THE DUALS OF BERGMAN SPACES IN SIEGEL DOMAINS OF TYPE II

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Abstract **:** *In every homogeneous Siegel domain of type* II *, for some real number* $p_0 > 2$, *we characterize the dual of the weighted Bergman space* $A^{p,r}$ *when* $1 \leq p < p_0$ *. In the symmetric case, we also characterize the dual of* $A^{p,r}$ *with* $p_1 < p < 1$ *for some* $p_1 \in (o,1)$, *and extend this to two homogeneous non symmetric Siegel domains of type* II.

Résumé : *Dans les domaines de Siegel homogènes de type* II, *nous caractérisons le dual de l'espace de Bergman avec poids* $A^{p,r}$ *, lorsque* $1 \leq p < p_0$ *, où le nombre réel* p_0 *dépend du domaine. Dans le cas où le domaine est symétrique, nous caractérisons également le dual de* $A^{p,r}$ *lorsque* $p_1 < p < 1$ *, avec* $p_1 \in (o,1)$ *, et nous généralisons ce résultat à deux domaines de Siegel homogènes et non symétriques de type* II.

I. INTRODUCTION.

Let D be a homogeneous Siegel domain of type II. Let $d\nu$ denote the Lebesgue measure on D and let H(D) be the space of holomorphic functions in D endowed with the topology of uniform convergence on compact subsets of D. The Bergman projection P of D is the orthogonal projection of $L^2(D, d\nu)$ onto its subspace $A^2(D)$ consisting of holomorphic functions. Moreover, P is the integral operator defined on $L^2(D, d\nu)$ by the Bergman kernel $B(\zeta, z)$ which, for D, was computed in [G].

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Let r be a real number. Since D is homogeneous, the function $\zeta \mapsto B(\zeta, \zeta)$ does not vanish on D, we can set:

$$L^{p,r}(D) = L^p\left(D, B^{-r}(\zeta,\zeta)d\nu(\zeta)\right), \quad 0 < p < \infty .$$

Let p be a positive number. The weighted Bergman space $A^{p,r}(D)$ is defined by

$$A^{p,r}(D) = L^{p,r}(D) \cap H(D) .$$

If r = 0, then $A^{p,r}(D)$ is simply denoted $A^p(D)$.

The weighted Bergman projection P_ε is the orthogonal projection of $L^{2,\varepsilon}(D)$ onto $A^{2,\varepsilon}(D)$. It is proved in [BT] that there exists a real number $\varepsilon_D < 0$ such that $A^{2,\varepsilon}(D) = \{0\}$ if $\varepsilon \le \varepsilon_D$ and that for $\varepsilon > \varepsilon_D$, P_ε is the integral operator defined on $L^{2,\varepsilon}(D)$ by the weighted Bergman kernel $c_\varepsilon B^{1+\varepsilon}(\zeta, z)$. In all our work, we shall assume that $\varepsilon > \varepsilon_D$.

The "norm" $\|.\|_{p,r}$ of $A^{p,r}(D)$, with $r > \varepsilon_D$, is defined by:

$$\|f\|_{p,r} = \left(\int_D |f(z)|^p B^{-r}(z,z)d\nu(z)\right)^{1/p}, \quad f \in A^{p,r}(D).$$

Let ρ be a positive integer. S.G. Gindikin [G] has defined a differential polynomial Λ_ρ in D that satisfies the property:

$$\left(\Lambda_\rho\right)_\zeta B(\zeta, z) = c_\rho B^{1+\rho}(\zeta, z) \qquad (\zeta, z \in D) .$$

A holomorphic function g in D is said to be a Bloch function in D if g satisfies the estimate:

$$\|g\|_* = \sup_{z \in D}\left\{\left|(\Lambda_\rho)_z g(z)\right| B^{-\rho}(z,z)\right\} < \infty .$$

Let $\mathcal{N} = \{g \in H(D) : \Lambda_\rho g = 0\}$. The Bloch space $\mathcal{B}_\rho$ of D is defined by :

$\mathcal{B}_\rho$ ={Bloch functions}/$\mathcal{N}$.

For $p \geq 1$, the space $\mathcal{C}^p_{\rho,r}(D)$ is the quotient space by $\mathcal{N}$ of the space of holomorphic functions g in D satisfying the estimate

$$\|g\|_{C^p_{\rho,r}(D)} = \left(\int_D \left| B^{-\rho}(z,z)\Lambda_\rho g(z) \right|^p B^{-r}(z,z)d\nu(z) \right)^{1/p} < \infty \ .$$

For r = 0, the space $\mathcal{C}^p_{\rho,r}(D)$ is simply denoted $\mathcal{C}^p_\rho(D)$.

In the upper half-plane, $\pi^+ = \{z \in C : \operatorname{Im} z > 0\}$, R. Coifman and R. Rochberg proved the following fact: the dual of the Bergman space $A^1(\pi^+)$ coincides with the Bloch space of holomorphic functions in π^+, and can be realized as the Bergman projection of $L^\infty(\pi^+)$. A few years later, D. Békollé in [B_4] carried out the same study on symmetric Siegel domains of type II. In fact, he proved that for homogeneous Siegel domains of type II associated with self-dual cones, the dual space of A^1 coincides with the Bloch space of holomorphic functions, and for symmetric Siegel domains, this space can be realized as the Bergman projection of L^∞. On the other hand, for bounded symmetric domains, K. Zhu ([Z] and [Z_1]) studied the dual of the Bergman spaces A^p with small exponents ($0 < p < 1$) and obtained that their dual is equal to the Bloch space.

Furthermore, in [B_6] , D. Békollé proved that when $p \in (4/3 , 4)$, the dual of $A^p(R^3 + i\Gamma)$, where Γ is the spherical cone in R^3, is equal to $A^{p'}(R^3 + i\Gamma)$ (p' is the conjugate exponent of p) and when $p \in (1 , 4/3]$, the dual of $A^p(R^3 + i\Gamma)$ coincides with the space $\mathcal{C}^{p'}_\rho(R^3 + i\Gamma)$ with $\rho = 1$.

The purpose of this work is to extend these results to the weighted Bergman space $A^{p,r}(D)$, $1 \le p < p_0$, in homogeneous Siegel domains of type II, where p_0 is a real number greater than 2 and depends on the domain D ; when D is symmetric, we also show that the dual of $A^{p,r}(D)$, $p_1 < p < 1$, where p_1 depends on D , is equal the Bloch space. This latter result is also true for two non symmetric domains.

The first aim of this work is to show that there exists $\rho_0 > 0$ such that whenever $\rho > \rho_0$,

1° when D is homogeneous and $0 < p \le 1$, $\mathcal{B}_\rho$ is isomorphic to a subspace of the dual space $(A^{p,r}(D))^*$ of $A^{p,r}(D)$ and the two spaces are isomorphic when p = 1 ;

2° the two spaces are equal when $p_1 < p < 1$ when D is symmetric, and the same is true for two particular non symmetric domains.

To show the first claim , let g be in $\mathcal{B}_\rho$ and consider the linear functional φ on $A^{p,r}(D)$ defined by

$$\varphi(f) = \int_D \overline{(\Lambda_\rho)g(z)} B^{-\rho}(z,z) f(z) B^{1-\frac{1+r}{p}}(z,z) d\nu(z) \qquad \left(f \in A^{p,r}(D)\right).$$

Since $\int_D |f(z)| B^{1-\frac{1+r}{p}}(z,z) d\nu(z) < \infty$ for all $f \in A^{p,r}(D)$, it follows that φ is bounded, hence belongs to $(A^{p,r}(D))^*$ and is represented by g .

Conversely, assume p = 1 . To obtain $\mathcal{B}_\rho = (A^{1,r}(D))^*$, let $\varphi \in (A^{1,r}(D))^*$; then there exists $b \in L^\infty(D)$ such that

$$\varphi(f) = \int_D \overline{b(z)} f(z) B^{-r}(z,z) d\nu(z), \quad f \in A^{1,r}(D).$$

Since $f(z) = c_{r,\rho} \int_D B^{1+r+\rho}(z,\zeta) f(\zeta) B^{-r-\rho}(\zeta,\zeta) d\nu(\zeta)$, if we set

$$g(z) = c_{r,\rho} \int_D B^{1+r+\rho}(z,\zeta) b(\zeta) B^{-r}(\zeta,\zeta) d\nu(\zeta) \, ,$$

we easily get :

$$\varphi(f) = \int_D \overline{g(z)} f(z) B^{-r-\rho}(z,z) d\nu(z) \, .$$

In fact, since $\rho > \rho_0$, g is a holomorphic function on D satisfying the estimate

$$\sup_{z \in D} \left\{ |g(z)| B^{-\rho}(z,z) \right\} \le c \|b\|_\infty .$$

On the other hand , by a lemma of Trèves [Tr], there exists $\tilde{h} \in H(D)$ such that $\Lambda_\rho \tilde{h} = g$.

Let h be the equivalence class of all holomorphic solutions of this equation. Then h belongs to $\mathcal{B}_\rho$, and the equality

$$\varphi(f) = \int_D \overline{\Lambda_\rho h(z)} f(z) B^{-r-\rho}(z,z) d\nu(z)$$

yields the equality $(A^{1,r}(D))^* = \mathcal{B}_\rho$.

We next assume that $0 < p < 1$. Let us now define two homogeneous Siegel domains D_0 and D_1. Set

$$V_0 = \left\{ \lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & 0 \\ \lambda_{13} & 0 & \lambda_{33} \end{pmatrix} \; : \; \lambda_{11} - \frac{\lambda_{12}^2}{\lambda_{22}} - \frac{\lambda_{13}^2}{\lambda_{33}} > 0 \;\; , \;\; \lambda_{22} > 0 \;\; , \;\; \lambda_{33} > 0 \right\} .$$

Observe that V_0 is a non self dual cone of rank 3. Define D_0 by $D_0 = R^5 + i V_0$. Then D_0 is a homogeneous,non symmetric, tubular domain. On the other hand, D_1 is the first example, due to Piateckii- Chapiro , of a homogeneous non symmetric Siegel domain of type II, and is defined as follows. Let $V_1 = \left\{ \lambda = (\lambda_{11}, \lambda_{12}, \lambda_{22}) \in R^3 : \lambda_{22} > 0, \lambda_{11} - \frac{\lambda_{12}^2}{\lambda_{22}} > 0 \right\}$ be the spherical cone in R^5 , and consider the V_1 - Hermitian form F_1 in C defined by

$$F_1 : C \times C \to C^3$$
$$(u,v) \mapsto \left(u\bar{v},0,0\right).$$

Then $D_1 = \left\{(z,u) \in C^3 \times C \ : \ \frac{z-\bar{z}}{2i} - F(u,u) \in V_1\right\}.$

The restrictions on D, p and ρ are as follows : D is either a symmetric Siegel domain of type II, or $D = D_0$, or $D = D_1$. Then there exists $p_1 \in (0,1)$ such that for all $p \in (p_1, 1)$, we have $(A^{p,r}(D))^* \subset \mathcal{B}_\rho$. Here are the main steps of the proof. Let φ be an element of $(A^{p,r}(D))^*$ and set $\alpha = \rho p + r + 1$, with ρ large enough. Let $f \in A^{p,r}(D)$. Then, in view of the molecular decomposition theorem [BT_1] , there exists $\{\lambda_i\} \in l^p$ such such that

$$f(z) = \sum_i \lambda_i B^{\frac{\alpha}{p}}(z,z_i) B^{\frac{1+r-\alpha}{p}}(z_i,z_i),$$

where $\{z_i\}$ is a lattice in D. Then we get

$$\varphi(f) = \int_D \varphi(B^{\frac{\alpha}{p}}(.,z) f(z) B^{1-\frac{\alpha}{p}}(z,z) dv(z).$$

Now, since $\varphi\overline{\left(B^{\frac{\alpha}{p}}(.,z)\right)}$ is holomorphic, then by the same lemma of Trèves, there exists g $\in \mathcal{B}_\rho$ such that $\Lambda_\rho g(z) = \varphi\overline{\left(B^{\frac{\alpha}{p}}(.,z)\right)}$ by the choice of α. Hence φ is represented by a Bloch function.

