# On surface-symmetric spacetimes with collisionless and charged matter

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**Abstract**

In this talk we present some global properties of cosmological solutions of the surface-symmetric Einstein equations coupled to collisionless and charged matter described by the Vlasov and Maxwell equations. This involves two issues which are the existence of a global time coordinate $t$ and the asymptotic behaviour of the solution at late times. For details concerning the proof of results stated in the following we refer to [1] and references therein.

## 1 Cosmological models with kinetic matter

The purpose of kinetic theory is to model the evolution of a collection of particles.

In cosmology the particles in the kinetic description are galaxies or even cluster of galaxies.

Cosmological spacetimes are those possessing a compact Cauchy hypersurface and data are given on a compact 3-manifold.

In kinetic theory the model is statistical and the particle system is described by a distribution function $f = f(t, x, p)$ which represents the density of particles at time $t$ in the position $x$ with momentum $p$.

## 2 The Einstein-Vlasov-Maxwell system

We consider a self-gravitating collisionless gas and suppose that:

- there are two species of charged particles, one of positive charge $+1$ and the other of negative charge $-1$

- all the particles have the same rest mass $m = 1$, and the 4-momentum of each particle is a future-pointing unit timelike vector.

Thus the number densities $f^+$ and $f^-$ for positive and negative charge species respectively, are non-negative functions supported on the mass shell

$$PM := \{g_{\alpha\beta}p^\alpha p^\beta = -1,\ p^0 > 0\},$$

a submanifold of the tangent bundle $TM$ of the space-time manifold $M$ with metric $g$ of signature $- + ++$.

On $PM$, $p^0 = \sqrt{-g^{00}}\sqrt{1 + g_{ab}p^a p^b}$.

We use coordinates $(t, x^a)$ with zero shift and corresponding canonical momenta $p^\alpha$ ; Greek indices always run from 0 to 3, and Latin ones from 1 to 3. The Einstein-Vlasov-Maxwell system reads as follows

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi(T_{\alpha\beta} + \tau_{\alpha\beta}) \tag{1}$$

$$\partial_t f^+ + \frac{p^a}{p^0}\partial_{x^a} f^+ - \frac{1}{p^0}(\Gamma^a_{\beta\gamma}p^\beta p^\gamma + F_\beta{}^{\ a}p^\beta)\partial_{p^a} f^+ = 0 \tag{2}$$

$$\partial_t f^- + \frac{p^a}{p^0}\partial_{x^a} f^- - \frac{1}{p^0}(\Gamma^a_{\beta\gamma}p^\beta p^\gamma - F_\beta{}^{\ a}p^\beta)\partial_{p^a} f^- = 0 \tag{3}$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \tag{4}$$

$$\nabla_\alpha F^{\alpha\beta} = J^\beta \tag{5}$$

$$T_{\alpha\beta} = -\int_{\mathbb{R}^3}(f^+ + f^-)p_\alpha p_\beta |g|^{1/2}\frac{dp^1dp^2dp^3}{p_0} \tag{6}$$

$$\tau_{\alpha\beta} = F_{\alpha\gamma}F_\beta{}^{\ \gamma} - \frac{g_{\alpha\beta}}{4}F^{\gamma\delta}F_{\gamma\delta} \tag{7}$$

$$J^\beta = \int_{\mathbb{R}^3}(f^+ - f^-)p^\beta |g|^{1/2}\frac{dp^1dp^2dp^3}{p_0} \tag{8}$$

where $|g|$ is the determinant of the metric $g$, $G_{\alpha\beta}$ the Einstein tensor, $\Lambda$ the cosmological constant, $F_{\alpha\beta}$ the electromagnetic field created by the charged particles, $J^\beta$ the total particle current density generated by the charged particles and $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$ the energy-momentum tensor for Vlasov and Maxwell matter respectively.

# 3 Some matter properties

- A computation in normal coordinates shows that $\nabla_\alpha J^\alpha = 0$. This is an expression of the conservation of charge.

