Mixed Hybrid Finite Volume Analysis of a 1D Diffusion Model

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Abstract: We present in this paper a numerical method using basic ideas of mixed hybrid finite element and finite volume methods. It is the reason why we call it "mixed hybrid finite volume method". Given a one-dimensional diffusion problem we develop the main aspects of this numerical approach. We show the connection with the classical mixed hybrid finite element method. We also show that "mixed hybrid finite volume" is an extension of the classical finite volume method at least for diffusion problems. We establish some error estimates with respect to a discrete version of H_0^1 norm for the potential and L^2 in $\|\cdot\|_\infty$norms for the velocity •

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Introduction

Numerical models play an increasing role in analyzing real life problems today and especially engineering problems. In many applications as underground water flow with transport of pollutants, the need of accurately computed water velocity (for a good understanding of the transport phenomenon) has focused the attention of engineers on Mixed Hybrid Finite Element Methods (MHFEM). These methods are based upon a simultaneous computation of the hydraulic potential and the velocity in the flow process, in such a way that the discrete

versions of these quantities converge, in adequate norms, to their exact counterparts respectively. The computation of the velocity is performed in such a way to preserve the physical law concerning the continuity of its normal component on each interface between two adjacent elements. Thus the mass conservation law is respected at the grid block level. Moreover the MHFEM-based velocity is a nice candidate to put into the discrete version of transport equation.
Although in much literature on MHFEM some authors (for instance [2], [1] and [6]) give a higher level mathematical presentation, some papers like [3] deal with a physical presentation of MHFEM available to engineers.
The objective of this paper is to present, through a diffusion model, a weak but efficient version of MHFEM constructed without any variational formulation, following only the basic ideas of finite volume method. This method could also be viewed as a variant of finite volume method as that will be shown later. Our paper is organized as it follows. In section 1 we describe the mathematical model problem. Section 2 deals with basic aspects of mixed hybrid finite volume through a discretization of the model problem. Section 3 is devoted to some mathematical properties of MHFVM solution: stability and error estimates. Section 4 concludes this work and gives some perspectives.

1 Mathematical model

We are dealing here with a mathematical model for a one-dimensional diffusion phenomenon governed by the following equation and boundary conditions:

$$-\frac{d}{dx}\left[D(x)\frac{d}{dx}u(x)\right] = f(x) \qquad in \qquad \Omega =]0\ ,\ 1[\tag{1}$$

$$u(0) = \alpha, \ \ u(1) = \beta \tag{2}$$

where α and β are in IR, $D(.)$ is the diffusion coefficient which is piecewise constant and such that there exists two real numbers D^- and D^+ satisfying

$$0 \prec D^- \leq D(x) \leq D^+ \quad \text{a.e. in} \quad \Omega \tag{3}$$

and where $f(.)$ is a source-term given in a suitable functional space.

The system (1)-(2) governs miscellaneous diffusion phenomena, for instance steady state one-phase flow in a porous medium. The existence and uniqueness of a variational solution $u \in H^1(\Omega) \subset C^0(\overline{\Omega})$ of (1)-(2) are ensured if $f(.)$ is given in $L^2(\Omega)$. Our aims in what follows are: (i) to carry out a mixed hybrid finite volume formulation of the problem (1)-(2), (ii) to show that this formulation generalizes classical finite volume method, (iii) to prove existence and uniqueness of a discrete solution, (iv) to show its connection with mixed hybrid finite element method, (v) to show the stability of the computed solution and give error estimates.

2 Mixed hybrid finite volume formulation of (1)-(2)

The notion of regular mesh plays a key role in what we intend to do in this section and the following ones. Let us define this notion before going on.

Definition 1
Let $\left\{x_{i+\frac{1}{2}}\right\}_{i=0}^{P}$ be a given sequence of points in $\overline{\Omega} = [0,1]$ such that

$$0 = x_{\frac{1}{2}} \prec x_{\frac{3}{2}} \prec ... \prec x_{P-\frac{1}{2}} \prec x_{P+\frac{1}{2}} = 1 \tag{4}$$

We set

$$\Omega_i = \left]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right[, \ h_i = x_{i-\frac{1}{2}} - x_{i+\frac{1}{2}} \text{ for } i = 1, ..., p \tag{5}$$

