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Characterization of some natural transformations between the bundle functors $T^A \circ T^*$ and $T^* \circ T^A$ on $\mathcal{M}f_m$.

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Abstract

In this paper, we characterize some natural transformations between the bundle functors $T^A \circ T^*$ and $T^* \circ T^A$ on $\mathcal{M}f_m$. In the particular case where $A = J_0^r(\mathbb{R}, \mathbb{R})$, we determine all natural transformations between the bundle functors $T^r \circ T^*$ and $T^* \circ T^r$ on $\mathcal{M}f_m$. These lifts of 1-forms are studied with application to the theory of presymplectic structures.

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Abstract. In this paper, we characterize some natural transformations between the bundle functors $T^A \circ T^*$ and $T^* \circ T^A$ on $\mathcal{M}f_m$. In the particular case where $A = J_0^r(\mathbb{R}, \mathbb{R})$, we determine all natural transformations between the bundle functors $T^r \circ T^*$ and $T^* \circ T^r$ on $\mathcal{M}f_m$. These lifts of 1-forms are studied with application to the theory of presymplectic structures.

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1. Introduction

By $\mathcal{M}f$ we denote the category of all smooth manifolds and all smooth maps and $\mathcal{M}f_m \subset \mathcal{M}f$ be the subcategory of m-dimensional manifolds and their local diffeomorphisms. Let A be a Weil algebra; it is a real commutative and finite dimensional algebra with unit, which is of the form $A = \mathbb{R} \cdot 1_A \oplus N_A$, where N_A is the ideal of nilpotent elements of A and $T^A : \mathcal{M}f \to \mathcal{M}f$ be the corresponding Weil functor, [5]. In particular, when A is the space of all r-jets of $\mathbb{R}^k$ into $\mathbb{R}$ with source $0 \in \mathbb{R}^k$ denoted by $J_0^r\left(\mathbb{R}^k, \mathbb{R}\right)$, the corresponding Weil functor is the functor of k-dimensional velocities of order r and denoted by T_k^r. For $k = 1$, it is called tangent functor of order r and denoted by T^r. For any manifold M, we consider each element of $T^A M$ in the form of an A-jet $j^A\varphi$, where $\varphi \in C^\infty(\mathbb{R}^n, M)$ and n the width of A. For a smooth map $f : M \to N$, the map $T^A f \in C^\infty(T^A M, T^A N)$ is defined by $T^A f\left(j^A\varphi\right) = j^A\left(f \circ \varphi\right)$.

Let M be a smooth manifold of dimension $m > 0$. For any $r \geq 1$, we consider the collection of canonical pairings (nondegenerates on the fibers)

$$\langle \cdot, \cdot \rangle_M : TM \times_M T^*M \to \mathbb{R} \text{ and } \langle \cdot, \cdot \rangle'_{T^rM} = \varsigma_r^1 \circ T^r\left(\langle \cdot, \cdot \rangle_M\right) : T^rTM \times_{T^rM} T^rT^*M \to \mathbb{R}$$

where ς_r^1 is a linear form on $J_0^r(\mathbb{R}, \mathbb{R})$ defined by $\varsigma_r^1\left(j_0^r\varphi\right) = \frac{1}{r!}\frac{d^r}{dt^r}\varphi(t)|_{t=0}$.

For each manifold M, there is a canonical diffeomorphism (see [3, 5])

$$\kappa_M^r : T^rTM \to TT^rM$$

which is an isomorphism of vector bundles

$$T^r\left(\pi_M\right) : T^rTM \to T^rM \quad \text{ and } \quad \pi_{TM}^r : TT^rM \to T^rM$$

such that $T(\pi_M^r)\circ\kappa_M^r = \pi_{TM}^r$. Let $\left(x^1,\cdots,x^m\right)$ be a local coordinate system of M, we introduce the coordinates $\left(x^i,\dot{x}^i\right)$ in TM, $\left(x^i,\dot{x}^i,x_\beta^i,\dot{x}_\beta^i\right)$ in T^rTM and $\left(x^i,x_\beta^i,\dot{x}^i,\widetilde{x}_\beta^i\right)$ in TT^rM. We have

$$\kappa_M^r\left(x^i,\dot{x}^i,x_\beta^i,\dot{x}_\beta^i\right) = \left(x^i,x_\beta^i,\dot{x}^i,\widetilde{x}_\beta^i\right)$$

with $\widetilde{x}_\beta^i = \dot{x}_\beta^i$. On the other hand, there is a canonical diffeomorphism ([2])

$$\alpha_M^r : T^*T^rM \to T^rT^*M$$

which is an isomorphism of vector bundles

$$\pi_{T^rM}^* : T^*T^rM \to T^rM \quad \text{and} \quad T^r\left(\pi_M^*\right) : T^rT^*M \to T^rM$$

dual of κ_M^r with respect to pairings $\langle\cdot,\cdot\rangle'_{T^rM} = \tau_r\circ T^r\left(\langle\cdot,\cdot\rangle_M\right)$ and $\langle\cdot,\cdot\rangle_{T^rM}$, i.e. for any $(u,u^*)\in T^rTM\oplus T^*T^rM$,

$$\langle\kappa_M^r(u),u^*\rangle_{T^rM} = \langle u,\alpha_M^r(u^*)\rangle'_{T^rM}$$

Let $\left(x^1,\cdots,x^m\right)$ be a local coordinates system of M, we introduce the coordinates $\left(x^i,p_j\right)$ in T^*M, $\left(x^i,p_j,x_\beta^i,p_j^\beta\right)$ in T^rT^*M and $\left(x^i,x_\beta^i,\pi_j,\pi_j^\beta\right)$ in T^*T^rM. We have:

$$\alpha_M^r\left(x^i,\pi_j,x_\beta^i,\pi_j^\beta\right) = \left(x^i,x_\beta^i,p_j,p_j^\beta\right) \quad \text{with} \left\{\begin{array}{lcl} p_j & = & \pi_j^r \\ p_j^\beta & = & \pi_j^{r-\beta}\end{array}\right.$$

So, α_M^r establishes a canonical isomorphism between T^*T^rM and T^rT^*M. It has a fundamental importance in the description of higher order Lagrangian and Hamiltonian formalisms (see [4]). By ε_M^r we denote the bundle map $\left(\alpha_M^r\right)^{-1}$. In particular, ε^r is a natural transformation between the functors $T^r\circ T^*$ and $T^*\circ T^r$ defined on the category $\mathcal{M}f_m$. For $r=1$, ε_M^1 is called *natural isomorphism of Tulczyjew over M*. This construction has been generalized in [7] for any Weil-Frobenius algebra defined below. In [9], the authors show that any Weil algebra has a Weil-Frobenius algebra structure if and only if there is a natural equivalence between the bundle functors $T^A\circ T^*$ and $T^*\circ T^A$ defined on $\mathcal{M}f_m$. The aim of this paper is to characterize all natural transformations $T^A\circ T^*\to T^*\circ T^A$, when A is a Weil algebra and we give some applications to the lifts of 1-forms. So, the main results of this paper are theorems 2, 3 and 4.

