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DUAL MIXED HYBRID FINITE ELEMENT METHOD: THE THEORY AND THE PRACTICE ON A MODEL PROBLEM. APPLICATION.

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Abstract *: We present in this work the theoretical and practical aspects of the Dual Mixed Hybrid Finite Element method for solving numerically fluid-flow in porous media. We develop an application of this method in numerical calculation of homogenized absolute permeability tensor ; this fact plays an important role in reservoir simulation.*

Résumé : *Dans ce travail, nous exposons des aspects théoriques et la mise en œuvre pratique de la méthode des éléments finis mixtes hybrides duaux pour la résolution numérique d'écoulements en milieux poreux. Nous développons aussi une application de cette méthode au calcul numérique du tenseur des perméabilités absolu homogénéisé; ce résultat joue un rôle essentiel en simulation de réservoir pétrolier.*

1. INTRODUCTION

The Dual Mixed Hybrid Finite Element (DMHFE) method is widely used in Structural Mechanics for solving numerically problems formulated in terms of Stress-Displacement (see, for instance: Oden and Lee[1], Pian and Tong [6], Pian [7]). Satisfactory approximate solutions are then given by this method in structural analysis problems.

But in fluid mechanics, numerical calculation of flows by DMHFE-method is only available to a few specialists.

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In several areas where one is interested in flows problem (hydraulics and petroleum engineering for instance), a better approximation of the velocity field is needed when solving numerically the flows. For such problems, application of the DMHFE-method leads to satisfactory results for both velocity and pressure fields in monophasic flows at least.

Our aim in this paper is to present the theoretical formulation and a practical discretization by DMHFE for stationary fluid flows in a porous medium with Dirichlet/Neumann boundary conditions. On the other hand we shall show how the computed velocity and pressure are used for the determination of homogenized absolute permeability tensor.

This work is divided as follows. In the next section we describe the model-problem, while the third one is devoted to the dual-mixed hybrid variational formulation of such a problem. For existence and uniqueness results we refer to our work in [4]. The fourth section deals with the practical aspects of DMHFE-method, and in the fifth section we develop an application to the computation of the homogenized permeability tensor. The last section is the conclusion of our work.

2. SETTING OF THE PROBLEM

We are interested in a two-dimensional stationary monophasic flow problem in a bounded (connected) open set Ω under a Dirichlet/Neumann boundary condition. We denote $\partial\Omega$ the boundary of Ω which is supposed sufficiently regular.

The general classical equations for such a problem with a source-term write :

(1) $$\vec{v} = -\frac{\overline{\overline{K}}}{\mu}(\vec{\mathrm{grad}}\ p - \vec{f}(x)) \qquad \text{(Darcy's law)},$$

(2) $$\mathrm{div}\vec{v} = \mathbf{S}(x) \qquad \text{(Equation of continuity)},$$

where $\vec{v}$ (the Darcy's velocity) and p (pressure field) are unknowns to determinate, while the problem data are : $\overline{\overline{K}}(x)$ (absolute permeability tensor), μ (viscosity of the fluid), $\vec{f}$ (volume density of external forces reduced to gravitational forces in the practice), **S** (volume density of distributed source).

To equations (1) and (2) are added the following mixed boundary conditions :

(3) $p = \psi$ on $\partial_p\Omega$ (Dirichlet condition)

(4) $\vec{v}.\vec{n}=g$ on $\partial_f\Omega$ (Neumann condition)

where $\partial_p\Omega$ and $\partial_f\Omega$ are two complementary parts of the boundary of Ω.

Remark 1. It may happen that $\partial_p\Omega = \emptyset$ or $\partial_f\Omega = \emptyset$. In the first case ($\partial_p\Omega = \emptyset$), one solves a problem reduced to the Neumann problem (1), (2) and (4), where $\partial_f\Omega = \partial\Omega$. In the second case (i.e. $\partial_f\Omega = \emptyset$), the problem turns out to be the Dirichlet problem (1), (2) and (3), where $\partial_p\Omega = \partial\Omega$. Since $\partial_p\Omega \cup \partial_f\Omega = \partial\Omega$ and $\partial\Omega \neq \emptyset$, one cannot have simultaneously $\partial_p\Omega = \emptyset$ and $\partial_f\Omega = \emptyset$.

Remark 2. The Einstein's convention is adopted in this section and the following ones, i.e.

$$a_i a_i = \sum_i a_i{}^2, \quad a_i b_i = \sum_i a_i b_i, \quad \frac{\partial a_i}{\partial x_i} = \sum_i \frac{\partial a_i}{\partial x_i}, \ldots$$

We make the following assumptions on the problem data :

- the absolute permeability tensor satisfies

(5) $K_{ij}(x) = K_{ji}(x) \quad \forall i, j = 1, 2 \quad$ a.e in Ω

(6) $K_{ij}(.) \in L^{\infty}(\Omega), \quad \forall i, j=1,2$

(7) $\exists\, \theta \in R_+^*, \quad K_{ij} a_i a_j \geq \theta a_i a_i, \ \forall a \in R^2,$ a.e in Ω,

- the components f_i of the vector -valued function $\vec{f}$ satisfy :

(8) $f_i(.) \in L^2(\Omega), \quad \forall i=1,2$

- the density of distributed source **S** satisfies

(9) $\mathbf{S(.)} \in L^2(\Omega)$

- if $\partial_p\Omega \neq \varnothing$, we assume that :

(10) $\psi \in H^{1/2}(\partial_p\Omega)$;

- if $\partial_f\Omega \neq \varnothing$, we assume that :

(11) $g \in L^2(\partial_f\Omega)$.

Before developing a practical DMHFE - discretization technique of (1)-(4), let us carry out the associated variational formulation.

3. DUAL MIXED HYBRID VARIATIONAL FORMULATION OF (1) - (4)

3.1 Preliminaries .

We consider a decomposition of the bounded open set Ω into a finite number of open non-empty subsets denoted S_i ; we denote S the generic name of these subsets and D_h the set of all the S_i , where h is a parameter representing the maximum of diameters of S_i :

$$\overline{\Omega} = \bigcup_i \overline{S}_i \,, \qquad S_i \cap S_j = \varnothing \quad \text{if } i \neq j .$$

The parameter h is by definition strictly positive and destinated to go to 0 . We denote ∂S_i the boundary of each S_i and, assuming this boundary is regular, i.e. piecewise differentiable, $\vec{n}$ the outer unit normal vector to S_i. It is important for the sequel to remark that ∂S_i is the collection of parts:

. $\Gamma_{ij} = \partial S_i \cap \partial S_j$, where $i \neq j$ and S_i adjacent to S_j ;

. $\Gamma_{ip} = \partial S_i \cap \partial_p\Omega$, i.e. intersection of ∂S_i with the pressure-given boundary ;

. $\Gamma_{if} = \partial S_i \cap \partial_f\Omega$, i.e. intersection of ∂S_i with the flux-given boundary.

3. 2 Construction of the dual mixed hybrid variational formulation of (1)-(4).

Let us recall a Green's result that plays an important role in this section.

Theorem 1 (Green). If Θ is an open non-empty bounded set in the space, with a regular boundary $\partial\Theta$, then :

$$\int_{\Theta} \overrightarrow{\mathrm{grad}}\, u.\vec{w}\,dx + \int_{\Theta} u\,\mathrm{div}\,\vec{w}\,dx = \int_{\partial\Theta} [u\,\vec{w}.\vec{n}]\,ds, \quad \forall u \in H^1(\Theta),\ \vec{w} \in H(\mathrm{div},\Theta),$$

where :

$\int_{\partial\Theta} [...]$ represents the duality pairing between $H^{-1/2}$ and $H^{1/2}$,

$H^1(\Theta)$ the usual Sobolev space of order 1 ,

$H(\mathrm{div},\Theta) = \{\, \vec{w} = (w_i) : w_i \in L^2(\Theta),\ \forall i,\ \text{and } \mathrm{div}\vec{w} \in L^2(\Theta)\}$ with distributional derivatives.