- It is known that $T_{\alpha\beta}$ satisfies the dominant energy condition i.e. $T_{\alpha\beta}V^\alpha W^\beta \geq 0$ for any two future-pointing timelike vectors $V^\alpha$ and $W^\alpha$.

Let us show that the same is true for the Maxwell tensor $\tau_{\alpha\beta}$.

Proving this is equivalent to show the weak-energy condition $\tau_{\alpha\beta}V^\alpha V^\beta \geq 0$ for all timelike vector $V^\alpha$, together with the property that $\tau_{\alpha\beta}V^\beta$ is non-spacelike for any future-pointing timelike vector $V^\alpha$.

The proof of the latter can be deduced from the following identities which hold since $F$ is antisymmetric :

$$\tau_{\alpha\nu}\tau^\nu_\beta = \frac{1}{4}(\tau^{\gamma\delta}\tau_{\gamma\delta})g_{\alpha\beta}, \qquad \tau_{\alpha\beta}\tau^{\alpha\beta} \geq 0$$

Contracting the first of these identities twice with $V^\alpha$ implies the following, using the second identity and the fact that $V^\alpha$ is timelike :

$$(V^\alpha\tau_{\alpha\nu})(\tau^\nu_\beta V^\beta) = \frac{1}{4}(\tau^{\gamma\delta}\tau_{\gamma\delta})g_{\alpha\beta}V^\alpha V^\beta \leq 0$$

and setting $P_\nu = V^\alpha\tau_{\alpha\nu}$, this means that $P_\alpha P^\alpha \leq 0$, that is $P_\alpha$ is non-spacelike as desired.

Now proving the weak-energy condition is equivalent to show that $\tau_{00}$ is non-negative since we can choose an orthonormal frame such that $V^\alpha$ is the timelike vector of the frame.

In such a frame $g_{00} = -1$ so that

$$\tau_{00} = \frac{1}{2}g^{ab}F_{0a}F_{0b} + \frac{1}{4}F^{ab}F_{ab} \geq 0$$

as the sum of spatial lengths of a vector and a tensor respectively.

# 4 Surface symmetry and time coordinates

Spacetime $M = \mathbb{R} \times S^1 \times F$, with $F$ a two-dimensional compact manifold (let $\tilde{F}$ be the covering manifold of $F$).

Areal coordinates :

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + t^2(d\theta^2 + \sin_k^2\theta d\varphi^2)$$

where $r \in [0,1]$, and the functions $\mu$ and $\lambda$ are periodic in $r$ with period 1

- plane symmetry $(k = 0)$ : $F = T^2$ and $G = E_2$ acts isometrically on $\tilde{F} = \mathbb{R}^2$, $\sin_k\theta = 1$, $(\theta,\varphi) \in [0,2\pi] \times [0,2\pi]$

- spherical symmetry $(k = 1)$ : $F = S^2$ and $G = SO(3)$ acts isometrically and without fixed points on $S^1 \times S^2$, $\sin_k\theta = \sin\theta$, $(\theta,\varphi) \in [0,\pi] \times [0,2\pi]$

- hyperbolic symmetry $(k = -1)$ : $\tilde{F} = H^2/\Gamma$ with $\Gamma$ a discrete group of isometries of $H^2$, $\sin_k\theta = \sinh\theta$, $(\theta,\varphi) \in [0,\infty) \times [0,2\pi]$.

### Symmetry leads to some simplifications

Due to symmetry $f^+$ and $f^-$ can be written as functions of

$$t, r, w := e^{\lambda} p^1, L := t^4 (p^2)^2 + t^4 \sin_k^2 \theta (p^3)^2,$$
$$\text{with } r, w \in \mathbb{R}; L \in [0, +\infty).$$

In surface symmetry the only non-zero components of $F$ are $F_{01}$ and $F_{23}$.