The family $F = \{\Omega_i\}_{i=1}^{P}$ defines a regular mesh of $\overline{\Omega}$ if the discontinuities of $D(.)$ do not belong to any Ω_i and there exists a real number $0 \prec \omega \leq 1$ such that

$$\omega h \leq h_i \leq h \qquad \forall\, 1 \leq i \leq P \tag{6}$$

where $h = \max\limits_{1\leq i\leq P} h_i \bullet$

Let us carry out now the mixed hybrid finite volume formulation of the system (1)-(2). Following the ideas presented in our earlier work (see [5],[6]), the system (1)-(2) is equivalent to

$$\frac{dq}{dx}(x) = f(x) \quad \text{in} \quad \Omega_i \quad \text{for} \quad i = 1, ..., P \quad \text{(Balance equation)} \tag{7}$$

$$q(x) = -D(x)\frac{du}{dx}(x) \quad \text{in} \quad \Omega_i \quad \text{for} \quad i = 1, ..., P \quad \text{(Darcy law)} \tag{8}$$

$$u(x^-_{i+\frac{1}{2}}) = u(x^+_{i+\frac{1}{2}}) \quad \text{for} \quad i = 1, ..., P-1 \quad \text{(Continuity of potential)} \tag{9}$$

$$q(x^-_{i+\frac{1}{2}}) = q(x^+_{i+\frac{1}{2}}) \quad \text{for} \quad i = 1, ..., P-1 \quad \text{(Continuity of flux)} \tag{10}$$

$$u(x_{\frac{1}{2}}) = \alpha \ , \quad u(x_{P+\frac{1}{2}}) = \beta \quad \text{(Boundary conditions)} \tag{11}$$

where

$$\phi(x^-_{i+\frac{1}{2}}) = \phi(x^+_{i+\frac{1}{2}}) \text{ means } \lim_{x \overset{\prec}{\to} x_{i+\frac{1}{2}}} \phi(x) = \lim_{x \overset{\succ}{\to} x_{i+\frac{1}{2}}} \phi(x) \tag{12}$$

For sake of clarity and commodity we set:

$$\phi^{\Omega_i}_{i+\frac{1}{2}} = \phi(x^-_{i+\frac{1}{2}}), \qquad \phi^{\Omega_i}_{i-\frac{1}{2}} = \phi(x^+_{i-\frac{1}{2}}), \qquad \phi^{\Omega_{i+1}}_{i+\frac{1}{2}} = \phi(x^+_{i+\frac{1}{2}}).$$

Let us assume that $f(.)$ is a sufficiently regular function, i.e. at least in $C^0(\overline{\Omega})$.When integrating (7) in Ω_i and using a Taylor expansion for (8), the system (7)-(11) yields obviously:

$$q^{\Omega_i}_{i+\frac{1}{2}} - q^{\Omega_i}_{i-\frac{1}{2}} = h_i \langle f \rangle_i \qquad \text{for } \; i = 1, ..., P \;\; \text{(balance equation)} \tag{13}$$

$$\left.\begin{array}{l} q^{\Omega_i}_{i+\frac{1}{2}} \approx \frac{Di}{h/2}\left[u_i - u^{\Omega_i}_{i+\frac{1}{2}}\right] \\ q^{\Omega_i}_{i-\frac{1}{2}} \approx \frac{Di}{h/2}\left[-u_i + u^{\Omega_i}_{i-\frac{1}{2}}\right] \end{array}\right\} \; for \; i = 1, ..., P \;\; (Darcy \;\; law) \tag{14}$$

$$\left.\begin{array}{lll} q^{\Omega_i}_{i+\frac{1}{2}} = q^{\Omega_{i+1}}_{i+\frac{1}{2}} & \text{(continuity of flux)} & \text{(i)} \\ u^{\Omega_i}_{i+\frac{1}{2}} = u^{\Omega_{i+1}}_{i+\frac{1}{2}} & \text{(continuity of potential)} & \text{(ii)} \end{array}\right\} \text{ for } \; i = 1, ..., P-1 \tag{15}$$

$$u^{\Omega_i}_{\frac{1}{2}} = \alpha \;\; \text{and} \;\; u^{\Omega_P}_{P+\frac{1}{2}} = \beta \qquad\qquad \text{(Boundary conditions)} \tag{16}$$

One naturally deduces from the system (13)-(16) what we call the "Mixed Hybrid Finite Volume" scheme which writes as it follows:

Find $\{U_i\}^P_{i=0}$, $\left\{U^{\Omega_i}_{i-\frac{1}{2}}, U^{\Omega_i}_{i+\frac{1}{2}}\right\}^P_{i=1}$ and $\left\{Q^{\Omega_i}_{i-\frac{1}{2}}, Q^{\Omega_i}_{i+\frac{1}{2}}\right\}^P_{i=1}$ such that

$$Q^{\Omega_i}_{i+\frac{1}{2}} - Q^{\Omega_i}_{i-\frac{1}{2}} = h_i \langle f \rangle_i \qquad \text{for } \; i = 1, ..., P \;\; \text{(Discrete balance equation)} \tag{17}$$

$$\left.\begin{array}{l} Q^{\Omega_i}_{i+\frac{1}{2}} \approx \frac{Di}{h/2}\left[U_i - U^{\Omega_i}_{i+\frac{1}{2}}\right] \\ Q^{\Omega_i}_{i-\frac{1}{2}} \approx \frac{Di}{h/2}\left[-U_i + U^{\Omega_i}_{i-\frac{1}{2}}\right] \end{array}\right\} \; for \; i = 1, ..., P \;\; (Discrete \;\; Darcy \;\; law) \tag{18}$$