To go further, let $h \in L^\infty(D)$. Take ρ large enough ; then the function $z \mapsto G(z) = \int_D B^{1+\rho}(z,\zeta)h(\zeta)d\nu(\zeta)$ satisfies the estimate $\sup_{z\in D}\left\{|G(z)|B^{-\rho}(z,z)\right\} \le c\|h\|_\infty$ and G is holomorphic in D. Hence, by the same lemma of Trèves, there exists a function g$\in$ $\mathcal{B}_\rho$ such that

(1) $(\Lambda_\rho)g = G$.

Now, let P be the operator from $L^\infty(D)$ into $\mathcal{B}_\rho$ which to each $h \in L^\infty(D)$ assigns the element g = Ph defined by (1). This operator P is called "Bergman projection" of $L^\infty(D)$ into $\mathcal{B}_\rho$ for the following reason : although P is not the integral operator $\mathcal{P}$ which is associated with the Bergman kernel $B(\varsigma , z)$, which has no meaning on $L^\infty(D)$, it is easy to show that for $h \in L^2 \cap L^\infty(D)$, the element Ph of $\mathcal{B}_\rho$ can be represented by $\mathcal{P}h$.

The second aim of this paper is to show that the "Bergman projection" P is bounded from $L^\infty(D)$ onto $\mathcal{B}_\rho$. In order to achieve this goal, we prove that for each real number ρ sufficiently large and for each $G\in H(D)$ such that

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$$\sup_{z\in D}\left\{|G(z)|B^{-\rho}(z,z)\right\} < \infty ,$$

one has the reproducing formula :

$$G(\zeta) = c_\rho \int_D B^{1+\rho}(\zeta, z)G(z)B^{-\rho}(z,z)dv(z) \qquad (z\in D).$$

Hence, for each $g \in \mathcal{B}_\rho$, if we set $h(z) = (\Lambda_\rho)g(z)B^{-\rho}(z,z)$, then $h\in L^\infty(D)$ and Ph = g.

The third aim of this work is to determine on a particular Siegel domain D_2 of type II, a kernel $K(\varsigma , z)$ that determines P in the following way : for each $h\in L^\infty(D_2)$, a representative of the element Ph of $\mathcal{B}_\rho$ is given by the function $\zeta \mapsto \int_D K(\zeta, z)h(z)dv(z)$. D. Békollé has determined such a kernel K in three different Siegel domains of type II, namely, the Cayley transform of the unit ball in C^n [B_1], the tube over the spherical cone ([B_2] and [B_3]), and finally, the tube over the cone of symmetric positive-definite matrices [B_5] . The domain we consider is

$$D_2 = \left\{(z, u) \in M_2 \times M_{r,2} : \frac{z - z^*}{2i} - u^* u \in V\right\},$$

where M_2 (resp. $M_{r,2}$ is the set of complex matrices of order 2 (resp. with r lines and 2 rows) , and V the cone of Hermitian positive-definite matrices of order 2. We determine a kernel B_0 such that

(2) $\qquad (B - B_0)((\zeta, v), (z, u)) \in L^1(D_2, dv(z, u)) \qquad ((\varsigma , v) \in D_2);$

(3) $\qquad \left(\Lambda_\rho\right)_\zeta B_0((\zeta, v), (z, u)) \equiv 0$.

Thus Ph can be represented by :

$$\mathcal{P} h(\zeta,v) = \int_{D_2} (B - B_0)((\zeta,v),(z,u))h(z,u)d\nu(z,u) \qquad (h\in L^{\infty}(D_2)).$$

Unfortunately, for Siegel domains of type II associated with cones of rank greater than 2, the determination of a kernel B_0 such that (2) and (3) simultaneously hold seems out of reach.

The fourth aim of this work is to prove that there exists $p_0 \in (2,\infty)$ such that whenever $p'_0 < p < p_0$ (p'_0 is the conjugate exponent of p_0) , the dual of $A^{p,r}(D)$ is equal to $A^{p',r}(D)$, and when $1 < p < p_0$, the dual of $A^{p,r}(D)$ is equal to $\mathcal{C}^{p'}_{\rho,r}(D)$ with $\rho > \rho_0$.

The plan of this work is as follows. In section II, we recall some preliminary results about affine-homogeneous Siegel domains of type II and we give precise statements of our results. In section III, we prove that $\mathcal{B}_\rho$ is contained in $(A^{p,r}(D))^*$ when $0 < p \leq 1$, and that $\mathcal{B}_\rho = (A^{1,r}(D))^*$ (Theorem II.7) ; under some additional assumptions on D, p and ρ, we prove that $\mathcal{B}_\rho = (A^{p,r}(D))^*$ (Theorem II.8). In section IV, we show that $P L^{\infty}(D) = \mathcal{B}_\rho$ (Theorem II.9). In section V, we determine a defining kernel B-B_0 of a representative of the Bergman projection of a bounded function in the particular domain D_0 (Theorem II.10). In section VI, we prove that $(A^{p,r}(D))^* = A^{p',r}(D)$ if $p'_0 < p < p_0$, and $(A^{p,r}(D))^* = \mathcal{C}^{p'}_{\rho,r}(D)$ if $1 < p < p_0$ (Theorem II.11).

For p = 1 and r = 0, the above results were first presented in [T]. In the sequel, as usual, the same letter c will denote constants that may be different from each other.

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II. STATEMENTS OF RESULTS.

Let $V \subset R^n$, $n \geq 3$, be an irreducible, open, convex homogeneous cone which contains no straight line. We first recall the canonical decomposition of V as stated in [G].

NOTATIONS. (i) At the j th step, j =1,2,..., the real line will be denoted by R_{jj} ; at the k th step, k = 1,2,..., R_k will stand for the n_k- dimensional Euclidean space.

(ii) Let $\Gamma \subset R^\sigma$ be a convex homogeneous cone which contains no straight line and let φ be a homogeneous Γ - bilinear symmetric form defined on $R^\tau \times R^\tau$. The real homogeneous Siegel domain $P = P(\Gamma, \varphi)$ is defined by

$$P = P(\Gamma,\varphi) = \{(y,t) \in R^\sigma \times R^\tau : \ y - \varphi(t,t) \in \Gamma\}.$$

We shall denote by V(P) the homogeneous cone defined by:

$$V(P) = \{(y,t,r) \in R^\sigma \times R^\tau \times R : \ r > 0 , (ry,t) \in P \}.$$

In order to describe the canonical decomposition of the cone V, we consider at the first step, the cone $V^{(1)} = (0,\infty) \subset R_{11}$. At the second step, we associate with $V^{(1)}$ and with a homogeneous $V^{(1)}$ - bilinear symmetric form $\varphi^{(2)}$ defined on R_2 , the real Siegel domain $P^{(2)} = P\left(V^{(1)},\varphi^{(2)}\right) \subset R_{11} \times R_2$ and then, the convex cone

$$V^{(2)} = V(P^{(2)}) \subset R_{11} \times R_2 \times R_{22} \ .$$

At the k th step, we associate with the cone $V^{(k-1)}$ and with a $V^{(k-1)}$ - bilinear symmetric form $\varphi^{(k)}$ defined on R_k , a real Siegel domain $P^{(k)} = P\left(V^{(k-1)},\varphi^{(k)}\right) \subset R_{11} \times R_2 \times \ldots \times R_k$

and the cone

$$V^{(k)} = V(P^{(k)}) \subset R_{11} \times R_2 \times R_{22} \times \ldots \times R_k \times R_{kk} .$$

It follows from the results of [G] that every homogeneous cone V which contains no straight line can be decomposed in the form $V^{(l)}$ (up to a linear isomorphism). The required number of steps to obtain V in this form is called the rank *l* of V, $V = V^{(l)}$; this yields the following decomposition of the space R^n that contains V :

(4) $$R^n = R_{11} \times R_2 \times R_{22} \times \ldots \times R_l \times R_{ll} , \quad n = l + \sum_{i=2}^{l} n_i .$$

Furthermore, the projection $\varphi_{ii}^{(k)}$ of $\varphi^{(k)}$ onto R_{ii} $(i < k)$ is a non-negative form. Thus $\varphi_{ii}^{(k)}$ is positive definite on a subspace R_{ik} of R_k with dim $R_{ik} = n_{ik}$. We then have:

(5) $$R_k = \prod_{i=1}^{k-1} R_{ik} , \; n_k = \sum_{i=1}^{k-1} n_{ik} .$$

We denote by G(V) the simply transitive group of linear transformations of V described in [G]. With respect to its canonical decomposition, the cone V can be described in the following quantitative manner : let $x \in V$ and let x_j , $j = 2,...,l$ (resp. x_{ii} , $i = 1,...,l$) denote the projection of x onto R_j (resp. R_{ii}) ; then there exists a unique transformation h $\in$ G(V) such that $(h(x))_j = 0$, $j = 1,...,l$. We set $\tilde{x} = h(x)$. The functions χ_j defined for $j = 1,...,l$, by $\chi_j(x) = \tilde{x}_{jj}$, $j = 1,...,l$, define the cone V in the following sense : a point x of R^n belongs to V if and only if $\chi_j(x) > 0$, $j = 1,...,l$.

Since the decomposition (4) of R^n yields in a natural way the following decomposition of C^n :

$$C^n = C_{11} \times C_2 \times C_{22} \times \ldots \times C_l \times C_{ll} ,$$

the functions χ_j , j = 1,...,l , can naturally be extended as rational functions on C^n .

Let $\rho = (\rho_1, \ldots, \rho_l) \in C^l$; we define the rational function $(z)^\rho$ on C^n by :

$$(z)^\rho = \prod_{j=1}^{l} \left(\chi_j(z)\right)^{\rho_j} \ , z \in C^n.$$

For i = 1,...,l , set $m_j = \sum_{i>j} n_{ji}$ and $d_i = -(1 + \frac{n_i + m_i}{2})$, and let d denote the vector of R^l whose components are d_i . In the sequel, e will denote the point of V whose components are $e_{ii} = 1$, $e_j = 0$, i = 1,...,l , j = 2,...,l

Let us recall the definition of the conjugate cone V^* of V. Consider the inner product < , > defined on R^n with respect to the canonical decomposition of R^n by :

$$<x,y> = \sum_i x_{ii} y_{ii} + 2 \sum_{i<j} \varphi_{ii}^{(j)} (x_i, y_j) .$$

Then V^* is defined by :

$$V^* = \left\{ x \in R^n : \ < x, y > \ > 0, \forall y \in \overline{V} - \{0\} \right\} .$$

The adjoint group $G^*(V)$ of G(V) with respect to < , > is the simply transitive group of linear transformations of V^*. The cone V^* is an irreducible, convex, homogeneous cone which contains no straight line, and it is also of rank l .