We are going to carry out, in the following lines, the dual mixed hybrid variational formulation of the system (1) - (4) . By virtue of (7) the tensor $\overline{\overline{K}}(x)/\mu$ is almost everywhere invertible (with the usual hypothesis that the viscosity μ is constant), and we denote $\overline{\overline{A}}(x)$ its inverse. Therefore equation (1) is equivalent to

$$\overline{\overline{A}}\vec{v} = -\overrightarrow{\mathrm{grad}}\,p + \vec{f} \tag{12}$$

(equation of fluid motion in Stokes' form at the macroscopic level).

Let $\vec{w}$ be a vector-valued function, sufficiently regular and defined in Ω. Multiplying (12) by $\vec{w}$ and integrating in Ω, one obtains :

$$\int_{\Omega} \overline{\overline{A}}\vec{v}.\vec{w}\,dx = -\int_{\Omega} \overrightarrow{\mathrm{grad}}\,p.\vec{w}\,dx + \int_{\Omega} \vec{f}.\vec{w}\,dx\,, \tag{13}$$

where all the functions are assumed sufficiently regular so that these integrals exist. The first integral term of the right hand of (13) can also write :

$$-\int_{\Omega} \overrightarrow{\text{grad}}\, p.\vec{w}\,dx = \sum_{i\in I} -\int_{S_i} \overrightarrow{\text{grad}}\, p.\vec{w}_i\,dx$$

where $\vec{w}_i$ denotes the restriction of $\vec{w}$ to each S_i and

$$I = \left\{ \text{integers } i \geq 0 : S_i \in D_h \right\}.$$

Assuming that :

$$P/_{S_i} \in H^1(S_i) \text{ and } \vec{w}_i \in H(\text{div}, S_i),$$

Theorem 1 (Green's formula) applies to the right hand side of the preceding equality and gives :

$$-\int_{\Omega} \overrightarrow{\text{grad}}\, p.\vec{w}\,dx = \sum_{i\in I} \left[\int_{S_i} p.\text{div}\vec{w}_i\,dx - \int_{\partial S_i} p\vec{w}_i.\vec{n}_i\,ds \right].$$

Therefore equation (13) is equivalent to

(14) $$\int_{\Omega} \overline{\overline{A}}\vec{v}.\vec{w}\,dx = \sum_{i\in I} \left[\int_{S_i} p.\text{div}\vec{w}_i\,dx - \int_{\partial S_i} p\vec{w}_i.\vec{n}_i\,ds \right] + \int_{\Omega} \vec{f}.\vec{w}\,dx.$$

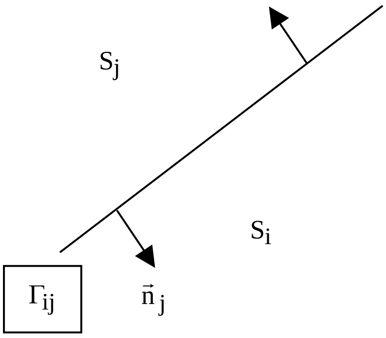
Let us now give a variational formulation of equation (2) : if u is a sufficiently regular and scalar-valued function defined in Ω, multiplying (2) by u and integrating in Ω, one obtains

(15) $$\int_{\Omega} u\,\text{div}\vec{v}\,dx = \int_{\Omega} \mathbf{S}u\,dx.$$

It is fundamental to remark that Darcy's velocity may be discontinuous on Γ_{ij} (the frontier between two adjacent elements S_i and S_j) :

this occurs when the permeability tensor $\overline{\overline{K}}(x)$ is discontinuous on Γ_{ij}. However we assume that at each point x of Γ_{ij},

$\vec{n}_i$



the two limits of $\vec{v}(x')$, when x' goes to x, with $x' \in S_i$ (resp. $x' \in S_j$) exist.. We denote those limits $\vec{v}_i$ and $\vec{v}_j$ respectively. On the other hand , the source distribution is, as supposed, given by a density **S**(.) which is a function defined in Ω, hence there is no concentrated source on Γ_{ij} .

Therefore equation (2) implies that almost everywhere on Γ_{ij} we have :

(16) $$\vec{v}_i \cdot \vec{n}_i + \vec{v}_j \cdot \vec{n}_j = 0 \qquad \text{(Continuity of the flux).}$$

Similarly, when an element S_i is such that the frontier $\Gamma_{if} \equiv \partial S_i \cap \partial_f \Omega$ is non empty , the continuity of the flux on that frontier writes :

(17) $$\vec{v}_i \cdot \vec{n}_i = g \text{ on } \Gamma_{if} ,$$

taking into account the Neumann condition, i.e. equation (4) .

From equations (16) and (17), one can deduce that a variational form of the continuity of flux writes

(18) $$\sum_i \int_{\partial S_i} \beta(\vec{v}_i \cdot \vec{n}_i) ds = \int_{\partial_f \Omega} \beta g ds, \quad \forall \beta \in \prod_i L^2(\partial S_i), \ \beta = 0 \text{ on } \partial_p \Omega ,$$

where $\vec{v}_i \cdot \vec{n}_i$ is assumed to be in $L^2(\partial S_i)$ for each $i \in I$. Indeed, the sum at the left hand side of (18) contains

. terms of the form $\int_{\Gamma_{ip}} \beta(\vec{v}_i \cdot \vec{n}_i) d\gamma$ which are equal to zero because $\beta=0$ on $\partial_p \Omega$ and $\Gamma_{ip} \subset \partial_p \Omega$;

. a sum of the form $\sum_{i,j \in I, i \neq j} \int_{\Gamma_{ij}} \beta(\vec{v}_i \vec{n}_i + \vec{v}_j \vec{n}_j) d\gamma$, which is null since for i and j in I, with $i \neq j$, $\vec{v}_i \cdot \vec{n}_i + \vec{v}_j \cdot \vec{n}_j = 0$ by virtue of (16);

. a sum of the form $\sum_{i \in I_f} \int_{\Gamma_{if}} \beta(\vec{v}_i.\vec{n}_i) d\gamma$, where

$I_f = \{ i \in I : \Gamma_{if} = \partial S_i \cap \partial_f \Omega \}$,

which is equal to $\int_{\partial_f \Omega} \beta g d\gamma$, since $\partial_f \Omega = \cup_i \Gamma_{if}$ and using (17).

Relations (18) and (15) combine to give :

$$\sum_{i \in I} \int_{\partial S_i} \beta(\vec{v}_i.\vec{n}_i) d\gamma - \int_{\Omega} u \operatorname{div} \vec{v} dx = \int_{\partial_f \Omega} \beta g d\gamma - \int_{\Omega} u \mathbf{S} dx,$$

otherwise, writing the integral in Ω, at the left hand side of the preceding equality, as sum of integrals in the subsets S_i one obtains :

$$(19) \qquad \sum_{i \in I} \left(\int_{\partial S_i} \beta(\vec{v}_i.\vec{n}_i) ds - \int_{S_i} u \operatorname{div} \vec{v} dx \right) = \int_{\partial_f \Omega} \beta g ds - \int_{\Omega} u \mathbf{S} dx,$$

where β and u are test functions. The preceding equality is a variational formulation of continuity equation in the subsets S_i taking into account the continuity of flux across the inter-element frontiers . Of course, equation (19) is formal in the sense that associated functional spaces are not yet specified. This specification will be our preoccupation in the following lines.

Let us recall here the variational formulation of Darcy's law, equation (14), in its formal setting :

$$(20) \qquad \int_{\Omega} \overline{\overline{A}} \vec{v}.\vec{w} dx + \left[\sum_{i \in I} \int_{\partial S_i} p \vec{w}_i . \vec{n}_i ds - \int_{S_i} p.\operatorname{div} \vec{w}_i dx \right] = \int_{\Omega} \vec{f}.\vec{w} dx ,$$

where $\vec{w}$ is a vector-valued test function .