$$\left.\begin{array}{l} Q^{\Omega_i}_{i+\frac{1}{2}} = Q^{\Omega_{i+1}}_{i+\frac{1}{2}} \\ U^{\Omega_i}_{i+\frac{1}{2}} = U^{\Omega_{i+1}}_{i+\frac{1}{2}} \end{array}\right\} \; for \; i = 1, ..., P \;\; (Continuity \;\; of \;\; flux \;\; and \;\; potential) \tag{19}$$

$$U^{\Omega_i}_{\frac{1}{2}} = \alpha \;\; \text{and} \;\; U^{\Omega_P}_{P+\frac{1}{2}} = \beta \qquad\qquad \text{(Boundary conditions)} \tag{20}$$

Remark 1

It is interesting to note that a simple elimination of $\left\{U^{\Omega_i}_{i-\frac{1}{2}}, U^{\Omega_i}_{i+\frac{1}{2}}\right\}^P_{i=1}$ using (18)-(19) leads to the classical finite volume method as presented in [4] for instance.

The converse is not true.Indeed the basic ideas involved in the equations (17)-(20) above are far to be the same as in the classical finite volume method. This shows that the mixed hybrid finite volume method can be viewed as a generalization of a classical finite volume method for diffusion problems•

We have the following obvious result.

Proposition 1
Let us set: $U_{i+\frac{1}{2}} \equiv U^{\Omega_i}_{i+\frac{1}{2}} = U^{\Omega_{i+1}}_{i+\frac{1}{2}}$ for $i = 1, ..., P-1$. Therefore the discrete unknowns $\{U_i\}_{i=1}^P$, $\left\{U_{i+\frac{1}{2}}\right\}_{i=1}^{P-1}$ obey to the following equations:

$$U_i = \frac{h_i^2}{4D_i}\langle f\rangle_i + \frac{1}{2}\left[U_{i+\frac{1}{2}} + U_{i-\frac{1}{2}}\right] \qquad for \quad i = 1, ..., P \tag{21}$$

and for i=1,...,P-1:

$$-\frac{D_i}{h_i}U_{i-\frac{1}{2}} + \left[\frac{D_i}{h_i} + \frac{D_{i+1}}{h_{i+1}}\right]U_{i+\frac{1}{2}} - \frac{D_{i+1}}{h_{i+1}}U_{i+\frac{3}{2}} = \frac{1}{2}\left[h_i\langle f\rangle_i + h_{i+1}\langle f\rangle_{i+1}\right] \tag{22}$$

with

$$U_{\frac{1}{2}} = \alpha \quad \text{and} \quad U_{P+\frac{1}{2}} = \beta \tag{23}$$

The system (22)-(23) satisfies obviously the discrete maximum principle, that is, if $\left[\langle f\rangle\right]_{i=1}^{P-1}$, α and β are ≥ 0 then $U_{i+\frac{1}{2}}$ is ≥ 0 for $i = 1, ..., P-1$. Thus there exists a unique solution for this system. One deduces then the existence and uniqueness of $\{U_i\}_{i=1}^P$ and $\left\{U^{\Omega_i}_{i-\frac{1}{2}}, U^{\Omega_i}_{i+\frac{1}{2}}\right\}_{i=1}^P$ via the equation (21) and system of relations (18) [i.e. the Darcy law] respectively. Hence, the Mixed Hybrid Finite Volume scheme (17)-(19) yields a unique discrete solution.

Let us prove that the mixed hybrid finite volume method is connected to the mixed hybrid finite element method. We denote $\mathbf{P}_1(\Omega_i)$ the space of polynomial functions defined in Ω_i whose degree is ≤ 1. Following [3] and [5], the mixed hybrid finite element formulation of the system (1)-(2) may be written as follows:

Find :

$\left\{\Phi^{\Omega_i}\right\}_{i=1}^P$ in $\prod_{i=1}^P X_h(\Omega_i)$, $\{\overline{U}_i\}_{i=1}^P$ in $\mathbf{R}^P$ and $\left\{\overline{U}^{\Omega_i}_{i-\frac{1}{2}}, \overline{U}^{\Omega_i}_{i+\frac{1}{2}}\right\}_{i=1}^P$ in $\left[\mathbf{R}^2\right]^P$

such that :

(i) for each $i \in \{1, ..., P\}$ *and* $\forall\, w \in X_h(\Omega_i)$ one has

$$\int_{\Omega_i}[D_i]^{-1}\Phi^{\Omega_i}(x)w(x)dx = \left[\overline{U}^{\Omega_i}_{i+\frac{1}{2}}w(x_{i+\frac{1}{2}}) - \overline{U}^{\Omega_i}_{i-\frac{1}{2}}w(x_{i-\frac{1}{2}})\right] - \int_{\Omega_i}\overline{U}_i w'(x)dx \tag{24}$$

$$\int_{\Omega_i}\Phi^{\Omega_i}(x)dx = \int_{\Omega_i}f(x)dx \tag{25}$$

where $X_h(\Omega_i) = \mathbf{P}_1(\Omega_i)$ is the so-called Raviart-Thomas space over Ω_i of lowest order;

(ii) Continuity of flux and potential across the mesh interfaces should be taken into account;

(iii) Boundary conditions must be involved.

