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REPUBLIC OF CAMEROON  
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DÉPARTEMENT DE MATHÉMATIQUES

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## ATTESTATION DE CORRECTION DU DOCTORAT/Ph.D

Nous soussignés, Pr. AYISSI Raoul Domingo, Président du jury, Pr. NANA Cyrille, Examineur du jury de la thèse de/ Doctorat/Ph.D. présenté par DEFO Hugues Olivier, Matricule 10V0559, intitulé : « Hankel and Toeplitz operators on vector-valued Bergman spaces » et soutenu en vue de l'obtention du diplôme de Doctorat/Ph.D en Mathématiques, attestons que cette thèse a été réalisée dans le cadre des exigences de l'Université de Yaoundé I, conformément aux normes académiques en vigueur, et que les corrections et recommandations demandées par le jury de soutenance en vue de l'amélioration de ce travail, ont été effectuées.

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CENTRE DE RECHERCHE ET DE FORMATION  
DOCTORALE EN SCIENCES TECHNOLOGIE  
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RESEARCH AND TRAINING UNIT FOR  
DOCTORATE IN MATHEMATICS,  
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APPLICATION

**DEPARTEMENT DE MATHEMATIQUES**

*DEPARTMENT OF MATHEMATICS*

**Laboratoire d'Analyse et Applications**

*Laboratory of Analysis and Applications*

# HANKEL AND TOEPLITZ OPERATORS ON VECTOR-VALUED BERGMAN SPACES

**THESIS**

Submitted to the graduate school in fulfilment of the requirements for the degree of Doctor of  
Philosophy/Doctorat PhD in Mathematics

**Option :** Classical Analysis

By

**Hugues Olivier DEFO**

Student registration number : 10V0559



Under the supervision of :

**Edgar TCHOUNDJA**

*Associate Professor  
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**Brett D. Wick**

*Professor  
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# ♣ DECLARATION OF ORIGINALITY



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This is to certify that to my knowledge, the content of this thesis is the product of my work. I confirm that this work has been done after the registration for the degree of PhD of Mathematics at the University of Yaounde I. I attest that I have not used inadmissible help of a third parties to produce this work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due to reference to the literature. I also attest that this work has not been previously included in a thesis or a dissertation submitted to this or any other institution for a degree or other purposes.

Date: April 3, 2023

Hugues Olivier DEFO

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## ♣ Résumé ♣

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Les opérateurs de Hankel et de Toeplitz sont deux classes naturelles d'opérateurs spécifiques sur des espaces de fonctions holomorphes. L'étude de ces opérateurs sur différents espaces de fonctions holomorphes n'est pas seulement motivée par les défis mathématiques qu'ils soulèvent, mais aussi par de nombreuses applications. La tâche principale de cette thèse est d'étudier la continuité et la compacité des opérateurs de Hankel et de Toeplitz agissant sur les espaces de Bergman à poids à valeurs vectorielles  $A_\alpha^p(\mathbb{B}_n, E)$  où  $p > 0$ ,  $\alpha > -1$  et  $E$  est un espace Banach complexe. Nous étudions d'abord la continuité du petit opérateur de Hankel  $h_b$  de  $A_\alpha^p(\mathbb{B}_n, E)$  dans  $A_\alpha^q(\mathbb{B}_n, F)$ , où  $b : \mathbb{B}_n \rightarrow \mathcal{L}(E, F)$  est un symbole holomorphe,  $E, F$  sont deux espaces de Banach complexes et  $0 < p, q < \infty$ . Pour  $1 < p, q < +\infty$ , la caractérisation de la continuité du petit opérateur de Hankel entre les espaces de Bergman à valeurs vectorielles  $A_\alpha^p(\mathbb{B}_n, E)$  et  $A_\alpha^q(\mathbb{B}_n, F)$  a été obtenue par Oliver [52]. Nous avons amélioré son résultat dans le cas  $1 < p < q < +\infty$ . Nous avons aussi caractérisé les symboles pour lesquels le petit opérateur de Hankel s'étend en un opérateur continu dans le cas  $0 < p, q \leq 1$ . Le cas  $p > 1$  et  $q = 1$  reste ouvert même dans le cas classique (lorsque  $E = F = \mathbb{C}$ ). Nous avons caractérisé les symboles holomorphes  $b$  pour lesquels le petit opérateur de Hankel  $h_b$  s'étend en un opérateur compact de  $A_\alpha^p(\mathbb{B}_n, E)$  dans  $A_\alpha^q(\mathbb{B}_n, F)$ , où  $1 < p \leq q < +\infty$  et  $E, F$  sont deux espaces de Banach réflexifs.

Dans la suite, si  $E$  et  $F$  sont deux espaces de Banach complexes, et  $\alpha > -1$ , on considère l'opérateur de Toeplitz  $T_b$  de symbole  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . On montre que pour deux espaces de Banach complexes  $E, F$  l'opérateur de Toeplitz  $T_b$  est continu de  $A_\alpha^2(\mathbb{B}_n, E)$  dans  $A_\alpha^2(\mathbb{B}_n, F)$  si et seulement si sa transformée de Berezin  $\tilde{b}$  est bornée sur  $\mathbb{B}_n$ . Lorsque  $E = F = \mathbb{C}^d$ , où  $d \geq 1$  est un entier, on montre que pour  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , l'opérateur de Toeplitz  $T_b$  est compact sur  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  si et seulement si pour tout vecteur unitaire  $e \in \mathbb{C}^d$ ,  $\tilde{b}(z)e \rightarrow 0$  faiblement dans  $\mathbb{C}^d$  quand  $|z| \rightarrow 1^-$ . Ce résultat stipule que la compacité des opérateurs de Toeplitz  $T_b$  de symbole  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , sur l'espace de Bergman à valeurs vectorielles  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  est entièrement déterminée par le comportement au bord de leurs transformées de Berezin.

**Mots et phrases clés:** Petit opérateur de Hankel, opérateur de Toeplitz, espace de Bochner-Lebesgue, espace de Bergman à valeurs vectorielles, espace de type Bloch à valeurs vectorielles, mesures de Carleson, transformée de Berezin, espace  $BMO$ .

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# ♣ Abstract ♣

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Hankel and Toeplitz operators are two natural classes of specific operators on spaces of holomorphic functions. The study of these operators on different holomorphic spaces is not only motivated by the mathematical challenges it raises, but also by many applications on applied mathematics and in theoretical physics. The main task of this thesis is to study the boundedness and the compactness of Hankel and Toeplitz operators acting on the weighted vector-valued Bergman spaces  $A_\alpha^p(\mathbb{B}_n, E)$  where  $p > 0$ ,  $\alpha > -1$  and  $E$  is a complex Banach space. We first study the boundedness of the little Hankel operator  $h_b$  from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$ , where  $b : \mathbb{B}_n \rightarrow \mathcal{L}(E, F)$  is a holomorphic operator-valued symbol,  $E, F$  are two complex Banach spaces and  $0 < p, q < \infty$ . For  $1 < p, q < \infty$ , the characterization of those holomorphic operator-valued symbols for which  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  was obtained by Oliver [52]. We have improved his result in the case  $1 < p < q < \infty$ . We have also characterized the boundedness of  $h_b$  in the case  $0 < p, q \leq 1$ . The case  $p > 1$  and  $q = 1$  is still open even in the classical case (when  $E = F = \mathbb{C}$ ). We have characterized those operator-valued holomorphic symbols  $b$  for which the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , is a compact operator where  $1 < p \leq q < \infty$  and  $E, F$  are reflexive Banach spaces.

Next, if  $E$  and  $F$  are two complex Banach spaces and  $\alpha > -1$ , we consider the Toeplitz operator  $T_b$ , with the symbol  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . We show that for two complex Banach spaces  $E, F$ , the Toeplitz operator  $T_b$  extends into a bounded from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$  if and only if its Berezin transform  $\tilde{b}$  is bounded on  $\mathbb{B}_n$ . When  $\mathcal{H} = \mathbb{C}^d$ , where  $d \geq 1$  is an integer, we proved that for  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , the Toeplitz operator  $T_b$  is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  if and only if for all unit vector  $e \in \mathbb{C}^d$ ,  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$ . This result states that the compactness of Toeplitz operators  $T_b$  with  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , on the vector-valued Bergman space  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  is completely determined by the boundary behaviour of the Berezin transform of  $b$ .

**Key words and phrases:** Little Hankel operator, Toeplitz operator, Bochner-Lebesgue space, vector-valued Bergman space, vector-valued Bloch type space, Carleson measures, Berezin transform, *BMO* space.

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# Dedication

*This work is dedicated to my mother **Ngandjou Micheline** who did not have time to see this work succeed.*

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# ♣ Introduction ♣

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Let  $p > 0$ , and  $n \geq 1$  be a nonnegative integer. Let  $\Omega$  be a region in  $\mathbb{C}^n$ . The classical Bergman space  $A^p(\Omega)$  consists of those holomorphic functions  $f$  on  $\Omega$  such that

$$\|f\|_{p,\nu} = \left( \int_{\Omega} |f(z)|^p d\nu(z) \right)^{1/p} < \infty,$$

where  $d\nu$  is the Lebesgue measure on  $\mathbb{C}^n$ .

The theory of Bergman spaces has been extensively studied and many results in operator theory, and function theory on Bergman spaces are now well-known (see for example [78] [77] [14]). Although there are still several difficult open problems on operator theory in Bergman spaces (boundedness criterion of the Toeplitz operator, boundedness criterion of the Hankel operator, boundedness estimates for the Bergman projection on tubular domains,  $\dots$ ), very little has been done in the case of vector-valued Bergman spaces.

For a Banach space  $E$ , the vector-valued Bergman space on  $\Omega$ , denoted by  $A^p(\Omega, E)$ , consists of all holomorphic functions  $f : \Omega \rightarrow E$  such that

$$\|f\|_{p,\nu,E} = \left( \int_{\Omega} \|f(z)\|_E^p d\nu(z) \right)^{1/p} < \infty.$$

Hankel and Toeplitz operators are important, natural classes of specific operators on spaces of holomorphic functions. Exploring their properties on a variety of spaces brings deeper understanding not only of the operators but also of the spaces in question, and provides interesting connections between operator theory and function theory. The study of these operators on different analytic spaces is not only motivated by the mathematical challenges it raises, but also by many applications on mathematics and in physics (see for example [55] for more information). Although the vector-valued Bergman spaces have similar properties as the scalar-valued Bergman spaces, this generalization makes the study of vector-valued Bergman spaces much more difficult. Sometimes even the most simple result in the classical case becomes more complicated in the vector-valued case.

There are many authors that are working on vector-valued Bergman spaces, see [4, 15, 18, 16, 17, 24, 25, 26, 27, 28, 42, 47, 52, 57, 58, 60, 68].

If  $P$  denotes the Bergman projection on  $A^2(\Omega)$ , the little Hankel operator of holomorphic symbol  $\varphi$ , denoted  $h_{\varphi}$ , and the Toeplitz operator of measurable symbol  $\psi$  denoted  $T_{\psi}$  are respectively defined by :

$$h_{\varphi}(f) = P(\varphi \bar{f}); \quad T_{\psi}(f) = P(\psi f).$$

The problem of characterizing those symbols for which these operators can be extended into bounded or compact operators between different classical Bergman spaces have been studied in [1, 2, 20, 61, 53, 76, 79].

The same problem in vector-valued settings has been studied recently in [26, 27, 28, 42, 47, 52, 57, 58, 60]. Hankel operators are operators having infinite Hankel matrices (i.e., matrices with entries depending only on the sum of the coordinates) with respect to some orthonormal basis. Finite matrices with this property were introduced by Hankel, who found interesting algebraic properties of their determinants. One of the first results on infinite Hankel matrices was obtained by Kronecker, who characterized Hankel matrices of finite rank as those whose entries are Taylor coefficients of rational functions. Since then, Hankel operators (or matrices) have found numerous applications in classical problems of analysis, such as moment problems, orthogonal polynomials, etc. Hankel operators admit various useful realizations, such as operators on spaces of analytic functions, integral operators on function spaces on  $(0, \infty)$ , operators on sequence spaces.

In 1957, Nehari described the bounded Hankel operators on the sequence space  $\ell^2$ . This description turned out to be very important and started the contemporary period of the study of Hankel operators. It turns out that for the need of applications (see [55, 51]) it is also important to consider vectorial Hankel operators, i.e., Hankel operators on spaces of vector-valued functions.

One of the most beautiful applications of Hankel operators is given in Chapter 15 of [55]. Vectorial Hankel operators are used in the theory of approximation by analytic matrix and operator functions. Chapter 15 of this book also gives a solution to the famous problem of whether a polynomially bounded operator on Hilbert space must be similar to a contraction. This problem remained open for a long time. In particular, it was one of the problems in a famous paper by Paul Halmos called "Ten problems in Hilbert space". Recently it has been solved in the negative with the help of vectorial Hankel operators (see [1]).

There are at least two reasons for the continuous and increasing interest in Toeplitz operators. On the one hand, Toeplitz operators are of importance in connection with a variety of problems in physics, probability theory, information and control theory, and several other fields (see [21, 51]). On the other hand, Toeplitz operators constitute one of the most important classes of non-selfadjoint operators and they are a fascinating example of the fruitful interplay between such topics as operator theory, function theory, and the theory of Banach algebras.

By now, there are many results that relate the boundedness and the compactness of an operator to its Berezin transform (see for example [6, 7, 60, 79]). For the compactness, it is well-known that a bounded operator  $T$  on  $A_0^2(\mathbb{B}_n)$  which can be written as a finite sum of finite products of Toeplitz operators with bounded symbol is compact if and only if its Berezin transform vanishes on the boundary of the unit disk  $\mathbb{D}$ . (Recall that  $\mathbb{D} = \mathbb{B}_1$  and  $A_0^2(\mathbb{B}_n)$  is the standard unweighted Bergman space on the unit ball in  $\mathbb{C}^n$ ). This result was extended by Raimondo to the spaces  $A_\alpha^2(\mathbb{B}_n)$  in [61] and Engliš was also extended the same result to Bergman spaces of bounded symmetric domains in [34].

There are also several results along these lines for more general operators than those that can be written as finite sums of finite products of Toeplitz operators with bounded symbols. In [35], Engliš proved that any compact operator on Bergman spaces is in the operator-norm topology closure of the set of finite sums of finite products of Toeplitz operators with bounded symbol (this is called the Toeplitz algebra and is denoted by  $\mathcal{T}_{p,\alpha}$ .) In [65], Suárez proved that an operator,  $T \in \mathcal{L}(A_0^p(\mathbb{B}_1))$  with  $1 < p < \infty$  is compact if and only if it is in the Toeplitz algebra  $\mathcal{T}_{p,\alpha}$  and its Berezin transform vanishes on the boundary  $\partial\mathbb{B}_n$  of the unit ball. This was extended to the weighted Bergman spaces  $A_\alpha^p(\mathbb{B}_n)$  in [49] by Suárez, Mitkovski and Wick. This result was also extended by Rahm and Wick to the vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, \mathbb{C}^d)$  (where  $d \geq 1$  is an integer) in [59]. For  $p = 2$ , Rahm proved in [60] that a finite sum of finite products of Toeplitz operators with  $d \times d$  matrix valued bounded symbol acting on  $A_\alpha^p(\mathbb{B}_n, \mathbb{C}^d)$  is compact if and only if its Berezin transform vanishes on the boundary of the ball. His result generalized the Raimondo's result in [61].

In this thesis, we are concerned with two main problems. The first problem is to characterize those operator-valued symbols for which the little Hankel operator extends into a bounded (respectively compact) operator from a vector-valued Bergman space to another vector-valued Bergman space. The second problem concerns the Toeplitz operators. More precisely, we study the boundedness and the compactness problem of Toeplitz operators with operator-valued symbol on some vector-valued Bergman spaces via the behaviour of its Berezin transform. We are interested in the extension of the results in [79] to the case of vector-valued functions.

The thesis is organized into five chapters.

## 0.1 Chapter 1: Preliminaries.

Since we are interested in functions which take their values in Banach spaces, it will be natural to recall some useful results on the theory of functional analysis, and the integration of vector-valued functions.

In **Chapter 1**, we collect some definitions, results and notations needed to follow the rest of the thesis. The first section of **Chapter 1** is devoted to some basic notions on functional analysis, integration of vector-valued functions and also on vector measures.

More concretely, in Section 1.1, we recall the Hahn-Banach extension theorem, we define the bidual of a Banach space and we introduce the canonical injection between a Banach space and its bidual. We also state the Uniform boundedness principle and some of its consequences. After that, we introduce the notion of weak topologies and we recall some notions on weak convergence and strong convergence for a sequence in some Banach space. We also state the well-known Banach-Alaoglu theorem and we define reflexive Banach spaces. In Subsection 1.1.2, we define and collect some needed results on bounded linear operators and compact operators on Banach spaces and we also collect some of their characterizations.

In this dissertation, we will often have to integrate vectors in some Banach space. It becomes important to give a sense of the integration of vector-valued functions. In the literature, there are many definitions of the integral of vector-valued functions. In Section 1.2, we adopt the definition introduced by Bochner in [19], since its definition is much similar to the definition of Lebesgue integral. Concretely, in Subsection 1.2.1, we recall some notions on measurability. Next, we define the vector-valued simple function and we introduce the Bochner integral for vector-valued simple functions. After having introduced the measurability of a vector-valued function with respect to a positive measure, we have defined the Bochner integral. Given a measure space  $(\Omega, \mathcal{A}, \mu)$  and a Banach space  $E$ , we have proved that a vector-valued function  $f : \Omega \rightarrow E$  is  $\mu$ -Bochner integrable if and only if the function  $z \mapsto \|f(z)\|_E$  is  $\mu$ -Lebesgue integrable. In Subsection 1.2.3, we introduced the Bochner-Lebesgue  $L^p_\mu(\Omega, E)$  spaces, where  $0 < p < \infty$ . We point out that when  $1 \leq p < \infty$ , the set  $L^p_\mu(\Omega, E)$  is a Banach space and that the space of vector-valued simple functions form a dense subspace of  $L^p_\mu(\Omega, E)$ . If  $0 < p < 1$ , we recall that  $L^p_\mu(\Omega, E)$  is a complete metric space. We next obtain some important results in the vector-valued Bochner-Lebesgue space. Namely, we state the dominated convergence theorem and its converse, continuity under integral symbol, differentiation under integral symbol, Fubini's theorem. In the last part of this section, we find the topological dual space of  $L^p_\mu(\Omega, E)$  for  $1 < p < \infty$ .

In Section 1.3, we introduce vector-measures. We know that in the classical case, the problem of finding symbols for which the little Hankel operator extends into a bounded operator on spaces of analytic functions

is often reduced to finding the topological dual of some of analytic functions spaces. It is well-known in the classical case (when  $E = \mathbb{C}$ ) that the topological dual space of the Lebesgue space  $L^p_\mu(\Omega)$  (with  $1 < p < \infty$ ) can be identified with the space  $L^{p'}_\mu(\Omega)$  where  $1/p + 1/p' = 1$  under the inner product  $\langle f, g \rangle = \int_\Omega f \bar{g} d\mu$ . But in the vector-valued settings, this identification is more difficult since the dual is identified with the space  $V^{p'}_\mu(\Omega, E^*)$  which is the space a  $\mu$ -continuous vector measures with  $p'$ -bounded variation (see Definition 1.3.16) which is larger than the space  $L^{p'}_\mu(\Omega, E^*)$ . In Subsection 1.3.2, we have proved that the topological dual space of  $L^p_\mu(\Omega, E)$  where  $1 < p < \infty$  can be identified with the space  $L^{p'}_\mu(\Omega, E^*)$  if and only if the dual space  $E^*$  of  $E$  has the Radon-Nikodým property (see Definition 1.3.10).

## 0.2 Chapter 2: Vector-valued Bergman spaces.

In **Chapter 2**, we place ourselves in the open unit ball  $\mathbb{B}_n$  of  $\mathbb{C}^n$  and we consider that the measure  $\mu$  is a standard probability measure on  $\mathbb{B}_n$ . In Section 2.1, we introduce the concept of 'strong holomorphy' and 'weak holomorphy' for vector-valued function, we remark that the two notions are equivalent on the unit ball. Given a complex Banach space  $E$ , the complex 'conjugate'  $\bar{E}$  of  $E$  is defined as the set all linear functional  $\bar{x}$  on  $E^*$  such that  $\bar{x}(x^*) := \overline{\langle x, x^* \rangle}_{E, E^*}$ . In Subsection 2.1.1, we discuss about the automorphism group and the Bergman metric. One of the most important classes of automorphisms of  $\mathbb{B}_n$  consists of involutions of  $\mathbb{B}_n$ , which are also called Möbius transformations. These automorphisms play an important role in the theory of holomorphic functions. The invariance of the Bergman metric under these automorphisms will be very useful in this thesis.

In Subsection 2.1.2, we define and collect some useful results on the classical Gamma and Beta functions. Subsection 2.1.3 is devoted to the important notions of differentiations in the unit ball  $\mathbb{B}_n$ . Namely, we define the concept of radial derivative, fractional derivative, for vector-valued functions in  $\mathbb{B}_n$ . We notice that these operators have the same properties as their analogues in the scalar-valued settings. In Subsection 2.1.4, we collect few results about harmonic and subharmonic functions in  $\mathbb{B}_n$ . Some estimates are given in Subsection 2.1.5.

For  $\alpha > -1$ , the weighted Lebesgue measure  $d\nu_\alpha$  is defined by  $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$ , where  $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$  is the normalizing constant such that  $\nu_\alpha(\mathbb{B}_n) = 1$ . Let  $E$  be a Banach space,  $\alpha > -1$  and  $0 < p < \infty$ . We recall that the Bochner-Lebesgue space  $L^p_\alpha(\mathbb{B}_n, E)$  consists of all functions  $f : \mathbb{B}_n \rightarrow E$  such that

$$\|f\|_{p, \alpha, E} = \left[ \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) \right]^{1/p} < \infty.$$

We are now ready to present one of the main topics of this thesis. Let  $E$  be a Banach space,  $\alpha > -1$  and  $0 < p < \infty$ . The vector-valued Bergman space on the unit ball  $A^p_\alpha(\mathbb{B}_n, E)$  is the closed subspace of the Bochner-Lebesgue space  $L^p_\alpha(\mathbb{B}_n, E)$  which consists of vector-valued holomorphic functions. Many standard properties of the theory of vector-valued Bergman spaces have been proved in the vector-valued settings in [16] and recently in [52]. Several results in vector-valued Bergman spaces, can be proved by reducing to the scalar-valued Bergman spaces. This passage is done by using the Hille's fundamental theorem (Proposition 1.2.24). In Subsection 2.2.1, we collect some needed estimates on the unit ball and the Bergman balls. Some results such as pointwise estimates, reproducing formula, density of vector-valued holomorphic polynomials and of vector-valued bounded holomorphic functions in the vector-valued Bergman space  $A^p_\alpha(\mathbb{B}_n, E)$ , where  $0 < p < \infty$ , are proven in Subsection 2.2.2. We also present some important estimates on vector-valued holomorphic functions in Subsection 2.2.3. Subsection 2.2.4 is devoted to the boundedness of the vector-valued Bergman type projections, the topological dual of the vector-valued Bergman space  $A^p_\alpha(\mathbb{B}_n, E)$ , where

$1 < p < \infty$ ,  $\alpha > -1$  and  $E$  is a Banach space. In Subsection 2.2.6, we recall some characterizations of vector-valued Bergman spaces and some decomposition of the unit ball. We mention that most of the results of this chapter already appear in [52] and are certainly known by experts. Since the problem of characterizing those holomorphic symbols for which the little Hankel operator extends into a bounded operator between vector-valued Bergman spaces is often reduced to the search for the topological dual space of certain holomorphic function spaces. In [4], Arregui and Blasco proved that for  $n = 1$ ,  $\alpha = 0$  and  $1 < p < \infty$ , the topological dual of  $A_\alpha^p(\mathbb{B}_n, E)$  can be identified with the space  $A_\alpha^{p'}(\mathbb{B}_n, E^*)$  under the pairing

$$\langle f, g \rangle_{\alpha, E} := \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z),$$

where  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $g \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ . We want to mention that this result is obtained without any restriction on the Banach space  $E$ . In Subsection 2.2.5, we give details proof of this result in the general case where  $n \geq 1$  is an integer and  $\alpha > -1$ , which is strongly based on the theory of vector-measures.

## 0.3 Chapter 3: Vector-valued Bloch type spaces.

The Bloch type spaces are very related with the Bergman spaces. In **Chapter 3**, we fix  $E$  to be a complex Banach space and we denote by  $\alpha > -1$  a real parameter. The scalar-valued Bloch space  $\mathcal{B}(\mathbb{B}_n)$  can be thought as the limit case of the scalar-valued Bergman space  $A_\alpha^p(\mathbb{B}_n)$  as  $p \rightarrow \infty$  (see [77]). In particular,  $\mathcal{B}(\mathbb{B}_n)$  can be naturally identified with the dual space of  $A_\alpha^1(\mathbb{B}_n)$ . The Bloch space is intimately related to the Bergman metric; it consists exactly of those holomorphic functions that are Lipschitz from  $\mathbb{B}_n$  with respect to the Bergman metric. The Bloch space has been studied much earlier than the Bergman space. Serious research on the Bloch space of the unit ball began with Timoney's papers [66, 67]. For more information on scalar-valued Bloch type spaces, we recommend [77].

Another important reason of studying Bloch type spaces is that these spaces appear naturally when we characterize the boundedness of Hankel operators. In fact, a classical theorem says that the little Hankel operator is bounded from  $A_\alpha^p(\mathbb{B}_n)$  into  $A_\alpha^q(\mathbb{B}_n)$ , for  $p \leq q$  if and only if its symbol belongs to some Bloch type space [3, 5]. The vector-valued Bloch space of the unit disk  $\mathbb{D}$ , (where  $\mathbb{D} = \mathbb{B}_1$ ),  $\mathcal{B}(\mathbb{D}, E)$  was defined in [4] as the set of all vector-valued holomorphic functions  $f : \mathbb{D} \rightarrow E$  for which

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(z)\|_E < \infty.$$

The generalization to the case of the unit ball is done in [52] where the derivative is replaced by the radial derivative  $\mathcal{N}$  to be defined later. The vector-valued Bloch type spaces play a similar role as their scalar-valued versions. In Section 3.1, we study the vector-valued Bloch space of the unit ball  $\mathcal{B}(\mathbb{B}_n, E)$ . Almost all results of this section have been taken in [52]. In [4], Arregui and Blasco proved that the topological dual of  $A_\alpha^1(\mathbb{D}, E)$  can be identified with the vector-valued Bloch space  $\mathcal{B}(\mathbb{D}, E^*)$ . This result was recently extended to the case of the unit ball  $\mathbb{B}_n$  in [52]. We also define the little vector-valued Bloch space  $\mathcal{B}_0(\mathbb{B}_n, E)$  as the set of all functions  $f \in \mathcal{B}(\mathbb{B}_n, E)$  such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \|\mathcal{N}f(z)\|_E = 0,$$

where  $\mathcal{N}f(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z)$  is the radial derivative operator.

It is important to note that  $\mathcal{B}_0(\mathbb{B}_n, E)$  is a closed subspace of  $\mathcal{B}(\mathbb{B}_n, E)$  in which vector-valued holomorphic polynomials are dense. Little vector-valued Bloch spaces play an important role in the characterization of the compactness of little Hankel operators on vector-valued Bergman spaces (see [26]). In Section 3.2, we extend those results of Section 3.1 to the more generalized vector-valued Bloch type space  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$ , where  $\gamma > 0$ . This section is part of [52, Section 3.2]. Section 3.3 is devoted to the vector-valued holomorphic Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, E)$ , where  $\gamma$  is a real number. Our approach here is to treat the vector-valued Lipschitz spaces as close relatives of the vector-valued Bloch space. For  $0 < \gamma < 1$ , we have that  $\mathcal{B}_\gamma(\mathbb{B}_n, E) = \Lambda_{1-\gamma}(\mathbb{B}_n, E)$ .

Many characterizations of vector-valued holomorphic Lipschitz spaces in terms of differential operators are developed in this section. These characterizations will play an important role in the proof of some results in the next chapter. Since  $\Lambda_\gamma(\mathbb{B}_n, E)$  is not separable (see [77] for the scalar case), we consider the little vector-valued Lipschitz space  $\Lambda_{\gamma,0}(\mathbb{B}_n, E)$  which is the subspace of  $\Lambda_\gamma(\mathbb{B}_n, E)$  consisting of those functions  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$  such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_E = 0,$$

where  $R^{\alpha,k}$  is the linear partial differential operator of order  $k$  defined in Proposition 2.1.21. We also prove (see Proposition 3.3.17) that  $\Lambda_{\gamma,0}(\mathbb{B}_n, E)$  is a closed subspace of  $\Lambda_\gamma(\mathbb{B}_n, E)$  in which the space  $\mathcal{P}(\mathbb{B}_n, E)$  of all  $E$ -valued holomorphic polynomials are dense. We finally show that for  $0 < p \leq 1$  and  $\alpha > -1$ , the dual of the vector valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$  can be identified with the vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, E^*)$ , where  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$  under the pairing

$$\langle f, g \rangle_{\alpha, E} = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z),$$

where  $D_k$  is the differential operator of order  $k$  defined by  $D_k := (2I + \mathcal{N}) \circ (3I + \mathcal{N}) \circ \dots \circ ((k + 1)I + \mathcal{N})$ ,  $k > \gamma$ , is an integer,  $g \in \Lambda_\gamma(\mathbb{B}_n, E^*)$  and  $f \in A_\alpha^p(\mathbb{B}_n, E)$ .

## 0.4 Chapter 4: Little Hankel operators on vector-valued Bergman spaces.

In **Chapter 4**, we give criteria for boundedness and compactness of the little Hankel operator with operator-valued holomorphic symbols on vector-valued Bergman spaces. Let  $E$  be a complex Banach space. We denote by  $\mathcal{H}(\mathbb{B}_n, E)$  the set of all  $E$ -valued holomorphic functions and  $H^\infty(\mathbb{B}_n, E)$  the subset of  $\mathcal{H}(\mathbb{B}_n, E)$  which consists of bounded functions. If  $0 < p, q < \infty$  are real parameters and  $E$  and  $F$  are complex Banach spaces, the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  with holomorphic operator-valued symbol  $b : \mathbb{B}_n \rightarrow \mathcal{L}(\overline{E}, F)$  is defined for  $z \in \mathbb{B}_n$  by

$$h_b(f)(z) := \int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w),$$

where  $\overline{E}$  is the complex conjugate of the Banach space  $E$  and  $f \in H^\infty(\mathbb{B}_n, E)$ . We assume that the symbol  $b \in A_\alpha^1(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . This condition is equivalent to the following:

$$\int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{E}, F)}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) < \infty, \quad \text{for every } z \in \mathbb{B}_n.$$

We also assume that our symbol  $b$  also satisfy the following condition

$$\int_{\mathbb{B}_n} \|b(z)\|_{\mathcal{L}(\overline{E}, F)} \log \left( \frac{1}{1 - |z|^2} \right) d\nu_\alpha(z) < \infty.$$

These conditions was introduced by Oliver in [52] to study the little Hankel operator between vector-valued Bergman spaces.

The boundedness properties of the little Hankel operator in the classical case (that is, when  $E = F = \mathbb{C}$ ) have been extensively studied and many results are now well known. For  $n > 1$ , a complete characterization has been obtained by Aline Bonami and Luo Luo in [20] in the case  $p \leq q$ . The case,  $1 < q < p < \infty$  was finally resolved by Pau and Zhao in [53]. Indeed, they showed that if  $b$  is a holomorphic symbol, the little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n)$  into  $A_\alpha^q(\mathbb{B}_n)$ , with  $1 < q < p < \infty$ , if and only if the holomorphic symbol  $b$  belongs to the weighted Bergman space  $A_\alpha^t(\mathbb{B}_n)$  where  $1/t = 1/q - 1/p$ .

We are here concerned with the question of characterizing the operator-valued holomorphic symbols  $b$  for which the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  where  $0 < p, q < \infty$ . In [1] Aleman and Constantin solved this problem for the particular case  $n = 1$ ,  $p = q = 2$  and  $E = F = \mathcal{H}$  where  $\mathcal{H}$  is a separable Hilbert space. More precisely, they showed that the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  into  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  if and only if the symbol  $b$  belongs to the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\mathcal{H}))$ . Their results extend clearly the one known in the classical case (when  $\mathcal{H} = \mathbb{C}$ ). On the first hand, Oliver [52] solved this problem in the more general case  $1 < p, q < \infty$ . Namely, he showed that for  $1 < p < \infty$ , the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if the symbol  $b$  belongs to the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  and this result clearly generalizes the one obtained by Aleman and Constantin in [1]. On the second hand, for  $1 < q < p < \infty$ , Oliver [52] showed that the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if  $b \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where  $1/t = 1/q - 1/p$ . This result also generalizes the corresponding result of Pau and Zhao in [53].

In Section 4.1, we recall some basic properties on little Hankel operators. For  $1 < p \leq q < \infty$ , Oliver [52] showed that the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if the symbol  $b$  belongs to the  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  with  $\gamma = 1 + (n + 1 + \alpha) \left( \frac{1}{q} - \frac{1}{p} \right)$ . His proof used the fact that the real parameter  $0 < \gamma < 1$ , but we remark that  $\gamma$  is not always positive. Indeed, since  $1/q - 1/p \in (-1, 0)$ , it follows that  $\gamma \in (-n - \alpha, 1)$ . Therefore, if  $\gamma \in (-(n + \alpha), 0)$ , the vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  is not interesting and does not make sense since the definition of the vector-valued  $\gamma$ -Bloch space introduced by Oliver only takes into account the case where  $\gamma > 0$ . In Section 4.2, we solved this problem by replacing the vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  with the vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . Our result is stated as follows.

**Theorem 0.4.1.** Let  $1 < p \leq q < \infty$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator if and only if  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ .

Moreover,

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}.$$

When  $0 < \gamma < 1$ , we know that  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F)) = \Lambda_{1-\gamma}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . Thus, Theorem 0.4.1 extends the Oliver's corresponding result.

In Section 4.3, we study the boundedness properties of little Hankel operators with operator-valued symbols.

We recall that for  $0 < p < \infty$  and a Banach space  $E$ , the weak Bochner-Lebesgue space  $L_\alpha^{p, \infty}(\mathbb{B}_n, E)$

consists of all vector-valued measurable functions  $f : \mathbb{B}_n \rightarrow E$  for which

$$\|f\|_{L_\alpha^{p,\infty}(\mathbb{B}_n, E)} = \left( \sup_{\lambda > 0} \lambda^p \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_E > \lambda\}) \right)^{1/p} < \infty.$$

The weak vector-valued Bergman space  $A_\alpha^{p,\infty}(\mathbb{B}_n, E)$  is defined by

$$A_\alpha^{p,\infty}(\mathbb{B}_n, E) = \mathcal{H}(\mathbb{B}_n, E) \cap L_\alpha^{p,\infty}(\mathbb{B}_n, E).$$

In Subsection 4.3.1, for two complex Banach spaces  $E, F$  we proved the following results.

**Theorem 0.4.2.** Let  $0 < p \leq 1$  and  $\alpha > -1$ . If the little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  for some (all) positive  $q < 1$ , then the symbol  $b$  is in  $\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  with  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ . Conversely, if  $b$  is in  $\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  with  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ , then the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator.

A direct consequence of Theorem 0.4.2 is the following.

**Corollary 0.4.3.** Suppose  $0 < p \leq 1$ , and  $\alpha > -1$ . The little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  for some (all) positive  $q < 1$  if and only if its symbol  $b$  belongs to  $\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ .

In Subsection 4.3.2, we proved the following theorem.

**Theorem 0.4.4.** Let  $0 < p \leq 1$ ,  $\alpha > -1$  and  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ . The little Hankel operator extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^1(\mathbb{B}_n, F)$  if and only if for some integer  $k > \gamma$ ,

$$\|\mathcal{N}^k b(w)\|_{\mathcal{L}(\overline{E}, F)} \lesssim \frac{1}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1}, \quad (1)$$

where  $w \in \mathbb{B}_n$  and  $\mathcal{N}^k = \mathcal{N} \circ \mathcal{N} \circ \dots \circ \mathcal{N}$  ( $k$ -times).

The condition (1) in Theorem 0.4.4 means that the symbol  $b$  is belong to the vector-valued logarithmic Lipschitz space given by Definition 4.3.3. These results also generalize those in [20] for  $0 < p, q \leq 1$ .

In Subsection 4.3.3, we prove the boundedness of the little Hankel operator  $h_b$  from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  for  $0 < p \leq q < \infty$  and  $1 < q < \infty$ . In Section 4.4 we summarize all the results obtained on the boundedness of the little Hankel operator and we give some estimates with loss on the boundedness of the little Hankel operator  $h_b$  from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  in the two remaining cases  $1 < p < \infty$ ,  $0 < q < 1$ , and  $1 < p < \infty$ ,  $q = 1$ . We recall that the problem of finding a necessary and sufficient condition for which the little Hankel operator  $h_b$  is a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  is still open for  $1 < p < \infty$ ,  $q = 1$  even in the classical case ( $E = F = \mathbb{C}$ ).

After the study of bounded little Hankel operators with holomorphic operator-valued symbols acting on vector-valued Bergman spaces, our next step is to look for a characterization of the compactness of the little Hankel operator on vector-valued Bergman spaces. More concisely, we are now concerned with the problem of characterizing those holomorphic operator-valued symbols  $b$  for which the little Hankel operator  $h_b$  extends into a compact operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$ , where  $E, F$  are Banach spaces and  $1 < p \leq q < \infty$ .

In the case  $n = 1$ ,  $E = F = \mathcal{H}$  ( $\mathcal{H}$  is a separable Hilbert space), and  $p = q = 2$ , Constantin [26] proved

that the little Hankel operator  $h_b$  is a compact operator from  $A_\alpha^2(\mathbb{B}_1, \mathcal{H})$  into  $A_\alpha^2(\mathbb{B}_1, \mathcal{H})$  if and only if the symbol  $b$  belongs to the little vector-valued Bloch space  $\mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(\mathcal{H}))$  (where  $\mathcal{K}(\mathcal{H})$  denote the space of all compact operators on  $\mathcal{H}$ ). Following this result, Oliver stated the conjecture (see [52, Chapter 7]): "For  $1 < p < \infty$ , the little Hankel  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^p(\mathbb{B}_n, F)$  is compact if and only if the symbol  $b$  belongs to the little vector-valued Bloch space  $\mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(E, F))$ , where  $\mathcal{K}(E, F)$  denotes the space of all compact operators from  $E$  to  $F$ ." We proved this conjecture in Section 4.5 with the additional condition that  $E$  and  $F$  are reflexive complex Banach spaces. More generally, we have obtained the following result.

