac{1}{a},a\,q\,|q),\ \gamma_{n+1}(a,\frac{1}{q\,a}|q)=q^2\,a^2\,\gamma_n(\frac{1}{a},a\,q\,|q),$$

$$\beta_{n-1}(a,\frac{1}{q\,a},c;q)=a\,\beta_n(\frac{1}{a},a\,q,c\,q;q),\ \gamma_{n+1}(a,\frac{1}{q\,a},c;q)=a^2\,\gamma_n(\frac{1}{a},a\,q,c\,q;q).$$

The previous equations used, together with (2.23), give:

Theorem 5.1 *The monic Little q-Jacobi (resp. monic Big q-Jacobi) polynomials and their respective first associated are related by*

$$p_n^{(1)}(x;a,\frac{1}{q\,a}|q) = a^n\,q^n\,p_n(\frac{x}{a\,q};\frac{1}{a},a\,q\,|q), \tag{5.16}$$

$$P_n^{(1)}(x;a,\frac{1}{q\,a},c;q) = a^n\,P_n(\frac{x}{a};\frac{1}{a},a\,q,c\,q;q). \tag{5.17}$$

5.4.2 The first associated $\mathcal{D}$-classical orthogonal polynomials

Since $\lim_{q\to 1}\mathcal{D}_q = \frac{d}{dx}$, from Equations (5.13) and (5.14), we recover by a limit process the factored form of the fourth-order differential equation satisfied by the first associated $P_{n-1}^{(1)}(x)$ of the (continuous) classical orthogonal polynomials P_{n-1} [Ronveaux, 1988],

$$\mathcal{Q}_{2,n-1}^{**c}\,\mathcal{Q}_{2,n-1}^{*c}\left[P_{n-1}^{(1)}(x)\right] = 0, \tag{5.18}$$

with

$$\mathcal{Q}_{2,n-1}^{*c} = \lim_{q\to 1}\frac{\mathcal{Q}_{2,n-1}^{*}}{q^2(q-1)^2x^2} = \phi\frac{d^2}{dx^2} + (2\phi' - \psi)\frac{d}{dx} + (\phi'' - \psi' + \lambda_n)\mathcal{I}_d,$$

$$\mathcal{Q}_{2,n-1}^{**c} = \frac{1}{4\phi(x)}\lim_{q\to 1}\frac{\mathcal{Q}_{2,n-1}^{**}}{q^2(q-1)^2x^2} = \phi\frac{d^2}{dx^2} + (\phi' + \psi)\frac{d}{dx} + (\psi' + \lambda_n)\mathcal{I}_d,$$

where

$$\lambda_n \equiv \lim_{q\to 1}\lambda_{n,0} = -n\,[(n-1)\frac{\phi''}{2} + \psi'].$$

5.4.3 The first associated $D_{q,\omega}$-classical orthogonal polynomials

In this subsection we apply the result of Theorem 4.2 to deduce the fourth-order difference equation satisfied by the $D_{q,\omega}$-classical orthogonal polynomials and then deduce the difference equation for classical orthogonal polynomials of a discrete variable.

In the first step we replace in (5.15), the polynomials ϕ (resp. ψ and $P_{n-1}^{(1)}$) by $\mathcal{T}_{\frac{\omega}{1-q}}\bar{\phi}$, $\mathcal{T}_{\frac{\omega}{1-q}}\bar{\psi}$ and $\mathcal{T}_{\frac{\omega}{1-q}}\bar{P}_{n-1}^{(1)}$, respectively, i.e.,

$$\phi = \mathcal{T}_{\frac{\omega}{1-q}}\bar{\phi},\ \psi = \mathcal{T}_{\frac{\omega}{1-q}}\bar{\psi},\ P_{n-1}^{(1)}(x;q) = \mathcal{T}_{\frac{\omega}{1-q}}\bar{P}_{n-1}^{(1)}(x;q,\omega) \tag{5.19}$$

and get an equation which multiplied by $\mathcal{T}_{\frac{-\omega}{1-q}}$, taking into account (4.23) and Proposition 3.5, gives

$$\mathcal{Q}_{2,n-1}^{**+}\frac{\mathcal{Q}_{2,n-1}^{*-}}{((q-1)x+\omega)^2}\left[\bar{P}_{n-1}^{(1)}(x;q,\omega)\right] = 0, \tag{5.20}$$

where

$$\mathcal{Q}_{2,n-1}^{*+} = \bar{\phi}_{[2]}\,A_{q,\omega}^2 - ((1+q)\bar{\phi}_{[1]} + \bar{\psi}_{[1]}\,\bar{t}_1 - \lambda_{n,0}\,\bar{t}_1^2)A_{q,\omega} + q\,(\bar{\phi} + \bar{\psi}\,\bar{t})\,\mathcal{I}_d,$$

$$\begin{aligned}\mathcal{Q}_{2,n-1}^{**+} &= (\bar{\phi}_{[3]} + \bar{\psi}_{[3]}\,\bar{t}_3)[q^2\,\bar{A}_1 + (1+q)\,\bar{\phi}_{[2]} + \bar{\psi}_{[2]}\,\bar{t}_2]A_{q,\omega}^2\\ &-[q^3\,\bar{A}_1\,(\bar{\phi}_{[2]} + \bar{\psi}_{[2]}\,\bar{t}_2) + \bar{A}_2\,(\bar{\phi}_{[2]} + q\,\bar{A}_1)]\,A_{q,\omega}\\ &+q\,\bar{\phi}_{[1]}\,[q^2\,\bar{A}_2 + (1+q)\,\bar{\phi}_{[3]} + \bar{\psi}_{[3]}\,\bar{t}_3)]\,\mathcal{I}_d.\end{aligned}$$

with the notations

$$\bar{A}_j(x) \equiv \bar{A}_j = (1+q)\bar{\phi}_{[j]} + \bar{\psi}_{[j]}\,\bar{t}_j - \lambda_{n,0}\,\bar{t}_j^2,\ \bar{\phi}_{[j]} \equiv \bar{\phi}(q^j\,x + \omega\,[j]_q),\ \bar{\psi}_{[j]} \equiv \bar{\psi}(q^j\,x + \omega\,[j]_q),$$

$$\bar{t}_j \equiv q^j\,\mathcal{T}_{\frac{-\omega}{1-q}}\,t(x) = q^j\,((q-1)x + \omega),\ \bar{t}_0 \equiv \bar{t}(x) = (q-1)x + \omega.$$

Since $\{P_n\}_{n\in\mathbb{N}}$ is $\mathcal{D}_q$-classical with respect to $\mathcal{L}$ (see (5.5)), it follows immediately from Theorem 4.2 that $\{\bar{P}_n\}_{n\in\mathbb{N}}$ (with $\bar{P}_n(x;q,\omega) = \mathcal{T}_{\frac{-\omega}{1-q}} P_n(x;q)$) is $D_{q,\omega}$-classical with respect to $\bar{\mathcal{L}} = \mathcal{T}_{\frac{-\omega}{1-q}}\mathcal{L}$, where the linear functional $\bar{\mathcal{L}}$ satisfies $D_{q,\omega}(\bar{\phi}\bar{\mathcal{L}}) = \bar{\psi}\bar{\mathcal{L}}$. Therefore (5.20) is the factored form of the fourth-order difference equation satisfied by the first associated $\bar{P}^{(1)}_{n-1}(x;q,\omega)$ of the $D_{q,\omega}$-classical orthogonal polynomial $\bar{P}_{n-1}(x;q,\omega)$.

5.4.4 The first associated Δ-classical orthogonal polynomials

We obtain the difference equation satisfied by the first associated $P^{(1)}_{n-1}$ of the polynomial of a discrete variable, P_{n-1}, orthogonal with respect to the classical linear functional $\tilde{\mathcal{L}}$ (with $\mathcal{L}$ satisfying $\Delta(\tilde{\phi}\tilde{\mathcal{L}}) = \tilde{\psi}\tilde{\mathcal{L}}$) [Atakishiyev et al., 1988], [Ronveaux et al. 1998a], [Foupouagnigni et al., 1998b] by limit processes ($\lim_{q\to 1,\,\omega\to 1} D_{q,\omega} = \Delta$) :

$$\mathcal{Q}^{**d}_{2,n-1}\mathcal{Q}^{*d}_{2,n-1}\left[P^{(1)}_{n-1}(x)\right] = 0, \tag{5.21}$$

where

$$\begin{aligned}\mathcal{Q}^{*d}_{2,n-1} &= \lim_{\omega\to 1,\,q\to 1}\mathcal{Q}^{*+}_{2,n-1}\\ &= \tilde{\phi}_{(2)}\mathcal{T}^2 - (2\tilde{\phi}_{(1)} + \tilde{\phi}_{(1)} - \lambda_n)\mathcal{T} + (\tilde{\phi}_{(0)} + \tilde{\phi}_{(0)})\mathcal{I}_d\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}^{**d}_{2,n-1} &= \lim_{\omega\to 1,\,q\to 1}\mathcal{Q}^{**+}_{2,n-1}\\ &= (\tilde{\phi}_{(3)} + \tilde{\psi}_{(3)})(\lambda_n - 2\tilde{\phi}_{(1)} - \tilde{\psi}_{(1)} - 2\tilde{\phi}_{(2)} - \tilde{\psi}_{(2)})\mathcal{T}^2\\ &\quad+[\tilde{\phi}_{(3)}(2\tilde{\phi}_{(2)} + 4\tilde{\phi}_{(1)} + 2\tilde{\psi}_{(1)} - 2\lambda_n) + \tilde{\phi}_{(2)}(2\tilde{\phi}_{(1)} + \tilde{\psi}_{(1)} + \tilde{\psi}_{(3)} - 2\lambda_n)\\ &\quad+2\tilde{\phi}_{(1)}(\tilde{\psi}_{(2)} + \tilde{\psi}_{(3)} - \lambda_n) + (\tilde{\psi}_{(2)} + \tilde{\psi}_{(3)})(\tilde{\psi}_{(1)} - \lambda_n) + \lambda_n(\lambda_n - \tilde{\psi}_{(1)})]\mathcal{T}\\ &\quad+\tilde{\phi}_{(1)}(\lambda_n - 2\tilde{\phi}_{(2)} - \tilde{\psi}_{(2)} - 2\tilde{\phi}_{(3)} - \tilde{\psi}_{(3)})\mathcal{I}_d,\end{aligned}$$

with the notations

$$\tilde{\phi}_{(j)} \equiv \tilde{\phi}(x+j),\ \tilde{\psi}_{(j)} \equiv \tilde{\psi}(x+j),\ \lambda_n = \lim_{q\to 1}\lambda_{n,0} = -n(\tilde{\psi}' + (n-1)\frac{\tilde{\phi}''}{2}).$$

The results given in this chapter (see Equations (5.12) and (5.14)), which agree with the ones obtained using the Stieltjes properties of the associated linear functional [Foupouagnigni et al., 98e19], can be used for connection problems (see [Askey, 1965, 1975], [Askey et al.. 1984], [Lewanowicz, 1995, 1996], [Godoy et al.. 1997a] [Area et al.. 1998b]) , expanding the first associated $P^{(1)}_{n-1}$ in terms of P_n, in the same spirit as in [Lewanowicz, 1995]: and also in order to represent finite modifications inside the Jacobi matrices of the q-classical starting family [Ronveaux et al., 1996]. We have also computed the coefficients of the fourth order q-difference equation satisfied by the first associated q-classical orthogonal polynomials appearing in the q-Hahn tableau. In particular, from the Big q-Jacobi polynomials, we derive by limit processes [Koekoek et al.. 1996] the fourth-order differential (resp. q-difference) equation satisfied by the first associated classical (resp. q-classical) orthogonal polynomials.

For the Little q-Jacobi polynomials, for example, the operators $\mathcal{Q}^{*}_{2,n-1}$ and $\mathcal{Q}^{**}_{2,n-1}$ are given below, with the notation: $\nu = q^n$.

$$\begin{aligned}\mathcal{Q}^{*}_{2,n-1} &= q\,x\left[(q^2x-1)\mathcal{G}_q^2 - \nu^{-1}(-\nu - a\,\nu + q^2\,x\,a\,b\,\nu^2 + q\,x)\mathcal{G}_q\right.\\ &\quad\left.+\,a(-1+b\,q\,x)\mathcal{I}_d\right],\\ \mathcal{Q}^{**}_{2,n-1} &= \nu^{-1}q^4x^2\left[qa(-1+bq^4x)\times\right.\\ &\quad(q^3xab\nu + q^3xab\nu^2 + q^2x\nu + q^2x - q\nu - qa\nu - \nu - a\nu)\mathcal{G}_q^2\\ &-\ \nu^{-1}(q^5x^2 + a\nu^2 + q\nu^2 - q^2x\nu^2 - q^3xab\nu^3 + q^7x^2a^2b^2\nu^3\\ &-\ q^3xa^2b\nu^3 - q^5xab\nu^3 + q^2a^2\nu^2 - q^5xab\nu^2 - q^5xa^2b\nu^2 + q^2a\nu^2\end{aligned}$$

$$
\begin{aligned}
- & \quad q^5xa^2b\nu^3 - q^2xa\nu - q^4xa\nu - q^2x\nu - q^4x\nu - q^3xa\nu + q^5x^2\nu \\
- & \quad q^3x\nu + q^7x^2a^2b^2\nu^4 + q^6x^2ab\nu - q^4xa^2b\nu^3 + qa^2\nu^2 - q^2xa\nu^2 \\
+ & \quad 2q^6x^2ab\nu^2 + q^6x^2ab\nu^3 + 2qa\nu^2 + \nu^2 - q^4xab\nu^3)\mathcal{T}_q \\
+ & \quad (-1 + qx)(q^4xab\nu + q^4xab\nu^2 + q^3x\nu + q^3x - q\nu \\
- & \quad qa\nu - \nu - a\nu)\mathcal{I}_d] \, .
\end{aligned}
$$

Chapter 6

Difference equations for the rth associated Laguerre-Hahn orthogonal polynomials

6.1 Introduction

Using the properties of the Stieltjes function of a given Laguerre-Hahn linear functional, we derive the single fourth-order difference equation satisfied by the rth associated $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials [Foupouagnigni et al., 1998d, 1998e]. We deduce by the limit process, $\lim_{q\to 1} \mathcal{D}_q = \frac{d}{dx}$, the fourth-order differential equation satisfied by the rth associated $\mathcal{D}$-Laguerre-Hahn orthogonal polynomials [Belmehdi et al., 1991].

Moreover, we use Theorem 4.2 to give the fourth-order difference equation satisfied by the rth associated $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials. Then follows, immediately, the fourth-order difference equation satisfied by the rth associated Δ-Laguerre-Hahn orthogonal polynomials [Letessier et al., 1996], [Foupouagnigni et al., 1998b, 1998c].

6.2 The associated $\mathcal{D}_q$-Laguerre-Hahn linear functional

6.2.1 The associated $\mathcal{D}_q$-Laguerre-Hahn linear functional is a $\mathcal{D}_q$-Laguerre-Hahn linear functional

Theorem 6.1 (Foupouagnigni et al., 1998e) *The associated of any integer order of the regular linear functional belonging to the $\mathcal{D}_q$-Laguerre-Hahn class belongs to the $\mathcal{D}_q$-Laguerre-Hahn class.*

The proof of the above theorem is given by induction on the order of association using the following proposition.

Proposition 6.1 (Foupouagnigni et al., 1998e) *Let $\mathcal{L}$ be a given regular linear functional; $\mathcal{L}^{(r)}$ the associated of order r of $\mathcal{L}$ and $S_r(\equiv S(\mathcal{L}^{(r)}))$ the Stieltjes function of $\mathcal{L}^{(r)}$.*

If S_r satisfies the $\mathcal{D}_q$-Riccati difference equation,

$$\begin{aligned}\phi(qx)\mathcal{D}_q S_r(x) &= G_r(x;q)S_r(x)\mathcal{G}_q S_r(x) + E_r(x;q)S_r(x) \\ &+ F_r(x;q)\mathcal{G}_q S_r(x) + H_r(x;q), \quad r \geq 0, \end{aligned} \tag{6.1}$$

where ϕ is a non-zero polynomial and E_r, F_r, G_r and H_r are polynomials in the variable x depending on q, then the same property holds for S_{r+1}:

$$\phi(qx)D_{q,\omega} S_{r+1}(x) = G_{r-1}(x;q)S_{r+1}(x)\mathcal{G}_q S_{r+1}(x)$$

$$+E_{r+1}(x;q)S_{r+1}(x) \quad (6.2)$$
$$+F_{r+1}(x;q)\mathcal{G}_q S_{r+1}(x) + H_{r+1}(x;q),$$

with

$$G_{r+1} = \frac{H_r}{\gamma_r}, \quad (6.3)$$

$$E_{r+1} = (qx-\beta_r)\frac{H_r}{\gamma_r} - F_r, \quad (6.4)$$

$$F_{r+1} = (x-\beta_r)\frac{H_r}{\gamma_r} - E_r. \quad (6.5)$$

$$H_{r+1} = -\phi(qx) + \gamma_r G_r - (qx-\beta_r)E_r - (x-\beta_r)F_r$$
$$+(x-\beta_r)(qx-\beta_r)\frac{H_r}{\gamma_r}. \quad (6.6)$$

Proof: Application of the $\mathcal{D}_q$-derivative rule

$$\mathcal{D}_q\left(\frac{f}{g}\right)(x) \equiv \frac{\frac{f(qx)}{g(qx)} - \frac{f(x)}{g(x)}}{(q-1)x} = \frac{g(x)\mathcal{D}_q f(x) - f(x)\mathcal{D}_q g(x)}{g(x)g(qx)}, \quad (6.7)$$

to (6.21) gives

$$\mathcal{D}_q S_r(x) = \frac{\gamma_r\left[1+\mathcal{D}_q S_{r+1}(x)\right]}{(qx-\beta_r+\mathcal{G}_q S_{r+1}(x))\,(x-\beta_r+S_{r+1}(x))}. \quad (6.8)$$

Using (6.21), (6.1) and (6.8), we obtain the $\mathcal{D}_q$-Riccati difference equation for S_{r+1}

$$\phi(qx)\mathcal{D}_q S_{r+1} = \frac{H_r}{\gamma_r} S_{r+1}\mathcal{G}_q S_{r+1}$$
$$+[(qx-\beta_r)\frac{H_r}{\gamma_r} - F_r]S_{r+1} - [(x-\beta_r)\frac{H_r}{\gamma_r} - E_r]\mathcal{G}_q S_{r+1}$$
$$-\phi(qx) + \gamma_r G_r - (qx-\beta_r)E_r - (x-\beta_r)F_r.$$

Identification of the previous difference equation with (6.2) completes the proof. □

Remark 6.1 *Use of (6.4)-(6.6) gives the following properties:*

i)

$$E_{r+1} - F_{r+1} - E_r + F_r = (q-1)x\,\frac{H_r}{\gamma_r}. \quad (6.9)$$

$$E_{r+1} + F_{r+1} + E_r + F_r = \frac{H_r}{\gamma_r}\left((1-q)x - 2\beta_r\right). \quad (6.10)$$

$$E_{r+1}F_{r+1} - E_rF_r = \phi(qx)\frac{H_r}{\gamma_r} + \frac{H_rH_{r+1}}{\gamma_r} - H_rG_r. \quad (6.11)$$

ii) Knowing polynomials ϕ, E_0, F_0, G_0, H_0, β_n and γ_n, $n \geq 0$, we can compute the coefficients E_i, F_i and H_i for all $i \geq 1$ using equations (6.4)-(6.6).

Note that the coefficients β_n and γ_n of the three-term recurrence relation (see (2.18)), for $\mathcal{D}_q$-semi-classical orthogonal polynomials of class one are given by Theorem 8.1.

Let $\mathcal{L}$ be a regular $\mathcal{D}_q$-Laguerre-Hahn linear functional. By Theorem 6.1, the rth associated of $\mathcal{L}$, $\mathcal{L}^{(r)}$, belongs to the $\mathcal{D}_q$-Laguerre-Hahn class and its Stieltjes function S_r satisfies the following $\mathcal{D}_q$-Riccati difference equation

$$\phi(qx)\mathcal{D}_q S_r(x) = G_r(x;q)S_r(x)\mathcal{G}_q S_r(x) + E_r(x;q)S_r(x)$$
$$+ F_r(x;q)\mathcal{G}_q S_r(x) + H_r(x;q), \quad r \geq 0,$$

where ϕ is a non-zero polynomial and E_r, F_r, G_r and H_r are polynomials in the variable x depending eventually on q. The following proposition proves that the degrees of the polynomials E_r, F_r, G_r and H_r are bounded.

Proposition 6.2 (Foupouagnigni et al., 1998e) *The polynomial coefficients E_r, F_r, G_r and H_r satisfy:*

$$\deg(H_r) \leq m - 1, \ \deg(E_r) \leq m \text{ and } \deg(F_r) \leq m, \quad r \geq 0, \tag{6.12}$$

where m is given by $m = \max\{\deg(E_0), \deg(F_0), \deg(H_0) + 1\}$.

Proof: For $r = 0$, (6.12) holds by hypothesis. Suppose that (6.12) holds up to a fixed integer r. Then using (6.4), we obtain

$$\deg(E_{r+1}) = \deg((qx - \beta_r)\frac{H_r}{\gamma_r} - F_r) \leq m, \tag{6.13}$$

by the above hypothesis. Likewise, using (6.5), we have $\deg(F_{r+1}) \leq m$. Finally use of (6.4) and the fact that the last two inequalities of (6.12) hold for any integer r, give

$$\deg(H_{r-1}) + 1 = \deg(F_{r+1} + E_{r+2}) \leq m.$$

□

Corollary 6.1 *Let $\mathcal{L}$ be a $\mathcal{D}_q$-semi-classical linear functional satisfying*

$$\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}, \tag{6.14}$$

where ϕ is any non-zero polynomial, ψ a polynomial of degree at least one, and E_r, F_r, G_r and H_r are defined by (6.1). Then the following properties hold:

$$\begin{aligned} \deg(H_r) &\leq \max\{\deg(\psi), \ \deg(\phi) - 1\} - 1 \quad \forall r \in \mathbb{N}, \\ \deg(E_r) &\leq \max\{\deg(\psi), \ \deg(\phi) - 1\} \quad \forall r \in \mathbb{N}, \\ \deg(F_r) &\leq \max\{\deg(\psi), \ \deg(\phi) - 1\} \quad \forall r \in \mathbb{N}. \end{aligned} \tag{6.15}$$

Proof: We shall give the proof by showing that

$$m = \max\{\deg(E_0), \deg(F_0), \deg(H_0) + 1\} \leq \max\{\deg(\psi), \deg(\phi) - 1\},$$

then use Proposition 6.2.

In fact, since $\mathcal{L}$ is $\mathcal{D}_q$-semi-classical satisfying (6.14), we deduce from Theorem 4.1 that $\mathcal{L}$ is a $\mathcal{D}_q$-Laguerre-Hahn linear functional and its Stieltjes function S_0 satisfies

$$\phi(qx)\mathcal{D}_q S_0 = G_0 S_0 \mathcal{G}_q S_0 + E_0 S_0 + F_0 \mathcal{G}_q S_0 + H_0,$$

where

$$\begin{aligned} &E_0(x;q) = \psi(x) - \mathcal{D}_q\phi(x), \\ &F_0(x;q) = G_0(x;q) = 0, \\ &H_0(x;q) = \mathcal{L}\theta_0\psi(x) - (\mathcal{D}_q\mathcal{L})\theta_0\phi(qx) - \mathcal{L}\theta_0\mathcal{D}_q\phi(x). \end{aligned} \tag{6.16}$$

From (6.16) results immediately

$$\deg(F_0) \leq \deg(E_0) \leq \max\{\deg(\psi), \ \deg(\phi) - 1\}. \tag{6.17}$$

It follows from (2.11) and (2.12) that

$$\deg(\mathcal{L}\theta_0\psi) \leq \deg(\psi) - 1, \ \deg(\mathcal{L}\theta_0\mathcal{D}_q\phi) \leq \deg(\phi) - 2. \tag{6.18}$$

To show that

$$\deg((\mathcal{D}_q\mathcal{L})\theta_0\phi(qx) \leq \deg(\phi) - 2, \tag{6.19}$$

we assume

$$\phi(qx) = \sum_{j=0}^{n} \phi_j x^j$$

and deduce that

$$\begin{aligned}\theta_0\phi(qx) &= \sum_{j=0}^{n-1} \phi_{j+1}x^j, \\ (\mathcal{D}_q\mathcal{L})\theta_0\phi(qx) &= \sum_{j=0}^{n-1} \tilde{\phi}_j x^j,\end{aligned}$$

with

$$\tilde{\phi}_j = \sum_{k=j}^{n-1} \phi_{k+1}\langle\mathcal{D}_q\mathcal{L}, x^{k-j}\rangle.$$

It turns out that

$$\tilde{\phi}_{n-1} = \phi_n\langle\mathcal{D}_q\mathcal{L}, 1\rangle = -\frac{1}{q}\phi_n\langle\mathcal{L}, \mathcal{D}_q^* 1\rangle = 0,$$

then $\deg((\mathcal{D}_q\mathcal{L})\theta_0\phi(qx)) \leq \deg(\phi) - 2$.

Using (6.18) and (6.19), we deduce that

$$\deg(H_0) \leq \max\{\deg(\psi),\ \deg(\phi) - 1\} - 1. \tag{6.20}$$

It results from (6.17) and (6.20) that

$$m = \max\{\deg(E_0),\ \deg(F_0),\ \deg(H_0) + 1\} \leq \max\{\deg(\psi),\ \deg(\phi) - 1\}.$$

The previous equation, combined with Proposition 6.2, completes the proof of the corollary. □

6.3 Fourth-order difference equation

Through the following steps, we will show that the rth associated Laguerre-Hahn orthogonal polynomials are solution of a single fourth-order linear difference equation with polynomial coefficients. To do this, we shall need the following identities giving relation between S_r and the associated orthogonal polynomials.

Lemma 6.1 (Sherman, 1933, Maroni, 1986a) *Let $\mathcal{L}$ be a given regular linear functional; $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal polynomials satisfying (2.18); $\mathcal{L}^{(r)}$ the rth associated of $\mathcal{L}$ and $S_r(\equiv S(\mathcal{L}^{(r)}))$ the Stieltjes function of $\mathcal{L}^{(r)}$; then, we have*

$$S_r(x) = \frac{-\gamma_r}{x - \beta_r + S_{r-1}(x)}, \quad \forall r \in \mathbb{N}, \tag{6.21}$$

where β_n and γ_n are defined in (2.18).