All manifolds and maps are assumed to be infinitely differentiable, we fix one Weil algebra A. For any $g\in C^\infty\left(\mathbb{R}^k,\mathbb{R}\right)$ and any multiindex $\beta=(\beta_1,\cdots,\beta_k)$, we denote by

$$D_\beta(g)(z) = \frac{1}{\beta!}\frac{\partial^{|\beta|}g}{(\partial z_1)^{\beta_1}\cdots(\partial z_k)^{\beta_k}}(z)$$

the partial derivative with respect to the multiindex β of g.

2. The natural transformations $T^A\circ T^*\to T^*\circ T^A$.

2.1. Preliminaries

For any $k\geq 2$, we denote by N_A^k the ideal of A generated by the products of k elements of N_A.

Proposition 2.1. *There is one and only one natural integer $h\geq 1$ such that, $N_A^h\neq 0$ and $N_A^{h+1}=0$. It is called the height of A.*

Proof. See [3, 5].

■

We put $e_0 = 1_A$, for each multiindex $\alpha \neq 0$ the vector $e_\alpha = j^A(x^\alpha)$ is an element of N_A. Therefore, for any $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ we have

$$j^A\varphi = \varphi(0) \cdot 1_A + \sum_{1 \leq |\alpha| \leq h} \tfrac{1}{\alpha!} \cdot D_\alpha(\varphi)(0)\, e_\alpha$$

It follows that the family $\{e_\alpha\}_{1\leq|\alpha|\leq h}$ generates the ideal N_A. We denote by B_A the set of all multiindices such that $\{e_\alpha\}_{\alpha \in B_A}$ is a basis of N_A and $\overline{B}_A$ its complementary with respect to the set of all multiindices $\mu \in \mathbb{N}^n$ such that $1 \leq |\mu| \leq h$. For $\beta \in \overline{B}_A$, we have $e_\beta = \sum_{\mu \in B_A} \lambda_\beta^\mu e_\mu$.

By this formula, we deduce that:

$$j^A\varphi = \varphi(0) \cdot 1_A + \sum_{\alpha \in B_A} \left[\tfrac{1}{\alpha!} \cdot D_\alpha(\varphi)(0) + \sum_{\beta \in \overline{B}_A} \frac{\lambda_\beta^\alpha}{\beta!} \cdot D_\beta(\varphi)(0) \right] e_\alpha \tag{1}$$

Corollary 2.2. *Let $\varphi, \psi \in C^\infty(\mathbb{R}^n, M)$, the following assertions are equivalent:*

(i) $j^A\varphi = j^A\psi$

(ii) $\varphi(0) = \psi(0) = x$ *and for any chart (U, x^i) of M in x we have:*

$$\tfrac{1}{\alpha!} D_\alpha(x^i \circ \varphi)(0) + \sum_{\beta \in \overline{B}_A} \tfrac{\lambda_\beta^\alpha}{\beta!} D_\beta(x^i \circ \varphi)(0) = \tfrac{1}{\alpha!} D_\alpha(x^i \circ \psi)(0) + \sum_{\beta \in \overline{B}_A} \tfrac{\lambda_\beta^\alpha}{\beta!} D_\beta(x^i \circ \psi)(0)$$

where $1 \leq i \leq m$ and $\alpha \in B_A$.

Remark 2.3. Let (U, x^i) be a local coordinate system of M, the local coordinate system $(\overline{x}^i, \overline{x}^i_\alpha)$ of T^AM over the open T^AU is such that,

$$\begin{cases} \overline{x}^i = x_0^i \\ \overline{x}^i_\alpha = x^i_\alpha + \sum_{\beta \in \overline{B}_A} \lambda_\beta^\alpha \cdot x^i_\beta \end{cases} \tag{2}$$

where $x_0^i(j^A\varphi) = x^i(\varphi(0))$ and $x^i_\alpha(j^A\varphi) = \frac{1}{\alpha!} \cdot D_\alpha(x^i \circ \varphi)(z)\,|_{z=0}$. It is called an adapted coordinate system associated to (U, x^i). In the sequel, the same symbol x^i will be used both for a function $U \to \mathbb{R}$ and for the composite function $T^AU \to U \to \mathbb{R}$. The latter function may also be written as the pullback $\pi^*_{A,U}(x^i)$.

2.2. The canonical isomorphisms between T^AE^* and $(T^AE)^*$

Let p be a linear form on A. The mapping $\widehat{p} : (a, b) \mapsto p(ab)$ is bilinear symmetric and satisfies

$$\widehat{p}(ab, c) = \widehat{p}(a, bc)$$

Definition 2.4. *We say that the linear form p is nondegenerate if the bilinear form $\widehat{p}$ is nondegenerate. The pair (A, p) is called a Weil-Frobenius algebra.*

We denote by $\mathcal{D}_m$ the category of vector bundles with m-dimensional base and vector bundle isomorphisms with identity as base maps. We denote by T^A, the covariant functor $T^A : \mathcal{D}_m \to \mathcal{VB}$ from the category $\mathcal{D}_m$ into the category $\mathcal{VB}$ of all vector bundles and their vector bundle homomorphisms, such that

$$T^A(E, M, \pi) = (T^AE, T^AM, T^A\pi) \quad \text{and} \quad T^A(id_M, f) = (id_{T^AM}, T^Af)$$

for any $\mathcal{D}_m$-objet (E, M, π) and $\mathcal{D}_m$-morphism (id_M, f) ([3]). For a linear form $p : A \to \mathbb{R}$ and the vector bundle (E, M, π), we consider the natural vector bundle morphism

$$\tau^p_{A,E} : T^AE^* \to (T^AE)^* \tag{3}$$

defined for any $j^A\varphi \in T^AE^*$ and $j^A\psi \in T^AE$ by:

$$\tau^p_{A,E}(j^A\varphi)(j^A\psi) = p\left(j^A(\langle \psi, \varphi \rangle_E)\right) \tag{4}$$

where $\langle\psi,\varphi\rangle_E : \mathbb{R}^n \to \mathbb{R},\ z \mapsto \langle\psi(z),\varphi(z)\rangle_E$ and $\langle\cdot,\cdot\rangle_E$ the canonical pairing. We have

Proposition 2.5. *For any $\mathcal{D}_m$-morphism $f : E_1 \to E_2$, the diagram*

$$\begin{array}{ccccc} & & T^A f^* & & \\ & T^A E_2^* & \rightarrow & T^A E_1^* & \\ \tau^p_{A,E_2} & \downarrow & & \downarrow & \tau^p_{A,E_1} \\ & \left(T^A E_2\right)^* & \rightarrow & \left(T^A E_1\right)^* & \\ & & \left(T^A f\right)^* & & \end{array}$$

commutes.