**Theorem 0.4.5.** Let  $1 < p \leq q < \infty$ , and  $E, F$  be two reflexive complex Banach spaces. The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a compact operator if and only if  $b \in \Lambda_{\gamma,0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$ , where  $\Lambda_{\gamma,0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$  denotes the little vector-valued Lipschitz space and  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ .

Observe that for  $p = q$  in the previous theorem, we have that  $\Lambda_{0,0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F)) = \mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$  is the little vector-valued Bloch space. So, under the assumptions of Theorem 0.4.5, we have that the little Hankel  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^p(\mathbb{B}_n, F)$  is a compact operator if and only if its symbol  $b$  is in  $\mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$ . This particular case is the Oliver's conjecture.

## 0.5 Chapter 5: Toeplitz operators on vector-valued Bergman spaces.

Zorboska [79] proved that for  $b \in BMO^1$  symbol, then the Toeplitz operator  $T_b$  is bounded on  $A_0^2(\mathbb{D})$  if and only if its Bezerin transtorm  $\tilde{b}$  is bounded on  $\mathbb{D}$ , and  $T_b$  is compact on  $A_0^2(\mathbb{D})$  if and only if  $\tilde{b}(z) \rightarrow 0$  as  $z \rightarrow \partial\mathbb{D}$ . This result was extended to the spaces  $A_\alpha^2(\mathbb{B}_n)$  in [70]. In **Chapter 5**, we extend results in [79] to vector-valued settings. Section 5.1 contains some notations and definitions. We recall that the scalar-valued Bergman space  $A_\alpha^2(\mathbb{B}_n)$  is a reproducing kernel Hilbert space with a kernel  $K^\alpha$  defined by

$$K^\alpha(z, w) = K_w^\alpha(z) := \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}},$$

where  $z, w \in \mathbb{B}_n$ . The normalized reproducing kernel  $k^\alpha$  is defined by

$$k_w^\alpha(z) := k^\alpha(z, w) = \|k_w^\alpha\|_{2,\alpha}^{-1} K_w^\alpha(z) = \frac{(1 - |w|^2)^{(n+1+\alpha)/2}}{(1 - \langle z, w \rangle)^{n+1+\alpha}},$$

where  $z, w \in \mathbb{B}_n$ . The Berezin transform of a function  $f \in L_\alpha^1(\mathbb{B}_n, E)$  is the function  $\tilde{f}$  defined by

$$\tilde{f}(z) := \int_{\mathbb{B}_n} f(w) |k_z^\alpha(w)|^2 d\nu_\alpha(w),$$

for  $z \in \mathbb{B}_n$ , and  $E$  is a Banach space. Given two Banach spaces  $E$  and  $F$ , and  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , we define the vectorial Toeplitz operator  $T_b$  by

$$T_b(f)(z) := \int_{\mathbb{B}_n} \frac{b(w)(f(w))}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w),$$

for  $z \in \mathbb{B}_n$ , and  $f \in H^\infty(\mathbb{B}_n, E)$ . The Berezin transform  $\widetilde{T_b}$  of the Toeplitz operator  $T_b$  is given by the following relation

$$\langle \widetilde{T_b}(z)(x), y^* \rangle_{F, F^*} = \langle T_b(k_z^\alpha x), k_z^\alpha y^* \rangle_{\alpha, F},$$

where  $x \in E$  and  $y^* \in F^*$ . The Berezin transform  $\widetilde{T}_b$  is also denoted  $\widetilde{b}$ . Section 5.2 is devoted to Carleson measures for vector-valued Bergman spaces. Let  $\mu$  be a finite positive Borel measure on  $\mathbb{B}_n$ . Recall that the Berezin transform  $\widetilde{\mu}$  of  $\mu$  is defined by

$$\widetilde{\mu}(z) := \int_{\mathbb{B}_n} |k_z^\alpha(w)|^2 d\mu(w), \quad z \in \mathbb{B}_n.$$

If  $p > 0$ ,  $\alpha > -1$  and  $E$  is a Banach space, we say that a positive finite Borel measure on  $\mathbb{B}_n$  is a Carleson measure for  $A_\alpha^p(\mathbb{B}_n, E)$  if there is a constant  $c_\mu > 0$  such that

$$\int_{\mathbb{B}_n} \|f(z)\|_E^p d\mu(z) \leq c_\mu \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z),$$

for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . The Toeplitz operator  $T_\mu$  induced by a finite measure  $\mu$  on  $\mathbb{B}_n$  is given by

$$T_\mu(f)(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\mu(w), \quad z \in \mathbb{B}_n.$$

We also keep the same definition of the Berezin transform of a finite positive Borel measure  $\mu$  used in the scalar-valued cases (see [78]). The main result of Section 5.2 is the following.

**Theorem 0.5.1.** Let  $\mu$  is a finite positive Borel measure on  $\mathbb{B}_n$ , and  $1 < p < \infty$ . Then the following are equivalent.

- (a)  $T_\mu$  is bounded on  $A_\alpha^p(\mathbb{B}_n, E)$ .
- (b)  $\widetilde{\mu}$  is a bounded function on  $\mathbb{B}_n$ .
- (c)  $\mu$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$ .

Moreover,

$$\|T_\mu\| \simeq \|\widetilde{\mu}\|_{L^\infty(\mathbb{B}_n, \mathbb{C})} \simeq \|\mu\|_{Carl}.$$

This result shows that that the boundedness of the Toeplitz operator  $T_\mu$  on the vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$ , where  $1 < p < \infty$  is intimately related both to the boundedness of the Berezin transform  $\widetilde{\mu}$  of  $\mu$  and to the notion of Carleson measures.

In Section 5.3, we introduce and we study the vector-valued *BMO* spaces. Let  $1 \leq p < \infty$ ,  $E$  a complex Banach space and  $f \in L_\alpha^p(\mathbb{B}_n, E)$ . We say that  $f$  is in  $BMO_\alpha^p(\mathbb{B}_n, E)$  if

$$\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w) - \widetilde{f}(z)\|_E^p d\nu_\alpha(w) < \infty,$$

where  $\varphi_z$  is the involution that interchanges 0 and  $z$  and  $\widetilde{f}$  is the Berezin transform of the function  $f$  defined by (5.10). Section 5.3 also includes the study of some needed properties of vector-valued *BMO* spaces. We conclude the section by extending the Li and Luecking theorem [45] to vector-valued settings. Namely, we show that if  $1 \leq p < \infty$  and  $E$  is a Banach space, then the Bergman projection  $P_\alpha : BMO_\alpha^p(\mathbb{B}_n, E) \rightarrow \mathcal{B}(\mathbb{B}_n, E)$  is a bounded operator. In Section 5.4, we let  $E, F$  be two Banach spaces and  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . The main result of Section 5.4 is stated as follow.

**Theorem 0.5.2.** Let  $E, F$  be two complex Banach spaces and  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . The Toeplitz operator  $T_b : A_\alpha^2(\mathbb{B}_n, E) \rightarrow A_\alpha^2(\mathbb{B}_n, F)$  is a bounded linear operator if and only if its Berezin transform  $\widetilde{b}$  is bounded in  $\mathbb{B}_n$ .

In Section 5.5 we use the following symbols and notations:  $\mathcal{H} = \mathbb{C}^d$ , where  $d \geq 1$  is an integer. The vectors  $\{e_i\}_{i=1}^d$  denote the standard orthonormal basis for  $\mathbb{C}^d$ . The letter  $e$  will always denote a unit vector in  $\mathbb{C}^d$ . If  $M$  is a  $d \times d$  matrix,  $\|M\|$  will denote any convenient matrix norm. Since all norms of matrices are equivalent in finite dimension, the exact norm used does not matter for quantitative considerations. We recall that the set  $\mathcal{L}(\mathbb{C}^d)$  can be identified with the set  $\mathcal{M}^d$ , the set  $d \times d$  matrix with coefficients in  $\mathbb{C}$ . Section 5.5 is devoted to the study of the compactness of Toeplitz operators with matrix-valued symbols acting on the  $\mathbb{C}^d$ -valued weighted Bergman space  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ . The main result of Section 5.5 is the following

**Theorem 0.5.3.** Let  $d \geq 1$  be an integer and  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$  such that  $\tilde{b} \in L^\infty(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ . The following are equivalent.

- (a)  $T_b$  is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ .
- (b)  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .
- (c)  $T_{b \circ \varphi_z} e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .
- (d)  $\|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and every  $p > 1$ .
- (e)  $\|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and some  $0 < p - 1 < \frac{\alpha + 1}{n + 1 + \alpha}$ .

More generally, in [60], Rahm showed that an operator  $T \in \mathcal{L}(A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d))$  which satisfies

$$\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_z e\|_{p, \alpha, \mathbb{C}^d} < \infty, \quad (2)$$

also satisfies the four conditions in Theorem 0.5.3. In Proposition 5.4.9, we prove that for  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , the Toeplitz operator  $T_b$  satisfies condition (2). By [60, Lemma 3.4], we know that  $L^\infty(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$  is an example of a class of symbols for which Toeplitz operators satisfies condition (2). From Theorem 0.5.3, we can also deduce that  $BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$  is another non trivial class of symbols for which condition (2) is satisfied by Toeplitz operators.

# Preliminaries

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In this chapter, we collect some results which will be used in later chapters. Some of the results are well-known and therefore will be stated without proofs.

## 1.1 Basic notions on functional analysis

The proof of most results in this section can be found in [22].

### 1.1.1 Hahn-Banach theorem, uniform boundedness principle, weak topologies and reflexive spaces.

Let  $E$  be a vector space over  $\mathbb{R}$ . We recall that a functional is a function defined on  $E$ , or on some subspace of  $E$ , with values in  $\mathbb{R}$ . The main result of this section concerns the extension of linear functional defined on a linear subspace of  $E$  to linear functional defined on  $E$ .

**Theorem 1.1.1 (Hahn-Banach).** Let  $p : E \rightarrow \mathbb{R}$  be a function satisfying

- (1)  $p(\lambda x) = \lambda p(x), \quad \forall x \in E \text{ and } \forall \lambda > 0,$
- (2)  $p(x + y) \leq p(x) + p(y), \quad \forall x, y \in E.$

Let  $G \subset E$  be a linear subspace and let  $g : G \rightarrow \mathbb{R}$  be a linear functional such that

$$(3) \quad g(x) \leq p(x), \quad \forall x \in G.$$

Under these assumptions, there exists a linear functional  $f$  defined on  $E$  that extends  $g$ , i.e.,  $g(x) = f(x), \quad \forall x \in G$ , and such that

$$(4) \quad f(x) \leq p(x), \quad \forall x \in E.$$

We now describe some simple applications of Theorem 1.1.1 to the case in which  $E$  is a normed vector space with norm  $\|\cdot\|$ .

**Corollary 1.1.2.** Let  $E$  be a (real) normed vector space over  $\mathbb{R}$  and  $G \subset E$  be a linear subspace of  $E$ . If  $g : G \rightarrow \mathbb{R}$  is a continuous linear functional, then there exists a linear functional  $f$  on  $E$  which extends  $g$  ( $f|_G = g$ ) such that  $\|f\| = \|g\|$ .

## 1.1. Basic notions on functional analysis

**Notation 1.1.3.** We denote by  $E^*$  the dual space of  $E$ , that is, the space of all continuous linear functionals on  $E$ ; the (dual) norm on  $E^*$  is defined by

$$\|f\|_{E^*} = \sup_{\|x\|_E=1} |\langle f, x \rangle_{E^*, E}|. \quad (1.1)$$

When there is no confusion we shall write  $\|f\|$  instead of  $\|f\|_{E^*}$ . Given  $f \in E^*$  and  $x \in E$  we also write  $f(x)$  to denote  $\langle f, x \rangle_{E^*, E}$ ; we say that  $\langle \cdot, \cdot \rangle_{E^*, E}$  is the scalar product for the duality  $E^*, E$ .

It is well known that  $E^*$  is a Banach space, i.e.,  $E^*$  is complete (even if  $E$  is not).

We denote by  $E_{\mathbb{R}}$  the vector space  $E$  over  $\mathbb{R}$  and  $E_{\mathbb{C}}$  the vector space  $E$  over  $\mathbb{C}$ .

- Lemma 1.1.4.**
1. Let  $f : E_{\mathbb{C}} \rightarrow \mathbb{C}$  be a linear functional such that  $f(x) = \alpha(x) + i\beta(x)$ , where  $\alpha(x), \beta(x) \in \mathbb{R}$ . Then  $\alpha$  and  $\beta$  are linear functionals on  $E_{\mathbb{R}}$  and for all  $x \in E_{\mathbb{C}}$ , we have that  $\beta(x) = -\alpha(ix)$ .
  2. For each linear functional  $\alpha$  on  $E_{\mathbb{R}}$ , the function  $f(x) = \alpha(x) - i\alpha(ix)$  defines a linear functional on  $E_{\mathbb{C}}$ .
  3. If  $f$  is a continuous linear functional on  $E_{\mathbb{C}}$ , then  $\alpha$  is a continuous linear functional on  $E_{\mathbb{R}}$  and  $\|f\| = \|\alpha\|$ .

In the sequel,  $E_{\mathbb{C}}$  will be simply denoted by  $E$  and will be called "complex vector space".

**Corollary 1.1.5.** Let  $E$  be a complex vector space and  $G \subset E$  be a linear subspace of  $E$ . If  $g : G \rightarrow \mathbb{C}$  is a continuous linear functional, then there exists a linear functional  $f \in E^*$  defined on all of  $E$  that extends  $g$  such that  $\|f\| = \|g\|$ .

**Corollary 1.1.6.** Let  $E$  be a complex normed vector space. For every  $x_0 \in E$  there exists  $f_0 \in E^*$  such that

$$\|f_0\| = \|x_0\| \quad \text{and} \quad \langle f_0, x_0 \rangle = \|x_0\|^2.$$

**Corollary 1.1.7.** For every  $x \in E$ , we have

$$\|x\| = \sup_{\|f\|_{E^*}=1} |\langle f, x \rangle| = \max_{\|f\|_{E^*}=1} |\langle f, x \rangle|. \quad (1.2)$$

**Corollary 1.1.8.** Let  $E$  be a complex normed vector space. Let  $G$  be a linear subspace of  $E$  which is not dense in  $E$  and  $x_0 \in E \setminus \overline{G}$ . Then there exists a continuous linear functional  $f$  defined on  $E$  such that  $f(x_0) = 1$ ,  $f|_G = 0$  and  $\|f\| = \frac{1}{d}$ , where  $d = \text{dist}(x_0, G) = \inf_{y \in G} \|x_0 - y\|$ .

**Corollary 1.1.9.** Let  $E$  be a complex normed vector space and  $G \subset E$  be a linear subspace of  $E$  such that every continuous linear functional on  $E$  which vanishes on  $G$  is vanishing on all  $E$ . Then  $G$  is dense in  $E$ .

Let  $E$  be a normed vector space and  $E^*$  be the topological dual of  $E$  with the norm

$$\|f\|_{E^*} = \sup_{\|x\|_E=1} |\langle f, x \rangle_{E^*, E}|.$$

## 1.1. Basic notions on functional analysis

The bidual  $E^{**}$  is the dual of  $E^*$  with the norm

$$\|\xi\|_{E^{**}} = \sup_{\|f\|_{E^*}=1} |\langle \xi, f \rangle_{E^{**}, E^*}|.$$

There is a canonical injection  $J : E \rightarrow E^{**}$  defined as follows: given  $x \in E$ , the map  $f \mapsto \langle f, x \rangle_{E^*, E}$  is a continuous linear functional on  $E^*$ ; thus it is an element of  $E^{**}$ , which we denote by  $Jx$ . We have that

$$\langle Jx, f \rangle_{E^{**}, E^*} = \langle f, x \rangle_{E^*, E} \quad \forall x \in E, \quad \forall f \in E^*.$$

It may happen that  $J$  is not surjective from  $E$  onto  $E^{**}$  (see Chapter 3 and 4 of [22]). However, it is convenient to identify  $E$  with a subspace of  $E^{**}$  using  $J$ .

**Definition 1.1.10.** Given two normed vector spaces  $E$  and  $F$ , we denote by  $\mathcal{L}(E, F)$  the space of continuous (=bounded) linear operators  $T$  from  $E$  to  $F$  equipped with the norm

$$\|T\|_{\mathcal{L}(E, F)} = \sup_{\|x\|_E=1} \|Tx\|_F.$$

As usual, one writes  $\mathcal{L}(E)$  instead of  $\mathcal{L}(E, E)$ . When  $F = \mathbb{K}$ , (where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ), we have  $\mathcal{L}(E, \mathbb{K}) = E^*$ .

**Theorem 1.1.11 (Banach-Steinhaus, Uniform boundedness principle).** Let  $E$  and  $F$  be two Banach spaces and let  $(T_i)_{i \in I}$  be a family (not necessarily countable) of continuous linear operators from  $E$  into  $F$ . Assume that

$$(1) \quad \sup_{i \in I} \|T_i x\|_F < \infty \quad \forall x \in E.$$

Then

$$(2) \quad \sup_{i \in I} \|T_i\|_{\mathcal{L}(E, F)} < \infty.$$

In other words, there exists a constant  $c > 0$  such that

$$\|T_i x\|_F \leq c \|x\|_E \quad \forall x \in E, \quad \forall i \in I.$$

A topology is a geometric structure defined on a set.

**Definition 1.1.12.** A *topology* on a set  $E$  is a collection  $\tau$  of subsets of  $E$  such that

(T<sub>1</sub>)  $\emptyset$  and  $E$  are in  $\tau$ .

(T<sub>2</sub>) Any union of subsets of  $\tau$  is in  $\tau$ .

(T<sub>3</sub>) The finite intersection of subsets in  $\tau$  are in  $\tau$ .

A set with a topology  $\tau$  is called a *topological space*. An element of  $\tau$  is called an *open set*.

**Example 1.1.13.** Let  $E$  be a set. The topology defined by  $\tau = \mathcal{P}(E)$ , where  $\mathcal{P}(E)$  is the set of all subsets of  $E$  is called the *discrete topology* on  $E$ .

## 1.1. Basic notions on functional analysis

Suppose  $X$  is a set without any structure and  $(Y_i)_{i \in I}$  is a collection of topological spaces. Let  $\{\varphi_i : X \rightarrow Y_i\}_{i \in I}$  be an indexed set of functions such that for every  $i \in I$ ,  $\varphi_i$  maps  $X$  into  $Y_i$ . Let  $\mathcal{T}$  denote the set of all topologies  $\tau$  on  $X$  such that  $\varphi_i$  is a continuous map for every  $i \in I$ . Then the intersection  $\bigcap_{\tau \in \mathcal{T}} \tau$  is again a topology and also belongs to  $\mathcal{T}$ . (see [22, Section 3.1 p. 55]). Clearly, it is the weakest (coarsest/initial) topology  $\tau_0$  such that each function  $\varphi_i : X \rightarrow Y_i$  is a continuous map.

Let  $E$  be a Banach space and let  $f \in E^*$ . We denote by  $\varphi_f : E \rightarrow \mathbb{K}$  the linear functional  $\varphi_f(x) = \langle f, x \rangle_{E^*, E}$ . As  $f$  runs through  $E^*$  we obtain a collection  $(\varphi_f)_{f \in E^*}$  of maps from  $E$  into  $\mathbb{K}$ . We now ignore the usual topology on  $E$  (associated to  $\| \cdot \|_E$ ) and define a new topology on the set  $E$  as follows:

**Definition 1.1.14.** The weak topology  $\sigma(E, E^*)$  on  $E$  is the coarsest topology associated to the collection  $(\varphi_f)_{f \in E^*}$  (in the sense of [22, Section 3.1 p. 55] with  $X = E$ ,  $Y_i = \mathbb{K}$ , for each  $i$ , and  $I = E^*$ ).

Since bounded linear functionals are continuous in the norm (strong) topology, it follows that the weak topology is coarser than the strong topology.

**Proposition 1.1.15.** ([22, Proposition 3.4 p. 57]) Let  $x_0 \in E$ ,  $\epsilon > 0$  and a finite set  $\{f_1, f_2, \dots, f_k\}$  in  $E^*$ . Let

$$V = V(f_1, f_2, \dots, f_k; \epsilon) = \{x \in E : |\langle f_i, x - x_0 \rangle| < \epsilon \quad \forall i = 1, 2, \dots, k\}.$$

Then  $V$  is a neighborhood of  $x_0$  for the topology  $\sigma(E, E^*)$ . Moreover, we obtain *basis of neighborhoods* of  $x_0$  for  $\sigma(E, E^*)$  by varying  $\epsilon, k$  and  $f_i \in E^*$ .

**Definition 1.1.16.** The sequence  $(x_n)$  in  $E$  converges strongly to  $x \in E$  if  $\|x_n - x\|_E \rightarrow 0$ , as  $n \rightarrow \infty$ .

**Notation 1.1.17.** If a sequence  $(x_n)$  in  $E$  converges to  $x$  in the weak topology  $\sigma(E, E^*)$  we shall write  $x_n \rightharpoonup x$ . To avoid any confusion, we shall sometimes say, " $x_n \rightharpoonup x$  weakly in  $\sigma(E, E^*)$ ."

In order to be totally clear we shall sometimes emphasize strong convergence by saying, " $x_n \rightarrow x$  strongly" meaning that  $\|x_n - x\|_E \rightarrow 0$ .

**Proposition 1.1.18.** ([22, Proposition 3.5]) Let  $(x_n)$  be a sequence in  $E$ . Then

- (i)  $[x_n \rightharpoonup x \text{ weakly in } \sigma(E, E^*)] \Leftrightarrow [\langle f, x_n \rangle_{E^*, E} \rightarrow \langle f, x \rangle_{E^*, E} \quad \forall f \in E^*]$ .
- (ii) If  $x_n \rightarrow x$  strongly, then  $x_n \rightharpoonup x$  weakly in  $\sigma(E, E^*)$ .
- (iii) If  $x_n \rightharpoonup x$  weakly in  $\sigma(E, E^*)$ , then  $(\|x_n\|_E)$  is bounded and  $\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|_E$ .
- (iv) If  $x_n \rightharpoonup x$  weakly in  $\sigma(E, E^*)$  and if  $f_n \rightarrow f$  strongly in  $E^*$  (i.e.,  $\|f_n - f\|_{E^*} \rightarrow 0$ ), then  $\langle f_n, x_n \rangle_{E^*, E} \rightarrow \langle f, x \rangle_{E^*, E}$ .

Now, we have two topologies on the dual space  $E^*$ .

- (a) The usual (strong) topology associated to the norm in  $E^*$ .
- (b) The weak topology  $\sigma(E^*, E^{**})$ , obtained by performing on  $E^*$  the construction above.

We are now going to define a third topology on  $E^*$  called weak\* topology and denoted by  $\sigma(E^*, E)$  (the  $*$  is here to remind us that this topology is defined on dual spaces). For every  $x \in E$  consider the linear functional

$\varphi_x : E^* \rightarrow \mathbb{K}$   
 $f \mapsto \varphi_x(f) := \langle f, x \rangle_{E^*, E}$ . As  $x$  runs through  $E$  we obtain a collection  $(\varphi_x)_{x \in E}$  of maps from  $E^*$  into  $\mathbb{K}$ .

## 1.1. Basic notions on functional analysis

**Definition 1.1.19.** The *weak\* topology* on  $\sigma(E^*, E)$  is the weakest topology on  $E^*$  for which all the mappings  $(\varphi_x)_{x \in E}$  are continuous.

Since  $E \subset E^{**}$ , it is clear that the topology  $\sigma(E^*, E)$  is the coarser than the topology  $\sigma(E^*, E^{**})$ ; i.e., the topology  $\sigma(E^*, E)$  has fewer open sets (resp. closed sets) than the topology  $\sigma(E^*, E^{**})$ , which has fewer open sets (resp. closed sets) than the strong topology.

**Proposition 1.1.20.** ([22, Proposition 3.12]) Let  $f_0 \in E^*$ ,  $\epsilon > 0$  and a finite set  $\{x_1, x_2, \dots, x_k\}$  in  $E$ . Let

$$V = V(x_1, x_2, \dots, x_k; \epsilon) = \{f \in E^* : |\langle f - f_0, x_i \rangle| < \epsilon \quad \forall i = 1, 2, \dots, k\}.$$

Then  $V$  is a neighborhood of  $f_0$  for the topology  $\sigma(E^*, E)$ . Moreover, we obtain *basis of neighborhoods* of  $f_0$  for  $\sigma(E^*, E)$  by varying  $\epsilon, k$  and  $x_i \in E$ .

**Notation 1.1.21.** If a sequence  $(f_n)$  in  $E^*$  converges to  $f$  in the weak\* topology we shall write  $f_n \xrightarrow{*} f$ .

**Proposition 1.1.22.** ([22, Proposition 3.13]) Let a sequence  $(f_n)$  be in  $E^*$ . Then

- (i)  $[f_n \xrightarrow{*} f \text{ in } \sigma(E^*, E)] \Leftrightarrow [\langle f_n, x \rangle_{E^*, E} \rightarrow \langle f, x \rangle_{E^*, E} \quad \forall x \in E]$ .
- (ii) If  $f_n \rightarrow f$  strongly, then  $f_n \rightarrow f$  in  $\sigma(E^*, E^{**})$ .  
If  $f_n \rightarrow f$  in  $\sigma(E^*, E^{**})$ , then  $f_n \xrightarrow{*} f$  in  $\sigma(E^*, E)$ .
- (iii) If  $f_n \xrightarrow{*} f$  in  $\sigma(E^*, E)$ , then  $(\|f_n\|)$  is bounded and  $\|f\| \leq \liminf_{n \rightarrow \infty} \|f_n\|$ .
- (iv) If  $f_n \xrightarrow{*} f$  in  $\sigma(E^*, E)$ , and if  $x_n \rightarrow x$  strongly in  $E$ , then  $\langle f_n, x_n \rangle_{E^*, E} \rightarrow \langle f, x \rangle_{E^*, E}$ .

Note that every map  $\varphi_f$  is continuous for the usual topology and therefore the weak topology is weaker than the usual topology.

**Definition 1.1.23.** The sequence  $(x_n)$  in  $E$  converges weakly to  $x \in E$  if for all  $x^* \in E^*$ ,  $|\langle x_n, x^* \rangle_{E, E^*} - \langle x, x^* \rangle_{E, E^*}| \rightarrow 0$ , as  $n \rightarrow \infty$ .

**Theorem 1.1.24 (Banach-Alaoglu-Bourbaki).** The closed unit ball

$$B_{E^*} = \{f \in E^* : \|f\|_{E^*} \leq 1\}$$

is compact in the weak\* topology  $\sigma(E^*, E)$ .

**Remark 1.1.25.** The compactness of  $B_{E^*}$  is one of the most important property of the weak\* topology.

**Definition 1.1.26.** Let  $E$  be a Banach space and let  $J : E \rightarrow E^{**}$  be the canonical injection from  $E$  into  $E^{**}$  (see Section 1.1.2). The space  $E$  is said to be reflexive if  $J$  is surjective, i.e.,  $J(E) = E^{**}$ .

When  $E$  is reflexive,  $E^{**}$  is usually identified with  $E$ .

**Remark 1.1.27.** Many spaces in analysis are reflexive. Clearly, finite-dimensional spaces are reflexive (since  $\dim E = \dim E^* = \dim E^{**}$ ). Also, Hilbert spaces are reflexive.

The next result is a very important property of reflexive spaces.

## 1.1. Basic notions on functional analysis

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**Theorem 1.1.28 (Kakutani).** Let  $E$  be a Banach space. Then  $E$  is reflexive if and only if

$$B_E = \{x \in E : \|x\|_E \leq 1\}$$

is compact in the weak topology  $\sigma(E, E^*)$ .

In connection with the compactness properties of reflexive spaces we also have the following two results:

**Theorem 1.1.29.** Assume that  $E$  is a reflexive Banach space and let  $(x_n)_n$  be a bounded sequence in  $E$ . Then there exists a subsequence  $(x_{n_k})_k$  that converges in the weak topology  $\sigma(E, E^*)$ .

The converse is also true, namely, we have

**Theorem 1.1.30 (Eberlein-Šmulian).** Assume that  $E$  is a Banach space such that every bounded sequence in  $E$  admits a weakly convergent subsequence (in  $\sigma(E, E^*)$ ). Then  $E$  is reflexive.

**Definition 1.1.31.** We say that a metric space  $E$  is separable if there exists a subset  $D \subset E$  that is countable and dense.

**Lemma 1.1.32.** If  $E$  is a Banach space and  $E^*$  is separable, then  $E$  is separable.

**Definition 1.1.33.** A subset  $\Lambda \subset E^*$  separate  $E$  if for all  $a, b \in E$ , we have

$$(\forall f \in \Lambda, f(a) = f(b)) \Rightarrow a = b.$$

**Lemma 1.1.34.** For all Banach set  $E$ , the set  $E^*$  separates  $E$ .

*Proof.* To show that  $E^*$  separate  $E$ , we fix  $a \in E$  and we assume that  $f(a) = 0$ , for all  $f \in E^*$ . Let us show that  $a = 0_E$ . By the Hahn-Banach theorem, there exists  $f_a \in E^*$  with  $\|f_a\| = 1$  and  $f_a(a) = \|a\|_E$ . By the assumption, we have that  $0 = f_a(a) = \|a\|_E$ . Thus,  $a = 0_E$ .  $\square$

## 1.1.2 Bounded and compact operators on Banach spaces

**Definition 1.1.35.** Let  $E$  and  $F$  be two Banach spaces. A linear map  $T : E \rightarrow F$  is called *continuous* if it maps convergent sequences into convergent ones, that is, if

$$x_n \longrightarrow x \quad \text{implies} \quad Tx_n \longrightarrow Tx.$$

**Definition 1.1.36.** Let  $E$  and  $F$  be Banach spaces. A linear map  $T : E \rightarrow F$  is *bounded* if there is a constant  $C > 0$  such that

$$\|Tx\|_F \leq C\|x\|_E, \quad \forall x \in E.$$

We recall that  $\mathcal{L}(E, F)$  is the space of all bounded linear operators from  $E$  into  $F$ . The set  $\mathcal{L}(E, F)$  is a Banach space whenever  $E$  is a vector space and  $F$  is a Banach space.

**Theorem 1.1.37.** ([44, Theorem 1 p. 160]) A linear map  $T : E \rightarrow F$  of one Banach space  $E$  into another  $F$  is continuous if and only if it is bounded.

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**Definition 1.1.38.** A bounded linear operator  $T : E \rightarrow F$  induces a linear operator  $T^* : F^* \rightarrow E^*$  as follows.

$$T^*(f)(x) = f(T(x)), \quad x \in E, f \in F^*.$$

We observe that the operator  $T^*$  is still bounded and  $\|T^*\| = \|T\|$ . The operator  $T^*$  is called the *adjoint* of  $T$ .

Throughout this section, and unless otherwise specified,  $E$  and  $F$  denote two Banach spaces.

**Definition 1.1.39.** A bounded operator  $T \in \mathcal{L}(E, F)$  is said to be *compact* if  $T(B_E)$  has compact closure in  $F$  (in the strong topology).

The set of all compact operators from  $E$  into  $F$  is denoted by  $\mathcal{K}(E, F)$ . For simplicity, one writes  $\mathcal{K}(E) = \mathcal{K}(E, E)$ .

**Remark 1.1.40.** Since every set with compact closure is bounded, it follows that every compact linear operator is bounded.

If  $E$  and  $F$  are finite-dimensional spaces, then every bounded linear operator from  $E$  into  $F$  is necessary compact.

If  $E$  is infinite-dimensional vector normed space, then the identity operator  $Id_E$  is not a compact operator.

**Proposition 1.1.41.** A bounded operator  $T \in \mathcal{L}(E, F)$  is compact if and only if every bounded sequence  $(x_n)_n$  in  $E$ , has a subsequence  $(x_{n_k})_k$  such that the sequence  $(T(x_{n_k}))_k$  converges (strongly) in  $F$ .

**Proposition 1.1.42.** Let  $T \in \mathcal{L}(E, F)$  and let  $B_E$  be the closed unit ball in  $E$ .

- (a) If  $T \in \mathcal{K}(E, F)$ , then  $T$  is continuous from  $B_E$  equipped with the weak topology into  $F$  equipped with the strong topology; consequently, for every sequence  $(x_n)_n$  in  $E$  which converges weakly to  $0_E$ , the sequence  $(T(x_n))_n$  converges strongly to  $0_F$ .
- (b) Suppose that  $E$  is reflexive. Then the operator  $T$  is in  $\mathcal{K}(E, F)$  if and only if for every sequence  $(x_n)_n$  of points of  $E$  converging weakly to  $0_E$ , the sequence  $(T(x_n))_n$  converges strongly to  $0_F$ . Furthermore, the set  $T(B_E)$  is compact (strongly) in  $F$  when  $T \in \mathcal{K}(E, F)$ .

**Definition 1.1.43.** An operator  $T \in \mathcal{L}(E, F)$  is said to be of *finite rank* if the range  $R(T)$  of  $T$  is *finite-dimensional* ( $\dim R(T) = \dim T(E) < \infty$ ).

**Proposition 1.1.44.** Every bounded linear operator  $T$  from  $E$  into  $F$  of finite-rank is a compact operator.

**Theorem 1.1.45.** The set  $\mathcal{K}(E, F)$  is a closed subspace of  $\mathcal{L}(E, F)$  (in the topology associated to the norm  $\|\cdot\|_{\mathcal{L}(E, F)}$ ).

**Corollary 1.1.46.** Let  $(T_n)_n$  be a sequence of finite-rank operators and let  $T \in \mathcal{L}(E, F)$  be such that  $\|T_n - T\|_{\mathcal{L}(E, F)} \rightarrow 0$ . Then  $T \in \mathcal{K}(E, F)$ .

## 1.2 Integration of vector-valued functions

We assume that the Lebesgue theory of scalar-valued functions is well known. We present here a theory of integration of functions with values in a complex Banach space, which is due to Bochner. In the following,

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$(\Omega, \mathcal{A}, \mu)$  will denote a measure space and  $(E, \|\cdot\|_E)$  will be the Banach space of values.

The principal problem of integrating vector-valued functions lies in the definition of the measurability of functions. The "measurability" notion introduced for positive-valued functions is not sufficient for vector-valued functions. It becomes necessary to introduce a " $\mu$ -measurability" notion which depends on both the  $\sigma$ -algebra  $\mathcal{A}$  in the set  $\Omega$  and the measure  $\mu : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$ . Throughout the thesis, we shall use the convention  $0 \cdot \infty = 0$ .

### 1.2.1 Some reminders on measurability and $\mu$ -measurability.

The interested reader can see [31] for more details of this section.

**Definition 1.2.1.** Let  $(\Omega_1, \mathcal{A})$  and  $(\Omega_2, \mathcal{B})$  be two measurable spaces. A function  $f : (\Omega_1, \mathcal{A}) \rightarrow (\Omega_2, \mathcal{B})$  is measurable if, for every  $B \in \mathcal{B}$ , we have  $f^{-1}(B) \in \mathcal{A}$ .

**Definition 1.2.2.** Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space. The measure  $\mu$  is finite if  $\mu(\Omega) < \infty$ .

The set  $A \subset \Omega$  is  $\sigma$ -finite with respect to the measure  $\mu$  if there exists pairwise disjoint sequence  $(A_i)_{i \geq 1} \in \mathcal{A}$  such that  $A = \cup_{i=1}^{\infty} A_i$  and  $\mu(A_i) < \infty$ , for every  $i \geq 1$ .

A set  $A \subset \Omega$  is a null set ( $\mu$ -null set) with respect to the measure  $\mu$  if there exists  $B \in \mathcal{A}$  such that  $A \subset B$  and  $\mu(B) = 0$ .

The measure space  $(\Omega, \mathcal{A}, \mu)$  is complete or the measure  $\mu$  is complete or  $\mathcal{A}$  is complete if

$$\forall A \subset \Omega, \forall B \subset \Omega, (A \subset B \text{ and } B \in \mathcal{A} \text{ and } \mu(B) = 0) \Rightarrow (A \in \mathcal{A}).$$

The following is an example of positive measure which is not complete.

**Example 1.2.3.** Let  $\Omega = \{a, b, c\}$ ,  $\mathcal{A} = \{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}\}$  and  $\mu : \mathcal{A} \rightarrow \overline{\mathbb{R}}_+$  such that

$$\begin{aligned}\mu(\emptyset) &= 0, \\ \mu(\{a, b\}) &= 0, \\ \mu(\{a, b, c\}) &= \infty, \\ \mu(\{c\}) &= \infty.\end{aligned}$$

$\mathcal{A}$  is a  $\sigma$ -algebra,  $\mu$  is a measure on it. However  $\mu$  is not complete since  $\{a\} \subset \{a, b\}$  and  $\mu(\{a, b\}) = 0$ , but  $\{a\} \notin \mathcal{A}$ .

In the cases that interest us, the target space of functions is the Banach space  $E$ . The  $\sigma$ -algebra  $\mathcal{B}$  on  $E$  will always be induced by the topological structure on  $E$  that is, the  $\sigma$ -algebra of Borel sets of  $E$ .

**Proposition 1.2.4.** Let  $(f_n)_{n \geq 1}$  be a sequence of measurable functions on  $\Omega$  with values in  $E$ . If  $(f_n)_{n \geq 1}$  converges pointwise to  $f$ , then  $f$  is measurable.

In order to define the integral of measurable function, we need first to introduce simple functions.

**Definition 1.2.5.** Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space and  $E$  a Banach space. A simple function  $s : \Omega \rightarrow E$  is any finite sum of the form

$$s = \sum_{i=1}^N a_i \chi_{A_i},$$

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where  $N$  is a nonnegative integer,  $(A_i)_{i=1}^N$  is a finite sequence of pairwise disjoint members of the  $\sigma$ -algebra  $\mathcal{A}$  such that  $\cup_{i=1}^N A_i = \Omega$ ,  $(a_i)_{i=1}^N \subset E$ ,  $A_i = s^{-1}(\{a_i\})$  and  $\chi_A : \Omega \rightarrow \{0, 1\}$  denotes the characteristic function of  $A \in \mathcal{A}$ .

**Notation 1.2.6.** In the sequel, we will denote by  $S(\Omega, E)$  the space of all simple functions  $s : \Omega \rightarrow E$ .