It is obvious that the integrals in the equations (19) - (20) exist if the following conditions are satisfied :

. $\vec{v}$ (Darcy's velocity to determinate) and $\vec{w}$ (associated test function) belong to

$$V = \left\{ \vec{\Phi} \in \prod_{i \in I} H(\operatorname{div}, S_i) : \vec{\Phi}_i . \vec{n}_i \in L^2(\partial S_i), \quad \forall i \in I \right\};$$

. $\vec{P}\big|S_i$ (unknown pressure in S_i) and $u\big|S_i$ (corresponding test function) are in $L^2(S_i)$ for each i ;

. $P|\partial S_i$ (unknown pressure on ∂S_i) belongs to $L^2(\partial S_i)$ with the constraints :

$$P\Big|\partial S_i = P\Big|\partial S_j \text{ on } \Gamma_{ij}\,, \quad \forall i, j \quad (i \neq j)$$

and

$$P\Big|\partial S_i = \psi \text{ on } \Gamma_{ip}\,, \ \forall i\,,$$

while the associated test function β satisfies :

$$\beta \in L^2(\Gamma), \text{ with } \beta = 0 \text{ on } \partial_p\Omega,$$

where we have set $\Gamma = \partial\Omega \cup (\bigcup\limits_{i,j}\Gamma_{ij})$.

The preceding conditions are supposed fulfilled.

Notations: From now on we denote q the restriction of p to $\bigcup\limits_{i} S_i$ and λ the restriction of p to Γ.

We then have :

(21) $$q \in L^2(\Omega) \text{ and } \lambda \in L^2(\Gamma) \text{ with } \lambda = \psi \text{ on } \partial_p\Omega.$$

If we set

(22) $$a(\vec{v},\vec{w}) = \sum_{i\in I} \int_{S_i} \overline{\overline{A}}(x)\,\vec{v}_i\,.\,\vec{w}_i\,dx, \quad \forall \vec{v},\vec{w} \in V$$

and

(23) $$b(\vec{w};q,\lambda) = \sum_{i\in I}\left(\int_{\partial S_i} \lambda\vec{w}_i.\vec{n}_i ds - \int_{S_i} q\,\mathrm{div}\,\vec{w}_i dx \right)$$

where $\vec{w} \in V, q \in L^2(\Omega)$ and $\lambda \in L^2(\Gamma)$, then the dual mixed hybrid variational formulation of (1)-(4) consists of finding :

(24) $$\vec{v} \in V,\ \ q \in L^2(\Omega),\ \ \lambda \in L^2(\Gamma) \quad \text{with} \quad \lambda = \psi \quad \text{on} \quad \partial_P\Omega\,,$$

satisfying :

(25) $$a(\vec{v},\vec{w})+b(\vec{w};q,\lambda)=\int_{\Omega}\vec{f}.\vec{w}\,dx,\quad \forall\vec{w}\in V\ ;$$

(26) $$b(\vec{v};u,\beta)=\int_{\partial_f\Omega} g\beta ds-\int_{\Omega}\mathbf{S}u\,dx,\ \forall u\in L^2(\Omega),\ \beta\in L^2(\Gamma),\ \text{with } \beta|\partial_p\Omega=0.$$

In conclusion , we have proved, for sufficiently smooth data, that if $(\vec{v},p)$ satisfies (1)-(4), then it also satisfies (24)-(26), where we have put down :

(27) $$\lambda = p \quad \text{on} \quad \bigcup_i \partial S_i,$$

(28) $$q = p \text{ on } \bigcup_i S_i.$$

The application of an abstract result due to Babuska-Brezzi shows that the variational problem (24)-(26) admits one and only one solution (see Njifenjou[4]). In [4], we have worked, for sake of clarity, with $\vec{f}\equiv 0$ and g $\equiv$ 0, that is neither external forces nor flows across the boundary part $\partial_f\Omega$. Moreover the variational problem (24)-(26) is still well-posed (existence and uniqueness of the solution hold) under weaker hypothesis on the physical data . The corresponding solution is called the weak solution and is considered as a meaningful physical one, since those weak assumptions appear nearer to experimental observations : one learns from physical measurements that the absolute permeability can be regarded as a L^∞ - function rather than a C^1- function on $\overline{\Omega}$.

Let us now show that the solution of the variational system (24)-(26) satisfies the partial differential equation (1)-(4) almost everywhere in Ω. We must mention that the derivations are understood in the distributional sense in this framework.

Theorem 2 : If the functions $\vec{v}\in V$, $q\in L^2(\Omega)$, $\lambda\in L^2(\Gamma)$ with $\lambda=\psi$ on $\partial_p\Omega$, satisfy the following variational equations:

(29) $$a(\vec{v},\vec{w}) + b(\vec{w};q,\lambda) = \int_{\Omega} \vec{f}.\vec{w}dx, \quad \forall \vec{w} \in V$$

(30) $$b(\vec{v},u,\beta) = \int_{\partial_f \Omega} g\beta d\gamma - \int_{\Omega} \mathbf{S} u dx, \ \forall u \in L^2(\Omega), \ \forall \beta \in L^2(\Gamma), \ \beta\big|\partial_p\Omega = 0,$$

then there exists $p \in H^1(\Omega)$ such that :

(i) $$p = q \text{ in } \bigcup_i S_i \text{ and on } \Gamma$$

(ii) $$\begin{cases} \vec{v} + \dfrac{\overline{\overline{K}}}{\mu}\left[\vec{\mathrm{grad}}\, p - \vec{f}\right] = 0 \\ \mathrm{div}\vec{v} = \mathbf{S}(x) \text{ a.e.in}\,\Omega \end{cases}$$

(iii) $$\begin{cases} p = \psi & \text{on} \quad \partial_p\Omega \\ \vec{v}.\vec{n} = g & \text{on} \quad \partial_f\Omega. \end{cases}$$

PROOF : Let us introduce the space $\prod_{i \in I} \underline{D}(S_i)$, where each $\underline{D}(S_i)$ is the set of infinitely derivable vector-valued functions with compact support in S_i.

It is obvious that $\prod_{i \in I} \underline{D}(S_i) \subset V$. Hence, the relation (29) holds for every $\vec{w}$ lying in $\prod_{i \in I} \underline{D}(S_i)$. We get, using the definitions of a (. , .) and b (. , . , .) given by (22) and (23) respectively :

$$\sum_{i \in I} \int_{S_i} \overline{\overline{A}}\vec{v}_i\vec{\varphi}_i dx + \sum_{i \in I}\left(\int_{\partial S_i} \lambda_i \vec{\varphi}_i.\vec{n}_i d\gamma - \int_{S_i} q_i \mathrm{div}\vec{\varphi}_i dx \right) = \sum_{i \in I} \int_{S_i} \vec{f}_i.\vec{\varphi}_i dx,$$

for every $\vec{w} \in \prod_i \underline{D}(S_i)$ or, which is equivalent :

$$\sum_{i \in I} \int_{S_i} \left(\overline{\overline{A}}\vec{v}_i + \vec{\mathrm{grad}}\, q_i - \vec{f}_i \right).\vec{w}_i dx = 0, \quad \forall \vec{w} \in \prod_{i \in I} \underline{D}(S_i)$$

where the symbol i means the restriction of the corresponding quantity to the subdomain S_i. The preceding equality implies :

(31) $$\overline{\overline{A}}\vec{v}_i + \vec{\text{grad}}\, q_i = \vec{f}_i \text{ a.e. in } \underline{D}'(S_i), \quad \forall i \in I .$$

This relation holds in the distributional sense. Since $\overline{\overline{A}}\vec{v}_i$ and $\vec{f}_i$ are in $\underline{L}^2(S_i)$, $\vec{\text{grad}}\, q_i$ lies in $\underline{L}^2(S_i)$ via the relation (31). We deduce that:

(32) $$\overline{\overline{A}}\vec{v}_i + \vec{\text{grad}}\, q_i = f_i \quad \text{a.e.} \quad \text{in} \quad S_i, \quad \forall i \in I$$

and

(33) $$q_i \in H^1(S_i), \qquad \forall i \in I$$

taking into account the assumption that $q_i \in L^2(S_i)$ for every $i \in I$.