Proof. Let $j^A\varphi \in T^A E_2^*$ and $j^A\psi \in T^A E_1$ over $T^A M$. We have:

$$\begin{array}{rcl} \left(T^A f\right)^* \circ \tau^p_{A,E_2}\left(j^A\varphi\right)\left(j^A\psi\right) & = & \left(\tau^p_{A,E_2}\left(j^A\varphi\right)\right)\left(T^A f\left(j^A\psi\right)\right) \\ & = & \left(\tau^p_{A,E_2}\left(j^A\varphi\right)\right)\left(j^A\left(f\circ\psi\right)\right) \\ & = & p\left(j^A\left(\langle f\circ\psi,\varphi\rangle_{E_2}\right)\right) \\ & = & p\left(j^A\left(\langle \psi, f^*\circ\varphi\rangle_{E_1}\right)\right) \end{array}$$

On the other hand,

$$\begin{array}{rcl} \tau^p_{A,E_1}\circ T^A f^*\left(j^A\varphi\right)\left(j^A\psi\right) & = & \tau^p_{A,E_1}\left(j^A\left(f^*\circ\varphi\right)\right)\left(j^A\psi\right) \\ & = & p\left(j^A\left(\langle \psi, f^*\circ\varphi\rangle_{E_1}\right)\right) \\ & = & \left(T^A f\right)^*\circ\tau^p_{A,E_2}\left(j^A\varphi\right)\left(j^A\psi\right) \end{array}$$

It follows that $\left(T^A f\right)^*\circ\tau^p_{A,E_2} = \tau^p_{A,E_1}\circ T^A f^*$. Thus $\tau^p_{A,E} : T^A E^* \to \left(T^A E\right)^*$ is a natural homomorphism of vector bundles.

■

Remark 2.6. (*Local expression of $\tau^p_{A,E}$*). Let $(\eta_1,\cdots,\eta_k)$ be a basis of local sections of E and $\left(\eta^1,\cdots,\eta^k\right)$ be the dual basis of local sections of $\pi_* : E^* \to M$. We have an adapted coordinate systems $\left(x^i,y^j\right)$ in E, $\left(x^i,u_j\right)$ in E^*, $\left(x^i,y^j,\overline{x}^i_\alpha,\overline{y}^j_\alpha\right)$ in $T^A E$, $\left(x^i,u_j,\overline{x}^i_\alpha,\overline{u}^\alpha_j\right)$ in $T^A E^*$ and $\left(x^i,w_j,\overline{x}^i_\alpha,\overline{w}^\alpha_j\right)$ in $\left(T^A E\right)^*$. Locally, we have

$$\tau^p_{A,E}\left(x^i,u_j,\overline{x}^i_\alpha,\overline{u}^\alpha_j\right) = \left(x^i,w_j,\overline{x}^i_\alpha,\overline{w}^\alpha_j\right) \text{ with } \left\{\begin{array}{rcl} w_j & = & u_j p_0 + \sum\limits_{\alpha\in B_A}\overline{u}^\alpha_j p_\alpha \\ \overline{w}^\alpha_j & = & \sum\limits_{\beta\in B_A}\overline{u}^{\beta-\alpha}_j p_\beta \end{array}\right.$$

where $p(e_\gamma) = p_\gamma$.

Theorem 2.7. *There is a bijective correspondence between the set of all the natural isomorphism of vector bundles $\tau_{A,E} : T^A E^* \to (T^A E)^*$ satisfying, for any $a,b\in A$*

$$\tau_{A,\mathbb{R}}(a)(b) = \tau_{A,\mathbb{R}}(1_A)(ab) \tag{5}$$

and the set of all the linear and nondegenerate maps of A.

Proof. For the first part, see [7]. Inversely, let $\tau_{A,E} : T^A E^* \to \left(T^A E\right)^*$ be the canonical vector bundle isomorphism verifying (1.5). The map $\tau_{A,\mathbb{R}} : A \to A^*$ denoted by $\overline{p}$ is a vector space isomorphism. It induces the linear map

$$\begin{array}{rccc} p: & A & \to & \mathbb{R} \\ & a & \to & \overline{p}(1_A)(a) \end{array}$$

We consider the bilinear symmetric map induced by p denoted $\widehat{p}$ and defined in the following way: $\widehat{p} : (a,b) \mapsto p(1_A)(ab)$. By the equality (1.5), it follows that $\widehat{p}$ is nondegenerate. Let $\tau^p_{A,E}$ be a natural transformation defined by p. For any vector space V, using the equation (1.5) we have $\tau^p_{A,V} = \tau_{A,V}$. The equality $\tau^p_{A,E} = \tau_{A,E}$ comes by calculation in local coordinates.

Remark 2.8. The theorem above, shows in particular that: a natural vector bundle morphisms $T^A E^* \to \left(T^A E\right)^*$ (satisfying (1.5)) is a natural equivalence if and only if A is a Weil-Frobenius algebra.

■

Example 2.9. (i) For $A = \mathbb{D}$, consider the linear map $p_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}$ given by

$$p_{\mathbb{D}}(j_0^1\varphi) = \frac{d}{dt}\left(\varphi(t)\right)|_{t=0}$$

We have the natural isomorphism $\tau^{p_{\mathbb{D}}}_{\mathbb{D},E} = I_E : TE^* \to (TE)^*$, called the Swap map of E.

(ii) For $A = J_0^r(\mathbb{R},\mathbb{R})$ and the linear form ς_r^1 is non degenerate, it induces the natural vector bundle isomorphism $I_E^r : T^r E^* \to (T^r E)^*$, ([6]). The local expression of I_E^r is of the form:

$$I_E^r(x^i, u_j, x^i_\beta, u^\beta_j) = (x^i, w_j, x^i_\beta, w^\beta_j) \quad \text{with} \quad \begin{cases} w_j &= u_j^r \\ w_j^\beta &= u_j^{r-\beta} \end{cases}$$

For an arbitrary linear map $p : A \to \mathbb{R}$ non necessarily nondegenerate, it induces the natural vector bundle morphism $\tau^p_{A,E} : T^A E^* \to \left(T^A E\right)^*$ over $\mathrm{id}_{T^A M}$ non necessarily bijective.