**Remark 1.2.7.** Every simple function is measurable. The set of simple functions is a vector space and the multiplication of a simple function by a characteristic function is also a simple function since  $\chi_A \chi_B = \chi_{A \cap B}$  for all  $A, B \in \mathcal{A}$ .

The following notion of measurability with respect to the positive measure  $\mu$  will be very important to define the integral of a vector-valued function.

**Definition 1.2.8.** Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space and  $E$  a Banach space. A function  $f : \Omega \rightarrow E$  is  $\mu$ -measurable if there exists a sequence of simple functions which converges  $\mu$ -almost everywhere ( $\mu$ -a.e.) to  $f$ .

**Notation 1.2.9.** We denote  $\mathcal{M}_\mu(\Omega, E)$  the space of all  $\mu$ -measurable functions from  $\Omega$  into  $E$ .

**Remark 1.2.10.** It is well-known that if  $f$  a measurable function, then  $f$  is  $\mu$ -measurable (see [64, Theorem 1.17] for positive functions). The converse is not true in general.

If the measure space  $(\Omega, \mathcal{A}, \mu)$  is not complete, we have the following counter example.

**Example 1.2.11.** If the measure space  $(\Omega, \mathcal{A}, \mu)$  is not complete, then there exists  $A \subset \Omega$  and  $B \subset \Omega$  such that  $A \subset B$ ,  $B \in \mathcal{A}$ ,  $A \notin \mathcal{A}$  and  $\mu(B) = 0$ . Let  $f = \chi_A$ , and  $s_n = \chi_B$ . Since  $\mu(B) = 0$ , the sequence  $(s_n)$  of simple functions converges to  $f$  on  $\mu$ -almost everywhere. But  $f$  is not measurable.

**Remark 1.2.12.** When the measure space is  $\sigma$ -finite and complete, the  $\mu$ -measurability implies the measurability (see [31]). This remark will allow us to only consider the case where the measure space  $(\Omega, \mathcal{A}, \mu)$  is  $\sigma$ -finite and complete.

As in the case of the measurability, one can ask whether the properties of the  $\mu$ -measurability goes to the limit. We will see that the answer is yes and the proof is very technical.

**Theorem 1.2.13.** ([31, Theorem 1.1.2]) Let  $(\Omega, \mathcal{A}, \mu)$  be a complete measure space and  $(f_k)_{k \geq 1} \subset \mathcal{M}_\mu(\Omega, E)$ . If  $(f_k)_{k \geq 1}$  converges  $\mu$ -a.e. to  $f$ , then  $f \in \mathcal{M}_\mu(\Omega, E)$ .

As a consequence of Theorem 1.2.13, we have the following.

**Corollary 1.2.14.** ([31, Corollary 1.1.1]) Let  $(\Omega, \mathcal{A}, \mu)$  be a complete  $\sigma$ -finite measure space and  $E$  be a separable Banach space. Then  $f : \Omega \rightarrow E$  is  $\mu$ -measurable if and only if  $f$  is measurable.

**Corollary 1.2.15.** ([31, Corollary 1.1.2]) Let  $(\Omega, \mathcal{A}, \mu)$  be a complete  $\sigma$ -finite measure space and  $E$  be a separable Banach space and  $D$  a countable dense subset of  $E$ . The function  $f : \Omega \rightarrow E$  is  $\mu$ -measurable if and only if for all  $d \in D$ , the function  $z \mapsto \|f(z) - d\|_E$  is measurable from  $\Omega$  into  $\mathbb{R}_+$ .

1.2.2 Integration of vector-valued  $\mu$ -measurable functions.

**Definition 1.2.16.** We say that the simple function

$$s = \sum_{i=1}^N a_i \chi_{A_i},$$

Bochner-integrable with respect to a positive measure  $\mu$  whenever

$$\sum_{i=1}^N \mu(A_i) \|a_i\|_E < \infty.$$

That is,  $\mu(A_i) < \infty$  whenever  $a_i \neq 0$ . In this case, we define the Bochner-integral of  $s : \Omega \rightarrow E$  over  $A \in \mathcal{A}$  by:

$$\int_A s(z) d\mu(z) = \int_{\Omega} \sum_{i=1}^N a_i \chi_{A_i \cap A}(z) d\mu(z) = \sum_{i=1}^N \mu(A_i \cap A) a_i.$$

Unless otherwise,  $(\Omega, \mathcal{A}, \mu)$  will be a complete  $\sigma$ -finite measure space. We will also assume that the positive measure  $\mu$  is such that for each simple function  $s = \sum_{i=1}^N a_i \chi_{A_i}$ ,  $a_i = 0$  whenever  $\mu(A_i) = \infty$ .

**Remark 1.2.17.** Every simple function  $s : \Omega \rightarrow E$  is  $\mu$ -Bochner integrable.

**Definition 1.2.18.** The function  $f : \Omega \rightarrow E$  is  $\mu$ -Bochner integrable if there exists a sequence of simple functions  $(s_k)_{k \geq 1} \subset S(\Omega, E)$  such that

- (i)  $s_k \rightarrow f$  a.e., as  $k \rightarrow \infty$ .
- (ii)  $\int_{\Omega} \|f(z) - s_k(z)\|_E d\mu(z) \rightarrow 0$ , as  $k \rightarrow \infty$ .

**Notation 1.2.19.** We denote by  $\mathcal{L}_{\mu}^1(\Omega, E)$  the set of all functions from  $\Omega$  into  $E$  which are  $\mu$ -Bochner integrable. A sequence of simple functions which satisfies assertions (i) and (ii) is called *approximant sequence* for  $f$ .

In practice, to verify that a vector-valued  $\mu$ -measurable function is integrable, we use the following result.

**Proposition 1.2.20.** ([31, Proposition 1.2.1]) Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space and  $E$  a Banach space. A function  $f : \Omega \rightarrow E$  is Bochner-integrable if and only if  $f$  is  $\mu$ -measurable and

$$\int_{\Omega} \|f(z)\|_E d\mu(z) < \infty.$$

Since we know how to define and to show that a vector-valued function is Bochner-integrable, we will give now a formal definition of the Bochner integral which will show us how to compute an integral of a vector-valued function.

**Definition 1.2.21.** ( $\mu$ -Bochner-integral) Let  $f \in \mathcal{L}_{\mu}^1(\Omega, E)$ , and  $(s_k)_{k \geq 1}$  an *approximant sequence* for  $f$ . The  $\mu$ -Bochner integral of  $f$  is defined by

$$\int_{\Omega} f(z) d\mu(z) = \lim_{k \rightarrow \infty} \int_{\Omega} s_k(z) d\mu(z).$$

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**Notation 1.2.22.** For  $A \in \mathcal{A}$ , and  $f \in \mathcal{L}_\mu^1(\Omega, E)$ , then  $\chi_A f$  is  $\mu$ -Bochner integrable and

$$\int_A f(z) d\mu(z) = \int_\Omega (\chi_A f)(z) d\mu(z).$$

**Proposition 1.2.23.** The limit in Definition 1.2.21 exists and it is independent of the choice of the approximant sequence for  $f$ .

*Proof.* Let  $f \in \mathcal{L}_\mu^1(\Omega, E)$ , and  $(s_n)_{n \geq 1}$  an approximant sequence for  $f$ , we note that

$$\begin{aligned} \left\| \int_\Omega (s_n - s_m)(z) d\mu(z) \right\|_E &\leq \int_\Omega \|s_n(z) - s_m(z)\|_E d\mu(z) \\ &\leq \int_\Omega \|s_n(z) - f(z)\|_E d\mu(z) + \int_\Omega \|s_m(z) - f(z)\|_E d\mu(z). \end{aligned}$$

Thus,  $(\int_\Omega (s_n(z) d\mu(z))_{n \geq 1}$  is a Cauchy sequence in the Banach space  $E$ . Its converges in  $E$  to some vector  $\ell_s$ . The limit obtained is independent of the approximant sequence choosen. Indeed, if  $(s_n)_{n \geq 1}$  and  $(t_n)_{n \geq 1}$  are two approximant sequences for  $f$  which integrals converges respectively to  $\ell_s$  and  $\ell_t$ . We have

$$\begin{aligned} \left\| \int_\Omega (s_n - t_n)(z) d\mu(z) \right\|_E &\leq \int_\Omega \|s_n(z) - t_n(z)\|_E d\mu(z) \\ &\leq \int_\Omega \|s_n(z) - f(z)\|_E d\mu(z) + \int_\Omega \|t_n(z) - f(z)\|_E d\mu(z). \end{aligned}$$

Thus,  $\ell_s - \ell_t = \lim_{n \rightarrow \infty} \int_\Omega (s_n - t_n)(z) d\mu(z) = 0$ . □

The next result exhibits a very strong property of Bochner integration.

**Proposition 1.2.24.** (Hille's theorem) Let  $E, F$  be two complex Banach spaces, and  $T : E \rightarrow F$  is a bounded linear operator. If  $f$  is  $\mu$ -Bochner integrable, then  $T(f)$  is  $\mu$ -Bochner integrable and we have

$$T \left( \int_\Omega f(z) d\mu(z) \right) = \int_\Omega T(f(z)) d\mu(z).$$

*Proof.* Let us first show that  $T(f)$  is  $\mu$ -Bochner integrable. Let  $(s_n)_{n \geq 1} \subset S(\Omega, E)$  be an approximant sequence for  $f$ . Then the sequence  $(T(s_n))_{n \geq 1}$  is a sequence of simple functions from  $\Omega$  into  $F$ . (If  $s = \sum_{i=1}^N a_i \chi_{A_i}$  is a simple function from  $\Omega$  into  $E$ , then  $T(s) = \sum_{i=1}^N T(a_i) \chi_{A_i}$  is a simple function from  $\Omega$  into  $F$ .) Since  $T$  is a bounded linear operator from  $E$  into  $F$ , then for every  $n \geq 1$ , we have

$$\|T(s_n) - T(f)\|_E \leq \|T\| \|s_n - f\|_E \rightarrow 0,$$

as  $n \rightarrow \infty$ . So, the sequence of simple functions  $(T(s_n))_{n \geq 1}$  satisfies the assertion (i) of Definition 1.2.18. We also have that

$$\int_\Omega \|T(s_n) - T(f)\|_E d\mu \leq \|T\| \int_\Omega \|s_n - f\|_E d\mu \rightarrow 0,$$

as  $n$  tends to infinity. Thus,  $(T(s_n))_{n \geq 1}$  also satisfies the assertion (ii) of Definition 1.2.18. It follows from (i) and (ii) that  $(T(s_n))_{n \geq 1}$  is an approximant sequence for  $T(f)$ . Let us prove that the following equality holds:

$$T \left( \int_\Omega f(z) d\mu(z) \right) = \int_\Omega T(f(z)) d\mu(z).$$

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Let  $(s_n)_{n \geq 1} \subset S(\Omega, E)$  be an approximant sequence for  $f$ . Then the sequence  $(T(s_n))_{n \geq 1} \subset S(\Omega, F)$  is an approximant sequence for  $T(f)$  and by Definition 1.2.21, we can write

$$\int_{\Omega} f(z) d\mu(z) = \lim_{n \rightarrow \infty} \int_{\Omega} s_n(z) d\mu(z),$$

and

$$\int_{\Omega} T(f(z)) d\mu(z) = \lim_{n \rightarrow \infty} \int_{\Omega} T(s_n(z)) d\mu(z).$$

Since  $T : E \rightarrow F$  is a bounded linear operator, we get that

$$T\left(\int_{\Omega} f(z) d\mu(z)\right) = \lim_{n \rightarrow \infty} T\left(\int_{\Omega} s_n(z) d\mu(z)\right).$$

So, to obtain the desired result, it suffices to prove that for every nonnegative integer  $n \geq 1$ , we have

$$T\left(\int_{\Omega} s_n(z) d\mu(z)\right) = \int_{\Omega} T(s_n(z)) d\mu(z).$$

Let  $n \geq 1$  be a nonnegative integer we know that  $s_n$  has the following form  $s_n = \sum_{i=1}^{N_n} a_i^n \chi_{A_i^n}$ , where  $(A_i^n)_{i=1}^{N_n}$  is a measurable partition of  $E$  and  $(a_i^n)_{i=1}^{N_n} \subset E$ . By using the fact that  $T$  is a linear operator, we write  $T(s_n) = \sum_{i=1}^{N_n} T(a_i^n) \chi_{A_i^n}$ . It follows that

$$\int_{\Omega} T(s_n) d\mu = \sum_{i=1}^{N_n} \mu(A_i^n) T(a_i^n),$$

and

$$T\left(\int_{\Omega} s_n d\mu\right) = T\left(\sum_{i=1}^{N_n} \mu(A_i^n) a_i^n\right) = \sum_{i=1}^{N_n} \mu(A_i^n) T(a_i^n) = \int_{\Omega} T(s_n) d\mu.$$

□

**Lemma 1.2.25.** Let  $f : \Omega \rightarrow E$  be a  $\mu$ -Bochner integrable vector-valued function. Then the inequality

$$\left\| \int_A f(z) d\mu(z) \right\|_E \leq \int_A \|f(z)\|_E d\mu(z)$$

holds for all  $A \in \mathcal{A}$ .

*Proof.* Let  $A \in \mathcal{A}$  and  $(s_n)_{n \geq 1} \subset S(\Omega, E)$  an approximate sequence for  $f$ . By duality, we have

$$\left\| \int_A f(z) d\mu(z) \right\|_E = \sup_{T \in E^*, \|T\|_{E^*} = 1} \left| T\left(\int_A f(z) d\mu(z)\right) \right|.$$

Now, using Proposition 1.2.24, and the fact that  $T$  is bounded, we obtain that

$$\left| T\left(\int_A f(z) d\mu(z)\right) \right| \leq \int_A \|T(f(z))\|_E d\mu(z) \leq \|T\|_{E^*} \int_A \|f(z)\|_E d\mu(z).$$

It then follows that

$$\left\| \int_A f(z) d\mu(z) \right\|_E \leq \int_A \|f(z)\|_E d\mu(z).$$

□

## 1.2. Integration of vector-valued functions

**Remark 1.2.26.** By using the integrability criterion in Proposition 1.2.20, we obtain

$$\mathcal{L}_\mu^1(\Omega, E) = \left\{ f : \Omega \rightarrow E \text{ measurable such that } \int_\Omega \|f(z)\|_E d\mu(z) < \infty \right\}.$$

For  $f \in \mathcal{L}_\mu^1(\Omega, E)$ , let  $N_1(f) = \int_\Omega \|f(z)\|_E d\mu(z)$ . It is well known that  $N_1$  is not a norm on  $\mathcal{L}_\mu^1(\Omega, E)$ . Indeed, if  $N_1(f) = 0$ , then  $f = 0_E$   $\mu$ -a.e.. As in the case of Lebesgue integrals, we consider the set  $\mathcal{N}$  to be the space of vector-valued measurable functions which are equal to zero  $\mu$ -a.e.

The relation  $f \sim g \Leftrightarrow f - g \in \mathcal{N} \Leftrightarrow f - g = 0_E$   $\mu$ -a.e. is an equivalence relation on  $\mathcal{L}_\mu^1(\Omega, E)$ . Now, let

$$L_\mu^1(\Omega, E) = \mathcal{L}_\mu^1(\Omega, E) /_{\sim \mu\text{-a.e.}} = \mathcal{L}_\mu^1(\Omega, E) /_{\mathcal{N}}$$

be the quotient space of  $\mathcal{L}_\mu^1(\Omega, E)$  by  $\mathcal{N}$ . It is made of classes of vector-valued measurable functions  $\Phi$  which are equal  $\mu$ -a.e.. Indeed,  $f, g \in \Phi \Rightarrow f - g = 0$   $\mu$ -a.e. Let  $\Phi \in L_\mu^1(\Omega, E)$ . If  $f, g \in \Phi$ , we have  $f = g$   $\mu$ -a.e., so  $\|f\|_E = \|g\|_E$   $\mu$ -a.e., and then  $\int_\Omega \|f\|_E d\mu = \int_\Omega \|g\|_E d\mu$ . We then define on  $L_\mu^1(\Omega, E)$  the function also denoted  $N_1$  by

$$N_1(\Phi) = N_1(f), \quad \text{for } f \in \Phi.$$

**Definition 1.2.27** (Bochner-Lebesgue space). The Bochner-Lebesgue space denoted  $L_\mu^1(\Omega, E)$  consists of all classes of vector-valued  $\mu$ -measurable functions  $\tilde{f} : \Omega \rightarrow E$  such that

$$N_1(\tilde{f}) = \int_\Omega \|\tilde{f}(z)\|_E d\mu(z) = \int_\Omega \|f(z)\|_E d\mu(z) < \infty,$$

for any  $f \in \tilde{f}$ .

**Proposition 1.2.28.** The Bochner-Lebesgue space  $L_\mu^1(\Omega, E)$  is a normed vector space endowed with the norm  $\|\tilde{f}\|_{1,\mu,E} = N_1(\tilde{f})$ .

In the sequel, we will not distinguish the class  $\tilde{f}$  with its representative  $f \in \tilde{f}$ .

### 1.2.3 Bochner-Lebesgue $L_\mu^p(\Omega, E)$ -spaces.

**Definition 1.2.29.** i) For  $0 < p < \infty$ , the Bochner-Lebesgue space  $L_\mu^p(\Omega, E)$  is defined by:

$$L_\mu^p(\Omega, E) = \left\{ f : \Omega \rightarrow E, \mu\text{-measurable such that } \int_\Omega \|f(z)\|^p d\mu(z) < \infty \right\} /_{\mathcal{N}}.$$

The space  $L_\mu^p(\Omega, E)$  is endowed with the norm

$$\|f\|_{p,\mu,E} := \|f\|_{L_\mu^p(\Omega,E)} = \left( \int_\Omega \|f(z)\|^p d\mu(z) \right)^{1/p}.$$

ii) For  $p = \infty$ , we define the Bochner-Lebesgue space  $L^\infty(\Omega, E)$  by :

$$L^\infty(\Omega, E) = \{ f : \Omega \rightarrow E, \mu\text{-measurable such that } \text{ess sup } \|f(z)\|_E < \infty \} /_{\mathcal{N}},$$

where

$$\|f\|_{\infty,E} := \text{ess sup } \|f(z)\|_E = \inf \{ C > 0 : \|f(z)\|_E \leq C \text{ for } \mu\text{-almost every } z \in \Omega \}.$$

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**Proposition 1.2.30.** For  $0 < p < 1$ , the Bochner-Lebesgue space  $L_\mu^p(\Omega, E)$  is a metric space endowed with the metric

$$d(f, g) = \|f - g\|_{p, \mu, E}^p.$$

In the sequel, we will write  $L_\mu^p(\Omega)$  instead of  $L_\mu^p(\Omega, \mathbb{C})$ .

**Theorem 1.2.31.** ([31, Theorem 1.3.1]) For each  $1 \leq p < \infty$ , the Bochner-Lebesgue space  $L_\mu^p(\Omega, E)$  is a Banach space.

**Proposition 1.2.32.** For each  $1 \leq p < \infty$ , the space of simple functions  $S(\Omega, E)$  form a dense subspace of  $L_\mu^p(\Omega, E)$ .

*Proof.* Let  $1 \leq p < \infty$ ,  $f \in L_\mu^p(\Omega, E)$ , and  $(t_n)_{n \geq 1} \subset S(\Omega, E)$  an approximant sequence for  $f$ . For  $n \geq 1$ , let

$$s_n = t_n \chi_{\{z \in E: \|t_n(z)\|_E \leq 2\|f(z)\|_E\}}.$$

As in Proposition 1.2.20,  $(s_n)_{n \geq 1}$  is a sequence of simple functions from  $\Omega$  to  $E$  which converges to  $f$   $\mu$ -a.e and satisfies  $\|s_n(\cdot) - f(\cdot)\|_E \leq 3\|f(\cdot)\|_E \in L_\mu^p(\Omega, \mathbb{C})$ , so by using the dominated convergence theorem for integral of real-valued functions, it follows that  $\|s_n(\cdot) - f(\cdot)\|_E \rightarrow 0$  in  $L_\mu^p(\Omega, \mathbb{C})$ , that is  $s_n \rightarrow f$  in  $L_\mu^p(\Omega, E)$  as  $n \rightarrow \infty$ .  $\square$

**Corollary 1.2.33.** ([31, Corollary 1.3.2]) If  $\Omega$  is an open set of  $\mathbb{C}^n$ , and  $E$  is a separable Banach space, then for each  $1 \leq p < \infty$ , the space  $L_\mu^p(\Omega, E)$  is separable.

**Lemma 1.2.34.** ([31, Corollary 1.3.3]) Suppose  $1 \leq p < \infty$  and  $\mu(\Omega) < \infty$ . If  $(f_k)_{k \geq 1}$  is a bounded sequence in  $L_\mu^p(\Omega, E)$  which converge  $\mu$ -a.e. to  $f$ , then  $f_k \rightarrow f$  in  $L_\mu^q(\Omega, E)$  as  $k \rightarrow \infty$ , for every  $1 \leq q < p$ .

**Corollary 1.2.35.** ([31, Corollary 1.4.1]) Let  $1 < p < \infty$  and  $E$  a reflexive separable Banach space. Then  $L_\mu^p(\Omega, E)$  is reflexive.

### 1.2.4 Some reminders on scalar-valued functions

In this thesis, we have used the identity

$$\int_\Omega \|f(z)\|_E^p d\mu(z) = p \int_0^\infty \mu(\{z \in \Omega : \|f(z)\|_E > \lambda\}) \lambda^{p-1} d\lambda,$$

where  $f : \Omega \rightarrow E$  is a  $\mu$ -measurable. The proof is obtained by replacing the modulus by the norm in the following lemma.

**Lemma 1.2.36.** ([78, Lemma 2.7]) Let  $0 < p < \infty$ . Suppose that  $f : \Omega \rightarrow \mathbb{C}$  is measurable. Then

$$\int_\Omega |f(z)|^p d\mu(z) = p \int_0^\infty \mu(\{z \in \Omega : |f(z)| > \lambda\}) \lambda^{p-1} d\lambda.$$

We first need to recall the following monotonicity convergence theorem.

**Theorem 1.2.37.** (Monotone Convergence Theorem) Let  $(f_k)_{k \geq 1}$  be an increasing sequence of nonnegative measurable functions, and let  $f = \lim_{k \rightarrow \infty} f_k$  a.e. Then

$$\lim_{k \rightarrow \infty} \int_\Omega f_k d\mu = \int_\Omega f d\mu.$$

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**Corollary 1.2.38.** Let  $(u_k)_{k \geq 1}$  be a sequence of nonnegative measurable functions, and let  $f = \sum_{k=1}^{\infty} u_k$ .

Then

$$\int_{\Omega} f d\mu = \sum_{k=1}^{\infty} \int_{\Omega} u_k d\mu.$$

The following result is referenced in [64].

**Proposition 1.2.39.** (Minkowski inequality for integrals) Let  $(\Omega_1, \mathcal{A}, \mu)$  and  $(\Omega_2, \mathcal{B}, \nu)$  be two  $\sigma$ -finite measure spaces. Let  $1 \leq p < \infty$ . If  $f : (\Omega_1 \times \Omega_2, \mathcal{A} \otimes \mathcal{B}) \rightarrow \mathbb{R}_+$  is measurable, then

$$\left( \int_{\Omega_1} \left( \int_{\Omega_2} f(x, y) d\nu(y) \right)^p d\mu(x) \right)^{\frac{1}{p}} \leq \int_{\Omega_2} \left( \int_{\Omega_1} f(x, y)^p d\mu(x) \right)^{\frac{1}{p}} d\nu(y).$$

**Remark 1.2.40.** If  $\Omega_2 = \{1, 2\}$ , and  $\nu = \delta_1 + \delta_2$  is a counting measure, then for

$$\begin{aligned} \Phi : [0, 1] &\rightarrow \{0, 1\} \\ f(x, y) &= \begin{cases} h(x) & \text{if } y = 1 \\ g(x) & \text{if } y = 2, \end{cases} \end{aligned}$$

then for  $1 \leq p < \infty$ , we have the usual Minkowski inequality

$$\|h + g\|_{p, \mu} \leq \|h\|_{p, \mu} + \|g\|_{p, \mu}.$$

**Theorem 1.2.41.** Let  $g$  be a positive integrable function over  $\Omega$ , and  $(f_k)_{k \geq 1}$  a sequence of measurable functions such that on  $\Omega$ ,

$$|f_k(x)| \leq g(x)$$

and such that  $f_k(x) \rightarrow f(x)$  a.e. Then

$$\lim_{k \rightarrow \infty} \int_{\Omega} f_k d\mu = \int_{\Omega} f d\mu.$$

In what will follow, we recall the Tonelli's and the Fubini's theorems for scalar-valued functions.

Given a scalar-valued function  $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{C}$  we denote by  $f_z : \Omega_1 \rightarrow \mathbb{C}$  the function defined by  $f_z(w) = f(z, w)$ , and  $f^w : \Omega_2 \rightarrow \mathbb{C}$  is the function defined by  $f^w(z) = f(z, w)$ .

**Theorem 1.2.42.** (Tonelli's theorem) Let  $(\Omega_1, \mathcal{A}, \mu)$  and  $(\Omega_2, \mathcal{B}, \nu)$  be two  $\sigma$ -finite measure spaces and let  $f : \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$  be a measurable function satisfying:

$$(a) \int_{\Omega_2} |f(z, w)| d\nu(w) < \infty \text{ for a.e. } z \in \Omega_1$$

and

$$(b) \int_{\Omega_1} d\mu(z) \int_{\Omega_2} |f(z, w)| d\nu(w) < \infty.$$

Then  $f \in L^1(\Omega_1 \times \Omega_2)$ .

**Theorem 1.2.43.** (Fubini's theorem) Let  $(\Omega_1, \mathcal{A}, \mu)$  and  $(\Omega_2, \mathcal{B}, \nu)$  be two  $\sigma$ -finite measure spaces. Assume that  $f \in L^1(\Omega_1 \times \Omega_2)$ . Then

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(i) for a.e.  $z \in \Omega_1$ ,  $f_z \in L^1(\Omega_2)$  and  $\int_{\Omega_2} f_z d\nu \in L^1(\Omega_1)$ .

and

(ii) for a.e.  $w \in \Omega_2$ ,  $f^w \in L^1(\Omega_1)$  and  $\int_{\Omega_1} f^w d\mu \in L^1(\Omega_2)$ .

Moreover, one has

$$\int_{\Omega_1} d\mu(z) \int_{\Omega_2} f(z, w) d\nu(w) = \int_{\Omega_2} d\nu(w) \int_{\Omega_1} f(z, w) d\mu(z) = \iint_{\Omega_1 \times \Omega_2} f(z, w) d\mu(z) d\nu(w).$$

**Theorem 1.2.44.** (Fubini-Tonelli) Let  $(\Omega_1, \mathcal{A}, \mu)$  and  $(\Omega_2, \mathcal{B}, \nu)$  be two  $\sigma$ -finite measure spaces. If  $f : \Omega_1 \times \Omega_2 \rightarrow [0, \infty]$  is a measurable function for  $\mathcal{A} \otimes \mathcal{B}$ , then

- i) For  $\mu$ -almost every  $z \in \Omega_1$ ,  $f_z : \Omega_2 \rightarrow [0, \infty]$  is measurable and, for  $\nu$ -almost every  $w \in \Omega_2$ ,  $f^w : \Omega_1 \rightarrow [0, \infty]$  is measurable.
- ii) The function  $z \mapsto \int_{\Omega_2} f_z d\nu$  and the function  $w \mapsto \int_{\Omega_1} f^w d\mu$  are both measurable.
- iii)  $\iint_{\Omega_1 \times \Omega_2} f d(\mu \times \nu) = \int_{\Omega_1} \left( \int_{\Omega_2} f_z d\nu \right) d\mu = \int_{\Omega_2} \left( \int_{\Omega_1} f^w d\mu \right) d\nu$ .

### 1.2.5 Dominated convergence theorem for vector-valued functions and its applications.

The following result is the dominated convergence theorem for vector-valued functions.

**Theorem 1.2.45.** ([31, Theorem 1.3.3]) Let  $1 \leq p < \infty$ ,  $E$  a Banach space and  $(f_k)_{k \geq 1}$  a sequence of  $E$ -valued  $\mu$ -measurable functions such that

- i)  $f_k \rightarrow f$ ,  $\mu$ -a.e. on  $\Omega$  as  $k \rightarrow \infty$ .
- ii) There exists  $g \in L^p_\mu(\Omega)$  such that, for all  $k \geq 1$ ,  $\|f_k(\cdot)\|_E \leq g(\cdot)$   $\mu$ -a.e.  
Then  $f_k \rightarrow f$  in  $L^p_\mu(\Omega, E)$ . In particular, in the case  $p = 1$ ,

$$\int_{\Omega} f_k d\mu \rightarrow \int_{\Omega} f d\mu.$$

The following theorem is known as the converse of the dominated convergence theorem.

**Theorem 1.2.46.** ([31, Theorem 1.3.4]) Suppose  $1 \leq p < \infty$ . If  $f_n \rightarrow f$  in  $L^p_\mu(\Omega, E)$  as  $n \rightarrow \infty$ , then there exists a subsequence  $(f_{n_k})_{k \geq 1}$  of  $(f_n)_{n \geq 1}$  and  $g \in L^p_\mu(\Omega, \mathbb{R}_+)$  such that:

- i)  $f_{n_k} \rightarrow 0$   $\mu$ -a.e. as  $k \rightarrow \infty$ .
- ii) For all  $k \geq 1$ ,  $\|f_{n_k}(\cdot)\|_E \leq g(\cdot)$   $\mu$ -a.e.
- iii)  $\forall \epsilon > 0$ , there exists  $A_\epsilon \subset \Omega$  such that  $\mu(A_\epsilon) < \epsilon$  and  $f_{n_k} \rightarrow f$  uniformly on  $\Omega \setminus A_\epsilon$ .

**Theorem 1.2.47.** ([31, Theorem 1.3.5]) Let  $I$  be a metric space and  $f : I \times \Omega \rightarrow E$ . If

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- i) for  $\mu$ -almost every  $z \in \Omega$ ,  $f(\cdot, z)$  is continuous at a point  $t_0 \in I$ ,
- ii) for all  $t \in I$ ,  $f(t, \cdot)$  is  $\mu$ -measurable,
- iii) there exists  $g \in L^1_\mu(\Omega)$  such that for all  $t \in I$  and for  $\mu$ -almost every  $z \in \Omega$ ,  $\|f(t, z)\|_E \leq g(z)$ , then  $F(t) = \int_\Omega f(t, z) d\mu(z)$  is continuous at  $t_0$ .

**Theorem 1.2.48.** ([31, Theorem 1.3.6]) Let  $I$  be an interval of  $\mathbb{R}$  and  $f : I \times \Omega \rightarrow E$ . If

- i) for all  $t \in I$ ,  $f(t, \cdot)$  is  $\mu$ -measurable,
- ii) there exists  $t_0 \in I$  such that  $f(t_0, \cdot) \in L^1_\mu(\Omega, E)$ , there exists  $A \subset \Omega$  such that  $\mu(\Omega \setminus A) = 0$  and
- iii) for all  $z \in A$ ,  $f(\cdot, z)$  is differentiable on  $I$ ,
- iv) there exists  $g \in L^1_\mu(\Omega)$  positive such that  $\forall t \in I, \forall z \in A$ ,  $\left\| \frac{\partial f}{\partial t}(t, z) \right\|_E \leq g(z)$ , then  $F(t) = \int_\Omega f(t, z) d\mu(z)$  is well-defined and is differentiable on  $I$  and we have

$$F'(t) = \int_\Omega \frac{\partial f}{\partial t}(t, z) d\mu(z).$$

### 1.2.6 Fubini's theorem for vector-valued functions.

In this section,  $E$  is still a Banach space and  $(\Omega_1, \mathcal{A}, \mu)$   $(\Omega_2, \mathcal{B}, \nu)$  are two  $\sigma$ -finite measure spaces. We endowed the product  $\Omega_1 \times \Omega_2$  with the  $\sigma$ -algebra  $\mathcal{A} \otimes \mathcal{B}$  spanned by  $\{A \times B, A \in \mathcal{A}, B \in \mathcal{B}\}$  and with the measure  $\mu \otimes \nu$ . Given a vector-valued function  $f : \Omega_1 \times \Omega_2 \rightarrow E$  we denote by  $f_z : \Omega_1 \rightarrow E$  the function defined by  $f_z(w) = f(z, w)$ , and  $f^w : \Omega_2 \rightarrow E$  is the function defined by  $f^w(z) = f(z, w)$ .

In what will follow, we give the vector-valued version of the Fubini's theorem. We first recall that the space  $L^1(\Omega_1 \times \Omega_2, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu, E)$  is the class of  $(\mu \otimes \nu)$ -measurable vector-valued functions  $f : \Omega_1 \times \Omega_2 \rightarrow E$  which are Bochner-integrable.

**Theorem 1.2.49.** ([31, Theorem 1.5.2]) Suppose that  $E$  is a Banach space and that  $(\Omega_1, \mathcal{A}, \mu)$  and  $(\Omega_2, \mathcal{B}, \nu)$  are two  $\sigma$ -finite measure spaces. If  $f \in L^1(\Omega_1 \times \Omega_2, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu, E)$ , then

- i) for  $\mu$ -almost every  $z \in \Omega_1$ ,  $f_z \in L^1(\Omega_2, \mathcal{B}, \nu, E)$  and for  $\nu$ -almost every  $w \in \Omega_2$ ,  $f^w \in L^1(\Omega_1, \mathcal{A}, \mu, E)$ ,
- ii) the function  $z \mapsto \int_{\Omega_2} f_z d\nu$  belongs to  $L^1(\Omega_1, \mathcal{A}, \mu, E)$  and the function  $w \mapsto \int_{\Omega_1} f^w d\mu$  belongs to  $L^1(\Omega_2, \mathcal{B}, \nu, E)$ ,

iii) we have

$$\iint_{\Omega_1 \times \Omega_2} f d(\mu \otimes \nu) = \int_{\Omega_1} \left( \int_{\Omega_2} f_z d\nu \right) d\mu = \int_{\Omega_2} \left( \int_{\Omega_1} f^w d\mu \right) d\nu.$$

## 1.3 Vector measures : topological dual of $L^p_\mu(\Omega, E)$ , with $1 < p < \infty$ .

### 1.3.1 Vector measures with bounded variation.

This section deals with basic straightforward properties of vector measures. The familiar notions of countable additivity and variation are introduced together with the concepts of additivity[29].

**Definition 1.3.1.** Let  $\mathcal{A}$  be a  $\sigma$ -algebra of subsets of a set  $\Omega$  and  $E$  be a Banach space.

A function  $G : \mathcal{A} \rightarrow E$  is called a *finitely additive vector measure*, if for every finite sequence  $(A_i)_{i=1}^N$  of pairwise disjoint members of  $\mathcal{A}$ , we have

$$G\left(\bigcup_{i=1}^N A_i\right) = \sum_{i=1}^N G(A_i).$$

If in addition,

$$G\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} G(A_i)$$

in the norm topology of  $E$  for every sequence  $(A_i)_{i=1}^{\infty}$  of pairwise disjoint members of  $\mathcal{A}$ , then  $G$  is called a *countably additive vector measure*.

**Example 1.3.2.** (A countably additive vector measure) Let  $\Omega = [0, 1]$  with the Lebesgues measure  $\lambda$ . Let  $T : L^1([0, 1]) \rightarrow E$  be a continuous linear operator. Define  $G(A) = T(\chi_A)$  ( $\chi_A$  denotes the indicator function of  $A$ ) for each Lebesgue measurable set  $A \subset [0, 1]$ . Then  $G$  is clearly finitely additive. Consequently, if  $(A_i)_{i=1}^{\infty}$  is a sequence of disjoint Lebesgue measurable subsets of  $[0, 1]$ , then

$$\begin{aligned} \lim_{m \rightarrow \infty} \left\| G\left(\bigcup_{i=1}^{\infty} A_i\right) - \sum_{i=1}^m G(A_i) \right\|_E &= \lim_{m \rightarrow \infty} \left\| G\left(\bigcup_{i=m+1}^{\infty} A_i\right) \right\|_E \\ &= \lim_{m \rightarrow \infty} \left\| T\left(\chi_{\bigcup_{i=m+1}^{\infty} A_i}\right) \right\|_E \\ &\leq \lim_{m \rightarrow \infty} \lambda\left(\bigcup_{i=m+1}^{\infty} A_i\right) \|T\| = 0. \end{aligned}$$

In the sequel, instead of countably additive vector measure, we will simply write vector measure.

**Definition 1.3.3.** Let  $G : \mathcal{A} \rightarrow E$  be a vector measure. The *variation* of  $G$  is the extended nonnegative set function  $|G|$  whose value on a set  $A \in \mathcal{A}$  is given by

$$|G|(A) = \sup_{\Pi} \sum_{B \in \Pi} \|G(B)\|_E,$$

where the supremum is taken over all partitions  $\Pi$  of  $A$  into a finite number of pairwise disjoint members of  $\mathcal{A}$ . If  $|G|(\Omega) < \infty$ , then  $G$  will be called a *measure of bounded variation*.

We now give an example of vector measure with bounded variation

**Example 1.3.4.** Let  $G$  be the vector measure defined in Example 1.3.2. Since  $\|G(A)\|_E \leq \|T\|\lambda(A)$  for all  $A \in \mathcal{A}$ , it follows that  $|G|(A) \leq \|T\|\lambda(A)$ . So,  $G$  is of bounded variation.

We have the following result.

**Proposition 1.3.5.** ([29, Proposition 9 p.3]) A vector measure of bounded variation is countably additive if and only if its variation is also countably additive.

Further elementary facts about the Bochner integral are collected next.

**Theorem 1.3.6.** ([29, Theorem 4 p.46]) If  $f$  is a  $\mu$ -Bochner integrable function, then

- (i)  $\lim_{\mu(A) \rightarrow 0} \int_A f d\mu = 0$ ;
- (ii) if  $(A_i)$  is a sequence of pairwise disjoint members of  $\mathcal{A}$  and  $A = \bigcup_{i=1}^\infty A_i$ , then

$$\int_A f d\mu = \sum_{i=1}^\infty \int_{A_i} f d\mu,$$

where the sum on the right is absolutely convergent;

- (iii) if  $G(A) = \int_A f d\mu$ , then  $G$  is of bounded variation and

$$|G|(A) = \int_A \|f\| d\mu$$

for all  $A \in \mathcal{A}$ .