Using as basis functions for $X_h(\Omega_i)$ the polynomial functions defined by

$$e^{\Omega_i}_{i+\frac{1}{2}}(x) = \frac{x - x_{i-\frac{1}{2}}}{h_i} \qquad \text{and} \qquad e^{\Omega_i}_{i-\frac{1}{2}} = \frac{x_{i+\frac{1}{2}} - x}{h_i}$$

and applying the so-called trapezoidal rule to the left hand of (24) one obtains the system of equations (18) i.e. the discrete Darcy law. Since (25) is equivalent to(17), this shows the connection between the mixed hybrid finite element method and the scheme (17)-(19) named "Mixed Hybrid Finite Volume Method".

3 Stability and Error Estimates

We should introduce a tool needed in our analysis of some mathematical properties of the discrete solution.

Lemma 1 (Discrete Poincaré inequality type)
Let v be a continuous function in $\overline{\Omega}$, with $v(0) = v(1) = 0$. Then we have

$$\sum_{i=1}^{P} h_i \left[\frac{v_{i-\frac{1}{2}} + v_{i+\frac{1}{2}}}{2}\right]^2 \leq [mes\,(\Omega)]^2 \sum_{i=1}^{P} \frac{1}{h_i} \left[v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}\right]^2$$

where $v_{i+\frac{1}{2}} = v(x_{i+\frac{1}{2}})$ for $i = 0, ..., P\bullet$

Terminology:
For any continuous function v in $\overline{\Omega}$, the quantity $\|v\|_F$ defined by

$$\|v\|_F = \left[\sum_{i=1}^{P} \frac{1}{h_i}\left(v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}$$

is the so-called discrete $H^1_0\,(\Omega)$ norm (see for instance [4]).

Proof of Lemma 1:
Let v be a continuous function in $\overline{\Omega}$, with $v(0) = v(1) = 0$. Then we have, with the notations previously introduced,
$v_{i-\frac{1}{2}} = -v_{\frac{1}{2}} + v_{\frac{3}{2}} - v_{\frac{3}{2}} + ... + v_{i-\frac{3}{2}} - v_{i-\frac{3}{2}} + v_{i-\frac{1}{2}}$
$v_{i+\frac{1}{2}} = -v_{\frac{1}{2}} + v_{\frac{3}{2}} - v_{\frac{3}{2}} + ... + v_{i-\frac{3}{2}} - v_{i-\frac{3}{2}} + v_{i-\frac{1}{2}} - v_{i-\frac{1}{2}} + v_{i+\frac{1}{2}}$
Combining linearly these two relations and using Minkowski inequality gives
$\left[\frac{1}{2}\left(v_{i-\frac{1}{2}} + v_{i+\frac{1}{2}}\right)\right]^2 \leq \left[\left|v_{\frac{3}{2}} - v_{\frac{1}{2}}\right| + \left|v_{\frac{5}{2}} - v_{\frac{3}{2}}\right| + ... + \left|v_{i-\frac{1}{2}} - v_{i-\frac{3}{2}}\right| + \left|v_{P+\frac{1}{2}} - v_{P-\frac{1}{2}}\right|\right]^2$

Integrating in Ω_i, summing over $i \in \{1, ..., P\}$ and applying Cauchy-Schwarz inequality yields $\sum_{i=1}^{P} h_i \left[\frac{1}{2}\left(v_{i-\frac{1}{2}} + v_{i+\frac{1}{2}}\right)\right]^2 \leq [mes\,(\Omega)]^2 \sum_{i=1}^{P} \frac{1}{h_i}\left[v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}\right]^2$

The lemma 1 is then proven.

We should give now a stability result for the mixed hybrid finite volume scheme (22)-(23). In this frame-work a non-restrictive assumption which we make is that $\alpha = \beta = 0$ i.e. $U_{\frac{1}{2}} = U_{P+\frac{1}{3}} = 0$.