Lemma 6.2 (Dzoumba, 1985) *Let $\mathcal{L}$ be a given regular linear functional; $\{P_n\}_{n\in\mathbb{N}}$ the corresponding monic orthogonal polynomials satisfying (2.18); $\mathcal{L}^{(r)}$ the associated of order r of $\mathcal{L}$; $\{P_n^{(r)}\}_{n\in\mathbb{N}}$ the orthogonal polynomials associated to $\mathcal{L}^{(r)}$ and $S_r(= S(\mathcal{L}^{(r)}))$ the Stieltjes function of $\mathcal{L}^{(r)}$. Then, the following identity holds:*

$$S_r = -\gamma_r \frac{P_n^{(r+1)} + S_{n+r+1}P_{n-1}^{(r+1)}}{P_{n+1}^{(r)} + S_{n+r+1}P_n^{(r)}}, \tag{6.22}$$

where β_n and γ_n are defined in (2.18).

We suppose that $\mathcal{L}$ is a regular linear functional belonging to the $\mathcal{D}_q$-Laguerre-Hahn class, that $\mathcal{L}^{(r)}$ is the rth associated of $\mathcal{L}$, and that $\{P_n^{(r)}\}_{n\in\mathbb{N}}$ is the family of monic polynomials, orthogonal with respect to $\mathcal{L}^{(r)}$. If S_r represents the Stieltjes function of $\mathcal{L}^{(r)}$, then by Theorem 6.1, for any integer r, S_r satisfies a $\mathcal{D}_q$-Riccati difference equation (see (6.1)). We first apply the difference operator $\mathcal{G}_q$ to (6.22) and obtain

$$\mathcal{G}_q\, S_r = -\gamma_r \frac{\mathcal{G}_q\, P_n^{(r+1)} + \mathcal{G}_q\, S_{n+r+1}\mathcal{G}_q\, P_{n-1}^{(r+1)}}{\mathcal{G}_q\, P_{n+1}^{(r)} + \mathcal{G}_q\, S_{n+r+1}\mathcal{G}_q\, P_n^{(r)}}. \tag{6.23}$$

Secondly, we apply the quotient rule (see (6.7)) to (6.22) and obtain

$$\begin{aligned}
&\left(P_{n+1}^{(r)} + S_{n+r+1}P_n^{(r)}\right)\left(\mathcal{G}_q P_{n+1}^{(r)} + \mathcal{G}_q\, S_{n+r+1}\mathcal{G}_q\, P_n^{(r)}\right)\frac{\mathcal{D}_q\, S_r}{\gamma_r}\\
= &\left(\mathcal{G}_q\, P_n^{(r+1)}\mathcal{D}_q\, P_n^{(r)} - \mathcal{G}_q\, P_{n+1}^{(r)}\mathcal{D}_q\, P_{n-1}^{(r+1)}\right) S_{n+r+1}\\
+ &\left(\mathcal{G}_q\, P_{n-1}^{(r+1)}\mathcal{D}_q\, P_{n+1}^{(r)} - \mathcal{G}_q\, P_n^{(r)}\mathcal{D}_q\, P_n^{(r+1)}\right)\mathcal{G}_q\, S_{n+r+1}\\
- &\left(\mathcal{G}_q\, P_n^{(r)}\mathcal{D}_q\, P_{n-1}^{(r+1)} - \mathcal{G}_q\, P_{n-1}^{(r+1)}\mathcal{D}_q\, P_n^{(r)}\right) S_{n+r+1}\mathcal{G}_q\, S_{n+r+1}\\
- &\ \mathcal{G}_q\, P_{n+1}^{(r)}\mathcal{D}_q\, P_n^{(r+1)} + \mathcal{G}_q\, P_n^{(r+1)}\mathcal{D}_q\, P_{n+1}^{(r)}\\
+ &\left(\mathcal{G}_q\, P_n^{(r+1)}\mathcal{G}_q\, P_n^{(r)} - \mathcal{G}_q\, P_{n-1}^{(r+1)}\mathcal{G}_q\, P_{n+1}^{(r)}\right)\mathcal{D}_q\, S_{n+r+1}.
\end{aligned} \tag{6.24}$$

Further, we replace S_r, $\mathcal{G}_qS_r$ and $\mathcal{D}_qS_r$, given by (6.22), (6.23) and (6.24), respectively, in (6.1) and obtain after taking into account (2.24), the $\mathcal{D}_q$-Riccati difference equation for S_{n+r+1}; an equation which when compared with

$$\begin{aligned}
\phi(qx)\mathcal{D}_q\, S_{n+r+1}(x) &= G_{n+r+1}(x;q)S_{n+r+1}(x)\mathcal{G}_q\, S_{n+r+1}(x)\\
&\quad + E_{n+r+1}(x;q)S_{n+r+1}(x)\\
&\quad + F_{n+r+1}(x;q)\mathcal{G}_q\, S_{n+r+1}(x) + H_{n+r+1}(x;q),
\end{aligned}$$

gives the following proposition:

Proposition 6.3 (Foupouagnigni et al., 1998e) *The coefficients of the $\mathcal{D}_q$-Riccati difference equation for S_{n+r+1} are given by*

$$\begin{aligned}
\pi_{n,r}\, E_{n+r+1} &= -\phi(qx)\left(\mathcal{G}_q\, P_n^{(r+1)}\mathcal{D}_q\, P_n^{(r)} - \mathcal{G}_q\, P_{n+1}^{(r)}\mathcal{D}_q\, P_{n-1}^{(r+1)}\right)\\
&\quad - E_r P_{n-1}^{(r+1)}\mathcal{G}_q\, P_{n+1}^{(r)} - F_r P_n^{(r)}\mathcal{G}_q\, P_n^{(r+1)}\\
&\quad + \frac{H_r}{\gamma_r} P_n^{(r)}\mathcal{G}_q\, P_{n+1}^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_{n-1}^{(r+1)}\mathcal{G}_q\, P_n^{(r+1)},
\end{aligned} \tag{6.25}$$

$$\begin{aligned}
\pi_{n,r}\, F_{n+r+1} &= \phi(qx)\left(\mathcal{G}_q\, P_n^{(r)}\mathcal{D}_q\, P_n^{(r+1)} - \mathcal{G}_q\, P_{n-1}^{(r+1)}\mathcal{D}_q\, P_{n+1}^{(r)}\right)\\
&\quad - E_r P_n^{(r+1)}\mathcal{G}_q\, P_n^{(r)} - F_r P_{n+1}^{(r)}\mathcal{G}_q\, P_{n-1}^{(r+1)}\\
&\quad + \frac{H_r}{\gamma_r} P_{n+1}^{(r)}\mathcal{G}_q\, P_n^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_n^{(r+1)}\mathcal{G}_q\, P_{n-1}^{(r+1)},
\end{aligned} \tag{6.26}$$

$$\begin{aligned}
\pi_{n,r}\, H_{n+r+1} &= -\phi(qx)\left(\mathcal{G}_q\, P_n^{(r+1)}\mathcal{D}_q\, P_{n+1}^{(r)} - \mathcal{G}_q\, P_{n+1}^{(r)}\mathcal{D}_q\, P_n^{(r+1)}\right)\\
&\quad - E_r P_n^{(r+1)}\mathcal{G}_q\, P_{n+1}^{(r)} - F_r P_{n+1}^{(r)}\mathcal{G}_q\, P_n^{(r+1)}\\
&\quad + \frac{H_r}{\gamma_r} P_{n+1}^{(r)}\mathcal{G}_q\, P_{n+1}^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_n^{(r+1)}\mathcal{G}_q\, P_n^{(r+1)},
\end{aligned} \tag{6.27}$$

$$\begin{aligned}\pi_{n-1,r}\, H_{n+r} &= -\phi(qx)\left(\mathcal{G}_q\, P_{n-1}^{(r+1)}\mathcal{D}_q\, P_n^{(r)} - \mathcal{G}_q\, P_n^{(r)}\mathcal{D}_q\, P_{n-1}^{(r+1)}\right)\\ &\quad -E_r P_{n-1}^{(r+1)}\mathcal{G}_q\, P_n^{(r)} - F_r P_n^{(r)}\mathcal{G}_q\, P_{n-1}^{(r+1)}\\ &\quad +\frac{H_r}{\gamma_r} P_n^{(r)}\mathcal{G}_q\, P_n^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_{n-1}^{(r+1)}\mathcal{G}_q\, P_{n-1}^{(r+1)},\end{aligned} \tag{6.28}$$

where $\pi_{n,r}$ is given by (2.24).

We combine (2.24) and (6.25)-(6.28) to obtain:

Theorem 6.2 (Foupouagnigni et al., 1998e) *The associated polynomials obey:*

$$\begin{aligned}\phi(qx)\mathcal{D}_q\, P_n^{(r)} &= -E_{n+r+1}\mathcal{G}_q\, P_n^{(r)} - F_r P_n^{(r)}\\ &\quad +\frac{H_{n+r}}{\gamma_{n+r}}\mathcal{G}_q\, P_{n+1}^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_{n-1}^{(r+1)}.\end{aligned} \tag{6.29}$$

$$\begin{aligned}\phi(qx)\mathcal{D}_q\, P_{n-1}^{(r+1)} &= -E_{n+r+1}\mathcal{G}_q\, P_{n-1}^{(r+1)} + E_r P_{n-1}^{(r+1)}\\ &\quad +\frac{H_{n+r}}{\gamma_{n+r}}\mathcal{G}_q\, P_n^{(r+1)} - \frac{H_r}{\gamma_r} P_n^{(r)},\end{aligned} \tag{6.30}$$

$$\begin{aligned}\phi(qx)\mathcal{D}_q\, P_{n+1}^{(r)} &= F_{n+r+1}\mathcal{G}_q\, P_{n+1}^{(r)} - F_r P_{n+1}^{(r)}\\ &\quad -H_{n+r+1}\mathcal{G}_q\, P_n^{(r)} + \frac{\gamma_r H_{r-1}}{\gamma_{r-1}} P_n^{(r+1)},\end{aligned} \tag{6.31}$$

$$\begin{aligned}\phi(qx)\mathcal{D}_q\, P_n^{(r+1)} &= F_{n+r+1}\mathcal{G}_q\, P_n^{(r+1)} + E_r P_n^{(r+1)}\\ &\quad -H_{n+r+1}\mathcal{G}_q\, P_{n-1}^{(r+1)} - \frac{H_r}{\gamma_r} P_{n+1}^{(r)}.\end{aligned} \tag{6.32}$$

Proof: We subtract the two equations obtained after multiplying (6.25) (and (6.28), respectively) by $\mathcal{G}_q P_n^{(r)}$, $\mathcal{G}_q P_{n+1}^{(r+1)}$ and obtain

$$\begin{aligned}&\pi_{n,r} E_{n+r+1}\mathcal{G}_q\, P_n^{(r)} - \pi_{n-1,r} H_{n+r}\mathcal{G}_q\, P_{n+1}^{(r)}\\ &= -\phi(qx)\left(\mathcal{G}_q P_n^{(r)}\mathcal{G}_q P_n^{(r+1)} - \mathcal{G}_q P_{n+1}^{(r)}\mathcal{G}_q P_{n-1}^{(r+1)}\right)\mathcal{D}_q\, P_n^{(r)}\\ &- \left(\mathcal{G}_q P_n^{(r)}\mathcal{G}_q P_n^{(r+1)} - \mathcal{G}_q P_{n+1}^{(r)}\mathcal{G}_q P_{n-1}^{(r+1)}\right) F_r\, P_n^{(r)}\\ &+ \left(\mathcal{G}_q P_n^{(r)}\mathcal{G}_q P_n^{(r+1)} - \mathcal{G}_q P_{n+1}^{(r)}\mathcal{G}_q P_{n-1}^{(r+1)}\right)\frac{\gamma_r H_{r-1}}{\gamma_{r-1}}\, P_{n-1}^{(r+1)}.\end{aligned} \tag{6.33}$$

Then use of the relation obtained from (2.24)

$$\mathcal{G}_q P_n^{(r)}\mathcal{G}_q P_n^{(r+1)} - \mathcal{G}_q P_{n+1}^{(r)}\mathcal{G}_q P_{n-1}^{(r+1)} = \pi_{n,r}, \tag{6.34}$$

and the fact that

$$\pi_{n-1,r} = \frac{\pi_{n,r}}{\gamma_{n+r}},$$

transform (6.33) in (6.29).

Again, we multiply both sides of (6.25) (and(6.28), respectively) by $\mathcal{G}_q P_{n-1}^{(r+1)}$. $\mathcal{G}_q P_n^{(r+1)}$ and obtain two equations which subtracted give

$$\begin{aligned}&\pi_{n,r} E_{n+r+1}\mathcal{G}_q\, P_{n-1}^{(r+1)} - \pi_{n-1,r} H_{n+r}\mathcal{G}_q\, P_n^{(r+1)}\\ &= -\phi(qx)\left(\mathcal{G}_q P_n^{(r)}\mathcal{G}_q P_n^{(r+1)} - \mathcal{G}_q P_{n+1}^{(r)}\mathcal{G}_q P_{n-1}^{(r+1)}\right)\mathcal{D}_q\, P_{n-1}^{(r+1)}\\ &+ \left(\mathcal{G}_q P_n^{(r)}\mathcal{G}_q P_n^{(r+1)} - \mathcal{G}_q P_{n+1}^{(r)}\mathcal{G}_q P_{n-1}^{(r+1)}\right) E_r\, P_{n-1}^{(r+1)}\\ &+ \left(\mathcal{G}_q P_n^{(r)}\mathcal{G}_q P_n^{(r+1)} - \mathcal{G}_q P_{n+1}^{(r)}\mathcal{G}_q P_{n-1}^{(r+1)}\right)\frac{H_r}{\gamma_r}\, P_n^{(r)}.\end{aligned} \tag{6.35}$$

Then use of (6.34) transforms (6.35) in (6.30).

Equations (6.31) and (6.32) are obtained in the same way by combining (6.26), (6.27) and (6.34). □

For the sake of simplicity and uniformity we shall present difference equations in terms of powers of the operator $A_{q,\omega}$ instead of $D_{q,\omega}$. This is possible because for $q \neq 1$ or $\omega \neq 0$, all powers of the operator $D_{q,\omega}$ can be expressed in terms of the powers of $A_{q,\omega}$ and conversely. To do this we present the following lemma (proved by solving system of equations).

Lemma 6.3 *The powers of the operators $D_{q,\omega}$ and $A_{q,\omega}$ are linked by the following relations:*

$$
\begin{aligned}
D^0_{q,\omega} &= A^0_{q,\omega} = \mathcal{I}_d, \\
((q-1)x+\omega)\, D_{q,\omega} &= A_{q,\omega} - \mathcal{I}_d, \\
((q-1)x+\omega)^2\, D^2_{q,\omega} &= q^{-1}\, A^2_{q,\omega} - [2]_q\, q^{-1}\, A_{q,\omega} + \mathcal{I}_d, \\
((q-1)x+\omega)^3\, D^3_{q,\omega} &= q^{-3}\, A^3_{q,\omega} - [3]_q\, q^{-3}\, A^2_{q,\omega} + [3]_q\, A_{q,\omega} - \mathcal{I}_d, \\
((q-1)x+\omega)^4\, D^4_{q,\omega} &= q^{-6}\, A^4_{q,\omega} - (q-1)\,[4]_q\, q^{-6}\, A^3_{q,\omega} + (1+q^2)[3]_q\, q^{-5}\, A^2_{q,\omega} \\
&\quad -[4]_q\, q^{-3}\, A_{q,\omega} + \mathcal{I}_d, \\
((q-1)x+\omega)^5\, D^5_{q,\omega} &= q^{-10}\, A^5_{q,\omega} - [5]_q\, q^{-10}\, A^4_{q,\omega} + (1+q^2)\,[5]_q\, q^{-9}\, A^3_{q,\omega} \\
&\quad -(1+q^2)\,[5]_q\, q^{-7}\, A^2_{q,\omega} + [5]_q\, q^{-4}\, A_{q,\omega} - \mathcal{I}_d,
\end{aligned}
$$

$$
\begin{aligned}
A^0_{q,\omega} &= D^0_{q,\omega} = \mathcal{I}_d, \\
A_{q,\omega} &= ((q-1)x+\omega)D_{q,\omega} + \mathcal{I}_d, \\
A^2_{q,\omega} &= q\,((q-1)x+\omega)^2\, D^2_{q,\omega} + (1+q)\,((q-1)x+\omega)D_{q,\omega} + \mathcal{I}_d, \\
A^3_{q,\omega} &= q^3\,((q-1)x+\omega)^3\, D^3_{q,\omega} + q[3]_q\,((q-1)x+\omega)^2\, D^2_{q,\omega} + [3]_q\,((q-1)x+\omega)D_{q,\omega} + \mathcal{I}_d, \\
A^4_{q,\omega} &= q^6\,((q-1)x+\omega)^4\, D^4_{q,\omega} + q^3\,(q-1)\,[4]_q\,((q-1)x+\omega)^3\, D^3_{q,\omega} \\
&\quad +q\,(1+q^2)\,[3]_q\,((q-1)x+\omega)^2 D^2_{q,\omega} - (q-1)\,[4]_q\,((q-1)x+\omega)\, D_{q,\omega} + \mathcal{I}_d, \\
A^5_{q,\omega} &= q^{10}\,((q-1)x+\omega)^5\, D^5_{q,\omega} + q^6\,[5]_q\,((q-1)x+\omega)^4\, D^4_{q,\omega} \\
&\quad +q^3\,(1+q^2)\,[5]_q\,((q-1)x+\omega)^3 D^3_{q,\omega} + q\,(1+q^2)\,[5]_q\,((q-1)x-\omega)^2 D^2_{q,\omega} \\
&\quad +[5]_q\,((q-1)x+\omega)\, D_{q,\omega} + \mathcal{I}_d.
\end{aligned}
$$

Remark 6.2 *If we take $\omega = 0$, $q \neq 1$ (resp. $\omega \neq 0$, $q = 1$) in the previous lemma, we find the link between the powers of the operators $\mathcal{D}_q$ and $\mathcal{G}_q$ (resp. D_ω and $\mathcal{T}_\omega$).*

Theorem 6.3 (Foupouagnigni et al., 1998e) *Let $\mathcal{L}$ be a regular linear functional belonging to the $\mathcal{D}_q$-Laguerre-Hahn class, $\mathcal{L}^{(r)}$ the rth associated of $\mathcal{L}$ and $\{P_n^{(r)}\}_{n\in\mathbb{N}}$ the family of monic polynomials, orthogonal with respect to $\mathcal{L}^{(r)}$. If S_r represents the Stieltjes function of $\mathcal{L}^{(r)}$, by Theorem 6.1. for any integer r, S_r satisfies a $\mathcal{D}_q$-Riccati difference equation (see (6.1)). The associated polynomials $P_n^{(r)}$ satisfy*

$$\mathcal{D}_{r,n}\left[P_n^{(r)}\right] = \mathcal{N}_{r+1,n-1}\left[P_{n-1}^{(r+1)}\right], \tag{6.36}$$

$$\bar{\mathcal{D}}_{r+1,n-1}\left[P_{n-1}^{(r+1)}\right] = \bar{\mathcal{N}}_{r,n}\left[P_n^{(r)}\right], \tag{6.37}$$

where the operators $\mathcal{D}_{r,n}$, $\mathcal{N}_{r+1,n-1}$, $\bar{\mathcal{D}}_{r+1,n-1}$ and $\bar{\mathcal{N}}_{r,n}$ are given by

$$\mathcal{D}_{r,n} = a_2\, \mathcal{G}_q^2 + a_1\, \mathcal{G}_q + a_0\, \mathcal{I}_d, \quad \mathcal{N}_{r+1,n-1} = \tilde{a}_1\, \mathcal{G}_q + \tilde{a}_0\, \mathcal{I}_d, \tag{6.38}$$

$$\bar{\mathcal{D}}_{r+1,n-1} = b_2\, \mathcal{G}_q^2 - b_1\, \mathcal{G}_q + b_0\, \mathcal{I}_d, \quad \bar{\mathcal{N}}_{r,n} = \tilde{b}_1\, \mathcal{G}_q + \tilde{b}_0\, \mathcal{I}_d. \tag{6.39}$$

The coefficients a_j, b_j, $\tilde{a}_j$ and $\tilde{b}_j$ are defined as

$$
\begin{aligned}
a_2 &= K_{3,0}(K_{1,1}\, K_{7,1} - K_{3,1}\, K_{8,1}), & b_2 &= K_{3,0}(K_{1,1}\, K_{7,1} - K_{3,1}\, K_{8,1}) \\
a_1 &= -K_{2,1}(K_{3,0}\, K_{7,1} + K_{1,0}\, K_{3,1}), & b_1 &= -K_{5,1}(K_{3,0}\, K_{7,1} + K_{1,0}\, K_{3,1}) \\
a_0 &= K_{3,1}(K_{2,0}\, K_{2,1} + K_{4,1}\, K_{6,0}), & b_0 &= K_{3,1}(K_{5,0}\, K_{5,1} + K_{4,0}\, K_{6,1}) \\
\tilde{a}_1 &= K_{4,1}(K_{3,0}\, K_{7,1} + K_{1,0}\, K_{3,1}), & \tilde{b}_1 &= K_{6,1}(K_{3,0}\, K_{7,1} + K_{1,0}\, K_{3,1}) \\
\tilde{a}_0 &= -K_{3,1}(K_{2,1}\, K_{4,0} + K_{4,1}\, K_{5,0}), & \tilde{b}_0 &= -K_{3,1}(K_{5,1}\, K_{6,0} + K_{6,1}\, K_{2,0}),
\end{aligned}
\tag{6.40}
$$

where the coefficients $K_{i,j}$ are given below with the notations:

$$\left\{\begin{array}{l} K_i \equiv K_{i,0}(x;r,n,q) = K_i(x;r,n,q), \\ K_{i,j} \equiv K_{i,j}(x;r,n,q) = \mathcal{G}_q^j K_i(x;r,n,q) = K_i(q^j x;r,n,q). \end{array}\right. \tag{6.41}$$

$$\begin{aligned} K_1 &= \frac{\phi(qx)}{(q-1)x} + E_{n+r+1}(x;q), \; K_2 = \frac{\phi(qx)}{(q-1)x} - F_r(x;q), \\ K_3 &= \frac{H_{n+r}(x;q)}{\gamma_{n+r}}, K_4 = \left\{\begin{array}{ll} \gamma_r \frac{H_{r-1}(x;q)}{\gamma_{r-1}} & \text{if } r \geq 1 \\ \gamma_0 G_0 & \text{if } r = 0 \end{array}\right., \\ K_5 &= \frac{\phi(qx)}{(q-1)x} + E_r(x;q), \; K_6 = -\frac{H_r(x;q)}{\gamma_r}, \\ K_7 &= \frac{\phi(qx)}{(q-1)x} - F_{n+r+1}(x;q), \; K_8 = -\gamma_{n+r+1}\frac{H_{n-r+1}(x;q)}{\gamma_{n+r+1}}. \end{aligned} \tag{6.42}$$

Proof: Use of the relation

$$(q-1)x\mathcal{D}_q P(x) = \mathcal{G}_q P(x) - P(x) \quad \forall P \in \mathbb{P},$$

transforms relations (6.29)-(6.32) in

$$K_1\, \mathcal{G}_q\, P_n^{(r)} = K_2\, P_n^{(r)} + K_3\, \mathcal{G}_q\, P_{n+1}^{(r)} + K_4\, P_{n-1}^{(r-1)}, \tag{6.43}$$

$$K_1\, \mathcal{G}_q\, P_{n-1}^{(r+1)} = K_5\, P_{n-1}^{(r-1)} + K_3\, \mathcal{G}_q\, P_n^{(r-1)} + K_6\, P_n^{(r)}, \tag{6.44}$$

$$K_7\, \mathcal{G}_q\, P_{n+1}^{(r)} = K_2\, P_{n+1}^{(r)} + K_8\, \mathcal{G}_q\, P_n^{(r)} + K_4\, P_n^{(r-1)}, \tag{6.45}$$

$$K_7\, \mathcal{G}_q\, P_n^{(r+1)} = K_5\, P_n^{(r-1)} + K_8\, \mathcal{G}_q\, P_{n-1}^{(r-1)} + K_6\, P_{n+1}^{(r)}, \tag{6.46}$$

where K_j are given by (6.42).