Corollary 2.10. *There is a bijective correspondence between the set of all the natural vector bundle morphisms $\tau_{A,E} : T^A E^* \to \left(T^A E\right)^*$ verifying* (1.5) *and the set A^*.*

For each $1 \le |\alpha| \le h$, we consider the linear map $\varsigma_A^\alpha : A \to \mathbb{R}$ defined by:

$$\varsigma_A^\alpha(j^A\varphi) = \tfrac{1}{\alpha!} D_\alpha\left(\varphi\right)(z)|_{z=0}$$

It induces the vector bundle morphism $\tau^\alpha_{A,E} : T^A E^* \to \left(T^A E\right)^*$ over $\mathrm{id}_{T^A M}$.
Let $\left(x^i, u^j\right)$ be an adapted local coordinate system of E, the local expression of the bundle map $\tau^\alpha_{A,E} : T^A E^* \to \left(T^A E\right)^*$ takes the form

$$\tau^\alpha_{A,E}\left(x^i, u_j, \overline{x}^i_\beta, \overline{u}^\beta_j\right) = \left(x^i, w_j, \overline{x}^i_\beta, \overline{w}^\beta_j\right) \text{ with } \begin{cases} w_j &= \overline{u}_j^\alpha \\ \overline{w}_j^\beta &= \overline{u}_j^{\alpha-\beta} \end{cases}$$

We denote by $*$ the covariant functor from $\mathcal{D}_m$ into $\mathcal{D}_m$ defined by:

$$*\,(E, M, \pi) = (E^*, M, \pi_*) \quad \text{and} \quad *\,(id_M, f) = (id_M, ({}^t f)^{-1})$$

Corollary 2.11. *All natural transformations of $T^A \circ * \to * \circ T^A$ verifying* (1.5) *are of the form*

$$p_0 \tau^0_{A,*} + \sum_{1\le|\alpha|\le h} p_\alpha \cdot \tau^\alpha_{A,*} \tag{6}$$

where p_0, p_α are the real numbers.

Proof. Let $\tau_A : T^A \circ * \to * \circ T^A$ be a natural transformations verifying (1.5), it induces a linear map $p : A \to \mathbb{R}$. This linear map has the form

$$p_0 \varsigma_A^0 + \sum_{1\le|\alpha|\le h} p_\alpha \varsigma_A^\alpha$$

So we have the result.

■

Corollary 2.12. *For all $k \ge 2$ and $r \ge 1$, do not exist a natural equivalence between $T^r_k E^*$ and $(T^r_k E)^*$ verifying* (1.5)*. In particular $J_0^r\left(\mathbb{R}^k, \mathbb{R}\right)$ is not a Weil-Frobenius algebra.*

Proof. See [9].

■

2.3. Main results.

For each manifold M, there is a canonical diffeomorphism (see [3, 5])

$$\kappa_M^A : T^A TM \to TT^A M$$

which is an isomorphism of vector bundles

$$T^A(\pi_M) : T^A TM \to T^A M \quad \text{and} \quad \pi_{T^A M} : TT^A M \to T^A M$$

such that, $\pi_{T^A M} \circ \kappa_M^A = T^A(\pi_M)$. In particular, for any $f \in C^\infty(M, N)$ we have

$$\kappa_N^A \circ T^A T f = TT^A f \circ \kappa_M^A$$

Let $p : A \to \mathbb{R}$ be a linear map, it induces the natural vector bundle morphism $\tau_{A,\cdot}^p : T^A \circ * \to * \circ T^A$. For any manifold M of dimension m, we consider the vector bundle morphism

$$\varepsilon_{A,M}^p = \left[\left(\kappa_M^A\right)^{-1}\right]^* \circ \tau_{A,TM}^p : T^A T^* M \to T^* T^A M.$$

It is clear that the family of maps $\left(\varepsilon_{A,M}^p\right)$ defines a natural transformation between the functors $T^A \circ T^*$ and $T^* \circ T^A$ on the category $\mathcal{M}f_m$ and denoted

$$\varepsilon_{A,*}^p : T^A \circ T^* \to T^* \circ T^A.$$

When p is nondegenerate, the mapping $\varepsilon_{A,M}^p$ is a vector bundle isomorphism over $id_{T^A M}$. In local coordinate system $\left\{x^1, \cdots, x^m\right\}$ of M, we introduce the coordinates $\left(x^i, \dot{x}^i\right)$ in TM, $\left(x^i, \pi_i\right)$ in T^*M, $(x^i, \dot{x}^i, \overline{x}_\beta^i, \overline{\dot{x}_\beta^i})$ in $T^A TM$, $(x^i, \pi_j, \overline{x}_\beta^i, \overline{\pi}_j^\beta)$ in $T^A T^* M$, $(x^i, \overline{x}_\beta^i, \dot{x}^i, \dot{\overline{x}}_\beta^i)$ in $TT^A M$ and $(x^i, \overline{x}_\beta^i, \overline{\xi}_j, \overline{\xi}_j^\beta)$ in $T^* T^A M$. We have:

$$\kappa_M^A\left(x^i, \dot{x}^i, \overline{x}_\beta^i, \overline{\dot{x}_\beta^i}\right) = \left(x^i, \overline{x}_\beta^i, \dot{x}^i, \dot{\overline{x}}_\beta^i\right)$$

with $\overline{\dot{x}_\beta^i} = \dot{\overline{x}}_\beta^i$. It follows that

$$\varepsilon_{A,M}^p\left(x^i, \pi_j, \overline{x}_\beta^i, \overline{\pi}_j^\beta\right) = \left(x^i, \overline{x}_\beta^i, \overline{\xi}_j, \overline{\xi}_j^\beta\right) \quad \text{with} \quad \begin{cases} \overline{\xi}_j &= \pi_j p_0 + \sum_{\mu \in B_A} \overline{\pi}_j^\mu p_\mu \\ \overline{\xi}_j^\beta &= \sum_{\mu \in B_A} \overline{\pi}_j^{\mu-\beta} p_\mu \end{cases} \tag{7}$$