Proposition 2(Stability of the discrete solution)

The unique solution of (22)-(23), denoted $\{U_i\}_{i=1}^{P}$, satisfies the inequality

$\sum_{i=1}^{P} \frac{1}{h_i}\left(U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}\right)^2 \leq \left[\frac{mes(\Omega)}{D^-}\right]^2 \left[\|f\|_{L^2(\Omega)}\right]^2$ •

Proof of Proposition 2:

Multiplying the relation (22) by $U_{i+\frac{1}{2}}$, summing over $i \in \{1, ..., P\}$ and reordering the terms of the right and the left hands yields

$\sum_{i=1}^{P} \frac{D_i}{h_i}\left(U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}\right)^2 = \sum_{i=1}^{P} \int_{\Omega_i} f(x)\left(\frac{U_{i-\frac{1}{2}}+U_{i+\frac{1}{2}}}{2}\right)dx$

A double application of Cauchy-Schwarz inequality to the right hand of the preceding equality gives

$\sum_{i=1}^{P} \frac{D_i}{h_i}\left(U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}\right)^2 \leq \|f\|_{L^2(\Omega)} \left[\sum_{i=1}^{P} h_i \left(\frac{U_{i-\frac{1}{2}}+U_{i+\frac{1}{2}}}{2}\right)^2\right]^{\frac{1}{2}}$

Using the lemma 1 and the assumption (3), it's easily seen that the proposition 2 follows •

Let us give now in the following proposition our main result.

Proposition 3 (Error estimates)

Setting:

$e_{i+\frac{1}{2}} = u(x_{i+\frac{1}{2}}) - U_{i+\frac{1}{2}},\ e_i = u(x_i) - U_i\ ,\ \ \hat{e}_{i+\frac{1}{2}} = [-Du'](x_{i+\frac{1}{2}}) - Q_{i+\frac{1}{2}},$

one has

$$\sum_{i=1}^{P} \frac{1}{h_i}\left[e_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\right]^2 \leq Ch^2 \tag{26}$$

(let us recall that $h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ for $i \in 1, ..., P$);

$$\sum_{i=1}^{P} \frac{1}{h_{i+\frac{1}{2}}} [e_{i+1} - e_i]^2 \leq Ch^2\ ,\ \text{ with } \quad h_{i+\frac{1}{2}} = \frac{h_{i+1} - h_i}{2} \tag{27}$$

and

$$(i)\ \sum_{i=1}^{P} h_i \left|\hat{e}_{i+\frac{1}{2}}\right|^2 \leq Ch^2 \qquad \textit{and} \qquad (ii)\ \max_{0\leq i\leq P}\left|\hat{e}_{i+\frac{1}{2}}\right| \leq Ch. \tag{28}$$

where C denoted miscellaneous constants not depending on h •

Proof of Proposition 3:

We are going to use some times in the proof the following simplified notations:

$q_{i+\frac{1}{2}} = q^{\Omega_i}_{i+\frac{1}{2}},\ q_{i-\frac{1}{2}} = q^{\Omega_i}_{i-\frac{1}{2}},\ Q_{i+\frac{1}{2}} = Q^{\Omega_i}_{i+\frac{1}{2}},\ Q_{i-\frac{1}{2}} = Q^{\Omega_i}_{i-\frac{1}{2}},$ for $i = 1, ..., P$

Let us prove first the estimate (26). The system from which we have derived the Mixed Hybrid Finite Volume (MHFV) scheme writes (see equations (7)-(11)):

$$q^{\Omega_i}_{i+\frac{1}{2}} - q^{\Omega_i}_{i-\frac{1}{2}} = h_i \langle f \rangle_i \qquad \text{for } \ i = 1, ..., P \ \text{ (Balance equation)} \tag{29}$$

$$\left.\begin{array}{l} q^{\Omega_i}_{i+\frac{1}{2}} \approx \frac{Di}{h_i/2}\left[u_i - u^{\Omega_i}_{i+\frac{1}{2}}\right] \\ q^{\Omega_i}_{i-\frac{1}{2}} \approx \frac{Di}{h_i/2}\left[-u_i + u^{\Omega_i}_{i-\frac{1}{2}}\right] \end{array}\right\} \ for\ i = 1, ..., P \ \ (Darcy\ \ law) \tag{30}$$

$$\left.\begin{array}{lll} q^{\Omega_i}_{i+\frac{1}{2}} = q^{\Omega_{i+1}}_{i+\frac{1}{2}} & \text{(continuity of flux)} & \text{(i)} \\ u^{\Omega_i}_{i+\frac{1}{2}} = u^{\Omega_{i+1}}_{i+\frac{1}{2}} & \text{(continuity of potential)} & \text{(ii)} \end{array}\right\} \text{ for } \ i = 1, ..., P-1 \tag{31}$$

$$u^{\Omega_i}_{\frac{1}{2}} = \alpha \ \text{ and } \ u^{\Omega_P}_{P+\frac{1}{2}} = \beta \qquad \text{(Boundary conditions)} \tag{32}$$

Taking into account the consistency error, (30) leads to

$$\left.\begin{array}{l} q^{\Omega_i}_{i+\frac{1}{2}} = \frac{D_i}{h_i/2}\left[u_i - u^{\Omega_i}_{i+\frac{1}{2}}\right] - E^{\Omega_i}_{i+\frac{1}{2}} \\ q^{\Omega_i}_{i-\frac{1}{2}} = \frac{D_i}{h_i/2}\left[-u_i + u^{\Omega_i}_{i-\frac{1}{2}}\right] + E^{\Omega_i}_{i-\frac{1}{2}} \end{array}\right\} \ \text{ for } \ i = 1, ..., P \tag{33}$$

where $E^{\Omega_i}_{i+\frac{1}{2}} = \frac{D_i h_i}{4} u''\left(\theta^g_{i+\frac{1}{2}}\right)$ and $E^{\Omega_i}_{i-\frac{1}{2}} = \frac{D_i h_i}{4} u''\left(\theta^d_{i-\frac{1}{2}}\right)$