In the first step, we solve equations (6.43) and (6.44) in terms of $\mathcal{G}_q\, P_{n+1}^{(r)}$ and $\mathcal{G}_q\, P_n^{(r+1)}$ and obtain

$$\mathcal{G}_q\, P_{n-1}^{(r)} = \frac{K_1\, \mathcal{G}_q\, P_n^{(r)} - K_2\, P_n^{(r)} - K_4\, P_{n-1}^{(r+1)}}{K_3}. \tag{6.47}$$

$$\mathcal{G}_q\, P_n^{(r+1)} = \frac{K_1\, \mathcal{G}_q\, P_{n-1}^{(r+1)} - K_5\, P_{n-1}^{(r-1)} - K_6\, P_n^{(r)}}{K_3}. \tag{6.48}$$

In the second step we apply the operator $\mathcal{G}_q$ to both sides of (6.45) and (6.46) and get

$$\begin{aligned} K_{7,1}\, \mathcal{G}_q^2\, P_{n+1}^{(r)} &= K_{2,1}\, \mathcal{G}_q\, P_{n-1}^{(r)} + K_{8,1}\, \mathcal{G}_q^2\, P_n^{(r)} + K_{4,1}\, \mathcal{G}_q\, P_n^{(r+1)}, \\ K_{7,1}\, \mathcal{G}_q^2\, P_n^{(r-1)} &= K_{5,1}\, \mathcal{G}_q\, P_n^{(r-1)} + K_{8,1}\, \mathcal{G}_q^2\, P_{n-1}^{(r+1)} + K_{6,1}\, \mathcal{G}_q\, P_{n+1}^{(r)}. \end{aligned}$$

Then, we replace $\mathcal{G}_q\, P_{n-1}^{(r)}$ and $\mathcal{G}_q\, P_n^{(r+1)}$ given by (6.47) and (6.48) respectively, in the two previous equations and obtain

$$\begin{aligned} \mathcal{G}_q^2\, P_{n+1}^{(r)} &= \frac{K_{8,1}}{K_{7,1}}\mathcal{G}_q^2\, P_n^{(r)} + \frac{K_1\, K_{2,1}}{K_3\, K_{7,1}}\mathcal{G}_q\, P_n^{(r)} - \frac{(K_2\, K_{2,1} + K_6\, K_{4,1})}{K_3\, K_{7,1}} P_n^{(r)} \\ &+ \frac{K_1\, K_{4,1}}{K_3 K_{7,1}}\, \mathcal{G}_q\, P_{n-1}^{(r+1)} - \frac{(K_4\, K_{2,1} + K_5\, K_{4,1})}{K_3\, K_{7,1}} P_{n-1}^{(r+1)}, \end{aligned} \tag{6.49}$$

$$\begin{aligned} \mathcal{G}_q^2\, P_n^{(r+1)} &= \frac{K_{8,1}}{K_{7,1}}\mathcal{G}_q^2\, P_{n-1}^{(r+1)} + \frac{K_1\, K_{5,1}}{K_3\, K_{7,1}}\mathcal{G}_q\, P_{n-1}^{(r+1)} - \frac{(K_5\, K_{5,1} + K_4\, K_{6,1})}{K_3\, K_{7,1}} P_{n-1}^{(r-1)} \\ &+ \frac{K_1\, K_{6,1}}{K_3 K_{7,1}}\, \mathcal{G}_q\, P_n^{(r)} - \frac{(K_6\, K_{5,1} + K_2\, K_{6,1})}{K_3\, K_{7,1}} P_n^{(r)}. \end{aligned} \tag{6.50}$$

In the third step we apply the operator $\mathcal{G}_q$ to both sides of (6.43) and (6.44) and obtain

$$\begin{aligned} K_{1,1}\,\mathcal{G}_q^2\,P_n^{(r)} &= K_{2,1}\,\mathcal{G}_q\,P_n^{(r)} + K_{3,1}\,\mathcal{G}_q^2\,P_{n+1}^{(r)} + K_{4,1}\,\mathcal{G}_q\,P_{n-1}^{(r+1)}, \\ K_{1,1}\,\mathcal{G}_q^2\,P_{n-1}^{(r+1)} &= K_{5,1}\,\mathcal{G}_q\,P_{n-1}^{(r+1)} + K_{3,1}\,\mathcal{G}_q^2\,P_n^{(r+1)} + K_{6,1}\,\mathcal{G}_q\,P_n^{(r)}. \end{aligned}$$

Finally, use of (6.49) and (6.50) transforms the two previous equations in

$$\begin{aligned} &K_{3,0}(K_{1,1}\,K_{7,1} - K_{3,1}\,K_{8,1})\mathcal{G}_q^2\,P_n^{(r)} - K_{2,1}(K_{3,0}\,K_{7,1} + K_{1,0}\,K_{3,1})\mathcal{G}_q\,P_n^{(r)} \\ &+K_{3,1}(K_{2,0}\,K_{2,1} + K_{4,1}\,K_{6,0})P_n^{(r)} = \\ &K_{4,1}(K_{3,0}\,K_{7,1} + K_{1,0}\,K_{3,1})\mathcal{G}_q\,P_{n-1}^{(r+1)} - K_{3,1}(K_{2,1}\,K_{4,0} + K_{4,1}\,K_{5,0})P_{n-1}^{(r+1)}, \end{aligned}$$

$$\begin{aligned} &K_{3,0}(K_{1,1}\,K_{7,1} - K_{3,1}\,K_{8,1})\mathcal{G}_q^2\,P_{n-1}^{(r+1)} - K_{5,1}(K_{3,0}\,K_{7,1} + K_{1,0}\,K_{3,1})\mathcal{G}_q\,P_{n-1}^{(r+1)} \\ &+K_{3,1}(K_{5,0}\,K_{5,1} + K_{4,0}\,K_{6,1})P_{n-1}^{(r+1)} = \\ &K_{6,1}(K_{3,0}\,K_{7,1} + K_{1,0}\,K_{3,1})\mathcal{G}_q\,P_n^{(r)} - K_{3,1}(K_{5,1}\,K_{6,0} + K_{6,1}\,K_{2,0})P_{n-1}^{(r+1)}. \end{aligned}$$

thus the proof of Theorem 6.3 is complete. □

After proving Theorem 6.3. we have now all ingredients to derive the single fourth-order difference equation satisfied by $P_n^{(r)}$.

In fact, we apply the operator $\mathcal{G}_q$ to both sides of (6.36) and eliminate $\mathcal{G}_q{}^2P_{n-1}^{(r+1)}$ in the equation obtained, by using (6.37) and obtain

$$\begin{aligned} c_3\,\mathcal{G}_q{}^3P_n^{(r)} &+ c_2\,\mathcal{G}_q{}^2P_n^{(r)} + c_1\,\mathcal{G}_q\,P_n^{(r)} + c_0\,P_n^{(r)} \\ &= \tilde{c}_1\,\mathcal{G}_q\,P_{n-1}^{(r+1)} + \tilde{c}_0\,P_{n-1}^{(r+1)}, \end{aligned} \tag{6.51}$$

with polynomials c_j and $\tilde{c}_j$ given by

$$\begin{aligned} c_3 &= b_2\,a_{2,1},\; c_2 = b_2\,a_{1,1},\; c_1 = b_2\,a_{0,1} - \tilde{b}_1\tilde{a}_{1,1} \\ c_0 &= -\tilde{b}_0\tilde{a}_{1,1},\; \tilde{c}_1 = b_2\tilde{a}_{0,1} - b_1\,\tilde{a}_{1,1},\; \tilde{c}_0 = -b_0\tilde{a}_{1,1}, \end{aligned}$$

where $\chi_{i,j} = \mathcal{G}_q^j\chi_i$ for $\chi_i \in \{a_i, b_i, \tilde{a}_i, \tilde{b}_i\}$.

By the same process, we apply the operator $\mathcal{G}_q$ to both sides of (6.51) and eliminate $\mathcal{G}_q{}^2P_{n-1}^{(r+1)}$ in the equation obtained, by using (6.37) and get

$$\begin{aligned} d_4\,\mathcal{G}_q{}^4P_n^{(r)} &+ d_3\,\mathcal{G}_q{}^3P_n^{(r)} + d_2\,\mathcal{G}_q{}^2P_n^{(r)} + d_1\,\mathcal{G}_q\,P_n^{(r)} + d_0\,P_n^{(r)} \\ &= \tilde{d}_1\,\mathcal{G}_q\,P_{n-1}^{(r+1)} + \tilde{d}_0\,P_{n-1}^{(r+1)}, \end{aligned} \tag{6.52}$$

with

$$\begin{aligned} d_4 &= b_2\,b_{2,1}a_{2,1},\; d_3 = b_2\,b_{2,1}a_{1,1},\; d_2 = b_2(a_{0,1}b_{2,1} - \tilde{a}_{1,1}\tilde{b}_{1,1}), \\ d_1 &= (\tilde{b}_1\,b_{1,1}\,\tilde{a}_{1,1} - \tilde{b}_1\,\tilde{a}_{0,1}\tilde{b}_{2,1}) - \tilde{b}_2\,\tilde{b}_{0,1}\tilde{a}_{1,1}),\; d_0 = (\tilde{a}_{1,1}\,b_{1,1} - \tilde{a}_{0,1}\,b_{2,1})\tilde{b}_0, \\ \tilde{d}_1 &= (b_1\,b_{1,1}\,\tilde{a}_{1,1} - b_1\,\tilde{a}_{0,1}b_{2,1}) - b_2\,b_{0,1}\tilde{a}_{1,1}),\; \tilde{d}_0 = (\tilde{a}_{1,1}\,b_{1,1} - \tilde{a}_{0,1}\,b_{2,1})b_0. \end{aligned}$$

We, therefore, deduce from (6.36), (6.51) and (6.52) the following result:

Theorem 6.4 (Foupouagnigni et al., 1998e) *The associated polynomials $P_n^{(r)}$, for any integer n and for any integer r. satisfies the single fourth-order difference equation*

$$\begin{vmatrix} a_2\,\mathcal{G}_q^2P_n^{(r)} + a_1\,\mathcal{G}_qP_n^{(r)} + a_0\,P_n^{(r)} & \tilde{a}_1 & \tilde{a}_0 \\ c_3\,\mathcal{G}_q^3P_n^{(r)} + c_2\,\mathcal{G}_q^2P_n^{(r)} + c_1\,\mathcal{G}_qP_n^{(r)} + c_0\,P_n^{(r)} & \tilde{c}_1 & \tilde{c}_0 \\ d_4\,\mathcal{G}_q^4P_n^{(r)} + d_3\,\mathcal{G}_q^3P_n^{(r)} + d_2\,\mathcal{G}_q^2P_n^{(r)} + d_1\,\mathcal{G}_qP_n^{(r)} + d_0\,P_n^{(r)} & \tilde{d}_1 & \tilde{d}_0 \end{vmatrix} = 0, \tag{6.53}$$

which by Lemma 6.3 can be written in the two different forms:

$$\sum_{j=0}^{4} I_j(r,n,q;x)\,\mathcal{G}_q^j\,P_n^{(r)}(x) = 0, \tag{6.54}$$

$$\sum_{j=0}^{4} I_j^*(r,n,q;x)\,\mathcal{D}_q^j\,P_n^{(r)}(x) = 0. \tag{6.55}$$

where $I_j(r,n,q;x)$, $I_j^*(r,n,q;x)$ are polynomials in the variable x and depending on r,n and q.

6.3.1 Fourth-order differential equation for $P_n^{(r)}$

We deduce from the previous results and by the limit process, $\lim_{q\to 1}\mathcal{D}_q = \frac{d}{dx}$, the fourth-order differential equation satisfied by the rth associated orthogonal polynomial of the $\mathcal{D}$-Laguerre-Hahn class [Belmehdi et al., 1991]. Moreover, we recover relations used in [Belmehdi et al., 1991] to derive the fourth-order differential equation satisfied by the rth associated $\mathcal{D}$-Laguerre-Hahn orthogonal polynomials.

From (6.55) and by the limit process we get

$$\sum_{j=0}^{4} I_j^*(r,n,1;x)\,\frac{d}{dx}^{j}\,P_n^{(r)}(x) = 0, \tag{6.56}$$

where $I_j^*(r,n,1;x) = \lim_{q\to 1} I_j^*(r,n,q;x)$.

To compare more easily the equations obtained from (6.29)-(6.32) by this limit process with those given in [Belmehdi et al., 1991], we state the following lemma:

Lemma 6.4 *If $E_r(x;1)$, $F_r(x;1)$ are the limit when $q\to 1$ of $E_r(x;q)$ and $F_r(x;q)$ respectively, we have*

$$E_{n+r+1}(x;1) - E_r(x;1) = F_{n+r+1}(x;1) - F_r(x;1) \quad \forall n\in\mathbb{N}. \tag{6.57}$$

Proof: We shall prove the lemma using the relation

$$E_{r+1}(x;1) - E_r(x;1) = F_{r+1}(x;1) - F_r(x;1),$$

easily derived by limit process from (6.9).

In fact, use of the previous relation gives:

$$\begin{aligned} E_{n+r+1}(x;1) - E_r(x;1) &= \sum_{j=0}^{n} E_{j+r+1}(x;1) - E_{j+r}(x;1) \\ &= \sum_{j=0}^{n} F_{j+r+1}(x;1) - F_{j+r}(x;1) \\ &= F_{n+r+1}(x;1) - F_r(x;1). \end{aligned}$$

□

When we take the limit of equations (6.29)-(6.32) as $q\to 1$, we obtain, taking into account the previous lemma [Magnus, 1984], [Dzoumba, 1985] [Belmehdi et al., 1994],

$$\begin{aligned} \phi\frac{d}{dx}P_n^{(r)} &= -\frac{C_{n+r+1}+C_r}{2}P_n^{(r)} + \frac{D_{n+r}}{\gamma_{n+r}}P_{n+1}^{(r)} + \frac{\gamma_r}{\gamma_{r-1}}D_{r-1}P_{n-1}^{(r+1)}, \\ \phi\frac{d}{dx}P_{n-1}^{(r+1)} &= -\frac{C_{n+r+1}-C_r}{2}P_{n-1}^{(r+1)} + \frac{D_{n+r}}{\gamma_{n+r}}P_n^{(r+1)} - \frac{D_r}{\gamma_r}P_n^{(r)}, \\ \phi\frac{d}{dx}P_{n-1}^{(r)} &= \frac{C_{n+r-1}-C_r}{2}P_{n-1}^{(r)} - D_{n+r-1}P_n^{(r)} + \frac{\gamma_r}{\gamma_{r-1}}D_{r-1}P_n^{(r+1)}, \\ \phi\frac{d}{dx}P_n^{(r+1)} &= \frac{C_{n+r-1}+C_r}{2}P_n^{(r+1)} - D_{n+r-1}P_{n-1}^{(r+1)} - \frac{D_r}{\gamma_r}P_{n+1}^{(r)}, \end{aligned}$$

where the polynomial coefficients C_r and D_r are given by

$$C_r \equiv C_r(x) = E_r(x;1) + F_r(x;1),\ D_r \equiv D_r(x) = \lim_{q\to 1} H_r(x;q). \tag{6.58}$$

Notice that the previous four differential equations, already known earlier [Magnus, 1984], [Dzoumba, 1985], are exactly those which allow Belmehdi et al. (1991) to derive the fourth-order differential equation satisfied by the associated orthogonal polynomial of the $\mathcal{D}$-Laguerre-Hahn class. The coefficients C_r and D_r, for the associated $\mathcal{D}$-classical orthogonal polynomials, are given by

$$C_r = (x - \beta_r)(2r\phi_2 + \psi_1) - \phi',\ \frac{D_r}{\gamma_r} = (2r-1)\phi_2 + \psi_1,$$

where ϕ and ψ are the polynomials appearing in the Pearson differential equation satisfied by the regular linear functional $\mathcal{L}$: $\mathcal{D}(\phi\mathcal{L}) = \psi\mathcal{L}$, with

$$\phi(x) = \phi_2\, x^2 + \phi_1\, x + \phi_0,\ \psi(x) = \psi_1\, x + \psi_0.$$

6.3.2 Fourth-order difference equation for the rth associated $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials

We deduce the difference equation satisfied by the associated $D_{q,\omega}$-Laguerre-Hahn class from Theorem 6.4.

Consider $\mathcal{L}$ a $\mathcal{D}_q$-Laguerre-Hahn linear functional and $\{P_n\}_{n\in\mathbb{N}}$ the corresponding family of monic orthogonal polynomials. Let $P_n^{(r)}$ and $\mathcal{L}^{(r)}$ be the rth associated of P_n and $\mathcal{L}$, respectively. The Stieltjes function S_r of $\mathcal{L}^{(r)}$ satisfies (6.1):

$$\begin{aligned}\phi(qx)\mathcal{D}_q S_r(x) &= G_r(x;q)S_r(x)\mathcal{G}_q S_r(x) + E_r(x;q)S_r(x)\\ &+ F_r(x;q)\mathcal{G}_q S_r(x) + H_r(x;q),\ r \geq 0,\end{aligned}$$

where ϕ, E_r, F_r, G_r and H_r are polynomials in x and depending on q. It follows from Theorem 6.4 that $P_n^{(r)}$ satisfies the fourth-order q-difference equation (6.54) where the polynomials $I_j(r,n,q;x)$ depend on the polynomial coefficients ϕ, E_r, F_r, G_r and H_r. To be more explicit, we denote $I_j(r,n,q;x) = I_j(r,n,q;x;\phi,E_r,F_r,G_r,H_r)$.

It results from Theorem 4.2 and Lemma 2.5 that the polynomials $\{\tilde{P}_n\}_{n\in\mathbb{N}}$, with $\tilde{P}_n = \mathcal{T}_{\frac{-\omega}{1-q}} P_n$, are orthogonal with respect to $\tilde{\mathcal{L}} = \mathcal{T}_{\frac{-\omega}{1-q}} \mathcal{L}$ and that the Stieltjes function $\tilde{S}_r$ of $\tilde{\mathcal{L}}^{(r)}$ satisfies

$$\begin{aligned}\tilde{\phi}(qx+\omega)D_{q,\omega}\tilde{S}_r(x) &= \tilde{G}_r(x;q,\omega)\tilde{S}_r(x)A_{q,\omega}\tilde{S}_r(x) + \tilde{E}_r(x;q,\omega)\,\tilde{S}_r(x)\\ &+ \tilde{F}_r(x;q,\omega)A_{q,\omega}\tilde{S}_r(x) + \tilde{H}_r(x;q,\omega),\ r \geq 0,\end{aligned}$$

where $\tilde{\phi} = \mathcal{T}_{\frac{-\omega}{1-q}}\phi$, and $\tilde{\Phi}(x;q,\omega) = \mathcal{T}_{\frac{-\omega}{1-q}}\Phi(x;q)$, $\Phi \in \{E_r, F_r, G_r, H_r\}$.

We state the following

Theorem 6.5 *The rth associated $\tilde{P}_n^{(r)}$ of the polynomial $\tilde{P}_n$ satisfies the fourth-order difference equation*

$$\sum_{j=0}^{4} I_j^g(r,n,q,\omega;x)\,A_{q,\omega}^j\,\tilde{P}_n^{(r)}(x) = 0. \tag{6.59}$$

where the polynomial coefficient $I_j^g(r,n,q,\omega;x)$ depending on $\tilde{\phi}$, $\tilde{E}_r$, $\tilde{F}_r$, $\tilde{G}_r$, $\tilde{H}_r$ and denoted $I_j^g(r,n,q,\omega;x) = I_j^g(r,n,q,\omega;x;\tilde{\phi},\tilde{E}_r,\tilde{F}_r,\tilde{G}_r,\tilde{H}_r)$, are given by

$$I_j^g(r,n,q,\omega;x;\tilde{\phi},\tilde{E}_r,\tilde{F}_r,\tilde{G}_r,\tilde{H}_r) = I_j(r,n,q;x - \frac{\omega}{1-q};\phi,E_r,F_r,G_r,H_r).$$

Proof: We replace in (6.54) $P_n^{(r)}$ by $\mathcal{T}_{\frac{\omega}{1-q}} \tilde{P}_n^{(r)}$, i.e.,

$$P_n^{(r)} = \mathcal{T}_{\frac{\omega}{1-q}} \tilde{P}_n^{(r)},$$

and obtain an equation which multiplied by the operator $\mathcal{T}_{\frac{-\omega}{1-q}}$ gives

$$\sum_{j=0}^{4} \mathcal{T}_{\frac{-\omega}{1-q}} I_j(r, n, q; x) \, \mathcal{T}_{\frac{-\omega}{1-q}} \mathcal{G}_q^j \, \mathcal{T}_{\frac{\omega}{1-q}} \tilde{P}_n^{(r)}(x) = 0.$$

We therefore use the relation (4.23): $\mathcal{T}_{\frac{-\omega}{1-q}} \mathcal{G}_q^j \, \mathcal{T}_{\frac{\omega}{1-q}} = A_{q,\omega}^j$ to transform the previous equation in

$$\sum_{j=0}^{4} I_j(r, n, q; x - \frac{\omega}{1-q}) \, A_{q,\omega}^j \, \tilde{P}_n^{(r)}(x) = 0.$$

We complete the proof by identifying the coefficients of $A_{q,\omega}^j \, \tilde{P}_n^{(r)}(x)$ in the previous equation with the ones of (6.59). □

6.3.3 Fourth-order difference equation for the rth associated Δ-Laguerre-Hahn orthogonal polynomials

From the fourth-order difference equation satisfied by the rth associated orthogonal polynomial of the $D_{q,\omega}$-Laguerre-Hahn class, we deduce, again, by the limit process the fourth-order difference equation satisfied by the rth associated orthogonal polynomial of the Δ-Laguerre-Hahn class [Foupouagnigni et al., 1998b]

$$\sum_{j=0}^{4} I_j^{\Delta}(r, n; x) \, \mathcal{T}^j \, \tilde{P}_n^{(r)}(x) = 0,$$

with

$$I_j^{\Delta}(r, n; x) = \lim_{\omega \to 1} \lim_{q \to 1} I_j^{g}(r, n, q, \omega; x; \phi, \tilde{E}_r, \tilde{F}_r, \tilde{G}_r, \tilde{H}_r).$$

6.4 Application of difference equations to classical situations

6.4.1 Coefficients E_r, F_r and H_r for classical situations

Here we suppose that the regular linear functional $\mathcal{L}$ satisfies the $\mathcal{D}_q$-Pearson linear functional equation, $\mathcal{D}_q(\phi \, \mathcal{L}) = \psi \, \mathcal{L}$, where ϕ is a polynomial of degree at most two, and ψ is a first-degree polynomial given by

$$\phi(x) = \phi_2 \, x^2 + \phi_1 \, x + \phi_0, \; \psi(x) = \psi_1 \, x + \psi_0, \; |\psi_1|(|\phi_3| + |\phi_2| + |\phi_1| + |\phi_0|) \neq 0.$$

It follows from Proposition 6.2 that H_r is constant and E_r and F_r are polynomials of degree at most one.

Let us compute first polynomials E_r, F_r, and $\frac{H_r}{\gamma_r}$ in terms of ϕ and ψ. The first $\mathcal{D}_q$-derivative of (6.4), (6.5) and the first and second $\mathcal{D}_q$-derivative of (6.6) give, respectively,

$$\mathcal{D}_q \, E_{r+1} = q \frac{H_r}{\gamma_r} - \mathcal{D}_q \, F_r, \quad r \geq 0, \tag{6.60}$$

$$\mathcal{D}_q \, F_{r+1} = \frac{H_r}{\gamma_r} - \mathcal{D}_q \, E_r, \quad r \geq 0, \tag{6.61}$$

$$\begin{aligned} qE_r + F_r = \; & -q\mathcal{G}_q \, \mathcal{D}_q \, \phi - (q^2 x - \beta_r) \mathcal{D}_q \, E_r - (qx - \beta_r) \mathcal{D}_q \, F_r \\ & + (1-q)(qx - \beta_r) \frac{H_r}{\gamma_r}, \; r \geq 0, \end{aligned} \tag{6.62}$$

$$q\mathcal{D}_q \, E_r + \mathcal{D}_q \, F_r = q \frac{H_r}{\gamma_r} - q^2 \, \phi_2, \; r \geq 0. \tag{6.63}$$

In the first step, we solve equations (6.60), (6.61) and (6.63), taking into account the initial conditions (6.16)

$$
\begin{aligned}
&H_0(x,q) = \mathcal{L}\theta_0\psi(x) - (\mathcal{D}_q\mathcal{L})\theta_0\phi(qx) - \mathcal{L}\theta_0\mathcal{D}_q\psi = (\psi_1 - \phi_2)\gamma_0,\\
&\mathcal{D}_q\, E_0(x,q,w) = \psi' - \mathcal{D}_q^2\phi(x) = \psi_1 - (1+q)\,\phi_2,\\
&F_0(x,q) = 0,
\end{aligned}
$$

and obtain [Foupouagnigni et al., 1998e]

$$
\begin{aligned}
\mathcal{D}_q\, E_r &= q^r\psi_1 + ([r]_q - [2]_q)\phi_2,\\
\mathcal{D}_q\, F_r &= q^{2-r}[r]_q\phi_2,\\
\frac{H_r}{\gamma_r} &= q^r\psi_1 + q^{-r}([2r]_q - 1)\phi_2.
\end{aligned}
\tag{6.64}
$$

In a second step, we compute the coefficients E_r and F_r using (6.62), (6.64) and the equation obtained after iterating (6.9):

$$
E_r - F_r = ((q-1)x)\sum_{k=0}^{r-1}\frac{H_k}{\gamma_k} + \psi - \mathcal{D}_q\,\phi
$$

and we get huge expressions for E_r and F_r. Finally, use of Maple V.4 and the simplification procedures for q-hypergeometric terms developped in [Böing et al., 1998] allow us to have readable expressions for E_r and F_r [Foupouagnigni et al., 1998e],

$$
\begin{aligned}
E_r(x;q) \;=\;& ((q^r - q^2)\,\phi_2 - q^r\,\psi_1\,(q-1))((q^r - q)\,(q^r\,x\,q - q^r\,x + x\,q^2 - x\,q)\,\phi_2\\
&+ (q-1)\,(q^r - q)\,\phi_1 + q^r\,(q-1)\,(q^r\,x\,q\,\psi_1 + q\,\psi_0 - q^r\,x\,\psi_1 - \psi_0))\Big/\\
&(q-1)^2\,(q^r - q)\,(q^r + q)\,\phi_2 + (q^r)^2\,\psi_1\,(q-1)),
\end{aligned}
\tag{6.65}
$$

$$
\begin{aligned}
F_r(x;q) \;=\;& (-1 + q^r)\,(q^r - q)\,(q^r\,x\,q^2 - q^r\,x\,q + x\,q^3 - x\,q^2)\,\phi_2{}^2 +\\
&(q^r\,(q-1)\,(q^r - q)\,\phi_1 - q^r\,(q-1)\,(q^r\,x\,q^2\,\psi_1 - \psi_0\,q^2 - q^r\,x\,q\,\psi_1 + q\,\psi_0))\phi_2\\
&+ \psi_1\,(q^r)^2\,(q-1)^2\,\phi_1)q\Big/(q^r\,(q-1)^2((q^r - q)\,(q^r + q)\,\phi_2 + (q^r)^2\,\psi_1\,(q-1))).
\end{aligned}
\tag{6.66}
$$

Remark 6.3 *1. For q-classical situations, coefficients K_2, K_4, K_6 and K_8 (see (6.42)) are constant with respect to the variable x.*

2. For $r = 0$, $K_4 = 0$, then it follows from (6.40), (6.42) and (6.64)-(6.66) that (6.36) and (6.37) (for $r = 0$ and for $\mathcal{D}_q$-classical situations) are, respectively, equivalent to equations (5.7) and (5.12).

3. When the regular functional $\mathcal{L}$ is $\mathcal{D}_q$-semi-classical, $K_4 = \gamma_0 G_0 = 0$ (for $r = 0$). This allows us to obtain the factored form of the fourth-order difference equation for the first associated $\mathcal{D}_q$-semi-classical orthogonal polynomials.

6.4.2 Results on general associated $\mathcal{D}_q$-classical orthogonal polynomials

The coefficients $I_j(r,n,q;x)$ (see (6.53)) can be computed using the algorithm described in (6.21)-(6.53). But this involves heavy computations due to huge expressions containing powers of q which need to be factored. To avoid these difficulties, we again used Maple V.4 to compute symbolically the coefficients $I_j(r,n,q;x)$ and to simplify common factors as was done for the associated classical discrete orthogonal polynomial [Foupouagnigni et al., 1997c] to obtain

Theorem 6.6 (Foupouagnigni et al., 1998c, 1998e) *The coefficients $I_j(r,n,q;x)$ of the fourth-order q-difference equation satisfied by the rth associated $\mathcal{D}_q$-classical orthogonal polynomials are* *given by*

$$
\begin{aligned}
I_4 &= K_{9,2}(K_{10,0}K_{10,1} - K_{12,0}K_{12,1}),\\
I_3 &= K_{10,2}\,(K_{12,0}\,(k_{2,3}K_{12,1} + K_{13,1}) - K_{10,0}\,K_{10,1}\,(K_{2,3} + K_{5,2})) + K_{9,1}\,K_{10,0}\,K_{12,2},\\
I_2 &= K_{10,1}\,(K_{10,2}\,(K_{10,0}\,K_{10,1} + K_{13,0} - K_{5,1}\,K_{12,0})\\
&\quad - K_{9,1}\,K_{10,0}) - K_{12,1}\,(K_{12,2}\,K_{13,0} + k_{11,2}\,K_{12,0}),\\
I_1 &= K_{10,0}\,K_{12,2}\,(k_{2,2}\,K_{12,0} + K_{13,0}) + K_{10,2}\,K_{12,0}\,(K_{9,0} - K_{10,0}\,K_{10,1}),\\
I_0 &= K_{9,-1}\,(K_{10,1}K_{10,2} - K_{12,1}K_{12,2}),
\end{aligned}
\tag{6.67}
$$

where the coefficients $K_{i,j}$ are obtained from (6.41), (6.42) and

$$
\begin{aligned}
K_9(x) &= K_7(qx)K_1(qx) - K_3(x)K_8(x), \quad K_{10}(x) = K_7(qx) + K_1(x),\\
K_{11}(x) &= K_2(qx)K_2(x) + K_4(x)K_6(x), \quad K_{12}(x) = K_2(qx) + K_5(x),\\
K_{13}(x) &= K_5(qx)K_5(x) + K_4(x)K_6(x), \quad K_{14}(x) = K_5(qx) + K_2(x),
\end{aligned}
$$

with coefficients E_n, F_n, and $\frac{H_n}{\gamma_n}$ given by (6.64)-(6.66).