**Corollary 1.3.7.** If  $f$  and  $g$  are  $\mu$ -Bochner integrable, and  $\int_A f d\mu = \int_A g d\mu$  for each  $A \in \mathcal{A}$ , then  $f = g$   $\mu$ -almost everywhere.

*Proof.* Set  $G(A) = \int_A (f - g) d\mu$ . Then  $G(A) = 0_E$  for each  $A \in \mathcal{A}$ . Therefore,  $|G|(A) = 0$  for each  $A \in \mathcal{A}$ . Then  $0 = |G|(\Omega) = \int_\Omega \|f - g\|_E d\mu$  and so  $\|f - g\|_E = 0$   $\mu$ -almost everywhere. This can happen only if  $f = g$   $\mu$ -almost everywhere.  $\square$

**Definition 1.3.8.** Let  $(\Omega, \mathcal{A}, \mu)$  be a measure space, and  $G : \mathcal{A} \rightarrow E$  be a vector measure. We say that  $G$  is  $\mu$ -continuous if

$$\lim_{\mu(A) \rightarrow 0} G(A) = 0_E.$$

**Example 1.3.9.** Let  $f \in L^1_\mu(\Omega, E)$  and  $G(A) = \int_A f d\mu$ , where  $A \in \mathcal{A}$ . Using (i) of Theorem 1.3.6, we have that  $G$  is a  $\mu$ -continuous vector measure.

The vector measure in Example 1.3.4 is also a  $\lambda$ -continuous vector measure.

From now until the end of this chapter,  $(\Omega, \mathcal{A}, \mu)$  is a finite measure space and  $E$  is a Banach space.

**Definition 1.3.10.** A Banach space  $E$  has the *Radon-Nikodým property* with respect to  $(\Omega, \mathcal{A}, \mu)$  if for each  $\mu$ -continuous vector measure  $G : \mathcal{A} \rightarrow E$  of bounded variation there exists  $g \in L^1_\mu(\Omega, E)$  such that

$$G(A) = \int_A g d\mu,$$

for all  $A \in \mathcal{A}$ .

A Banach space  $E$  has *Radon-Nikodým property* if  $E$  has the Radon-Nikodým property with respect to every finite measure space.

**Definition 1.3.11.** A bounded linear operator  $T : L^1_\mu(\Omega) \longrightarrow E$  is Riesz representable (or simply representable) if there exists  $g \in L^\infty(\Omega, E)$  such that

$$T(f) = \int_\Omega fg d\mu,$$

for all  $f \in L^1_\mu(\Omega)$ .

The details in the following examples are in [29, Chapter 4].

**Example 1.3.12.** (Spaces with Radon-Nikodým property)

- (a) Hilbert spaces have Radon-Nikodým property (see [29, Corollary 4 p.100]).
- (b) Separable dual spaces have Radon-Nikodým property (see [29, Theorem 1 p.79]).
- (c) Reflexive Banach spaces have Radon-Nikodým property (see [29, Corollary 4 p.82]).

The fundamental connection between representable operators on  $L^1_\mu(\Omega)$  and vector measure with Radon-Nikodým derivatives is contained in the following straightforward lemma:

**Theorem 1.3.13.** ([29, Lemma 4 p.62]) Let  $T : L^1_\mu(\Omega) \longrightarrow E$  be a bounded linear operator. For  $A \in \mathcal{A}$ ,  $G(A)$  defined by

$$G(A) = T(\chi_A)$$

is a  $\mu$ -continuous vector measure with bounded variation. We have that  $T$  is representable if and only if there exists  $g \in L^1_\mu(\Omega, E)$  such that

$$G(A) = \int_A g d\mu,$$

for all  $A \in \mathcal{A}$ . In this case,  $g \in L^\infty(\Omega, E)$  and

$$T(f) = \int_\Omega fg d\mu,$$

for all  $f \in L^1_\mu(\Omega)$ . Moreover  $\|g\|_{\infty, E} = \|T\|$ .

**Remark 1.3.14.** The set function  $G$  defined in the previous lemma is a  $\mu$ -continuous vector measure with bounded variation. In fact,  $G$  is countably additive (by the linearity and the continuity of  $T$ ). Since  $\|G(A)\|_E \leq \|T\|\mu(A)$ , it follows that  $G$  is a  $\mu$ -continuous vector measure with bounded variation.

The next theorem cements the connection between the Radon-Nikodým theorem and the Riesz representation theorem.

**Theorem 1.3.15.** ([29, Theorem 5 p.63]) Let  $E$  be a Banach space and  $(\Omega, \mathcal{A}, \mu)$  be a finite measure space. Then  $E$  has Radon-Nikodým property with respect to  $(\Omega, \mathcal{A}, \mu)$  if and only if each  $T \in \mathcal{L}(L^1_\mu(\Omega), E)$  is representable.

### 1.3.2 Vector measures with bounded $p$ -variation.

**Definition 1.3.16.** Let  $1 < p < \infty$  and  $E$  be a Banach space. A  $\mu$ -continuous vector measure  $G : \mathcal{A} \rightarrow E$  is said to have *bounded  $p$ -variation* if

$$\|G\|_p := \sup_{\Pi} \left( \sum_{A \in \Pi} \frac{\|G(A)\|_E^p}{\mu(A)^{p-1}} \right)^{1/p} < \infty,$$

where the supremum is taken over all finite partitions  $\Pi$  of  $\Omega$ . We denote by  $V^p_\mu(\Omega, E)$  the space of all  $\mu$ -continuous vector measures with bounded  $p$ -variation.

For  $p = 1$ , we denote by  $V^1_\mu(\Omega, E)$  the space all  $\mu$ -continuous vector measures with bounded variation equipped with the norm  $\|G\|_1 = |G|(\Omega)$ .

For  $p = \infty$ , we denote by  $V^\infty(\Omega, E)$  the space of all  $\mu$ -continuous vector measures  $G$  for which there exists a constant  $C > 0$  such that  $\|G(A)\|_E \leq C\mu(A)$ , for all  $A \in \mathcal{A}$ . Its norm is given by

$$\|G\|_\infty = \sup_{A \in \mathcal{A}} \frac{\|G(A)\|_E}{\mu(A)}.$$

As in  $L^p$  spaces, we have the following inclusion.

**Lemma 1.3.17.** Let  $1 \leq p < \infty$ .

$$V^p_\mu(\Omega, E) \subset V^1_\mu(\Omega, E).$$

**Lemma 1.3.18.** ([18, Lemma 2.8 for  $p = q$ ]) Let  $E$  be a Banach space and  $1 < p < \infty$ . The following are equivalent.

- (a)  $G \in V^p_\mu(\Omega, E)$ .
- (b) There exists a nonnegative  $\varphi \in L^p_\mu(\Omega)$  such that  $|G|(A) = \int_A \varphi d\mu$ , for all  $A \in \mathcal{A}$ . Moreover if  $|G|(A) = \int_A \varphi d\mu$  for all  $A \in \mathcal{A}$  for some  $\varphi \geq 0$ ,  $\varphi \in L^p_\mu(\Omega)$  then  $\|G\|_p = \|\varphi\|_{p,\mu}$ .

**Theorem 1.3.19.** ([18, Lemma 2.9 for  $p = q$ ]) Let  $E$  be a Banach space and  $1 < p < \infty$ . The following hold.

- (i) The Bochner-Lebesgue space  $L^p_\mu(\Omega, E)$  is isometrically embedded into  $V^p_\mu(\Omega, E)$ .
- (ii) We have  $V^p_\mu(\Omega, E) = L^p_\mu(\Omega, E)$  if and only if  $E$  has the Radon-Nikodým property.

*Proof.* (i) Let  $f \in L^p_\mu(\Omega, E)$ . For each  $A \in \mathcal{A}$ , The set function  $F(A) = \int_A f d\mu$  is a  $\mu$ -continuous countably vector measure with bounded  $p$ -variation and  $\|F(A)\|_E \leq \|f\|_{p,\mu,E} \mu(A)^{\frac{1}{p}}$ , for all  $A \in \mathcal{A}$ . Since  $|F|(A) = \int_A \|f\|_E d\mu$  for all  $A \in \mathcal{A}$ , by Lemma 1.3.19 we have that  $\varphi = \|f\|_E$   $\mu$ -a.e. and  $\|F\|_p = \|\varphi\|_{p,\mu} = \|f\|_{p,\mu,E}$ .

(ii) We first assume that  $V^p_\mu(\Omega, E) = L^p_\mu(\Omega, E)$ . Let  $T : L^1_\mu(\Omega) \rightarrow E$  be a bounded linear operator. By Theorem 1.3.15 we need to show that  $T$  is representable. For each  $A \in \mathcal{A}$ , consider  $G(A) := T(\chi_A)$ . By using Remark 1.3.14, we see that  $G \in V^\infty_\mu(\Omega, E) \subset V^p_\mu(\Omega, E) = L^p_\mu(\Omega, E)$ . By Lemma 1.3.18, there exists a function  $f \in L^p_\mu(\Omega)$  such that  $F(f) = \int_A f d\mu$  for all  $A \in \mathcal{A}$ . Thus, Theorem 1.3.13 shows that  $T(\psi) = \int_\Omega \psi f d\mu$  for all  $\psi \in L^1_\mu(\Omega)$ .

Conversely, let us assume that  $E$  has Radon-Nikodým property and let  $F \in V^p_\mu(\Omega, E) \subset V^1_\mu(\Omega, E)$ . We shall prove that  $F \in L^p_\mu(\Omega, E)$ . By the assumption,  $F$  is a  $\mu$ -continuous vector measure of bounded variation It follows that there exists Bochner integrable function  $g \in L^1_\mu(\Omega, E)$  such that  $F(A) = \int_A g d\mu$  for all  $A \in \mathcal{A}$ .

### 1.3. Vector measures : topological dual of $L^p_\mu(\Omega, E)$ , with $1 < p < \infty$ .

So by Theorem 1.3.6, we have  $|F|(A) = \int_A \|g\|_E d\mu$  for all  $A \in \mathcal{A}$ . Using Lemma 1.3.18, we get a nonnegative function  $\varphi \in L^p_\mu(\Omega)$  such that  $|F|(A) = \int_A \varphi d\mu$  for all  $A \in \mathcal{A}$  and  $\|F\|_p = \|\varphi\|_{p,\mu}$ . It follows that  $\varphi = \|g\|_E$   $\mu$ -a.e. Thus,  $F \in L^p_\mu(\Omega, E)$ .  $\square$

**Lemma 1.3.20.** ([18, Lemma 2.10 for  $p = q$ ]) Let  $E$  be a Banach space and  $1 < p < \infty$ . If  $F \in V^p_\mu(\Omega, E)$  then

$$\begin{aligned} \|F\|_p &\simeq \sup \left\{ \sum_{i=1}^N |\alpha_i| \|F(A_i)\|_E : \sum_{i=1}^N \alpha_i \chi_{A_i} \in L^{p'}_\mu(\Omega), \left\| \sum_{i=1}^N \alpha_i \chi_{A_i} \right\|_{p',\mu} = 1 \right\} \\ &= \sup \left\{ \sum_{i=1}^N |\alpha_i| |F|(A_i) : \sum_{i=1}^N \alpha_i \chi_{A_i} \in L^{p'}_\mu(\Omega), \left\| \sum_{i=1}^N \alpha_i \chi_{A_i} \right\|_{p',\mu} = 1 \right\}. \end{aligned}$$

**Definition 1.3.21.** *i)* Let  $G : \mathcal{A} \rightarrow E$  be a vector measure with bounded variation. If  $s = \sum_{i=1}^N \alpha_i \chi_{A_i}$  is a simple function such that  $(A_i)_{i=1,2,\dots,N}$  forms a partition of  $\Omega$  and  $\alpha_i \in \mathbb{C}$ , we define the integral of the simple function  $s$  over  $\Omega$  with respect to  $G$  by:

$$\int_\Omega s(x) dG(x) = \sum_{i=1}^N \alpha_i G(A_i) \quad (1.3)$$

*ii)* Let  $G : \mathcal{A} \rightarrow E^*$  be a vector measure with bounded variation. If  $s = \sum_{i=1}^N x_i \chi_{A_i}$  is a simple function, such that  $(A_i)_{i=1,2,\dots,N}$  form a partition of  $\Omega$  and  $x_i \in E$ , we define the integral of the simple function  $s$  over  $\Omega$  by

$$\int_\Omega s dG = \int_\Omega \langle dG, s \rangle_{E^*,E} = \sum_{i=1}^N \langle G(A_i), x_i \rangle_{E^*,E} = \sum_{i=1}^N G(A_i)(x_i). \quad (1.4)$$

In particular, if  $\phi = \sum_{k=1}^N \lambda_k \chi_{B_k}$  is a scalar-valued simple function, we have that

$$\begin{aligned} \int_\Omega \phi(z) x dG(z) &= \int_\Omega \sum_{k=1}^N \lambda_k x \chi_{B_k} dG(z) = \sum_{k=1}^N G(B_k)(\lambda_k x) \\ &= \sum_{k=1}^N \lambda_k G(B_k)(x) \\ &= \sum_{k=1}^N \lambda_k \mu_x(B_k), \end{aligned}$$

where we used the notation  $x dG(z) = dG(z)(x) = \langle dG(z), x \rangle_{E^*,E}$ ,  $\mu_x(B_k) = G(B_k)(x)$  and  $\mu_x$  is a scalar-valued measure. We note that

$$\langle x^*, x \rangle_{E^*,E} = \overline{\langle x, x^* \rangle_{E,E^*}}.$$

**Remark 1.3.22.** Given an  $E$ -valued simple measurable function  $f = \sum_{k=1}^N x_k \chi_{A_k}$  and a  $E^*$ -valued measure  $G$ , we denote by

$$\langle f, G \rangle = \sum_{k=1}^N \langle x_k, x_k^* \rangle_{E,E^*},$$

where  $x_k^* = G(A_k)$ .

If  $G$  is an  $E$ -valued measure with bounded variation, and  $\phi = \sum_{k=1}^N \lambda_k \chi_{B_k}$  is a scalar-valued simple function,

then we write

$$\int_\Omega \phi dG = \sum_{k=1}^N \lambda_k G(B_k).$$

Since  $\|\int_\Omega \phi dG\| \leq \|G\|_1 \|\phi\|_\infty$ , using the density of simple functions we extend the definition of  $\int_\Omega \phi dG$  for any integrable function  $\phi$ .

Let  $1 < p < \infty$  and  $E$  be a Banach space. It is well-known that if  $E = \mathbb{C}$ , the dual  $(L^p_\mu(\Omega))^*$  of  $L^p_\mu(\Omega)$  is identified with the space  $L^{p'}_\mu(\Omega)$ . In [39, Section 3], Gretskey and Uhl proved that  $(L^p_\mu(\Omega, E))^*$  cannot be identified with  $L^{p'}_\mu(\Omega, E^*)$  for all Banach spaces. In particular, they proved that for  $p = 2$  and  $E = \ell^1$ , the dual space  $L^2(\ell^1)$  cannot be identified with  $L^2(\ell^\infty) = L^2((\ell^1)^*)$ . The obvious question now is "when can  $(L^p_\mu(\Omega, E))^*$  be identified with  $L^{p'}_\mu(\Omega, E^*)$  ?"

**Theorem 1.3.23.** Let  $1 < p < \infty$  and  $E$  be a Banach space. Then  $V^{p'}_\mu(\Omega, E^*)$  can be identified with  $(L^p_\mu(\Omega, E))^*$ .

*Proof.* Let  $F \in V^{p'}_\mu(\Omega, E^*)$ . Consider the linear functional

$$\phi_F : L^p_\mu(\Omega, E) \longrightarrow \mathbb{C}$$

$$\sum_{i=1}^N x_i \chi_{A_i} \mapsto \phi_F \left( \sum_{i=1}^N x_i \chi_{A_i} \right) = \sum_{i=1}^N \langle x_i, F(A_i) \rangle_{E, E^*}.$$

It is clear that

$$\left| \phi_F \left( \sum_{i=1}^N x_i \chi_{A_i} \right) \right| \leq \sum_{i=1}^N \|x_i\|_E \|F(A_i)\|_{E^*}.$$

Therefore, by Lemma 1.3.20 there exists a constant  $C > 0$  such that  $\|\phi_F\|_{(L^p_\mu(\Omega, E))^*} \leq C \|F\|_p$ .

Now, let  $\phi \in (L^p_\mu(\Omega, E))^*$ . For each  $A \in \mathcal{A}$ , consider the set function  $F_\phi(A) : E \longrightarrow \mathbb{C}$  defined by  $F_\phi(A)(x) := \phi(x \chi_A)$ . It is clear that  $F_\phi(A)$  is linear (since  $\phi$  is linear). Observe that  $F_\phi$  is well-defined and is countably additive (by the continuity of  $\phi$ ). We have that  $\phi = \phi_{F_\phi}$  on simple functions. Since

$$|F_\phi(A)(x)| \leq \|\phi\|_{(L^p_\mu(\Omega, E))^*} \|x \chi_A\|_{p, \mu, E} = \|x\|_E \mu(A)^{\frac{1}{p}} \|\phi\|_{(L^p_\mu(\Omega, E))^*},$$

we have that  $\|F_\phi(A)\|_{E^*} \leq \mu(A)^{\frac{1}{p}} \|\phi\|_{(L^p_\mu(\Omega, E))^*}$ . It follows that  $F_\phi$  is a  $\mu$ -continuous  $E^*$ -valued measure with

bounded variation and by Lemma 1.3.18, there is a constant  $C > 0$  such that

$$\begin{aligned}
 \|F_\phi\|_{p'} &\leq C \sup \left\{ \sum_{i=1}^N |\alpha_i| \|F_\phi(A_i)\|_{E^*} : \sum_{i=1}^N \alpha_i \chi_{A_i} \in L_\mu^p(\Omega), \left\| \sum_{i=1}^N \alpha_i \chi_{A_i} \right\|_{p,\mu} = 1 \right\} \\
 &= C \sup \left\{ \sum_{i=1}^N |\langle \alpha_i x_i, F_\phi(A_i) \rangle_{E,E^*}| : \sum_{i=1}^N \alpha_i \chi_{A_i} \in L_\mu^p(\Omega), \left\| \sum_{i=1}^N \alpha_i \chi_{A_i} \right\|_{p,\mu} = 1, \|x_i\|_E = 1 \ \forall i \right\} \\
 &= C \sup \left\{ \sum_{i=1}^N |\langle y_i, F_\phi(A_i) \rangle_{E,E^*}| : \sum_{i=1}^N y_i \chi_{A_i} \in L_\mu^p(\Omega, E), \left\| \sum_{i=1}^N y_i \chi_{A_i} \right\|_{p,\mu,E} = 1 \right\} \\
 &= C \sup \left\{ \sum_{i=1}^N \langle y_i e^{-\theta_i}, F_\phi(A_i) \rangle_{E,E^*} : \sum_{i=1}^N y_i e^{-\theta_i} \chi_{A_i} \in L_\mu^p(\Omega, E), \left\| \sum_{i=1}^N y_i e^{-\theta_i} \chi_{A_i} \right\|_{p,\mu,E} = 1 \right\} \\
 &= C \sup \left\{ \left| \sum_{i=1}^N \langle z_i, F_\phi(A_i) \rangle_{E,E^*} \right| : \sum_{i=1}^N z_i \chi_{A_i} \in L_\mu^p(\Omega, E), \left\| \sum_{i=1}^N z_i \chi_{A_i} \right\|_{p,\mu,E} = 1 \right\} \\
 &= C \sup \left\{ \left| \phi_{F_\phi} \left( \sum_{i=1}^N z_i \chi_{A_i} \right) \right| : \sum_{i=1}^N z_i \chi_{A_i} \in L_\mu^p(\Omega, E), \left\| \sum_{i=1}^N z_i \chi_{A_i} \right\|_{p,\mu,E} = 1 \right\} \\
 &= C \|\phi\|_{(L_\mu^p(\Omega,E))^*}.
 \end{aligned}$$

□

**Corollary 1.3.24.** Let  $1 < p < \infty$  and  $E$  be a Banach space. Then  $(L_\mu^p(\Omega, E))^* = L_\mu^{p'}(\Omega, E^*)$  if and only if  $E^*$  has Radon-Nikodým property.

# Vector-valued Bergman spaces

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Results of this chapter extend those in [77, Chapter 2] to the vector-valued setting. The proof of some of them can be found in [52].

Let  $\mathbb{C}$  denote the set of complex numbers. Throughout this thesis, we fix a nonnegative integer  $n$  and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the  $n$ -dimensional Euclidean space. For

$$z = (z_1, \dots, z_n), \quad w = (w_1, \dots, w_n),$$

in  $\mathbb{C}^n$ , we define the inner product of  $z$  and  $w$  by

$$\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n},$$

where  $\overline{w_k}$  is the complex conjugate of  $w_k$ . The resulting norm is then

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Endowed with the above inner product,  $\mathbb{C}^n$  becomes a Hilbert space whose canonical basis consists of the following vectors

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad e_n = (0, \dots, 0, 1).$$

The open unit ball in  $\mathbb{C}^n$  is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}. \tag{2.1}$$

The boundary of  $\mathbb{B}_n$  will be denoted by  $\mathbb{S}_n$ , and is called the unit sphere in  $\mathbb{C}^n$ . Thus,

$$\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}. \tag{2.2}$$

We will occasionally need the closed unit ball

$$\overline{\mathbb{B}}_n = \{z \in \mathbb{C}^n : |z| \leq 1\}. \tag{2.3}$$

## 2.1. Vector-valued holomorphic functions.

When  $\alpha > -1$ , the weighted Lebesgue measure  $d\nu_\alpha$  in  $\mathbb{B}_n$  is defined by

$$d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z), \quad z \in \mathbb{B}_n, \quad (2.4)$$

where  $d\nu$  is the Lebesgue measure in  $\mathbb{C}^n$  and

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

is the normalizing constant so that  $d\nu_\alpha$  becomes a probability measure on  $\mathbb{B}_n$ . A function defined on the unit ball  $\mathbb{B}_n$  will be called a vector-valued function when it takes its values in some vector space.

If we do not say the contrary, in this chapter  $E$  will stand for any complex Banach space and  $\alpha > -1$  will be a real parameter. We will write  $a \lesssim b$  when there is a positive constant  $C$  such that  $a \leq Cb$ . If both  $a \lesssim b$  and  $b \lesssim a$ , then we write  $a \simeq b$ .

## 2.1 Vector-valued holomorphic functions.

**Definition 2.1.1** (Strongly holomorphic function). If  $E$  is a complex Banach space, a vector-valued function  $f : \mathbb{B}_n \rightarrow E$  (a  $E$ -valued function) is said to be *strongly holomorphic* in  $\mathbb{B}_n$  if for every  $z \in \mathbb{B}_n$  and for every  $k \in \{1, \dots, n\}$ , the limit

$$\lim_{\lambda \rightarrow 0} \frac{f(z + \lambda e_k) - f(z)}{\lambda}$$

exists in  $E$ , where  $\lambda \in \mathbb{C} \setminus \{0\}$ . When  $f$  is strongly holomorphic, we use the notation

$$\frac{\partial f}{\partial z_k}(z)$$

to denote the above limit and call it the *partial derivative of  $f$  with respect to  $z_k$* .

Equivalently, a vector-valued function  $f : \mathbb{B}_n \rightarrow E$  is strongly holomorphic if

$$f(z) = \sum_m a_m z^m, \quad a_m \in E, \quad z \in \mathbb{B}_n. \quad (2.5)$$

Here, the summation is over all multi-indexes

$$m = (m_1, \dots, m_n),$$

where each  $m_k$  is a nonnegative integer and

$$z^m = z_1^{m_1} \dots z_n^{m_n}.$$

The series in (2.5) is called the Taylor series of  $f$  at the origin, As in the scalar case, it converges absolutely and uniformly [77] on compact sets of the form

$$r\overline{\mathbb{B}}_n := \{z \in \mathbb{C}^n : |z| \leq r\}, \quad r \in ]0, 1[.$$

## 2.1. Vector-valued holomorphic functions.

If we put

$$f_k(z) = \sum_{|m|=k} a_m z^m$$

for each integer  $k \geq 1$ , where

$$|m| = m_1 + \cdots + m_n,$$

then the Taylor series of  $f$  can be rewritten as

$$f(z) = \sum_{k=0}^{\infty} f_k(z).$$

This is called the *vector-valued homogeneous expansion* of  $f$ ; each  $f_k$  is a vector-valued homogeneous polynomial of degree  $k$ . Both the Taylor series and the vector-valued homogeneous expansions of  $f$  are uniquely determined by  $f$ . Given a vector-valued strongly holomorphic function  $f : \mathbb{B}_n \rightarrow E$ , all higher order partial derivatives exist and are still strongly holomorphic. For a multi-index  $m = (m_1, \dots, m_n)$ , we will use the notation

$$\partial^m f := \frac{\partial^m f}{\partial z^m} = \frac{\partial^{|m|} f}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}.$$

For a multi-index  $m$ , we will also use the following definition

$$m! = m_1! \cdots m_n!.$$

In particular, we have the multinomial formula

$$(z_1 + \cdots + z_n)^N = \sum_{|m|=N} \frac{N!}{m!} z^m. \quad (2.6)$$

**Definition 2.1.2.** A vector-valued function  $f : \mathbb{B}_n \rightarrow E$  is *weakly holomorphic* if for every  $x^* \in E^*$ , the scalar-valued function  $x^*(f) : \mathbb{B}_n \rightarrow \mathbb{C}$  is holomorphic in the usual sense.

**Definition 2.1.3.** A vector-valued function is said to be *locally bounded* if it is bounded on every compact subset of  $\mathbb{B}_n$ .

**Remark 2.1.4.** We recall (see [8, 43]) that a vector-valued function is strongly holomorphic if and only if it is weakly holomorphic. In the rest of this thesis, we will omit the word 'strongly' and we will only say holomorphic function to mean a strongly holomorphic and weakly holomorphic function.

The space of all  $E$ -valued holomorphic functions on  $\mathbb{B}_n$  will be denoted by  $\mathcal{H}(\mathbb{B}_n, E)$ . We will also denote by  $H^\infty(\mathbb{B}_n, E)$  the space of all bounded  $E$ -valued holomorphic functions.

We denote by  $E^n := E \times \cdots \times E$  the Banach space product of  $n$  copies of  $E$  with the norm

$$\|x\|_{E^n} = \left( \sum_{i=1}^n \|x_i\|_E^2 \right)^{1/2}, \quad x = (x_1, \dots, x_n) \in E^n. \quad (2.7)$$

We will write  $\mathbb{B}_n(E)$  to denote the open ball of  $E^n$  of radius 1, that is,

$$\mathbb{B}_n(E) = \{x \in E^n : \|x\|_{E^n} < 1\}, \quad (2.8)$$

and we denote by  $\mathbb{B}(E) = \mathbb{B}_1(E)$ . It is clear that  $\mathbb{B}_n = \mathbb{B}_n(\mathbb{C})$ .

## 2.1. Vector-valued holomorphic functions.

In the sequel, we will need the notion of 'conjugate' of a complex Banach space.

We will use the following definition and notation which can be found in [52]. If  $E$  is a complex Banach space, then each vector  $x \in E$  can be looked as an element of the bidual  $E^{**}$  of  $E$ . That is, the norm of a vector  $x \in E$  is equal to the norm of  $x \in E^{**}$  (consequence of the Hahn-Banach theorem). Thus,

$$\|x\|_E = \sup_{x^* \in E^*, \|x^*\|_{E^*}=1} |\langle x, x^* \rangle_{E, E^*}|.$$

Let  $x \in E$ ,  $x^* \in E^*$  and  $\lambda \in \mathbb{C}$ . We define

$$(\lambda x^*)(x) := \overline{\lambda} x^*(x).$$

We also use the notation

$$\langle x, x^* \rangle_{E, E^*} = x^*(x)$$

to represent the 'inner product' in the complex Banach space  $E$ . We have the following identities

$$\langle \lambda x, x^* \rangle_{E, E^*} = \lambda \langle x, x^* \rangle_{E, E^*} = \langle x, \overline{\lambda} x^* \rangle_{E, E^*},$$

so that we have a regular rule of an inner product. The *complex conjugate*  $\overline{x}$  of  $x \in E$ , is the linear functional on  $E^*$  defined by

$$\overline{x}(x^*) = \overline{\langle x, x^* \rangle_{E, E^*}},$$

for every  $x^* \in E^*$ . Therefore,

$$\overline{E} = \{\overline{x} : x \in E\}$$

is called the *complex conjugate of the Banach space*  $E$ . With the norm defined by

$$\|\overline{x}\| := \sup_{\|x^*\|_{E^*}=1} |\overline{x}(x^*)|,$$

$\overline{E}$  becomes a Banach space. Moreover, we have that  $\|x\|_E = \|\overline{x}\|_{\overline{E}}$  for any  $x \in E$ , so that  $E$  and  $\overline{E}$  are isometrically anti-isomorphic.

When  $\mu$  is the counting measure, the Bochner-Lebesgue space is denoted by  $\ell^p(E)$ , which consists of sequences  $\{x_k\}_{k \geq 1}$  of  $E$  such that

$$\|\{x_k\}_{k \geq 1}\|_{\ell^p(E)} = \left( \sum_{k=1}^{\infty} \|x_k\|_E^p \right)^{1/p} < \infty \quad \text{if } 0 < p < \infty, \quad (2.9)$$

and if  $p = \infty$ ,  $\|\{x_k\}_{k \geq 1}\|_{\ell^\infty(E)} = \sup_{k \geq 1} \|x_k\|_E < \infty$ .

Another important example is when  $\mu = \nu_\alpha$ , for  $\alpha > -1$ , where  $\nu_\alpha$  is defined in (2.4). So we consider The weighted Bochner-Lebesgue space  $L_\alpha^p(\mathbb{B}_n, E) := L_{\nu_\alpha}^p(\mathbb{B}_n, E)$  which consists of  $\nu_\alpha$ -measurable functions  $f : \mathbb{B}_n \rightarrow E$  such that

$$\|f\|_{p, \alpha, E} := \|f\|_{L_\alpha^p(\mathbb{B}_n, E)} = \left( \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) \right)^{1/p} < \infty, \quad 0 < p < \infty. \quad (2.10)$$

In the sequel, when  $E = \mathbb{C}$ ,  $L_\alpha^p(\mathbb{B}_n, \mathbb{C})$  is denoted by  $L_\alpha^p(\mathbb{B}_n)$  which is the classical Lebesgue space and the norm is denoted by  $\|\cdot\|_{p, \alpha}$ .

### 2.1.1 The automorphism group and the Bergman metric.

When we are working on the unit ball  $\mathbb{B}_n$ , we need to introduce the Bergman metric. To that, it is first important to introduce the automorphisms of  $\mathbb{B}_n$  which are also important in what will follow. The characterization of the automorphisms in  $\mathbb{B}_n$  is well-known and we refer to [63] and [77] for more details.

A mapping  $F : \mathbb{B}_n \rightarrow \mathbb{B}_n$  is said to be bi-holomorphic if  $F$  is bijective, holomorphic and  $F^{-1}$  is holomorphic. The automorphism group of  $\mathbb{B}_n$ , denoted by  $\text{Aut}(\mathbb{B}_n)$ , consists of all *bi-holomorphic* mappings of  $\mathbb{B}_n$ . Traditionally, bi-holomorphic mappings are also called *automorphisms*. It is clear that  $\text{Aut}(\mathbb{B}_n)$  is a group with composition being the group operation. Basically, any automorphism consist of unitary transformations of  $\mathbb{C}^n$  and involutions. Zhu [77] proved that an automorphism  $\varphi$  of  $\mathbb{B}_n$  is a unitary transformation of  $\mathbb{C}^n$  if and only if  $\varphi(0) = 0$ .

Recall that to every  $z$  in the unit disk  $\mathbb{B}_1$  corresponds an automorphism  $\varphi_z$  of the unit disk that interchanges  $z$  and 0, namely for  $w$  in the unit disk,  $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$ . The same can be done in the unit ball  $\mathbb{B}_n$  of  $\mathbb{C}^n$ . Fix  $a \in \mathbb{B}_n$ . Let  $P_a$  be the orthogonal projection of  $\mathbb{C}^n$  onto the subspace  $[a]$  generated by  $a$ , and let  $Q_a = I - P_a$  the projection onto the orthogonal complement of  $[a]$ . Explicitly,  $P_0 = 0$  and

$$P_a(z) = \frac{\langle z, a \rangle}{|a|^2} a, \quad \text{if } a \neq 0,$$

where  $z \in \mathbb{C}^n$ .

**Definition 2.1.5** (Möbius transform). The *Möbius transform (involution)*  $\varphi_a$  of  $\mathbb{B}_n$  is defined by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle},$$

where  $z \in \mathbb{B}_n$ , and  $s_a = \sqrt{1 - |a|^2}$ .

It is obvious that each  $\varphi_a$  is a holomorphic mapping from  $\mathbb{B}_n$  into  $\mathbb{C}^n$ . In [63, Theorem 2.2.5] or in [77, Theorem 1.4], we have a complete characterization of automorphisms in  $\mathbb{B}_n$ , that is, every  $\varphi \in \text{Aut}(\mathbb{B}_n)$  is of the form

$$\varphi = U \circ \varphi_a = \varphi_b \circ V,$$

where  $U$  and  $V$  are unitary transformations of  $\mathbb{C}^n$ , and  $\varphi_a$  and  $\varphi_b$  are involutions.

From [63, Theorem 2.2.2], we have the following formulas. For each  $a \in \mathbb{B}_n$  the mapping  $\varphi_a$  satisfies

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}, \quad z \in \mathbb{B}_n, \quad (2.11)$$

and

$$\varphi_a \circ \varphi_a(z) = z, \quad z \in \mathbb{B}_n. \quad (2.12)$$

**Lemma 2.1.6.** ([77, Lemma 1.3]) Suppose  $a \in \mathbb{B}_n$ . Then

$$1 - \langle \varphi_a(z), \varphi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)} \quad (2.13)$$

for all  $z$  and  $w$  on the closed unit ball  $\bar{\mathbb{B}}_n$ .

In particular, each  $\varphi_a$  is an automorphism of  $\mathbb{B}_n$  that interchanges the points 0 and  $a$ .

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The Bergman distance between two points  $z, w \in \mathbb{B}_n$ , is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad (2.14)$$

where  $\varphi_z$  is the involutive automorphism of  $\mathbb{B}_n$  that interchanges 0 and  $z$ .

A calculation shows that  $|\varphi_z(w)| = \tanh \beta(z, w)$ , for every  $z, w \in \mathbb{B}_n$ . It is well-known that the Bergman metric is invariant under automorphisms, that is,

$$\beta(\varphi(z), \varphi(w)) = \beta(z, w), \quad (2.15)$$

for all  $z, w \in \mathbb{B}_n$  and  $\varphi \in \text{Aut}(\mathbb{B}_n)$ .

We recall that  $d\nu$  is the normalized volume measure on  $\mathbb{B}_n$ , so  $\nu(\mathbb{B}_n) = 1$ . The surface measure on  $\mathbb{S}_n$  is denoted by  $d\sigma$  and is normalized so that  $\sigma(\mathbb{S}_n) = 1$ .

**Lemma 2.1.7.** ([77, Lemma 1.8]) The measures  $\nu$  and  $\sigma$  are related by the following relation:

$$\int_{\mathbb{B}_n} f(z) d\nu(z) = 2n \int_0^1 r^{2n-1} \int_{\mathbb{S}_n} f(r\zeta) d\sigma(\zeta). \quad (2.16)$$

For  $z \in \mathbb{B}_n$  and  $r > 0$  we let  $D(z, r)$  denote the Bergman ball centred at  $z$  of radius  $r$ , that is,

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}. \quad (2.17)$$

The volume of the Bergman ball is computed in the following lemma.

**Lemma 2.1.8.** ([77, Lemma 1.23]) For  $z \in \mathbb{B}_n$  and  $r > 0$  we have

$$\nu(D(z, r)) = \frac{R^{2n}(1 - |z|^2)^{n+1}}{(1 - R^2|z|^2)^{n+1}},$$

where  $R = \tanh(r)$ . In particular, for any  $r > 0$ , there exist constants  $c_r > 0$  and  $C_r > 0$  such that

$$c_r(1 - |z|^2)^{n+1} \leq \nu(D(z, r)) \leq C_r(1 - |z|^2)^{n+1},$$

for all  $z \in \mathbb{B}_n$ .

For more general real  $\alpha > -1$ , we have the following estimate of  $\nu_\alpha(D(z, r))$ .

**Lemma 2.1.9.** ([77, Lemma 1.24]) Let  $\alpha > -1$ , and  $r > 0$ . There exist constants  $C > 0$  and  $c > 0$  such that

$$c(1 - |z|^2)^{n+1+\alpha} \leq \nu_\alpha(D(z, r)) \leq C(1 - |z|^2)^{n+1+\alpha},$$

for all  $z \in \mathbb{B}_n$ .

Given  $\varphi \in \text{Aut}(\mathbb{B}_n)$ , we denote by  $J_{\mathbb{C}}\varphi(z)$  the determinant of the complex  $n \times n$  matrix  $\varphi'(z)$  (see [77, p.7]) and call it the complex Jacobian of  $\varphi$  at  $z$ . If we identify  $\mathbb{B}_n$  (in the natural way) with the unit ball in the  $2n$ -dimensional real Euclidean space  $\mathbb{R}^{2n}$ , then the mapping  $\varphi$  induces a real Jacobian determinant which we denote by  $J_{\mathbb{R}}\varphi(z)$  and we have that  $J_{\mathbb{R}}\varphi(z) = |J_{\mathbb{C}}\varphi(z)|^2$ , for all  $z \in \mathbb{B}_n$ .

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**Lemma 2.1.10.** ([63, Theorem 2.2.6]) For each  $\varphi \in \text{Aut}(\mathbb{B}_n)$  we have

$$J_{\mathbb{R}}\varphi_a(z) = \left( \frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{n+1}, \quad (2.18)$$

where  $a = \varphi^{-1}(0)$ .

For integration on the unit sphere  $\mathbb{S}_n$ , we will use the following formula.

**Lemma 2.1.11.** ([63, Proposition 1.4.7]) For  $f \in L^1(\mathbb{S}_n, E)$  we have

$$\int_{\mathbb{S}_n} f(\zeta) d\sigma(\zeta) = \int_{\mathbb{S}_n} d\sigma(\zeta) \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}\zeta) d\theta. \quad (2.19)$$

The proof of the following change of variables formula is the same as in the scalar case, see [77, Proposition 1.13].