At this stage we have proved that :

(34) $$q_i\big|\partial S_i \quad \text{exists and lies in } H^{1/2}(\partial S_i), \text{ for all } i \in I .$$

We are going to prove that $q \in H^1(\Omega)$. Since $q_i \equiv q\big|S_i$ is lying in $H^1(S_i)$ for all $i \in I$, it suffices to prove that for two adjacent subdomains S_i and S_j, denoting Γ_{ij} their common frontier, we have :

$$q_i = q_j \text{ on } \Gamma_{ij} .$$

For this purpose we consider $\vec{w} = (\vec{w}_i)_{i \in I}$ an element of V. For each i, let us multiply (32) by $\vec{w}_i$ and integrate in S_i. We obtain then :

$$\int_{S_i} \overline{\overline{A}}\vec{v}_i.\vec{w}_i dx + \int_{S_i} \vec{\text{grad}} q_i.\vec{w}_i dx = \int_{S_i} \vec{f}_i.\vec{w}_i dx, \quad \forall \vec{w}_i \in H(\text{div}, S_i).$$

Integration by parts of the second term of the left hand side of the preceding relation and summation on i yield :

$$\sum_{i \in I}\left(\int_{S_i} \overline{\overline{A}}\vec{v}_i.\vec{w}_i dx + \int_{\partial S_i} q_i \vec{w}_i.\vec{n}_i d\gamma - \int_{S_i} q_i \,\text{div}\vec{w}_i dx \right) = \int_{\Omega} \vec{f}.\vec{w} dx, \quad \forall \vec{w} \in V .$$

Comparing this relation with the relation (29), we deduce that :

(35) $$\sum_{i \in I} \int_{\partial S_i} (q_i - \lambda_i)\vec{w}_i.\vec{n}_i d\gamma = 0, \quad \forall \vec{w} \in V = \prod_{i \in I} H(\text{div}, S_i) .$$

This relation holds in particular for all $\vec{w} \in H(\mathrm{div},\Omega)$.

On the other hand we have :

(36) $$\vec{w}_i.\vec{n}_{i\to j} = \vec{w}_j.\vec{n}_{j\to i} \text{ on } \Gamma_{ij}, \quad \forall \vec{w} \in H(\mathrm{div},\Omega),$$

where $\vec{n}_{i\to j}\,(\text{resp.}\,\vec{n}_{j\to i})$ is the unit normal vector of $\Gamma_{ij} = \overline{S}_i \cap \overline{S}_j$ oriented from S_i to S_j (resp. S_j to S_i). Thus we deduce from (35) that :

$$\sum_{i,j\in I} \int_{\Gamma_{ij}} \left[(q_i - \lambda_i) - (q_j - \lambda_j)\right] w_i.n_{i\to j}\, d\gamma = 0, \quad \forall \vec{w} \in H(\mathrm{div},\Omega),\ \vec{w}.\vec{n} = 0 \text{ on } \partial\Omega.$$

Taking into account the fact that $\lambda_i = \lambda_j$ on Γ_{ij}, we get :

$$\sum_{i,j\in I} \int_{\Gamma_{ij}} (q_i - q_j)\vec{w}_i.\vec{n}_{i\to j}\, d\gamma, \ \forall \vec{w} \in H(\mathrm{div},\Omega),\ \vec{w}.\vec{n} = 0 \text{ on } \partial\Omega.$$

One obviously deduces that

$$q_i = q_j \text{ on } \Gamma_{ij}, \quad \forall \Gamma_{ij} = \overline{S}_i \cap \overline{S}_j$$

and therefore

(37) $$q \in H^1(\Omega).$$

Now, we shall prove that $\vec{v}$ is in $H(\mathrm{div},\Omega)$. We know by assumption that $\vec{v} \in \prod_{i\in I} H(\mathrm{div},S_i) \equiv V$ and $\overline{\Omega} = \bigcup_{i\in I} \overline{S}_i$; thus proving that $\vec{v} \in H(\mathrm{div},\Omega)$ is equivalent to proving that

(38) $$\vec{v}_i.\vec{n}_i + \vec{v}_j.\vec{n}_j = 0 \text{ on } \Gamma_{ij}, \quad \forall i,j \in I, \quad i \neq j,$$

where $\vec{n}_i\ (\text{resp.}\,\vec{n}_j)$ represents the unit outer normal to ∂S_i (resp.∂S_j). In what follows our purpose is to prove relation (38). If the test-function u is chosen identically equal to zero (i.e. $u \equiv 0$ in Ω) in (30), one gets :

(39) $$\sum_{i\in I} \int_{\partial S_i} \beta_i \vec{v}_i.\vec{n}_i\, d\gamma = \int_{\partial_f\Omega} \beta g\, d\gamma, \ \forall \beta \in L^2(\Gamma),\ \beta = 0 \text{ on } \partial_p\Omega.$$

Denoting:

$$\Gamma_{if} = \partial_f\Omega \cap \partial S_i,$$

when S_i is adjacent to the boundary $\partial_f \Omega$, (39) written in terms of Γ_{ij} and Γ_{if} becomes:

(40) $$\sum_{i,j\in I}\int_{\Gamma_{ij}} \beta\left(\vec{v}_i.\vec{n}_i + \vec{v}_j.\vec{n}_j\right)d\gamma + \sum_{i\in I_f}\int_{\Gamma_{if}} \beta \vec{v}_i.\vec{n}_i d\gamma = \sum_{i\in I_f}\int_{\Gamma_{if}} \beta g d\gamma,$$

$$\forall \beta \in L^2(\Gamma) : \beta = 0 \text{ on } \partial_p \Omega.$$

If we choose β in (40) such that for each (i,j) :

$$\beta = 0 \text{ on } \Gamma - \Gamma_{ij}$$

we get :

$$\int_{\Gamma_{ij}} \beta\left(\vec{v}_i.\vec{n}_i + \vec{v}_j.\vec{n}_j\right)d\gamma = 0, \ \forall \beta \in L^2(\Gamma_{ij}) .$$

Then, relation (38) follows, i.e.

$$\vec{v}_i.\vec{n}_i + \vec{v}_j.\vec{n}_j = 0 .$$

Let us prove now that :

(41) $$q = \lambda \text{ on } \Gamma \equiv \bigcup_i \partial S_i .$$

We have proved the following relation (see equation (32)) :

$$\overline{\overline{A}}\vec{v}_i + \vec{\mathrm{grad}}\, q_i = f_i \text{ a.e. in } S_i, \quad \forall i \in I.$$

Let $\vec{w} \in \prod_{i\in I} H(\mathrm{div}, S_i)$ and $\vec{w}_i$ the restriction of $\vec{w}$ to S_i. We get from the preceding relation, after multiplying by $\vec{w}_i$ and integration in S_i :

$$\int_{S_i} \overline{\overline{A}}\vec{v}_i.\vec{w}_i dx + \int_{\partial S_i} q_i \vec{w}_i.\vec{n}_i d\gamma - \int_{S_i} q_i \mathrm{div}\vec{w}_i dx = \int_{S_i} \vec{f}_i.\vec{w}_i dx, \quad \forall i \in I .$$