Notice that coefficients $I_j(r,n,q;x)$, are given in appendix III, for some $\mathcal{D}_q$-classical orthogonal polynomials.

6.4.3 Fourth-order differential equation for the rth associated $\mathcal{D}$-classical orthogonal polynomials

From the relation $\lim_{q\to 1}\mathcal{D}_q = \frac{d}{dx}$ and by the limit process, we recover using Maple V Release 4 the fourth-order differential equation satisfied by the rth associated classical continuous orthogonal polynomials (see [Belmehdi et al., 1991], [Zarzo et al., 1993]) [Lewanowicz, 1995], [Foupouagnigni et al., 1998e]). This equation is given in terms of the factored form of the fourth-order differential equation satisfied by the first associated classical continuous orthogonal polynomials as already done earlier [Lewanowicz, 1995].

$$
\mathcal{O}^c(r,n;x)P_n^{(r)}(x) = 0, \tag{6.68}
$$

where

$$
\mathcal{O}(r,n,q;x) = \sum_{j=0}^{4} I_j(r,n,q;x)\,\mathcal{G}_q^j
$$

and

$$
\begin{aligned}
\mathcal{O}^c(r,n;x) &= \frac{1}{2\phi(x)\,\eta(r,n)}\lim_{q\to 1}\frac{\mathcal{O}(r,n,q;x)}{q^2(q-1)^2x^2}\\
&= \mathcal{Q}_{2,n}^{**c}\mathcal{Q}_{2,n}^{*c} + (1-r)\zeta(n,r)\mathcal{Q}_{2,n}^{c},
\end{aligned}
\tag{6.69}
$$

with

$$
\begin{aligned}
\mathcal{Q}_{2,n}^{c} &= 2\phi\frac{d^2}{dx^2} + 3\phi'\frac{d}{dx} - n(n-2)\phi''\mathcal{I}_d,\\
\zeta(r,n) &= ((n+r-1)\phi'' + 2\psi'),\\
\eta(r,n) &= (n+1)((n-2r-2)\phi'' + 2\psi').
\end{aligned}
$$

$\mathcal{Q}_{2,r}^{**c}$ and $\mathcal{Q}_{2,n}^{*c}$ are given by (5.13) and (5.14).

6.4.4 Fourth-order difference equation for the rth associated Δ-classical orthogonal polynomials

We first deduce the fourth-order difference equation for the rth associated $D_{q,\omega}$-classical orthogonal polynomials using Theorems 4.2 and 6.6, then deduce the difference equation for the rth associated Δ-classical orthogonal polynomial by the limit process: $\lim_{\omega\to 1, q\to 1} D_{q,\omega} = \Delta$.

We assume that $\{P_n\}_{n\in\mathbb{N}}$, orthogonal with respect to $\mathcal{L}$, is $\mathcal{D}_q$-classical with $\mathcal{L}$ satisfying $\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}$ where ϕ and ψ are polynomials of degree at most two and degree one, respectively. The rth associated $P_n^{(r)}$ satisfies Theorem 6.6. It yields from Theorem 4.2 and 6.6 that $\{\bar{P}_n\}_{n\in\mathbb{N}}$, with $\bar{P}_n = \mathcal{T}_{\frac{-\omega}{1-q}} P_n$, is orthogonal with respect to $\bar{\mathcal{L}} = \mathcal{T}_{\frac{-\omega}{1-q}}\mathcal{L}$ and $\bar{\mathcal{L}}$ is $D_{q,\omega}$-classical satisfying $D_{q,\omega}(\bar{\phi}\bar{\mathcal{L}}) = \bar{\psi}\bar{\mathcal{L}}$, where $\bar{\phi} = \mathcal{T}_{\frac{-\omega}{1-q}}\phi$ and $\bar{\psi} = \mathcal{T}_{\frac{-\omega}{1-q}}\psi$. Again, we use Theorem 4.2 and 6.6 to conclude that the rth associated $\bar{P}_n^{(r)}$ of the $D_{q,\omega}$-classical orthogonal polynomial $\bar{P}_n$ satisfies the fourth-order difference equation

$$\sum_{j=0}^{4} I_j(r,n,q,\omega;x;\bar{\phi},\bar{\psi})\, A_{q,\omega}{}^{j}\, \bar{P}_n^{(r)}(x) = 0.$$

We, therefore, use the limit process to state the following:

Theorem 6.7 (Foupouagnigni et al., 1998c) *Let $\tilde{P}_n$ be the classical orthogonal polynomials of a discrete variable associated with the linear functional $\tilde{\mathcal{L}}$ satisfying $\Delta(\tilde{\phi}) = \tilde{\psi}\tilde{\mathcal{L}}$. Then, the rth associated $\tilde{P}_n^{(r)}$ of $\tilde{P}_n$ satisfies the fourth-order difference equation*

$$\sum_{j=0}^{4} I_j^{\Delta}(r,n;x)\, \mathcal{T}^{j}\, \tilde{P}_n^{(r)}(x) = 0,$$

where the coefficients I_j^{Δ} are given by

$$\begin{aligned}
I_4^{\Delta} &= K_{9,2}(K_{10,0}K_{10,1} - K_{12,0}K_{12,1}),\\
I_3^{\Delta} &= K_{10,2}\,(K_{12,0}\,(k_{2,3}\,K_{12,1} + K_{13,1}) - K_{10,0}\,K_{10,1}\,(K_{2,3} + K_{5,2})) + K_{9,1}\,K_{10,0}\,K_{12,2},\\
I_2^{\Delta} &= K_{10,1}\,(K_{10,2}\,(K_{10,0}\,K_{10,1} + K_{13,0} - K_{5,1}\,K_{12,0})\\
&\quad - K_{9,1}\,K_{10,0}) - K_{12,1}\,(K_{12,2}\,K_{13,0} + k_{11,2}\,K_{12,0}),\\
I_1^{\Delta} &= K_{10,0}\,K_{12,2}\,(k_{2,2}\,K_{12,0} - K_{13,0}) + K_{10,2}\,K_{12,0}\,(K_{9,0} - K_{10,0}\,K_{10,1}),\\
I_0^{\Delta} &= K_{9,-1}\,(K_{10,1}K_{10,2} - K_{12,1}K_{12,2}),
\end{aligned} \tag{6.70}$$

with the notation: $K_{i,j} \equiv k_i(x+j)$. Coefficients k_j read as:

$$\begin{aligned}
k_1(x) &= \tilde{\phi}(x+1) + \tilde{E}_{n+r+1}(x),\; k_2(x) = \tilde{\phi}(x+1) - \tilde{F}_r(x),\; k_3(x) = \frac{\tilde{H}_{n+r}}{\tilde{\gamma}_{n+r}},\\
k_4(x) &= \begin{cases} \tilde{\gamma}_r \frac{\tilde{H}_{r-1}}{\tilde{\gamma}_{r-1}} & \text{if } r \geq 1\\ 0 & \text{if } r = 0\end{cases},\; k_5(x) = \tilde{\phi}(x+1) + \tilde{E}_r(x),\; k_6(x) = -\frac{\tilde{H}_r}{\tilde{\gamma}_r},\\
k_7(x) &= \tilde{\phi}(x+1) - \tilde{F}_{n+r+1}(x),\; k_8(x) = -\tilde{H}_{n+r+1},\; k_9(x) = k_7(x+1)k_1(x+1) - k_3(x)k_8(x),\\
k_{10}(x) &= k_7(x+1) - k_1(x),\; k_{11}(x) = k_2(x+1)k_2(x) + k_4(x)k_6(x),\; k_{12}(x) = k_2(x+1) + k_5(x),\\
k_{13}(x) &= k_5(x+1)k_5(x) + k_4(x)k_6(x),\; k_{14}(x) = k_5(x+1) + k_2(x),
\end{aligned}$$

with

$$\begin{aligned}
\tilde{E}_r(x,\tilde{\phi},\tilde{\psi}) &= \lim_{\omega\to 1, q\to 1} E_r(x - \frac{\omega}{1-q}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\phi}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\psi})\\
&= (\tilde{\phi}_2\, n - 2\,\tilde{\phi}_2 + \tilde{\psi}_1)\, x\\
&\quad + \frac{(\tilde{\phi}_2\, n - 2\,\tilde{\phi}_2 + \tilde{\psi}_1)\,(\tilde{\phi}_2\, n^2 - \tilde{\phi}_2 + \tilde{\phi}_1\, n + \tilde{\psi}_1\, n - \tilde{\phi}_1 + \tilde{\psi}_0)}{2\,(n-1)\,\tilde{\phi}_2 + \tilde{\psi}_1},\\
\tilde{F}_r(x,\tilde{\phi},\tilde{\psi}) &= \lim_{\omega\to 1, q\to 1} F_r(x - \frac{\omega}{1-q}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\phi}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\psi})\\
&= \tilde{\phi}_2\, x\, n\\
&\quad - \frac{n\,(\tilde{\phi}_2\,\tilde{\phi}_1 + 3\,\tilde{\phi}_2^2 + \tilde{\phi}_2^2\, n^2 - 4\,\tilde{\phi}_2^2\, n - 2\,\tilde{\phi}_2\,\tilde{\psi}_1 - \tilde{\phi}_2\,\tilde{\phi}_1\, n + \tilde{\psi}_1\, n\,\tilde{\phi}_2 + \tilde{\phi}_2\,\tilde{\psi}_0 - \tilde{\psi}_1\,\tilde{\phi}_1)}{2\,(n-1)\,\tilde{\phi}_2 + \tilde{\psi}_1},\\
\tilde{H}_r = &= \lim_{\omega\to 1, q\to 1} H_r(x - \frac{\omega}{1-q}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\phi}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\psi}) = ((2r-1)\tilde{\phi}_2 + \tilde{\psi}_1)\tilde{\gamma}_r.
\end{aligned}$$

It should be mentioned that E_r, F_r and H_r are given by (6.64), (6.65) and (6.66), respectively. The coefficients β_n, γ_n are given in Lemma 7.1.

$$
\begin{aligned}
\tilde{\beta}_n &\equiv \tilde{\beta}_n(\tilde{\phi},\tilde{\psi}) = \lim_{\omega\to 1, q\to 1} \left(\beta_n(\mathcal{T}_{\frac{\omega}{1-q}}\tilde{\phi}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\psi}) + \frac{\omega}{1-q}\right) \\
&= -\frac{\tilde{\phi}_2\,(\tilde{\psi}_1 + 2\,\tilde{\phi}_1)\,n^2 - (\tilde{\psi}_1 + 2\,\tilde{\phi}_1)\,(-\tilde{\psi}_1 + \tilde{\phi}_2)\,n - \tilde{\psi}_0\,(-\tilde{\psi}_1 + 2\,\tilde{\phi}_2)}{(\tilde{\psi}_1 + 2\,\tilde{\phi}_2\,n)\,(2\,(n-1)\,\tilde{\phi}_2 + \tilde{\psi}_1)},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\gamma}_n &\equiv \tilde{\gamma}_n(\tilde{\phi},\tilde{\psi}) = \lim_{\omega\to 1, q\to 1} \gamma_n(\mathcal{T}_{\frac{\omega}{1-q}}\tilde{\phi}, \mathcal{T}_{\frac{\omega}{1-q}}\tilde{\psi}) \\
&= -((n-2)\,(n-1)^4\,\tilde{\phi}_2^4 \\
&\quad +(4\,(n-2)\,(n-1)^2\,\tilde{\phi}_0 + (n-1)^2\;\; 3\,\tilde{\psi}_1\,n^2 + 2\,n\,\tilde{\psi}_0 - 8\,\tilde{\psi}_1\,n - 4\,\tilde{\psi}_0 + 5\,\tilde{\psi}_1))\,\tilde{\phi}_2^3 + \\
&\quad (-(n-2)\,(n-1)^2\,\tilde{\phi}_1^2 - \tilde{\psi}_1\,(n-2)\,(n-1)^2\,\tilde{\phi}_1 + 4\,\tilde{\psi}_1\,(n-1)\,(-3+2\,n)\,\tilde{\phi}_0 \\
&\quad + (\tilde{\psi}_0 + \tilde{\psi}_1\,n - \tilde{\psi}_1)\,(n\,\tilde{\psi}_0 - 2\,\tilde{\psi}_0 + 4\,\tilde{\psi}_1 - 7\,\tilde{\psi}_1\,n + 3\,\tilde{\psi}_1\,n^2))\tilde{\phi}_2^2 \\
&\quad +(-\tilde{\psi}_1\,(n-1)\,(-3+2\,n)\,\tilde{\phi}_1^2 - \tilde{\psi}_1\;\; -5\,\tilde{\psi}_1\,n + 3\,\tilde{\psi}_1 - 2\,\tilde{\psi}_0 + n\,\tilde{\psi}_0 + 2\,\tilde{\psi}_1\,n^2)\,\tilde{\phi}_1 \\
&\quad + \tilde{\psi}_1^2\,(-6+5\,n)\,\tilde{\phi}_0 + \tilde{\psi}_1\,(\tilde{\psi}_0 - \tilde{\psi}_1\,n - \tilde{\psi}_1)^2)\tilde{\phi}_2 - \tilde{\psi}_1^2\,(n-1)\,\tilde{\phi}_1^2 - \tilde{\psi}_1^2\,(\tilde{\psi}_0 + \tilde{\psi}_1\,n - \tilde{\psi}_1)\,\tilde{\phi}_1 \\
&\quad + \tilde{\phi}_0\,\tilde{\psi}_1^3)n \Big/ (((2\,n-1)\,\tilde{\phi}_2 + \tilde{\psi}_1)\,((-2+2\,n)\,\tilde{\phi}_2 + \tilde{\psi}_1\;\,^2\,((-3+2\,n)\,\tilde{\phi}_2 + \tilde{\psi}_1)).
\end{aligned}
$$

Remark 6.4 *The coefficients $I_0^{\Delta}(r,n,x)$, as well as operators $\mathcal{D}_{r,n}$, $\mathcal{N}_{r,n}$, $\tilde{\mathcal{D}}_{r,n}$ and $\tilde{\mathcal{N}}_{r,n}$ (see Theorem 6.3), are given in Appendix II for all classical orthogonal polynomial of a discrete variable. They are obviously deduced from those of q-classical case by Theorem 4.2.*

Chapter 7

Three-term recurrence relation coefficients for classical situations

7.1 Introduction

We describe the method used to compute the coefficients β_n and γ_n for the $\mathcal{D}_q$-classical case.

This method, already used in [Koepf et al., 1996] but for classical continuous and classical discrete cases, consists to derive from the second order difference equation satisfied by $\{P_n\}_{n\in\mathbb{N}}$ (3.54) a system of equations satisfied by $T_{n,1}$, $T_{n,2}$ and $\lambda_{n,0}$ [Foupouagnigni et al., 1998a], then solve these equations and deduce coefficients β_n and γ_n.

7.2 Three-term recurrence relation coefficients for $\mathcal{D}_q$-classical situations

7.2.1 Coefficients $T_{n,1}$ and $T_{n,2}$

Let $\mathcal{L}$ be a $\mathcal{D}_q$-classical linear functional satisfying

$\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}$, where ϕ is of degree at most two and ψ a first-degree polynomial i.e.,

$$\phi(x) = \phi_2\, x^2 + \phi_1\, x + \phi_0, \psi(x) = \psi_1\, x + \psi_0, \; |\psi_1|(|\phi_2| + |\phi_1| + |\phi_0|) \neq 0. \tag{7.1}$$

It follows from Theorem 3.1 that $\{P_n\}_{n\in\mathbb{N}}$ satisfies

$$\phi\, \mathcal{D}_q \mathcal{D}_{\frac{1}{q}}\, P_n + \psi\, \mathcal{D}_q\, P_n + \lambda_{n,0}\, P_n = 0 \qquad \forall n \in \mathbb{N}, \tag{7.2}$$

with $\lambda_{n,0}$ given by (3.55).

Use of the expansions (see [Foupouagnigni et al., 1998a], see also (8.8)),

$$P_n(x) = \sum_{j=0}^{n} T_{n,j}\, x^{n-j}, \; \mathcal{D}_q x^n = [n]_q\, x^{n-1}$$

allows us to write (7.2) as

$$\sum_{j=0}^{n} d_{n,j}\, x^{n-j} = 0. \tag{7.3}$$

We compute the first three coefficients $d_{n,j}$ and obtain, with $\rho = q^n$,

$$d_{n,0} = q^4\, \rho\, (q-1)^2\, \lambda_{n,0} - q^1\, (\rho - 1)\, (-\rho + q)\, \phi_2 + q^4\, \rho\, (\rho - 1)\, (q - 1)\, \psi_1,$$

$$\begin{aligned}
d_{n,1} &= (2\,q^3\,(-\rho+q)\,(-\rho+q^2)\,\phi_1 - 2\,q^3\,\rho\,(q-1)\,(-\rho+q)\,\psi_0)\,\mathrm{A}(n) \\
&\quad + (2\,q^4\,\rho\,(q-1)^2\,\lambda_{n,0} + 2\,q^2\,(-\rho+q^2)\,(q^3-\rho)\,\phi_2 - 2\,q^2\,\rho\,(q-1)\,(-\rho+q^2)\,\psi_1)\,\mathrm{B}(n) \\
&\quad - 2\,q^4\,(\rho-1)\,(-\rho+q)\,\phi_0 \\
d_{n,2} &= (q^4\,\rho\,(q-1)^2\,\lambda_{n,0} + q^3\,(-\rho+q)\,(-\rho+q^2)\,\phi_2 - q^3\,\rho\,(q-1)\,(-\rho+q)\,\psi_1)\,\mathrm{A}(n) \\
&\quad - q^4\,(\rho-1)\,(-\rho+q)\,\phi_1 + q^4\,\rho\,(\rho-1)\,(q-1)\,\psi_0.
\end{aligned}$$

We solve the equations $d_{n,0}=0$, $d_{n,1}=0$ and $d_{n,2}=0$ in terms of $\lambda_{n,0}$, $T_{n,1}$ and $T_{n,2}$ and get

$$\begin{aligned}
\lambda_{n,0} &= \frac{(\rho-1)\,(-\phi_2\,\rho+\rho\,\psi_1-q\,\rho\,\psi_1+\phi_2\,q)}{\rho\,(q-1)^2} = -[n]_q\,(\psi_1+[n-1]_{\frac{1}{q}}\,\frac{\phi_2}{q}), \\
T_{n,1} &= \frac{(-1+\rho)\,q\,(q-1)\,(q-\rho)\,\phi_1 - \rho\,q\,(q-1)^2\,(-1+\rho)\,\psi_0}{(q-1)^2\,(q^2\,\phi_2-\psi_1\,q\,\rho^2+\psi_1\,\rho^2-\phi_2\,\rho^2)}. \\
T_{n,2} &= \frac{1}{2}(2\,q^2\,(-1+\rho)\,(q-1)^3\,(q+\rho)\,(q-\rho)^2\,\phi_0\,\phi_2 \\
&\quad - 2\,q^2\,(-1+\rho)\,(q-1)^2\,(-\rho+q^2)\,(q-\rho)^2\,{\phi_1}^2 \\
&\quad + 2\,q^2\,\rho\,(-1+\rho)\,(q-1)^3\,(q-\rho)\,(-2\,\rho+q^2+q)\,\psi_0\,\phi_1 \\
&\quad - 2\,\rho^2\,q^2\,(-1+\rho)\,(q-1)^4\,(q-\rho)\,\psi_1\,\phi_0 - 2\,\rho^2\,q^2\,(-1+\rho)\,(q-1)^4\,(q-\rho)\,{\psi_0}^2)\Big/ \\
&\quad ((q+1)\,(q-1)^4\,(-\psi_1\,q\,\rho^2+\psi_1\,\rho^2+\phi_2\,q^3-\phi_2\,\rho^2)\,(q^2\,\phi_2-\psi_1\,q\,\rho^2+\psi_1\,\rho^2-\phi_2\,\rho^2)).
\end{aligned}$$

7.2.2 Coefficients β_n and γ_n for $\mathcal{D}_q$-classical orthogonal polynomials

We use the following identities already given in [Foupouagnigni, 1998a] (see also (8.8))

$$\beta_n = T_{n,1} - T_{n+1,1}, \; \gamma_n = T_{n,2} - T_{n+1,2} - \beta_n\,T_{n,1} \tag{7.4}$$

to compute the coefficients β_n, γ_n and get:

Lemma 7.1 (Medem, 1996) *The coefficients β_n and γ_n of TTRR satisfied by the polynomials $\{P_n\}_{n\in\mathbb{N}}$ (see (2.18)), orthogonal with respect to the $\mathcal{D}_q$-classical linear functional $\mathcal{L}$, satisfying $\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}$, where ϕ and ψ are defined in (7.1), are given by:*

$$\begin{aligned}
\beta_n(q,\phi,\psi) &= -\rho((-(q+1)\,(-1+\rho)\,(-q+\rho)\,\phi_1 - (q-1)\,(-\rho\,q^2+q-q\,\rho+\rho^2)\,\psi_0)\,\phi_2 \\
&\quad - \rho\,(q-1)\,(q+1)\,(-1+\rho)\,\psi_1\,\phi_1 - \rho^2\,(q-1)^2\,\psi_0\,\psi_1)\Big/ \\
&\quad ((-1+\rho)\,(\rho+1)\,\phi_2 + \rho^2\,(q-1)\,\psi_1)(-(-q+\rho)\,(q+\rho)\,\phi_2 - \rho^2\,(q-1)\,\psi_1), \\
\gamma_n(q,\phi,\psi) &= -(-1+\rho)\,((-\rho+q^2)\,\phi_2 - (q-1)\,\rho\,\psi_1)((-q+\rho)^2\,(q+\rho)^2\,\phi_0\,{\phi_2}^2 \\
&\quad + (-q\,\rho\,(-q+\rho)^2\,{\phi_1}^2 - q\,\rho\,(q-1)\,(-q+\rho)^2\,\psi_0\,\phi_1 \\
&\quad + 2\,\rho^2\,(q-1)\,(-q+\rho)\,(q+\rho)\,\psi_1\,\phi_0 + q^2\,\rho^2\,(q-1)^2\,{\psi_0}^2)\phi_2 \\
&\quad -\rho^2\,q\,(q-1)(-q+\rho)\,\psi_1\,{\phi_1}^2 - q\,\rho^3\,(q-1)^2\,\psi_0\,\psi_1\,\phi_1 \\
&\quad + \rho^4\,(q-1)^2\,{\psi_1}^2\,\phi_0)\rho\,q\Big/(((-q+\rho^2)\,\phi_2 + \rho^2\,(q-1)\,\psi_1)\times \\
&\quad ((q-\rho)\,(q+\rho)\,\phi_2 - \rho^2\,(q-1)\,\psi_1)^2\,((q^3-\rho^2)\,\phi_2 - \rho^2\,(q-1)\,\psi_1)).
\end{aligned}$$

7.3 Three-term recurrence relation coefficients for $\mathcal{D}$-classical situations

7.3.1 Coefficients $\tilde{T}_{n,1}$ and $\tilde{T}_{n,2}$

If we denote by $\tilde{T}_{n,1}$ and $\tilde{T}_{n,2}$ the coefficients $T_{n,1}$ and $T_{n,2}$ when the linear functional $\mathcal{L}$ is $\mathcal{D}$-classical satisfying $\mathcal{D}(\phi\mathcal{L}) = \psi\mathcal{L}$, we obtain $\tilde{T}_{n,1}$ and $\tilde{T}_{n,2}$ by limit process [Koepf et al., 1996]:

$$\tilde{T}_{n,1} = \lim_{q\to 1} T_{n,1} = \frac{n\,((n-1)\,\phi_1 + \psi_0)}{2(n-1)\phi_2 + \psi_1},$$

$$\begin{aligned}\tilde{T}_{n,2} &= \lim_{q\to 1} T_{n,2} \\ &= \frac{1}{2}n(2\,(n-1)^2\,\phi_0\,\phi_2 + (n-2)\,(n-1)^2\,{\phi_1}^2 + \psi_0\,(n-1)\,(2\,n-3)\,\phi_1 \\ &+ \psi_1\,(n-1)\,\phi_0 + {\psi_0}^2\,(n-1)))\,/(((2\,n-3)\,\phi_2 + \psi_1)\,((2\,n-2)\,\phi_2 + \psi_1)).\end{aligned}$$

We, therefore, use (7.4) to deduce coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ and get [Lesky, 1985], [Koepf et al., 1996]....

$$\tilde{\beta}_n = \lim_{q\to 1} \beta_n = -\frac{(2\,n\,(n-1)\,\phi_1 - 2\,\psi_0)\,\phi_2 + 2\,\phi_1\,\psi_1\,n + \psi_0\,\psi_1}{(\psi_1 + 2\,\phi_2\,n)\,((2\,n-2)\,\phi_2 + \psi_1)},$$

$$\begin{aligned}\tilde{\gamma}_n &= \lim_{q\to 1} \gamma_n = -n(4\,\phi_0\,(n-2)\,(n-1)^2\,{\phi_2}^3 + (-(n-2)\,(n-1)^2\,{\phi_1}^2 \\ &+ 4\,(n-1)\,(2\,n-3)\,\psi_1\,\phi_0 + (n-2)\,{\psi_0}^2)\,{\phi_2}^2 + (-\psi_1\,(n-1)\,(2\,n-3)\,{\phi_1}^2 \\ &+ (2-n)\,\psi_0\,\psi_1\,\phi_1 + (-6+5\,n)\,{\psi_1}^2\,\phi_0 + {\psi_0}^2\,\psi_1)\,\phi_2 \\ &\quad + {\psi_1}^2\,(-n+1)\,{\phi_1}^2 - \psi_0\,\phi_1\,{\psi_1}^2 + \phi_0\,{\psi_1}^3)\Big/ \\ &\quad (((2\,n-1)\,\phi_2 + \psi_1)\,((2\,n-2)\,\phi_2 + \psi_1)^2((2\,n-3)\,\phi_2 + \psi_1)).\end{aligned}$$

7.4 Three-term recurrence relation coefficients for Δ-classical situations

We state the following:

Lemma 7.2 (Salto, 1995, Koepf et al., 1996) *Let $\{P_n\}_{n\in\mathbb{N}}$ be a family of monic polynomials, orthogonal with respect to the Δ-classical linear functional $\bar{\mathcal{L}}$ satisfying $\Delta(\bar{\phi}\bar{\mathcal{L}}) = \bar{\psi}\bar{\mathcal{L}}$. If $\bar{\beta}_n$ and $\bar{\gamma}_n$ are the coefficients of TTRR satisfied by $\{P_n\}_{n\in\mathbb{N}}$, then, they are given by*

$$\begin{aligned}\bar{\beta}_n &= -\frac{\bar{\phi}_2\,(\bar{\psi}_1 + 2\,\bar{\phi}_1)\,n^2 - (\bar{\psi}_1 + 2\,\bar{\phi}_1)\,(-\bar{\psi}_1 + \bar{\phi}_2)\,n - \bar{\psi}_0\,(-\bar{\psi}_1 + 2\,\bar{\phi}_2)}{(\bar{\psi}_1 + 2\,\bar{\phi}_2\,n)\,(2\,(n-1)\,\bar{\phi}_2 + \bar{\psi}_1)}. \\ \bar{\gamma}_n &= -((n-2)\,(n-1)^4\,\bar{\phi}_2^4 \times \\ &\quad -(4\,(n-2)\,(n-1)^2\,\bar{\phi}_0 + (n-1)^2\,(3\,\bar{\psi}_1\,n^2 - 2\,n\,\bar{\psi}_0 - 8\,\bar{\psi}_1\,n - 4\,\bar{\psi}_0 + 5\,\bar{\psi}_1))\,\bar{\phi}_2^3 \\ &\quad +(-(n-2)\,(n-1)^2\,\bar{\phi}_1^2 - \bar{\psi}_1\,(n-2)\,(n-1)^2\,\bar{\phi}_1 + 4\,\bar{\psi}_1\,(n-1)\,(-3+2\,n)\,\bar{\phi}_0 \\ &\quad + (\bar{\psi}_0 + \bar{\psi}_1\,n - \bar{\psi}_1)\,(n\,\bar{\psi}_0 - 2\,\bar{\psi}_0 + 4\,\bar{\psi}_1 - 7\,\bar{\psi}_1\,n + 3\,\bar{\psi}_1\,n^2))\bar{\phi}_2^2 \\ &\quad +(-\bar{\psi}_1\,(n-1)\,(-3+2\,n)\,\bar{\phi}_1^2 - \bar{\psi}_1\,(-5\,\bar{\psi}_1\,n + 3\,\bar{\psi}_1 - 2\,\bar{\psi}_0 + n\,\bar{\psi}_0 + 2\,\bar{\psi}_1\,n^2)\,\bar{\phi}_1 \\ &\quad + \bar{\psi}_1^2\,(-6+5\,n)\,\bar{\phi}_0 + \bar{\psi}_1\,(\bar{\psi}_0 + \bar{\psi}_1\,n - \bar{\psi}_1)^2)\bar{\phi}_2 - \bar{\psi}_1^2\,(n-1)\,\bar{\phi}_1^2 - \bar{\psi}_1^2\,(\bar{\psi}_0 + \bar{\psi}_1\,n - \bar{\psi}_1)\,\bar{\phi}_1 \\ &\quad + \bar{\phi}_0\,\bar{\psi}_1^3)n\Big/(((2\,n-1)\,\bar{\phi}_2 + \bar{\psi}_1)\,((-2+2\,n)\,\bar{\phi}_2 + \bar{\psi}_1)^2\,((-3+2\,n)\,\bar{\phi}_2 + \bar{\psi}_1)).\end{aligned}$$

The corresponding coefficients $\bar{T}_{n,1}$ and $\bar{T}_{n,2}$ are deduced by the same way [Koepf et al., 1996].