Example 2.13. (i) When $A = \mathbb{D}$ and $p_\mathbb{D} : \mathbb{D} \to \mathbb{R}$, $j_0^1\varphi \mapsto \frac{d}{dt}\left(\varphi(t)\right)|_{t=0}$ we have the natural isomorphism of Tulczyjew $\varepsilon_M : TT^*M \to T^*TM$, (see [5]). For the linear map $p_0\left(j_0^1\gamma\right) = \gamma(0)$, we obtain the natural vector bundle morphisms ε_M^0 such that locally,

$$\varepsilon_M^0\left(x^i, \pi_i, \dot{x}^i, \dot{\pi}_i\right) = \left(x^i, \dot{x}^i, \pi_i, 0\right).$$

(ii) If $A = J_0^1(\mathbb{R}^p, \mathbb{R})$ and $p_{J_0^1(\mathbb{R}^p,\mathbb{R})} : J_0^1(\mathbb{R}^p, \mathbb{R}) \to \mathbb{R}$, $j_0^1\varphi \mapsto \varphi(0) + \sum_{i=1}^{p} \frac{\partial\varphi}{\partial x^i}(0)$, we have the natural vector bundle morphism $\varepsilon_{p,M}^1 : T_p^1 T^* M \to T^* T_p^1 M$ defined in [12]. In local coordinate,

$$\varepsilon_{p,M}^1\left(x^i, \pi_i, x_\beta^i, \pi_i^\beta\right) = \left(x^i, x_\beta^i, \xi_i, \xi_i^\beta\right) \text{ with } \begin{cases} \xi_i &= \sum_{|\alpha|=1} \pi_i^\alpha \\ \xi_i^\beta &= \pi_i \end{cases}$$

(iii) If $A = J_0^r(\mathbb{R}, \mathbb{R})$, and $p_{J_0^r(\mathbb{R},\mathbb{R})} : J_0^r(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, $j_0^r\varphi \mapsto \frac{1}{r!} \cdot \frac{d^r}{dt^r}(\varphi(t))|_{t=0}$, we have the natural vector bundle isomorphism $\varepsilon_M^r : T^r T^* M \to T^* T^r M$ defined in [2].

(iv) When $A = J_0^r(\mathbb{R}^k, \mathbb{R})$ and the linear form on $J_0^r(\mathbb{R}^k, \mathbb{R})$ defined by

$$p_{J_0^r(\mathbb{R}^k,\mathbb{R})}\left(j_0^r\varphi\right) = \sum_{|\alpha|=r} \frac{1}{\alpha!} D_\alpha(\varphi)(z)|_{z=0}.$$

We deduce the natural transformations $\varepsilon_{k,M}^r : T_k^r T^* M \to T^* T_k^r M$ such that locally

$$\varepsilon_{k,M}^r\left(x^i, \pi_i, x_\beta^i, \pi_i^\beta\right) = \left(x^i, x_\beta^i, \xi_i, \xi_i^\beta\right) \quad \text{where} \quad \begin{cases} \xi_i &= \sum_{|\alpha|=r} \pi_i^\alpha \\ \xi_i^\beta &= \sum_{|\alpha|=r} \pi_i^{\alpha-\beta} \end{cases}$$

Let D be a derivation of A, for any real number t, $D_t = \exp(tD) \in \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of all automorphisms of A. It is a Lie subgroup of Lie group $GL(A)$. The map $D_t : A \to A$ is an automorphism of A, it induces a natural transformation $\widetilde{D}_{t,M} : T^A M \to T^A M$. On the other hand, the multiplication of the tangent vectors of M by reals is a map $\mathfrak{m}_{TM} : \mathbb{R} \times TM \to TM$. Applying the Weil functor T^A, we obtain $T^A(\mathfrak{m}_{TM}) : A \times T^A TM \to T^A TM$. Let $c \in A$, we put

$$\mathrm{af}_M(c) = \kappa_M^A \circ T^A(\mathfrak{m}_{TM})(c, \cdot) \circ (\kappa_M^A)^{-1},$$

it is a natural tensor of type $(1,1)$ on $T^A M$, called affinor. In [5], one shows that, all natural transformations $T \circ T^A \to T \circ T^A$ are of the form $\mathrm{af}(c) + T\left(\widetilde{D}_t\right)$, where $t \in \mathbb{R}$.

Theorem 2.14. *Let (A, p) be a Weil-Frobenius algebra. All natural transformations $\theta_A : T^A \circ T^* \to T^* \circ T^A$ are of the form*

$$T^*\left(\widetilde{D}_t\right) \circ \varepsilon_A^p + (\mathrm{af}(c))^* \circ \varepsilon_A^p \tag{8}$$

where $c \in A$, $t \in \mathbb{R}$ and D a derivation of A.

Proof. Let $\theta_A : T^A \circ T^* \to T^* \circ T^A$ be a natural transformation, $\theta_A \circ (\varepsilon_A^p)^{-1} = \varphi_{A,p} : T^* \circ T^A \to T^* \circ T^A$ is a natural transformation. We obtain a natural transformation $\varphi_{A,p}^* : T \circ T^A \to T \circ T^A$, it exists a derivation D of A and $c \in A$ such that $\varphi_{A,p}^* = \mathrm{af}(c) + T\left(\widetilde{D}_t\right)$, for a real number t. We obtain $\theta_A = T^*\left(\widetilde{D}_t\right) \circ \varepsilon_A^p + (\mathrm{af}(c))^* \circ \varepsilon_A^p$.

■

Corollary 2.15. *Let (A, p) be a Weil-Frobenius algebra. All natural isomorphisms on a manifold M, $T^A T^* M \to T^* \circ T^A M$ are of the form*

$$T^*(\widetilde{D}_{t,M}) \circ \varepsilon_{A,M}^p$$

where $t \in \mathbb{R}$ and D a derivation of A.

Corollary 2.16. *All natural morphisms $TT^*M \to T^*TM$ are of the form*

$$aT^*(F_{t,M}) \circ \varepsilon_M + b\varepsilon_M + c\varepsilon_M^0$$

where $F_{t,M}$ is a one parameter subgroup of the Euler vector field on TM, a, b, c are real numbers and $t \neq 0$.

Proof. We recall that $\mathbb{D} \simeq \mathbb{R}^2$, the structure of Weil algebra is given by:

$$(x_0, x_1) \cdot (y_0, y_1) = (x_0 y_0, x_0 y_1 + x_1 y_0)$$

Let D be a derivation of $\mathbb{R}^2$. The natural transformation $\widetilde{D}_t$ associated is given by:

$$\widetilde{D}_{t,M} = \alpha F_{t,M}.$$

On the other hand, any affinor is of the form $\beta \mathrm{id}_{TTM} + c \cdot \mathrm{af}_M(e_1)$, with $e_1 = (0,1)$. It follows that the natural morphism

$$\theta_M : TT^*M \to T^*TM$$

is given by:

$$\theta_M = \alpha T^*(F_{t,M}) \circ \varepsilon_M + (\alpha+\beta)\varepsilon_M + b\varepsilon_M^0,$$

because $(\mathrm{af}_M(e_1))^* \circ \varepsilon_M = \varepsilon_M^0$.