Combining linearly these two equations and thanks to (29) one obtains

$$u_i = \frac{h_i^2}{4D_i}\left[\langle f \rangle_i + E^{\Omega_i}_{i+\frac{1}{2}} + E^{\Omega_i}_{i-\frac{1}{2}}\right] + \frac{1}{2}\left[u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}\right] \ \text{ for } \ i = 1, ..., P \tag{34}$$

Therefore one deduces from (31) and (33) that that

$$\begin{array}{c} \frac{D_i}{h_i}\left[u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}\right] + \frac{D_{i+1}}{h_{i+1}}\left[u_{i+\frac{1}{2}} - u_{i+\frac{3}{2}}\right] = \\ \frac{1}{2}\left[h_i \langle f \rangle_i + h_{i+1}\langle f \rangle_{i+1}\right] + \left[E^{\Omega_i}_{i-\frac{1}{2},i+\frac{1}{2}} + E^{\Omega_{i+1}}_{i+\frac{1}{2},i+\frac{3}{2}}\right] \quad \text{for } \quad i = 1, ..., P-1 \end{array} \tag{35}$$

where we have set for $i = 1, ..., P$

$$E^{\Omega_i}_{j-\frac{1}{2},j+\frac{1}{2}} = \frac{D_j h_j}{8} u''\left(\theta^d_{j-\frac{1}{2}}\right) - \frac{D_j h_j}{8} u''\left(\theta^g_{j+\frac{1}{2}}\right) \tag{36}$$

with, thanks to the assumption (6),

$$\left| E^{\Omega_i}_{j-\frac{1}{2},j+\frac{1}{2}} \right| \leq Ch \tag{37}$$

Remarking that the equation (22) is equivalent to

$$\begin{array}{c} \frac{D_i}{h_i}\left[U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}\right] + \frac{D_{i+1}}{h_{i+1}}\left[U_{i+\frac{1}{2}} - U_{i+\frac{3}{2}}\right] = \\ \frac{1}{2}\left[h_i \left\langle f \right\rangle_i + h_{i+1} \left\langle f \right\rangle_{i+1}\right] \qquad \text{for} \quad i = 1, ..., P-1 \end{array} \tag{38}$$

and combining linearly this equation with (35) one sees that the errors $e_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} - U_{i+\frac{1}{2}}$ $\quad i = 1, ..., P-1$ satisfy the following system of equations:

For $\; i = 1, ..., P-1$

$$\frac{D_i}{h_i}\left[e_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\right] + \frac{D_{i+1}}{h_{i+1}}\left[e_{i+\frac{1}{2}} - e_{i+\frac{3}{2}}\right] = \left[E^{\Omega_i}_{i-\frac{1}{2},i+\frac{1}{2}} + E^{\Omega_{i+1}}_{i+\frac{1}{2},i+\frac{3}{2}}\right] \tag{39}$$

Multiplying (39) by $e_{i+\frac{1}{2}}$, summing over $i = 1, ..., P-1$ and reordering the terms yields

$$\begin{array}{c} \sum\limits_{i=1}^{P} \frac{D_i}{h_i}\left[e_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\right]^2 = \\ \sum\limits_{i=1}^{P-1} e_{i+\frac{1}{2}} E^{\Omega_i}_{i-\frac{1}{2},i+\frac{1}{2}} - \sum\limits_{i=1}^{P-1} e_{i+\frac{1}{2}} E^{\Omega_{i+1}}_{i+\frac{1}{2},i+\frac{3}{2}} = \\ \sum\limits_{i=1}^{P} E^{\Omega_i}_{i-\frac{1}{2},i+\frac{1}{2}}\left[e_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\right] \end{array}$$

Thanks to (37) Cauchy-Schwarz inequality one deduces

$$\sum_{i=1}^{P} \frac{D_i}{h_i}\left[e_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\right]^2 \leq Ch \left\{ \sum_{i=1}^{P} \frac{D_i}{h_i}\left[e_{i+\frac{1}{2}} - e_{i-\frac{1}{2}}\right]^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{P} \frac{h_i}{D_i} \right\}^{\frac{1}{2}}$$

Remarking that $\sum\limits_{i=1}^{P} h_i = mes\,(\Omega)$ and using the assumption(3) yields the estimate (26).