Chapter 8

Laguerre-Freud equations for the recurrence coefficient of the semi-classical orthogonal polynomials of class one

8.1 Introduction

We assume that $\mathcal{L}$ is a regular linear functional satisfying

$$\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}, \tag{8.1}$$

with polynomials ϕ and ψ given by

$$\phi(x) = \sum_{j=0}^{t} \phi_j x^j, \quad \psi(x) = \sum_{j=0}^{p} \psi_j x^j, \quad p \geq 1, \ |\phi_t||\psi_p| \neq 0. \tag{8.2}$$

We suppose that (8.1) is not reducible and that the class of the linear functional $\mathcal{L}$, $\mathrm{cl}(\mathcal{L})$ is $\mathrm{cl}(\mathcal{L}) \equiv s = \max\{\deg(\phi) - 2, \deg(\psi) - 1\}$. $\{P_n\}_{n\in\mathbb{N}}$ which is a family of monic polynomials orthogonal with respect to $\mathcal{L}$, satisfies the TTRR:

$$\begin{cases} P_{n+1}(x) = (x - \beta_n)P_n(x) - \gamma_n P_{n-1}(x), & n \geq 1, \\ P_0(x) = 1, P_1(x) = x - \beta_0, \end{cases} \tag{8.3}$$

where β_n and γ_n are complex numbers with $\gamma_n \neq 0 \quad \forall n \in \mathbb{N}$.

When $\mathcal{L}$ is $\mathcal{D}$, $\mathcal{D}_q$ or Δ-classical, the coefficients β_n and γ_n can be given explicitly in terms of polynomials ϕ and ψ appearing in (8.1) (see the previous chapter).

But if $\mathcal{L}$ is $\mathcal{D}$, $\mathcal{D}_q$ or Δ-semi-classical of class $s > 0$, it is very difficult to give the coefficients β_n and γ_n explicitly in terms of the polynomials ϕ and ψ.

We propose a method which enables us to compute them recursively when the linear functional $\mathcal{L}$ is $\mathcal{D}_q$-semi-classical of class $s = 1$. Then, we use Theorem 4.2 to extend this result to the Δ-classical orthogonal polynomials.

This method consists to derive two non-linear equations satisfied by β_n and γ_n, called Laguerre-Freud equations.

8.2 Starting the Laguerre-Freud equations

The initial form of the Laguerre-Freud equations is obtained by applying both sides of (8.1) to the polynomials P_nP_n and P_nP_{n+1}, respectively

$$\begin{aligned}\langle D_{q,\omega}(\phi\mathcal{L}), P_nP_n\rangle &= \langle\psi\mathcal{L}, P_nP_n\rangle,\\ \langle D_{q,\omega}(\phi\mathcal{L}), P_nP_{n+1}\rangle &= \langle\psi\mathcal{L}, P_nP_{n+1}\rangle.\end{aligned}$$

Then, we apply the rules (3.6) and (3.12) to the previous equations and obtain

$$\langle\mathcal{L}, \phi\mathcal{D}_{\frac{1}{q}}P_n\mathcal{G}_{\frac{1}{q}}P_n\rangle + \langle\mathcal{L}, \phi\mathcal{D}_{\frac{1}{q}}P_n\, P_n\rangle = -q\,\langle\psi\mathcal{L}, P_nP_n\rangle, \tag{8.4}$$

$$\langle\mathcal{L}, \phi\mathcal{D}_{\frac{1}{q}}P_{n+1}\mathcal{G}_{\frac{1}{q}}P_n\rangle + \langle\mathcal{L}, \phi\mathcal{D}_{\frac{1}{q}}P_n\, P_{n+1}\rangle = -q\,\langle\psi\mathcal{L}, P_nP_{n+1}\rangle. \tag{8.5}$$

The respective right-hand sides of the previous equations are given by

Lemma 8.1 (Belmehdi et al., 1994)

$$\left\{\begin{aligned}\langle\psi\mathcal{L}, P_nP_n\rangle &= [\psi(\beta_n) + \psi_2(\gamma_n + \gamma_{n+1})]I_{0,n},\\ \langle\psi\mathcal{L}, P_nP_{n+1}\rangle &= [\psi_1 + \psi_2(\beta_n + \beta_{n+1})]\gamma_{n+1}I_{0,n}.\end{aligned}\right. \tag{8.6}$$

Proof: Using the three-term recurrence relation (8.3), we first derive the relation

$$I_{0,n+1} = \gamma_{n+1}I_{0,n} \quad \forall n \in \mathbb{N} \tag{8.7}$$

and then use it together with (8.3) to prove the lemma. In fact, use of (8.3) and (8.7) give

$$\begin{aligned}I_{0,n+1} &= \langle\mathcal{L}, P_{n+1}P_{n+1}\rangle\\ &= \langle\mathcal{L}, P_{n+1}((x-\beta_n)P_n - \gamma_nP_{n-1})\rangle\\ &= \langle\mathcal{L}, P_{n+1}xP_n\rangle\\ &= \langle\mathcal{L}, P_n(P_{n+2} + \beta_{n+1}P_{n+1} + \gamma_{n+1}P_n)\rangle\\ &= \gamma_{n+1}\langle\mathcal{L}, P_nP_n\rangle\\ &= \gamma_{n+1}I_{0,n}.\end{aligned}$$

Using (8.7) we obtain

$$\begin{aligned}\langle\psi\mathcal{L}, P_nP_n\rangle &= \psi_0\langle\mathcal{L}, P_nP_n\rangle + \psi_1\langle\mathcal{L}, xP_nP_n\rangle + \psi_2\langle\mathcal{L}, x^2P_nP_n\rangle\\ &= \psi_0I_{0,n} + \psi_1\langle\mathcal{L}, (P_{n+1} + \beta_nP_n + \gamma_nP_{n-1})P_n\rangle\\ &\quad + \psi_2\langle\mathcal{L}, (P_{n+1} + \beta_nP_n + \gamma_nP_{n-1})^2\rangle\\ &= \psi_0I_{0,n} + \psi_1\beta_nI_{0,n} + \psi_2(I_{0,n+1} - \beta_n^2I_{0,n} + \gamma_n^2I_{0,n-1})\\ &= [\psi(\beta_n) + \psi_2(\gamma_n + \gamma_{n+1})]I_{0,n}.\end{aligned}$$

$$\begin{aligned}\langle\psi\mathcal{L}, P_nP_{n-1}\rangle &= \psi_0\langle\mathcal{L}, P_nP_{n+1}\rangle + \psi_1\langle\mathcal{L}, xP_nP_{n+1}\rangle + \psi_2\langle\mathcal{L}, x^2P_{n-1}P_n\rangle\\ &= \psi_1\langle\mathcal{L}, (P_{n+1} + \beta_nP_n + \gamma_nP_{n-1})P_{n+1}\rangle\\ &\quad + \psi_2\langle\mathcal{L}, (P_{n+2} + \beta_{n+1}P_{n+1} + \gamma_{n+1}P_n)(P_{n+1} + \beta_nP_n + \gamma_nP_{n-1})\rangle\\ &= \psi_1I_{0,n+1} + \psi_2(\beta_{n+1}I_{0,n+1} + \gamma_{n+1}\beta_n)I_{0,n}\\ &= [\psi_1 + \psi_2(\beta_n + \beta_{n+1})]\gamma_{n+1}I_{0,n}\end{aligned}$$

□

In order to express all terms of (8.4) and (8.5) in terms of β_n and γ_n, we need to expand the polynomials P_n in the basis $\{x^n\}_{n\in\mathbb{N}}$ with coefficients depending on β_n and γ_n.

8.3 Intermediate coefficients

8.3.1 Coefficients $T_{n,j}$

Lemma 8.2 (Foupouagnigni et al., 1998a) *All basic coefficients $T_{n,i}$ in the expansion of*

$$P_n(x) = \sum_{i=0}^{n} T_{n,i} x^{n-i} \tag{8.8}$$

can be computed recursively from the relations:

$$\begin{aligned} T_{1,1} &= -\beta_0, \\ T_{n,0} &= 1, \quad n \geq 0, &(8.9)\\ T_{n+1,1} &= T_{n,1} - \beta_n, \quad n \geq 1, &(8.10)\\ T_{n+1,j} &= T_{n,j} - \beta_n T_{n,j-1} + \gamma_n T_{n-1,j-2}, \quad 2 \leq j \leq n, &(8.11)\\ T_{n+1,n+1} &= -\beta_n T_{n,n} - \gamma_n T_{n-1,n-1}, \quad n \geq 1. &(8.12) \end{aligned}$$

Proof: We use the relation (8.8) and the three-term recurrence relation (8.3) to obtain

$$\sum_{i=0}^{n} T_{n,i} x^{n+1-i} = \sum_{i=0}^{n+1} T_{n+1,i} x^{n+1-i} + \beta_n \sum_{j=0}^{n} T_{n,j} x^{n-j} + \gamma_n \sum_{k=0}^{n-1} T_{n-1,k} x^{n-1-k}.$$

We replace the variable j and k in the previous equation by $j-1$ and $k-2$, respectively, to obtain

$$\sum_{i=0}^{n} T_{n,i} x^{n+1-i} = \sum_{i=0}^{n+1} T_{n+1,i} x^{n+1-i} + \beta_n \sum_{j=1}^{n+1} T_{n,j-1} x^{n+1-j} + \gamma_n \sum_{k=2}^{n+1} T_{n-1,k-2} x^{n+1-k},$$

an equation which is equivalent to

$$\begin{aligned} &(T_{n+1,0} - T_{n,0})x^{n+1} + (T_{n+1,1} - T_{n,1} + \beta_n T_{n,0})x^n \\ &+ \sum_{k=2}^{n} (T_{n+1,k} - T_{n,k} + \beta_n T_{n,k-1} + \gamma_n T_{n-1,k-2})x^{n+1-k} - T_{n+1,n+1} + \beta_n T_{n,n} \\ &+ \gamma_n T_{n-1,n-1} = 0. \end{aligned}$$

From the relation $P_1 = T_{1,0}x + T_{1,1} = x - \beta_0$, it follows that $T_{1,0} = 1$ and $T_{1,1} = -\beta_0$. We complete the proof by identifying to zero all coefficients of the polynomial on the left hand-side of the previous equation. □

Corollary 8.1 *Using Lemma 8.2, we compute the coefficients $T_{n,j}$ $j = 0,3$ as:*

$$\begin{aligned} T_{n+1,1} &= -\sum_{i=0}^{n} \beta_i, \quad n \geq 0, \\ T_{n+1,2} &= \sum_{0 \leq i < j \leq n} \beta_i \beta_j - \sum_{i=1}^{n} \gamma_i, \quad n \geq 1, \\ T_{n+1,3} &= -\sum_{0 \leq i < j < k \leq n} \beta_i \beta_j \beta_k + \sum_{1 \leq i < j \leq n} (\gamma_i \beta_j + \beta_i \gamma_j) + \beta_0 \sum_{i=1}^{n} \gamma_i \\ &\quad - \sum_{i=1}^{n} \beta_{i-1} \gamma_i, \quad n \geq 2. \end{aligned} \tag{8.13}$$

All other terms can be computed in the same way, but for class $s = 1$, only these 3 terms will be used.

Let us emphasise that the two terms $T_{n,1}$ and $T_{n,2}$ are already given in [Chihara, 1978]; the computation of the higher order coefficients allows to generate Laguerre-Freud equations for any arbitrary class $s > 1$. These coefficients play the role (but in a simpler way) of the Turán determinants introduced in [Belmehdi et al., 1994] showing the interest of Laguerre-Freud equations.

8.3.2 Coefficients B_n^k

The coefficients B_n^k appear from the action of the linear functional $\mathcal{L}$ on the polynomial $x^{n+k}\,P_n$

$$B_n^k = \langle \mathcal{L}, x^{n+k}\,P_n\rangle, \tag{8.14}$$

with the initial condition

$$B_n^0 = \langle \mathcal{L}, x^n\,P_n\rangle = \langle \mathcal{L}, P_n\,P_n\rangle = I_{0,n}.$$

From the relation $0 = \langle \mathcal{L}, P_{n+k}\,P_n.\rangle; k \geq 1$ and (8.8) we deduce that

$$B_n^k = -\sum_{i=1}^{k} T_{n+k,i}B_n^{k-i}.$$

We use the previous equation to compute, recursively, the coefficient B_n^k. In particular, we have:

$$\begin{aligned} B_n^1 &= -T_{n+1,1}I_{0,n}, \\ B_n^2 &= (T_{n+1,1}T_{n+2,1} - T_{n+2,2})I_{0,n}, \\ B_n^3 &= [T_{n+1,1}(T_{n+3,2} - T_{n+2,1}T_{n+3,1}) + T_{n+3,1}T_{n+2,2} - T_{n+3,3}]\,I_{0,n}. \end{aligned} \tag{8.15}$$

Notice that the connection between B_n^k and the coefficients $C_{j,k}^n$ introduced in [Belmehdi et al., 1994],

$$x^{n+k}P_n(x) = \sum_{j=0}^{2n+k} C_{j,n}^{n+k}P_j(x),$$

is obviously

$$B_n^k = C_{0,n}^{n+k}I_{0,0}.$$

8.3.3 Structure relations

We first recall the structure relation (3.84):

$$\phi\,\mathcal{D}_{\frac{1}{q}}P_n = \sum_{j=n-s-1}^{n+t-1} \xi_{n,j}\,P_j, \ , \ n > s.$$

with $t = \deg(\phi)$. $\xi_{n,n-s-1} \neq 0$, $n > s+1$ and then apply the linear functional $\mathcal{L}$ to both sides of the equation obtained when multiplying the previous one by P_j and get

$$\xi_{n,j}I_{0,j} = \langle \phi\mathcal{L}, P_j\,\mathcal{D}_{\frac{1}{q}}P_n\rangle.\ n-s-1 \leq j \leq n+t-1. \tag{8.16}$$

Then, using (8.2), (8.14) and the previous equation we get

$$\begin{aligned} \xi_{n,j}I_{0,j} &= \langle \phi\mathcal{L}, P_j\,\mathcal{D}_{\frac{1}{q}}P_n\rangle \\ &= \langle \mathcal{L}, P_j\,\sum_{i=0}^{t}\phi_i x^i)\sum_{k=1}^{n}[n+1-k]_{\frac{1}{q}}\,T_{n,k-1}\,x^{n-k}\rangle \\ &= \sum_{k=0}^{n-t-1} c_k(q)\langle \mathcal{L}, x^k P_j\rangle \\ &= \sum_{k=j}^{n-t-1} c_k(q)\langle \mathcal{L}, x^k P_j\rangle \\ &= \sum_{k=j}^{n-t-1} c_k(q)B_j^{k-j}. \end{aligned}$$

with

$$c_k(q) = \sum_{i+j=k,\, i\leq t,\, j\leq n-1}^{n+t-1} \phi_i \, [j+1]_{\frac{1}{q}} \, T_{n,n-j-1}. \tag{8.17}$$

Thus,

$$\xi_{n,j} I_{0,j} = \sum_{k=j}^{n+t-1} c_k(q) B_j^{k-j}. \tag{8.18}$$

Once (8.18) is derived, we are now able to compute the coefficients $\xi_{n,i}$ in terms of β_n, γ_n and the polynomials ϕ and ψ, by using (8.2) (8.15) and (8.18). To be more precise, we assume that the linear functional $\mathcal{L}$ is of class at most one, this implies that

$$\phi(x) = \sum_{j=0}^{3} \phi_j \, x^j, \;\; \psi(x) = \sum_{j=0}^{2} \psi_j \, x^j,$$

with

$$(|\phi_0| + |\phi_1| + |\phi_2| + \phi_3)(|\psi_1| + |\psi_2|) \neq 0.$$

We use the method described above to compute the coefficients $\xi_{n,j}$, $n-1 \leq j \leq n+2$ and get

$$\begin{aligned}
\xi_{n,n+2} &= [n]_{\frac{1}{q}} \phi_3, \\
\xi_{n,n+1} &= q^{1-n}\{[[n]_n(\beta_n + \beta_{n+1}) - T_{n,1}]\phi_3 + [n]_q \phi_2\}, \\
\xi_{n,n} &= q^{1-n}\{[n]_q \phi_1 + [[n]_q \beta_n - T_{n,1}]\phi_2 + [[n]_q(\gamma_n + \gamma_{n+1} + \beta_n^2) \\
&\quad + T_{n,1}^2 - \beta_n T_{n,1} - (1+q) T_{n,2}]\phi_3\}, \\
\xi_{n,n-1} &= q^{1-n}\{[n]_q \phi_0 - T_{n,1}\phi_1 - [T_{n,1}^2 - (1+q)T_{n,2} + [n]_q \gamma_n]\phi_2 \\
&\quad - [-T_{n,1}^3 + (1+q)T_{n,2} - \gamma_n T_{n,1} - [3]_q T_{n,3} + [n]_q \gamma_n(\beta_{n-1} + \beta_n)]\phi_3\}.
\end{aligned}$$

The search for $\xi_{n,n-2}$ requires the constant $T_{n,4}$ which is huge and needs heavy computation. To get rid of this difficulty we, again, use (3.12), (8.1) and (8.16) to get

$$\begin{aligned}
\xi_{n,j} I_{0,j} &= \langle \phi\mathcal{L}, P_j \mathcal{D}_{\frac{1}{q}} P_n \rangle \\
&= \langle \phi\mathcal{L}, \mathcal{D}_{\frac{1}{q}}[A_{q,\omega} P_j P_n] - \mathcal{D}_{\frac{1}{q}} A_{q,\omega} P_j \, P_n \rangle \\
&= -q\langle \mathcal{D}_q(\phi\mathcal{L}), \mathcal{G}_q P_j \, P_n \rangle - q\langle \phi\mathcal{L}, \mathcal{D}_q P_j \, P_n \rangle \\
&= -q\langle \mathcal{L}, \psi \mathcal{G}_q P_j \, P_n \rangle - q\langle \phi\mathcal{L}, \mathcal{D}_q P_j \, P_n \rangle \\
&= -q\langle \mathcal{L}, (\psi \mathcal{G}_q P_j + \phi\mathcal{D}_q P_j) P_n \rangle,
\end{aligned}$$

hence

$$\xi_{n,j} I_{0,j} = -q\langle \mathcal{L}, (\psi A_{q,\omega} P_j + \phi \mathcal{D}_q P_j) P_n \rangle. \tag{8.19}$$

It follows immediately that

$$\xi_{n,n-2} I_{0,n-2} = -q\langle \mathcal{L}, (q^{n-2}\psi_2 + [n-2]_q \phi_3) x^n P_n \rangle.$$

We use (3.63) and (8.7) to simplify the expression of $\xi_{n,n-2}$ and get

$$\xi_{n,n-2} = -q^{n-2}(\psi_2 + [n-2]_{\frac{1}{q}} \frac{\phi_3}{q}) \gamma_n \gamma_{n-1}. \tag{8.20}$$

In the same way, we compute another expression for $\xi_{n,n-1}$ which we denote by $\xi^{-}_{n,n-1}$

$$\begin{aligned}
\xi^{-}_{n,n-1} I_{0,n-1} &= -q\langle \mathcal{L}, (\psi A_{q,\omega} P_{n-1} + \phi \mathcal{D}_q P_{n-1}) P_n \rangle \\
&= -q\langle \mathcal{L}, [(\psi_0 + \psi_1 x + \psi_2 x^2)(q^{n-1} x^{n-1} + q^{n-2} T_{n-1,1} x^{n-2}) \\
&\quad + (\phi_0 + \phi_1 x + \phi_2 x^2 + \phi_3 x^3)([n-1]_q x^{n-2} + [n-2]_q T_{n-1,1} x^{n-3})] P_n \rangle
\end{aligned}$$

and get, after simplifications,

$$\begin{aligned}\xi^{+}_{n,n-1} &= q[n-1]_q\,\gamma_n T_{n,1}\phi_3 - q([n-2]_q T_{n-1,1} + [n-1]_q\beta_n)\gamma_n\phi_3 \\ &\quad - q[n-1]_q\gamma_n\phi_2 + q^n\gamma_n T_{n,1}\psi_2 - (q\psi_1 + q\psi_2\beta_n + T_{n-1,1}\psi_2)\gamma_n q^{n-1}.\end{aligned}$$

8.4 Final form of the Laguerre-Freud equations

We prove the following theorem which is the main result of this chapter.