■

Let $(e_0, \cdots, e_r)$ the canonical basis of $A = J_0^r(\mathbb{R},\mathbb{R})$. For $0 \le \alpha \le r$ and a manifold M, we put:

$$\begin{cases} \varepsilon_M^0 &= \left[(\kappa_M^r)^{-1}\right]^* \circ \tau_{A,TM}^0 \\ \varepsilon_M^\alpha &= \left[(\kappa_M^r)^{-1}\right]^* \circ \tau_{A,TM}^\alpha \end{cases}$$

Consider the linear map $\phi_\alpha : J_0^r(\mathbb{R},\mathbb{R}) \to J_0^r(\mathbb{R},\mathbb{R})$ defined by

$$\begin{cases} \phi_\alpha(e_0) &= 0 \\ \phi_\alpha(e_{\beta+1}) &= \frac{(\alpha+\beta)!}{\alpha!\beta!} e_{\alpha+\beta} \end{cases}$$

is a derivation, it induces a one parameter subgroup of a vector field on T^rM denoted by $\phi_{\alpha,M}^t : T^rM \to T^rM$.

Proposition 2.17. *Any derivation $\phi : J_0^r(\mathbb{R},\mathbb{R}) \to J_0^r(\mathbb{R},\mathbb{R})$ is of the form*

$$\phi = \sum_{\beta=1}^{r} a_\beta \cdot \phi_\beta$$

where $a_1, \cdots, a_r$ are real numbers.

Proof. For any $\alpha = 0, \cdots, r$, we have $e_0 \cdot e_\alpha = e_\alpha$, therefore $\phi(e_\alpha)\cdot e_0 + \phi(e_0)\cdot e_\alpha = \phi(e_\alpha)$. It follows that

$$\phi(e_0)\cdot e_\alpha = 0, \quad \forall \alpha = 0, \cdots, r$$

So that, $\phi(e_0) = 0$. We put,

$$\phi(e_1) = \sum_{\beta=0}^{r} a_\beta e_\beta$$

with $a_0, a_1, \cdots, a_r$ are the real numbers. Using the relation $e_1 \cdot e_1 = 2e_2$, we have

$$\phi(e_2) = \phi(e_1)\cdot e_1 = \sum_{\beta=0}^{r-1} (\beta+1) a_\beta e_{\beta+1}$$

By the same way, $e_2 \cdot e_1 = 3e_3$, it follows that, $3\phi(e_3) = \phi(e_2)\cdot e_1 + \phi(e_1)\cdot e_2$. Now

$$\begin{aligned} \phi(e_2)\cdot e_1 &= \sum_{\beta=0}^{r-2} (\beta+1)(\beta+2) a_\beta e_{\beta+2} \\ \phi(e_1)\cdot e_2 &= \sum_{\beta=0}^{r-2} \tfrac{(\beta+1)(\beta+2)}{2} a_\beta e_{\beta+2} \end{aligned}$$

We deduce that,

$$\phi(e_2)\cdot e_1 + \phi(e_1)\cdot e_2 = \sum_{\beta=0}^{r-2} 3\tfrac{(\beta+1)(\beta+2)}{2} a_\beta e_{\beta+2}$$

So,

$$\phi(e_3) = \sum_{\beta=0}^{n-2} \tfrac{(\beta+1)(\beta+2)}{2} a_\beta e_{\beta+2}$$

Looking the expressions of $\phi(e_1)$, $\phi(e_2)$ and $\phi(e_3)$ we put

$$\phi(e_\alpha) = \sum_{\beta=0}^{r-\alpha+1} \tfrac{(\alpha+\beta-1)!}{(\beta-1)!\alpha!} a_\beta e_{\alpha+\beta-1}$$

By induction, using the relation $e_\alpha \cdot e_1 = (\alpha+1)\, e_{\alpha+1}$, we obtain,

$$(\alpha+1)\,\phi(e_{\alpha+1}) = \phi(e_\alpha)\cdot e_1 + \phi(e_1)\cdot e_\alpha$$

Now,

$$\phi(e_\alpha)\cdot e_1 = \sum_{\beta=0}^{r-\alpha+1} \tfrac{(\alpha+\beta-1)!}{(\beta-1)!\alpha!} a_\beta e_{\alpha+\beta-1}\cdot e_1 = \sum_{\beta=0}^{r-\alpha} \tfrac{(\alpha+\beta)!}{(\beta-1)!\alpha!} a_\beta e_{\alpha+\beta}$$

$$\phi(e_1)\cdot e_\alpha = \sum_{\beta=0}^{r} a_\beta e_\beta \cdot e_\alpha = \sum_{\beta=0}^{r-\alpha} \tfrac{(\alpha+\beta)!}{\beta!\alpha!} a_\beta e_{\alpha+\beta}$$

We deduce that

$$\phi(e_\alpha)\cdot e_1 + \phi(e_1)\cdot e_\alpha = \sum_{\beta=0}^{r-\alpha} \tfrac{(\alpha+1)(\alpha+\beta)!}{\beta!\alpha!} a_\beta e_{\alpha+\beta}$$

Thus,

$$\phi(e_{\alpha+1}) = \sum_{\beta=0}^{r-\alpha} \tfrac{(\alpha+\beta)!}{\beta!\alpha!} a_\beta e_{\alpha+\beta}$$

On the other hand, $\phi(e_r) = a_0 e_{r-1} + a_1 e_r$ and $e_r \cdot e_1 = 0$. So that $\phi(e_r)\cdot e_1 + \phi(e_1)\cdot e_r = 0$. As

$$\begin{array}{rcl} \phi(e_r)\cdot e_1 & = & r a_0 e_r \\ \phi(e_1)\cdot e_r & = & a_0 e_r \end{array}$$

It follows that $a_0 = 0$. So that, for any $\alpha = 0, \cdots, r-1$, we have

$$\phi(e_{\alpha+1}) = \sum_{\beta=1}^{r-\alpha} a_\beta \tfrac{(\alpha+\beta)!}{\beta!\alpha!} e_{\alpha+\beta} = \sum_{\beta=1}^{r-\alpha} a_\beta \phi_\beta(e_{\alpha+1})$$

Thus, we obtain the result.