After summation on $i \in I$, one gets :

$$\sum_{i\in I}\int_{S_i} \overline{\overline{A}}\vec{v}_i.\vec{w}_i dx + \sum_{i\in I}\left(\int_{\partial S_i} q_i \vec{w}_i.\vec{n}_i d\gamma - \int_{S_i} q_i \mathrm{div}\vec{w}_i dx\right) = \int_{\Omega} \vec{f}.\vec{w} dx,$$

$$\forall \vec{w} \in V \equiv \prod_{i\in I} H(\mathrm{div}, S_i)$$

Comparing this relation with (29), we deduce that

$$\sum_{i\in I}\int_{\partial S_i} \left(q_i - \lambda_i\right)\vec{w}_i.\vec{n}_i d\gamma = 0, \ \forall \vec{w} \in \prod_{i\in I} H(\mathrm{div}, S_i)$$

and therefore :

$$q_i = \lambda_i \text{ on } \partial S_i, \qquad \forall i \in I,$$

which means

$$q = \lambda \text{ on } \Gamma \equiv \bigcup_{i\in I} \partial S_i \, .$$

Hence (41) is proved. This result has as consequence the following relation that we are going to prove :

(42) $$\overline{\overline{A}}\vec{v} + \vec{\text{grad}}\, q = \vec{f} \text{ a.e. in } \Omega.$$

Relation (29) rewrites, taking into account (41) :

$$\sum_{i \in I} \int_{S_i} \overline{\overline{A}}\vec{v}_i.\vec{w}_i dx + \sum_{i \in I} \left(\int_{\partial S_i} q_i \vec{w}_i.\vec{n}_i d\gamma - \int_{S_i} q_i \text{div}\vec{w}_i dx \right) = \int_{\Omega} \vec{f}.\vec{w} dx,$$

$$\forall \vec{w} \in V \equiv \prod_{i \in I} H(\text{div}, S_i)$$

or, which is equivalent,

$$\sum_{i\in I} \int_{S_i} \overline{\overline{A}}\vec{v}_i.\vec{w}_i dx + \sum_{i} \int_{S_i} \vec{\text{grad}}\, q_i.\vec{w}_i dx = \int_{\Omega} \vec{f}.\vec{w} dx, \; \forall \vec{w} \in V \, .$$

This relation remains true in particular for $\vec{w} = \vec{\varphi} \in \underline{D}(\Omega)$; this implies (42) since $q \in H^1(\Omega)$ (see relation (37)) and thus $\vec{\text{grad}}\, q \in \underline{L^2}(\Omega)$ on one hand, and $\vec{v} \in H(\text{div}, \Omega)$ and thus $\vec{v} \in \underline{L^2}(\Omega)$ on the other hand.

We are going to show now that

(43) $$\text{div}\vec{v} = \mathbf{S}(x) \text{ a.e. in } \Omega \, .$$

Taking $\beta = 0$ in equation (30) leads to

$$\sum_{i\in I} \int_{S_i} u_i \text{div}\vec{v}_i dx = \int_{\Omega} \mathbf{S}(x) u(x) dx, \quad \forall u \in L^2(\Omega) \, .$$

$\vec{v} \in H(\text{div}, \Omega)$ implies $\text{div}\vec{v} \in L^2(\Omega)$ and therefore the preceding equation is equivalent to

$$\int_{\Omega} \left(\text{div}\vec{v} - \mathbf{S}(x)\right) u(x) dx = 0, \quad \forall u \in L^2(\Omega) \, ,$$

which means

$$\operatorname{div}\vec{v} = \mathbf{S}(x) \text{ a.e. in } \Omega,$$

and then (43) is proved.

We know (by assumption) that $\lambda = \psi$ on $\partial_P \Omega$, and (by reasoning) that $q = \lambda$ on $\partial_P\Omega$; therefore it is obvious that :

(44) $$q = \psi \text{ on } \partial_P\Omega.$$

We are going now to prove

(45) $$\vec{v}.\vec{n} = g \text{ on } \partial_f\Omega..$$

In relation (40), for each $\Gamma_{if} \subset \partial_f\Omega$ and for all β such that :

(46) $$\beta \in L^2(\Gamma), \beta = 0 \text{ on } \Gamma - \Gamma_{if}$$

one gets :

(47) $$\int_{\Gamma_{if}} (\vec{v}_i.\vec{n}_i)\beta d\gamma = \int_{\Gamma_{if}} g\beta d\gamma \ ,$$

and this implies

(48) $$\vec{v}_i.\vec{n}_i = g \text{ on } \Gamma_{if}, \quad \forall i \in I_f \ ,$$

or, which is equivalent,

(49) $$\vec{v}.\vec{n} = g \text{ on } \partial_f\Omega.$$

Hence (45) is established.

5. Practical Approach for DMHFE-Discretization

We deal now with the practical aspects of the dual mixed hybrid finite element discretization of the system of equations (1)-(4) . The approach that we present in this section is based upon ideas of Chavent and Roberts [3]. It leads to a physically comprehensive linear system where we have made use of Raviart and Thomas [8] finite elements of lowest degree.

5.1 Practical Construction of the Discrete Approximate Problem

The domain Ω is divided into open non-empty subdomains S. Each subdomain is a triangle or a quadrangle . In the sequel we adopt the same notation as in the preceding section : ∂S is the boundary of S, D_h is the set of the subdomains, . . . We introduce also new symbols : E the set of all the edges, E(S) the set of the edges associated to S, . . .

In each subdomain S, Darcy's law and the continuity equation write respectively :

$$(50)\qquad \begin{aligned} &(i)\quad \vec{v} = -\frac{\overline{\overline{K}}}{\mu}\left(\vec{\mathrm{grad}}\,p - \vec{f}\right) \\ &(ii)\quad \mathrm{div}\,\vec{v} = \mathbf{S} \end{aligned} \qquad \text{in } S.$$

In what follows we shall carry out the discrete version of Darcy's law (equation$(50)_{(i)}$).

Setting $\overline{\overline{A}} = \left(\overline{\overline{K}}/\mu\right)^{-1}$, equation $(50)_{(i)}$ is equivalent to

$$(51)\qquad \overline{\overline{A}}\,\vec{v} = -\left(\overrightarrow{\mathrm{grad}}\,p - \vec{f}\right) \text{ in } S.$$

By the principle of virtual works, the preceding equation is equivalent to

$$(52)\ \int_S \overline{\overline{A}}\,\vec{v}.\vec{w}\,dx = \int_S P_S.\mathrm{div}\,\vec{w}\,dx - \int_{\partial S} P_{\partial S}\,\vec{w}.\vec{n}\,ds + \int_S \vec{f}.\vec{w}\;dx,\quad \forall \vec{w}\ \text{test} - \text{function}$$

after integration by parts. A variational approximation of this equation consists to write :

$$(53)\qquad \int_S \overline{\overline{A}}\,\vec{v}_S.\vec{w}_h\,dx = \int_S P_S.\mathrm{div}\,\vec{w}_h\,dx - \int_{\partial S} P_{\partial S}\,\vec{w}_h.\vec{n}\,ds + \int_S \vec{f}.\vec{w}_h\,dx\,,$$

where $\vec{w}_h$ lies in a finite-dimensional test-function space V^S (which is defined later), and where $\vec{v}_S$ and P_S are approximate values in S of the velocity v and the pressure p respectively, while $P_{\partial S}$ is an approximation of p on the boundary ∂S.

We suppose, for sake of simplicity, that P_S is a constant (equal to the mean pressure in S) and on each edge $e \subset \partial S$, $P_{\partial S}$ is a constant denoted $P_{\partial S, e}$ in the sequel. With these notations, and using the divergence-theorem, equation (53) writes :

$$(54) \quad \int_S \overline{\overline{A}}\vec{v}_S.\vec{w}_h \, dx = \sum_{e \in E(S)} \left[\left(P_S - P_{\partial S,e}\right) \int_e \vec{w}_h.\vec{n} \, d\gamma \right] + \int_S \vec{f}.\vec{w}_h \, dx, \ \forall \vec{w}_h \in V^S ,$$

where E(S) is the set of edges included in ∂S.