Theorem 8.1 *The coefficients β_n and γ_n of the three-term recurrence relation*

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), n \geq 1, P_0(x) = 1, P_1(x) = x - \beta_0$$

satisfied by the $\mathcal{D}_q$-semi-classical orthogonal polynomials of class at most one, $\{P_n\}_{n\in\mathbb{N}}$, can be computed recursively from the two non-linear equations

$$\begin{cases} (\psi_2 + [2n]_{\frac{1}{q}}\frac{\phi_3}{q})(\gamma_n + \gamma_{n+1}) = F_1(q,;\beta_0,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n), \\ (\psi_2 + [2n+1]_{\frac{1}{q}}\frac{\phi_3}{q})\beta_{n+1}\gamma_{n-1} = F_2(q,;\beta_0,\ldots,\beta_n;\gamma_1,\ldots,\gamma_{n+1}). \end{cases} \tag{8.21}$$

ϕ_j and ψ_j are the coefficients of the polynomials ϕ and ψ appearing in the Pearson equation, $\mathcal{D}_q(\phi\mathcal{L}) = \psi\mathcal{L}$, satisfied by the regular linear functional $\mathcal{L}$. F_1 is a polynomial of $2n+1$ variables and of degree 2 and F_2 a polynomial of $2n+2$ variables and of degree 3, with the initial conditions

$$\beta_0 = \frac{\langle\mathcal{L},x\rangle}{\langle\mathcal{L},1\rangle},\ \psi_2\gamma_1 = -\psi(\beta_0). \tag{8.22}$$

Proof: In the first step we use the structure relation (3.84) to transform Equations (8.4) and (8.5) as

$$\begin{aligned}&\xi_{n,n-2}\langle\mathcal{L},\mathcal{G}_{\frac{1}{q}}P_n P_{n-2}\rangle + \xi^{+}_{n,n-1}\langle\mathcal{L},\mathcal{G}_{\frac{1}{q}}P_n P_{n-1}\rangle + (1+q^{-n})\,\xi_{n,n}\,I_{0,n} \\ &\qquad = -q\langle\mathcal{L},\psi\,P_n\,P_n\rangle,\end{aligned} \tag{8.23}$$

$$\begin{aligned}&\xi_{n+1,n-1}\langle\mathcal{L},\mathcal{G}_{\frac{1}{q}}P_n P_{n-1}\rangle + \xi_{n+1,n}\,q^{-n}\,I_{0,n} + \xi_{n,n-1}\,I_{0,n+1} \\ &\qquad = -q\langle\mathcal{L},\psi\,P_n\,P_{n+1}\rangle.\end{aligned} \tag{8.24}$$

In the second step we compute $\langle\mathcal{L},\mathcal{G}_{\frac{1}{q}}P_n P_{n-1}\rangle$ and $\langle\mathcal{L},\mathcal{G}_{\frac{1}{q}}P_n P_{n-2}\rangle$ using (8.8) and get

$$\begin{aligned}\langle\mathcal{L},\mathcal{G}_{\frac{1}{q}}P_n P_{n-1}\rangle &= \langle\mathcal{L},(q^{-n}x^n + q^{1-n}T_{n,1}x^{n-1})P_{n-1}\rangle \\ &= q^{-n}B^1_{n-1} + q^{1-n}T_{n,1}I_{0,n-1} \\ &= q^{-n}(q-1)T_{n,1}I_{0,n-1}, \\ \langle\mathcal{L},\mathcal{G}_{\frac{1}{q}}P_n P_{n-2}\rangle &= \langle\mathcal{L},(q^{-n}x^n + q^{1-n}T_{n,1}x^{n-1} + q^{2-n}T_{n,2}x^{n-2})P_{n-2}\rangle \\ &= q^{-n}B^2_{n-2} + q^{1-n}T_{n,1}B^1_{n-2} + q^{2-n}T_{n,2}I_{0,n-2}.\end{aligned}$$

In the third step we use (8.6) and the previous equations to simplify (8.23) and (8.24) and obtain:

$$\begin{aligned}&-q^{2n+2}(\psi_2 + [2n]_{\frac{1}{q}}\frac{\phi_3}{q})(\gamma_n + \gamma_{n+1}) = [q^2[2n]_q\beta_n^2 + q(q+q^{2n})(T^2_{n,1} - \beta_n T_{n,1}) \\ &-(q+1)(q^2+q^{2n})T_{n,2}]\phi_3 + [q^2[2n]_q\beta_n - q(q+q^{2n})T_{n,1}]\phi_2 + q^2[2n]_q\phi_1 + q^{2n+2}\beta_n^2 \\ &+q^{2n}(q-1)\psi_2[qT^2_{n,1} - q\beta_n T_{n,1} - (1+q)T_{n,2}] + q^{2n+1}[q\beta_n - (q-1)T_{n,1}]\psi_1 + q^{2n+2}\psi_0,\end{aligned} \tag{8.25}$$

$$\begin{aligned}&-q^{2n+1}(\psi_2 + [2n+1]_{\frac{1}{q}}\frac{\phi_3}{q})\gamma_{n+1}\beta_{n+1} = \{\beta_n^3 + (q-1)\beta_n^2 T_{n,1} \\ &+[(q+2)\gamma_n - (q-1)T^2_{n,1} + (q^2-1)T_{n,2} + (1+[2n+1]_q)\gamma_{n+1}]\beta_n \\ &+[(q^2-1)T_{n,1} + [3]_q\beta_{n-1}]\gamma_n - T^3_{n,1} - [(q+2)T_{n,2} - (1+q^{2n+1})\gamma_{n+1}]T_{n,1} - [3]_q T_{n,3}\}\phi_3 \\ &+\{\beta_n^2 + (q-1)\beta_n T_{n,1} + (q+1)\gamma_n + T^2_{n,1} - (q+1)T_{n,2} + [2n+1]_q\gamma_{n+1}\}\phi_2 + (\beta_n - T_{n,1})\phi_1 \\ &+[n+1]_q\phi_0 + q^{2n+1}\beta_n\gamma_{n+1}\psi_2 - q^{2n}(q-1)\gamma_{n+1}T_{n,1}\psi_2 + q^{2n+1}\gamma_{n+1}\psi_1.\end{aligned} \tag{8.26}$$

The first initial condition is obtained by applying the linear functional $\mathcal{L}$ to $P_1 = x - \beta_0$ while the second comes from the application of both sides of (8.1) to the polynomial P_0P_0.

In fact, it follows from (8.1) that

$$\langle \psi\mathcal{L}, P_0P_0\rangle = \langle \mathcal{D}_q(\phi\mathcal{L}), P_0P_0\rangle = -\frac{1}{q}\langle \phi\mathcal{L}, \mathcal{D}_{\frac{1}{q}} P_0P_0\rangle = 0.$$

The previous equation used together with (8.6) gives $\psi(\beta_0) + \gamma_1\psi_2 = 0$.

We complete the proof of the theorem by saying that:

1. For any non-zero integer n, the coefficients $\psi_2 + [j]_{\frac{1}{q}}\frac{\phi_3}{q}$, $j = 2n, 2n+1$ of the right-hand sides of the two previous equations, thanks to the fact that the $\mathcal{D}_q$-semi-classical linear functional $\mathcal{L}$ is regular (see (3.89)), are non-zero (except if $\phi_3 = \psi_2 = 0$).

2. The left-hand sides of the previous equations contain only constants, sums and products of coefficients β_j and γ_j. Polynomials F_1 and F_2 are obtained by replacing $T_{n,j}$, $j = 1,2,3$ in equations (8.25) and (8.26) by (8.13).

□

Notice that we can also obtain the second Laguerre-Freud Equation (8.26) by identification of the two expressions $\xi_{n,n-1}$ and $\xi^+_{n,n-1}$.

Equation (8.25) gives, linearly, γ_{n+1} in terms of β_j, $j = 0,n$ and γ_j, $j = 1,n$; when (8.26) gives β_{n+1} in terms of β_j, $j = 0,n$, γ_j, $j = 1,n$ and the previous γ_{n+1} via the non-linear term $(\psi_2 + [2n+1]_{\frac{1}{q}}\frac{\phi_3}{q})\beta_{n+1}\gamma_{n+1}$.

The fact that β_{n+1} is not obtained linearly (except for the classical case) in terms of the previous β_j and γ_j exemplify the fundamental barrier between semi-classical of class $s > 0$ and classical situation in which both ϕ_3 and ψ_2 are zero. For $\mathcal{D}_q$-semi-classical of class $s > 0$, both relations (8.25) and (8.26) must be used simultaneously, starting with the initial values given by (8.22). In the classical situation Equations (8.25) and (8.26) can be decoupled.

8.4.1 Laguerre-Freud equations for $\mathcal{D}_q$-classical orthogonal polynomials

When we take $\phi_3 = \psi_2 = 0$ in Equations (8.25) and (8.26), we obtain the Laguerre-Freud equations for $\mathcal{D}_q$-classical orthogonal polynomials:

$$\begin{aligned} q^{2n}(\psi_1 + [2n]_{\frac{1}{q}}\frac{\phi_2}{q})\beta_n &+ [(1+q^{2n-1})\phi_2 + (q-1)q^{2n-1}\psi_1]\sum_{j=0}^{n-1}\beta_j \\ &+ [2n]_q\phi_1 + q^{2n-1}\psi_0 = 0, \end{aligned} \tag{8.27}$$

$$\begin{aligned} q^{2n+1}(\psi_1 + [2n+1]_{\frac{1}{q}}\frac{\phi_2}{q})\gamma_{n+1} &+ (1+q)\phi_2\sum_{j=0}^{n}\gamma_j \\ &= -(\sum_{j=0}^{n-1}\beta_j)^2\phi_2 + [(q-1)\phi_2\beta_n - \phi_1]\sum_{j=0}^{n-1}\beta_j \\ &+(q-1)\phi_2\sum_{0\le i<j\le n-1}\beta_i\beta_j - \phi_2\beta_n^2 - \phi_1\beta_n - [n+1]_q\phi_0. \end{aligned} \tag{8.28}$$

Remark 8.1 *Using Maple V.4 and the simplification procedures for q-hypergeometric terms developped in [Böing et al., 1998], we have solved (8.27) with the initial condition* $\beta_0 = -\frac{\psi_0}{\psi_1}$ *to get* β_n.

Taking into account the β_n *obtained above, we have solved (8.28) with the initial condition* $\gamma_1 = -\frac{\phi(\beta_0)}{\phi_2+q\psi_1}$ *to get* γ_n. *Obviously the coefficients* β_n *and* γ_n *obtained coincide with the ones given in Lemma 7.1.*

8.5 Applications to $\mathcal{D}$, D_ω and $D_{q,\omega}$-semi-classical orthogonal polynomials of class one

8.5.1 Laguerre-Freud equations for $\mathcal{D}$-semi-classical orthogonal polynomials of class one

We obtain these equations by computing the limits of (8.25) and (8.26) as $q \to 1$ to obtain [Belmehdi et al., 1994], [Foupouagnigni et al., 1998a]

$$\begin{aligned}
\psi(\beta_n) &+ 4\phi_3 \sum_{i=1}^{n-1} \gamma_i + 2\sum_{i=0}^{n-1} \theta_{\beta_n}\phi(\beta_i) = -(\psi_2 + 2n\phi_3)(\gamma_n + \gamma_{n+1}),\\
\sum_{i=0}^{n} \phi(\beta_i) &+ 3\phi_3 \sum_{i=1}^{n} \gamma_i(\beta_{i-1} + \beta_i) + \left[(2n+1)\gamma_{n+1} + 2\sum_{i=1}^{n} \gamma_i\right] \phi_2 + 2\gamma_{n+1}(n\beta_n + \sum_{i=0}^{n} \beta_i)\phi_3\\
&+ [\psi_1 + \psi_2\beta_n]\gamma_{n+1} = -[\psi_2 + (2n+1)\phi_3]\beta_{n+1}\gamma_{n+1}.
\end{aligned} \tag{8.29}$$

where

$$\theta_a\phi(x) = \frac{\phi(x) - \phi(a)}{x - a},$$

with the initial conditions

$$\beta_0 = \frac{\langle \mathcal{L}, x\rangle}{\langle \mathcal{L}, 1\rangle},\ \psi_2\gamma_1 = -\psi(\beta_0).$$

8.5.2 Laguerre-Freud equations for D_ω-semi-classical orthogonal polynomials of class one

It follows from Theorem 4.2 that the Laguerre-Freud equations for $\mathcal{D}_q$ semi-classical linear functional of class one is obtained just by replacing β_j (resp. ϕ and ψ) in (8.25) and (8.26) by $\beta_j - \frac{\omega}{1-q}$ ($\mathcal{T}_{\frac{\omega}{1-q}}\phi$ and $\mathcal{T}_{\frac{\omega}{1-q}}\psi$ respectively). For this reason, we need to control the behaviour of $T_{n,1}$, $T_{n,2}$ and $T_{n,3}$ when β_j is replaced by $\beta_j - \frac{\omega}{1-q}$.

Lemma 8.3 *If the coefficients $\bar{T}_{n,j}$, $j = 1,2,3$ represent the coefficients $T_{n,j}$, $j = 1,2,3$ in which β_j is replaced by $\beta_j - \frac{\omega}{1-q}$, then, they are related by*

$$\begin{aligned}
\bar{T}_{n,1} &= T_{n,1} + \frac{n\omega}{1-q},\\
\bar{T}_{n,2} &= T_{n,2} + \frac{(n-1)\omega}{1-q} T_{n,1} + \frac{\omega^2}{(1-q)^2}\binom{n}{2},\\
\bar{T}_{n,3} &= T_{n,3} + \frac{(n-2)\omega}{1-q} T_{n,2} + \frac{\omega^2}{(1-q)^2}\binom{n-1}{2} T_{n,1} + \frac{\omega^3}{(1-q)^3}\binom{n}{3}.
\end{aligned} \tag{8.30}$$

Proof:

The proof follows immediately from (8.13). □

We replace ϕ and ψ in (8.25) and (8.26) by $\mathcal{T}_{\frac{\omega}{1-q}}\phi$ and $\mathcal{T}_{\frac{\omega}{1-q}}\psi$, respectively (and implicitly β_j by $\beta_j - \frac{\omega}{1-q}$), taking into account the previous lemma and obtain the Laguerre-Freud equations for the recurrence coefficients of the $D_{q,\omega}$-semi-classical [Azatassou et al, 1998] orthogonal polynomials $\{P_n\}_{n\in\mathbb{N}}$. These polynomials are orthogonal with respect to the linear functional $\mathcal{L}$, of class at most one, satisfying $D_{q,\omega}(\phi\mathcal{L}) = \psi\mathcal{L}$. Hence we take the limit of these two equations as $q \to 1$ and obtain:

Theorem 8.2 (Foupouagnigni et al., 1998a) *The coefficients β_j and γ_j of the three-term recurrence relation,*

$$P_{n+1} = (x - \beta_n)P_n - \gamma_n P_{n-1},\ n \geq 0,\ P_{-1} = 0,\ P_0 = 1,$$

satisfied by the monic polynomials $\{P_n\}_{n\in\mathbb{N}}$, orthogonal with respect to the D_ω-semi-classical linear functional $\mathcal{L}$, of class at most one, satisfying $D_\omega(\phi\mathcal{L}) = \psi\mathcal{L}$, are given by

$$\begin{aligned}\psi(\beta_n) &+ 4\phi_3\sum_{i=1}^{n-1}\gamma_i + 2\sum_{i=0}^{n-1}\theta_{\beta_n}\phi(\beta_i) + \omega\sum_{i=0}^{n-1}\theta_{\beta_n}\psi(\beta_i) + 2\binom{n}{3}\omega^2\phi_3 \\ &+ \binom{n}{2}\omega^2\psi_2 = -(\psi_2 + 2n\phi_3)(\gamma_n + \gamma_{n+1}),\end{aligned} \tag{8.31}$$

where

$$\theta_a\phi(x) = \frac{\phi(x)-\phi(a)}{x-a}$$

and

$$\begin{aligned}\sum_{i=0}^{n}\phi(\beta_i) &+ \left[(2n+1)\gamma_{n+1} + 2\sum_{i=1}^{n}\gamma_i\right]\phi_2 + \\ &+ 3\phi_3\sum_{i=1}^{n}\gamma_i(\beta_{i-1}+\beta_i) + 2\gamma_{n+1}(n\beta_n + \sum_{i=0}^{n}\beta_i)\phi_3 \\ &+ n\omega\psi_2\gamma_{n+1} - \binom{n+1}{2}\omega\phi_1 + \left[-n\omega\sum_{i=0}^{n}\beta_i + \binom{n+1}{3}\omega^2\right]\phi_2 \\ &- \omega\left[\sum_{0\le i<j\le n}\beta_i\beta_j + n\sum_{i=0}^{n}\beta_i\beta_i + (2n-1)\sum_{i=1}^{n}\gamma_i + n\gamma_{n+1}\right]\phi_3 \\ &+ \left[\binom{n}{2}\omega^2\sum_{i=0}^{n}\beta_i - \binom{n+1}{4}\omega^3\right]\phi_3 + [\psi_1 + \psi_2\beta_n]\gamma_{n+1} \\ &= -[\psi_2 + (2n+1)\phi_3]\beta_{n+1}\gamma_{n+1},\end{aligned} \tag{8.32}$$

with the initial conditions

$$\beta_0 = \frac{\langle\mathcal{L},x\rangle}{\langle\mathcal{L},1\rangle},\quad \psi_2\gamma_1 = -\psi(\beta_0).$$

8.5.3 Laguerre-Freud equations for D_ω-classical orthogonal polynomials

The Laguerre-Freud equations obtained in (8.31) and (8.32) contain, obviously, the classical cases when $\psi_2 = \phi_3 = 0$. We use the notation of [Salto, 1995] so that we can compare more easily with the results therein.

$$\phi(x) = ax^2 + bx + c \quad\text{and}\quad \psi(x) = px + q.$$

Equations (8.31) and (8.32) reduce to:

$$\psi(\beta_n) + 2a\sum_{i=0}^{n-1}\beta_i + 2nb + 2na\beta_n = -n\omega p, \tag{8.33}$$

$$\begin{aligned}\sum_{i=0}^{n}\phi(\beta_i) &+ \left[(2n+1)\gamma_{n+1} + 2\sum_{i=1}^{n}\gamma_i\right]a - \binom{n+1}{2}\omega b \\ &+ \left[-n\omega\sum_{i=0}^{n}\beta_i + \binom{n+1}{3}\omega^2\right]a = -p\gamma_{n+1}.\end{aligned} \tag{8.34}$$

Rewriting the second equation with $n \to n-1$ and subtracting we get:

$$\begin{aligned}&\phi(\beta_n) + [p + (2n+1)a]\gamma_{n+1} - [p + (2n-3)a]\gamma_n \\ &-n\omega b - an\omega\beta_n - a\omega\sum_{i=0}^{n-1}\beta_i + a\omega^2\binom{n}{2} = 0.\end{aligned} \tag{8.35}$$

Using symbolic computation with Maple V.4 we have checked positively that for the classical discrete orthogonal polynomials, the coefficients β_n and γ_n, given explicitly in terms of polynomials ϕ and ψ (see Lemma 7.2), are solutions of Equations (8.33) and (8.34) (with $\omega = 1$).

Equations (8.33) and (8.35) are exactly the ones derived in the thesis [Salto, 1995] taking into account the D_ω derivative of the linear functional given by definition 2.20 and the one used in [Salto, 1995]. Let us remark, however, that in [Salto, 1995] the γ_n equation is obtained using the so-called D_ω representation, expanding a classical orthogonal polynomial P_n as a sum of (see (2.62)) (maximum three) $D_\omega P_i(i = n+1, n, n-1)$. This technique cannot be extended to the class 1, because of the non-existence of such a representation for semi-classical orthogonal polynomials of class $s > 0$.

8.6 Applications to generalised Charlier and generalised Meixner polynomials of class one

8.6.1 Laguerre-Freud equations for the generalised Meixner polynomial of class one

These polynomials with ℓ parameters were introduced in [Ronveaux, 1986] in order to show the quasi-orthogonality character of the D_ω derivative (with $\omega = 1$). The weight ρ is given by:

$$\rho(i) = \frac{\mu^i}{(i!)^\ell} \prod_{j=1}^{\ell} \Gamma(i + \alpha_j), \quad (0 < \mu < 1, \alpha_j > 0), i = 0, 1, 2, \ldots \tag{8.36}$$

Generalised Meixner polynomials are denoted by $m_n^{(\vec{\alpha},\mu)}$, where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$, which reduce, of course, to the well-known classical Meixner polynomials when $\vec{\alpha}$ is the scalar α ($\ell = 1$).

If $\ell = 2$, $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$, the weight ρ obeys

$$\Delta(\phi\rho) = \psi\rho.$$

with

$$\phi(x) = x^2 \text{ and } \psi(x) = (\mu - 1)x^2 + (\alpha_1 + \alpha_2)\mu x + \mu\alpha_1\alpha_2. \tag{8.37}$$

The family is, therefore, discrete semi-classical of class one.

In fact, we have

$$\phi(x) = x\,\phi_0(x), \quad \psi(x) - \phi_0(x) = (x+1)\psi_{0,1}(x) + r_{0,1},$$

with

$$\phi_0(x) = x, \quad \psi_{0,1}(x) = (\mu - 1)x + \mu(\alpha_1 + \alpha_2 - 1), \quad r_{0,1} = \mu(\alpha_1 - 1)(\alpha_2 - 1). \tag{8.38}$$

Since the only root of ϕ is zero, it follows from Proposition 2.5, and the fact that $r_{0,1} \neq 0$ (for $(\alpha_1 - 1)(\alpha_2 - 1) \neq 0$), that for $\ell = 2$ and for $(\alpha_1 - 1)(\alpha_2 - 1) \neq 0$, the generalised Meixner polynomial is of class one.

Of course, when $\alpha_1 = 1$ (or $\alpha_2 = 1$), the class reduce to 0 and we obtain the classical Meixner polynomials $m_n^{(\alpha_2,\mu)}$ [Nikiforov et. al., 1991]. In particular for $\alpha_1 = \alpha_2 = 1$, the generalised Meixner polynomials of class 1 reduces to the particular case of the Meixner polynomials, called discrete Laguerre polynomials and denoted [Chihara, 1978]

$$la_n(x) = m_n^{(1,\mu)}(x). \tag{8.39}$$

We have checked, positively, the Laguerre-Freud equations when $\omega \to 1$ with the known β_n, γ_n of the classical Meixner polynomials and the discrete Laguerre polynomials.

It should be noted that for $\ell = 2$ and for arbitrary positive α_1 and α_2, the weight given by Equation (8.36), is not a polynomial modification of the Meixner weight, except when α_1 or α_2 is an integer.

Replacing in Equations (8.31) and (8.32) ω by one and polynomials ϕ and ψ given by Equation (8.38), we obtain the Laguerre-Freud equations for the generalised Meixner polynomial of class $s = 1$:

$$\begin{aligned}(1-\mu)(\gamma_n+\gamma_{n+1}) &= (\mu-1)(\binom{n}{2}+\beta_n^2)+((1+\mu)n \\ &\quad +\mu(\alpha_1+\alpha_2))\,\beta_n+(1+\mu)\sum_{i=0}^{n-1}\beta_i \\ &\quad +\mu(\alpha_1+\alpha_2)\,n+\mu\alpha_1\alpha_2, \qquad (8.40)\\ (1-\mu)(\beta_n+\beta_{n+1})\gamma_{n+1} &= -n\sum_{i=0}^{n}\beta_i+((1+\mu)n+\mu(\alpha_1+\alpha_2)+1)\,\gamma_{n+1} \\ &\quad +\binom{n+1}{3}+\sum_{i=0}^{n}\beta_i^2+2\sum_{i=1}^{n}\gamma_i, \qquad (8.41)\end{aligned}$$

with initial values

$$\beta_0=\frac{M_1}{M_0}=\frac{\mu\alpha_1\,\alpha_2\;{}_2F_1(1+\alpha_1,1+\alpha_2;2;\mu)}{{}_2F_1(\alpha_1,\alpha_2;1;\mu)},\quad \gamma_1=\frac{\psi(\beta_0)}{1-\mu}. \qquad (8.42)$$

8.6.2 Laguerre-Freud equations for generalised Charlier polynomial of class one

The generalised Charlier polynomials introduced in [Hounkonnou et al., 1998] are discrete semi-classical orthogonal polynomials associated with the weight

$$\rho(x)=\frac{\mu^x}{(x!)^N},\ N\geq 1,\ (\mu>0), x=0,1,2,\ldots. \qquad (8.43)$$

The generalised Charlier weight ρ is semi-classical and satisfies the Pearson equation

$$\Delta(\phi\rho)=\psi\rho,$$

with

$$\phi(x)=x^N \quad\text{and}\quad \psi(x)=\mu-x^N. \qquad (8.44)$$

If $N=2$, the orthogonal polynomial family associated to the weight $\rho(x)=\frac{\mu^x}{(x!)^2}$ is discrete semi-classical of class one (and called generalised Charlier polynomials of class one).

Replacing in Equations (8.31) and (8.32) ω by one and the polynomials ϕ and ψ given by Equation (8.44) (but with $N=2$), we obtain the Laguerre-Freud equations for the generalised Charlier polynomials of class one:

$$\begin{aligned}\gamma_n+\gamma_{n+1} &= -\binom{n}{2}-\beta_n^2+n\beta_n+\sum_{i=0}^{n-1}\beta_i+\mu, \qquad (8.45)\\ (\beta_n+\beta_{n+1})\gamma_{n+1} &= -n\sum_{i=0}^{n}\beta_i+n\,\gamma_{n+1}+\binom{n+1}{3}+\sum_{i=0}^{n}\beta_i^2+2\sum_{i=1}^{n}\gamma_i+\gamma_{n+1}, \qquad (8.46)\end{aligned}$$

with initial values

$$\begin{aligned}\beta_0 &= \frac{M_1}{M_0}=\frac{\sqrt{\mu}I_1(2\sqrt{\mu})}{I_0(2\sqrt{\mu})}=\frac{\mu\,{}_0F_1(2;\mu)}{{}_0F_1(1;\mu)}, \qquad (8.47)\\ \gamma_1 &= \mu-\beta_0^2,\end{aligned}$$

where $I_0(x)$ and $I_1(x)$ are the modified Bessel functions of order 0 and 1, respectively.

Remark 8.2 *The polynomials P_n have been computed for the generalised Meixner and Charlier polynomials of class one, up to $n = 10$ from β_n, γ_n generated by the Laguerre-Freud equations given above and also from the Hankel representation of polynomials (see (2.6)) which requires the computation of the moments M_j up to $j = 19$. These moments were computed from the moment recurrence relation for the generalised Meixner and Charlier polynomials of class one, respectively:*

$$\begin{aligned}(1-\mu)M_{k+2} &= \alpha_1\alpha_2\mu M_k + (\alpha_1+\alpha_2)\mu M_{k+1} \\ &\quad -\sum_{j=1}^{k}(-1)^j\binom{k}{j}M_{k+2-j}, \\ M_{k+2} &= \mu M_k - \sum_{j=1}^{k}(-1)^j\binom{k}{j}M_{k+2-j}.\end{aligned}$$

The polynomial coefficients in both approaches are written in terms of M_0 and M_1 using the initial values of the Laguerre-Freud recurrence given by Equations (8.42) and (8.47). The polynomials obtained in these two ways coincide, of course, and the Laguerre-Freud approach is obviously more efficient.

8.6.3 Asymptotic behaviour

In the first step we compute numerically, up to $n = 100000$, the coefficients β_n and γ_n Using (8.40) and (8.41), for several values of the coefficients α_1. α_2 and μ and the result of the plot for all cases indicates that the sequences $\frac{\gamma_n}{n^2}$ and $\frac{\beta_n}{n}$ are convergent. Assuming that they converge, their limits, $a(\mu)$ and $b(\mu)$

$$a(\mu) = \frac{\mu}{(1-\mu)^2}, \quad b(\mu) = \frac{1+\mu}{1-\mu}, \tag{8.48}$$

are obtained using Maple V.4 and Equations (8.40), (8.41) with the approximations:
$\gamma_n \cong a(\mu)\, n^2$ and $\beta_n \cong b(\mu)\, n$, for n large.

In the same way, but with β_n and γ_n replaced by $\beta_n - b(\mu)n$ and $\gamma_n - a(\mu)n^2$, respectively, using numerical and symbolic computation with Maple V.4 and analysis of Equations (8.40) and (8.41) [Foupouagnigni et al., 1998f], we observe the asymptotic behaviour for the coefficients β_n and γ_n.

The same process, applied to (8.45) and (8.46), allows to observe the asymptotic behaviour for the generalised Charlier polynomials of class one. We, therefore, give the following conjecture about the asymptotic behaviour for the generalised Charlier and Meixner polynomials of class one. These results, obtained by the Laguerre-Freud equations with Maple V.4, are under investigation [Foupouagnigni et al., 1998f] in order to give a suitable proof.

Conjecture 8.1 *The coefficients β_n and γ_n of the three-term recurrence relation satisfied by the monic generalised Meixner polynomials of class one obey:*

$$\lim_{n\to\infty}\left(\beta_n - \frac{1-\mu}{1-\mu}n - \frac{\mu(\alpha_1+\alpha_2-1)}{1-\mu}\right) = 0, \quad \lim_{n\to\infty}\left(\gamma_n - \frac{\mu(n+\alpha_1-1)(n+\alpha_2-1)}{(1-\mu)^2}\right) = 0,$$

and those of the three-term recurrence relation satisfied by the monic generalised Charlier polynomials of class one obey:

$$\lim_{n\to\infty}(\beta_n - n) = 0, \quad \lim_{n\to\infty}(\gamma_n - \mu) = 0.$$

It should be mentioned that the coefficients β and γ_n of the generalised Meixner polynomials of class 1, are known when α_1 or α_2 is an integer [Ronveaux et al., 1998b]. They obviously confirm the asymptotic behaviour of the coefficients β and γ_n stated in the previous conjecture.