We shall denote by χ_j^* the defining functions of V^* . We have the following :

$$n_{ij}^* = n_{ij}(V^*) = n_{l-j+1,l-i+1} \qquad (1 \le i < j \le l).$$

For $\rho \in C^l$, we define ρ^* by $\rho_i^* = \rho_{l-i+1}$, i = 1,...,l , and we also define the function

$(z)_*^{\rho^*}$ on C^n by :

$$(z)_*^{\rho^*} = \prod_{j=1}^{l} \left(\chi_j^*(z)\right)^{\rho_j^*} .$$

The Siegel domain of type II, associated with the homogeneous cone V of R^n and a V-Hermitian, homogeneous form $F : C^m \times C^m \to C^n$, is defined by :

$$D=D(V,F) = \left\{(z,u) \in C^n \times C^m \;:\; \frac{z-\bar{z}}{2i} - F(u,u) \in V\right\} .$$

The domain D is then an affine-homogeneous domain. Let F_{ii} denote the projection of F onto C_{ii}, and $C^{(i)}$ the complex subpace of C^m on which F_{ii} is positive definite. Set $q_i = \dim_C C^{(i)}$; then $m = \sum_{i=1}^{l} q_i$ and $C^m = \prod_{i=1}^{l} C^{(i)}$. We shall denote by q the vector of N^l whose components are q_i, $i = 1,...,l$.

We now recall the following two expressions of the Bergman kernel B((ζ,v),(z,u)) of D=D(V,F) :

II.1 PROPOSITION [G] . The Bergman kernel B((ζ,v),(z,u)) of D is given by

$$\begin{aligned} B\big((\zeta,v),(z,u)\big) &= c\left(\frac{\zeta-\bar{z}}{2i} - F(v,u)\right)^{2d-q} \\ &= c\int_{V^*} \exp\left(-<\lambda, \frac{\zeta-\bar{z}}{2i} - F(v,u)>\right)(\lambda)_*^{-d^*+q^*} d\lambda. \end{aligned}$$

NOTATIONS. For $\alpha = (\alpha_1,\ldots,\alpha_l) \in R^l$, the notation $1+\alpha$ stands for the vector $(1+\alpha_1,\ldots,1+\alpha_l)$. Let $\alpha' = (\alpha'_1,\ldots,\alpha'_l) \in R^l$; we set $\alpha\alpha' = (\alpha_1\alpha'_1,\ldots,\alpha_l\alpha'_l)$ and by

$\alpha > \alpha'$, we mean that $\alpha_i > \alpha'_i$ for all $i \in \{1,...,l\}$. For (ζ,v) and (z,u) in $D \subset C^n \times C^m$, we let b denote the kernel

$$b((\zeta,v),(z,u)) = \left(\frac{\zeta-\bar{z}}{2i} - F(v,u)\right)^{2d-q} .$$

Notice that $B = cb$. Moreover, b^{α} and $b^{1+\alpha}$, $\alpha \in R^l$, will denote the expressions :

$$b^{\alpha}((\zeta,v),(z,u)) = \left(\frac{\zeta-\bar{z}}{2i} - F(v,u)\right)^{(2d-q)\alpha}$$

and

$$b^{1+\alpha}((\zeta,v),(z,u)) = \left(\frac{\zeta-\bar{z}}{2i} - F(v,u)\right)^{2d-q+(2d-q)\alpha} .$$

Let r be a vector of R^l . For $p \in (0,\infty)$, we set $L^{p,r}(D) = L^p(D, b^{-r}(z,z)dv(z))$ and define the weighted Bergman space $A^{p,r}(D)$ by $A^{p,r}(D) = L^{p,r}(D) \cap H(D)$. We equip $A^{p,r}(D)$ with the $L^{p,r}(D)$ - " norm" $\| \ \|_{p,r}$. The weighted Bergman projection P_r is the orthogonal projection of $L^{2,r}(D)$ onto $A^{2,r}(D)$. Recall (cf. [BT]) that $A^{2,r}(D) = \{0\}$ when $r_i \le \frac{n_i+2}{2(2d-q)_i}$ for some $i \in \{1,...,l\}$ and otherwise, P_r is equal to the integral operator defined on $L^{2,r}(D))$ by the weighted Bergman kernel $c_r\, b^{1+r}((\zeta,v),(z,u))$.

Let us now state some prerequisite results :

II.2 THEOREM [BT]. Let α and ε be in R^l and $(\zeta,v) \in D$. Then we have :

$$\int_D \left| b^{1+\alpha}((\zeta,v),(z,u)) \right| b^{-\varepsilon}((z,u),(z,u))dv(z,u) < \infty$$

if and only if $\varepsilon_i > \frac{n_i + 2}{2(2d-q)_i}$ and $\alpha_i - \varepsilon_i > \frac{n_i}{-2(2d-q)_i}$, i = 1,...,*l*. In this case, the following equality holds :

$$\int_D \left| b^{1+\alpha}((\zeta,v),(z,u)) \right| b^{-\varepsilon}(z,u),(z,u))d\nu(z,u) = c_{\alpha,\varepsilon} b^{\alpha-\varepsilon}((\zeta,v),(\zeta,v)) .$$

We shall need the following reproducing formulas which, indeed, improve those obtained in [BT] (Theorem II.6.1, p.225), and whose proof, based on ideas of [BBR], will be given in the appendix:

II.3 THEOREM. Let r be a vector of R^l such that $r_i > \frac{n_i + 2}{2(2d-q)_i}$ for all i = 1,...,*l* and p a real number such that $1 \leq p < \min \left\lfloor \frac{n_i - 2(2d-q)_i(1+r_i)}{n_i} \right\rfloor$. Then for all $\varepsilon \in R^l$ such that $\varepsilon_i > \frac{n_i + 2}{2(2d-q)_i}\frac{p-1}{p} + \frac{r_i}{p}$ (i = 1,...,*l*), the reproducing formula $P_\varepsilon f = f$ holds for all f $\in A^{p,r}(D)$.

II.4 PROPOSITION [BT]. Let $\alpha \in R^l$ be such that $\alpha_i \geq 0$, i = 1,...,*l* . Then :

$$\left| b^{\alpha}((\zeta,v),(z,u)) \right| \leq c_\alpha b^{\alpha}((\zeta,v),(\zeta,v))$$

and

$$\left| b^{\alpha}\left((\zeta,v)+(\zeta',v'),(z,u)+(z',v')\right) \right| \leq c_\alpha b^{\alpha}((\zeta,v),(\zeta,v)) ,$$

for all (ζ,v), (ζ',v'), (z,u) and (z',u) in D.

II.5 LEMMA [R] . For all $f \in A^{p,r}(D)$ (p > 0) , we have the estimate :

The duals of Bergman spaces

$$|f(z,u)|^p \le cb^{1+r}((z,u),(z,u))\|f\|_{p,r}^p .$$

DEFINITION : A vector $\rho \in R^l$ is a V-integral vector if $(\lambda)_*^{\rho^*}$ is a polynomial in λ.

In the sequel, ρ will be a V-integral vector. We associate to ρ the differential polynomial $(\Lambda_\rho)_\zeta$ in C^n in the following way : for $\lambda \in C^n$,

$$(\Lambda_\rho)_\zeta \exp(<\lambda,\zeta>) = (\lambda)_*^{\rho^*} \exp(<\lambda,\zeta>) \qquad (\zeta \in C^n).$$

Let us now recall the following lemma due to F. Trèves [Tr] which is crucial in our work.

II.6 LEMMA [Tr]. For each holomorphic function G in D, there exists a holomorphic function g in D such that $(\Lambda_\rho)_\zeta\, g(\zeta,v)=G(\zeta,v)$ for all $(\zeta,v) \in D$.

DEFINITION : A function $g \in H(D)$ is a Bloch function if :

$$\|g\|_* = \sup_{(z,u)\in D}\left\{ \left|(\Lambda_\rho)_z g(z,u)\right| b^{-\frac{\rho}{-2d+q}}((z,u),(z,u))\right\} < \infty .$$

Set $\mathcal{N}$ = $\{g \in H(D) : (\Lambda_\rho)g \equiv 0\}$. We define the Bloch space $\mathcal{B}_\rho$ and the space $\mathcal{C}_{\rho,r}^p$ (D) in the following manner, where we set $\sigma = \frac{\rho}{-2d+q}$:

$$\mathcal{B}_\rho = \{\text{Bloch functions in D}\}/\mathcal{N} ,$$

while $\mathcal{C}_{\rho,r}^p$ (D) is the quotient space:

$$\left\{ f \in H(D) : \|f\|_{\mathcal{C}^p_{\rho,r}} = \left(\int_D \left| (\Lambda_\rho)_z f(z,u) \right|^p b^{-p\sigma - r}((z,u),(z,u)) d\nu(z,u) \right)^{\frac{1}{p}} < \infty \right\} / \mathcal{N}.$$

These two spaces have the following topological property:

LEMMA : ($\mathcal{B}_\rho$, $\|\cdot\|_*$) and ($\mathcal{C}^p_{\rho,r}$ (D) , $\|\cdot\|_{\mathcal{C}^p_{\rho,r}(D)}$) are complex Banach spaces.

Our first two results read as follows:

II.7 THEOREM. Let D be a homogeneous Siegel domain of type II. Let p be a real number and r a vector of R^l such that $0 < p \leq 1$ and $r_i > \frac{n_i + 2}{2(2d - q)_i}$, i = 1,...,l . Then the following assertions hold:

(i) $\mathcal{B}_\rho$ is isomorphic to a subspace of $(A^{p,r}(D))^*$,

(ii) $\mathcal{B}_\rho$ is equal to $A^{1,r}(D))^*$) if $\rho_i > \frac{n_i}{2}, i = 1,..., l,$ with equivalent norms.

The duality $(A^{1,r}(D)), \mathcal{B}_\rho$) is given by

$$(f,g) = \int_D \overline{(\Lambda_\rho)_z g(z,u)} b^{-\sigma}((z,u),(z,u)) f(z,u) b^{-r}((z,u),(z,u)) d\nu(z,u) ,$$

where $\sigma = \frac{\rho}{-2d + q}$.

II.8 THEOREM : Let $D \subset \mathbb{C}^N$ be either a symmetric Siegel domain of type II, or $D = D_0$, or $D = D_1$. Let $p_1 = \frac{2N}{2N+1}$ and let r be a vector of R such that $r_i > \frac{n_i + 2}{2(2d-q)_i}$, i = 1,...,$l$.

Then for all $p \in (p_1, 1)$ and all $\rho \in R^l$ such that

$$\rho_i > -\frac{n_i + 2}{2} + (-2d+q)_i + \frac{1+2r_i}{p}(-2d+q)_i, \quad i = 1,\dots,l,$$

we have $\mathcal{B}_\rho = (A^{p,r}(D))^*$, with equivalent norms.

Moreover, the duality $(A^{p,r}(D)), \mathcal{B}_\rho)$ is given by

$$(f,g) = \int_D \overline{(\Lambda_\rho)_z\, g(z,u)}\, b^{-\sigma}((z,u),(z,u))\, f(z,u)\, b^{1-\frac{1+r}{p}}((z,u),(z,u))\, dv(z,u),$$

with $\sigma = \frac{\rho}{-2d+q}$.