**Proposition 2.1.12** (Change of variables formula). Let  $\alpha > -1$  and  $f \in L^1_{\alpha}(\mathbb{B}_n, E)$ . Then

$$\int_{\mathbb{B}_n} f \circ \varphi(z) d\nu_{\alpha}(z) = \int_{\mathbb{B}_n} f(z) \frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - \langle z, a \rangle|^{2(n+1+\alpha)}} d\nu_{\alpha}(z),$$

where  $\varphi \in \text{Aut}(\mathbb{B}_n)$  and  $a = \varphi(0)$ .

### 2.1.2 Gamma and Beta functions.

**Definition 2.1.13.** (1) The *Gamma function* denoted by  $\Gamma$  is the function defined for a complex number  $z$  by:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \text{with } \mathcal{R}e z > 0. \quad (2.20)$$

(2) We let  $\mathcal{B}$  denote the *Beta function* which is the function defined for  $(z, w) \in \mathbb{C} \times \mathbb{C}$  by

$$\mathcal{B}(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt \quad \text{where } (\mathcal{R}e z, \mathcal{R}e w) \in (0, \infty) \times (0, \infty). \quad (2.21)$$

These functions satisfy the following properties.

**Proposition 2.1.14.**

(1) For each complex number  $z$  with  $\mathcal{R}e z \in (0, \infty)$ , we have that  $\Gamma(z+1) = z\Gamma(z)$ . In particular, if  $n \in \mathbb{N}^*$ , we have  $\Gamma(n+1) = n!$ .

(2) Given two complex numbers  $z, w \in \mathbb{C}$  with  $(\mathcal{R}e z, \mathcal{R}e w) \in (0, \infty) \times (0, \infty)$  we have

$$\mathcal{B}(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

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(3) For each  $x \in (0, \infty)$ , we have:  $\Gamma(x+1) \simeq \sqrt{2\pi x} \left(\frac{x}{e}\right)$  as  $x \rightarrow \infty$  (Stirling formula). In particular, for  $\lambda > 0$ , we have:  $\frac{\Gamma(x+\lambda)}{\Gamma(x+1)} \simeq (1+x)^{\lambda-1}$  as  $x \rightarrow \infty$ .

(4) For  $|z| < 1$  and  $\lambda > 0$  we have

$$(1-z)^{-\lambda} = \sum_{k=0}^{\infty} \frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)} z^k. \quad (2.22)$$

The following lemma is very important in many situations.

**Lemma 2.1.15.** ([63, Proposition 1.4.8]) Suppose  $m = (m_1, \dots, m_n)$  is a multi-index of nonnegative integers. Then

$$\int_{\mathbb{S}_n} |\zeta^m|^2 d\sigma(\zeta) = \frac{(n-1)!m!}{(n-1+|m|)!}, \quad (2.23)$$

and

$$\int_{\mathbb{B}_n} |z^m|^2 d\nu_\alpha(z) = \frac{m!\Gamma(n+1+\alpha)}{\Gamma(n+|m|+\alpha+1)}. \quad (2.24)$$

### 2.1.3 Several differentiations in $\mathbb{B}_n$ .

A very important concept of differentiation on the unit ball is that of the *radial derivative*  $\mathcal{N}$ , which is based on the usual partial derivative of a holomorphic function. Thus, for a holomorphic vector-valued function  $f: \mathbb{B}_n \rightarrow E$  we write

$$\mathcal{N}f(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z), \quad z \in \mathbb{B}_n. \quad (2.25)$$

If the homogeneous expansion of the vector-valued holomorphic function  $f$  is given by

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \quad (2.26)$$

where  $f_k$  are the homogeneous holomorphic polynomial of degree  $k$  with coefficients on  $E$ , then

$$\mathcal{N}f(z) = \sum_{k=0}^{\infty} k f_k(z) = \sum_{k=1}^{\infty} k f_k(z), \quad z \in \mathbb{B}_n. \quad (2.27)$$

This is called the radial derivative of  $f$  because

$$\mathcal{N}f(z) = \lim_{r \rightarrow 0} \frac{f(z+rz) - f(z)}{r}, \quad z \in \mathbb{B}_n. \quad (2.28)$$

Here,  $r$  is a real parameter, so that  $z+rz$  is a radial variation of the point  $z$ . Given a holomorphic function  $f: \mathbb{B}_n \rightarrow E$ , we have

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{N}f(tz)}{t} dt, \quad z \in \mathbb{B}_n. \quad (2.29)$$

With the help of the homogeneous expansion, we can introduce a class of fractional radial derivatives

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on the space  $\mathcal{H}(\mathbb{B}_n, E)$ . Thus, for each real parameter  $t$ , we define the operator

$$\mathcal{N}^t : \mathcal{H}(\mathbb{B}_n, E) \longrightarrow \mathcal{H}(\mathbb{B}_n, E)$$

as follows.

$$\mathcal{N}^t f(z) = \sum_{k=1}^{\infty} k^t f_k(z), \quad \text{where } f(z) = \sum_{k=0}^{\infty} f_k(z). \quad (2.30)$$

The inverse  $\mathcal{N}_t$  of the operator  $\mathcal{N}^t$  in  $\mathcal{H}(\mathbb{B}_n, E)/E$  is given by

$$\mathcal{N}_t f(z) = \mathcal{N}^{-t} f(z) = \sum_{k=0}^{\infty} k^{-t} f_k(z), \quad (2.31)$$

If we equip the space  $\mathcal{H}(\mathbb{B}_n, E)$  with the topology of uniform convergence on compact sets, then the operators  $\mathcal{N}^t$  and  $\mathcal{N}_t$  are both continuous on  $\mathcal{H}(\mathbb{B}_n, E)$ .

**Definition 2.1.16.** For any two real parameters  $\alpha$  and  $t$ , such that neither  $n + \alpha$ , nor  $n + \alpha + t$  is a negative integer, we define the invertible operator  $R^{\alpha,t} : \mathcal{H}(\mathbb{B}_n, E) \rightarrow \mathcal{H}(\mathbb{B}_n, E)$  as follows.

If  $f(z) = \sum_{k=0}^{\infty} f_k(z)$  is the homogeneous expansion of  $f$ , then

$$R^{\alpha,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)}{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)} f_k(z). \quad (2.32)$$

$R^{\alpha,t}$  is invertible and the inverse of  $R^{\alpha,t}$  denoted by  $R_{\alpha,t}$  is given by

$$R_{\alpha,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+t)\Gamma(n+1+k+\alpha)}{\Gamma(n+1+\alpha)\Gamma(n+1+k+\alpha+t)} f_k(z). \quad (2.33)$$

The proof of the two following lemmas also follows from the definition of these operators, see [71].

**Lemma 2.1.17.** Suppose neither  $n + \alpha$  nor  $n + \alpha + t$  is a negative integer. Then

$$R_{\alpha,t} = R^{\alpha+t,-t}.$$

**Lemma 2.1.18.** Suppose  $\alpha$ ,  $t$ , and  $\lambda$  are real parameters such that none of  $n + \lambda$ ,  $n + \lambda + t$ , and  $n + \lambda + \alpha + t$  is a negative integer. Then

$$R^{\lambda,t} R^{\lambda+t,\alpha} = R^{\lambda,\alpha+t}.$$

The proof of the following lemma and proposition are in [77].

**Lemma 2.1.19.** Suppose neither  $n + s$  nor  $n + s + t$  is a negative integer. If  $\beta = s + N$  for some positive integer  $N$ , then there exists a one-variable polynomial  $h$  of degree  $N$  such that

$$R^{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+\beta}} = \frac{h(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{n+1+\beta+t}}, \quad z, w \in \mathbb{B}_n.$$

There also exists a one variable polynomial  $q$  of degree  $N$  such that

$$R_{s,t} \frac{1}{(1 - \langle z, w \rangle)^{n+1+\beta+t}} = \frac{q(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{n+1+\beta}}, \quad z, w \in \mathbb{B}_n.$$

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We recall that the scalar-valued Bergman kernel function is given by

$$K^\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n.$$

The main advantage of the operators  $R^{\alpha,t}$  and  $R_{\alpha,t}$  is that they interact well with the Bergman kernel functions. This is made precise by the following result.

**Proposition 2.1.20.** Suppose neither  $n + \alpha$  nor  $n + \alpha + t$  is a negative integer. Then the operator  $R^{\alpha,t}$  is the unique continuous linear operator on  $\mathcal{H}(\mathbb{B}_n, E)$  satisfying

$$R^{\alpha,t} \left( \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right) = \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}},$$

for all  $z, w \in \mathbb{B}_n$  and  $x \in E$ . Similarly, the operator  $R_{\alpha,t}$  is the unique continuous linear operator on  $\mathcal{H}(\mathbb{B}_n, E)$  satisfying

$$R_{\alpha,t} \left( \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} \right) = \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha}},$$

for all  $z, w \in \mathbb{B}_n$  and  $x \in E$ .

The proof of the following proposition is not quite different from the proof in [77, Proposition 1.15]. The interested reader can find it in [9].

**Proposition 2.1.21.** Suppose  $N$  is a positive integer and  $\alpha$  is a real such that  $n + \alpha$  is not a negative integer. Then  $R^{\alpha,N}$  as an operator acting on  $\mathcal{H}(\mathbb{B}_n, E)$  is a linear partial differential operator of order  $N$  with polynomial coefficients, that is

$$R^{\alpha,N} f(z) = \sum_{m \in \mathbb{N}^n, |m| \leq N} p_m(z) \frac{\partial^{|m|} f}{\partial z^m}(z),$$

where each  $p_m$  is a polynomial and  $z \in \mathbb{B}_n$ .

The following theorem is used to prove that those differentiation operators defined above maps holomorphic functions on holomorphic functions.

**Theorem 2.1.22** (Theorem 1.5.6 of [63]). Let  $f_k : \mathbb{B}_n \rightarrow E$  be such that

- (a) for each  $k = 0, 1, 2, \dots$ ,  $f_k$  is a homogeneous polynomial of degree  $k$  ( $f_k(rz) = r^k f_k(z)$ ), and
- (b)  $\sup_{k \in \mathbb{N}} \|f_k(z)\|_E < \infty$ , for every  $z \in \mathbb{B}_n$ .

The series  $\sum_{k=0}^{\infty} f_k(z)$  converges uniformly on every compact subset of  $\mathbb{B}_n$  and the sum of the series is holomorphic on  $\mathbb{B}_n$ .

### 2.1.4 Harmonic and Subharmonic functions.

In this section, we collect few results on harmonic and subharmonic functions in the unit ball that will be needed later in the book. Each result in this section comes from [52].

A harmonic function is a continuous function that fulfills the mean value property. It is well known that

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any holomorphic function in  $\Omega$  is harmonic in  $\Omega$ . In particular, if  $f \in \mathcal{H}(\mathbb{B}_n, E)$ , then  $f$  satisfies the mean value property

$$f(0) = \int_{\mathbb{S}_n} f(r\zeta) d\sigma(\zeta). \quad (2.34)$$

Thus, for every  $0 < R < 1$ , the following version of the mean value property holds

$$f(0) = \frac{1}{\nu_\alpha(R\mathbb{B}_n)} \int_{R\mathbb{B}_n} f(w) d\nu_\alpha(w). \quad (2.35)$$

A function  $f : \mathbb{B}_n \rightarrow [-\infty, \infty)$  is said to be upper semi-continuous if

$$\limsup_{z \rightarrow z_0} f(z) \leq f(z_0)$$

for every  $z_0 \in \mathbb{B}_n$ . An upper semi-continuous function  $f : \mathbb{B}_n \rightarrow [-\infty, \infty)$  is said to be subharmonic if

$$f(z) \leq \int_{\mathbb{S}_n} f(z + r\zeta) d\sigma(\zeta)$$

for all  $z \in \mathbb{B}_n$  and  $0 \leq r < 1 - |z|$ . Sometimes, this property is called the sub-mean value property. It is interesting to notice that if  $f$  is subharmonic, we also have

$$f(0) \leq \frac{1}{\nu_\alpha(R\mathbb{B}_n)} \int_{R\mathbb{B}_n} f(w) d\nu_\alpha(w), \quad (2.36)$$

for any  $0 < R < 1$ , which is the volume version of the sub-mean value property. In our case, the volume version will be more convenient.

The next result is well known, see [62, Lemma 6.4.1].

**Lemma 2.1.23.** ([52, Lemma 1.5.1]) If  $f : \mathbb{B}_n \rightarrow E$  is a holomorphic function from the open unit ball  $\mathbb{B}_n$  to the complex Banach space  $E$ , then the function  $z \mapsto \log \|f(z)\|_E$  is subharmonic.

The following theorem can also be proved by using the scalar-valued case of the sub-mean property, see the proof of Proposition 2.3 of [59] for example.

**Theorem 2.1.24.** ([52, Theorem 1.5.2]) If  $f : \mathbb{B}_n \rightarrow E$  is a holomorphic function from the open unit ball  $\mathbb{B}_n$  to the complex Banach space  $E$ , then the function  $z \mapsto \|f(z)\|_E^p$  is subharmonic for every  $0 < p < \infty$ . In particular, for  $0 < p < \infty$ ,  $\alpha > -1$  and  $r > 0$ , there exists a constant  $C > 0$  such that

$$\|f(z)\|_E^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_E^p d\nu_\alpha(w),$$

where  $f \in \mathcal{H}(\mathbb{B}_n, E)$ ,  $z \in \mathbb{B}_n$  and  $D(z, r)$  is defined by (2.17)

### 2.1.5 Some interesting inequalities.

The following result is classical and very important for some estimates of integrals.

**Theorem 2.1.25.** ([63, Proposition 1.4.10]) Let  $\beta \in \mathbb{R}$  and  $\alpha > -1$ . Then the integrals

$$I_\beta(z) = \int_{\mathbb{S}_n} \frac{d\sigma(\zeta)}{|1 - \langle z, \zeta \rangle|^{n+\beta}}, \quad z \in \mathbb{B}_n,$$

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and

$$J_{\alpha,\beta}(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+\beta}}, \quad z \in \mathbb{B}_n,$$

have the following asymptotic properties.

(i) If  $\beta = 0$ , then

$$I_\beta(z) \simeq J_{\alpha,\beta}(z) \simeq \log \frac{1}{1 - |z|^2}.$$

(ii) If  $\beta > 0$ , then

$$I_\beta(z) \simeq J_{\alpha,\beta}(z) \simeq \frac{1}{(1 - |z|^2)^\beta}.$$

(iii) If  $\beta < 0$ , then

$$I_\beta(z) \simeq J_{\alpha,\beta}(z) \simeq 1.$$

The best constant  $C$  in the previous estimates has been recently obtained in [46]. We also have a similar integral estimate with an extra unbounded factor  $\beta(z, w)$ . We give a proof here.

**Lemma 2.1.26.** Let  $\alpha > -1$ ,  $t > 0$  and let

$$\mathcal{J}_{\alpha,t}(z) := \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha \beta(z, w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+t}} d\nu(w),$$

then  $\mathcal{J}_{\alpha,t}(z) \lesssim (1 - |z|^2)^{-t}$ , for every  $z \in \mathbb{B}_n$ .

*Proof.* Since  $\beta(0, z) = \frac{1}{2} \log \frac{1 + |\varphi_0(z)|}{1 - |\varphi_0(z)|}$ , we know that

$$\beta(z, w) = \beta(\varphi_z(z), \varphi_z(w)) = \beta(0, \varphi_z(w)) \lesssim (1 - |\varphi_z(w)|^2)^\gamma = \frac{(1 - |z|^2)^\gamma (1 - |w|^2)^\gamma}{|1 - \langle z, w \rangle|^{2\gamma}}$$

for some small  $\gamma < 0$  that will be determined later. Then, by Theorem 2.1.25, we have that

$$\begin{aligned} \mathcal{J}_{\alpha,t}(z) &\lesssim (1 - |z|^2)^\gamma \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\alpha+\gamma}}{|1 - \langle z, w \rangle|^{n+1+\alpha+t+2\gamma}} d\nu(w) \\ &\lesssim (1 - |z|^2)^\gamma (1 - |z|^2)^{-(t+\gamma)} = (1 - |z|^2)^{-t} \end{aligned}$$

provided that  $\alpha + \gamma > -1$  and  $\gamma + t > 0$ . These constraints are fulfilled if we choose  $\gamma$  such that

$$\max\{-t, -(1 + \alpha)\} < \gamma < 0$$

and the result is proved. □

**Lemma 2.1.27.** ([77, Lemma 2.20]) For each  $r > 0$  there exists a positive constant  $C_r$  such that

$$C_r^{-1} \leq \frac{(1 - |a|^2)}{(1 - |z|^2)} \leq C_r$$

and

$$C_r^{-1} \leq \frac{(1 - |a|^2)}{|1 - \langle a, z \rangle|} \leq C_r,$$

for all  $a$  and  $z$  in  $\mathbb{B}_n$  with  $\beta(a, z) < r$ . Moreover, if  $r$  is bounded above, then we may choose  $C_r$  to be independent of  $r$ .

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**Lemma 2.1.28.** ([77, (2.20) p. 63]) For each  $R > 0$ , there exists a constant  $C_R > 0$  such that

$$C_R^{-1} \leq \frac{|1 - \langle z, u \rangle|}{|1 - \langle z, v \rangle|} \leq C_R$$

for all  $z \in \mathbb{B}_n$  and all  $u, v \in \mathbb{B}_n$  with  $\beta(u, v) \leq R$ .

**Corollary 2.1.29.** ([77, Corollary 2.21]) Suppose  $r_1 > 0$ ,  $r_2 > 0$  and  $r_3 > 0$ . Then, there exists a constant  $C > 0$  such that

$$C^{-1} \leq \frac{v_\alpha(D(z, r_1))}{v_\alpha(D(w, r_2))} \leq C \quad (2.37)$$

for all  $z$  and  $w$  in  $\mathbb{B}_n$  with  $\beta(z, w) \leq r_3$ .

**Definition 2.1.30.** A sequence  $\{a_k\}_k$  of points in  $\mathbb{B}_n$  is called a separated sequence (in the Bergman metric) if there exists a constant  $\delta > 0$  such that  $\beta(a_i, a_j) > \delta$  for any  $i \neq j$ .

We also need the following well known discrete version of Theorem 2.1.25.

**Lemma 2.1.31.** ([52, Lemma 1.4.5]) Let  $\{z_k\}_k$  be a separated sequence in  $\mathbb{B}_n$ , and let  $n < t < s$ . Then there exists a positive constant  $C$  such that

$$\sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^t}{|1 - \langle z, z_k \rangle|^s} \leq C(1 - |z|^2)^{t-s}, \quad z \in \mathbb{B}_n.$$

Lemma 2.1.31 can be deduced from Theorem 2.1.25 after noticing that, if a sequence  $\{z_k\}_k$  is separated, then there is a constant  $r > 0$  such that the Bergman metric balls  $D(z_k, r)$  are pairwise disjoint.

## 2.2 Vector-valued Bergman spaces.

In this section, we present the theory of the weighted vector-valued Bergman spaces in the unit ball of  $\mathbb{C}^n$ . Main topics covered include integral representations, Bergman-type projections, characterizations in terms of various derivatives. The integral representation formulas developed in this section, together with the fractional differential and fractional integral operators introduced will play a very important role in subsequent chapters.

### 2.2.1 Pointwise estimate and reproducing formulas on vector-valued Bergman spaces.

We recall that for  $\alpha > -1$ , the weighted Lebesgue measure  $d\nu_\alpha$  is defined by

$$d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z),$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

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is the normalizing constant so that  $d\nu_\alpha$  is a probability measure on  $\mathbb{B}_n$ .

For  $\alpha > -1$ , the weighted vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$  are defined to be

$$A_\alpha^p(\mathbb{B}_n, E) := \mathcal{H}(\mathbb{B}_n, E) \cap L_\alpha^p(\mathbb{B}_n, E), \quad 0 < p < \infty.$$

It is not hard to see that  $A_\alpha^p(\mathbb{B}_n, E) \subset A_\alpha^1(\mathbb{B}_n, E)$  for all  $1 < p < \infty$ .

When  $\alpha = 0$  and  $E = \mathbb{C}$ , we simply write  $A^p(\mathbb{B}_n)$  instead of  $A_0^p(\mathbb{B}_n, \mathbb{C})$ . These are the standart (unweighted) Bergman spaces.

We will use the notation

$$\|f\|_{p,\alpha,E} = \left[ \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) \right]^{1/p},$$

for every  $f \in L_\alpha^p(\mathbb{B}_n, E)$ . If  $E = \mathbb{C}$ , we simply write

$$\|f\|_{p,\alpha} := \|f\|_{p,\alpha,\mathbb{C}} = \left[ \int_{\mathbb{B}_n} |f(z)|^p d\nu_\alpha(z) \right]^{1/p}.$$

Recall that when  $1 \leq p < \infty$  the Bochner-Lebesgue space  $L_\alpha^p(\mathbb{B}_n, E)$  is a Banach space with the norm  $\|\cdot\|_{p,\alpha,E}$ . If  $0 < p < 1$ , the Bochner-Lebesgue space  $L_\alpha^p(\mathbb{B}_n, E)$  is a complete metric space with the following distance:

$$d(f, g) = \|f - g\|_{p,\alpha,E}^p.$$

In the special case when  $p = 2$  and  $\mathcal{H}$  is a Hilbert space, the Bochner-Lebesgue space  $L_\alpha^2(\mathbb{B}_n, \mathcal{H})$  is a Hilbert space under the inner product given by

$$\langle f, g \rangle_{\alpha,\mathcal{H}} = \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{\mathcal{H}} d\nu_\alpha(z),$$

where  $f, g \in L_\alpha^2(\mathbb{B}_n, \mathcal{H})$  and  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  is the inner product on  $\mathcal{H}$ .

According to the sub-mean property in Theorem 2.1.24, for a function  $f \in \mathcal{H}(\mathbb{B}_n, E)$ , for every  $0 < p < \infty$ , and  $r > 0$  we have that

$$\|f(z)\|_E^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_E^p d\nu_\alpha(w),$$

for  $z \in \mathbb{B}_n$ . An immediate consequence is that any vector-valued Bergman function  $f$  satisfies the following pointwise estimate

$$\|f(z)\|_E \leq \frac{C}{(1 - |z|^2)^{(n+1+\alpha)/p}} \|f\|_{p,\alpha,E},$$

for some positive constant  $C$  independent of the point  $z \in \mathbb{B}_n$  and the function  $f$ .

As in the scalar case, one can get the constant  $C$  to be one.

**Theorem 2.2.1.** ([77, Theorem 1.5.2]) Let  $0 < p < \infty$  and  $\alpha > -1$ . Then

$$\|f(z)\|_E \leq \frac{\|f\|_{p,\alpha,E}}{(1 - |z|^2)^{(n+1+\alpha)/p}},$$

for every  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $z \in \mathbb{B}_n$ .

One important consequence is that, for any  $z \in \mathbb{B}_n$ , the point evaluation

$$\begin{aligned} T_z : A_\alpha^p(\mathbb{B}_n, E) &\longrightarrow E \\ f &\longmapsto f(z) \end{aligned}$$

is a bounded linear operator for all  $1 \leq p < \infty$ . This allows one to prove that the weighted vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$  is a closed subspace of  $L_\alpha^p(\mathbb{B}_n, E)$  and, therefore, it is a Banach space. We use

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the following notation

$$\langle f, g \rangle_{\alpha, E} = \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{E, E^*} d\nu_{\alpha}(z). \quad (2.38)$$

to represent the "inner product" for  $f \in A_{\alpha}^2(\mathbb{B}_n, E)$  and  $g \in A_{\alpha}^2(\mathbb{B}_n, E^*)$ . When we refer to the classical weighted scalar-valued Bergman spaces, we will denote  $\langle \cdot, \cdot \rangle_{\alpha} = \langle \cdot, \cdot \rangle_{\alpha, \mathbb{C}}$  for the inner product in  $L_{\alpha}^2(\mathbb{B}_n)$ . Recall that in the weighted scalar-valued Bergman spaces,  $A_{\alpha}^2(\mathbb{B}_n)$  is also a reproducing kernel Hilbert space but in the vector-valued scheme, this is not clear, but we still have an integral representation of each weighted vector-valued Bergman function that we still call it, the reproducing kernel formula.

**Proposition 2.2.2** (Reproducing kernel formula). Let  $\alpha > -1$  and  $f \in A_{\alpha}^1(\mathbb{B}_n, E)$ . Then

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha}(w),$$

for any  $z \in \mathbb{B}_n$ .

*Proof.* Let  $f \in A_{\alpha}^1(\mathbb{B}_n, E) \subset \mathcal{H}(\mathbb{B}_n, E)$ . Using (2.35) and letting  $R \rightarrow 1$ , by the dominated convergence theorem, we obtain

$$f(0) = \int_{\mathbb{B}_n} f(w) d\nu_{\alpha}(w).$$

Now, fix  $z \in \mathbb{B}_n$ . If we replace  $f$  by  $f \circ \varphi_z$  and we make a change of variables (Proposition 2.1.12) we obtain that

$$f(z) = \int_{\mathbb{B}_n} f \circ \varphi_z(w) d\nu_{\alpha}(w) = \int_{\mathbb{B}_n} f(w) \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} d\nu_{\alpha}(w).$$

Since the function  $w \mapsto f(w)$  is in  $A_{\alpha}^1(\mathbb{B}_n, E)$ , the function  $w \mapsto f(w)(1 - \langle w, z \rangle)^{n+1+\alpha}$  is also in  $A_{\alpha}^1(\mathbb{B}_n, E)$ . Replacing again  $f(w)$  by  $f(w)(1 - \langle w, z \rangle)^{n+1+\alpha}$  we get that

$$f(z)(1 - |z|^2)^{n+1+\alpha} = (1 - |z|^2)^{n+1+\alpha} \int_{\mathbb{B}_n} f(w) \frac{(1 - \langle w, z \rangle)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)}} d\nu_{\alpha}(w).$$

Then simplifying we arrive at the desired reproducing kernel formula.  $\square$

We have some important consequences which are the following (recall the definition of  $R^{\alpha, t}$  and  $R_{\alpha, t}$  given in Subsection 2.1.3.

**Proposition 2.2.3.** Let  $\alpha > -1$ ,  $f \in A_{\alpha}^1(\mathbb{B}_n, E)$  and  $t > 0$ . Then

$$R^{\alpha, t} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} d\nu_{\alpha}(w)$$

and

$$R_{\alpha, t} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha+t}(w),$$

for any  $z \in \mathbb{B}_n$ .

*Proof.* By using the reproducing formula in Proposition 2.2.2 and applying  $R^{\alpha, t}$  under the integral sign, we get the first identity using Proposition 2.1.20. The case  $R_{\alpha, t}$  follows the same proof but applying the reproducing formula in Proposition 2.2.2 with  $\alpha + t$  instead of  $\alpha$ .  $\square$

### 2.2.2 Density of vector-valued holomorphic polynomials and bounded holomorphic functions on $A_\alpha^p(\mathbb{B}_n, E)$ .

The following density property is extremely useful in many situations.

**Lemma 2.2.4.** Let  $\alpha > -1$ . The space  $\mathcal{P}(\mathbb{B}_n, E)$  of all vector-valued holomorphic polynomials is dense in  $A_\alpha^p(\mathbb{B}_n, E)$ , for any  $1 \leq p < \infty$ .

*Proof.* Let  $1 \leq p < \infty$ . Given a function  $f \in A_\alpha^p(\mathbb{B}_n, E)$ , let  $f_\rho(z) = f(\rho z)$  be its dilation, where  $0 < \rho < 1$ . Each function  $f_\rho$  is analytic in a larger disk, so it can be approximated uniformly in  $\mathbb{B}_n$  by polynomials, the partial sums of its Taylor series. Thus it will be enough to prove that  $f$  can be approximated in  $A_\alpha^p(\mathbb{B}_n, E)$  norm by its dilations, i.e.,  $\|f - f_\rho\|_{p,\alpha,E} \rightarrow 0$  as  $\rho \rightarrow 1$ . First, we recall that the integral means

$$M_p(r, f) := \left( \int_{\mathbb{S}_n} \|f(r\zeta)\|_E^p d\sigma(\zeta) \right)^{1/p}, \quad 0 \leq r < 1,$$

are increasing with  $r$ , see [77, Corollary 4.21], and observe that  $M_p(r, f_\rho) = M_p(r\rho, f)$ .

Therefore,

$$M_p^p(r, f - f_\rho) \leq 2^p(M_p^p(r, f) + M_p^p(r, f_\rho)) \leq 2^{p+1}M_p^p(r, f).$$

But the assumption that  $f \in A_\alpha^p(\mathbb{B}_n, E)$  is equivalent to saying that  $M_p^p(r, f)$  is integrable over the interval  $[0, 1)$  with respect to the measure  $2nc_\alpha r^{2n-1}(1-r^2)^\alpha dr$  see (2.16), and it is clear that  $f_\rho(z) \rightarrow f(z)$  uniformly on compact subsets of  $\mathbb{B}_n$  as  $\rho \rightarrow 1$ , which implies that  $M_p^p(r, f - f_\rho) \rightarrow 0$  for each  $r \in [0, 1)$ . Thus by (2.16), and the Lebesgue dominated convergence theorem (Theorem 1.2.45), we conclude that

$$\|f - f_\rho\|_{p,\alpha,E}^p = 2nc_\alpha \int_0^1 M_p^p(r, f - f_\rho) r^{2n-1} (1-r^2)^\alpha dr \rightarrow 0,$$

as  $\rho \rightarrow 1$ , which completes the proof.  $\square$

**Lemma 2.2.5 (Density).** Suppose that  $0 < p < \infty$ . Then the space of all bounded vector-valued holomorphic functions  $H^\infty(\mathbb{B}_n, E)$  is dense in  $A_\alpha^p(\mathbb{B}_n, E)$ .

*Proof.* We are going to give the proof for  $0 < p < 1$ , since the case  $1 \leq p < \infty$  is [52, Lemma 2.1.4]. Given a function  $f \in A_\alpha^p(\mathbb{B}_n, E)$ , let  $f_\rho$  defined for  $z \in \mathbb{B}_n$  by  $f_\rho(z) := f(\rho z)$ , where  $0 < \rho < 1$ . The function  $f_\rho$  is holomorphic in the set  $\{z \in \mathbb{B}_n : |z| < 1/\rho\}$  hence is bounded on  $\mathbb{B}_n$ . We first recall that the integral means

$$M_p(r, f) := \int_{\mathbb{S}_n} \|f(r\zeta)\|_E^p d\sigma(\zeta), \quad 0 \leq r < 1$$

are increasing with  $r$ , see [77, Corollary 4.21]. Since  $M_p(r, f_\rho) = M_p(r\rho, f)$ , we have by Minkowski's inequality that

$$M_p^p(r, f_\rho - f) \leq M_p^p(r, f) + M_p^p(r, f_\rho) \leq 2M_p^p(r, f).$$

By the formula (2.16) in Lemma 2.1.7, we get

$$\|f - f_\rho\|_{p,\alpha,E}^p = 2nc_\alpha \int_0^1 M_p^p(r, f_\rho - f) (1-r^2)^\alpha r^{2n-1} dr. \quad (2.39)$$

Since  $f \in A_\alpha^p(\mathbb{B}_n, E)$ , we have that the function  $M_p^p(r, f)$  is integrable over the interval  $[0, 1)$  with respect to the measure  $2n(1-r^2)^\alpha r^{2n-1} dr$ . It is also clear that  $f_\rho \rightarrow f$  on any compact subsets of  $\mathbb{B}_n$  which implies

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that  $M_p^p(r, f_\rho - f) \rightarrow 0$  for each  $r \in [0, 1)$  as  $\rho \rightarrow 1$ . Applying the dominated convergence theorem in (2.39), we obtain that  $\|f - f_\rho\|_{p,\alpha,E}^p \rightarrow 0$ , as  $\rho \rightarrow 1$ .  $\square$

**Corollary 2.2.6.** For  $0 < p \leq 1$ , the following inclusion is dense

$$A_\alpha^2(\mathbb{B}_n, E) \subset A_\alpha^p(\mathbb{B}_n, E).$$

**Lemma 2.2.7.** Let  $t > 0$ . Then

$$\int_{\mathbb{B}_n} f(z) \cdot \overline{g(z)} d\nu_\alpha(z) = \int_{\mathbb{B}_n} R^{\alpha,t} f(z) \cdot \overline{g(z)} d\nu_{\alpha+t}(z),$$

for every  $f \in \mathcal{P}(\mathbb{B}_n, E)$  and  $g \in \mathcal{P}(\mathbb{B}_n, \mathbb{C})$ .

*Proof.* Let  $f \in \mathcal{P}(\mathbb{B}_n, E)$  and  $g \in \mathcal{P}(\mathbb{B}_n, \mathbb{C})$  be polynomials. Then, by Proposition 2.2.3, Fubini's theorem and the integral representation in Proposition 2.2.2, we have that

$$\begin{aligned} \int_{\mathbb{B}_n} R^{\alpha,t} f(z) \cdot \overline{g(z)} d\nu_{\alpha+t}(z) &= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{f(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} \right) \cdot \overline{g(z)} d\nu_{\alpha+t}(z) \\ &= \int_{\mathbb{B}_n} f(w) \left( \int_{\mathbb{B}_n} \frac{\overline{g(z)} d\nu_{\alpha+t}(z)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} \right) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} f(w) \cdot \overline{g(w)} d\nu_\alpha(w). \end{aligned}$$

Since  $f$  and  $g$  are polynomials we can apply Theorem 2.1.25 to see that the assumption of Fubini's theorem is fulfilled.  $\square$

Let  $w \in \mathbb{B}_n$ . Recall that the scalar-valued Bergman reproducing kernels is defined as

$$K_w^\alpha(z) := \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z \in \mathbb{B}_n.$$

Using Theorem 2.1.25, we get that

$$\|K_w^\alpha\|_{p,\alpha} \simeq (1 - |w|^2)^{-\frac{(p-1)}{p}(n+1+\alpha)}, \quad 1 < p < \infty.$$

In particular, for  $p = 2$ , we have that

$$\|K_w^\alpha\|_{2,\alpha} = (1 - |w|^2)^{-\frac{1}{2}(n+1+\alpha)}. \quad (2.40)$$

So, for  $1 < p < \infty$ , we recall also the scalar-valued  $p$ -normalized reproducing kernels

$$k_{w,p}^\alpha(z) = \frac{K_w^\alpha(z)}{\|K_w^\alpha\|_{p,\alpha}}, \quad z \in \mathbb{B}_n.$$

For  $p = 2$ , we write

$$k_w^\alpha(z) := k_{w,2}^\alpha(z) = \frac{K_w^\alpha(z)}{\|K_w^\alpha\|_{2,\alpha}}, \quad z \in \mathbb{B}_n.$$

### 2.2.3 Some useful estimates

**Lemma 2.2.8.** Let  $E$  be a Banach space. Suppose  $p > 0$ ,  $\alpha > -1$ ,  $0 < r < 1$ , and  $m = (m_1, \dots, m_n)$  is a multi-index of nonnegative integers. Then there exists a positive constant  $C$  such that

$$\left\| \frac{\partial^m f}{\partial z^m}(z) \right\|_E \leq C \|f\|_{p,\alpha,E}$$

for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and all  $z \in \mathbb{B}_n$  with  $|z| \leq r$ .

*Proof.* Fix some real  $\delta \in (r, 1)$  and apply Proposition 2.2.2 in the special case  $\alpha = 0$ . We have that

$$f_\delta(z) = \int_{\mathbb{B}_n} \frac{f_\delta(w) d\nu(w)}{(1 - \langle z, w \rangle)^{n+1}}, \quad z \in \mathbb{B}_n,$$

where  $f_\delta(z) = f(\delta z)$ . It follows that

$$f(\delta z) = \int_{\mathbb{B}_n} \frac{f(\delta w) d\nu(w)}{(1 - \langle z, w \rangle)^{n+1}}, \quad z \in \mathbb{B}_n. \quad (2.41)$$

Making the change of variables  $\zeta = \delta w$ , and replacing  $\delta z$  by  $z$ , in we get

$$f(z) = \delta^2 \int_{|\zeta| < \delta} \frac{f(\zeta) d\nu(\zeta)}{(\delta^2 - \langle z, \zeta \rangle)^{n+1}}, \quad |z| < \delta. \quad (2.42)$$

We differentiate under the integral sign (Theorem 1.2.48), and we get a positive  $C$  (which depend only on  $n$ ,  $|m|$  and  $\delta$ ) such that

$$\left\| \frac{\partial^m f}{\partial z^m}(z) \right\|_E \leq C \sup\{\|f(w)\|_E : |w| \leq \delta\},$$

for all  $|z| \leq \delta$ . Which reduces the proof of the lemma in the case  $|m| = 0$ , therefore, the desired result then follows from Theorem 2.2.1.  $\square$

**Corollary 2.2.9.** For each  $p > 0$ , the weighted vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$  is closed in  $L_\alpha^p(\mathbb{B}_n, E)$ .

*Proof.* Suppose  $\{f_n\}$  is a sequence in  $A_\alpha^p(\mathbb{B}_n, E)$  such that

$$\lim_{n \rightarrow +\infty} \|f_n - f\|_{p,\alpha,E} = 0$$

for some  $f \in L^p(\mathbb{B}_n, d\nu_\alpha)$ . Then by Theorem 1.2.46, there exists a subsequence  $\{f_{n_k}(z)\}$  of  $\{f_n(z)\}$  which converges to  $f(z)$  for almost every  $z$  in  $\mathbb{B}_n$ . Also,  $\{f_n\}$  is a Cauchy sequence in  $A_\alpha^p(\mathbb{B}_n, E)$ , so by Theorem 2.2.1, the sequence  $\{f_n(z)\}$  is uniformly Cauchy on each set  $\{z \in \mathbb{B}_n : |z| \leq r\}$ , and must converge to a holomorphic function there, where  $0 < r < 1$ . Since  $r$  is arbitrary,  $\{f_n(z)\}$  converges to a holomorphic function  $g(z)$  on  $\mathbb{B}_n$ . By the uniqueness of pointwise limits, we have  $f(z) = g(z)$  for almost all  $z$  in  $\mathbb{B}_n$ . Since the function  $g$  is holomorphic, we have  $f = g$  on  $\mathbb{B}_n$ . This shows that  $f$  is holomorphic in  $\mathbb{B}_n$  and hence  $f \in A_\alpha^p(\mathbb{B}_n, E)$ .  $\square$

It follows that the weighted vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$ , with topology inherited from the Bochner-Lebesgue space  $L_\alpha^p(\mathbb{B}_n, E)$ , is a Banach space when  $1 \leq p < \infty$ , and is a complete metric space when  $0 < p < 1$ .