Chapter 9

Conclusion and perspectives

We first list our main contributions to the theory of orthogonal polynomials, then give some open problems which can be investigated as the continuation of this work.

9.1 Conclusion

Chapter 1 introduces the work while Chapter 2 recalls some known materials on orthogonal polynomials.

The main results of Chapter 3 are theorems 3.1 and 3.2. Theorem 3.1 gives a general caracterisation of classical orthogonal polynomials. This result gives a more general caracterisation of classical orthogonal polynomials, and is valid for classical orthogonal polynomials of a continuous variable, classical orthogonal polynomials of a discrete variable and also for q-classical polynomials. It constitutes a unified theory for classical orthogonal polynomials.

Theorem 3.2 caracterises the semi-classical orthogonal polynomials. It gives some links between the semi-classical aspect of the orthogonal polynomials, the quasi-orthogonal aspect of the derivative of these orthogonal polynomials and the structure relations satisfied by these polynomials.

In Chapter 4, we study the properties of the formal Stieltjes function. We mention two results. The first is the theorem 4.1, stating that the affine $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials and the $D_{q,\omega}$-semi-classical orthogonal polynomials are the same. This result is used to obtain the coefficients of the affine $D_{q,\omega}$-Riccati difference equation and the coefficients of the fourth-order difference equation satisfied by the associated Laguerre-Hahn orthogonal polynomials.

The second result is theorem 4.2. It proves that the $D_{q,\omega}$-Laguerre-Hahn orthogonal polynomials can be deduced, using a suitable change of variable, from the $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials. This result is very interesting and could have lot of applications. We have used it to deduce the coefficients of the fourth-order difference equation satisfied by the rth associated Δ-Laguerre-Hahn orthogonal polynomials from the coefficients of the fourth-order difference equation satisfied by the rth associated $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials. The Laguerre-Freud equations for the recurrence coefficients of the D_ω-semi-classical orthogonal polynomials of class 1 have also been obtained using theorem 4.2.

In Chapter 5 we use a result by Suslov [Suslov, 1989] to obtain the factored form of the fourth-order difference equation satisfied by the first associated $\mathcal{D}_q$-classical orthogonal polynomials. We have again used theorem 4.2 to deduce the factored form of the fourth-order difference equation satisfied by the first associated Δ-classical orthogonal polynomials.

We mention that equation (5.12) can be used to obtain some families of classical orthogonal polynomials for which the first associated is still classical. These families:

1- For classical continuous orthogonal polynomials we note the Grosjean polynomials [Ronveaux et al., 1996] of the first kind G_n^α for which the first associated is a Grosjean polynomial of the second kind g_n^α [Grosjean, 1985, 1986], i.e.,

$$(G_n^\alpha)^{(1)} = g_n^{-\alpha},\ -1 < \alpha < 0,$$

where $G_n^\alpha(x) = P_n^{(\alpha,-1-\alpha)}(x)$, $-1 < \alpha < 0$ and $g_n^\alpha(x) = P_n^{(\alpha,1-\alpha)}(x)$, $-1 < \alpha < 2$. $P_n^{(\alpha,\beta)}$ represents the monic Jacobi polynomials with the parameters α and β.

2- For the classical orthogonal polynomials of a discrete variable, we note that the first associated of the monic Hahn polynomial $H_n(x,\alpha,\beta,N)$ with $\alpha+\beta+1=0$ is classical and is related to the Hahn family by [Area et al., 1996]

$$H_n(x,\alpha,\beta,N)^{(1)} = H_n(x-\alpha-1,-\alpha,1+\alpha,N-1).$$

3- For the q-classical polynomials, we have already pointed out the situations for which the first associated little q-Jacobi polynomials $p_n(x;a,b|q)$ and big q-Jacobi polynomials $P_n(x;a,b,c;q)$ are still classical.

The monic little q-Jacobi (resp. monic big q-Jacobi) polynomials and their respective first associated are related by

$$\begin{aligned} p_n^{(1)}(x;a,\frac{1}{q\,a}|q) &= a^n\,q^n\,p_n(\frac{x}{a\,q};\frac{1}{a},a\,q\,|q), \\ P_n^{(1)}(x;a,\frac{1}{q\,a},c;q) &= a^n\,P_n(\frac{x}{a};\frac{1}{a},a\,q,c\,q;q). \end{aligned}$$

In Chapter 6 we have proved (see theorem 6.1 and proposition 6.1) that the associated of any integer order of the Laguerre-Hahn linear functional is a Laguerre-Hahn linear functional. We also gave upper bounds for the degrees of coefficients E_r, F_r, G_r and H_r of the $\mathcal{D}_q$-Riccati difference equation satisfied by the Stieltjes function S_r of the rth associated $\mathcal{L}^{(r)}$ of $\mathcal{L}$ (see proposition 6.2).

Theorem 6.3 gives fondamental relations which lead to the fourth-order difference equation for the rth associated $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials

$$\sum_{j=0}^{4} I_j(r,n,q;x)\mathcal{G}_q^j\,P_n^{(r)} = 0.$$

given in theorem 6.4. Theorem 6.3 and 6.4 are valid for $\mathcal{D}$-Laguerre-Hahn orthogonal polynomials (by limit process) and for Δ-Laguerre-Hahn orthogonal polynomials (via theorem 4.2). We have also given explicitly coefficients E_r, F_r, G_r, H_r and $I_j(r,n,q;x)$ for classical situations.

Chapter 7 contains known materials needed for this work.

The main result of Chapter 8 is theorem 8.1 which shows that it is possible to compute recursively via two non-linear equations, coefficients β_n and γ_n of the three-term recurrence relation satisfied by the $\mathcal{D}_q$-semi-classical orthogonal polynomials of class one. This new result (theorem 8.1) is used, together with theorem 4.2 and lemma 8.3, to obtain Theorem 8.2 giving the Laguerre-Freud equations for the recurrence coefficients of the D_ω-semi-classical orthogonal polynomials of class one.

Using theorem 8.2, we have given a conjecture about the asymptotic behaviour of the coefficients β_n and γ_n of the three-term recurrence relation satisfied by the generalised Charlier and generalised Meixner polynomials of class 1.

9.2 Perspectives

As the continuation of this work, many investigations can be done:

1. Theorem 4.2 proves that the Δ-Laguerre-Hahn orthogonal polynomials can be obtained from the $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials. In principle, this result means that any result obtained for the $\mathcal{D}_q$-Laguerre-Hahn orthogonal polynomials can be extended to the Δ-Laguerre-Hahn orthogonal polynomials. It will be interesting to see how these results are extended and see their consequences in the applications of orthogonal polynomials.

2. It might be possible to simplify and writte the fourth-order difference equation for the $\mathcal{D}_q$-classical orthogonal polynomials in the compact form as was done for $\mathcal{D}$-classical orthogonal polynomials (see (6.69)) [Lewanowicz, 1995].

3. One can use the fourth-order difference equation satisfied by the rth associated $\mathcal{D}_q$-classical orthogonal polynomials $P_n^{(r)}$

$$\mathbf{M}(r,n,q;x)\,P_n^{(r)} = \sum_{j=0}^{4} I_j(r,n,q;x)\mathcal{G}_q^j\,P_n^{(r)} = 0,$$

to expand the rth associated $P_n^{(r)}$ in the basis $\{P_n\}_{n\in\mathbb{N}}$

$$P_n^{(r)}(x) = \sum_{j=0}^{n} C(n,j)\,P_j,$$

as was done for $\mathcal{D}$-classical orthogonal polynomials and Δ-classical orthogonal polynomials (see [Lewanowicz, 1996,1997], [Area et al. ,1998a, 1998b], [Godoy et al., 1996], [Askey 1965,1975],[Askey et al, 1984] ...).

4. The fourth-order difference equation can be established for the general Laguerre-Hahn orthogonal polynomials. We mention for example that Bangerezako [Bangerezako, 1998], had derived the fourth-order difference equation for the Laguerre-Hahn polynomials orthogonal on special non-uniform lattices (snul).

5. The Laguerre-Freud equations for class $s > 1$ can be obtained by mimicking the approach developed in this thesis. This generalisation is already under investigation [Azatassou et al., 1998].

6. The conjecture obtained using the Laguerre-Freud equations need to be proved and extended to the semi-classical orthogonal polynomials of class $s > 1$. For this purpose, It may be hepful to have a look at the papers giving the proof of Freud's conjecture (see the Introduction).

Chapter 10

Appendices

10.1 Appendix I

10.1.1 About $\mathcal{D}$-classical orthogonal polynomials

We give the polynomials ϕ and ψ appearing in the Pearson equation satisfied by the weight ρ ($\mathcal{D}(\phi\rho) = \psi\rho$) defining the classical orthogonal polynomials of a continuous variable [Chihara, 1978], [Nikiforov et al., 1983, 1991], [Szegö, 1939].

1. Jacobi $P_n^{(\alpha,\beta)}(x)$
$$\phi(x) = 1 - x^2,\ \psi(x) = -(\alpha+\beta+2)\,x + \beta - \alpha$$
2. Laguerre $L_n^{\alpha}(x)$
$$\phi(x) = x,\ \psi(x) = -x + \alpha + 1$$
3. Hermite $H_n(x)$
$$\phi(x) = 1,\ \psi(x) = -2\,x$$
4. Bessel $B(x)$
$$\phi(x) = x^2,\ \psi(x) = -2\,(x+1)$$

10.1.2 About Δ-classical orthogonal polynomials

We give the polynomials ϕ and ψ appearing in the Pearson equation satisfied by the weight ρ ($\Delta(\phi\rho) = \psi\rho$) defining the classical orthogonal polynomials of a discrete variable [Chihara, 1978], [Nikiforov et al., 1983, 1991], [Szegö, 1939].

1. Hahn $h_n^{(\alpha,\beta)}(x)$
$$\phi(x) = x\,(N+\alpha-x),\ \psi(x) = -(\alpha+\beta+2)\,x + (\beta+1)(N-1)$$
2. Meixner $m_n^{(\nu,\mu)}(x)$
$$\phi(x) = x,\ \psi(x) = \mu\nu + (\mu-1)\,x$$
3. Krawtchouk $k_n^{(p)}(x)$
$$\phi(x) = x,\ \psi(x) = \frac{(N\,p - x)}{1-p}$$
4. Charlier $c_n^{(\mu)}(x)$
$$\phi(x) = x,\ \psi(x) = \mu - x$$

10.1.3 About q polynomials

We give the polynomials ϕ and ψ appearing in the Pearson equation satisfied by the weight ρ ($\mathcal{D}_q(\phi\rho) = \psi\rho$) defining the polynomials appearing in the q-Hahn tableau. [Koekoek et al, 1996], [Koornwinder, 1994]. Notice that these polynomials ϕ and ψ were already given case by case in [Medem, 1996] and [Ivan et al, 1998].

1. Big q-Jacobi $P_n(x; a, b, c; q)$

$$\phi(x) = a\,c\,q - (a + c)\,x + \frac{x^2}{2},\ \psi(x) = \frac{c\,q - x + a\,q\,(1 - (b + c)\,q + b\,q\,x)}{(-1 + q)\,q}$$

2. Little q-Jacobi $p_n(x; a, b|q)$

$$\phi(x) = \frac{x\,(x - 1)}{q},\ \psi(x) = \frac{1 - x + a\,q\,(b\,q\,x - 1)}{(-1 + q)\,q}$$

3. Stieltjes-Wigert $S_n(x; q)$

$$\phi(x) = \frac{x}{q},\ \psi(x) = \frac{q\,x - 1}{(-1 + q)\,q}$$

4. q-Meixner $m_n(x; b, c; q)$

$$\phi(x) = -b\,c + \frac{c\,x}{q},\ \psi(x) = \frac{c + q - b\,c\,q - q\,x}{(1 - q)\,q}$$

5. Alternative q-Charlier $K_n(x; a; b)$

$$\phi(x) = \frac{x\,(1 - x)}{q},\ \psi(x) = \frac{-1 + x\,(1 + a\,q)}{(-1 + q)\,q}$$

6. Little q-Laguerre/Wall $L_n^{(a)}(x; q)$

$$\phi(x) = \frac{x\,(1 - x)}{q},\ \psi(x) = \frac{-1 + a\,q + x}{(-1 + q)\,q}$$

7. q-Charlier $U_n^{(a)}(x; q)$

$$\phi(x) = a - (1 + a)\,x + x^2,\ \psi(x) = \frac{1 + a - x}{q - 1}$$

8. Discrete q-Hermite $h_n(q; x)$

$$\phi(x) = x^2 - 1,\ \psi(x) = \frac{x}{1 - q}$$

10.2 Appendix II

10.2.1 Results on general associated classical discrete polynomials

We use Theorem 4.2 and 6.3 to obtain operators $\mathcal{D}_{r,n}$, $\mathcal{N}_{r+1,n-1}$, $\bar{\mathcal{D}}_{r+1,n-1}$ and $\bar{\mathcal{N}}_{r,n}$ for the classical orthogonal polynomials of a discrete variable (see Foupouagnigni et. al. 1998b]. These basic operators (see (6.38) and (6.39)) and the coefficients of the fourth order difference equation for associated classical discrete orthogonal polynomials (see (6.7)) are written down in each case (for notations see [Nikiforov et al.,1991]).

Charlier case $C_n^{\mu}(x), \ \ \mu > 0$

$$\phi(x) = x, \quad \psi(x) = -x + \mu,$$

$$\begin{aligned}\mathcal{D}_{r,n} &= \mu\,(2+x)\,\mathcal{T}^2 - (2+x-r)\,(x-n-r+1+\mu)\,\mathcal{T}\\ &\quad -(-3\,x-2+3\,r-x^2+2\,x\,r-r^2+r\,\mu)\,\mathcal{I}_d,\\ \mathcal{N}_{r+1,n-1} &= -r\,\mu\,(x-n-r+1+\mu)\,\mathcal{T} + r\,\mu\,(\mu+2+x-r)\,\mathcal{I}_d,\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{D}}_{r+1,n-1} &= \mu\,(2+x)\,\mathcal{T}^2 - \mu\,(x-n-r+1+\mu)\,\mathcal{T} + \mu\,(-r+\mu)\,\mathcal{I}_d,\\ \bar{\mathcal{N}}_{r,n} &= -(-x+n+r-1-\mu)\,\mathcal{T} - (\mu+x+1-r)\,\mathcal{I}_d,\end{aligned}$$

$$\begin{aligned}I_0(r,n,x) &= \mu\,(1+x)\,(-2+n+2\,R),\\ I_1(r,n,x) &= (2\,x\,\mu+2\,R+4\,\mu-2\,R^3+n\,R-3\,n\,R^2-n^2\,R),\\ I_2(r,n,x) &= (2\,x\,\mu+2\,R+4\,\mu-5\,\mu\,n-2\,x\,\mu\,n-4\,\mu\,x\,R+4\,R^3-10\,\mu\,R-6\,R^2,\\ &\quad -4\,n\,R+6\,n\,R^2+4\,n^2\,R-n^2+n^3),\\ I_3(r,n,x) &= +(-2\,x\,\mu-4\,R-6\,\mu-2\,n-2\,R^3+6\,R^2+7\,n\,R-3\,n\,R^2-n^2\,R+2\,n^2),\\ I_4(r,n,x) &= \mu\,(4+x)\,(n+2\,R),\end{aligned}$$

where R is given by $R = r - x - \mu - 2$.

Meixner case $M_n^{(\nu,\mu)}(x), \ \ \nu > 0, \ \ 0 < \mu < 1$

$$\phi(x) = x; \ \psi(x) = (\mu-1)x + \mu\,\nu,$$

$$\begin{aligned}\mathcal{D}_{r,n} &= \mu\,(x+2)\,(x+1+\nu)\,(\mu-1)\mathcal{T}^2\\ &\quad +(-2-x+r)\,(1+x-r-n+\mu\,\nu+x\,\mu+r\,\mu+n\,\mu+\mu)\,(\mu-1)\mathcal{T}\\ &\quad -(-r\,\mu+r\,\mu\,\nu+r^2\,\mu-3\,x-r^2+2\,x\,r-x^2-2+3\,r)\,(\mu-1)\mathcal{I}_d,\end{aligned}$$

$$\begin{aligned}\mathcal{N}_{r+1,n-1} &= +r\,(\nu+r-1)\,(1+x-r-n+\mu\,\nu+x\,\mu+r\,\mu+n\,\mu+\mu)\mu\mathcal{T}\\ &\quad -r\,(\nu+r-1)\,(\mu\,\nu+x\,\mu+x-r+r\,\mu+2)\mu\mathcal{I}_d,\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{D}}_{r+1,n-1} &= \mu\,(x+2)\,(x+1+\nu)\mathcal{T}^2\\ &\quad -(x+1+\nu+r)\,(1+x-r-n+\mu\,\nu+x\,\mu+r\,\mu+n\,\mu+\mu)\,\mu\mathcal{T}\\ &\quad +(r-r\,\nu+x\,\mu+r\,\mu+r^2\,\mu+\mu\,\nu^2+x^2\,\mu+2\,x\,\mu\,\nu+2\,r\,\mu\,\nu\\ &\quad +2\,x\,\mu\,r+\mu\,\nu-r^2)\mu\mathcal{I}_d,\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{N}}_{r,n} &= -(\mu-1)\,(1+x-r-n+\mu\,\nu+x\,\mu-r\,\mu+n\,\mu+\mu)\mathcal{T}\\ &\quad +(\mu-1)\,(x\,\mu+\mu+\mu\,\nu+r\,\mu-r+x+1)\mathcal{I}_d,\end{aligned}$$

$$I_0(r,n,x) = -\mu\,(-3\,\mu+M+2\,R-3)\,(x+\nu)\,(x+1),$$

$$\begin{aligned}I_1(r,n,x) &= -6\,\mu^2\,x-2\,x^2\,\mu^2-4\,\mu^2-4\,\mu^2\,\nu-2\,x\,\mu^2\,\nu-3\,\mu\,R^2-3\,\mu\,M\,R-6\,x\,\mu\\ &\quad -2\,x^2\,\mu-4\,\mu-4\,\mu\,\nu-2\,x\,\mu\,\nu-3\,M\,R-3\,R^2+M^2\,R+3\,R^2\,M+2\,R^3,\end{aligned}$$

$$
\begin{aligned}
I_2(r,n,x) &= -4R^3-6\mu^2-4M^2R-9\mu^2\nu-14\mu^2x-4\mu R-2\mu M-5M-M^3\\
&\quad -6\mu-4x^2\mu^2-4x^2\mu-4x\mu\nu-4x\mu^2\nu-14x\mu-9\mu\nu+2+16Rx\mu+10R\mu\nu\\
&\quad +12\mu R^2+4Rx^2\mu+4Rx\mu\nu+8\mu Mx+12\mu MR+12MR+2\mu^3+4M^2-10\mu^2R\\
&\quad +5M\mu\nu-6R^2M+2x^2\mu M-5\mu^2M+2x\mu\nu M-10R+12R^2+4\mu M^2,
\end{aligned}
$$

$$
\begin{aligned}
I_3(r,n,x) &= 2R^3+M^2R+6\mu^2\nu+10\mu^2x+24\mu R+12\mu M+6M+2x^2\mu^2\\
&\quad +2x^2\mu+2x\mu\mu+2x\mu^2\nu+10x\mu+6\mu\nu-4-9\mu R^2-9\mu MR-9NR-4\mu^3\\
&\quad -2M^2+12\mu^2R+3R^2M+6\mu^2M+12R-9R^2-2\mu M^2,
\end{aligned}
$$

$$
I_4(r,n,x) = -(x+4)(x+3+\nu)(-\mu+M+2R-1)\mu,
$$

where $R=r-x-2-\mu(r+x+\nu)$, and $M=(n+1)(1-\mu)$.

Krawtchouk case $k_n^{(p)}(x),\ \ p>0,\ q>0,\ p+q=1$

$$
\phi(x)=x,\ \psi(x)=\frac{1}{q}((1-q)N-x),
$$

$$
\begin{aligned}
\mathcal{D}_{r,n} &= (q-1)(x+2)(-x-1+N)\mathcal{T}^2\\
&\quad +(-2-x+r)(-2q-2xq+qN-N+x+r+1+n)\mathcal{T}\\
&\quad +(-3xq-2q-Nrq+r-r^2-x^2q+rN+2xqr+2rq)\mathcal{I}_d,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}_{r+1,n-1} &= (q-1)(N-r+1)(-2q-2xq+qN-N+x+r+1+n)r\mathcal{T}\\
&\quad +(q-1)(N-r+1)(qN-N-2q-2xq+r+x)r\mathcal{I}_d,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathcal{D}}_{r+1,n-1} &= q(q-1)(x+2)(-x-1+N)\mathcal{T}^2\\
&\quad +(q-1)(N-x-r-1)(-2q-2xq+qN-N+x+r+1+n)\mathcal{T}\\
&\quad -(q-1)(-2xqN+qN^2-qN+2rq+2xqr+x^2q+xq-Nrq+2rN\\
&\quad -r^2-r-2xr-N^2+N+2Nx-x^2-x)\mathcal{I}_d,
\end{aligned}
$$

$$
\bar{\mathcal{N}}_{r,n} = -(-x-r-qN+2q+2xq-n-1+N)\mathcal{T}-(-2xq-2q+qN+r+x-N+1)\mathcal{I}_d,
$$

$$
\begin{aligned}
I_0(r,n,x) &= q(1+x)(x-N)(q-1)(2R+n),\\
I_1(r,n,x) &= (6xq+9nq-4Nq-12q^2\\
&\quad -4xq^3N+2n^2-2xNq+2x^2q-8q^3N+12xq^3+4x^2q^3+8q^3\\
&\quad -9nq^2+4q-2n-4R-3n^2q+12q^2N+6xq^2N-18xq^2-6x^2q^2\\
&\quad -12nqR-3nR^2-2R^3-18Rq^2-12qR^2-n^2R+7nR+18Rq+6R^2),\\
I_2(r,n,x) &= -(10xq-8nq-6Nq-42q^2\\
&\quad -5nNq^2+8xq^2n-4xq^3N+2x^2nq^2+5nNq+2xNqn+n^2\\
&\quad -2xNq+2x^2q-12q^3N+20xq^3+4x^2q^3-2xNq^2n+28q^3+6nq^2\\
&\quad +14q-2R-8xqn-n^3-4n^2q+18q^2N+6xq^2N-30xq^2-2x^2nq\\
&\quad -6x^2q^2-12nqR-4x^2qR-6nR^2-4R^3+12Rq^2-12qR^2-4n^2R\\
&\quad +4nR-12Rq+6R^2-10q^2NR-16xqR+16xq^2R-4xq^2NR\\
&\quad +4xNqR+10NqR+4x^2q^2R),
\end{aligned}
$$

$$
\begin{aligned}
I_3(r,n,x) &= -(10\,x\,q + n\,q - 6\,N\,q - 42\,q^2 - 4\,x\,q^3\,N - 2\,x\,N\,q + 2\,x^2\,q - 12\,q^3\,N \\
&\quad +20\,x\,q^3 + 4\,x^2\,q^3 + 28\,q^3 - 3\,n\,q^2 + 14\,q - 2\,R - n^2\,q + 18\,q^2\,N + 6\,x\,q^2\,N \\
&\quad -30\,x\,q^2 - 6\,x^2\,q^2 + 3\,n\,R^2 + 2\,R^3 - 6\,R\,q^2 + n^2\,R - n\,R + 6\,R\,q), \\
I_4(r,n,x) &= q\,(4+x)\,(x+3-N)\,(q-1)\,(n-2+4\,q+2\,R).
\end{aligned}
$$

where R is given by $R = r + x - 2xq + qN - 5q - N + 2$.