REMARK: In the symmetric case, R.R. Coifman and R. Rochberg ([CR], p. 43-44) stated that $A^{p,r}(D))$ and $A^{1,-1+\frac{1+r}{p}}(D)$ have the same dual. The proof of Theorem II.8 relies on their atomic decomposition theorem . For a proof of the atomic decomposition theorem, cf. [BT_1].

Theorems II.7 and II.8 will be proved in section III.

Let $h \in L^\infty(D)$ and let ρ be a V-integral vector such that $\rho_i > \frac{n_i}{2}$ (i = 1,...,l) . Then, in view of Theorem II.2, the following function G is holomorphic in D:

$$G(\zeta, v) = \int_D b^{1+\sigma}((\zeta,v),(z,u))\, h(z,u)\, dv(z,u),$$

where $\sigma = \frac{\rho}{-2d+q}$. By Lemma II.6, there exists $\tilde{g} \in H(D)$ such that $(\Lambda_\rho)_z \tilde{g}(z,u) = G(z,u)$. Let g be the equivalence class of all holomorphic solutions of this equation. Then g $\in \mathcal{B}_\rho$; hence, we can define an operator P from $L^\infty(D)$ into $\mathcal{B}_\rho$ in the following way :

Ph = g.

P is called the "Bergman projection" of $L^\infty(D)$ into $\mathcal{B}_\rho$. Let us justify this name : the Bergman projection $\mathcal{P}$ is usually the integral operator defined on $L^2(D)$ by :

$$\mathcal{P}h(\zeta,v) = \int_D B((\zeta,v),(z,u))h(z,u)d\nu(z,u) \qquad (h \in L^2(D)).$$

Now, let h $\in L^2 \cap L^\infty(D)$; since $(\Lambda_\rho)_\zeta b((\zeta,v),(z,u)) = c_\rho b^{1+\sigma}((\zeta,v),(z,u))$ (recall $\sigma = \frac{\rho}{-2d+q}$), one easily obtains that $\mathcal{P}h$ is a representative of the element Ph = g of $\mathcal{B}_\rho$.

Our third result reads as follows:

II.9 THEOREM. Let D be a homogeneous Siegel domain of type II. Let ρ be a V-integral vector such that $\rho_i > \frac{n_i}{2}$ (i=1,...,*l*). Then $PL^\infty(D) = \mathcal{B}_\rho$ and P has a bounded right inverse.

Theorem II.9 will be proved in section IV .

Our fourth result reads as follows :

II-10 THEOREM: Let $D_2 = \left\{(z,u) \in M_2 \times M_{r2} \;\middle|\; \frac{z - z^*}{2i} - u^* u \in V\right\}$ and $\rho = (5,5)$. Then there exists a kernel b_0 in D_2 such that :

(i) with respect to (ζ , v) , $b_0 ((\zeta , v) , (z ,u))$ is holomorphic in D_2 and $(\Lambda_\rho)_\zeta\, b_0 ((\zeta , v),(z,u)) \equiv 0$,

(ii) for all $(\zeta , v) \in D_0$, $(b- b_0) ((\zeta , v) , (z ,u)) \in L^1 (D_2 , d\nu (z,u))$.

Hence, for each $h \in L^\infty(D_2)$, the function $\boldsymbol{\wp} h$ defined on D_2 by

$$\boldsymbol{\wp} h(\zeta, v) = \int_{D_2} \left(b - b_0\right)\left((\zeta, v), (z, u)\right) h(z, u) d\nu(z, u)$$

is a representative of Ph.

This result will be proved in section V.

Our fifth result focuses on the case p > 1 and reads as follows:

II-11 THEOREM : Let D be a homogeneous Siegel domain of type II. Let p be a real number and r a vector of $\mathbb{R}^l$ such that $r_i > \frac{n_1 + 2}{2(2d - q)_i}$ $(i = 1,..., l)$ and

$$1 < p < \min_i \left\{ \frac{2n_i + 2 - 2(2d - q)_i r_i}{n_i} \right\}.$$

Then we have the following assertions.

(i) If $\max_i \left\{ \frac{2n_i + 2 - 2(2d - q)_i r_i}{n_i + 2 - 2(2d - q)_i r_i} \right\} < p < \min_i \left\{ \frac{2n_i + 2 - 2(2d - q)_i r_i}{n_i} \right\}$, then $(A^{p,r}(D))^* = A^{p',r}(D)$ (p' is the conjugate exponent of p).

(ii) $(A^{p,r}(D))^* = \mathcal{C}^{p'}_{\rho,r}(D)$ whenever $\rho_i > \frac{n_i}{2}$, i = 1,...,l, with equivalent norms.

The duality $(A^{p,r}(D), \mathcal{C}^{p'}_{\rho,r}(D))$ is given by

$$(f,h) = \int_D b^{-\sigma}((z,u),(z,u))\overline{\left(\Lambda_\rho\right)_z h(z,u)} f(z,u) b^{-r}((z,u),(z,u))d\nu(z,u),$$

where $\sigma = \frac{\rho}{-2d+q}$.

This theorem will be proved in section VI.

III. PROOFS OF THEOREMS II.7 AND II.8.

III.1 PROOF OF THEOREM II.7

(i) Let $g \in \mathcal{B}_\rho$ and consider the linear functional φ defined on $A^{p,r}(D)$ by :

$$\varphi(f) = \int_D \overline{\left(\Lambda_\rho\right)_\zeta g(\zeta,v)} b^{-\sigma}((\zeta,v),(\zeta,v)) f(\zeta,v) b^{1-\frac{1+r}{p}}((\zeta,v),(\zeta,v))$$

where $\sigma = \frac{\rho}{-2d+q}$ and $0 < p \le 1$. By Lemma II.5, we have the estimate :

$$\int_D |f(\zeta,v)| b^{1-\frac{1+r}{p}}((\zeta,v),(\zeta,v)d\nu(\zeta,v))$$
$$\le c\left(\int_D |f(\zeta,v)|^p b^{-r}((\zeta,v),\zeta,v))\right)d\nu(\zeta,v)\|f\|^{1-p}_{p,r}$$
$$\le c\|f\|_{p,r}.$$

Hence φ possesses the following property:

$|\varphi(f)| \le c\|g\|_*\|f\|_{p,r}$,

and thus $\|\varphi\| \le c\|g\|_*$. Therefore $\varphi \in (A^{p,r}(D))^*$. This proves the inclusion of $\mathcal{B}_\rho$ into

(ii) Let p = 1, and let φ be in $(A^{1,r}(D))^*$. Then by the Hahn-Banach theorem, there exists a bounded function k in D such that

$$\varphi(f) = \int_D \overline{k(z,u)} f(z,u) b^{-r}\big((z,u),(z,u)\big) d\nu(z,u).$$

On the other hand, by Theorem II.3, we have the following reproducing formula for every f $\in A^{1,r}(D)$:

$$f(z,u) = \int_D b^{1+r+\sigma}\big((z,u),(\zeta,v)\big) f(\zeta,v) b^{-r-\sigma}\big((\zeta,v),(\zeta,v)\big) d\nu(\zeta,v).$$

Therefore by the Fubini theorem, we easily get

$$\varphi(f) = \int_D \left(\int_D b^{1+r+\sigma}\big((\zeta,v),(z,u)\big) \overline{k(\zeta,v)} b^{-r}\big((\zeta,v),(\zeta,v)\big) d\nu(\zeta,v) \right)$$
$$f(z,u) b^{-r-\sigma}\big((z,u),(z,u)\big) d\nu(z,u).$$

Set g(z,u) = $\int_D b^{1+r+\sigma}\big((z,u),(\zeta,v)\big) k(\zeta,v) b^{-r}\big((\zeta,v),(\zeta,v)\big) d\nu(\zeta,v)$. Observe next that $g \in H(D)$ and that by Theorem II.2, the following estimate holds :

$$|g(z,u)| \leq C \|k\|_\infty b^{\sigma}\big((z,u),(z,u)\big) \qquad \big((z,u) \in D\big),$$

since $\sigma_i > \frac{n_i}{-2(2d-q)_i}, i = 1,\ldots,l.$ Then in view of Lemma II.6, there exists h in $\mathcal{B}_\rho$ such that $\Lambda_\rho h = g$. Therefore

$$\varphi(f) = \int_D \overline{(\Lambda_\rho)_z h(z,u)} f(z,u) b^{-r-\sigma}\big((z,u),(z,u)\big) d\nu(z,u)$$

and thus, φ is represented by the element g of $\mathcal{B}_\rho$. This proves the reverse inclusion of $(A^{1,r}(D))^*$ in $\mathcal{B}_\rho$. □

We shall now prove another reproducing formula whose proof relies on the dominated convergence theorem. In view of this formula, the Bergman weighted projection P_r also reproduces functions which satisfy certain uniform estimate.

III.2 PROPOSITION : Let r and ε be two vectors of R^l such that $\varepsilon_i > \frac{n_i}{-2(2d-q)_i}$ and $r_i > \frac{n_i+2}{2(2d-q)_i} + \varepsilon_i$, $i = 1,...,l$.. Let G be in H(D) such that $\sup_{z \in D} \left\{ |G(z)| b^{-\varepsilon}(z,z) \right\} < \infty$. Then $P_r G = G$.

PROOF : Consider the sequence $\{G_n\}$ defined by

$$G_n(z) = G\left(z + \frac{ie}{n}\right) b^{\alpha}\left(\frac{z}{n}, ie\right),$$

where the positive exponent α is to be specified later. There exists c > 0 such that for every z in D and every positive integer n :

$$\left| G\left(z + \frac{ie}{n}\right) \right| \leq c b^{\varepsilon}\left(z + \frac{ie}{n}, z + \frac{ie}{n}\right)$$
$$\leq c b^{\varepsilon}\left(\frac{ie}{n}, \frac{ie}{n}\right),$$

where the latter inequality is yielded by Proposition II.4 since $\varepsilon_i > 0$. Then , when we keep n fixed, the function $z \mapsto G\left(z + \frac{ie}{n}\right)$ is bounded and the following inequality holds :

$$\int_D |G_n(z)|^2 b^{-r}(z,z) d\nu(z) \leq C_n \int_D \left| b^{2\alpha}\left(\frac{z}{n}, ie\right) \right| b^{-r}(z,z) d\nu(z).$$

We choose α so that $2\alpha_i - 1 - r_i > \frac{n_i}{-2(2d-q)_i}$ for all i = 1,...,l. Hence by Theorem II.2, the latest integral converges and furthermore, $P_r\, G_n = G_n$ for all n .

On the other hand, by Proposition II.4, since $\alpha_i > 0$ and $\varepsilon_i > 0$, we have :

$$\left|G_n(z)\right| b^{-\varepsilon}(z,z) \leq \left|G(z + \frac{ie}{n}\right| b^{-\varepsilon}(z,z) \leq c.$$

Also, by Theorem II.2, under our assumptions $\varepsilon_i > \frac{n_i}{-2(2d-q)_i}$ and $r_i > \frac{n_i + 2}{2(2d-q)_i} + \varepsilon_i$, $i = 1,...,l$, we get :

$$\int_D \left|b^{1+r}(z,\zeta)\right| b^{-r+\varepsilon}(\zeta,\zeta) d\nu(\zeta) < \infty.$$

Henceforth, by the dominated convergence theorem, we easily obtain the equality $P_r G = G$.