■

Theorem 2.18. *All natural vector bundle morphisms $T^r T^* M \to T^* T^r M$ are of the form*

$$\sum_{\alpha=1}^{r} a_\alpha T^*\left(\phi^t_{\alpha,M}\right) \circ \varepsilon^r_M + \sum_{\beta=0}^{r-1} b_\beta \varepsilon^\beta_M$$

where a_α, b_β, t are real numbers.

Proof. Any derivation $\phi : J^r_0(\mathbb{R},\mathbb{R}) \to J^r_0(\mathbb{R},\mathbb{R})$ is a $\mathbb{R}$-linear combination of the maps ϕ_α. The rest of the proof comes from the formula $\varepsilon^\alpha_M = (\mathrm{af}_M(e_\alpha))^* \circ \varepsilon^r_M$, for any $\alpha = 0, \cdots r-1$.

■

Corollary 2.19. *All natural isomorphisms on a manifold M, $T^r T^* M \to T^* \circ T^r M$ are of the form*

$$\sum_{\alpha=1}^{r} a_\alpha T^*\left(\phi^t_{\alpha,M}\right) \circ \varepsilon^r_M$$

where $a_\alpha, t \in \mathbb{R}$.

3. Applications: Lifts of 1-forms to Weil bundles revisited

In this section, we fix the linear map $p : A \to \mathbb{R}$ and $\varepsilon^p_{A,*}$ the natural transformation $T^A \circ T^* \to T^* \circ T^A$ such that: for any manifold M, $\varepsilon^p_{A,M} = [(\kappa^A_M)^{-1}]^* \circ \tau^p_{A,TM}$.

3.1. Prolongations of 1-forms

Let $\omega \in \Omega^1(M)$, we put:

$$\omega^{(p)} = \varepsilon^p_{A,M} \circ T^A \omega \tag{9}$$

$\omega^{(p)}$ is a 1-form on $T^A M$. If locally $\omega = \omega_i dx^i$ then we have:

$$\omega^{(p)} = \left(\omega_i p_0 + \sum_{\gamma \in B_A} \overline{\omega}^{(\gamma)}_i p_\gamma \right) dx^i + \sum_{\beta \in B_A} \left(\sum_{\mu \in B_A} \overline{\omega}^{(\mu-\beta)}_i p_\mu \right) d\overline{x}^i_\beta \tag{10}$$

with

$$\begin{cases} \overline{\omega}^{(\gamma)}_i & = & \omega^{(\gamma)}_i + \sum_{\nu \in \overline{B}_A} \lambda^\gamma_\nu \omega^{(\nu)}_i \\ \overline{\omega}^{(\mu-\beta)}_i & = & \omega^{(\mu-\beta)}_i + \sum_{\alpha \in \overline{B}_A} \lambda^\mu_\alpha \omega^{(\alpha-\beta)}_i \end{cases} \tag{11}$$

Definition 3.1. *The differential form $\omega^{(p)}$ defined on $T^A M$ is called p-prolongation of ω from M to $T^A M$*

Example 3.2. (i) **Case where $A = \mathbb{D}$. (see** [4]**)**

(a) For the linear map $p = 1_{\mathbb{D}} : \mathbb{D} \to \mathbb{R}$, $\quad j^1_0\gamma \mapsto \gamma(0)$ the local expression of $\omega^{(1_{\mathbb{D}})}$ is given by:

$$\omega^{(1_{\mathbb{D}})} = \omega_i dx^i$$

The 1-form $\omega^{(1_{\mathbb{D}})}$ coincide with the vertical lift of ω from M to TM.

(b) For $p = p_{\mathbb{D}}$ as defined in example 2, we have $p_0 = 0$ and $p_1 = 1$, so

$$\omega^{(p_{\mathbb{D}})} = \frac{\partial \omega_i}{\partial x^k} \dot{x}_k dx^i + \omega_i d\dot{x}^i$$

The 1-form $\omega^{(p_{\mathbb{D}})}$ coincide with the complete lift of ω from M to TM.

(ii) **Case where $A = J^r_0(\mathbb{R}^k, \mathbb{R})$.** For the linear map $p = \varsigma^k_\alpha : j^r_0 g \mapsto \frac{1}{\alpha!} D_\alpha(g(t))|_{t=0}$ we have $p_\gamma = 0$ for $\gamma \neq \alpha$ and $p_\alpha = 1$. So using the equation (2.2) we deduce that:

$$\omega^{\left(\varsigma^k_\alpha\right)} = \omega^{(\alpha)}_i dx^i + \sum_{1 \leq |\beta| \leq r} \omega^{(\alpha-\beta)}_i dx^i_\beta = \sum_{0 \leq |\beta| \leq r} \omega^{(\alpha-\beta)}_i dx^i_\beta$$

Thus $\omega^{\left(\varsigma^k_\alpha\right)}$ coincide with the α-prolongation of differential form ω from M to $T^r_k M$ defined in [10].

(iii) **General case.** For the linear map $p = \varsigma^\alpha_A : j^A\varphi \mapsto \frac{1}{\alpha!} D_\alpha(\varphi(z))|_{z=0}$ with $\alpha \in B_A$ and $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ we have: $p_\gamma = 0$ for $\gamma \neq \alpha$ and $p_\alpha = 1$. Thus

$$\omega^{(\varsigma^\alpha_A)} = \overline{\omega}^{(\alpha)}_i dx^i + \sum_{\beta \in B_A} \overline{\omega}^{(\alpha-\beta)}_i d\overline{x}^i_\beta$$

The differential form $\omega^{(\varsigma^\alpha_A)}$ coincides with the α-prolongation of differential form defined in [3].