Indeed setting $h := g + e_0 \otimes e_0 - e_1 \otimes e_1$, with $e_0 = e^{-\mu} \frac{\partial}{\partial t}$ and $e_1 = e^{-\lambda} \frac{\partial}{\partial r}$, the mapping $X^\beta \mapsto h^\beta_\alpha X^\alpha$ is the orthogonal projection on the tangent space of the orbit, and since the vector $Y_\sigma := F_{\alpha\beta}(e_0)^\alpha h^\beta_\sigma$ is invariant under the symmetry group, it vanishes. This implies that $F_{02} = F_{03} = 0$. Similarly, replacing $e_0$ by $e_1$ in the expression of $Y_\sigma$ yields $F_{12} = F_{13} = 0$.

# 5 The coupled EVM system in surface symmetry

The hypotheses above are used to write down the equations in surface symmetry:

$$\partial_t f^+ + \frac{e^{\mu-\lambda} w}{\sqrt{1 + w^2 + L/t^2}} \partial_r f^+ - (\dot{\lambda} w + e^{\mu-\lambda} \mu' \sqrt{1 + w^2 + L/t^2} - e^{\lambda+\mu} E) \partial_w f^+ = 0 \tag{9}$$

$$\partial_t f^- + \frac{e^{\mu-\lambda} w}{\sqrt{1 + w^2 + L/t^2}} \partial_r f^- - (\dot{\lambda} w + e^{\mu-\lambda} \mu' \sqrt{1 + w^2 + L/t^2} + e^{\lambda+\mu} E) \partial_w f^- = 0 \tag{10}$$

$$e^{-2\mu}(2t\dot{\lambda} + 1) + k - \Lambda t^2 = 8\pi t^2 \rho \tag{11}$$

$$e^{-2\mu}(2t\dot{\mu} - 1) - k + \Lambda t^2 = 8\pi t^2 p \tag{12}$$

$$\mu' = -4\pi t e^{\lambda+\mu} j \tag{13}$$

$$e^{-2\lambda} \left(\mu'' + \mu'(\mu' - \lambda')\right) - e^{-2\mu} \left(\ddot{\lambda} + (\dot{\lambda} - \dot{\mu})(\dot{\lambda} + \frac{1}{t})\right) + \Lambda = 4\pi q \tag{14}$$

$$\partial_r (t^2 e^{\lambda} E) = t^2 e^{\lambda} a \tag{15}$$

$$\partial_t (t^2 e^{\lambda} E) = -t^2 e^{\mu} b \tag{16}$$

**Matter quantities**

$$\rho(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\sqrt{1+w^2+L/t^2}(f^+ + f^-)(t,r,w,L)dLdw$$
$$+\frac{1}{2}(e^{2\lambda}E^2+ct^{-4}) = e^{-2\mu}(T_{00}+\tau_{00})(t,r), \tag{17}$$
$$p(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\sqrt{1+w^2+L/t^2}}(f^+ + f^-)(t,r,w,L)dLdw$$
$$-\frac{1}{2}(e^{2\lambda}E^2+ct^{-4}) = e^{-2\lambda}(T_{11}+\tau_{11})(t,r), \tag{18}$$

$$j(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty} w(f^+ + f^-)(t,r,w,L)dLdw = -e^{\lambda+\mu}T_{01}(t,r), \tag{19}$$
$$q(t,r) := \frac{\pi}{t^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{L}{\sqrt{1+w^2+L/t^2}}(f^+ + f^-)(t,r,w,L)dLdw$$
$$+(e^{2\lambda}E^2+ct^{-4}) = \frac{2}{t^2}(T_{22}+\tau_{22})(t,r), \tag{20}$$
$$a(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}(f^+ - f^-)(t,r,w,L)dLdw, \tag{21}$$
$$b(t,r) := \frac{\pi}{t^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w}{\sqrt{1+w^2+L/t^2}}(f^+ - f^-)(t,r,w,L)dLdw. \tag{22}$$

# 6 Initial data, regularity of solutions

Initial data are prescribed at some time $t = t_0 > 0$,

$$f^+(t_0,r,w,L) = \overset{\circ}{f}{}^+(r,w,L),\ f^-(t_0,r,w,L) = \overset{\circ}{f}{}^-(r,w,L),$$
$$\lambda(t_0,r) = \overset{\circ}{\lambda}(r),\ \mu(t_0,r) = \overset{\circ}{\mu}(r),\ E(t_0,r) = \overset{\circ}{E}(r)$$

and we study the existence and behaviour of the corresponding solution for $t \in [t_0,+\infty)$.