REMARK : This result extends the one obtained in [B_2] for symmetric Siegel domains of type II, via the bounded circular realization of the domain. Our proof is straight and includes all homogeneous Siegel domains of type II.

III.3 COROLLARY : Let p $\in$ (0, 1] . Let α and r be vectors of R^l such that $\alpha_i > \frac{n_i}{-2(2d-q)_i} + 1 + r_i$ and $r_i > \frac{n_i + 2}{2(2d-q)_i}$ for all $i = 1,...,l$, Then for all $\varphi \in (A^{p,r}(D))^*$ and for all z_0 in D, we have

$$\int_D b^{\frac{\alpha}{p}}(z_0,z)\varphi\overline{\left(b^{\frac{\alpha}{p}}(.,z)\right)} b^{1-\frac{\alpha}{p}}(z,z) d\nu(z) = \varphi\overline{\left(b^{\frac{\alpha}{p}}(.,z_0)\right)}.$$

PROOF : By Proposition III.2, it is sufficient to show that

$$(6) \qquad \sup_{z\in D} \left|\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right)\right| b^{\frac{1+r-\alpha}{p}}(z,z) < \infty .$$

But since $\varphi \in (A^{p,r}(D))^*$, we obtain that

$$\left|\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right)\right| \le c \left\| b^{\frac{\alpha}{p}}(.,z)\right\|_{p,r} \le c' b^{\frac{\alpha-1-r}{p}}(z,z),$$

where the latest estimate follows from Theorem II.2. This proves Corollary III.3 . □

III.4 PROOF OF THEOREM II.8

Let $D \subset C^N$ be a symmetric Siegel domain of type II, or let D be equal to D_0 or D_1 . Remark that for $p \in \left(\frac{2N}{2N+1},1\right)$ and $r \in R^l$ satisfying $r_i > \frac{n_i+2}{2(2d-q)_i}$, the domain D satisfies the hypotheses of the molecular decomposition theorem [BT_1] for functions in Bergman spaces $A^{p,r}$ (D) ; more precisely, for such values of p and r, there exist constants c = c(p,r) and C = C(p,r) such that for every $f \in A^{p,r}(D)$, there exists an l^p -sequence $\{\lambda_i\}$ such that

$$f(z) = \sum_{i=0}^{\infty} \lambda_i b^{\frac{\alpha}{p}}(z,z_i) b^{\frac{1+r-\alpha}{p}}(z_i,z_i) \qquad (z\in D) \quad ,$$

where $\{z_i\}$ is a lattice in D and the following estimate holds :

$$c\|f\|^p_{p,r} \le \sum |\lambda_i|^p < C\|f\|^p_{p,r} \qquad .$$

Moreover, the vector α of R^{l} is defined as follows. Let ρ denote a V-integral vector in R^{l} such that when we set $\sigma = \frac{\rho}{-2d+q}$, the following condition is satisfied:

$$\sigma_i > \frac{n_i + 2}{2(2d-q)_i} + 1 + \frac{1+2r_i}{p} \qquad i = 1,\dots,l.$$

The vector α is given by $\alpha = \sigma p+1+r$.

Let $\varphi \in (A^{p,r}(D))^*$. For $f \in A^{p,r}(D)$, define the sequence $\{f_N\}$ of functions in D by

$$(7) \qquad f_N(z) = \sum_{i=0}^{N} \lambda_i b^{\frac{\alpha}{p}}(z,z_i) b^{\frac{1+r-\alpha}{p}}(z_i,z_i) \qquad (z \in D).$$

Then $\{f_N\}$ converges to f in $A^{p,r}(D)$. Hence $\varphi(f_N)$ goes to $\varphi(f)$ as N tends to infinity. Now, in view of Corollary III.3, we get :

$$\overline{\varphi\left(b^{\frac{\alpha}{p}}(.,z_i)\right)} = \int_D b^{\frac{\alpha}{p}}(z_i,z)\overline{\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right)} b^{1-\frac{\alpha}{p}}(z,z)dv(z)) \qquad (i \in Z_+),$$

and combining this with (7) yields for every positive integer N :

$$\overline{\varphi(f_N)} = \int_D \overline{f_N(z)}\,\overline{\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right)} b^{1-\frac{\alpha}{p}}(z,z)dv(z)..$$

This easily implies that for every positive integer N :

$$(8) \qquad \varphi(f_N) = \int_D f_N(z)\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right) b^{1-\frac{\alpha}{p}}(z,z)dv(z).$$

Now, let N tend to infinity in identity (8). On the one hand, $\varphi(f_N)$ goes to $\varphi(f)$. On the other hand, it follows from (6) that

$$\left|\int_D (f_N - f)(z)\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right) b^{1-\frac{\alpha}{p}}(z,z)d\nu(z)\right| \leq C\int_D \left|(f_N - f)(z)\right| b^{1-\frac{1+r}{p}}(z,z)d\nu(z)$$

$$\leq C'\int_D \left|(f_N - f)(z)\right|^p b^{-r}(z,z)d\nu(z)$$

where the latest inequality follows from Lemma II.5. Henceforth, since $\{f_N\}$ converges to f in $A^{p,r}(D)$, we conclude that the right hand side of (8) tends to $\int_D f(z)\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right) b^{1-\frac{\alpha}{p}}(z,z)d\nu(z))$. We have then proved that for every f in $A^{p,r}(D)$:

$$\varphi(f) = \int_D f(z)\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right) b^{1-\frac{\alpha}{p}}(z,z)d\nu(z).$$

Now,by Lemma II.6, since the function $z \mapsto \overline{\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right)}$ is holomorphic in D, there exists g in H(D) such that $\Lambda_\rho g(z) = \overline{\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right)}$. Furthermore, with the choice of α and with $\sigma = \frac{\alpha - 1 - r}{p}$, we deduce from (6) that for every $z \in D$, the following estimate holds :

$$\sup_{z\in D}\left|\varphi\left(b^{\frac{\alpha}{p}}(.,z)\right)\right| b^{-\sigma}(z,z) < \infty.$$

Hence g is a Bloch function in D, and g clearly represents the bounded linear functional $\varphi\in(A^{p,r}(D))^{*}$ in the following manner :

$$\varphi(f) = \int_D \overline{\Lambda_\rho g(z)}\, b^{-\sigma}((z),(z)) f(z) b^{1-\frac{1+r}{p}}(z,z)d\nu(z).$$

Moreover, $\|g\|_{*} \le c\|\varphi\|$. This proves the inclusion of $(A^{p,r}(D))^{*}$ in $\mathcal{B}_\rho$. Theorem II.8 is entirely proved. □

IV. PROOF OF THEOREM II.9.

Since $PL^{\infty}(D) \subset \mathcal{B}_\rho$, let us prove that $\mathcal{B}_\rho \subset PL^{\infty}(D)$. Let $g \in \mathcal{B}_\rho$ and set $h(\zeta,v) = \left(\Lambda_\rho\right)_\zeta g(\zeta,v) b^{-\sigma}\left((\zeta,v),(\zeta,v)\right)$ with $\sigma = \dfrac{\rho}{-2d+q}$. Then $h \in L^{\infty}(D)$. We are going to show that $Ph = g$. It suffices to prove the equality :

$$\left(\Lambda_\rho\right)_\zeta g(\zeta,v) = \int_D b^{1+\sigma}\left((\zeta,v),(z,u)\right)\left(\Lambda_\rho\right)_\zeta g(z,u) b^{-\sigma}\left((z,u),(z,u)\right)d\nu(z,u).$$

Setting $\varepsilon = \sigma$, this equality follows from Proposition III.2. To complete the proof, let us determine a right inverse of P. Define R as follows:

$$Rg(\zeta,v) = \left(\Lambda_\rho\right)_\zeta g(\zeta,v) b^{-\sigma}\left((\zeta,v),(\zeta,v)\right) \qquad \left(g\in\mathcal{B}_\rho\right).$$

Then $PRg = g$ and $Rg \in L^{\infty}(D)$. Furthemore $\|Rg\|_{L^{\infty}(D)} \le \|g\|_{*}$ and thus $\|R\| \le 1$. This completes the proof of Theorem II.9. □

V. PROOF OF THEOREM II.10.

V.1 Preliminaries on D_2.

The cone $V = \{z \in M_2 , z \text{ Hermitian} , z > 0\}$ is of rank 2 and linearly equivalent to the spherical cone in R^4 ; moreover, it is self-conjugate with respect to the inner product

$$< \zeta ,z > = \zeta_{11} z_{11} + \zeta_{12} z_{12} + \zeta_{21} z_{21} + \zeta_{22} z_{22} ,$$

where $\zeta = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}$ and $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ belong to M_2. Set $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We also have $n_{12} = 2$, $n_1 = 0$, $n_2 = 2$, $d = (-2,-2)$. The functions that define the cone V are defined by

$$\chi_1(\zeta) = \zeta_{11} - \frac{\zeta_{12}\zeta_{21}}{\zeta_{22}} \quad , \quad \chi_2(\zeta) = \zeta_{22} \qquad\qquad (\zeta \in M_2).$$

Let $F: M_{r,2} \times M_{r,2} \to M_2$ be defined by $F(v,u) = u^* v$ where $u = (u_1, u_2)$ and $v = (v_1 , v_2)$ are in $M_{r,2}$ with v_1 ,v_2 in C^r (i = 1,2) . Hence $F_{11}(v,u) = u_1^* v_1$, $F_{22}(v,u) = u_2^* v_2$ and it is clear that F_{ii} is concentrated on $C^r \times C^r$ and is positive definite on C^r (i = 1,2). Therefore q = (r,r).

Let $\rho = (5,5)$ be a V-integral vector. It is straightforward that

$$\left(\Lambda_\rho\right)_\zeta = \left(\frac{\partial^2}{\partial\zeta_{11}\partial\zeta_{22}} - \frac{\partial^2}{\partial\zeta_{12}\partial\zeta_{21}} \right)^5$$

and

$$\left(\Lambda_\rho\right)_\zeta B\big((\zeta,v),(z,u)\big) = c_\rho \ B^{1+\frac{\rho}{-2d+q}}\big((\zeta,v),(z,u)\big) ,$$

where B is the Bergman kernel of D_2. It then follows from Theorem II.2 that $\left(\Lambda_\rho\right)_\zeta B\big((\zeta,v),(z,u)\big)$ is integrable with respect to (z,u) in D_2 since $\rho = (\rho_1, \rho_2)$, $\rho_1 > 0$, $\rho_2 > 1$. We are now ready to state the following lemma whose proof is just computational.

V.2 LEMMA : Let α_1 and α_2 be two integers such that $\alpha_1 \in \{-4,-3,-2,-1,0\}$ or $\alpha_2 \in \{-3,-2,-1,0,1\}$. Then $\left(\Lambda_\rho\right)_\zeta \chi_2^{-\alpha_2}\chi_1^{-\alpha_1}\left(\frac{\zeta - z^*}{2i} - u^* v\right) \equiv 0$, where (z,u) and (ζ,v) belong to D_2.

The proof of the next lemma is somewhat lengthy and will be given in the appendix.