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When  $p = 2$  and  $E = \mathcal{H}$ , where  $\mathcal{H}$  is a complex Hilbert space, the space  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  is a Hilbert space. Moreover, if

$$f(z) = \sum_{m \in \mathbb{N}^n} a_m z^m$$

is the Taylor expansion of  $f$ , then by (2.24),

$$\|f\|_{2,\alpha,\mathcal{H}}^2 = \int_{\mathbb{B}_n} \|f(z)\|_{\mathcal{H}}^2 d\nu_\alpha(z) = \sum_{m \in \mathbb{N}^n} \frac{m! \Gamma(n + \alpha + 1)}{\Gamma(n + |m| + \alpha + 1)} \|a_m\|_{\mathcal{H}}^2. \quad (2.43)$$

If  $\mathcal{H}$  is separable with an orthonormal basis  $\{e_k\}_{k \in \mathbb{N}}$  then

$$\left( \sqrt{\frac{\Gamma(n + |m| + \alpha + 1)}{m! \Gamma(n + \alpha + 1)}} z^m e_k \right)_{m \in \mathbb{N}^n, k \in \mathbb{N}} \quad (2.44)$$

is an orthonormal basis of  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ .

**Remark 2.2.10.** We mention that (2.43) is not longer true for Banach spaces. Indeed, for  $n = 1$  and  $\alpha = 0$  let  $2 < p < \infty$  and let  $\ell^p = \{\{x_k\}_k : \sum_{k=1}^\infty |x_k|^p < \infty\}$  be the Banach space of scalar-valued sequences and let  $\{e_k\}$  be the canonical basis of  $\ell^p$ . If

$$f(z) = \sum_{k \in \mathbb{N}} e_k z^k,$$

then  $f \in A^2(\mathbb{B}_1, \ell^p)$ , indeed, for any  $k \geq 1$  and  $z \in \mathbb{B}_1$  we have that  $f_k(z) = \langle f(z), e_k \rangle_{\ell^p} = z^k$ . Thus

$$\|f(z)\|_{\ell^p} = \left( \sum_{k=1}^\infty |f_k(z)|^p \right)^{1/p} = \left( \sum_{k=1}^\infty (|z|^p)^k \right)^{1/p} = \frac{|z|}{(1 - |z|^p)^{\frac{1}{p}}}$$

It follows that

$$\|f\|_{2,0,\ell^p}^2 = \int_{\mathbb{B}_1} \|f(z)\|_{\ell^p}^2 d\nu(z) = \int_{\mathbb{B}_1} \frac{|z|^2 d\nu(z)}{(1 - |z|^p)^{\frac{2}{p}}} = 2 \int_0^1 \frac{r^3 dr}{(1 - r^p)^{\frac{2}{p}}} < \infty,$$

since  $p > 2$ . But

$$\sum_{k=1}^\infty \frac{\|e_k\|_{\ell^p}}{k+1} = \infty.$$

The following lemma is critical for many problems concerning the weighted vector-valued Bergman spaces  $A_\alpha^p(\mathbb{B}_n, E)$  whenever  $0 < p \leq q < \infty$ .

**Lemma 2.2.11.** Let  $0 < p \leq q < \infty$ . Then

$$\int_{\mathbb{B}_n} \|f(z)\|_E^q (1 - |z|^2)^{\left(\frac{1}{p} - \frac{1}{q}\right)q(n+1+\alpha)} d\nu_\alpha(z) \leq \|f\|_{p,\alpha,E}^q,$$

for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$ .

*Proof.* Since

$$\|f(z)\|_E^q = \|f(z)\|_E^p \|f(z)\|_E^{q-p}.$$

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By Theorem 2.2.1, we have that

$$\|f(z)\|_E^{q-p} \leq \frac{\|f\|_{p,\alpha,E}^{q-p}}{(1-|z|^2)^{\frac{q-p}{p}(n+1+\alpha)}}.$$

It follows that

$$\begin{aligned} \int_{\mathbb{B}_n} \|f(z)\|_E^q (1-|z|^2)^{\left(\frac{1}{p}-\frac{1}{q}\right)q(n+1+\alpha)} d\nu_\alpha(z) &= \int_{\mathbb{B}_n} \|f(z)\|_E^q (1-|z|^2)^{\left(\frac{1}{p}-\frac{1}{q}\right)q(n+1+\alpha)} d\nu_\alpha(z) \\ &\leq \|f\|_{p,\alpha,E}^{q-p} \times \\ &\quad \int_{\mathbb{B}_n} \|f(z)\|_E^p (1-|z|^2)^{\left(\frac{q}{p}-1\right)(n+1+\alpha)+\left(-\frac{q}{p}+1\right)(n+1+\alpha)} d\nu_\alpha(z) \\ &= \|f\|_{p,\alpha,E}^{q-p} \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) \\ &= \|f\|_{p,\alpha,E}^q \end{aligned}$$

□

In the case where  $q = 1$ , we have the following version

**Corollary 2.2.12.** Let  $0 < p \leq 1$ . Then

$$\int_{\mathbb{B}_n} \|f(z)\|_E (1-|z|^2)^{\left(\frac{1}{p}-1\right)(n+1+\alpha)} d\nu_\alpha(z) \leq \|f\|_{p,\alpha,E},$$

for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$ .

The proof of the following technical result can be found in [20].

**Lemma 2.2.13.** Let  $\beta, \delta > 0$ . For all  $w \in \mathbb{B}_n$ , we have

$$I_\alpha(w) = \int_{\mathbb{B}_n} \left| \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) \right|^\delta \frac{(1 - |w|^2)^\beta}{|1 - \langle z, w \rangle|^{n+1+\alpha+\beta}} d\nu_\alpha(z) \leq C,$$

where  $C$  is independent of  $w$  and  $\log$  is the principal determination of the logarithm.

**Definition 2.2.14.** Let  $0 < p < \infty$ .

- a) The weak Bochner-Lebesgue space  $L_\alpha^{p,\infty}(\mathbb{B}_n, E)$  consists of all vector-valued measurable functions  $f : \mathbb{B}_n \rightarrow E$  for which

$$\|f\|_{L_\alpha^{p,\infty}(\mathbb{B}_n, E)} = \left( \sup_{\lambda > 0} \lambda^p \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_E > \lambda\}) \right)^{1/p} < \infty.$$

- b) The weak vector-valued Bergman space  $A_\alpha^{p,\infty}(\mathbb{B}_n, E)$  is defined by

$$A_\alpha^{p,\infty}(\mathbb{B}_n, E) = \mathcal{H}(\mathbb{B}_n, E) \cap L_\alpha^{p,\infty}(\mathbb{B}_n, E).$$

In the sequel, we will use the fact that the weak vector-valued Bergman space  $A_\alpha^{1,\infty}(\mathbb{B}_n, E)$  is continuously embedded in the vector-valued Bergman space  $A_\alpha^q(\mathbb{B}_n, E)$ , with  $0 < q < 1$ . We justify this in the following lemma.

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**Lemma 2.2.15.** If  $0 < q < 1$ , then the identity  $i : L_\alpha^{1,\infty}(\mathbb{B}_n, E) \hookrightarrow L_\alpha^q(\mathbb{B}_n, E)$  is continuous in the sense that there exists a constant  $C(q) > 0$  such that for every  $f \in L_\alpha^{1,\infty}(\mathbb{B}_n, E)$ , we have

$$\|f\|_{q,\alpha,E} \leq C(q) \|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}.$$

*Proof.* Let  $f \in L_\alpha^{1,\infty}(\mathbb{B}_n, E)$ .

The result is evident if  $f = 0$ . Let now assume that  $f \neq 0$ . We have that

$$\begin{aligned} \|f\|_{q,\alpha,E}^q &= \int_{\mathbb{B}_n} \|f(z)\|_E^q d\nu_\alpha(z) \\ &= q \int_0^\infty t^{q-1} \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_E > t\}) dt \\ &= q \int_0^{\|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}} t^{q-1} \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_E > t\}) dt \\ &\quad + q \int_{\|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}}^\infty t^{q-1} \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_E > t\}) dt \\ &\leq q \int_0^{\|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}} t^{q-1} \nu_\alpha(\mathbb{B}_n) dt + q \int_{\|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}}^\infty t^{q-2} (t \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_E > t\})) dt \\ &\leq [t^q]_0^{\|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}} + q \int_{\|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}}^\infty t^{q-2} (t \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_E > t\})) dt \\ &\leq \|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}^q + \|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)} \left[ \frac{q}{q-1} t^{q-1} \right]_{\|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}}^\infty \\ &= \left(1 + \frac{q}{1-q}\right) \|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}^q = \frac{1}{(1-q)} \|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}^q, \end{aligned}$$

where in the second line we have used Lemma 1.2.36. Therefore, we obtain that

$$\|f\|_{q,\alpha,E} \leq \left(\frac{1}{1-q}\right)^{1/q} \|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, E)}.$$

□

**Corollary 2.2.16.** For  $0 < p \leq 1$ , the following inclusion is dense

$$L_\alpha^2(\mathbb{B}_n, E) \subset L_\alpha^p(\mathbb{B}_n, E).$$

*Proof.* Let  $f \in L_\alpha^2(\mathbb{B}_n, E)$ ,  $g = \|f\|_E^p$  and  $r = 2/p > 1$ . Using Hölder's inequality, We get

$$\int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) = \int_{\mathbb{B}_n} g(z) d\nu_\alpha(z) \leq \left( \int_{\mathbb{B}_n} g(z)^r d\nu_\alpha(z) \right)^{1/r} = \left( \int_{\mathbb{B}_n} \|f(z)\|_E^2 d\nu_\alpha(z) \right)^{p/2}.$$

Since simple functions are dense in  $L_\alpha^p(\mathbb{B}_n, E)$  and are also dense in  $L_\alpha^2(\mathbb{B}_n, E)$ , the desired result follow at once. □

### 2.2.4 Boundedness of vector-valued Bergman type projections.

The Bergman projection operator  $P_\alpha$ , is the integral type operator given by

$$P_\alpha(f)(z) := \int_{\mathbb{B}_n} \frac{f(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad (2.45)$$

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for any  $f \in L^1_\alpha(\mathbb{B}_n, E)$  and  $z \in \mathbb{B}_n$ . It is not hard to see that  $P_\alpha f \in \mathcal{H}(\mathbb{B}_n, E)$ , for every  $f \in L^1_\alpha(\mathbb{B}_n, E)$ . Indeed, Proposition 1.2.24 implies that  $\langle P_\alpha f, x^* \rangle_{E, E^*} = P_\alpha(\langle f, x^* \rangle_{E, E^*})$ , for any  $f \in L^1_\alpha(\mathbb{B}_n, E)$  and  $x^* \in E^*$ . Since  $P_\alpha$  acting on scalar-valued functions is holomorphic, we have that  $P_\alpha$  is weakly holomorphic and then vector-valued holomorphic. Moreover, it is easy to see by Fubini theorem and Theorem 2.1.25 that  $P_\gamma : L^1_\alpha(\mathbb{B}_n, E) \rightarrow A^1_\alpha(\mathbb{B}_n, E)$  is a bounded linear operator, for all  $\gamma > \alpha$ . We have a more general result on the boundedness of integral type operators acting on  $L^p_\alpha(\mathbb{B}_n, E)$ , for  $1 \leq p < \infty$ .

**Theorem 2.2.17.** Let  $1 \leq p < \infty$ ,  $\alpha, \gamma > -1$  and  $\beta$  a real parameter. If

$$-p\beta < \alpha + 1 < p(\gamma + 1), \quad (2.46)$$

then the operators defined by

$$Sf(z) = (1 - |z|^2)^\beta \int_{\mathbb{B}_n} \frac{f(w) d\nu_\gamma(w)}{(1 - \langle z, w \rangle)^{n+1+\gamma+\beta}}, \quad z \in \mathbb{B}_n,$$

and

$$Tf(z) = (1 - |z|^2)^\beta \int_{\mathbb{B}_n} \frac{f(w) d\nu_\gamma(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}}, \quad z \in \mathbb{B}_n$$

are both bounded on  $L^p_\alpha(\mathbb{B}_n, E)$ .

*Proof.* Let  $f \in L^p_\alpha(\mathbb{B}_n, E)$ . If  $p = 1$ , it directly follows by the Tonelli's theorem and Theorem 2.1.25. Indeed,

$$\begin{aligned} \int_{\mathbb{B}_n} \|Tf(z)\|_E d\nu_\alpha(z) &\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \|f(w)\|_E \frac{(1 - |z|^2)^{\alpha+\beta} (1 - |w|^2)^\gamma}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} d\nu(w) d\nu(z) \\ &= \int_{\mathbb{B}_n} \|f(w)\|_E (1 - |w|^2)^\gamma \left( \int_{\mathbb{B}_n} \frac{(1 - |z|^2)^{\alpha+\beta} d\nu(z)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} \right) d\nu(w) \\ &\lesssim \int_{\mathbb{B}_n} \|f(w)\|_E (1 - |w|^2)^\gamma (1 - |w|^2)^{\alpha-\gamma} d\nu(w) \lesssim \|f\|_{1, \alpha, E}, \end{aligned}$$

taking into account that  $\gamma - \alpha > 0$  and  $\alpha + \beta > -1$  which is true by the assumption (2.46).

Now suppose  $1 < p < \infty$  and let  $\epsilon > 0$  that will be specified later. By Lemma 1.2.25, we have that

$$\begin{aligned} \|Tf(z)\|_E^p &\lesssim (1 - |z|^2)^{p\beta} \left( \int_{\mathbb{B}_n} \|f(w)\|_E \frac{(1 - |w|^2)^\gamma}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} d\nu(w) \right)^p \\ &= (1 - |z|^2)^{p\beta} \left( \int_{\mathbb{B}_n} \|f(w)\|_E \frac{(1 - |w|^2)^{\frac{\alpha+\epsilon}{p} + \gamma - \frac{\alpha+\epsilon}{p}}}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} d\nu(w) \right)^p. \end{aligned}$$

The last integral over  $p$ , using the Hölder's inequality, is less or equal than

$$\left( \int_{\mathbb{B}_n} \|f(w)\|_E^p \frac{(1 - |w|^2)^{\alpha+\epsilon} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} \right) \left( \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{(\gamma p - \alpha - \epsilon)/(p-1)} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} \right)^{p-1}.$$

Then, if

$$\frac{\gamma p - \alpha - \epsilon}{p - 1} > -1, \quad (2.47)$$

and

$$\gamma + \beta - \frac{\gamma p - \alpha - \epsilon}{p - 1} > 0, \quad (2.48)$$

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using Theorem 2.1.25, it follows that

$$\left( \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{(\gamma p - \alpha - \epsilon)/(p-1)} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} \right)^{p-1} \lesssim (1 - |z|^2)^{\gamma - \beta(p-1) - \alpha - \epsilon}.$$

So, we get

$$\|Tf(z)\|_E^p \lesssim (1 - |z|^2)^{\gamma + \beta - \alpha - \epsilon} \left( \int_{\mathbb{B}_n} \|f(w)\|_E^p \frac{(1 - |w|^2)^{\alpha + \epsilon} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} \right).$$

Finally, applying Tonelli's theorem and Theorem 2.1.25 again, if

$$\gamma + \beta - \epsilon > -1, \quad (2.49)$$

we obtain that

$$\begin{aligned} \int_{\mathbb{B}_n} \|Tf(z)\|_E^p d\nu_\alpha(z) &\lesssim \int_{\mathbb{B}_n} \|f(w)\|_E^p (1 - |w|^2)^\epsilon \left( \int_{\mathbb{B}_n} \frac{d\nu_{\gamma+\beta-\epsilon}(z)}{|1 - \langle z, w \rangle|^{n+1+\gamma+\beta}} \right) d\nu_\alpha(w) \\ &\lesssim \int_{\mathbb{B}_n} \|f(w)\|_E^p (1 - |w|^2)^\epsilon (1 - |w|^2)^{-\epsilon} d\nu_\alpha(w) \lesssim \|f\|_{p,\alpha,E}^p. \end{aligned}$$

It only remains to show that (2.47), (2.48) and (2.49) are fulfilled. For this we need to take  $\epsilon > 0$  such that

$$\max\{0, \gamma - \alpha - \beta(p-1)\} < \epsilon < \min\{\gamma + \beta + 1, p(\gamma + 1) - (\alpha + 1)\}$$

which is clearly possible by the assumption (2.46). Since  $\|S(f)\|_E \leq \|T(f)\|_E$ , the boundedness of  $T$  on  $L_\alpha^p(\mathbb{B}_n, E)$  also shows that the operator  $S$  is bounded on  $L_\alpha^p(\mathbb{B}_n, E)$ .  $\square$

The following theorem singles out a very important case of Theorem 2.2.17.

**Theorem 2.2.18.** Let  $1 \leq p < \infty$  and  $\alpha, \gamma > -1$ . If  $\alpha + 1 < p(\gamma + 1)$ , then the Bergman projection  $P_\gamma : L_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^p(\mathbb{B}_n, E)$  defined in (2.45) is a bounded linear operator.

*Proof.* It is a direct consequence of Theorem 2.2.17 (use  $S$  with  $\beta = 0$ ).  $\square$

**Proposition 2.2.19.** For any  $1 < p < \infty$ , if  $p'$  is the conjugate exponent of  $p$ , then

$$\langle P_\alpha f, g \rangle_{\alpha,E} = \langle f, P_\alpha g \rangle_{\alpha,E},$$

for every  $f \in L_\alpha^p(\mathbb{B}_n, E)$  and  $g \in L_\alpha^{p'}(\mathbb{B}_n, E^*)$ .

*Proof.* Let  $f \in L_\alpha^p(\mathbb{B}_n, E)$  and  $g \in L_\alpha^{p'}(\mathbb{B}_n, E^*)$ . Then by Proposition 1.2.24 and Fubini's theorem, we have that

$$\begin{aligned} \langle P_\alpha f, g \rangle_{\alpha,E} &= \int_{\mathbb{B}_n} \langle P_\alpha f(z), g(z) \rangle_{E,E^*} d\nu_\alpha(z) = \int_{\mathbb{B}_n} g(z) (P_\alpha f(z)) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{g(z) (f(w))}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{g(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(z) \right) (f(w)) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} (P_\alpha g(w)) (f(w)) d\nu_\alpha(w) = \int_{\mathbb{B}_n} \langle f(w), P_\alpha g(w) \rangle_{E,E^*} d\nu_\alpha(w) \\ &= \langle f, P_\alpha g \rangle_{\alpha,E}. \end{aligned}$$

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Note that the application of Fubini's theorem is correct because of the Hölder's inequality and the boundedness of the scalar positive Bergman operator  $P_\alpha^+$ .  $\square$

Observe that Theorem 2.2.18 and Proposition 2.2.2 show that  $P_\alpha$  is a bounded projection of  $L_\alpha^p(\mathbb{B}_n, E)$  onto  $A_\alpha^p(\mathbb{B}_n, E)$  for any  $1 < p < \infty$  (that is  $P_\alpha : L_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^p(\mathbb{B}_n, E)$  is a bounded linear operator such that  $P_\alpha \circ P_\alpha = P_\alpha$  and  $P_\alpha L_\alpha^p(\mathbb{B}_n, E) = A_\alpha^p(\mathbb{B}_n, E)$ ).

As a consequence of that fact, we will obtain results on duality of the weighted vector-valued Bergman spaces. It is known that, for any  $1 < p < \infty$ , the dual space of  $L_\alpha^p(\mathbb{B}_n, E)$  can be identified with the space  $V^{p', \alpha}(\mathbb{B}_n, E^*)$  (see Theorem 1.3.23) where  $V_\alpha^{p'}(\mathbb{B}_n, E^*)$  is given by Definition 1.3.16 and  $p'$  is the conjugate exponent of  $p$ . If  $E^*$  has the Radon-Nikodym property with respect to the measure  $\nu_\alpha$ , see [29, p.98] (this condition is satisfied if, for example  $E^*$  is separable [29, p.79]), then the space  $V_\alpha^{p'}(\mathbb{B}_n, E^*)$  becomes  $L_\alpha^{p'}(\mathbb{B}_n, E^*)$ , where  $p'$  is the conjugate exponent of  $p$ . Therefore we identify the dual space of  $L_\alpha^p(\mathbb{B}_n, E)$  with the space  $L_\alpha^{p'}(\mathbb{B}_n, E^*)$ , where  $p'$  is the conjugate exponent of  $p$ , with some restriction on the Banach space  $E$ . Fortunately, for the weighted vector-valued Bergman spaces, we have the same duality structure without any restriction on the Banach space  $E$ . The case of the unit disc is proved in [4, Theorem 3.9] and the same proof work for the unit ball  $\mathbb{B}_n$ . For sake of the completeness, we will give the detail of the proof for the unit ball. Before that, we will introduce some preliminary notion.

**Definition 2.2.20.** (Bergman projection induced by vector measures) Let  $G \in V_\alpha^1(\mathbb{B}_n, E)$ . We define the Bergman projection of the vector measure  $G$  as the vector-valued analytic function in the unit ball  $\mathbb{B}_n$  given by:

$$P_\alpha G(z) = \int_{\mathbb{B}_n} \frac{dG(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \in E.$$

Since  $\sup_{w \in \mathbb{B}_n} \frac{1}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \leq \frac{1}{(1 - |z|)^{n+1+\alpha}}$ , it follows that  $P_\alpha G(z)$  is well defined. Actually since the series

$$K_z^\alpha(w) = \sum_{m \in \mathbb{N}^n} \frac{\Gamma(n+1+|m|+\alpha)}{m! \Gamma(n+1+\alpha)} z^m \bar{w}^m,$$

is absolutely convergent for each  $z \in \mathbb{B}_n$ , we have

$$P_\alpha G(z) = \sum_{m \in \mathbb{N}^n} x_m z^m,$$

where  $x_m = \frac{\Gamma(n+1+|m|+\alpha)}{m! \Gamma(n+1+\alpha)} \int_{\mathbb{B}_n} \bar{w}^m dG(w)$ .

We recall that the definition of the space  $V_\alpha^p(\mathbb{B}_n, E)$  is given in Definition 1.3.16.

**Theorem 2.2.21.** Let  $E$  be a complex Banach space and  $1 < p < \infty$ . Then the Bergman projection  $P_\alpha$  is bounded from  $V_\alpha^p(\mathbb{B}_n, E)$  onto  $A_\alpha^p(\mathbb{B}_n, E)$ .

*Proof.* Since  $G \in V_\alpha^p(\mathbb{B}_n, E)$ , by Lemma 1.3.18, there exists a nonnegative  $\phi \in L_\alpha^p(\mathbb{B}_n)$  such that  $d|G| = \phi d\nu_\alpha$  and  $\|\phi\|_{p, \alpha} = \|G\|_p$ . Now, for each  $z \in \mathbb{B}_n$ , we have

$$\|P_\alpha G(z)\|_E = \left\| \int_{\mathbb{B}_n} K^\alpha(z, w) dG(w) \right\|_E \leq \int_{\mathbb{B}_n} |K^\alpha(z, w)| d|G|(w) = \int_{\mathbb{B}_n} |K^\alpha(z, w)| \phi(w) d\nu_\alpha(w),$$

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where  $K^\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}$ . Now, to finish the proof, let us recall that if

$$P_\alpha^+ f(z) = \int_{\mathbb{B}_n} |K^\alpha(z, w)| f(w) d\nu_\alpha(w),$$

then  $P_\alpha^+ : L_\alpha^p(\mathbb{B}_n) \rightarrow L_\alpha^p(\mathbb{B}_n)$  defines a bounded operator for every  $1 < p < \infty$  (see for instance [77] or [75] for a proof). Therefore

$$\|P_\alpha G\|_{p,\alpha,E} \leq \|P_\alpha^+(\phi)\|_{L_\alpha^p} \leq C\|\phi\|_{L_\alpha^p} = C\|G\|_p.$$

□

Theorem 2.2.21, as in the scalar-valued case will be useful to get the duality result for vector-valued Bergman spaces.

### 2.2.5 Topological dual of $A_\alpha^p(\mathbb{B}_n, E)$ , with $1 < p < \infty$ .

We recall that the definition of the space  $V_\alpha^p(\mathbb{B}_n, E)$  is given in the Subsection 1.3.2.

**Theorem 2.2.22** (Duality of  $A_\alpha^p(\mathbb{B}_n, E)$  with  $p > 1$ ). Let  $1 < p < \infty$  and let  $E$  be a complex Banach space. If  $p'$  is the conjugate exponent of  $p$ , then  $(A_\alpha^p(\mathbb{B}_n, E))^*$  is isometrically isomorphic to  $A_\alpha^{p'}(\mathbb{B}_n, E^*)$ .

*Proof.* Let us define the linear operator  $J : A_\alpha^{p'}(\mathbb{B}_n, E^*) \rightarrow (A_\alpha^p(\mathbb{B}_n, E))^*$  defined by

$$J(g)(f) = \int_{\mathbb{B}_n} \langle g(z), f(z) \rangle_{E^*,E} d\nu_\alpha(z).$$

It follows from the Hölder's inequality that  $J$  is bounded with  $\|J\| \leq 1$ . Let us see that  $J$  is injective. Indeed, assume that  $g(z) = \sum_{\beta \in \mathbb{N}^n} x_\beta^* z^\beta \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$  is such that  $J(g) = 0$ . Hence if  $u_m(z) = \frac{\Gamma(n+1+|m|+\alpha)}{m!\Gamma(n+1+\alpha)} z^m$  then, for any  $m \in \mathbb{N}^n$  and  $x \in E$ , we have

$$\begin{aligned} 0 = J(g)(u_m \otimes x) &= \int_{\mathbb{B}_n} \langle g(z), u_m \otimes x(z) \rangle_{E^*,E} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle g(z), u_m(z)x \rangle_{E^*,E} d\nu_\alpha(z) \\ &= \left\langle \frac{\Gamma(n+1+|m|+\alpha)}{m!\Gamma(n+1+\alpha)} \int_{\mathbb{B}_n} g(z) \bar{z}^m d\nu_\alpha(z), x \right\rangle_{E^*,E} \\ &= \langle x_m^*, x \rangle_{E^*,E}, \end{aligned}$$

where  $u_m \otimes x$  is defined by  $(u_m \otimes x)(z) = u_m(z)x$ . It follows that all the Taylor coefficients of  $g$  are equal to zero. We conclude that  $g = 0$ .

Let us show that  $J$  is surjective. Given  $\xi \in (A_\alpha^p(\mathbb{B}_n, E))^*$ , the Hahn-Banach theorem gives an extension  $\tilde{\xi} \in (L_\alpha^p(\mathbb{B}_n, E))^*$  with the same norm. Using duality (see Theorem 1.3.24 and Definition 1.3.21) there exists a vector-valued measure  $G \in V_\alpha^{p'}(\mathbb{B}_n, E^*)$ , with  $p'$ -bounded variation equal to  $\|\tilde{\xi}\|$ , for which  $\tilde{\xi}(\phi) = \int_{\mathbb{B}_n} \langle dG, \phi \rangle_{E^*,E}$  for every  $\phi \in L_\alpha^p(\mathbb{B}_n, E)$ . Define  $g = P_\alpha G$ . From Theorem 2.2.21 we get  $g \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ . Let us see that  $J(g) = \xi$ . For  $f \in \mathcal{P}(\mathbb{B}_n, E)$  we can write  $f(z) = \sum_{|\beta| \leq N} x_\beta z^\beta$ , where each  $x_\beta \in E$ ,  $N \in \mathbb{N}$  and

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$z \in \mathbb{B}_n$ . By using the Fubini's theorem , it follows that

$$\begin{aligned}
J(g)(f) &= \int_{\mathbb{B}_n} \langle g(z), f(z) \rangle_{E^*, E} d\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} \langle P_\alpha G(z), f(z) \rangle_{E^*, E} d\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} \left\langle \int_{\mathbb{B}_n} K^\alpha(z, w) dG(w), f(z) \right\rangle_{E^*, E} d\nu_\alpha(z) \\
&= \sum_{|\beta| \leq N} \int_{\mathbb{B}_n} \left\langle \int_{\mathbb{B}_n} K^\alpha(z, w) dG(w), x_\beta z^\beta \right\rangle_{E^*, E} d\nu_\alpha(z) \\
&= \sum_{|\beta| \leq N} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} K^\alpha(z, w) \overline{z^\beta} \langle dG(w), x_\beta \rangle_{E^*, E} d\nu_\alpha(z) \\
&= \sum_{|\beta| \leq N} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} K^\alpha(z, w) z^\beta d\nu_\alpha(z) \langle dG(w), x_\beta \rangle_{E^*, E} \\
&= \sum_{|\beta| \leq N} \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} K^\alpha(w, z) z^\beta d\nu_\alpha(z) \right) \langle dG(w), x_\beta \rangle_{E^*, E} \\
&= \sum_{|\beta| \leq N} \int_{\mathbb{B}_n} \overline{w^\beta} \langle dG(w), x_\beta \rangle_{E^*, E} \\
&= \int_{\mathbb{B}_n} \langle dG(w), \sum_{|\beta| \leq N} x_\beta w^\beta \rangle_{E^*, E} \\
&= \int_{\mathbb{B}_n} \langle dG(w), f(w) \rangle_{E^*, E} \\
&= \xi(f).
\end{aligned}$$

□

**Remark 2.2.23.** For  $1 < p < \infty$  and  $\alpha > -1$ , the topological dual space of  $A_\alpha^p(\mathbb{B}_n, E)$  is identified with the space  $A_\alpha^{p'}(\mathbb{B}_n, E^*)$  under the pairing

$$\langle f, g \rangle_{\alpha, E} := \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z), \quad (2.50)$$

where  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $g \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ .

**Corollary 2.2.24.** Let  $1 < p < \infty$  and  $E$  be a reflexive Banach space. Then the weighted vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$  is a reflexive Banach space.

*Proof.* Assume that  $E$  is a reflexive Banach space. Then  $E^{**} \cong E$ . It follows by Theorem 2.2.22 that  $(A_\alpha^p(\mathbb{B}_n, E))^* \cong A_\alpha^{p'}(\mathbb{B}_n, E^*)$  and  $(A_\alpha^{p'}(\mathbb{B}_n, E^*))^* \cong A_\alpha^p(\mathbb{B}_n, E)$ . That is,  $A_\alpha^p(\mathbb{B}_n, E)$  is isometrically isomorphic to  $(A_\alpha^p(\mathbb{B}_n, E))^{**}$ . □

### 2.2.6 Several characterizations of $A_\alpha^p(\mathbb{B}_n, E)$ and Decomposition of $\mathbb{B}_n$ .

In this section, we present various characterizations of  $A_\alpha^p(\mathbb{B}_n, E)$  in terms of fractional differential operators and higher order derivatives.

We also need to recall the notion of (invariant) gradient. Let  $E$  be a Banach space and  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . Then

$$\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0), \quad z \in \mathbb{B}_n. \quad (2.51)$$

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and we call  $\|\tilde{\nabla}f(z)\|_{E^n}$  the invariant gradient of  $f$  at  $z \in \mathbb{B}_n$ , and  $\|\nabla f(z)\|_{E^n}$  is the holomorphic gradient of  $f$  at  $z \in \mathbb{B}_n$ , where  $\nabla$  is the "complex" gradient

$$\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right). \quad (2.52)$$

One of the most important property of the invariant gradient is the following:

$$\|\tilde{\nabla}(f \circ \varphi)(z)\|_{E^n} = \|\tilde{\nabla}(f) \circ \varphi(z)\|_{E^n}, \quad (2.53)$$

for all  $\varphi \in \text{Aut}(\mathbb{B}_n)$  and  $z \in \mathbb{B}_n$ .

In the scalar case, it is easy to see [77, Lemma 2.14] that

$$(1 - |z|^2)|\mathcal{N}f(z)| \leq (1 - |z|^2)|\nabla f(z)| \leq |\tilde{\nabla}f(z)|,$$

where  $\mathcal{N}f$  denotes the radial derivative of  $f$ . In the vector-valued case, the first inequality is still obvious using just Cauchy-Schwarz, see [52, (3.1.3)]. The second inequality is not trivial anymore. It is an example of a problem whose proof is simple in the scalar-valued setting and becomes much more difficult in the vector-valued setting.

**Lemma 2.2.25.** ([52, Theorem 3.2.6]) Let  $E$  be a Banach space and  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . For any  $z \in \mathbb{B}_n$ , we have

$$(1 - |z|^2)\|\mathcal{N}f(z)\|_E \leq (1 - |z|^2)\|\nabla f(z)\|_{E^n} \leq \|\tilde{\nabla}f(z)\|_{E^n}.$$

**Theorem 2.2.26.** Suppose  $0 < p < \infty$  and  $E$  is a Banach space. Then for every  $f \in \mathcal{H}(\mathbb{B}_n, E)$ , the following conditions are equivalent:

- (a)  $f \in A_\alpha^p(\mathbb{B}_n, E)$ .
- (b)  $\|\tilde{\nabla}f(z)\|_{E^n}$  is in  $L_\alpha^p(\mathbb{B}_n)$ .
- (c)  $(1 - |z|^2)\|\nabla f(z)\|_{E^n}$  is in  $L_\alpha^p(\mathbb{B}_n)$ .
- (d)  $(1 - |z|^2)\|\mathcal{N}f(z)\|_E$  is in  $L_\alpha^p(\mathbb{B}_n)$ .

*Proof.* Lemma 2.2.25 shows that (b) implies (c), and (c) implies (d).

To prove that (a) implies (b), we fix  $\beta > \alpha$  and observe (using Lemma 2.2.8 ) that there exists a constant  $c_1 > 0$  such that

$$\|\nabla g(0)\|_{E^n}^p \leq c_1 \int_{\mathbb{B}_n} \|g(w)\|_E^p d\nu_\beta(w)$$

for all  $g \in \mathcal{H}(\mathbb{B}_n, E)$ . Let  $g = f \circ \varphi_z$ , where  $z \in \mathbb{B}_n$ ; and  $\varphi_z$  is the holomorphic mapping of  $\mathbb{B}_n$  that interchanges 0 and  $z$ . Making a change of variables according to Proposition 2.1.12, we obtain

$$\|\tilde{\nabla}f(z)\|_{E^n}^p \leq c_1 (1 - |z|^2)^{n+1+\beta} \int_{\mathbb{B}_n} \frac{\|f(w)\|_E^p d\nu_\beta(w)}{|1 - \langle z, w \rangle|^{2(n+1+\beta)}}.$$

An application of Fubini theorem and part (ii) of Theorem 2.1.25 then gives

$$\int_{\mathbb{B}_n} \|\tilde{\nabla}f(z)\|_{E^n}^p d\nu_\alpha(z) \leq c_2 \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z),$$

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for some positive constant  $c_2$  and all  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . To prove that (d) implies (a), we assume that  $f \in \mathcal{H}(\mathbb{B}_n, E)$  is such that the function  $(1 - |z|^2)\|\mathcal{N}f(z)\|_E$  is in  $L^p_\alpha(\mathbb{B}_n)$ . Let  $\beta$  be a sufficiently large positive constant. Then by Proposition 2.2.2, we have

$$\mathcal{N}f(z) = \int_{\mathbb{B}_n} \frac{\mathcal{N}f(w)d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}, \quad z \in \mathbb{B}_n.$$

Since  $\mathcal{N}f(0) = 0$ , we have

$$\int_{\mathbb{B}_n} \mathcal{N}f(w) \left( \frac{1}{(1 - \langle z, w \rangle)^{n+1+\beta}} - 1 \right) d\nu_\beta(w), \quad z \in \mathbb{B}_n.$$

It follows that

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{N}f(tz)}{t} dt = \int_{\mathbb{B}_n} \mathcal{N}f(w)L(z, w)\nu_\beta(w),$$

where the kernel

$$L(z, w) = \int_0^1 \left( \frac{1}{(1 - \langle z, w \rangle)^{n+1+\beta}} - 1 \right) \frac{dt}{t}$$

satisfies

$$|L(z, w)| \leq \frac{c_3}{|1 - \langle z, w \rangle|^{n+\beta}},$$

for all  $z, w \in \mathbb{B}_n$  (see [77]). Thus,

$$\|f(z) - f(0)\|_E \leq c_4 \int_{\mathbb{B}_n} \frac{(1 - |w|^2)\|\mathcal{N}f(w)\|_E}{|1 - \langle z, w \rangle|^{n+\beta}} \nu_{\beta-1}(w). \quad (2.54)$$

Now if  $1 \leq p < \infty$  and  $\beta$  is large enough so that

$$0 < \alpha + 1 < p\beta,$$

then Theorem 2.2.17 shows that

$$\int_{\mathbb{B}_n} \|f(z) - f(0)\|_E^p d\nu_\alpha(z) \leq c_5 \int_{\mathbb{B}_n} \left( (1 - |z|^2)\|\mathcal{N}f(z)\|_E \right)^p d\nu_\alpha(z).$$

The case  $0 < p < 1$  calls for a different proof. We start from the inequality (2.54). Now assume  $\beta$  sufficiently large so that we can set

$$\beta = \frac{n+1+\beta'}{p} - (n+1),$$

with  $-1 < \alpha + p < \beta'$ . By Corollary 2.2.12, we have

$$\|f(z) - f(0)\|_E^p \leq c_6 \int_{\mathbb{B}_n} \left\| \frac{\mathcal{N}f(w)}{(1 - \langle z, w \rangle)^{n+\beta}} \right\|_E^p d\nu_{\beta'}(w),$$

where  $c_6 > 0$  is a constant independent of  $f$ . Using Fubini's theorem and part (ii) of Theorem 2.1.25, we then obtain

$$\int_{\mathbb{B}_n} \|f(z) - f(0)\|_E^p d\nu_\alpha(z) \leq c_7 \int_{\mathbb{B}_n} \left( (1 - |z|^2)\|\mathcal{N}f(z)\|_E \right)^p d\nu_\alpha(z).$$

This completes the proof of the theorem. □

**Remark 2.2.27.** We see that the proof of the above theorem is similar to the proof of Theorem 2.16 of [77] where we replace the modulus by the norm.

We also note that the proof of Theorem 2.2.26 actually produces equivalent norms on  $A^p_\alpha(\mathbb{B}_n, E)$  in terms

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of the radial derivative, the gradient, and the invariant gradient of  $f$ .

The interested reader can repeat the proof of the scalar-valued version of the following theorem in [77] by replacing the modulus by the norm.