Let us prove (27) now. All the previous notations are conserved. From the continuity of the flux across the mesh interfaces one has

$$q^{\Omega_i}_{i+\frac{1}{2}} = q^{\Omega_{i+1}}_{i+\frac{1}{2}}$$

Let us set:

$$q_{i+\frac{1}{2}} \equiv q_{i+\frac{1}{2}}^{\Omega_i} = q_{i+\frac{1}{2}}^{\Omega_{i+1}}$$

The system of equations (13)-(16) leads to

$$q_{i+\frac{1}{2}} - q_{i-\frac{1}{2}} = h_i \langle f \rangle_i \qquad for \;\; i = 1, ..., P \tag{40}$$

with

$$q_{i+\frac{1}{2}} = -\frac{2D_i D_{i+1}}{h_{i+1}D_i + h_i D_{i+1}} [u_{i+1} - u_i] + E^d_{i+\frac{1}{2}} - E^g_{i+\frac{1}{2}} \quad for \;\; i = 1, ..., P \tag{41}$$

where for $i = 0, ..., P$ one has set

$$\left.\begin{aligned} E^d_{i+\frac{1}{2}} &= \tfrac{2D_i D_{i+1} h^2_{i+1}}{4[h_{i+1}D_i + h_i D_{i+1}]} u'' \left(\xi^d_{i+\frac{1}{2}}\right) \\ E^g_{i+\frac{1}{2}} &= \tfrac{2D_i D_{i+1} h^2_{i}}{4[h_{i+1}D_i + h_i D_{i+1}]} u'' \left(\xi^g_{i+\frac{1}{2}}\right) \end{aligned}\right\} \tag{42}$$

with the following conventions

$$h_0 = h_{P+1} = 0, \quad D_0 = D_{P+1} = 1 \tag{43}$$

$$u_0 = u_{\frac{1}{2}} \quad \text{and} \quad u_{P+1} = u_{P+\frac{1}{2}} \tag{44}$$

Reasoning the same way on the system of equations (17)-(19) yields

$$Q_{i+\frac{1}{2}} - Q_{i-\frac{1}{2}} = h_i \langle f \rangle_i \qquad \text{for} \;\; i = 1, ..., P \tag{45}$$

with

$$Q_{i+\frac{1}{2}} = -\frac{2D_i D_{i+1}}{h_{i+1}D_i + h_i D_{i+1}} [U_{i+1} - U_i] \;\text{ for }\; i = 1, ..., P \tag{46}$$

$$U_0 = \alpha \quad and \quad U_{P+1} = \beta \quad (see\ (23) \quad and \ (44) \ ) \tag{47}$$

Note that the system (45)-(47) is nothing than the classical centered finite volume scheme.

Combining (40) and (45), and taking into account (41) and (46), one can see that the quantities $\{e_i\}_{i=1}^P$,where $e_i = u_i - U_i$, verify the following relations

$$\frac{2D_{i-1}D_i}{h_{i-1}D_i+h_iD_{i-1}}\left[e_{i+1}-e_i\right]+\frac{2D_iD_{i+1}}{h_{i+1}D_i+h_iD_{i+1}}\left[e_i-e_{i+1}\right]=$$
$$\left[E^g_{i+\frac{1}{2}}-E^d_{i+\frac{1}{2}}\right]-\left[E^g_{i-\frac{1}{2}}-E^d_{i-\frac{1}{2}}\right] \qquad \text{for } \ i=1,...,P \tag{48}$$

with

$$e_0=e_{P+1}=0 \tag{49}$$

Multiplying (48) by e_i, summing over $i\in\{1,...,P\}$ and reordering the terms yields

$$\sum_{i=1}^{P}\frac{2D_iD_{i+1}}{h_{i+1}D_i+h_iD_{i+1}}\left[e_i-e_{i+1}\right]^2=\sum_{i=0}^{P}E^d_{i+\frac{1}{2}}\left[e_{i+1}-e_i\right]-\sum_{i=0}^{P}E^g_{i+\frac{1}{2}}\left[e_{i+1}-e_i\right] \tag{50}$$

From the assumption (3) and (6) one deduces

$$\sum_{i=1}^{P}\frac{2D_iD_{i+1}}{h_{i+1}D_i+h_iD_{i+1}}\left[e_i-e_{i+1}\right]^2\geq\frac{\left(D^-\right)^2}{D^+}\sum_{i=0}^{P}\frac{1}{h_{i+\frac{1}{2}}}\left[e_{i+1}-e_i\right]^2 \tag{51}$$

where

$$h_{i+\frac{1}{2}}=\frac{h_i+h_{i+1}}{2} \qquad for \ \ i=0,...,P$$

On the other hand it follows from (41), (3) and Cauchy-Schwarz inequality that

$$\left|\sum_{i=0}^{P}E^d_{i+\frac{1}{2}}\left[e_{i+1}-e_i\right]-\sum_{i=0}^{P}E^g_{i+\frac{1}{2}}\left[e_{i+1}-e_i\right]\right|\leq Ch\left(\sum_{i=0}^{P}\frac{1}{h_{i+\frac{1}{2}}}\left[e_{i+1}-e_i\right]^2\right)^{\frac{1}{2}} \tag{52}$$

where C is a constant depending only on $\Omega,\omega,\ D^-,\ D^+ \ and \ \max_{x\in\overline{\Omega}}\left|u''(x)\right|.$

From (50)-(52) one deduces that

$$\left(\sum_{i=0}^{P}\frac{1}{h_{i+\frac{1}{2}}}\left[e_{i+1}-e_i\right]^2\right)^{\frac{1}{2}}\leq Ch$$

Hence (27) is proven.