A solution $(f^+, f^-, \lambda, \mu, E)$ is being said to be regular if $f^\pm \in C^1(I\times\mathbb{R}^2\times[0,\infty[)$, $f^\pm \geq 0$, supp$f^\pm(t,r,.,.)$ is compact, uniformly in $r$ and locally uniformly in $t$, the functions $\lambda, \dot{\lambda}, \mu, \mu', E$ are in $C^1(I\times\mathbb{R})$, and $f^\pm, \lambda, \mu, E$ are periodic in $r$ with period 1.

# 7 Some remarks

- If the subsystem (9)-(12), (16) is satisfied as well as equations (13) and (15) for $t = t_0$ then (13)-(15) hold for all $t$.

It is then enough to concentrate on the subsystem (9)-(12), (16).

On the other hand equations (13) and (15) are being considered as constraint equations on initial data (at $t = t_0$).

- To solve the Maxwell constraint equation (15) (for $t = t_0$) we need to impose the following condition, because $(e^{\overset{\circ}{\lambda}}\overset{\circ}{E})(r)$ is periodic in $r$ with period 1

$$\int_0^1 \int_{-\infty}^{\infty} \int_0^{\infty} e^{\overset{\circ}{\lambda}}(\overset{\circ}{f}{}^{+} - \overset{\circ}{f}{}^{-})(r, w, L)dLdwdr = 0,$$

This is the reason why it is necessary to consider a model with more than one species of particles, otherwise the integral $\int_0^1 \int_{-\infty}^{\infty} \int_0^{\infty} e^{\overset{\circ}{\lambda}}\overset{\circ}{f}(r, w, L)dLdwdr$ would never vanish, except if $\overset{\circ}{f}$ is identically zero.

# 8 Local existence theorem

An iterative scheme is used to prove the following

**Theorem 1** *Let $\overset{\circ}{f}{}^{\pm} \in C^1(\mathbb{R}^2 \times [0, \infty[)$ with $\overset{\circ}{f}{}^{\pm}(r + 1, w, L) = \overset{\circ}{f}{}^{\pm}(r, w, L)$ for $(r, w, L) \in \mathbb{R}^2 \times [0, \infty[$, $\overset{\circ}{f}{}^{\pm} \geq 0$, and $w_0 := w_0^+ + w_0^-$, $L_0 := L_0^+ + L_0^-$ with*

$$w_0^{\pm} := \sup\{|w||(r, w, L) \in \mathrm{supp}\overset{\circ}{f}{}^{\pm}\} < \infty$$
$$L_0^{\pm} := \sup\{F|(r, w, L) \in \mathrm{supp}\overset{\circ}{f}{}^{\pm}\} < \infty$$

*Let $\overset{\circ}{\lambda}$, $\overset{\circ}{E} \in C^1(\mathbb{R})$, $\overset{\circ}{\mu} \in C^2(\mathbb{R})$ with $\overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r + 1)$, $\overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r + 1)$, $\overset{\circ}{E}(r) = \overset{\circ}{E}(r + 1)$ for $r \in \mathbb{R}$, and*

$$\overset{\circ}{\mu}{}'(r) = -\frac{4\pi^2}{t_0} e^{\overset{\circ}{\lambda}+\overset{\circ}{\mu}} \int_{-\infty}^{\infty} \int_0^{\infty} w(\overset{\circ}{f}{}^{+} + \overset{\circ}{f}{}^{-})(r, w, L)dLdw, \quad r \in \mathbb{R},$$

$$\partial_r(t_0^2 e^{\overset{\circ}{\lambda}}\overset{\circ}{E}) = \pi e^{\overset{\circ}{\lambda}} \int_{-\infty}^{\infty} \int_0^{\infty} (\overset{\circ}{f}{}^{+} - \overset{\circ}{f}{}^{-})(r, w, L)dLdw, \quad r \in \mathbb{R}.$$