V.3. LEMMA : The following functions are integrable in D_2 :

(a) $\quad |u_2|^4 \chi_1^{-(5+r)}\chi_2^{-(8+r)}\left(\frac{ie - z^*}{2i}\right)$;

(b) $\quad |u_2|^5 \chi_1^{-(5+r)}\chi_2^{-(9+r)}\left(\frac{ie - z^*}{2i}\right)$;

(c) $\quad |u_2|^4 z_{12}\chi_1^{-(5+r)}\chi_2^{-(9+r)}\left(\frac{ie - z^*}{2i}\right)$;

(d) $\quad |u_2|^4 z_{21}\chi_1^{-(5+r)}\chi_2^{-(9+r)}\left(\frac{ie - z^*}{2i}\right)$;

(e) $\quad |u_2|^5 z_{21}\chi_1^{-(5+r)}\chi_2^{-(9+r)}\left(\frac{ie - z^*}{2i}\right)$;

(f) $\quad |u_1||u_2|^4 \chi_1^{-(5+r)} \chi_2^{-(9+r)}\left(\frac{ie-z^*}{2i}\right)$;

(g) $\quad |u_1||u_2|^5 \chi_1^{-(5+r)} \chi_2^{-(9+r)}\left(\frac{ie-z^*}{2i}\right)$;

(h) $\quad |u_1||u_2|^4 \chi_1^{-(5+r)} \chi_2^{-(9+r)}\left(\frac{ie-z^*}{2i}\right)$.

We shall also use the following lemma.

V.4. LEMMA : Let (ζ, v) be in D_0. Then there exists two positive constants c_1 and c_2 such that for all $(z,u) \in D_2$, the following estimates hold :

a) $\quad c_1\left|\chi_2\left(\frac{ie-z^*}{2i}\right)\right| \le \left|\chi_2\left(\frac{\zeta-z^*}{2i}-u^*v\right)\right| \le c_2\left|\chi_2\left(\frac{ie-z^*}{2i}\right)\right|$;

b) $\quad c_1\left|\chi_1\left(\frac{ie-z^*}{2i}\right)\right| \le \left|\chi_1\left(\frac{\zeta-z^*}{2i}-u^*v\right)\right| \le c_2\left|\chi_1\left(\frac{ie-z^*}{2i}\right)\right|$.

PROOF : This is a straightforward consequence of the following lemma due to A. Koranyi:

LEMMA ([CR] , cf. also [BT_1]) Let D be a symmetric Siegel domain of type II and let d denote the Bergman distance on D_2. Then there exists a constant C_D such that for all ζ , z , z' in D , $\left|\frac{B(\zeta,z)}{B(\zeta,z')}-1\right| \le C_D d(z,z')$ whenever $d(z,z') \le 10$.

We are now ready to handle the proof of Theorem II.10 :

PROOF OF THEOREM II.10. We can define b_0 in the following manner :

$$(b-b_0)((\zeta,v),(z,u)) = \left(\chi_1^{-(4+r)}\left(\frac{\zeta-z^*}{2i}-u^*v\right)-\chi_1^{-(4+r)}\left(\frac{\zeta^{(1)}-z^*}{2i}-u^*v^{(1)}\right)\right)$$

$$\left[\chi_2^{-(4+r)}\left(\frac{\zeta-z^*}{2i}-u^*v\right)-\chi_2^{-(4+r)}\left(\frac{ie-z^*}{2i}\right)\right.$$

$$-(4+r)\chi_2^{-(5+r)}\left(\frac{ie-z^*}{2i}\right)\left(\chi_2^{-(4+r)}\left(\frac{\zeta-z^*}{2i}-u^*v\right)-\chi_2^{-(4+r)}\left(\frac{ie-z^*}{2i}\right)\right)$$

$$-\alpha\chi_1^{-(6+r)}\left(\frac{ie-z^*}{2i}\right)\left(\chi_2\left(\frac{ie-z^*}{2i}\right)-\chi_2\left(\frac{\zeta-z^*}{2i}-u^*v\right)\right)^2$$

$$\left.-\beta\chi_2^{-(7+r)}\left(\frac{ie-z^*}{2i}\right)\left(\chi_2\left(\frac{ie-z^*}{2i}\right)-\chi_2\left(\frac{\zeta-z^*}{2i}-u^*v\right)\right)^3\right],$$

where $\alpha = \frac{(4+r)(5+r)}{2}, \beta = \frac{(4+r)(5+r)(6+r)}{2}$ and $\left(\zeta^{(1)},v^{(1)}\right) = \left(\begin{pmatrix} i & 0 \\ 0 & \zeta_{22}\end{pmatrix}(0,v_2)\right).$

Then b_0 $((\zeta,v),(z,u))$ is a holomorphic function of (ζ,v) in D_2. Furthermore, in view of Lemma V.2 and the fact that $\chi_1\left(\frac{\zeta^{(1)}-z^*}{2i}-u^*v^{(1)}\right)$ does not depend on ζ_{11} , ζ_{21} and ζ_{12} , we have $\left(\Lambda_\rho\right)_\zeta b_0((\zeta,v),(z,u)) \equiv 0.$ This proves the first assertion.

Using the identity $a^{-(n+1)} - b^{-(n+1)} = (b-a)\sum_{k=0}^{n} a^{-(k+1)}b^{-(n+1-k)}$ and in view of Lemma V.4 , we have the estimate :

$$(b-b_0)((\zeta,v),(z,u))$$
$$\leq c\left|\chi_1^{-(5+r)}\chi_2^{-(8+r)}\left(\frac{ie-z^*}{2i}\right)\right|\left(\left(\left|\frac{\zeta_{22}-i}{2i}\right|+|u_2|^4|v_2|^2\right)\left|\frac{\zeta_{11}-i}{2i}\right|+|u_1||v_1|+\frac{1}{4}\left|\chi_2\left(\frac{ie-z^*}{2i}\right)\right|\right.$$
$$\left.\left(|\zeta_{12}\zeta_{21}|+|\zeta_{12}||z_{21}|+|\zeta_{21}z_{12}|+2|\zeta_{12}||v_2|+2|\zeta_{12}||u_2||v_1|+2|v_1||z_{21}||u_2|+4|u_2||u_1||v_1||v_2|\right)\right).$$

Hence in view of Lemma V.3 , $(b-b_0)$ $((\zeta,v),(z,u))$ is in L^1 $(D_2, d\nu(z,u))$. This completes the proof of Theorem II.10. □

VI. PROOF OF THEOREM II.11.

VI.I. THEOREM [BT] . Let ε and r be in R^l such that $\varepsilon_i > \frac{1}{(2d-q)_i}$ and

$r_i > \frac{n_i + 2}{2(2d-q)_i}$, $i=1,\ldots,l$. Then P_ε is bounded from $L^{p,r}$ (D) into $A^{p,r}$ (D) if

$$\max_{i=1,\ldots,l} \left\{1, \frac{2n_i + 2 - 2(2d-q)_i r_i}{n_i + 2 - 2(2d-q)_i \varepsilon_i}\right\} < p < \min_{i=1,\ldots,l} \frac{2n_i + 2 - 2(2d-q)_i r_i}{n_i}.$$

I.2 PROOF OF THEOREM II.11. (i) In view of Theorems II.3 and VI.1, we have $P_r f = f$ for every $f \in A^{p,r}$ (D), and $\|P_r g\|_{p,r} \le c_{p,r}\|g\|_{p,r}$. One easily obtains the announced results from the Hahn-Banach and Riesz representation theorems.

(ii) Let $\varphi \in (A^{p,r}(D))^*$. By the Hahn- Banach theorem , there exists $g \in L^{p',r}$ (D) such that for all $f \in A^{p,r}(D)$, we have

$$\varphi(f) = \int_D \overline{g(z,u)} f(z,u) b^{-r}((z,u),(z,u)) d\nu(z,u).$$

Set $\sigma = \frac{\rho}{-2d+q}$. By Theorem II.3, $P_{\sigma+r} f = f$, $f \in A^{p,r}$ (D) , then

$$\begin{aligned}\varphi(f) &= \int_D \overline{g(z,u)} P_{\sigma+r} f(z,u) b^{-r}((z,u),(z,u)) d\nu(z,u) \\ &= \int_D \overline{T_{\sigma+r,r} g(z,u)} f(z,u) b^{-r}((z,u),(z,u)) d\nu(z,u)\end{aligned}$$

where $T_{\sigma+r,r}$ is the adjoint operator of $P_{\sigma+r}$ with respect to the inner product

$$< f , g >_{2,r} = \int_D f(z,u)\overline{g(z,u)} b^{-r}((z,u),(z,u)) d\nu(z,u),$$

and it is expressed in the following way :

$$T_{\sigma+r,r}f(w) = b^{-\sigma}(w,w)\int_D b^{1+\sigma+r}(w,(z,u))f(z,u)b^{-r}((z,u),(z,u))d\nu(z,u)$$

$$(w \in D).$$

Since $\rho_i > \frac{n_i}{2}$ and $p' > \max\left(\frac{2n \;\; + 2 - 2(2d-q)_i r_i}{n_i + 2 - 2(2d-q)_i r_i}\right)$, it follows from Theorem VI.1 that $\|T_{\sigma+r,r}g\|_{p',r} \leq c_{p,r}\|g\|_{p',r}$.

Clearly, the function $z \mapsto b^{\sigma}(z,z)\, T_{\sigma+r,r}g(z)$ belongs to H(D) , then by Lemma II.6 , there exists h $\in$H(D) such that $\Lambda_\rho h(z) = b^{\sigma}(z,z)\, T_{\sigma+r,r}g(z)$ for every z in D. Then h $\in \mathcal{C}^{p'}_{\rho,r}(D)$ and $\|h\|_{\mathcal{C}^{p'}_{\rho,r}(D)} \leq c\|\varphi\|$. Thus, finally, h represents φ in the following way :

$$\varphi(f) = \int_D b^{-\sigma}((z,u),(z,u))\overline{(\Lambda_\rho)h(z,u)}f(z,u)b^{-r}((z,u),(z,u))d\nu(z,u).$$

Conversely, let h $\in \mathcal{C}^{p'}_{\rho,r}(D)$; then the function $z \mapsto b^{-\sigma}(z,z)\,\Lambda_\rho h(z)$ belongs to $L^{p',r}(D)$ and the linear functional φ defined by

$$\varphi(f) = \int_D b^{-\sigma}((z,u),(z,u))\overline{(\Lambda_\rho)h(z,u)}f(z,u)b^{-r}((z,u),(z,u))d\nu(z,u)$$

$$(f \in A^{p,r}(D))$$

is such that $\|\varphi\| \leq c\|h\|_{\mathcal{C}^{p'}_{\rho,r}(D)}$. Hence $\varphi \in (A^{p,r}(D))^*$. Therefore this proves assertion (ii) and Theorem II.11 is entirely proved.