Let $\left\{\vec{\Phi}_e\right\}_{e \in E(S)}$ be a basis of V^S (usually called local basis). Then (54) is equivalent to

$$(55) \quad \int_S \overline{\overline{A}}\vec{v}_S.\vec{\Phi}_e \, dx = \sum_{e' \in E(S)} \left[\left(P_S - P_{\partial S,e'}\right) \int_{e'} \vec{\Phi}_e.\vec{n} \, d\gamma \right] + \int_S \vec{f}.\vec{\Phi}_e \, dx, \ \forall e \in E(S) .$$

The summation in the right hand side of (55) can be considerably simplified if one assumes that :

$$(56) \qquad \int_{e'} \vec{\Phi}_e.\vec{n} \, d\gamma = \delta_{ee'} \, (= \text{Kronecker symbol}), \ \forall e, e' \in E(S) .$$

With the preceding assumptions, equation (55) is equivalent to

$$(57) \qquad \sum_{e' \in E(S)} d(S)_{e'} \int_S \overline{\overline{A}}\vec{\Phi}_{e'}.\vec{\Phi}_e dx = P_S - P_{\partial S,e} + \int_S \vec{f}.\vec{\Phi}_e \, dx, \qquad \forall e \in E(S) ,$$

where we have made use of the decomposition of $\vec{v}_S$ on the basis $\left\{\vec{\Phi}_{e'}\right\}_{e' \in E(S)}$, i.e.

$$(57bis) \qquad \vec{v}_S = \sum_{e' \in E(S)} d(S)_{e'} \vec{\Phi}_{e'}$$

with $d(S)_{e'}$ = unknown components to determine.

In what follows we shall give explicitely the components of the basis vector-valued functions $\left\{\vec{\Phi}_{e'}\right\}_{e' \in E(S)}$.

Since calculations on computer with polynomials are easy, the basis functions are required to have polynomial components. If $\left\{\vec{\Phi}_{e'}\right\}_{e' \in E(S)}$ are polynomial functions of higher degree, the left hand side of (57) turns out to be difficult to calculate. Hence, these basis-functions are required to be of the lowest-degree.

A result of convergence (of the numerical solution to the exact solution) due to Raviart - Thomas [8] shows that functions of local basis $\left\{\vec{\Phi}_{e'}\right\}_{e'\in E(S)}$ are of the form (ax_1 + b, ax_2 + c) if S is a triangle and (ax_1 + b, cx_2 + d) if S is a quadrangle.

Introducing the concept of reference Finite Element (see figures 1 and 2 below), (56) implies that :

- on a reference triangle T, we have

(58) $\vec{\Phi}_I(\xi) = (\xi_1, \xi_2 - 1),\ \vec{\Phi}_{II}(\xi) = (\xi_1 - 1, \xi_2),\ \vec{\Phi}_{III}(\xi) = (\xi_1, \xi_2)$;

- on a reference square C, we have

(59)
$\vec{\Phi}_I(\xi) = (\xi_1, 0),\ \vec{\Phi}_{II}(\xi) = (\xi_1 - 1, 0),\ \vec{\Phi}_{III}(\xi) = (0, \xi_2),\ \vec{\Phi}_{IV}(\xi) = (0, \xi_2 - 1).$

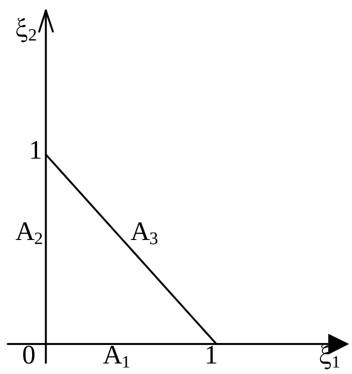


Figure 1: Reference Triangle

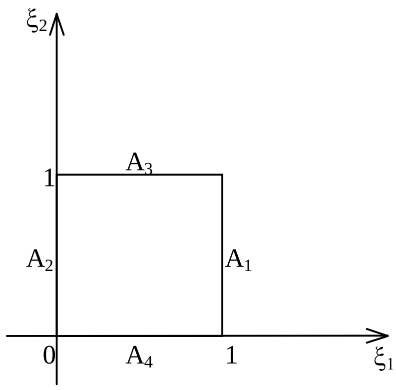


Figure 2: Reference Square

We must mention that the reference finite element is used for calculations of the integrals in (57).

Let us introduce the matrix $\overline{\overline{M}}^{S} = (M^{S}_{ee'})$ and the vector $\vec{b}^{S} = (b_{S,e})$ defined by the following relations, for each cell-grid S :

$$M^{S}_{ee'} = \int_{S} \overline{\overline{A}}(x)\vec{\Phi}_{e}.\vec{\Phi}_{e'}dx \text{ and } \quad b^{S}_{e} = \int_{S} \vec{f}(x).\vec{\Phi}_{e}(x)dx, \ \forall e, e' \in E(S) .$$

Since $\overline{\overline{M}}^{S} = (M^{S}_{ee'})$ is symmetrical, relation (57) can rewrite in the following vectorial form, for each cell-grid S :

(60) $$\overline{\overline{M}}^{S} \vec{v}^{S} = P_{S} \vec{\prod}^{S} - \vec{P}^{\partial S} + \vec{b}^{S}$$

where $\vec{P}^{\partial S} = (P_{\partial S,e})_{e \in E(S)}$ and

$$\vec{\prod}^{S} = \begin{cases} (1,1,1) \text{ if S is a triangle} \\ (1,1,1,1) \text{if S is a quadrangle} \end{cases} .$$

$\overline{\overline{M}}^{S} = (M^{S}_{ee'})$ can prove to be positive-definite, thus invertible. Multiplying (60) by $\left(\overline{\overline{M}}^{S}\right)^{-1}$, the discrete Darcy's law follows, i.e.

(61) $$\vec{v}^{S} = \left(\overline{\overline{M}}^{S}\right)^{-1} \left[P_{S} \vec{\prod}^{S} - \vec{P}^{\partial S}\right] + \left(\overline{\overline{M}}^{S}\right)^{-1} \vec{b}^{S}, \ \forall S \in D_{h} .$$

Integrating the continuity equation (equation (51)) in S, where the exact velocity $\vec{v}$ is replaced by $\vec{v}^{S}$, applying the divergence theorem and using (56) and (57bis), one gets :

(62) $$\sum_{e \in E(S)} d(S)_{e} = \int_{S} \mathbf{S}(x)dx, \ \forall S \in D_{h} .$$

Remark 1 : One can easily prove that the components $\{d(S)_{e}\}$ of the discrete Darcy velocity in the cell-grid S are the flow-rates across the edges e of S.

To summarize, the flow in each cell-grid S can be described (approximately) by equations (61)-(62). The description of the flow in the whole porous medium Ω will be complete if one takes into account two main features :

(i) the no-jump condition at the inter-block frontiers e for the pressure and the flow-rate :

$$\left.\begin{array}{l}(63)\ P_{\partial S,e} = P_{\partial S',e} \\ (64)\ d(S)_e + d(S')_e = 0\end{array}\right\} \quad \text{on } e = \partial S \cap \partial S' ;$$

(ii) the mixed-boundary conditions on the pressure and the flux, i.e.

$$P_{\partial S,e} = \frac{1}{|e|}\int_e \psi d\gamma, \forall e = \partial S \cap \partial_p \Omega , \tag{65}$$

$$d(S)_e = \frac{1}{|e|}\int_e g d\gamma, \ \forall e = \partial S \cap \partial_f \Omega . \tag{66}$$

Remark 2 : We suppose in what precedes that the grid is regular in the sense that :

(i) $\forall S, S' \in D_h$, $S \cap S' =$ an edge or a vertex or $\varnothing$

(ii) $\forall e \in E_h$ (set of edges), $(e \subset \partial\Omega) \Rightarrow (e \subset \partial_p \Omega \text{ or } e \subset \partial_f \Omega)$.

We recall that $\partial\Omega = \partial_p \Omega \cup \partial_f \Omega$ and $\partial_p \Omega \cap \partial_f \Omega$.