Hahn case $h_n^{(\alpha,\beta)}(x,N)$ $\alpha > -1,\ \beta > -1$,

$$\phi(x) = x(N+\alpha-x), \quad \psi(x) = (\beta+1)(N-1) - (\alpha+\beta+2)x$$

The r th associated $P_n^{(r)}$ of the Hahn polynomials, with $n + r \leq N$, is annihilated by the following difference operator, by a decomposition already used in the r associated Meixner case (see [Lewanowicz, 1997]).

$$M_n^{(r)} \equiv \sum_{j=0}^{4} I_j(r,n,x)\,\mathcal{T}^j = \bar{\mathcal{D}}^{**}_{1,n}\,\bar{\mathcal{D}}^{*}_{1,n} + (r-1)\sum_{j=0}^{4} \bar{I}_j(r,n,x)\,\mathcal{T}^j, \tag{10.1}$$

where from (5.21) ,

$$
\begin{aligned}
\bar{\mathcal{D}}^{*}_{1,n} &= (x+2)\,(\alpha+N-x-2)\mathcal{T}^2 \\
&\quad +(7+n\,(3+n+\alpha) - 3\,N + 6\,x - (\alpha+2\,N)\,x + 2\,x^2 + \beta\,(3+n+x-N))\mathcal{T} \\
&\quad +(x+\beta+1)\,(N-x-1)\mathcal{I}_d,
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathcal{D}}^{**}_{1,n} &= (x+4+\beta)\,(N-x-4)\,(20+3\,n+n^2-8\,N-4\,(N-4)\,x+4\,x^2 \\
&\quad +\beta\,(6+n-2\,N+2\,x)+\alpha\,(n-2\,x-2))\,\mathcal{T}^2 \\
&\quad +\big(360+141\,n+56\,n^2+6\,n^3+n^4-260\,N-45\,N\,n-15\,N\,n^2+44\,N^2 \\
&\quad -2\,(52+3\,n\,(3+n)-20\,N)\,(N-5)\,x \\
&\quad +2\,(3\,n\,(3+n)+152+4\,(-15+N)\,N)\,x^2-16\,(N-5)\,x^3+8\,x^4 \\
&\quad +\alpha^2\,(n-2-x)\,(n-2\,x-2)+\beta^2\,(3+n+x-N)\,(n+8-2\,N+2\,x)+\alpha(\\
&\quad 2\,n^3+2\,n^2-3\,n^2\,x-2\,(x+1)\,(38-12\,N+23\,x-4\,N\,x+4\,x^2) \\
&\quad +n\,(35-15\,N+21\,x-6\,N\,x+6\,x^2) \\
&\quad +\beta\,(2\,n^2+7\,n-3\,N\,n+4\,(N-x-4)\,(x+1)))+\beta(2\,n^3 \\
&\quad +n^2\,(17-3\,N+3\,x)+n\,(80+39\,x+6\,x^2-24\,N-6\,N\,x) \\
&\quad +2\,(N-x-4)\,(-23+9\,N+4\,N\,x-17\,x-4\,x^2))\big)\,\mathcal{T} \\
&\quad +(x+1)\,(x+1-N-\alpha)\,(-n\,(3+n)+\alpha\,(4-n+2\,x)-\beta\,(n+8-2\,N+2\,x) \\
&\quad -40+12\,N-24\,x+4\,N\,x-4\,x^2\big)\,\mathcal{I}_d,
\end{aligned}
$$

$$\bar{I}_0(r,n,x) = -2(x+1)\,(x+1-N)\,(x+1-N-\alpha)\,(x+\beta+1)\,(r+\beta+n+1+\alpha),$$

$$
\begin{aligned}
\bar{I}_1(r,n,x) = {} & (r+\beta+n+1+\alpha)(840\,x-24\,r\,N\,\alpha+18\,\alpha\,N\,\beta+4\,\alpha\,\beta\,r+74\,n\,\beta \\
& -234\,N\,\beta+58\,\beta\,r+58\,\alpha\,r+50\,n\,\alpha-30\,N\,\beta^2-10\,\alpha\,r^2+14\,r\,\beta^2 \\
& -24\,N\,r^2+20\,n^2+58\,r^2+36\,\beta^2+90\,N^2+2\,r^4+270\,\beta-420\,N-150\,\alpha \\
& +40\,n+600\,x^2+192\,x^3+24\,x^4+58\,r\,n-42\,\alpha\,\beta+66\,N\,\alpha+14\,\beta\,r^2 \\
& +12\,\alpha^2-24\,r\,N\,\beta+56\,\beta\,x\,n+48\,\beta\,N^2+6\,N^2\,\beta^2+4\,\alpha\,r^3+2\,r^2\,\beta^2+4\,\beta\,r^3 \\
& +6\,x^2\,\beta^2+348\,\beta\,x-252\,x\,\alpha-132\,x^2\,\alpha+156\,x^2\,\beta+30\,\beta^2\,x+48\,x\,r^2 \\
& +24\,\beta\,x^3-24\,x^3\,\alpha+12\,x^2\,r^2+6\,x^2\,\alpha^2+18\,x\,\alpha^2-10\,r\,\alpha^2+2\,\alpha^2\,r^2
\end{aligned}
$$

$$
\begin{aligned}
&- 600\,N\,x - 6\,r\,N\,\alpha\,\beta + 4\,\alpha\,r^2\,\beta - 6\,r^2\,N\,\beta - 6\,N\,\beta^2\,r + 24\,\beta\,N^2\,x \\
&- 12\,x^2\,\alpha\,\beta + 12\,x^2\,\beta\,r + 12\,x^2\,\alpha\,r - 12\,\beta^2\,N\,x - 48\,x\,\alpha\,\beta + 84\,\alpha\,N\,x \\
&- 204\,\beta\,N\,x + 24\,N\,x^2\,\alpha + 48\,x\,\alpha\,r + 48\,x\,\beta\,r - 48\,\beta\,N\,x^2 - 6\,x\,\alpha\,r^2 \\
&- 12\,x\,N\,r^2 + 6\,x\,\beta\,r^2 + 6\,\beta^2\,x\,r - 12\,\beta\,N\,x\,r - 12\,\alpha\,N\,x\,r + 12\,\alpha\,N\,x\,\beta \\
&- 6\,x\,\alpha^2\,r + 24\,n^2\,x + 48\,n\,x + 24\,N^2\,x^2 - 12\,N\,n^2 + 16\,n\,x^2 + 8\,n^2\,x^2 \\
&- 48\,N\,x^3 - 24\,N\,n - 288\,N\,x^2 + 14\,n\,\beta^2 + r^2\,n + 4\,r^3\,n + 3\,r^2\,n^2 + r\,n^2 \\
&+ 9\,\beta\,n^2 - 10\,n\,\alpha^2 - 3\,\alpha\,n^2 + \alpha^2\,n^2 + n^2\,\beta^2 + n^3\,\beta + n^3\,\alpha + r\,n^3 + 96\,x\,N^2 \\
&- 8\,N\,n^2\,x - 16\,N\,n\,x - 6\,N\,\beta^2\,n + 3\,\beta^2\,r\,n + 7\,r^2\,n\,\beta + 15\,r\,n\,\beta + 4\,\alpha\,n\,\beta \\
&- 4\,N\,\beta\,n^2 + 4\,\beta\,r\,n^2 + 3\,\alpha^2\,r\,n + 7\,\alpha\,r^2\,n + 4\,\alpha\,r\,n^2 + 2\,\alpha\,\beta\,n^2 + 6\,\alpha\,r\,n\,\beta \\
&- 6\,N\,\alpha\,n\,\beta - 6\,N\,r\,n\,\beta - 32\,N\,\beta\,n - 9\,\alpha\,r\,n - 24\,N\,\alpha\,n - 24\,N\,r\,n \\
&+ 6\,\beta^2\,x\,n + 4\,\beta\,x\,n^2 - 6\,x\,\alpha^2\,n - 4\,x\,\alpha\,n^2 + 40\,x\,\alpha\,n + 12\,x^2\,\alpha\,n \\
&+ 12\,x^2\,\beta\,n + 12\,x^2\,r\,n + 48\,x\,r\,n - 6\,x\,\alpha\,r\,n + 6\,x\,r\,n\,\beta - 12\,\alpha\,N\,x\,n \\
&- 12\,\beta\,N\,x\,n - 12\,x\,N\,r\,n + 450),
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_2(r,n,x) = {} & -2\,(r + \beta + n + 1 + \alpha)(1540\,x - 30\,r\,N\,\alpha + 19\,\alpha\,N\,\beta + 4\,\alpha\,\beta\,r \\
&+ 140\,n\,\beta - 326\,N\,\beta + 88\,\beta\,r + 88\,\alpha\,r + 60\,n\,\alpha - 36\,N\,\beta^2 - 13\,\alpha\,r^2 \\
&+ 17\,r\,\beta^2 - 30\,N\,r^2 + 131\,n^2 + 88\,r^2 + 54\,\beta^2 + 133\,N^2 + 2\,r^4 + 475\,\beta \\
&- 770\,N - 295\,\alpha + 238\,n + 858\,x^2 + 220\,x^3 + 22\,x^4 + 88\,r\,n - 55\,\alpha\,\beta \\
&+ 103\,N\,\alpha + 17\,\beta\,r^2 + 24\,\alpha^2 - 30\,r\,N\,\beta + 76\,\beta\,x\,n + 55\,\beta\,N^2 + 6\,N^2\,\beta^2 \\
&+ 4\,\alpha\,r^3 + 2\,r^2\,\beta^2 + 4\,\beta\,r^3 + 6\,x^2\,\beta^2 + 489\,\beta\,x - 369\,x\,\alpha - 153\,x^2\,\alpha \\
&+ 177\,x^2\,\beta + 36\,\beta^2\,x + 60\,x\,r^2 + 22\,\beta\,x^3 - 22\,x^3\,\alpha + 12\,x^2\,r^2 + 6\,x^2\,\alpha^2 \\
&+ 24\,x\,\alpha^2 - 13\,r\,\alpha^2 + 2\,\alpha^2\,r^2 - 858\,N\,x - 6\,r\,N\,\alpha\,\beta + 4\,\alpha\,r^2\,\beta - 6\,r^2\,N\,\beta \\
&- 6\,N\,\beta^2\,r + 22\,\beta\,N^2\,x - 10\,x^2\,\alpha\,\beta + 12\,x^2\,\beta\,r + 12\,x^2\,\alpha\,r - 12\,\beta^2\,N\,x \\
&- 50\,x\,\alpha\,\beta + 98\,\alpha\,N\,x - 232\,\beta\,N\,x + 22\,N\,x^2\,\alpha + 60\,x\,\alpha\,r + 60\,x\,\beta\,r \\
&- 44\,\beta\,N\,x^2 - 6\,x\,\alpha\,r^2 - 12\,x\,N\,r^2 + 6\,x\,\beta\,r^2 + 6\,\beta^2\,x\,r - 12\,\beta\,N\,x\,r \\
&- 12\,\alpha\,N\,x\,r + 10\,\alpha\,N\,x\,\beta - 6\,x\,\alpha^2\,r + 12\,n^3 + 80\,n^2\,x + 3\,n^4 + 160\,n\,x \\
&+ 22\,N^2\,x^2 - 40\,N\,n^2 + 32\,n\,x^2 + 16\,n^2\,x^2 - 44\,N\,x^3 - 80\,N\,n - 330\,N\,x^2 \\
&+ 19\,n\,\beta^2 + 9\,r^2\,n + 4\,r^3\,n + 7\,r^2\,n^2 + 9\,r\,n^2 + 35\,\beta\,n^2 - 11\,n\,\alpha^2 - 5\,\alpha\,n^2 \\
&+ 2\,\alpha^2\,n^2 + 2\,n^2\,\beta^2 + 5\,n^3\,\beta + 5\,n^3\,\alpha + 5\,r\,n^3 + 110\,x\,N^2 - 16\,N\,n^2\,x \\
&- 32\,N\,n\,x - 6\,N\,\beta^2\,n + 3\,\beta^2\,r\,n + 7\,r^2\,n\,\beta + 26\,r\,n\,\beta + 8\,\alpha\,n\,\beta - 8\,N\,\beta\,n^2 \\
&+ 8\,\beta\,r\,n^2 + 3\,\alpha^2\,r\,n + 7\,\alpha\,r^2\,n + 8\,\alpha\,r\,n^2 + 4\,\alpha\,\beta\,n^2 + 6\,\alpha\,r\,n\,\beta - 6\,N\,\alpha\,n\,\beta \\
&- 6\,N\,r\,n\,\beta - 46\,N\,\beta\,n - 4\,\alpha\,r\,n - 30\,N\,\alpha\,n - 30\,N\,r\,n + 6\,\beta^2\,x\,n + 8\,\beta\,x\,n^2 \\
&- 6\,x\,\alpha^2\,n - 8\,x\,\alpha\,n^2 + 44\,x\,\alpha\,n + 12\,x^2\,\alpha\,n + 12\,x^2\,\beta\,n + 12\,x^2\,r\,n \\
&+ 60\,x\,r\,n - 6\,x\,\alpha\,r\,n + 6\,x\,r\,n\,\beta - 12\,\alpha\,N\,x\,n - 12\,\beta\,N\,x\,n - 12\,x\,N\,r\,n \\
&+ 1093),
\end{aligned}
$$

$$
\begin{aligned}
\bar{I}_3(r,n,x) = {} & (r + \beta + n + 1 + \alpha)(2760\,x - 36\,r\,N\,\alpha + 30\,\alpha\,N\,\beta + 4\,\alpha\,\beta\,r \\
&+ 150\,n\,\beta - 486\,N\,\beta + 118\,\beta\,r + 118\,\alpha\,r + 94\,n\,\alpha - 42\,N\,\beta^2 - 16\,\alpha\,r^2 \\
&+ 20\,r\,\beta^2 - 36\,N\,r^2 + 100\,n^2 + 118\,r^2 + 72\,\beta^2 + 210\,N^2 + 2\,r^4 + 810\,\beta \\
&- 1380\,N - 570\,\alpha + 200\,n + 1320\,x^2 + 288\,x^3 + 24\,x^4 + 118\,r\,n - 102\,\alpha\,\beta \\
&+ 174\,N\,\alpha + 20\,\beta\,r^2 + 36\,\alpha^2 - 36\,r\,N\,\beta + 80\,\beta\,x\,n + 72\,\beta\,N^2 + 6\,N^2\,\beta^2 \\
&+ 4\,\alpha\,r^3 + 2\,r^2\,\beta^2 + 4\,\beta\,r^3 + 6\,x^2\,\beta^2 + 732\,\beta\,x - 588\,x\,\alpha - 204\,x^2\,\alpha \\
&+ 228\,x^2\,\beta + 42\,\beta^2\,x + 72\,x\,r^2 + 24\,\beta\,x^3 - 24\,x^3\,\alpha + 12\,x^2\,r^2 + 6\,x^2\,\alpha^2 \\
&+ 30\,x\,\alpha^2 - 16\,r\,\alpha^2 + 2\,\alpha^2\,r^2 - 1320\,N\,x - 6\,r\,N\,\alpha\,\beta + 4\,\alpha\,r^2\,\beta - 6\,r^2\,N\,\beta
\end{aligned}
$$

$$
\begin{aligned}
&- 6\,N\,\beta^2\,r + 24\,\beta\,N^2\,x - 12\,x^2\,\alpha\,\beta + 12\,x^2\,\beta\,r + 12\,x^2\,\alpha\,r - 12\,\beta^2\,N\,x \\
&- 72\,x\,\alpha\,\beta + 132\,\alpha\,N\,x - 300\,\beta\,N\,x + 24\,N\,x^2\,\alpha + 72\,x\,\alpha\,r + 72\,x\,\beta\,r \\
&- 48\,\beta\,N\,x^2 - 6\,x\,\alpha\,r^2 - 12\,x\,N\,r^2 + 6\,x\,\beta\,r^2 + 6\,\beta^2\,x\,r - 12\,\beta\,N\,x\,r \\
&- 12\,\alpha\,N\,x\,r + 12\,\alpha\,N\,x\,\beta - 6\,x\,\alpha^2\,r + 2250 + 56\,n^2\,x + 112\,n\,x + 24\,N^2\,x^2 \\
&- 28\,N\,n^2 + 16\,n\,x^2 + 8\,n^2\,x^2 - 48\,N\,x^3 - 56\,N\,n - 432\,N\,x^2 + 20\,n\,\beta^2 + r^2\,n \\
&+ 4\,r^3\,n + 3\,r^2\,n^2 + r\,n^2 + 17\,\beta\,n^2 - 16\,n\,\alpha^2 - 11\,\alpha\,n^2 + \alpha^2\,n^2 + n^2\,\beta^2 + n^3\,\beta \\
&+ n^3\,\alpha + r\,n^3 + 144\,x\,N^2 - 8\,N\,n^2\,x - 16\,N\,n\,x - 6\,N\,\beta^2\,n + 3\,\beta^2\,r\,n \\
&+ 7\,r^2\,n\,\beta + 21\,r\,n\,\beta + 4\,\alpha\,n\,\beta - 4\,N\,\beta\,n^2 + 4\,\beta\,r\,n^2 + 3\,\alpha^2\,r\,n + 7\,\alpha\,r^2\,n \\
&+ 4\,\alpha\,r\,n^2 + 2\,\alpha\,\beta\,n^2 + 6\,\alpha\,r\,n\,\beta - 6\,N\,\alpha\,n\,\beta - 6\,N\,r\,n\,\beta - 44\,N\,\beta\,n \\
&- 15\,\alpha\,r\,n - 36\,N\,\alpha\,n - 36\,N\,r\,n + 6\,\beta^2\,x\,n + 4\,\beta\,x\,n^2 - 6\,x\,\alpha^2\,n - 4\,x\,\alpha\,n^2 \\
&+ 64\,x\,\alpha\,n + 12\,x^2\,\alpha\,n + 12\,x^2\,\beta\,n + 12\,x^2\,r\,n + 72\,x\,r\,n - 6\,x\,\alpha\,r\,n \\
&+ 6\,x\,r\,n\,\beta - 12\,\alpha\,N\,x\,n - 12\,\beta\,N\,x\,n - 12\,x\,N\,r\,n),
\end{aligned}
$$

$$
\tilde{I}_4(r,n,x) = -2(x+4)\,(x+4-N)\,(x+4-N-\alpha)\,(x+4+\beta)\;(r+\beta+n+1+\alpha).
$$

10.3 Appendix III

We give the coefficients $I_j(r,n,q;x)$ of the fourth-order difference equation satisfied by the rth associated $\mathcal{D}_q$-classical orthogonal polynomials. ϕ_j and ψ_j are the coefficients of the polynomials ϕ and ψ, both related to the q-Pearson equation: $\mathcal{D}_q(\phi\,\mathcal{L}) = \psi\,\mathcal{L}$.

Coefficients $I_j(r,n,q;x)$ for some q-classical orthogonal polynomials

For the discrete q-Hermite and Stieltjes-Wigert cases [see Koekoek et al., 1996], we compute the coefficients $I_j(r,n,q;x)$ using the results given in Theorem 6.6 and obtain after cancelling common factors the following results with the notations: $\nu = q^r$ and $\rho = q^n$.

1. Discrete q-Hermite case ($\phi(x) = x^2 - 1,\ \psi(x) = \frac{x}{1-q}$).

$$
\begin{aligned}
I_0(r,n,q;x) &= (x^2\,\rho\,q^8 + q^7\,x^2 - q^3\,\nu\,\rho - q^2\,\nu\,\rho - \nu\,\rho\,q - \nu\,\rho) \\
&\quad (q\,x - 1)\,(q\,x + 1), \\
I_1(r,n,q;x) &= (q^{15}\,x^6 - \nu^3\,\rho^2 + 2\,\nu^2\,\rho^2\,q^{10}\,x^2 + 2\,\nu^2\,\rho\,q^9\,x^2 - q^8\,\nu\,\rho\,x^4 \\
&\quad + q^4\,\nu^2\,\rho\,x^2 + q^4\,\rho^2\,x^2\,\nu^3 + 2\,q^6\,x^2\,\nu^2\,\rho^2 + 2\,\nu^2\,\rho^2\,q^8\,x^2 - 2\,\nu\,\rho\,q^{12}\,x^4 \\
&\quad + 2\,\nu^2\,\rho^2\,x^2\,q^7 - 2\,\nu\,\rho\,q^{13}\,x^4 + 2\,\nu^2\,\rho\,q^8\,x^2 + 2\,\nu^2\,\rho^2\,q^9\,x^2 - \nu\,\rho^2\,q^{13}\,x^4 \\
&\quad - q^{11}\,x^4\,\nu\,\rho - q^{10}\,x^4\,\nu\,\rho + 2\,\nu^2\,\rho\,q^6\,x^2 - q^9\,\nu\,\rho\,x^4 + 2\,x^2\,\rho\,\nu^2\,q^5 \\
&\quad + \nu^2\,\rho^2\,q^{11}\,x^2 + \nu^2\,\rho\,q^{10}\,x^2 - \nu^3\,\rho^2\,q^9\,x^2 + \nu^2\,\rho^2\,q^5\,x^2 - \nu^3\,\rho^2\,q^8\,x^2 \\
&\quad + \nu^3\,\rho^2\,x^2\,q^5 - \nu\,\rho^2\,q^{14}\,x^4 + 2\,\nu^2\,\rho\,x^2\,q^7 - \nu\,q^{12}\,x^4 - q^{11}\,x^4\,\nu + x^6\,\rho\,q^{16} \\
&\quad - \nu^3\,\rho^2\,q^6 - 3\,\nu^3\,\rho^2\,q^4 - 4\,\nu^3\,\rho^2\,q^3 - 3\,\nu^3\,\rho^2\,q^2 - 2\,\nu^3\,\rho^2\,q - 2\,\nu^3\,\rho^2\,q^5) \\
&\quad \rho^{-1}\nu^{-2}q^{-3}, \\
I_2(r,n,q;x) &= -(q^{17}\,x^6 + 3\,\nu^2\,\rho^2\,q^{10}\,x^2 + 3\,q^6\,x^2\,\nu^2\,\rho^2 + 4\,\nu^2\,\rho^2\,q^8\,x^2 \\
&\quad - 2\,\nu\,\rho\,q^{12}\,x^4 + 4\,\nu^2\,\rho^2\,x^2\,q^7 - 2\,\nu\,\rho\,q^{13}\,x^4 + 4\,\nu^2\,\rho^2\,q^9\,x^2 - 2\,\nu\,\rho^2\,q^{13}\,x^4 \\
&\quad - q^{11}\,x^4\,\nu\,\rho - q^{10}\,x^4\,\nu\,\rho + 4\,\nu^2\,\rho^3\,q^8\,x^2 + 4\,\nu^2\,\rho^3\,q^9\,x^2 + 2\,\nu^2\,\rho^2\,q^{11}\,x^2 \\
&\quad + 2\,\nu^2\,\rho^2\,q^5\,x^2 - 2\,\nu\,\rho^2\,q^{14}\,x^4 - \nu^3\,\rho^3 - \nu\,x^4\,q^{15}\,\rho^2 - \nu\,\rho^3\,q^{17}\,x^4 \\
&\quad + \nu^2\,\rho^3\,q^{15}\,x^4 - \nu\,q^{16}\,\rho^3\,x^4 + \nu^2\,\rho^2\,q^{14}\,x^4 - \nu\,\rho^2\,q^{12}\,x^4 - \nu\,\rho\,q^{14}\,x^4 \\
&\quad - 2\,\nu\,\rho^3\,q^{14}\,x^4 + \nu^2\,q^{13}\,\rho^3\,x^4 - 2\,\nu^3\,\rho^3\,q^{10}\,x^2 + 4\,\nu^2\,\rho^3\,q^{10}\,x^2 \\
&\quad + 2\,\nu^2\,\rho^3\,q^{12}\,x^2 - 2\,\nu^3\,\rho^3\,q^7\,x^2 + 3\,\nu^2\,q^{11}\,\rho^3\,x^2 - \nu\,\rho\,x^4\,q^{15} - 2\,\nu^3\,\rho^3\,q^8\,x^2 \\
&\quad + 2\,\nu^2\,\rho^3\,q^6\,x^2 - 2\,\nu^3\,\rho^3\,q^9\,x^2 + 3\,\nu^2\,\rho^3\,q^7\,x^2 - 2\,\nu\,\rho^3\,q^{15}\,x^4 + \nu^2\,\rho^2\,q^{12}\,x^4
\end{aligned}
$$

$$
\begin{aligned}
& - \nu \rho^2 q^{11} x^4 - \nu \rho^3 q^{12} x^4 - \nu \rho^2 q^{16} x^4 - \nu q^{13} \rho^3 x^4 + q^{18} x^6 \rho + q^{19} x^6 \rho^2 \\
& - 5 \nu^3 \rho^3 q^3 - 4 \nu^3 \rho^3 q^2 - 4 \nu^3 \rho^3 q^5 - 5 \nu^3 \rho^3 q^4 - \nu^3 \rho^3 q^7 - 2 \nu^3 \rho^3 q^6 \\
& - 2 \nu^3 \rho^3 q + q^{20} x^6 \rho^3)\nu^{-2} \rho^{-2} q^{-5}, \\
I_3(r,n,q;x) \quad = \quad & (-\nu^3 \rho^2 + 2 \nu^2 \rho^2 q^{10} x^2 + 2 \nu^2 \rho q^9 x^2 + q^6 x^2 \nu^2 \rho^2 \\
& + 2 \nu^2 \rho^2 q^8 x^2 + 2 \nu^2 \rho^2 x^2 q^7 - 2 \nu \rho q^{13} x^4 + 2 \nu^2 \rho q^8 x^2 + 2 \nu^2 \rho^2 q^9 x^2 \\
& + 2 \nu^2 \rho q^6 x^2 + x^2 \rho \nu^2 q^5 + 2 \nu^2 \rho^2 q^{11} x^2 + 2 \nu^2 \rho q^{10} x^2 - \nu^3 \rho^2 q^8 x^2 \\
& - \nu \rho^2 q^{14} x^4 + 2 \nu^2 \rho x^2 q^7 + q^{20} x^6 - \nu x^4 q^{15} \rho^2 - 2 \nu \rho q^{14} x^4 - \nu \rho x^4 q^{15} \\
& - \nu q^{12} x^4 - \nu^3 \rho^2 q^6 - 3 \nu^3 \rho^2 q^4 - 4 \nu^3 \rho^2 q^3 - 3 \nu^3 \rho^2 q^2 - 2 \nu^3 \rho^2 q \\
& - 2 \nu^3 \rho^2 q^5 - \nu q^{13} x^4 - \nu \rho q^{16} x^4 - \nu \rho q^{18} x^4 - \nu \rho q^{17} x^4 - \nu^3 \rho^2 x^2 q^7 \\
& + \nu^2 \rho q^{11} x^2 + \nu^3 \rho^2 q^{11} x^2 + \nu^3 \rho^2 q^{12} x^2 + \nu^2 \rho^2 q^{12} x^2 + q^{21} x^6 \rho) \\
& \nu^{-2} \rho^{-1} q^{-6}, \\
I_4(r,n,q;x) \quad = \quad & (x^2 \rho q^6 + q^5 x^2 - q^3 \nu \rho - q^2 \nu \rho - \nu \rho q - \nu \rho) \\
& (q^4 x - 1)(q^4 x + 1) q^{-6}.
\end{aligned}
$$

2. Stieltjes-Wigert case ($\phi(x) = \frac{x}{q}$, $\psi(x) = \frac{qx-1}{q(q-1)}$)

$$
\begin{aligned}
I_0(r,n,q;x) \quad = \quad & (q^3 x \nu \rho + x \nu q^2 + q + 1) q x, \\
I_1(r,n,q;x) \quad = \quad & -(q^4 \rho^2 x^3 \nu^3 + q^4 x^2 \nu^2 \rho^2 + q^3 x^3 \nu^3 \rho + 2 q^3 \nu^2 \rho x^2 \\
& + q^3 x \nu \rho - q^2 x + x \nu q^2 + q^2 \nu^2 x^2 + q^2 x^2 \nu^2 \rho + \nu \rho q^2 x + q + q x \nu \\
& + q x^2 \nu^2 \rho + \nu \rho q x + 1 + x \nu + x) q, \\
I_2(r,n,q;x) \quad = \quad & (1 + 2q + 2q^4 \nu^2 \rho x^2 + q^6 x^2 \nu^2 \rho^2 + q^2 \nu^2 x^2 + 2 q x \nu \\
& - 2 q^2 x + x^2 \rho \nu^2 q^5 + 2 \nu^2 \rho^2 q^5 x^2 - 2 q^3 x + q^3 \nu^2 \rho x^2 + q^4 x^2 \nu^2 \rho^2 \\
& + 2 \nu \rho q^2 x + q^2 + 2 q^3 x \nu \rho + 2 x \nu q^2 + 2 q^3 x^2 \nu^2 + 2 q^3 x \nu + q^4 x^2 \nu^2 \\
& - \nu q^3 x^2 - \nu q^4 x^2 + q^4 x^3 \nu^3 + \nu^3 \rho^2 q^6 x^3 + \nu^3 \rho^3 q^7 x^3 + \nu^3 \rho q^5 x^3 \\
& - \nu \rho q^4 x^2 - \nu q^5 x^2 \rho + 2 q^4 x \nu \rho), \\
I_3(r,n,q;x) \quad = \quad & -(\nu^3 \rho^2 x^3 q^7 + \nu^3 \rho q^6 x^3 + \nu^2 \rho q^6 x^2 + \nu^2 \rho^2 q^5 x^2 \\
& + x^2 \rho \nu^2 q^5 + q^4 x \nu \rho + q^4 x + 2 q^4 \nu^2 \rho x^2 + q^3 x^2 \nu^2 + q^3 x \nu + q^3 x \nu \rho \\
& - q^2 x + x \nu q^2 + \nu \rho q^2 x + q + q x \nu + 1), \\
I_4(r,n,q;x) \quad = \quad & (\nu \rho q^2 x + q + q x \nu + 1) q^3 x.
\end{aligned}
$$

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