REMARK : Let r and p satisfy the hypotheses of assertion (i) of Theorem II.11, and let ρ be a vector of R^l such that $\rho_i > \frac{n_i}{2}$ for all i = 1,...,l. Then $\mathcal{C}^{p}_{\rho,r}(D)$ is equivalent to

$A^{p,r}(D)$ in the following manner : every equivalence class in $\mathcal{C}^{p}_{\rho,r}(D)$ contains one and only one representative f belonging to $A^{p,r}(D)$, and conversely, every $f \in A^{p,r}(D)$ is a representative of an equivalence class in $\mathcal{C}^{p}_{\rho,r}(D)$. In fact, first let $g \in \mathcal{C}^{p}_{\rho,r}(D)$. Set $h(z)=b^{\sigma}(z,z)\Lambda_{\rho}\, g(z)$ and $f = P_r\, h$; then by Theorem VI.I, f belongs to $A^{p,r}(D)$ under our assumptions on p, since h belongs to $L^{p,r}(D)$. Therefore $\Lambda_{\rho} f = P_{r+\sigma}(\Lambda_{\rho}\, g)$. On the other hand, $\Lambda_{\rho}\, g \in A^{p,\, \sigma p+r}(D)$; then $P_{r+\sigma}(\Lambda_{\rho}\, g) = \Lambda_{\rho}\, g$. Hence $\Lambda_{\rho}\, g = \Lambda_{\rho} f$ and then, f is a representative of the class g in $\mathcal{C}^{p}_{\rho,r}(D)$.

Next, let $f \in A^{p,r}(D)$ be such that $\Lambda_{\rho} f \equiv 0$. Then $f = P_r f$, and this implies that $T_{r+\sigma,r} f(z) = b^{-\sigma}(z,z)\, \Lambda_{\rho} f(z) \equiv 0$. By the Fubini theorem, one easily obtains that

$$P_r\left(T_{r+\sigma,r} f\right)(\zeta)$$
$$= c\int_D f(\eta) b^{-r}(\eta,\eta)\left(\int_D b^{1+r+\sigma}(z,\eta) b^{1+r}(\zeta,z) b^{-r-\sigma}(z,z)\, d\nu(z)\right) d\nu(\eta)$$
$$= c\int_D f(z) b^{-r}(\eta,\eta) b^{1+r}(\zeta,\eta)\, d\nu(\eta),$$

where the latest equality follows from the reproducing formula $P_{r+\sigma}\left(b^{1+r}(\cdot\,,\zeta)\right)(\eta) = b^{1+r}(\eta\,,\zeta)$ for all η and ζ in D, since the function $b^{1+r}(\cdot\,,\zeta)$ belongs to $A^{p,r}(D)$ and $\rho_i > \frac{n_i}{2}$ for all $i = 1,...,l$. Finally, we conclude that $P_r(T_{r+\sigma,r} f) = P_r(f) = f$, and since $T_{r+\sigma,r} f \equiv 0$, this implies that $f \equiv 0$. Hence each class in $\mathcal{C}^{p}_{\rho,r}(D)$ has only one representative in $A^{p,r}(D)$.

Conversely, let $f \in A^{p,r}(D)$; then $P_r\, f = f$ and $\Lambda_\rho f(\zeta) b^{-\sigma}(\zeta,\zeta) = T_{r+\sigma,r} f(\zeta)$, for all ζ in D. Hence, by Theorem VI.1, $T_{r+\sigma,r}$ f belongs to $L^{p,r}(D)$ since $p > \max_i \left\{ \frac{2n_i + 2 - 2(2d-q)_i r_i}{n_i + 2 - 2(2d-q)_i r_i} \right\}$ and $\rho_i > \frac{n_i}{2}$ for all i = 1,...,*l*. Therefore f is a representative of a class in $\mathcal{C}^p_{\rho,r}(D)$.

APPENDIX

PROOF OF LEMMA V. 3 .

We first remark that for all $(z, u) \in D_2$:

$$(*) \qquad |u_2|^2 < \left| \chi_2 \left(\frac{ie - z^*}{2i} \right) \right|.$$

By this remark (*), we have the estimate :

$$|u_2|^4 \left| \chi_1^{-(5+r)} \chi_2^{-(8+r)} \left(\frac{ie - z^*}{2i} \right) \right| \le \left| \chi_1^{-(5+r)} \chi_2^{-(6+r)} \left(\frac{ie - z^*}{2i} \right) \right|.$$

This latest function is integrable by Theorem II.2. Hence (a) is proved . The same estimate permits to conclude that (b) is true . For the next three integrals, by the previous remark (*), it is sufficient to prove that for a suitable choice of δ ($\delta = 0$ for (c) and (d) and $\delta = 0.5$ for (e)), the following function is integrable in D_2 :

$$g_\delta(z,u) = \bar{z}_{12} \chi_1^{-(5+r)} \chi_2^{-(7+r+\delta)} \left(\frac{ie^* - z}{2i} \right).$$

To this aim, we shall use computational techniques due to Békollé in [B_2] . Let α_1 and α_2 be two real numbers such that $0 < \alpha_1 < 1$ and $1 < \alpha_2 < 3 + 2\delta$, and consider the sets D_3 and D_4 defined by :

$$D_3 = \left\{(z,u) \in D_2 : \left|g_\delta(z,u)\right| \le \left|\chi_1^{-(4+r+\alpha_1)}\chi_2^{-(4+r+\alpha_2)}\left(\frac{ie-z^*}{2i}\right)\right|\right\}$$

$$D_4 = D_2 \setminus D_3.$$

In D_1 , $|g_\delta (z, u)|$ is dominated by an integrable function in D_2 by our assumption on α_1 and α_2 and Theorem II. 2. In D_4 , it is obvious that

$$\left|g_\delta(z,u)\right| \le c\left|h_\delta(z,u)\right|^2,$$

where $h_\delta(z,u)$ is equal to :

$$h_\delta(z,u) = \bar{z}_{12}\chi_1^{(6+r-\alpha_1)/2}\chi_2^{-(10-\alpha_2+r+2\delta)/2}\left(\frac{ie-z^*}{2i}\right).$$

From the very definition of χ_1 and χ_2 , we have :

$$h_\delta(z,u) = c\frac{\partial}{\partial\bar{z}}\left\{\chi_1^{-(r-\alpha_1+4)/2}\chi_2^{-(8-\alpha_2+r+2\delta)/2}\left(\frac{ie-z^*}{2i}\right)\right\}.$$

Therefore one easily gets :

$$h_\delta(z,u) = c\int_V \exp\left(-<\lambda,\frac{ie-z^*}{2i}>\right)\lambda_{12}\chi_1^{*(4-\alpha_2+r+2\delta)/2}(\lambda)\chi_2^{*(r-\alpha_1)/2}(\lambda)d\lambda,$$

where $\chi_1^*(\lambda) = \lambda_{22} - \frac{\lambda_{12}\lambda_{21}}{\lambda_{11}}$ and $\chi_2^*(\lambda) = \lambda_{11}$ are two defining functions of $V^* = V$.

Hence, by the Plancherel-Gindikin formula [BT] with $C^m = M_{r2}$, we get :

$$\int_{D_2} \left|h_\delta(z,u)\right|^2 d\nu(z,u) = c\int_{C^m} d\nu(u)$$
$$\left(\int_V \exp(<\lambda, e>) \left|\lambda_{12}\right|^2 \chi_1^{*\,2-\alpha_2+r+2\delta} \chi_2^{*\,r-\alpha_1+r-2}(\lambda) \exp(-2<\lambda, u*u> d\lambda \right)$$
$$= c' \int_V \exp(\lambda_{11}+\lambda_{22})(\lambda_{11})^{\alpha_2-\alpha_1-2\delta-4} \left(\lambda_{11}\lambda_{22} - \left|\lambda_{12}\right|^2 \right)^{2-\alpha_2+2\delta} \left|\lambda_{12}\right|^2 d\lambda,$$

since $\int_{C^m} \exp(-2<\lambda, u*u>) d\nu(u) = c(\lambda)_*^{-q^*} = c\left(\chi_1^*(\lambda)\chi_2^*(\lambda) \right)^{-r}$.

By a polar coordinate change of variables, the integration with respect to λ_{12} converges if $\alpha_2 < 3 + 2\delta$, and we have :

$$\int_{D_2} \left|h_\delta(z,u)\right|^2 d\nu(z,u) = \int_0^\infty \int_0^\infty \exp(-(\lambda_{11}+\lambda_{22}))\lambda_{11}^{-\alpha_1} \lambda_{22}^{4-\alpha_2+2\delta} d\lambda_{11} d\lambda_{22}.$$

Henceforth, this integral converges by our assumption on α_1 and α_2 . This proves assertions (c), (d) and (e) .

Following the same pattern, it is sufficient to prove that the function $f_\delta(z, u)$ defined by :

$$f_\delta(z,u) = \left|u_1\right| \chi_1^{-(5+r)} \chi_2^{-(6+r+\delta)} \left(\frac{ie - z*}{2i} \right)$$

is integrable in D_2 (for (f), take $\delta = 1$; for (g), take $\delta = 0.5$, and for (h), take $\delta = 0$). The techniques used before imply that it is sufficient to prove that if $0 < \alpha_1 < 1$ and $1 < \alpha_2 < 2 + 2\delta$, we have :

$$|f_\delta(z,u)| \le |K_\delta(z,u)|^2$$

where $K_\delta(z,u) = c\left|u_1\right| \left(\chi_1^{-\frac{6-\alpha_1+r}{2}} \chi_2^{-\frac{8-\alpha_2+r+2\delta}{2}} \right) \left(\frac{ie - z*}{2i} \right)$ is square integrable in D_2. It is clear that :

$$K_\delta(z,u) = c|u_1|\int_V \exp\left(-<\lambda\frac{ie-z^*}{2i}>\right)\left(\chi_1^{*\frac{4-\alpha_2+r+2\delta}{2}}\chi_2^{*\frac{2-\alpha_1+r}{2}}\right)(\lambda)d\lambda\,;$$

then :

$$\int_{D_2} |K_\delta(z,u)|^2 d\nu(z,u)$$

$$= c\iint \exp(-<\lambda, e>|u_1|^2\chi_1^{*2-\alpha_2+r+2\delta}(\lambda)\chi_2^{*-\alpha_1+r}(\lambda)d\lambda d\nu(u)$$

$$= c\int_V \exp\left(-\left(\lambda_{11}+\lambda_{22}\right)\right)\lambda_{22}\lambda_{11}^{-2+\alpha_2-2\delta-\alpha_1}\left(\lambda_{11}\lambda_{22}.-|\lambda_{12}|^2\right)^{1-\alpha_1+2\delta}d\lambda$$

$$= c\int_0^\infty\int_0^\infty \exp\left(-(\lambda_{11}+\lambda_{22})\right)\lambda_{11}^{-\alpha_1}\lambda_{22}^{3+2\delta-\alpha_2}d\lambda_{11}d\lambda_{22}.$$

This converges by our assumption on α_1 and α_2 . □

PROOF OF THEOREM II.3.