The system of equations (61)-(66) is linear and constitute a discrete version of (1)-(6). The unknowns, in this system, are: the flow-rate $\{d(S)_e\}$ (resp. the pressure $\{P_{\partial S,e}\}$) across (resp. on) the edge of each cell-grid, $\{P_S\}$ the pressure $\{P_S\}$ in the cell-grid.

Each triangle corresponds to seven unknowns and each quadrangle corresponds to nine unknowns. Therefore the total number of unknowns is $7*N_T + 9*N_Q$, where N_T is the number of triangles and N_Q is the number of quadrangles.

5.2 Resolution of the discrete problem (61)-(66).

Let us recall that E_h = set of the edges, and D_h = set of the blocks (or cell-grids). In what follows we denote E_h^{int} the set of interior edges.

We can see from system (63) that if we set

$$\lambda_e \equiv P_{\partial S,e} = P_{\partial S',e}, \ \forall e \in E_h^{int},$$

the system (61)-(66) reduces to the following equivalent form :

$$(67) \qquad d(S)_e = P_S \Delta_e^S - \sum_{e'} \left(\overset{=S}{M} \right)^{-1}_{ee'} \lambda_{e'}, \quad \forall S$$

$$(68) \qquad \sum_{e} d(S)_e = \int_S \mathbf{S}(x)dx, \quad \forall S$$

$$(69) \qquad d(S)_e + d(S')_e = 0, \qquad \forall e \in E_h^{int}$$

$$(70) \qquad \lambda_e = \frac{1}{|e|} \int_e \psi d\gamma, \ \forall e \subset \partial_p \Omega$$

$$(71) \qquad d(S)_e = \frac{1}{|e|} \int_e g d\gamma, \quad \forall e \subset \partial_f \Omega .$$

Let us present the first step of resolution of the equations (67) - (71).

<u>First step</u>. Elimination of the unknowns $\{d(S)_e\}$.

Reasoning by substitution, the system (67) allows to eliminate the unknowns $\{d(S)_e\}$ in (68), (69) and (71). Therefore the obtained system writes, if we put :

$$\Delta^S = \sum_{e \in E(S)} \Delta_e^S ,$$

$$(72)_1 \qquad \Delta^S P_S - \sum_{e'} \Delta_{e'}^S \lambda_{e'} = \int_S \mathbf{S}(x)dx, \quad \forall S$$

$(72)_2$

$$\sum_{\{S \in D_h : e \in E(S)\}} \Delta_e^S P_S = \sum_{\{S \in D_h : e \in E(S)\}} \left(\sum_{e' \in E(S)} \left(\overset{=S}{M} \right)^{-1}_{ee'} \lambda_{e'} \right), \quad \forall e \in E_h^{int}$$

$$(72)_3 \qquad \Delta_e^S P_S - \sum_{e'} \left(\overline{\overline{M}}^S \right)^{-1}_{ee'} \lambda_{e'} = \frac{1}{|e|} \int_e g\,d\gamma, \quad \forall e \in E_h, \quad e \subset \partial_f \Omega$$

$$(72)_4 \qquad \lambda_e = \frac{1}{|e|} \int_e \psi\,d\gamma, \ \forall e \in E_h, \ e \subset \partial_p \Omega.$$

The relations $(72)_1$-$(72)_4$ form a linear algebraic system with unknowns P_S and λ_e, for all $S \in D_h$ and $e \in E_h$.

<u>Second step.</u> Elimination of the unknowns P_S .

The system $(72)_1$ gives :

$$P_S = \frac{1}{\Delta S} \sum_{e'} \Delta_{e'}^S \lambda_{e'} + \frac{1}{\Delta S} \int_S \mathbf{S}(x)dx, \quad \forall S \in D_h .$$

Multiplying the preceding relation by Δ_e^S gives :

$$\Delta_e^S P_S = \sum_{e'} \frac{\Delta_e^S \Delta_{e'}^S}{\Delta S} \lambda_{e'} + \frac{\Delta_e^S}{\Delta S} \int_S \mathbf{S}(x)dx, \quad \forall S \in D_h .$$

Substituting in equations $(72)_2$-$(72)_3$ one obtains :

$$(73) \qquad \sum_{\{S : e \in E(S)\}} \left(\sum_{e'} \left[\left(\overline{\overline{M}} \right)^{-1}_{ee'} - \left(\overline{\overline{N}}^S \right)_{ee'} \right] \lambda_{e'} \right)$$

$$= \sum_{\{S : e \in E(S)\}} \frac{\Delta_e^S}{\Delta S} \int_S \mathbf{S}(x)dx + 1_{\partial_f \Omega}(e) \left[\frac{1}{|e|} \int_e g\,d\gamma \right]$$

for all $e \in E_h$, e being an interior edge or $e \subset \partial_f\Omega$, where we have put

$$(74) \qquad 1_{\partial_f \Omega}(e) = \begin{cases} 1 \text{ if } e \in \partial_f \Omega \\ 0 \text{ in other cases} \end{cases} \quad \text{and} \quad \left(\overline{\overline{N}}^S \right)_{ee'} = \frac{\Delta_e^S \Delta_{e'}^S}{\Delta^S} .$$

We obtain finally a linear square system with unknowns $\{\lambda_e\}$. The matricial form of this system is :

$$(75) \qquad \begin{bmatrix} \overline{\overline{G}} & \overline{\overline{H}} \\ 0 & \overline{\overline{I}} \end{bmatrix} \begin{pmatrix} \vec{\Lambda}_1 \\ \vec{\Lambda}_2 \end{pmatrix} = \begin{pmatrix} \vec{R}_1 \\ \vec{R}_2 \end{pmatrix},$$

or, what is the same,

(76) $$\begin{cases} \overline{\overline{G}}\vec{\Lambda}_1 + \overline{\overline{H}}\vec{\Lambda}_2 = \vec{R}_1 \\ \vec{\Lambda}_2 = \vec{R}_2 . \end{cases}$$

where $\overline{\overline{G}}$ is a square matrix that can prove to be positive definite, $\overline{\overline{H}}$ is a rectangular matrix and $\overline{\overline{I}}$ the unit (square) matrix. One can then solve (73) by a classical algorithm (Gauss for instance). At this stage one can consider that the quantities $\{ \lambda_e \}$ are known.

<u>Third Step</u>. Calculation of P_S (pressure in each cell-grid S) and $d(S)_e$ (flow-rate across each edge e).

For each $S \in D_h$, one gets P_S directly from relation $(72)_1$:

(77) $$P_S = \frac{1}{\Delta S}\left\{ \sum_{e' \in E(S)} \Delta^S_{e'} \lambda_{e'} + \int_S \mathbf{S}(x)dx \right\} .$$

Finally one gets the flow-rate $d(S)_e$, for each $e \in E_h$, either from (71), i.e.

(78) $$d(S)_e = \frac{1}{|e|}\int_e g d\gamma ,$$

if e is an edge such that $e \subset \partial_f \Omega$, or from (67), i.e.

(79) $$d(S)_e = \Delta^S_e P_S - \sum_{e' \in E(S)} \left(\overline{\overline{M}}^S \right)^{-1}_{ee'} \lambda_{e'} ,$$

if e = interior edge or $e \subset \partial_p \Omega$.

4.3 – Results of convergence.

Before giving the order of convergence let us recall the definition of some Sobolev spaces and associated standard norms. We start with some preliminaries. We set $\alpha = (\alpha_1, \alpha_2)$, where α_1 and α_2 are non-negative integers; we denote $|\alpha|$ the sum of α_1 and α_2 and $D^\alpha u$ the partial derivatives (in the distributional sense) of u defined as follows :

$$D^{\alpha}u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} .$$

<u>Convention</u> : when $\alpha = (0,0)$, we have by convention $D^{\alpha}u = u$.