3.2. The symplectomorphisms $\varepsilon^p_{A,M} : T^A T^* M \to T^* T^A M$

Let Ω be a 2 form on M. It induces the vector bundle morphism $\Omega^\sharp : TM \to T^*M$. We put:

$$\left(\Omega^\sharp\right)^{(p)} = \varepsilon^p_{A,M} \circ T^A\left(\Omega^\sharp\right) \circ \left(\kappa^A_M\right)^{-1} \tag{12}$$

The T^AM-morphism of vector bundles $\left(\Omega^\sharp\right)^{(p)} : TT^AM \to T^*T^AM$ defines a differential form $\Omega^{(p)}$ on T^AM of degree 2 called p-prolongation of Ω from M to T^AM. If locally $\Omega = \Omega_{ij}dx^i \wedge dx^j$ then:

$$\begin{cases} \Omega^{(p)} = \Omega_{ij}p_0 dx^i \wedge dx^j + \sum\limits_{\alpha\in B_A} p_\alpha \left(\sum\limits_{\beta\in B_A} \overline{\Omega}^{(\alpha-\beta)}_{ij}\right) dx^i \wedge d\overline{x}^j_\beta \\ + \sum\limits_{\mu,\beta\in B_A}\left(\sum\limits_{\alpha\in B_A} p_\alpha \overline{\Omega}^{(\alpha-\beta-\mu)}_{ij}\right) d\overline{x}^i_\mu \wedge d\overline{x}^j_\beta \end{cases} \tag{13}$$

Example 3.3. In the particular case where $A = J^r_0\left(\mathbb{R}^k, \mathbb{R}\right)$ and $p = \varsigma^k_\alpha$ we have:

$$\Omega^{\left(\varsigma^k_\alpha\right)} = \Omega^{(\alpha-\beta-\mu)}_{ij} dx^i_\mu \wedge dx^j_\beta$$

It coincides with the α-prolongation of Ω from M to $T^r_k M$ defined in [10].

Example 3.4. If Ω_M is a Liouville 2-form on T^*M defined in local coordinates system $\left(x^i, \pi_j\right)$ by:

$$\Omega_M = dx^i \wedge d\pi_i,$$

then we have:

$$\Omega^{(p)}_M = p_0 dx^i \wedge d\pi_i + \sum_{\alpha\in B_A} p_\alpha dx^i \wedge d\overline{\pi}^\alpha_i + \sum_{\alpha,\beta\in B_A} p_\alpha d\overline{x}^i_\beta \wedge d\overline{\pi}^{\alpha-\beta}_i \tag{14}$$

It is clear that $d\left(\Omega^{(p)}_M\right) = 0$. Thus the 2-form $\Omega^{(p)}_M$ defines a presymplectic structure on T^AT^*M. It is symplectic form if p is nondegenerate.

Theorem 3.5. *The vector bundle morphisms $\varepsilon^p_{A,M} : T^AT^*M \to T^*T^AM$ is a symplectomorphism between the pre-symplectic manifolds $\left(T^AT^*M, \Omega^{(p)}_M\right)$ and $\left(T^*T^AM, \Omega_{T^AM}\right)$. Where Ω_{T^AM} is a Liouville 2-form on T^*T^AM*

Proof. The expression in local coordinate of Liouville 2-form on T^*T^AM is given by:

$$\Omega_{T^AM} = dx^i \wedge d\pi_i + \sum_{\alpha\in B_A} d\overline{x}^i_\alpha \wedge d\overline{\pi}^\alpha_i$$

$$\begin{aligned} \left(\varepsilon^p_{A,M}\right)_*\left(\Omega_{T^AM}\right) &= \sum_{\alpha\in B_A\cup\{0\}} d\left(\overline{x}^i_\alpha \circ \varepsilon^p_{A,M}\right) \wedge d\left(\overline{\pi}^\alpha_i \circ \varepsilon^p_{A,M}\right) \\ &= dx^i \wedge d\left(p_0\pi_i + \sum_{\alpha\in B_A} p_\alpha \overline{\pi}^\alpha_i\right) + \sum_{\beta,\alpha\in B_A} p_\alpha d\overline{x}^i_\beta \wedge d\overline{\pi}^{\alpha-\beta}_i \\ &= p_0 dx^i \wedge d\pi_i + \sum_{\alpha\in B_A} p_\alpha dx^i \wedge d\overline{\pi}^\alpha_i + \sum_{\beta,\alpha\in B_A} p_\alpha d\overline{x}^i_\beta \wedge d\overline{\pi}^{\alpha-\beta}_i \end{aligned}$$

Thus $\left(\varepsilon^p_{A,M}\right)_*\left(\Omega_{T^AM}\right) = \Omega^{(p)}_M$.

■

Remark 3.6. (i) In particular, when $p = \varsigma^1_r$ we obtain the results of [2].

(ii) When (A, p) is a Weil-Frobenius algebra, the bundle T^AT^*M has a canonical symplectic structure determined by $\left(\varepsilon^p_{A,M}\right)_*\left(\Omega_{T^AM}\right) = \Omega^{(p)}_M$. More precisely, in [9], the authors show that: for any Weil algebra A the bundle T^AT^*M has the canonical symplectic structure if and only if A is a Weil-Frobenius algebra.

References

1. Abraham, R. and Marsden, J., E. *Foundations of mechanics*, second edition Library of congress cataloging in publication data, October 1987.
2. Cantrijn, F., Crampin, M.,Sarlet W., and Saunders, D., *The canonical isomorphism between T^kT^* and T^*T^k.* C.R. Acad. Sci. Paris, t. **309** (1989), série II, 1509–1514.
3. Gancarzewicz, J. , Mikulski, W. and Pogoda, Z., *Lifts of some tensor fields and connections to product preserving functors*, Nagoya Math. J. **135** (1994), 1–41.
4. Gràcia, X., Pons, J., M., and Romàn-Roy, N., *Higher order Lagrangian systems: Geometric structures, dynamics, and constraints*, J. Math. Phy., **32**, No., 10 (1991), 2744-2763.
5. Kolar, I., Michor, P. and Slovak, J., *Natural operations in differential geometry*, Springer-Verlag. 1993.
6. Kouotchop Wamba, P., M., *Canonical Poisson-Nijenhuis structures on higher order tangent bundles*, Annales Polonici Mathematici **111** 1 (2014), 21–37.
7. Kouotchop Wamba, P., M. and Ntyam, A., *Prolongations of Dirac structures related to Weil bundles*, Lobatchevskii journal of mathematics, **35** (2014), N° 2, pp 106–121.
8. Kurek, J., *Natural affinors in higher order cotangent bundle*, Archivum Mathematicum (BRNO), Tomus **28** (1992), 175–180.
9. Miroslav Doupovec and Miroslav Kureš *Some geometric constructions on Frobenius Weil bundles*, Differential geometry and its applications **35** (2014), 143–149.
10. Morimoto, A., *Lifting of some type of tensors fields and connections to tangent bundles of p^r-velocities*, Nagoya Math., J. **40** (1970), 13–31.
11. Tomáš J. , *Some classification problems on natural bundles related to Weil bundles*. Proc. Conf. Dif. Geom. (2001), University Valencia, published by World Scientific, pp. 297–310.
12. Wouafo Kamga, J., *Global prolongation of geometric objets to some jet spaces*, International centre for theoretical physics, Trieste, Italy, november 1997 .

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