**Theorem 2.2.28.** Suppose  $0 < p < \infty$ ,  $k$  is a positive integer and  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . Then  $f \in A_\alpha^p(\mathbb{B}_n, E)$  if and only if the functions

$$(1 - |z|^2)^k \frac{\partial^m f}{\partial z^m}(z), \quad |m| = k,$$

all belong to  $L_\alpha^p(\mathbb{B}_n, E)$ .

### Decomposition of the unit ball $\mathbb{B}_n$ .

Many technics in analysis involve covering lemmas, namely, ways to decompose the domain into special nice pieces. We now present a useful decomposition of the open unit ball  $\mathbb{B}_n$  into Bergman metric balls.

**Lemma 2.2.29.** ([77, Lemma 2.22]) Given any positive number  $R$  and natural number  $M$ , there exists a natural number  $N$  such that every Bergman metric ball of radius  $r$ , where  $r \leq R$ , can be covered by  $N$  Bergman metric balls of radius  $r/M$ .

Recall that for  $r > 0$  and  $z \in \mathbb{B}_n$ , the set

$$D(z, r) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}$$

is the Bergman metric ball at  $z$ .

**Theorem 2.2.30.** ([77, Theorem 2.23]) There exists a positive integer  $N$  such that for any  $0 < r \leq 1$  we can find a sequence  $\{a_k\}_k$  in  $\mathbb{B}_n$  with the following properties:

- (i)  $\mathbb{B}_n = \cup_{k=1}^\infty D(a_k, r)$ .
- (ii) The sets  $D(a_k, r/4)$  are mutually disjoint.
- (iii) Each point  $z \in \mathbb{B}_n$  belongs to at most  $N$  of the sets  $D(a_k, 4r)$ .

**Notation 2.2.31.** The constant  $r$  in Theorem 2.2.30 is called *separation constant* for the sequence  $\{a_k\}_k$  and the sequence  $\{a_k\}_k$  in Theorem 2.2.30 is called *r-lattice* in the Bergman metric.

# Vector-valued Bloch type spaces

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We discuss in this chapter some properties of vector-valued Bloch type spaces. We use here many arguments of scalar-valued Bloch type spaces. Many results in this chapter are also developed in [52].

## 3.1 Vector-valued Bloch space.

In this section, we study the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, E)$  and the little Bloch space  $\mathcal{B}_0(\mathbb{B}_n, E)$ . The Bloch space is prominent among Möbius invariant function spaces. In fact, it is the largest possible space of holomorphic functions whose (semi-)norm is invariant under the action of the automorphism group. We discuss here some properties of the vector-valued Bloch space. It is well-known that Bochner type spaces [78] play an important role in characterizing bounded Hankel operators on Bergman spaces [20, 52, 78].

In the classical function theory of the open unit disk  $\mathbb{D} = \mathbb{B}_1$  in the complex plane, the Bloch space is a central object of study.

**Definition 3.1.1.** The vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, E)$  is defined as the space of vector-valued holomorphic functions  $f \in \mathcal{H}(\mathbb{B}_n, E)$  such that

$$\|f\|_{\mathcal{B}(\mathbb{B}_n, E)} := \sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|\nabla f(z)\|_{E^n} < \infty.$$

$\|\cdot\|_{\mathcal{B}(\mathbb{B}_n, E)}$  is only a semi-norm, with  $\|f\|_{\mathcal{B}(\mathbb{B}_n, E)} = 0$  if and only if  $f$  is constant. We obtain a norm on  $\mathcal{B}(\mathbb{B}_n, E)$  by putting

$$\|f\| = \|f(0)\|_E + \|f\|_{\mathcal{B}(\mathbb{B}_n, E)}.$$

With this norm, we have the following result whose proof is obtained by replacing the modulus by the norm in the proof of the scalar-valued version.

**Proposition 3.1.2.** Endowed with the norm  $\|f\| = \|f(0)\|_E + \|f\|_{\mathcal{B}(\mathbb{B}_n, E)}$ , the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, E)$  becomes a Banach space.

**Lemma 3.1.3.** ([52, Lemma 3.1.1]) Suppose  $t \geq 1$ . If

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^\beta}, \quad z \in \mathbb{B}_n,$$

### 3.1. Vector-valued Bloch space.

for some  $g \in A_\alpha^1(\mathbb{B}_n, E)$ , then

$$\|\nabla f(z)\|_{E^n} \lesssim \int_{\mathbb{B}_n} \frac{\|R^{\alpha,t}g(w)\|_E d\nu_{\alpha+t}(w)}{|1 - \langle z, w \rangle|^{\beta+1}},$$

for every  $z \in \mathbb{B}_n$ .

Sometimes it will be very useful to have other characterizations of the vector-valued Bloch space.

**Theorem 3.1.4.** Suppose  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . Then the following conditions are equivalent:

- (a)  $f \in \mathcal{B}(\mathbb{B}_n, E)$ .
- (b)  $\sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|\mathcal{N}f(z)\|_E < \infty$ , where  $\mathcal{N}$  is the radial derivative given by (2.25).
- (c)  $f = P_\alpha g$  for some  $g \in L^\infty(\mathbb{B}_n, E)$ .

*Proof.* (a)  $\Rightarrow$  (b): By the Cauchy-Schwarz inequality, for every  $z \in \mathbb{B}_n$ , we have

$$(1 - |z|^2) \|\mathcal{N}f(z)\|_E \leq (1 - |z|^2) \|\nabla f(z)\|_{E^n}.$$

Thus,

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|\mathcal{N}f(z)\|_E \leq \sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|\nabla f(z)\|_{E^n} < \infty.$$

(b)  $\Rightarrow$  (c): Consider the function

$$g(z) := \frac{c_{\alpha+1}}{c_\alpha} (1 - |z|^2) \int_{\mathbb{B}_n} \frac{f(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+2+\alpha}}, \quad z \in \mathbb{B}_n.$$

We can rewrite

$$g(z) = \frac{c_{\alpha+1}}{c_\alpha} (1 - |z|^2) \int_{\mathbb{B}_n} \frac{1 - \langle z, w \rangle + \langle z, w \rangle}{(1 - \langle z, w \rangle)^{n+2+\alpha}} f(w) d\nu_\alpha(w),$$

and break the integral into two. The result is that

$$g(z) = \frac{c_{\alpha+1}}{c_\alpha} \left[ (1 - |z|^2) f(z) + \frac{(1 - |z|^2) \mathcal{N}f(z)}{n + 1 + \alpha} \right].$$

Since  $\sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|\mathcal{N}f(z)\|_E < \infty$ , to conclude that  $g \in L^\infty(\mathbb{B}_n, E)$ , it remains to show that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|f(z)\|_E < \infty.$$

But the identity

$$f(z) - f(0) = \int_0^1 \frac{\mathcal{N}f(tz)}{t} dt \tag{3.1}$$

shows that  $f$  grows at most as fast as  $-\log(1 - |z|^2)$ . Since  $\sup_{z \in \mathbb{B}_n} (1 - |z|^2) \log \frac{1}{1 - |z|^2} < \infty$ , it follows that  $g \in L^\infty(\mathbb{B}_n, E)$ . By Fubini's theorem and the reproducing property of  $P_\alpha$  and  $P_{\alpha+1}$ , we have that

$$\begin{aligned}
P_\alpha g(z) &= \int_{\mathbb{B}_n} \frac{g(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \\
&= \frac{c_{\alpha+1}}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \int_{\mathbb{B}_n} \frac{f(u) d\nu_\alpha(u)}{(1 - \langle w, u \rangle)^{n+2+\alpha}} d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \int_{\mathbb{B}_n} \frac{f(u)}{(1 - \langle w, u \rangle)^{n+2+\alpha}} d\nu_\alpha(u) d\nu_{\alpha+1}(w) \\
&= \int_{\mathbb{B}_n} f(u) \int_{\mathbb{B}_n} \frac{(1 - \langle w, z \rangle)^{-(n+1+\alpha)}}{(1 - \langle u, w \rangle)^{n+2+\alpha}} d\nu_{\alpha+1}(w) d\nu_\alpha(u) \\
&= \int_{\mathbb{B}_n} f(u) (1 - \langle u, z \rangle)^{-(n+1+\alpha)} d\nu_\alpha(u) = f(z).
\end{aligned}$$

(c)  $\Rightarrow$  (a): We finally assume that  $f = P_\alpha g$  for some  $g \in L^\infty(\mathbb{B}_n, E)$ . For  $z \in \mathbb{B}_n$ , we have

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w).$$

Now, differentiating under the integral sign we obtain that

$$\frac{\partial f}{\partial z_k} f(z) = (n + 1 + \alpha) \int_{\mathbb{B}_n} \frac{\bar{w}_k g(w)}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w),$$

and this implies that

$$\nabla f(z) = (n + 1 + \alpha) \int_{\mathbb{B}_n} \frac{\bar{w} g(w)}{(1 - \langle z, w \rangle)^{n+2+\alpha}} d\nu_\alpha(w).$$

By using Theorem 2.1.25, we get that

$$\|\nabla f(z)\|_{E^n} \leq \frac{\sqrt{n}(n + 1 + \alpha)}{(1 - |z|^2)} \|g\|_{\infty, E}.$$

Thus,

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|\nabla f(z)\|_E \lesssim \|g\|_{\infty, E} < \infty.$$

□

If  $f \in \mathcal{H}(\mathbb{B}_n, E)$  and  $s, t$  are such that neither  $n + s$  nor  $n + s + t$  is a negative integer, we define the functions

$$f_{s,t}(z) := (1 - |z|^2)^t R^{s,t} f(z), \quad z \in \mathbb{B}_n. \quad (3.2)$$

We will see that these functions play an important role on the vector-valued Bloch and Bergman spaces. Next result is an example.

**Theorem 3.1.5.** Let  $f \in \mathcal{H}(\mathbb{B}_n, E)$ ,  $\alpha > -1$ ,  $t > 0$  and  $0 < p < \infty$ . If neither  $n + s$  nor  $n + s + t$  is a negative integer, then  $f \in A_\alpha^p(\mathbb{B}_n, E)$  if and only if the function  $f_{s,t} \in L_\alpha^p(\mathbb{B}_n, E)$  (defined in (3.2)). Moreover,  $\|f\|_{p,\alpha,E} \simeq \|f_{s,t}\|_{p,\alpha,E}$ .

*Proof.* First assume that  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . Let  $\beta := s + N$ , where  $N$  is a sufficiently large positive integer. We have the following integral representation

$$f(z) = \int_{\mathbb{B}_n} \frac{f(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}, \quad z \in \mathbb{B}_n$$

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Applying the operator  $R^{s,t}$  inside the integral and using Lemma 2.1.19 and Lemma 2.2.7, we have

$$R^{s,t}f(z) = \int_{\mathbb{B}_n} \frac{f(w)h(\langle z, w \rangle)d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+t}} = \int_{\mathbb{B}_n} \frac{R^{\beta,t}f(w)h(\langle z, w \rangle)d\nu_{\beta+t}(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+t}}, \quad (3.3)$$

where  $h$  is a one-variable polynomial of degree  $N$ , and  $z \in \mathbb{B}_n$ . By using the first integral of (3.3), we have that

$$\|R^{s,t}f(z)\|_E \lesssim \int_{\mathbb{B}_n} \frac{\|f(w)\|_E d\nu_\beta(w)}{|1 - \langle z, w \rangle|^{n+1+\beta+t}},$$

and so

$$\|f_{s,t}(z)\|_E \lesssim (1 - |z|^2)^t \int_{\mathbb{B}_n} \frac{\|f(w)\|_E d\nu_\beta(w)}{|1 - \langle z, w \rangle|^{n+1+\beta+t}},$$

for all  $z \in \mathbb{B}_n$ .

If  $p \geq 1$ , and  $N$  is large enough so that

$$\alpha + 1 < p(\beta + 1),$$

then it follows by Theorem 2.2.17 that

$$\int_{\mathbb{B}_n} \|f_{s,t}(z)\|_E^p d\nu_\alpha(z) \lesssim \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z),$$

If  $0 < p < 1$ , we write

$$\beta = \frac{n + 1 + \alpha'}{p} - (n + 1).$$

Here we assume that  $N$  is large enough such that  $\alpha < \alpha'$ . Let

$$g_z(w) = \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\beta+t}}, \quad w \in \mathbb{B}_n.$$

Then by the assumption, we have  $g_z \in A_{\alpha'}^p(\mathbb{B}_n, E)$ . Using (3.3) and Corollary 2.2.12, we have

$$\begin{aligned} \|R^{s,t}f(z)\|_E^p &\lesssim \left( \int_{\mathbb{B}_n} \|g_z(w)\|_E d\nu_\beta(w) \right)^p \\ &\lesssim \left( \int_{\mathbb{B}_n} \|g_z(w)\|_E (1 - |w|^2)^{(n+1+\alpha')(\frac{1}{p}-1)} d\nu_{\alpha'}(w) \right)^p \\ &\lesssim \int_{\mathbb{B}_n} \|g_z(w)\|_E^p d\nu_{\alpha'}(w) \\ &\lesssim \int_{\mathbb{B}_n} \frac{\|f(w)\|_E^p d\nu_{\alpha'}(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha'+pt}}. \end{aligned}$$

An application of Fubini's theorem in combination of Theorem 2.1.25 shows that

$$\int_{\mathbb{B}_n} \|f_{s,t}(z)\|_E^p d\nu_\alpha(z) \leq c_5 \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z),$$

where  $c_5$  is a positive constant independent of  $f$ .

Next assume that  $f_{s,t} \in L_\alpha^p(\mathbb{B}_n, E)$ . Since  $R^{s,t}$  and  $R^{\beta,t}$  are comparable, see [77, p. 54], the function  $f_{\beta,t}$  also belongs to  $L_\alpha^p(\mathbb{B}_n, E)$ . By [77, Corollary 2.3], Fubini's theorem and the reproducing formula in Proposition

### 3.1. Vector-valued Bloch space.

2.2.2 we have that

$$\begin{aligned} P_\beta f_{\beta,t}(z) &= \int_{\mathbb{B}_n} \frac{f_{\beta,t}(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}} \\ &= \frac{c_\beta}{c_{\beta+t}} \int_{\mathbb{B}_n} \frac{R^{\beta,t} f(w) d\nu_{\beta+t}(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}} \\ &= \frac{c_\beta}{c_{\beta+t}} \lim_{r \rightarrow 1} \int_{\mathbb{B}_n} \frac{f_r(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}} = \frac{c_\beta}{c_{\beta+t}} f(z). \end{aligned}$$

It follows that

$$f(z) = \int_{\mathbb{B}_n} \frac{R^{\beta,t} f(w) d\nu_{\beta+t}(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}. \quad (3.4)$$

If  $1 \leq p < \infty$ , then by Theorem 2.2.18, we have that

$$\|f\|_{p,\alpha,E} = \frac{c_{\beta+t}}{c_\beta} \|P_\beta f_{\beta,t}\|_{p,\alpha,E} \leq \frac{c_{\beta+t}}{c_\beta} \|P_\beta\| \|f_{\beta,t}\|_{p,\alpha,E},$$

showing that  $f \in A_\alpha^p(\mathbb{B}_n, E)$ .

When  $0 < p < 1$ , By (3.4), we have

$$\|f(z)\|_E \lesssim \int_{\mathbb{B}_n} \|g_z(w)\|_E (1 - |w|^2)^{\beta+t} d\nu(w), \quad z \in \mathbb{B}_n$$

where

$$g_z(w) = \frac{R^{\beta,t} f(w)}{(1 - \langle w, z \rangle)^{n+1+\beta}}, \quad w \in \mathbb{B}_n.$$

We assume that  $N$  is large enough so that

$$\beta + t = \frac{n+1+\alpha'}{p} - (n+1),$$

for some  $\alpha' > -1$ . Observe that

$$(n+1+\beta)p = n+1+\alpha' - pt = n+1+\alpha + (\alpha' - pt - \alpha),$$

and we assume that  $N$  is so large that

$$\alpha' - pt - \alpha > 0.$$

Then by the assumption, we have  $g_z \in A_{\alpha'}^p(\mathbb{B}_n, E)$  ( $R^{\beta,t} f \in A_{\alpha+pt}^p(\mathbb{B}_n, E) \subset A_{\alpha'}^p(\mathbb{B}_n, E)$ ). Therefore using Corollary 2.2.12, we have

$$\begin{aligned} \|f(z)\|_E^p &\lesssim \left( \int_{\mathbb{B}_n} \|g_z(w)\|_E (1 - |w|^2)^{\beta+t} d\nu(w) \right)^p \\ &\lesssim \left( \int_{\mathbb{B}_n} \|g_z(w)\|_E (1 - |w|^2)^{(n+1+\alpha')(\frac{1}{p}-1)} d\nu_{\alpha'}(w) \right)^p \\ &\lesssim \int_{\mathbb{B}_n} \|g_z(w)\|_E^p d\nu_{\alpha'}(w) \\ &= \int_{\mathbb{B}_n} \frac{\|R^{\beta,t} f(w)\|_E^p}{|1 - \langle z, w \rangle|^{n+1+\alpha'-pt}} d\nu_{\alpha'}(w). \end{aligned}$$

Using Fubini's Theorem and Theorem 2.1.25, we deduce that

$$\int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) \lesssim \int_{\mathbb{B}_n} \|f_{\beta,t}(z)\|_E^p d\nu_\alpha(z).$$

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Thus, the proof of the theorem is completed.  $\square$

It is not always easy to show that a given function belongs to the vector-valued Bloch space. The following theorem give equivalent conditions for a function to be in the vector-valued Bloch space.

**Theorem 3.1.6.** ([52, Theorem 3.1.3]) Suppose  $t > 0$ . If  $f \in \mathcal{H}(\mathbb{B}_n, E)$ , then the following conditions are equivalent:

- (i) The function  $f$  is in  $\mathcal{B}(\mathbb{B}_n, E)$ .
- (ii) We have  $f = P_\alpha g$  for some  $g \in L^\infty(\mathbb{B}_n, E)$ .
- (iii) The function  $f_{\alpha,t}(z) = (1 - |z|^2)^t R^{\alpha,t} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ .

Moreover,

$$\|f\|_{\mathcal{B}(\mathbb{B}_n, E)} \simeq \|g\|_{\infty, E} \simeq \|f_{\alpha,t}\|_{\infty, E}.$$

Now we continue with some properties of vector-valued Bloch spaces.

**Lemma 3.1.7.** Let  $0 < p < \infty$ . Then  $\mathcal{B}(\mathbb{B}_n, E) \subset A_\alpha^p(\mathbb{B}_n, E)$ .

*Proof.* This is immediate from Theorem 3.1.5 and Theorem 3.1.6.  $\square$

Next result gives a pointwise estimate for vector-valued Bloch functions.

**Proposition 3.1.8.** ([52, Proposition 3.1.5]) If  $f \in \mathcal{B}(\mathbb{B}_n, E)$ , then there exists  $C > 0$  such that

$$\|f(z)\|_E \leq C \|f\|_{\mathcal{B}(\mathbb{B}_n, E)} \log \frac{1}{1 - |z|^2},$$

for any  $z \in \mathbb{B}_n$ .

To finish this section, we recall that  $\mathcal{B}(\mathbb{B}_n, E^*)$  is the topological dual space of  $A_\alpha^1(\mathbb{B}_n, E)$ . This generalize the similar result in the scalar-valued case.

**Theorem 3.1.9.** ([52, Theorem 3.1.6]) Let  $E$  be a Banach space. The space  $(A_\alpha^1(\mathbb{B}_n, E))^*$  can be identified with  $\mathcal{B}(\mathbb{B}_n, E^*)$  under the integral pairing given by

$$\langle f, g \rangle_{\alpha, E} = \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z), \quad (3.5)$$

where  $f \in \mathcal{P}(\mathbb{B}_n, E)$  which is a dense subspace of  $A_\alpha^1(\mathbb{B}_n, E)$  and  $g \in \mathcal{B}(\mathbb{B}_n, E^*)$ .

A completely different proof of Theorem 3.1.9 in the unit disk can be seen in [4, Corollary 3.17].

#### 3.1.1 Little vector-valued Bloch space.

The vector-valued Bloch space is not separable. In this section, we discuss a separable subspace of the vector-valued Bloch space, the little vector-valued Bloch space. As in the scalar-valued case, this space is useful in the study of the compactness of operators on vector-valued Bergman spaces.

### 3.2. Vector-valued $\gamma$ -Bloch space.

The little vector-valued Bloch space  $\mathcal{B}_0(\mathbb{B}_n, E)$  is the subspace of  $\mathcal{B}(\mathbb{B}_n, E)$  consisting of those functions  $f : \mathbb{B}_n \rightarrow E$  for which

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \|\mathcal{N}f(z)\|_E = 0.$$

The following proposition is a particular case of Proposition 3.3.17 in page 84.

**Proposition 3.1.10.** Let  $f \in \mathcal{B}(\mathbb{B}_n, E)$ . The following conditions are equivalent:

- (i)  $f \in \mathcal{B}_0(\mathbb{B}_n, E)$ .
- (ii)  $\lim_{s \rightarrow 1^-} \|f - f_s\|_{\mathcal{B}(\mathbb{B}_n, E)} = 0$ , where  $f_s$  is the dilation function defined for  $z \in \mathbb{B}_n$  by  $f_s(z) := f(sz)$ .
- (iii)  $f$  belongs to the closure of  $\mathcal{P}(\mathbb{B}_n, E)$ , where  $\mathcal{P}(\mathbb{B}_n, E)$  is the set of all holomorphic polynomials on the unit ball with coefficients in  $E$ .

## 3.2 Vector-valued $\gamma$ -Bloch space.

In this section we are going to define and characterize the more general type of vector-valued Bloch spaces, the  $\gamma$ -Bloch spaces  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$ .

**Definition 3.2.1.** For  $\gamma > 0$ ,  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$  is the space of vector-valued holomorphic functions  $f \in \mathcal{H}(\mathbb{B}_n, E)$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \left\| \frac{\partial f}{\partial z_k}(z) \right\|_E < \infty,$$

for any  $1 \leq k \leq n$ . It is simple to see that the vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$  consists of all vector-valued holomorphic functions  $f \in \mathcal{H}(\mathbb{B}_n, E)$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \|\mathcal{N}f(z)\|_E < \infty.$$

It is clear that  $\mathcal{B}_1(\mathbb{B}_n, E) = \mathcal{B}(\mathbb{B}_n, E)$ . Therefore, the vector-valued  $\gamma$ -Bloch spaces with  $\gamma > 0$  extend the vector-valued Bloch space. As in the scalar case, it is well-known that  $f \in \mathcal{H}(\mathbb{B}_n, E)$  belongs to  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$  if and only if

$$\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)} := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \|\nabla f(z)\|_{E^n} < \infty,$$

where

$$\|\nabla f(z)\|_{E^n} := \left( \sum_{i=1}^n \left\| \frac{\partial f}{\partial z_i}(z) \right\|_E^2 \right)^{1/2}.$$

Taking  $\|f\| = \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)} + \|f(0)\|_E$  we have a norm. With this norm, it can be shown that  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$  is a Banach space (see [77] for the scalar-valued case).

**Definition 3.2.2.** Let  $\gamma > 0$ . The little vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_{\gamma,0}(\mathbb{B}_n, E)$  is the subspace of  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$  consisting of functions  $f$  such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\gamma \|\mathcal{N}f(z)\|_E = 0.$$

We also have some equivalent characterizations of these spaces more suitable for our purpose.

### 3.2. Vector-valued $\gamma$ -Bloch space.

**Theorem 3.2.3.** ([52, Theorem 3.2.1]) Let  $E$  be a Banach space. Let  $\gamma > 0$ ,  $\beta > -1$  such that  $\gamma + \beta > 0$ . If  $f \in \mathcal{H}(\mathbb{B}_n, E)$  then the following conditions are equivalent:

- (i)  $f \in \mathcal{B}_\gamma(\mathbb{B}_n, E)$ .
- (ii) There exists a function  $g \in L^\infty(\mathbb{B}_n, E)$  such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+\beta+\gamma}}, \quad z \in \mathbb{B}_n.$$

- (iii) The function  $(1 - |z|^2)^\gamma R^{\beta+\gamma-1,1} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ . Moreover,

$$\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)} \simeq \|g\|_{\infty, E} \simeq \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \|R^{\beta+\gamma-1,1} f(z)\|_E.$$

Next lemma gives more general criteria of functions in  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$ .

**Lemma 3.2.4.** ([52, Lemma 3.2.2]) Let  $\alpha > -1$  and  $\gamma > 0$  such that  $\alpha - \gamma > -2$ . If  $f \in \mathcal{H}(\mathbb{B}_n, E)$  then the following conditions are equivalent:

- (a) The function  $(1 - |z|^2)^\gamma R^{\alpha,1} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ ;
- (b) The function  $(1 - |z|^2)^{\gamma+t} R^{\alpha,t+1} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$  for some  $t \geq 0$ .

Moreover,

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \|R^{\alpha,1} f(z)\|_E \simeq \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{\gamma+t} \|R^{\alpha,t+1} f(z)\|_E.$$

As a corollary we get the following result which is actually the most used in what follows.

**Corollary 3.2.5.** Let  $\alpha > -1$  and  $\gamma > 0$  such that  $\alpha - \gamma > -2$ . If  $f \in \mathcal{H}(\mathbb{B}_n, E)$  then  $f \in \mathcal{B}_\gamma(\mathbb{B}_n, E)$  if and only if the function  $(1 - |z|^2)^{\gamma+t} R^{\alpha,t+1} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ , for some  $t \geq 0$ . Moreover,

$$\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)} \simeq \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{\gamma+t} \|R^{\alpha,t+1} f(z)\|_E.$$

*Proof.* Notice that with the same notations of above theorems, if we set  $\alpha := \beta + \gamma - 1$ , condition  $\beta > -1$  is equivalent to say that  $\alpha - \gamma > -2$  and condition  $\gamma + \beta > 0$  is equivalent to say that  $\alpha > -1$ . This means that, in fact, under these conditions, statement (iii) of Theorem 3.2.3 is equivalent to statement (a) of Lemma 3.2.4. So, Theorem 3.2.3 and Lemma 3.2.4 give the result.  $\square$

As a consequence we also have some basic properties of these spaces.

**Corollary 3.2.6.** Let  $\gamma > 0$  and  $0 < p < \infty$ . Then  $\mathcal{B}_\gamma(\mathbb{B}_n, E) \subset A_\alpha^p(\mathbb{B}_n, E)$  for all  $\alpha > -1$  with  $\alpha > p(\gamma - 1) - 1$ .

*Proof.* Let  $f \in \mathcal{B}_\gamma(\mathbb{B}_n, E)$  and  $0 < p < \infty$  and  $\alpha > -1$ . By Theorem 3.1.5 and Theorem 3.2.3, we have that

$$\begin{aligned} \|f\|_{p, \alpha, E} &\simeq \left( \int_{\mathbb{B}_n} (1 - |z|^2)^{\gamma p + (1-\gamma)p} \|R^{\alpha,1} f(z)\|_E^p d\nu_\alpha(z) \right)^{1/p} \\ &\lesssim \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)} \left( \int_{\mathbb{B}_n} (1 - |z|^2)^{\alpha + (1-\gamma)p} d\nu(z) \right)^{1/p}. \end{aligned}$$

The condition  $\alpha > p(\gamma - 1) - 1$  ensures that the last integral is finite.  $\square$

### 3.2. Vector-valued $\gamma$ -Bloch space.

Then we also have the following pointwise estimate for vector-valued  $\gamma$ -Bloch functions.

**Proposition 3.2.7.** ([52, Proposition 3.2.5]) Suppose  $\gamma > 1$ . If  $f \in \mathcal{B}_\gamma(\mathbb{B}_n, E)$  then there exists  $C > 0$  such that

$$\|f(z)\|_E \leq \frac{C\|f\|}{(1-|z|^2)^{\gamma-1}},$$

where  $\|f\| = \|f(0)\|_E + \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)}$  and  $z \in \mathbb{B}_n$ .

Note that in the particular case of  $z = 0$  we have that  $\|f(0)\|_E \lesssim \|f\|$ , for any  $\gamma > 0$ . We have already seen the pointwise estimate of vector-valued  $\gamma$ -Bloch functions for the particular case of  $\gamma = 1$  in Proposition 3.1.8.

We need another characterization of the  $\gamma$ -Bloch spaces in terms of the invariant gradient. Before that we need two new results about the vector-valued invariant gradient.

**Lemma 3.2.8.** ([52, Theorem 3.2.7]) Suppose  $\beta > 0$  and  $t \geq 0$ . If

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\alpha(w)}{(1-\langle z, w \rangle)^\beta}, \quad z \in \mathbb{B}_n,$$

for some  $g \in A_\alpha^1(\mathbb{B}_n, E)$ , then

$$\|\tilde{\nabla} f(z)\|_{E^n} \leq \sqrt{2}\beta(1-|z|^2)^{1/2} \int_{\mathbb{B}_n} \frac{\|R^{\alpha,t}g(w)\|_E d\nu_{\alpha+t}(w)}{|1-\langle z, w \rangle|^{\beta+\frac{1}{2}}}$$

for every  $z \in \mathbb{B}_n$ .

Finally, the last characterization of the vector-valued  $\gamma$ -Bloch space is the following.

**Theorem 3.2.9.** ([52, Theorem 3.2.8]) Let  $E$  be a Banach space. If  $f \in \mathcal{H}(\mathbb{B}_n, E)$  and  $\gamma > \frac{1}{2}$ , then  $f \in \mathcal{B}_\gamma(\mathbb{B}_n, E)$  if and only if the function

$$(1-|z|^2)^{\gamma-1} \|\tilde{\nabla} f(z)\|_{E^n}$$

is in  $L^\infty(\mathbb{B}_n, E)$ . Moreover,  $\|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)} \simeq \sup_{z \in \mathbb{B}_n} (1-|z|^2)^{\gamma-1} \|\tilde{\nabla} f(z)\|_{E^n}$ .

To finish this section we have the following estimates for functions in the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, E)$ , you can follow the same proof of [4, Corollary 5.3], but we include another simpler proof for completeness.

**Corollary 3.2.10.** ([52, Corollary 3.2.9]) If  $f \in \mathcal{B}(\mathbb{B}_n, E)$ , then there exists  $C > 0$  such that

$$\|f(z) - f(w)\|_E \leq C\beta(z, w)\|f\|_{\mathcal{B}(\mathbb{B}_n, E)},$$

for any  $z, w \in \mathbb{B}_n$ .

In the case where  $0 < \gamma < 1$ , it is more convenient to introduce the vector-valued Lipschitz spaces.

## 3.3 Vector-valued Lipschitz spaces.

In this section, we introduce and study some properties of the vector-valued holomorphic Lipschitz spaces  $\Lambda_\gamma(\mathbb{B}_n, E)$  and we see, as in the scalar-valued case, that when  $0 < \gamma \leq 1$ , we have that  $\mathcal{B}_\gamma(\mathbb{B}_n, E) = \Lambda_{1-\gamma}(\mathbb{B}_n, E)$ . Our approach here is the same as in [77] and in [71]. Some results like integral representation, estimates in terms of various derivatives are developed in this section.

**Definition 3.3.1.** Let  $0 < \gamma < 1$ . The *vector-valued Lipschitz space*  $\Lambda_\gamma(\mathbb{B}_n, E)$  is the space of vector-valued holomorphic function  $f : \mathbb{B}_n \rightarrow E$  such that

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n, E)} = \sup \left\{ \frac{\|f(z) - f(w)\|_E}{|z - w|^\gamma} : z, w \in \mathbb{B}_n, z \neq w \right\} < \infty.$$

As in the scalar-valued case, we have that a vector-valued holomorphic function  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$  if and only if

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{1-\gamma} \|\mathcal{N}f(z)\|_E < \infty.$$

It is clear that each space  $\Lambda_\gamma(\mathbb{B}_n, E)$  contains vector-valued holomorphic polynomials. Endowed with the norm  $\|f\| = \|f(0)\|_E + \|f\|_{\Lambda_\gamma(\mathbb{B}_n, E)}$ ,  $\Lambda_\gamma(\mathbb{B}_n, E)$  becomes a Banach space (see [77] for the scalar-valued case).

The proof of the following theorem is similar to the proof of [77, Theorem 7.9] in the scalar-valued case.

**Theorem 3.3.2.** Suppose  $0 < \gamma < 1, \beta > -1$  and  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . Then the following conditions are equivalent:

- (a)  $f$  is in  $\Lambda_\gamma(\mathbb{B}_n, E)$ .
- (b)  $f$  is in the ball algebra and its boundary values satisfy

$$\sup \left\{ \frac{\|f(\zeta) - f(\xi)\|_E}{|\zeta - \xi|^\gamma} : \zeta, \xi \in \mathbb{S}_n, \zeta \neq \xi \right\} < \infty.$$

- (c) The function  $(1 - |z|^2)^{1-\gamma} \|\mathcal{N}f(z)\|_E$  is bounded in  $\mathbb{B}_n$ .
- (d) There exists a function  $g \in L^\infty(\mathbb{B}_n, E)$  such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\gamma}},$$

for all  $z \in \mathbb{B}_n$ .

- (e) The function  $(1 - |z|^2)^{1-\gamma} \|\nabla f(z)\|_{E^n}$  is bounded in  $\mathbb{B}_n$ .

**Remark 3.3.3.** Let  $\gamma \in (0, 1)$ . As consequence of Theorem 3.3.2, we have that

$$\mathcal{B}_{1-\gamma}(\mathbb{B}_n, E) = \Lambda_\gamma(\mathbb{B}_n, E)$$

with equivalent norms. Therefore, the following pointwise estimate holds

$$\|f(z) - f(w)\|_E \leq C \|f\| |z - w|^{1-\gamma},$$

### 3.3. Vector-valued Lipschitz spaces.

for all  $f \in \mathcal{B}_\gamma(\mathbb{B}_n, E)$ , and  $z, w \in \mathbb{B}_n$ . In particular, for  $w = 0$ , we have that

$$\|f(z)\|_E \leq C \|f\| |z|^{1-\gamma},$$

where  $\|f\| = \|f(0)\|_E + \|f\|_{\mathcal{B}_\gamma(\mathbb{B}_n, E)}$ , and  $|z| \geq \frac{1}{2}$ . The previous inequality means that the point evaluation is a bounded linear functional on the vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, E)$ , with a norm that is uniformly bounded on each compact subset of  $\mathbb{B}_n$ . We have  $\mathcal{B}_\gamma(\mathbb{B}_n, E) \subset H^\infty(\mathbb{B}_n, E)$ , the space of all vector-valued bounded holomorphic functions on  $\mathbb{B}_n$ .

In the section, we extend the theory of vector-valued Lipschitz spaces  $\Lambda_\gamma(\mathbb{B}_n, E)$  to the full range  $\gamma \in \mathbb{R}$ .

**Definition 3.3.4.** Let  $\gamma \in \mathbb{R}$ . We let  $\Lambda_\gamma(\mathbb{B}_n, E)$  to be the space of vector-valued holomorphic functions  $f : \mathbb{B}_n \rightarrow E$  for which, there exists an integer  $k > \gamma$  such that

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n, E)} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|\mathcal{N}^k f(z)\|_E < \infty,$$

where  $\mathcal{N}^k = \mathcal{N} \circ \mathcal{N} \circ \dots \circ \mathcal{N}$   $k$ -times and with the norm

$$\|f\|_{\gamma, E} = \|f(0)\|_E + \|f\|_{\Lambda_\gamma(\mathbb{B}_n, E)},$$

the space  $\Lambda_\gamma(\mathbb{B}_n, E)$  becomes a Banach space. The space  $\Lambda_\gamma(\mathbb{B}_n, E)$  will be called the *vector-valued holomorphic Lipschitz space*.

We first prove that the definition of  $\Lambda_\gamma(\mathbb{B}_n, E)$  is independent of the integer  $k$  used.

**Lemma 3.3.5.** Suppose  $f$  is in  $\mathcal{H}(\mathbb{B}_n, E)$ . Then the following conditions are equivalent:

- (a) There exists some nonnegative integer  $k > \gamma$  such that the function

$$(1 - |z|^2)^{k-\gamma} \mathcal{N}^k f(z)$$

is bounded in  $\mathbb{B}_n$ .

- (b) For every nonnegative integer  $k > \gamma$  the function

$$(1 - |z|^2)^{k-\gamma} \mathcal{N}^k f(z)$$

is bounded in  $\mathbb{B}_n$ .