For a non-negative integer m, we set :

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : D^{\alpha}u \in L^2(\Omega),\ \forall\alpha \text{ such that } |\alpha| \le m \right\}.$$

According to the preceding convention, we have $H^0(\Omega) = L^2(\Omega)$ .The space $H^m(\Omega)$ is a Hilbert space equipped with the scalar product :

$$\langle u, v\rangle = \sum_{\alpha:|\alpha|\le m} \int_{\Omega} D^{\alpha}u D^{\alpha}v dx, \qquad \forall u, v \in H^m(\Omega)$$

which associated norm is :

$$\|u\|_{m,\Omega} = \left(\sum_{\alpha:|\alpha \le m|} \int_{\Omega} \left|D^{\alpha}u\right|^2 dx \right)^{1/2} .$$

We shall make use of the following functional space :

$$H(\mathrm{div},\Omega) = \left\{ \vec{w} : \Omega \to IR^2 ;\ \vec{w} \in \underline{L}^2(\Omega),\ \mathrm{div}\vec{w} \in L^2(\Omega) \right\}$$

with its standard norm :

$$\|\vec{w}\|_{H(\mathrm{div},\Omega)} = \left(\int_{\Omega} |\vec{w}|^2 dx + \int_{\Omega} |\mathrm{div}\vec{w}|^2 dx \right)^{1/2} .$$

As it follows from Theorem 2, the solution $(\vec{v}, q, \lambda)$ of the variational problem (24)-(26) satisfies:

(i) $\vec{v} \in H(\mathrm{div},\Omega)$;

(ii) $q \in H^1(\Omega)$, with $q = \lambda$ on Γ .

Roberts and Thomas [9] have shown the following estimate :

$$\| q - q_h \|_{0,\Omega} + \| \vec{v} - \vec{v}_h \|_{H(\mathrm{div},\Omega)} \le C(\vec{v}, q)h .$$

For sufficiently regular permeability coefficients with a square grid, Chavent and Roberts [3] have established that

$$\| q - q_h \|_{0,\Omega} + \|\vec{v} - \vec{v}_h\|_{H(div,\Omega)} \le C^{te} h \left(\|q\|_{2,\Omega} + \|\vec{v}\|_{H(div,\Omega)} \right).$$

5. APPLICATION: COMPUTATION OF THE HOMOGENIZED ABSOLUTE PERMEABILITY TENSOR

5.1 – Introduction.

The calculation of the homogenized absolute permeability by empirical formulae are revealed not satisfactory. In the petroleum litterature the modern approach consists to use the so-called numerical methods based upon a numerical resolution of partial differential equations.

In the following lines we show how mixed hybrid finite element method can apply for computing the homogenized absolute permeability following the modern approach.

5.2 – Homogenized absolute permeability tensor.

Let $\Omega \subset IR^n$ ($1 \le n \le 3$ in the practice) be a heterogeneous porous medium with a periodic micro-structure. The permeability tensor $\overline{\overline{K}}$ associated to Ω is a spatially varying function.

If we suppose that $\overline{\overline{K}}$ depends only the local variable denoted y, one can prove (see Njifenjou[5]) that the homogenized permeability coefficients are given by:

(80) $$K^*_{ij} = \frac{1}{|Y|} \int_Y \overline{\overline{K}}(y) \vec{grad} P_i . \vec{grad} P_j dy, \quad \forall i, j = 1,..., n,$$

where Y is the reference period of the medium, and where the functions P_k, $k = 1,..., n$, are solutions of the family of partial differential equations :

$$(81)_1 \qquad -\operatorname{div}\left[\overline{\overline{K}}(y)\overrightarrow{\operatorname{grad}}P_k\right] = 0 \text{ in } IR^n$$

$$(81)_2 \qquad P_k(y) - y_k \text{ is } Y - \text{periodic.}$$

Applying a Green theorem to the integral in the right hand side of (80), one gets:

$$(82) \qquad \int_Y \overline{\overline{K}}(y)\overrightarrow{\operatorname{grad}}P_i.\overrightarrow{\operatorname{grad}}P_j dy = \int_Y \operatorname{div}\left(P_j \overline{\overline{K}}\overrightarrow{\operatorname{grad}}P_i\right)dy - \int_Y P_j \operatorname{div}\left(\overline{\overline{K}}\overrightarrow{\operatorname{grad}}P_i\right)dy .$$

Taking into account the equation $(81)_1$, it is obvious that the second integral in the right hand side of (82) is equal to zero ; applying the divergence theorem to the first integral, we obtain the

Theorem

$$(83) \quad K^*_{ij} = \frac{-1}{|Y|} \int_{\partial Y} P_j(y)\left[\vec{v}_i(y).\vec{n}\right]d\gamma, \quad \forall i, j = 1,..., n, \text{ where } \vec{v}_i = -\overline{\overline{K}}\overrightarrow{\operatorname{grad}}P_i .$$

It is now clear that a numerical resolution of $(81)_1$-$(81)_2$ by the dual mixed hybrid finite element method (see section 4) yields approximations of the pressure

$$(84) \qquad P_j \approx \text{constant} = \lambda_j(e) \text{ on each edge } e \subset \partial Y, \ \forall j = 1,..., n$$

and the total flow-rate

$$(85) \qquad \int_e \vec{v}_i.\vec{n}d\gamma \approx d^i_e, \quad \forall e \subset \partial Y, \ \forall i = 1,..., n .$$

We deduce from what precedes that

$$(86) \qquad K^*_{ij} \approx -\frac{1}{|Y|} \sum_{\{e:e\subset\partial Y\}} \lambda^j_e d^i_e , \quad \forall i, j = 1,..., n.$$

6. Conclusion.

In this paper we have presented the theoretical as well as the practical aspects of the Dual Mixed Hybrid Finite Element (DMHFE) on a model flow problem. The use of Raviart-Thomas space of lowest degree allows to obtain a simple linear system and then a weak cost of computation.

We must mention that the originality of this paper lies in two directions :

- a proof of the equivalence between (1)-(4) and (24)-(26) ;

-the presentation of a physically comprehensive discrete version of the so-called dual mixed hybrid variational formulation.

In this work we have also shown an application dealing with computation of the homogenized permeability tensor of a periodic heterogeneous medium.

REFERENCES :

[1] I. Babuska, J. T. Oden, and J. K. Lee : Mixed-hybrid finite element approximations of second ordre elliptic boundary value problems. Comput. Methods, APPL. Mech. Engrg. (1977), 11, pp. 175-206.

[2] F. Brezzi : On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, RAIRO Analyse Num. (1974), 122, pp. 129-151.

[3] G. Chavent and J. E. Roberts : A unified presentation of mixed and mixed-hybrid finite element, and finite differences for the determination of velocities in waterflow problems. Rapport INRIA (1989), Rocquencourt 78153 Le Chesnay (France).

[4] A. Njifenjou : Eléments finis mixtes hybrides duaux et homogénéisation des paramètres pétrophysiques. Application à l'étude numérique d'écoulements en milieu poreux. Thèse de Doctorat de l'Université de Paris VI, 1993 (France).

[5] A. Njifenjou : Expression en termes d'énergie pour la perméabilité absolue effective. Application au calcul numérique d'écoulement en milieu poreux. Revue de l'Institut Français du Pétrole, vol. 49, N° 4, Juillet-Août 1994, pp. 345-358.

[6] T. H. H. Pian and P. Tong : Basis of finite element methods for continua. Int. J. Numerical Method Eng. 1 (1969), pp. 3-28.

[7] T. H. H. Pian : Finite element formulation by variational principle with relaxed continuity requirements. The mathematical foundations of the finite element method with application to partial differential equations. pp. 671-687. Academic Press, New York (1973).

[8] P. A. Raviart and J. M. Thomas : Primal hybrid finite element method for second order elliptic equations. Université de Paris VI. Labo. d' Analyse numérique. Enreg. N° 75021.

[9] J. E. Roberts and J. M. Thomas : Mixed and hybrid finite element methods. Rapport INRIA (1987), Rocquencourt 78153 Le Chesnay (France).

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