It remains to prove (28) for achieving the proof of Proposition 3. Subtracting

(18) from (33) one obtains

$$q^{\Omega_i}_{i+\frac{1}{2}} - Q^{\Omega_i}_{i+\frac{1}{2}} = \frac{D_i}{h/2}\left[\left(u^{\Omega_i}_i - U^{\Omega_i}_i\right) + \left(U^{\Omega_i}_{i+\frac{1}{2}} - u^{\Omega_i}_{i+\frac{1}{2}}\right)\right] - E_{i+\frac{1}{2}} \tag{53}$$

$$q^{\Omega_i}_{i-\frac{1}{2}} - Q^{\Omega_i}_{i-\frac{1}{2}} = \frac{D_i}{h/2}\left[\left(u^{\Omega_i}_i - U^{\Omega_i}_i\right) + \left(u^{\Omega_i}_{i-\frac{1}{2}} - U^{\Omega_i}_{i-\frac{1}{2}}\right)\right] + E_{i-\frac{1}{2}} \tag{54}$$

It's clear by continuity of the exact and the discrete potential across mesh interfaces (see (15) and (19)) that one has

$$e_{i+\frac{1}{2}} = u^{\Omega_i}_{i+\frac{1}{2}} - U^{\Omega_i}_{i+\frac{1}{2}} \quad \textit{and} \quad e_{i-\frac{1}{2}} = u^{\Omega_i}_{i-\frac{1}{2}} - U^{\Omega_i}_{i-\frac{1}{2}}$$

Setting for $i = 1, ..., P$

$$\hat{e}_{i+\frac{1}{2}} = q^{\Omega_i}_{i+\frac{1}{2}} - Q^{\Omega_i}_{i+\frac{1}{2}} \quad \textit{and} \quad \hat{e}_{i-\frac{1}{2}} = u^{\Omega_i}_{i-\frac{1}{2}} - U^{\Omega_i}_{i-\frac{1}{2}}$$

we get by combining linearly (53) and (54) that

$$\hat{e}_{i+\frac{1}{2}} + \hat{e}_{i-\frac{1}{2}} = \frac{D_i}{h_i/2}\left[e_{i-\frac{1}{2}} - e_{i+\frac{1}{2}}\right] + \left[E_{i-\frac{1}{2}} - E_{i+\frac{1}{2}}\right] \quad \forall\ 1 \le i \le P \tag{55}$$

On the other hand it results from (13) and (17) that

$$\hat{e}_{i+\frac{1}{2}} + \hat{e}_{i-\frac{1}{2}} = 0 \quad \forall\ 1 \le i \le P \tag{56}$$

It follows from (55) and (56) that for $i = 1, ..., P$

$$h_i\left[\hat{e}_{i+\frac{1}{2}}\right] \le 2\max\left\{\left(D^-\right)^2, \frac{1}{4}\right\}\left(\frac{1}{h_i}\left[e_{i-\frac{1}{2}} - e_{i+\frac{1}{2}}\right]^2 + h_i\left[E_{i-\frac{1}{2}} - E_{i+\frac{1}{2}}\right]^2\right)$$

Summing over $i = 1, ..., P$ the right and the left hands of the preceding inequality, taking into account (26), and remarking that

$$\sum_{i=1}^{P} h_i\left[E_{i-\frac{1}{2}} - E_{i+\frac{1}{2}}\right]^2 \le Ch^2$$

yields

$$\sum_{i=1}^{P} h_i\left[\hat{e}_{i+\frac{1}{2}}\right]^2 \le Ch^2$$

Thus (28) -(i) is proven. One deduces from this last inequality (28)-(ii) by using (56), that is,

$$\hat{e}_{i+\frac{1}{2}} = \hat{e}_{i-\frac{1}{2}} \quad for\ i = 1, ..., P$$

Proposition 3 is then completely proven.

4 Conclusion and perspectives

The work we have presented here is a first step in our research activities concerning mixed hybrid finite volume analysis of engineering problems in general and underground-water flow in particular. The theoretical results obtained from our analysis are satisfactory: see stability and convergence of the discrete solution in Propositions 2 and 3. This situation encourages us to deal with mixed hybrid finite volume analysis of stationary and time-dependent diffusion-convection problems in multi-dimension. A 2-D mixed hybrid finite volume simulator for underground-water flow is under-development with the purpose of its validation on a real site. In this connection some results can be found in [7] [8] [9].

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