*Then there exists a unique, right maximal, regular solution $(f^+, f^-, \lambda, \mu, E)$ of (9)-(16) with $(f^+, f^-, \lambda, \mu, E)(t_0) = (\overset{\circ}{f}{}^{+}, \overset{\circ}{f}{}^{-}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{E})$ on a time interval $[t_0, T_{\max}[$ with $T_{\max} \in ]t_0, \infty]$.*

# 9 Continuation criteria and global existence to the future

**Proposition 1** *Let $(f^+, f^-, \lambda, \mu, E)$ be a right maximal regular solution obtained in the local existence theorem. If*

*$\sup\{|w||(r,w,L) \in \mathrm{supp} f^+\}$, $\sup\{|w||(r,w,L) \in \mathrm{supp} f^-\}$, $\sup\{\mu(t,r)|r \in \mathbb{R},\ t \in [t_0, T_{max}[\}$, and $\sup\{(e^\lambda E)(t,r)|r \in \mathbb{R},\ t \in [t_0, T_{max}[\}$ are finite then $T_{max} = \infty$.*

Actually it can be proven that the boundedness of $\mu$ implies the other conditions in the latter proposition. It is therefore enough to obtain a bound on $\mu$ in order to show that the solution can be extended on the whole time interval $[t_0, \infty)$. This follows from a series of estimates that we do not present here. We can thus state:

**Theorem 2** *For initial data as in the local existence theorem with $t_0^2 > 1/\Lambda$ in the case of spherical symmetry, the solution of the surface-symmetric Einstein-Vlasov-Maxwell system with positive cosmological constant, written in areal coordinates, exist for all $t \in [t_0, \infty[$ where $t$ denotes the area radius of the surfaces of symmetry of the induced spacetime.*

# 10 Spatially homogeneous solutions

These correspond to LRS (locally rotationally symmetric) models of Bianchi type I and type III and Kantowski-Sachs type for plane, hyperbolic and spherical symmetry respectively.

- spacetime : a manifold $G \times I$, , $I$ being an open interval and $G$ a simply connected three-dimensional Lie group.

- metric

$$ds^2 = -d\tau^2 + g_{ij} e^i \otimes e^j,$$

$\{e_i\}$ is a left invariant frame and $\{e^i\}$ the dual coframe.

- The Einstein constraint equations are

$$R - k_{ij}k^{ij} + (k_{ij}g^{ij})^2 = 16\pi(T_{00} + \tau_{00}) + 2\Lambda \tag{23}$$

$$\nabla^i k_{ij} = -8\pi T_{0j}. \tag{24}$$

- The Einstein evolution equations are

$$\partial_t g_{ij} = -2k_{ij} \tag{25}$$

$$\begin{aligned}\partial_t k_{ij} = R_{ij} + &(k_{lm}g^{lm})k_{ij} - 2k_{il}k_j^l - 8\pi(T_{ij} + \tau_{ij}) \\ &- 4\pi(T_{00} + \tau_{00}) + 4\pi(T_{lm} + \tau_{lm})g^{lm}g_{ij} - \Lambda g_{ij},\end{aligned} \tag{26}$$

- The Vlasov equations are

$$\begin{aligned}\partial_\tau f^+ + [2k_j^i v^j - &(1 + g_{rs}v^r v^s)^{-1/2}\gamma^i_{mn}v^m v^n \\ &- (F_0{}^i + F_j{}^i \frac{v^j}{v^0})]\partial_{v^i} f^+ = 0\end{aligned} \tag{27}$$

$$\begin{aligned}\partial_\tau f^- + [2k_j^i v^j - &(1 + g_{rs}v^r v^s)^{-1/2}\gamma^i_{mn}v^m v^n \\ &+ (F_0{}^i + F_j{}^i \frac{v^j}{v^0})]\partial_{v^i} f^- = 0,\end{aligned} \tag{28}$$