We first prove the following lemma:

A LEMMA**.** Let $p\in[1,\infty)$ and let r be a vector of R^l such that $r_i \geq -1$ ($i = 1,...,l$). Let $f \in A^{p,r}(D)$. Then for every $(y,u) \in V \times C^m$ such that $y - F(u,u) \in V$, the holomorphic function $f_{y,u}$ defined on $R^n + iV$ by

$$f_{y,u}(z) = f(z+iy,u)$$

belongs to the Hardy space $H^p(R^n+iV)$. Moreover, there is a constant $C = C(p,r)$ such that for all $f \in A^{p,r}(D)$ and $(y,u) \in V \times C^m$ satisfying $y - F(u,u) \in V$, then

$$\|f_{y,u}\|^p_{H^p(R^n+iV)} \leq Cb^{1+r}((iy,u),(iy,u))\|f\|^p_{p,r}.$$

Proof of Lemma A. First take $(y,u) = (e,0)$. Let P be a polydisc centered at $(ie,0)$ with closure contained in D. Let $(R_1,...,R_{n+m})$ be the multiradius of P. Then :

(1) $$|f(x+ie,0)|^p \le C\int_{P+(x,0)} |f(\xi+i\eta,v)|^p d\xi d\eta d\nu(v) .$$

If P' is the projection of P on $iV \times C^m$, we obtain :

(2) $$\int_{P+(x,0)} |f(\xi+i\eta,v)|^p d\xi d\eta d\nu(v) \le \int_{|\xi-x|<R_1} \left(\int_{P'} |f(\xi+i\eta,v)|^p d\eta d\nu(v)\right) d\xi.$$

Combining (1) and (2) yields

$$\int_{R^n} |f(x+ie,0)|^p dx \le C(R_1)^n \int_{R^n} \left(\int_{P'} |f(\xi+i\eta,v)|^p d\eta d\nu(v)\right) d\xi.$$

Since $\overline{P'}$ is a compact set contained in $\{(i\eta,v)\in iV\times C^m : \eta - F(u,u)\in V\}$, there are two positive constants c_r and c'_r such that $c'_r \le b^{-r}((\zeta+i\eta,v),(\zeta+i\eta,v)) \le c_r$ for all $(i\eta,v) \in$ P' and $\xi \in R^n$. Hence, for every $f \in H(D)$, the following holds :

$$\begin{aligned}&\int_{R^n} |f(x+ie,0)|^p dx \\ &\le C_r \int_{R^n} \left(\int_{P'} |f(\xi+i\eta,v)|^p b^{-r}((\xi+i\eta,v),(\xi+i\eta,v)) d\eta d\nu(v)\right) d\xi \\ &\le C_r \|f\|^p_{p,r}.\end{aligned}$$

Next assume that (iy,u) is an arbitrary point of D. Since D is affine-homogeneous, there is g $\in$ G(D) such that $g^{-1}(x+iy,u) = (x+ie,0)$ for every x in R^n . Set h = fog. Then

$$\begin{aligned}&\int_{R^n} |f(x+iy,u)|^p dx = \int_{R^n} |h(x+ie)|^p dx \\ &\le C_r \int_D |h(\xi+i\eta,v)|^p b^{-r}((\xi+i\eta,v),(\xi+i\eta,v)) d\xi d\eta d\nu(v).\end{aligned}$$

Make the change of variables $\xi = \xi'$, $(i\eta,v) = g^{-1}(i\eta', v')$. Then the equality

$$|\det g|^{-2} = cb((x+iy,u),(x+iy,u))$$

yields that

$$b\big((\xi + i\eta, v), (\xi + i\eta, v)\big) = b\big((\xi' + i\eta', v'), (\xi' + i\eta', v')\big)\left|\det \quad g\right|^2$$

$$= cb\big((\xi' + i\eta', v'), (\xi' + i\eta', v')\big)b^{-1}\big((x + iy, u), (x + iy, u)\big).$$

Furthermore, for $r \in R^l$, one can prove that

$$b^{-r}\big((\xi + i\eta, v), (\xi + i\eta, v)\big) = c_r b^{-r}\big((\xi' + i\eta', v'), (\xi' + i\eta', v')\big)b^r\big((x + iy, u), (x + iy, u)\big).$$

Hence:

$$\int_{R^n} |f(x + iy, u)|^p dx \le c'_r \; b^{1+r}\big((x + iy, u), (x + iy, u)\big)\|f\|^p_{p,r}.$$

So,

$$\left\| f_{y,u} \right\|^p_{H^p(R^n + iV)} = \sup_{\eta \in V} \int_{R^n} |f(x + i(y + \eta), u)|^p dx$$

$$\le c'_r \sup_{\eta \in V} \int_{R^n} b^{1+r}\big((x + i(y + \eta), u), (x + i(y + \eta), u)\big)\|f\|^p_{p,r}$$

$$= c'_r \, b^{1+r}\big((x + iy, u), (x + iy, u)\big)\|f\|^p_{p,r}.$$

The last equality follows from Proposition II.4 since $r_i \geq -1$ ($i = 1,...,l$). □

The following corollary can be deduced from results in [SW] :

B <u>COROLLARY</u>. Let $p \in [1, \infty)$ and let r be a vector of R^l such that $r_i \geq -1$ ($i = 1,...,l$). Let $(iy,u) \in D$ and $y' \in V$. Then for all $f \in A^{p,r}(D)$,

$$\int_{R^n} |f(x + i(y + y'), u)|^p dx \le \int_{R^n} |f(x + iy)|^p dx$$

and

$$\lim_{\substack{y'\to 0\\ y'\in\Gamma}} \int_{\mathbb{R}^n} |f(x+i(y+y'),u)-f(x+iy,u)|^p dx = 0.$$

PROOF OF THEOREM II.3 : Let ε be a vector of R^l such that $\varepsilon_i > \frac{n_i+2}{2(2d-q)_i}$ (i = 1,...,l).

Recall that for every $f \in A^{2,\varepsilon}(D)$, P_ε f is given by the following formula:

$$P_\varepsilon f(z,u) = c_\varepsilon \int_D b^{1+\varepsilon}((z,u),(\zeta,v))) f(\zeta,v) b^{-\varepsilon}((\zeta,v),\zeta,v)) d\nu(\zeta,v) \qquad ((z,u) \in D).$$

Hence P_ε extends as an operator on $L^{p,r}(D)$ if

$$\int_D \left| b^{1+\varepsilon}((z,u),(\zeta,v))\right|^{p'} b^{-\left(\varepsilon-\frac{r}{p}\right)p'}((\zeta,v),\zeta,v)) d\nu(\zeta,v) < \infty$$

when p > 1 (resp. if

$$\sup_{(\zeta,v)\in D} \left\{ \left| b^{1+\varepsilon}((z,u),(\zeta,v))\right| b^{-(\varepsilon-r)}((\zeta,v),\zeta,v)) \right\} < \infty$$

when p = 1) for all (z,u) ∈ D. By Theorem II.2 (resp. by Proposition II.4) , this is the case if and only if

$$\varepsilon_i > \frac{n_i+2}{2(2d-q)_i}\frac{p-1}{p} + \frac{r_i}{p} \text{ (i = 1,...,}l\text{) and } p < \min_{i=1,\dots,l} \left\{ \frac{n_i - 2(2d-q)_i(1+r_i)}{n_i} \right\} \text{ when } p > 1$$

[resp. if $1+\varepsilon_i \geq \varepsilon_i - r_i \geq 0$ (i = 1,...,l) when p = 1]. For p = 1, this reduces to the two conditions $r_i \geq -1$ and $\varepsilon_i \geq r_i$ (i = 1,...,l).

It suffices to show that for every (ζ ,v) ∈ D, under our assumptions on ε, r and p , the bounded linear functional $\varphi_{(\zeta,v)}$ defined on $A^{p,r}(D)$ by

$$\varphi_{(\zeta,v)}(f) = f(\zeta,v) - c_\varepsilon \int_D b^{1+\varepsilon}((\zeta,v),(z,u)) f(z,u) b^{-\varepsilon}((z,u),(z,u)) d\nu(z,u)$$

which is identically 0 on $A^{p,r}(D) \cap A^{2,\varepsilon}(D)$, vanishes identically on $A^{p,r}(D)$. The desired conclusion follows from the following lemma :

C LEMMA : Let ε and r be two vectors of R^{l} such that $\varepsilon_i > \frac{n_i+2}{2(2d-q)_i}$ and $r_i > \frac{n_i+2}{2(2d-q)_i}$ ($i = 1,...,l$). Then for every $p \in [1,\infty)$, the subspace $A^{p, r}(D) \cap A^{2, \varepsilon}(D)$ is dense in $A^{p, r}(D)$.

Proof of Lemma C. We use the following notations : for $z = (x+iy) \in D$, we write $\frac{z}{n}$ for $\left(\frac{x+iy}{n}, \frac{u}{\sqrt{n}}\right)$, and we write ie for (ie,0). Let $f \in A^{p, r}(D)$. Let α be a positive number to be chosen later. Consider the sequence $\{f_q\}$ defined by

$$f_q(z) = c_\alpha f\left(z + \frac{ie}{q}\right) b^\alpha\left(\frac{z}{q}, ie\right) \qquad (z \in D),$$

where $c_\alpha = b^{-\alpha}(0, ie)$. We will show that for α large , $f_q \in A^{p, r}(D) \cap A^{2, \varepsilon}(D)$ for every positive integer q, and that $\lim_{q \to \infty} \|f - f_q\|_{p, r} = 0$.

The function $z \mapsto f\left(z + \frac{ie}{q}\right)$ is bounded by Lemma II.5 and Proposition II.4. Take α big enough so that the function $z \mapsto b^\alpha\left(\frac{z}{q}, ie\right) = c_\alpha b^\alpha(z, qie)$ belongs to $A^{p, r}(D) \cap A^{2, \varepsilon}(D)$ for every $q \in N$. Next, by the Minkowski inequality, we have the estimate :

$$\|f - f_q\|_{p,r} \leq \left[\int_D |f(z)|^p \left| c_\alpha b^\alpha\left(\frac{z}{q}, ie\right) - 1\right|^p d\nu(z)\right]^{\frac{1}{p}}$$

$$+ \left[\int_D \left| c_\alpha b^\alpha\left(\frac{z}{q}, ie\right)\right|^p \left| f\left(z + \frac{ie}{q}\right) - f(z)\right|^p b^{-r}(z,z)d\nu(z)\right]^{\frac{1}{p}}.$$

By Lemma II.4, there is a positive constant A_α such that for all $z \in D$ and $q \in N$, we have $\left| c_\alpha b^\alpha\left(\frac{z}{q}, ie\right)\right| \leq A_\alpha$. Hence, by the dominated convergence theorem, the first integral on the right side of the estimate goes to 0 as q goes to infinity. Now, to study the second integral, it suffices to prove that

$$I_q = \int_D \left| f\left(z + \frac{ie}{q}\right) - f(z)\right|^p b^{-r}(z,z)d\nu(z)$$

goes to 0 when q goes to infinity. Set z = (x+iy, u) $\in$ D and obtain that I_q is equal to

$$\int_{R^n}\left[\int_{V+F(u,u)}\left(\int_{R^n}\left| f\left(x + iy\left(+\frac{e}{q}\right), u\right) - f(x+iy,u)\right|^p dx\right)(y - F(u,u))^{-(2d-q)r} dy\right] d\nu(u).$$

Observe that Corollary B yields the following two facts :

$$\int_{R^n}\left| f\left(x + i\left(y + \frac{e}{q}\right), u\right) - f(x+iy,u)\right|^p dx \leq 2^p \int_{R^n} |f(x+iy)|^p dx$$

and

$$\lim_{q\to\infty}\int_{R^n}\left| f\left(x + i\left(y + \frac{e}{q}\right), u\right) - f(x+iy,u)\right|^p dx = 0.$$

On the other hand, since $f \in A^{p,r}(D)$, the function $(y,u) \mapsto \int_{R^n} |f(x+iy,u|^p dx$

is integrable on $\{(y,u) : y \in V+F(u,u) , u \in C^m\}$ with respect to the measure $(y-F(u,u))^{-(2d-q)r}\, dy d\nu(u)$. Hence, by the dominated convergence theorem, it follows that I_q goes to 0 as q tends to infinity. Lemma C is proved. □

Thus Theorem II.3 is entirely proved.

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