- The Maxwell equations

$$C^i_{ij} F^{j0} = J^0 \tag{29}$$

$$\partial_\tau F^{0i} - (trk) F^{0i} + C^k_{kj} F^{ji} = J^i, \tag{30}$$

where

$$T_{00} + \tau_{00} = \int (f^+ + f^-)(\tau, v)(1 + g_{rs} v^r v^s)^{1/2} (\det g)^{1/2} dv$$
$$+ F_{0\gamma} F_0{}^\gamma + \frac{1}{4} F_{\gamma\delta} F^{\gamma\delta} \tag{31}$$

$$T_{0i} = \int (f^+ + f^-)(\tau, v) v_i (\det g)^{1/2} dv \tag{32}$$

$$T_{ij} + \tau_{ij} = \int (f^+ + f^-)(\tau, v) v_i v_j (1 + g_{rs} v^r v^s)^{-1/2} (\det g)^{1/2} dv$$
$$+ F_{i\gamma} F_j{}^\gamma - \frac{g_{ij}}{4} F_{\gamma\delta} F^{\gamma\delta}, \tag{33}$$

with $v := (v^1, v^2, v^3)$ and $dv := dv^1 dv^2 dv^3$.

$\gamma^i_{mn} = \frac{1}{2} g^{ik}(-C^l_{nk} g_{ml} + C^l_{km} g_{nl} + C^l_{mn} g_{kl})$, $C^i_{jk}$ are the structure constants of the Lie algebra of $G$.

$$J^0 = \int (f^+ - f^-)(\tau, v)(\det g)^{1/2} dv$$

$$J^i = \int (f^+ - f^-)(\tau, v) v^i (1 + g_{rs} v^r v^s)^{-1/2} (\det g)^{1/2} dv.$$

We can prove the following:

**Theorem 3** *Let $f^\pm(0, v)$ be a nonnegative $C^1$ function with compact support. Let $(g_{ij}(0), k_{ij}(0), f^+(0, v), f^-(0, v), F^{0i}(0))$ be an initial data set for the evolution equations (25), (26), the Vlasov equations (27), (28) and the Maxwell equation (30), which has Bianchi symmetry and satisfies the constraint equations (23), (24), and (29). Then the corresponding solution of the Einstein-Vlasov-Maxwell system is a future complete spacetime for causal trajectories.*

# 11 Completeness in the inhomogeneous case

A bootstrap argument is used to prove the major part of the following:

**Theorem 4** *Consider any solution of Einstein-Vlasov-Maxwell system with positive cosmological constant in surface symmetry written in areal coordinates, with initial data as in the global existence theorem. Let $\delta$ be a positive constant and suppose the following inequalities hold:*

$$|t_0 \dot{\lambda}(t_0) - 1| \leq \delta, \ |(e^{-\lambda} \mu')(t_0)| \leq \delta, \ |(e^\lambda E)(t_0)| \leq \delta \tag{34}$$

$$|\Lambda t_0^2 e^{2\mu(t_0)} - 3 - 3k e^{2\mu(t_0)}| \leq \delta, \ \bar{w}(t_0) \leq \delta, \ c \leq \delta, \tag{35}$$

*where $\bar{w}(t)$ denotes the maximum of $w$ over the support of $f^+(t)$ or $f^-(t)$. Then if $\delta$ is sufficiently small, the following properties hold at late times:*

$$t\dot{\lambda} - 1 = O(t^{-2}),\ e^{-\lambda}\mu' = O(t^{-2}),\ e^{\lambda}E = O(t^{-2}), \tag{36}$$

$$\Lambda t^2 e^{2\mu} - 3 - 3ke^{2\mu} = O(t^{-3}),\ \bar{w} = O(t^{-1}). \tag{37}$$

*Furthermore the spacetime is future complete for causal trajectories.*

## Main reference

[1] S. B. Tchapnda, On surface-symmetric spacetimes with collisionless and charged matter, gr-qc/0607140 (*Ann. Henri Poincaré*, to appear)