*Proof.* Suppose  $k$  is a nonnegative integer with  $k > \gamma$ . Let  $k' = k + 1$ . If the function  $(1 - |z|^2)^{k'-\gamma} \mathcal{N}^{k'} f(z)$  is bounded in  $\mathbb{B}_n$ , then the function  $\mathcal{N}^{k'} f(z) \in A_\beta^1(\mathbb{B}_n, E)$ , with  $\beta = k' - \gamma$  and by Proposition 2.2.2 we have

$$\mathcal{N}^{k'} f(z) = \int_{\mathbb{B}_n} \frac{\mathcal{N}^{k'} f(w)}{(1 - \langle z, w \rangle)^{n+\beta+1}} d\nu_\beta(w).$$

Since  $\mathcal{N}^{k'} f(0) = 0$ , we obtain that

$$\mathcal{N}^{k'} f(z) = \int_{\mathbb{B}_n} \mathcal{N}^{k'} f(w) \left( \frac{1}{(1 - \langle z, w \rangle)^{n+\beta+1}} - 1 \right) d\nu_\beta(w).$$

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Hence

$$\begin{aligned}
 \mathcal{N}^k f(z) - \mathcal{N}^k f(0) &= \int_0^1 \frac{\mathcal{N}^{k'} f(tz)}{t} dt \\
 &= \int_0^1 \int_{\mathbb{B}_n} \mathcal{N}^{k'} f(w) \left( \frac{1}{(1-t\langle z, w \rangle)^{n+\beta+1}} - 1 \right) d\nu_\beta(w) \frac{dt}{t} \\
 &= \int_{\mathbb{B}_n} \mathcal{N}^{k'} f(w) \int_0^1 \left( \frac{1}{(1-t\langle z, w \rangle)^{n+\beta+1}} - 1 \right) \frac{dt}{t} d\nu_\beta(w) \\
 &= \int_{\mathbb{B}_n} \mathcal{N}^{k'} f(w) L(z, w) d\nu_\beta(w),
 \end{aligned}$$

where

$$L(z, w) = \int_0^1 \left( \frac{1}{(1-t\langle z, w \rangle)^{n+\beta+1}} - 1 \right) \frac{dt}{t}.$$

By [77] there is a constant  $C > 0$  such that

$$|L(z, w)| \leq \frac{C}{|1 - \langle z, w \rangle|^{n+\beta}}.$$

It follows that

$$\begin{aligned}
 \|\mathcal{N}^k f(z)\|_E &= \left| \int_{\mathbb{B}_n} \mathcal{N}^{k'} f(w) L(z, w) d\nu_\beta(w) \right| \\
 &\leq \int_{\mathbb{B}_n} \|\mathcal{N}^{k'} f(w)\|_E |L(z, w)| d\nu_\beta(w) \\
 &\leq C \int_{\mathbb{B}_n} \frac{\|\mathcal{N}^{k'} f(w)\|_E}{|1 - \langle z, w \rangle|^{n+\beta}} d\nu_\beta(w) \\
 &\lesssim \int_{\mathbb{B}_n} \frac{d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+k-\gamma}} \\
 &\lesssim \frac{1}{(1 - |z|^2)^{k-\gamma}}.
 \end{aligned}$$

Thus,  $(1 - |z|^2)^{k-\gamma} \|\mathcal{N}^k f(z)\|_E$  is bounded in  $\mathbb{B}_n$ . Conversely, if the function  $(1 - |z|^2)^{k-\gamma} \mathcal{N}^k f(z)$  is bounded in  $\mathbb{B}_n$ , then the function  $\mathcal{N}^k f \in A_{k-\gamma}^1(\mathbb{B}_n, E)$  and by Proposition 2.2.2 we have

$$\mathcal{N}^k f(z) = c_{k-\gamma} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{k-\gamma} \mathcal{N}^k f(w) d\nu(w)}{(1 - \langle z, w \rangle)^{n+1+k-\gamma}}.$$

Taking the radial derivative in both sides, we get

$$\mathcal{N}^{k'} f(z) = c' c_{k-\gamma} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{k-\gamma} \mathcal{N}^k f(w) \langle z, w \rangle d\nu(w)}{(1 - \langle z, w \rangle)^{n+1+k'-\gamma}}.$$

It follows, by Theorem 2.1.25, that

$$\|\mathcal{N}^{k'} f(z)\|_E \lesssim \int_{\mathbb{B}_n} \frac{d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+k'-\gamma}} \lesssim \frac{1}{(1 - |z|^2)^{k'-\gamma}}.$$

Thus,  $(1 - |z|^2)^{k'-\gamma} \|\mathcal{N}^{k'} f(z)\|_E$  is bounded in  $\mathbb{B}_n$ .

Therefore, the function  $(1 - |z|^2)^{k-\gamma} \mathcal{N}^k f(z)$  is bounded in  $\mathbb{B}_n$  if and only if the function  $(1 - |z|^2)^{k+1-\gamma} \mathcal{N}^{k+1} f(z)$  is bounded in  $\mathbb{B}_n$ , where  $k$  is any nonnegative integer satisfying  $k > \gamma$ . This clearly proves the desired result.  $\square$

### 3.3. Vector-valued Lipschitz spaces.

For  $\gamma = 0$ , we write  $\Lambda_0(\mathbb{B}_n, E) = \mathcal{B}(\mathbb{B}_n, E)$ .

**Lemma 3.3.6.** Let  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . The following conditions are equivalent:

(a) There exists a nonnegative integer  $k > \gamma$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E < \infty.$$

(b) For every nonnegative integer  $k > \gamma$  we have

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E < \infty.$$

*Proof.* It is clear that (b)  $\Rightarrow$  (a). So to complete the proof, we will prove that (a)  $\Rightarrow$  (b). Suppose that there exists an integer  $k > \gamma$  such that

$$c := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E < \infty.$$

We want to prove that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha, k+1} f(z)\|_E < \infty.$$

Since  $c < \infty$ , then  $f \in A_{\beta}^1(\mathbb{B}_n, E)$ , where  $\beta = \alpha + N$ , and  $N$  is a sufficiently large integer. Indeed, if  $N$  is large enough so that  $\beta + \gamma > -1$ , by Theorem 3.1.5, we have that

$$\begin{aligned} \|f\|_{1, \beta, E} &\simeq \int_{\mathbb{B}_n} (1 - |z|^2)^k \|R^{\alpha, k} f(z)\|_E d\nu_{\beta}(z) \\ &= \int_{\mathbb{B}_n} [(1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E] (1 - |z|^2)^{\gamma} d\nu_{\beta}(z) \\ &\lesssim \int_{\mathbb{B}_n} (1 - |z|^2)^{\beta+\gamma} d\nu(z) \\ &< \infty. \end{aligned}$$

By using Lemma 2.1.19, we have that

$$R^{\alpha, k+1} f(z) = \int_{\mathbb{B}_n} \frac{f(w) h(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{n+1+\beta+k+1}} d\nu_{\beta}(w),$$

where  $h$  is a one-variable polynomial of degree  $N$  and  $\beta = \alpha + N$ . Applying Lemma 2.2.7 it follows that

$$R^{\alpha, k+1} f(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha, k} f(w) h(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{n+1+\beta+k+1}} d\nu_{\beta+k}(w).$$

If  $N$  is large enough such that  $\beta + \gamma > -1$ , we have

$$\begin{aligned} \|R^{\alpha, k+1} f(z)\|_E &\lesssim \int_{\mathbb{B}_n} \frac{[(1 - |w|^2)^{k-\gamma} \|R^{\alpha, k} f(w)\|_E] (1 - |w|^2)^{\beta+\gamma}}{|1 - \langle z, w \rangle|^{n+1+\beta+\gamma+(k+1-\gamma)}} d\nu(w) \\ &\lesssim \frac{c}{(1 - |z|^2)^{k+1-\gamma}}. \end{aligned}$$

Therefore, we have that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha, k+1} f(z)\|_E \lesssim c < \infty.$$

### 3.3. Vector-valued Lipschitz spaces.

Also, if  $k$  is a nonnegative integer with  $k > \gamma$  such that

$$c' := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha, k+1} f(z)\|_E < \infty,$$

then

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E < \infty.$$

Applying Proposition 2.2.3 and Lemma 2.2.7 we have that

$$R^{\alpha, k} f(z) = \int_{\mathbb{B}_n} \frac{f(w)h(\langle z, w \rangle) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+k}} = \int_{\mathbb{B}_n} \frac{R^{\alpha, k+1} f(w)h(\langle z, w \rangle) d\nu_{\beta+k+1}(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+k}},$$

where  $h$  is a one-variable polynomial of degree  $N$  and  $\beta = \alpha + N$ , and  $z \in \mathbb{B}_n$ . By using Theorem 2.1.25 and the fact that  $\beta + \gamma > -1$ , it follows that

$$\begin{aligned} \|R^{\alpha, k} f(z)\|_E &\lesssim \int_{\mathbb{B}_n} \frac{[(1 - |w|^2)^{k+1-\gamma} \|R^{\alpha, k+1} f(w)\|_E] (1 - |w|^2)^{\beta+\gamma} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\beta+k}} \\ &= c' \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\beta+\gamma} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\beta+\gamma+(k-\gamma)}} \\ &\lesssim \frac{c'}{(1 - |z|^2)^{k-\gamma}}. \end{aligned}$$

Since  $z \in \mathbb{B}_n$  is arbitrary, we obtain that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E \lesssim c' < \infty.$$

□

**Corollary 3.3.7.** Let  $0 < t < \infty$ ,  $\gamma_0 \in \mathbb{R}$  and  $\alpha$  such that  $\alpha + \gamma_0 t > -1$ . Then  $\Lambda_{\gamma_0}(\mathbb{B}_n, E) \subset A_\alpha^t(\mathbb{B}_n, E)$ .

*Proof.* Let  $k > \gamma_0$ . Applying Theorem 3.1.5, for  $f \in \Lambda_{\gamma_0}(\mathbb{B}_n, E)$ , we have that

$$\begin{aligned} \|f\|_{t, \alpha, E}^t &\simeq \int_{\mathbb{B}_n} [(1 - |z|^2)^k \|R^{\alpha, k} f(z)\|_E]^t d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} [(1 - |z|^2)^{k-\gamma_0} \|R^{\alpha, k} f(z)\|_E]^t (1 - |z|^2)^{\gamma_0 t} d\nu_\alpha(z) \\ &\lesssim \|f\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, E)} \int_{\mathbb{B}_n} (1 - |z|^2)^{\alpha+\gamma_0 t} d\nu(z) < \infty. \quad \square \end{aligned}$$

**Theorem 3.3.8.** Suppose  $s$  and  $\gamma$  are real such that neither  $n + s$  nor  $n + s + \gamma$  is a negative integer. Then  $R^{s, \gamma}$  map  $\Lambda_\gamma(\mathbb{B}_n, E)$  onto  $\mathcal{B}(\mathbb{B}_n, E)$ .

*Proof.* Suppose  $f \in \mathcal{H}(\mathbb{B}_n, E)$ . Then the function  $R^{s, \gamma} f$  is in the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, E)$  if and only if the function  $\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^k \|N^k(R^{s, \gamma} f)(z)\|_E < \infty$ , where  $k$  is any positive integer, (see Lemma 3.3.5). If  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ , then the function

$$g(z) = (1 - |z|^2)^{k-\gamma} N^k f(z)$$

is in  $L^\infty(\mathbb{B}_n, E)$ , where  $k > \gamma$  is a positive integer. Let  $N$  be a sufficiently large positive integer such that

$$k - \gamma + \beta = s + N > -1.$$

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Then we use Proposition 2.2.2 to write

$$\mathcal{N}^k f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k-\gamma+\beta}}, \quad z \in \mathbb{B}_n.$$

Applying Lemma 2.1.19, we obtain a polynomial  $h$  such that

$$R^{s,\gamma} \mathcal{N}^k f(z) = \int_{\mathbb{B}_n} \frac{g(w) h(\langle z, w \rangle) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k+\beta}}, \quad z \in \mathbb{B}_n.$$

Since all the radial differential operators commute, we have

$$(1 - |z|^2)^k R^{s,\gamma} \mathcal{N}^k f(z) = (1 - |z|^2)^k \mathcal{N}^k R^{s,\gamma} f(z), \quad z \in \mathbb{B}_n.$$

By Theorem 2.1.25, the function  $(1 - |z|^2)^k \mathcal{N}^k R^{s,\gamma} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ . On the other hand, if  $g \in \mathcal{B}(\mathbb{B}_n, E)$ , we search for a function  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$  such that  $R^{s,\gamma} f = g$ . Since  $g \in \mathcal{B}(\mathbb{B}_n, E)$ , then  $R^{s,\gamma} f$  will belong to the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, E)$ . By Lemma 2.1.17, the function  $R_{s+\gamma, -\gamma} f \in \mathcal{B}(\mathbb{B}_n, E)$ . We fix a sufficiently large positive integer  $N$  such that the real  $\beta = N + s + \gamma > -1$ . By part (c) of Theorem 3.1.4, there exists a function  $g \in L^\infty(\mathbb{B}_n, E)$  such that

$$R_{s+\gamma, -\gamma} f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta}}.$$

We apply the operator  $R^{s+\gamma, -\gamma}$  to both sides and use Lemma 2.1.19 to obtain that

$$f(z) = \int_{\mathbb{B}_n} \frac{p(\langle z, w \rangle) g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\gamma}},$$

where  $p$  is a polynomial. An easy computation then shows that

$$\mathcal{N}^k f(z) = \int_{\mathbb{B}_n} \frac{q(\langle z, w \rangle) g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+k-\gamma}},$$

where  $k$  is any positive integer greater than  $\gamma$  and  $q$  is another polynomial. By Theorem 2.1.25, the function  $(1 - |z|^2)^{k-\gamma} \mathcal{N}^k f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ , namely,  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ .  $\square$

Given a positive integer  $k$ , we define the differential operator  $D_k$  by

$$D_k := (2I + \mathcal{N}) \circ (3I + \mathcal{N}) \circ \dots \circ ((k+1)I + \mathcal{N}), \quad (3.6)$$

where  $I$  is the identity operator and  $\mathcal{N}$  is the differential operator given in (2.25).

In the sequel, we denote by  $\mathcal{P}(\mathbb{B}_n, E)$  the space of all vector-valued holomorphic polynomials. The proof of the following lemma is similar as in the scalar case in [40], we omit the details.

**Lemma 3.3.9.** For all  $f \in \mathcal{P}(\mathbb{B}_n, E)$  and  $g \in \mathcal{P}(\mathbb{B}_n, E^*)$ , we have the following identity

$$\int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z),$$

where  $c_k$  is a positive constant depending only on the integer  $k$ . The above identities are valid for vector-valued holomorphic functions when both sides make sense.

The following lemma will be very useful in the sequel.

### 3.3. Vector-valued Lipschitz spaces.

**Lemma 3.3.10.** Let  $\{a_k\}$  a sequence of positive numbers. For any positive integer  $k$ , let  $M_k$  the differential operator of order  $k$  defined by

$$M_k := (a_0 I + \mathcal{N}) \circ (a_1 I + \mathcal{N}) \circ \dots \circ (a_{k-1} I + \mathcal{N}). \quad (3.7)$$

Then a vector-valued holomorphic function  $f$  belongs to  $\Lambda_\gamma(\mathbb{B}_n, E)$  if and only if there exists an integer  $k > \gamma$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|M_k f(z)\|_E < \infty.$$

*Proof.* Let us assume first that  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ , and we prove the desired estimate on  $M_k$ . By assumption, there exists an integer  $k > \gamma$  and a positive constant  $C$  such that

$$\|\mathcal{N}^k f(z)\|_E \leq C(1 - |z|^2)^{\gamma-k},$$

for any  $z \in \mathbb{B}_n$ . It is enough to prove that the following inequality

$$\|\mathcal{N}^j f(z)\|_E < C(1 - |z|^2)^{\gamma-k},$$

holds for  $0 \leq j < k$ , since the assumption give the case  $j = k$ . For  $g \in \mathcal{H}(\mathbb{B}_n, E)$  and  $z = rz'$ , where  $r = |z|$ , and  $z'$  is in the unit sphere. We have

$$\mathcal{N}g(rz') = r\partial_r g(rz').$$

Thus,

$$g(rz') - g(z'/2) = \int_{\frac{1}{2}}^r \mathcal{N}g(sz') \frac{ds}{s}.$$

Now, for  $g \in \mathcal{H}(\mathbb{B}_n, E)$  such that  $\|\mathcal{N}g(z)\|_E \leq C(1 - |z|^2)^{\gamma-k}$ . We have that

$$\begin{aligned} \|g(rz') - g(z'/2)\|_E &\leq 2 \int_{\frac{1}{2}}^r \|\mathcal{N}g(sz')\|_E ds \\ &\leq 4C \int_{\frac{1}{2}}^r (1 - s^2)^{\gamma-k} s ds \\ &= -2C \int_{\frac{1}{2}}^r -2s(1 - s^2)^{\gamma-k} ds \\ &= \left[ \frac{-2C}{\gamma - k + 1} (1 - s^2)^{\gamma-k+1} \right]_{\frac{1}{2}}^r \\ &= \frac{-2C}{\gamma - k + 1} \left\{ (1 - r^2)^{\gamma-k+1} - (1 - \frac{1}{4})^{\gamma-k+1} \right\}. \end{aligned}$$

Now, if  $\gamma - k + 1 < 0$ , then

$$\|g(rz') - g(z'/2)\|_E \leq \frac{-2C}{\gamma - k + 1} (1 - r^2)^{\gamma-k} = C_{k,\gamma} (1 - |z|^2)^{\gamma-k}.$$

If  $\gamma - k + 1 > 0$ , then

$$\begin{aligned} \|g(rz') - g(z'/2)\|_E &\leq \frac{2C}{\gamma - k + 1} \left\{ (1 - \frac{1}{4})^{\gamma-k+1} - (1 - r^2)^{\gamma-k+1} \right\} \\ &\leq \frac{2C}{\gamma - k + 1} (1 - \frac{1}{4})^{\gamma-k+1} = C'_{k,\gamma} \\ &\leq C'_{k,\gamma} (1 - |z|^2)^{\gamma-k}, \end{aligned}$$

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where the last inequality is justified using the fact that  $(1 - |z|^2)^{\gamma-k} > 1$ . It then follows that

$$\|g(z)\|_E \leq C(1 - |z|^2)^{\gamma-k}.$$

Now, we use this fact inductively for  $g = \mathcal{N}^k f$ , then  $g = \mathcal{N}^{k-1} f, \dots$  to conclude. Conversely, assume that there exists an integer  $k > \gamma$  and a positive constant  $C$  such that

$$\|M_k f(z)\|_E \leq C(1 - |z|^2)^{\gamma-k},$$

for any  $z \in \mathbb{B}_n$ . To conclude, it is sufficient to prove that for a fixed positive real  $a$ , the inequality

$$\|ag(z) + \mathcal{N}g(z)\|_E \leq C(1 - |z|^2)^{\gamma-k} \quad (3.8)$$

implies the inequality

$$\|\mathcal{N}g(z)\|_E \leq C(1 - |z|^2)^{\gamma-k},$$

for any function  $g \in \mathcal{H}(\mathbb{B}_n, E)$ . Choose a real  $\beta$  such that  $\beta + \gamma - k > -1$ . By the assumption (3.8), we have that

$$\int_{\mathbb{B}_n} \|ag(z) + \mathcal{N}g(z)\|_E (1 - |z|^2)^\beta d\nu(z) < \infty.$$

Thus, for any  $z \in \mathbb{B}_n$ , we have

$$ag(z) + \mathcal{N}g(z) = c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + \mathcal{N}g(w)]}{(1 - \langle z, w \rangle)^{n+1+\beta}} (1 - |w|^2)^\beta d\nu(w).$$

Then, differentiating under the integral sign, we obtain that for all  $1 \leq i \leq n$ , we get

$$\partial_{z_i} [ag(z) + \mathcal{N}g(z)] = (n + 1 + \beta)c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + \mathcal{N}g(w)]\bar{w}_i}{(1 - \langle z, w \rangle)^{n+2+\beta}} (1 - |w|^2)^\beta d\nu(w).$$

Therefore,

$$\mathcal{N}(ag(z) + \mathcal{N}g(z)) = (n + 1 + \beta)c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + \mathcal{N}g(w)]\langle z, w \rangle}{(1 - \langle z, w \rangle)^{n+2+\beta}} (1 - |w|^2)^\beta d\nu(w).$$

Applying (3.8), and Theorem 2.1.25 we get that for all  $1 \leq i \leq n$ ,

$$\begin{aligned} \|\mathcal{N}(ag(z) + \mathcal{N}g(z))\|_E &\leq Cc_\beta \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\gamma-k+\beta}}{|1 - \langle z, w \rangle|^{n+1+\gamma-k+\beta+(k-\gamma+1)}} d\nu(w) \\ &\leq C(1 - |z|^2)^{\gamma-k-1}. \end{aligned}$$

Thus, the derivative of  $ag(z) + \mathcal{N}g(z)$  is bounded by  $(1 - |z|^2)^{\gamma-k-1}$ . So, to prove the inequality above, we are reduced to consider smooth functions  $\phi$  of one variable  $r \in [0, 1)$ , and to prove that the inequality

$$\|\psi'(r)\|_E \leq C(1 - r)^{\gamma-k-1},$$

with  $\psi(r) = a\phi(r) + r\phi'(r)$ , implies that

$$\|r\phi'(r)\|_E \leq C(1 - r)^{\gamma-k}$$

(here,  $\phi(r) = g(rz')$ ). Now, differentiating  $\psi$ , we obtain  $\psi'(s) = (a + 1)\phi'(s) + s\phi''(s)$ . Multiplying both

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sides of the previous inequality by  $s^a$ , we obtain that  $s^a\psi'(s) = (a+1)s^a\phi'(s) + s^{a+1}\phi''(s) = [s^{a+1}\phi'(s)]'$ . Then integrating the equality above on  $[0, r]$ , we obtain that

$$\phi'(r) = \frac{1}{r^{a+1}} \int_0^r s^a \psi'(s) ds.$$

Therefore, the desired estimate follows at once, since  $k > \gamma$ .  $\square$

**Remark 3.3.11.** We shall use extensively this lemma for two particular classes of differential operators: the class  $D_k$ , and the class  $L_k$ , corresponding to the choice  $a_j = n + \alpha + j + 1$ . For this choice, we have

$$(a_j I + \mathcal{N})(1 - \langle z, w \rangle)^{-n-\alpha-j-1} = \frac{n + \alpha + j + 1}{(1 - \langle z, w \rangle)^{n+\alpha+j+2}},$$

and inductively,

$$L_k(1 - \langle z, w \rangle)^{-n-\alpha-1} = \frac{c_k}{(1 - \langle z, w \rangle)^{n+\alpha+k+1}}.$$

The proof of Lemma 3.3.10 allows us to define an equivalent norm of  $f$  in terms of  $M_k f$ . Particularly, we will write the equivalent norm of  $f$  in terms of  $D_k f$  and  $L_k f$ . More precisely, we have the following result:

**Corollary 3.3.12.** Let  $D_k$  a differential operator of order  $k$  defined in (3.6) and  $L_k$  a differential operator of order  $k$  defined in Remark 3.3.11. For  $f \in \mathcal{H}(\mathbb{B}_n, E)$ , the following assertions are equivalent.

- (1)  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ .
- (2) There exists an integer  $k > \gamma$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k f(z)\|_E < \infty.$$

- (3) There exists an integer  $k > \gamma$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|L_k f(z)\|_E < \infty.$$

Moreover, the following are equivalent

$$\begin{aligned} \|f\|_{\gamma, E} &\simeq \|f(0)\|_E + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k f(z)\|_E \\ &\simeq \|f(0)\|_E + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|L_k f(z)\|_E. \end{aligned}$$

**Theorem 3.3.13.** Let  $\gamma \in \mathbb{R}$  and  $\beta > -1$  such that  $n + \beta - \gamma$  is not a negative integer. For  $f \in \mathcal{H}(\mathbb{B}_n, E)$ , the following conditions are equivalent.

- (i) The function  $f$  is in  $\Lambda_\gamma(\mathbb{B}_n, E)$ .
- (ii) There exists a function  $g \in L^\infty(\mathbb{B}_n, E)$  such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\gamma}}.$$

*Proof.* (i)  $\Rightarrow$  (ii) First assume that  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ . By Theorem 3.3.8 the function  $R^{\beta-\gamma, \gamma} f$  is in the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, E)$ . According to the integral representation in Theorem 3.1.4, there exists

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a function  $g \in L^\infty(\mathbb{B}_n, E)$  such that

$$R^{\beta-\gamma, \gamma} f(z) := P_\beta g(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+(\beta-\gamma)+\gamma}}. \quad (3.9)$$

Applying the operator  $R_{\beta-\gamma, \gamma}$  to both sides of (3.9) and using Proposition 2.1.20 we conclude that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\gamma}}, \quad z \in \mathbb{B}_n.$$

(ii)  $\Rightarrow$  (i). Let  $g \in L^\infty(\mathbb{B}_n, E)$  such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\gamma}}, \quad z \in \mathbb{B}_n.$$

For any nonnegative integer  $k > \gamma$ , we have

$$\mathcal{N}^k f(z) = \int_{\mathbb{B}_n} \frac{p(\langle z, w \rangle) g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+k-\gamma}}, \quad (3.10)$$

where  $p$  is a polynomial of degree  $k$  and  $z \in \mathbb{B}_n$ . Using Theorem 2.1.25 in (3.10), we get

$$\|\mathcal{N}^k f(z)\|_E \lesssim \frac{1}{(1 - |z|^2)^{k-\gamma}},$$

that is  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ . □

**Theorem 3.3.14.** Let  $\gamma$  and  $t$  be two real such that  $t > \gamma$ . Let  $s$  be a real such that neither  $n + s$  nor  $n + s + t$  is a negative integer. For  $f \in \mathcal{H}(\mathbb{B}_n, E)$ , the following conditions are equivalent.

- (i) The function  $f$  is in  $\Lambda_\gamma(\mathbb{B}_n, E)$ .
- (ii) The function  $(1 - |z|^2)^{t-\gamma} R^{s,t} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ .

*Proof.* (i)  $\Rightarrow$  (ii) First assume that  $f$  is in  $\Lambda_\gamma(\mathbb{B}_n, E)$ . By Theorem 3.3.13, there exists  $g \in L^\infty(\mathbb{B}_n, E)$  such that

$$f(z) = \int_{\mathbb{B}_n} \frac{g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta-\gamma}},$$

where  $z \in \mathbb{B}_n$ , and  $\beta$  is a sufficiently large positive real such that  $\beta - \gamma = s + N$ , for some positive integer  $N$ . We have, using Lemma 2.1.19

$$R^{s,t} f(z) = \int_{\mathbb{B}_n} \frac{p(\langle z, w \rangle) g(w) d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+\beta+t-\gamma}}, \quad z \in \mathbb{B}_n.$$

using Theorem 2.1.25

$$\|R^{s,t} f(z)\|_E \lesssim \frac{1}{(1 - |z|^2)^{t-\gamma}},$$

that is  $(1 - |z|^2)^{t-\gamma} R^{s,t} f(z) \in L^\infty(\mathbb{B}_n, E)$ .

(ii)  $\Rightarrow$  (i) Conversely, we assume that the function  $g(z) := (1 - |z|^2)^{t-\gamma} R^{s,t} f(z)$  is in  $L^\infty(\mathbb{B}_n, E)$ . Let  $\beta$  be a

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sufficiently large positive real. The function  $R^{s,t}f \in A_{t-\gamma+\beta}^1(\mathbb{B}_n, E)$ . Indeed, we have that

$$\begin{aligned} R^{s,t}f(z) &= \int_{\mathbb{B}_n} \frac{R^{s,t}f(w)}{(1 - \langle z, w \rangle)^{n+1+t-\gamma+\beta}} d\nu_{t-\gamma+\beta}(w) \\ &= \int_{\mathbb{B}_n} \frac{g(w)(1 - |w|^2)^{\gamma-t}}{(1 - \langle z, w \rangle)^{n+1+t-\gamma+\beta}} d\nu_{t-\gamma+\beta}(w) \\ &= c_{t-\gamma+\beta} \int_{\mathbb{B}_n} \frac{g(w)(1 - |w|^2)^\beta}{(1 - \langle z, w \rangle)^{n+1+t-\gamma+\beta}} d\nu(w). \end{aligned} \quad (3.11)$$

Using Theorem 2.1.25, it follows that

$$\|R^{s,t}f(z)\|_E \lesssim \frac{1}{(1 - |z|^2)^{t-\gamma}}, \quad z \in \mathbb{B}_n.$$

By (3.11) we have

$$R^{s,t}f(z) = \frac{c_{t-\gamma+\beta}}{c_\beta} \int_{\mathbb{B}_n} \frac{g(w)d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+t-\gamma+\beta}}, \quad z \in \mathbb{B}_n.$$

If  $k > \gamma$  is a nonnegative integer, we observe that there exists a polynomial  $p$  such that

$$\mathcal{N}^k R^{s,t}f(z) = \frac{c_{t-\gamma+\beta}}{c_\beta} \int_{\mathbb{B}_n} \frac{g(w)d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k+t+\beta-\gamma}}, \quad z \in \mathbb{B}_n. \quad (3.12)$$

Since  $\beta$  is sufficiently large positive real, let  $N$  be a sufficiently large positive integer such that  $\beta - \gamma = s + N$ . apply the operator  $R_{s,t}$  to both sides of (3.12), and using Lemma 2.1.19, we get

$$R_{s,t}\mathcal{N}^k R^{s,t}f(z) = \frac{c_{t-\gamma+\beta}}{c_\beta} \int_{\mathbb{B}_n} \frac{p(\langle z, w \rangle)g(w)d\nu_\beta(w)}{(1 - \langle z, w \rangle)^{n+1+k+\beta-\gamma}}, \quad (3.13)$$

where  $p$  is a one-variable polynomial of degree  $N$  and  $z \in \mathbb{B}_n$ . Since all radial differential operators commute, we have

$$R_{s,t}\mathcal{N}^k R^{s,t} = \mathcal{N}^k. \quad (3.14)$$

Applying Theorem 2.1.25 in (3.13) and using 3.14, we obtain

$$\|\mathcal{N}^k f(z)\|_E \lesssim \frac{1}{(1 - |z|^2)^{k-\gamma}}, \quad z \in \mathbb{B}_n.$$

That is  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ . □

**Remark 3.3.15.** According to Theorem 3.3.14, given  $\gamma \geq 0$ , we observe that the vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, E)$  consists of functions  $f \in \mathcal{H}(\mathbb{B}_n, E)$  such that for some nonnegative integer  $k > \gamma$ , we have

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n, E)} \simeq \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E < \infty.$$

**Definition 3.3.16.** The *little vector-valued Lipschitz space*  $\Lambda_{\gamma,0}(\mathbb{B}_n, E)$  is the subspace of  $\Lambda_\gamma(\mathbb{B}_n, E)$ , which consists of functions  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$  such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_E = 0. \quad (3.15)$$

When  $\gamma = 0$  and  $k = 1$ , then

$$\Lambda_{0,0}(\mathbb{B}_n, E) = \mathcal{B}_0(\mathbb{B}_n, E).$$

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The little vector-valued Lipschitz space  $\Lambda_{\gamma,0}(\mathbb{B}_n, E)$  is the closure of the set of vector-valued polynomial  $\mathcal{P}(\mathbb{B}_n, E)$ . We have the following important result.

**Proposition 3.3.17.** Let  $\gamma \geq 0$  and  $f \in \Lambda_\gamma(\mathbb{B}_n, E)$ . The following conditions are equivalent.

- (i)  $f \in \Lambda_{\gamma,0}(\mathbb{B}_n, E)$ .
- (ii)  $\lim_{s \rightarrow 1^-} \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, E)} = 0$ , where  $f_s$  is the dilation function defined for  $z \in \mathbb{B}_n$  by  $f_s(z) := f(sz)$ .
- (iii)  $f$  belongs to the closure of  $\mathcal{P}(\mathbb{B}_n, E)$ , where  $\mathcal{P}(\mathbb{B}_n, E)$  is the space of vector-valued holomorphic polynomials.

*Proof.* (i)  $\Rightarrow$  (ii). Suppose that  $\frac{1}{2} < r < s < 1$ , and let  $f_s(z) = f(sz)$ ,  $z \in \mathbb{B}_n$ . By the definition, we have:

$$\begin{aligned}
 \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, E)} &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}(f - f_s)(z)\|_E \\
 &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}f_s)(z)\|_E \\
 &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_E \\
 &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z) \\
 &\quad + \chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_E \\
 &\leq \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_E \\
 &\quad + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_E,
 \end{aligned}$$

where  $\chi_r$  is the characteristic function of the set  $\{|z| \leq r\}$ . We first have the following estimate:

$$\begin{aligned}
 \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_E &\leq \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_E \\
 &\quad + \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_E \\
 &= \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z)\|_E \\
 &\leq \sup_{r^2 < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z)\|_E.
 \end{aligned}$$

Secondly, we have following estimate,

$$\begin{aligned}
 \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_E &\leq \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_E \\
 &\quad + \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_E \\
 &= \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_E \\
 &\quad + \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(sz)\|_E.
 \end{aligned}$$

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Using the change of variables  $w = sz$ , we then obtain

$$\begin{aligned} \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(sz)\|_E &= \sup_{rs < |w| < s} \left(1 - \frac{|w|^2}{s^2}\right)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_E \\ &= \sup_{rs < |w| < s} \frac{1}{s^{2(k-\gamma)}} (s^2 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_E \\ &\leq 2^{2(k-\gamma)} \sup_{r^2 < |w| < 1} (1 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_E. \end{aligned}$$

It follows by using the assumption that

$$\begin{aligned} \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, E)} &\leq C_\gamma \sup_{r^2 < |w| < 1} (1 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_E \\ &\quad + \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_E, \end{aligned}$$

with  $C_\gamma = 1 + 2^{2(k-\gamma)}$ . Since  $(R^{\alpha,k} f)(sz) \rightarrow (R^{\alpha,k} f)(z)$  in  $E$  uniformly on the compact set  $\{|z| \leq r\}$  as  $s \rightarrow 1^-$ , we have

$$\lim_{s \rightarrow 1^-} \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_E = 0.$$

It follows that

$$\lim_{s \rightarrow 1^-} \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, E)} \leq C_\gamma \limsup_{|w| \rightarrow 1^-} (1 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_E = 0.$$

(ii)  $\Rightarrow$  (iii). Given  $\epsilon > 0$ , by the assumption, there exists  $s_0 \in (0, 1)$  such that

$$\|f - f_{s_0}\|_{\Lambda_\gamma(\mathbb{B}_n, E)} < \epsilon. \quad (3.16)$$

Further note that  $f_{s_0} \in \mathcal{H}(\frac{1}{s_0}\mathbb{B}_n, E)$  and  $1 < \frac{2}{1+s_0} < \frac{1}{s_0}$ . From this, and by using Taylor's formula, it follows that for each  $m \in \mathbb{N}$ , there exists a  $E$ -valued polynomial  $p_m$  such that

$$\lim_{m \rightarrow \infty} \sup_{z \in \frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - p_m(z)\|_E = 0.$$

Therefore, there exists  $m_0 \in \mathbb{N}$  such that

$$\sup_{z \in \frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - p_m(z)\|_E < \epsilon, \quad (3.17)$$

for  $m \geq m_0$ . By the Cauchy's inequality, there exists a constant  $c_{s_0} > 0$  such that for each  $i = 1, \dots, n$  we have

$$\sup_{z \in \overline{\mathbb{B}}_n} \left\| \frac{\partial f_{s_0}}{\partial z_i}(z) - \frac{\partial p_m}{\partial z_i}(z) \right\|_E \leq c_{s_0} \sup_{z \in \frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - p_m(z)\|_E. \quad (3.18)$$

Suppose  $k$  is a nonnegative integer with  $k > \gamma$ . By using (3.18) and Proposition 2.1.21, there is a constant  $c = c(s_0, n, \alpha, k)$  such that

$$\sup_{z \in \overline{\mathbb{B}}_n} \|(R^{\alpha,k} f_{s_0})(z) - (R^{\alpha,k} p_{m_0})(z)\|_E \leq c \sup_{z \in \frac{2}{1+s_0}\overline{\mathbb{B}}_n} \|f_{s_0}(z) - p_{m_0}(z)\|_E. \quad (3.19)$$

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It follows from (3.19) and (3.17) that

$$\begin{aligned} \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}(f_{s_0} - p_{m_0})(z)\|_E &\leq \sup_{z \in \mathbb{B}_n} \|(R^{\alpha,k}f_{s_0})(z) - (R^{\alpha,k}p_{m_0})(z)\|_E \\ &\leq c \sup_{z \in \frac{2}{1+s_0}\mathbb{B}_n} \|f_{s_0}(z) - p_{m_0}(z)\|_E < c\epsilon. \end{aligned}$$

Thus

$$\|f_{s_0} - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n, E)} < c\epsilon. \quad (3.20)$$

Using (3.16) and (3.20), it follows that

$$\begin{aligned} \|f - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n, E)} &\leq \|f - f_{s_0}\|_{\Lambda_\gamma(\mathbb{B}_n, E)} + \|f_{s_0} - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n, E)} \\ &< \epsilon + c\epsilon = (1 + c)\epsilon. \end{aligned}$$

(iii)  $\Rightarrow$  (i). Let  $f$  in the closure of the set of vector-valued polynomial  $\mathcal{P}(\mathbb{B}_n, E)$  in  $\Lambda_\gamma(\mathbb{B}_n, E)$ . There exists a sequence of vector-valued polynomials  $\{p_m\}$  in  $\mathcal{P}(\mathbb{B}_n, E)$  such that

$$\lim_{m \rightarrow \infty} \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, E)} = 0. \quad (3.21)$$

Let us prove that for each  $k > \gamma$ ,

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z)\|_E = 0.$$

Let  $k > \gamma$ . We have that

$$\begin{aligned} \|(R^{\alpha,k}f)(z)\|_E &\leq \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}p_m)(z)\|_E + \|(R^{\alpha,k}p_m)(z)\|_E \\ &\leq \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}p_m)(z)\|_E + \|R^{\alpha,k}p_m\|_{\infty, E}, \end{aligned}$$

where  $\|R^{\alpha,k}p_m\|_{\infty, E} = \max_{z \in \mathbb{B}_n} \|(R^{\alpha,k}p_m)(z)\|_E$ . It follows that for each  $m \in \mathbb{N}$ , we have

$$\begin{aligned} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z)\|_E &\leq (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}p_m)(z)\|_E \\ &\quad + (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}p_m\|_{\infty, E} \\ &\leq \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, E)} + (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}p_m\|_{\infty, E}. \end{aligned}$$

Letting  $|z| \rightarrow 1^-$ , we obtain that

$$\limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}f(z)\|_E \leq \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, E)},$$

for each  $m \in \mathbb{N}$ . Now, letting  $m \rightarrow \infty$  on both sides of the previous inequality, it follows by (3.21) that

$$\limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}f(z)\|_E = 0.$$

□

**Remark 3.3.18.** One of consequence of Proposition 3.3.17 is that, given  $\gamma \geq 0$ , the little vector-valued Lipschitz space  $\Lambda_{\gamma,0}(\mathbb{B}_n, E)$  is a closed subspace of the vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, E)$ .

We present now our first important contribution in this thesis. Mainly, we show that we can identify

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the topological dual space of the weighted vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$  for  $0 < p \leq 1$  with the vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, E^*)$ , where  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$ .

**Theorem 3.3.19.** Let  $0 < p \leq 1$ . The space  $(A_\alpha^p(\mathbb{B}_n, E))^*$  can be identified with  $\Lambda_\gamma(\mathbb{B}_n, E^*)$  with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$  under the pairing

$$\langle f, g \rangle_{\alpha, E} = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z), \quad (3.22)$$

where  $D_k$  is defined by (3.6),  $k > \gamma$ , is an integer, where  $c_k$  is the positive constant in Lemma 3.3.9,  $g \in \Lambda_\gamma(\mathbb{B}_n, E^*)$  and  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . Moreover,

$$\|g\|_{\Lambda_\gamma(\mathbb{B}_n, E^*)} \simeq \sup_{\|f\|_{A_\alpha^p(\mathbb{B}_n, E)}=1} |\langle f, g \rangle_{\alpha, E}|.$$

*Proof.* We first suppose that  $g \in \Lambda_\gamma(\mathbb{B}_n, E^*)$ , with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$ . Given a positive integer  $k > \gamma$ , we consider the functional

$$\begin{aligned} \wedge_g : A_\alpha^p(\mathbb{B}_n, E) &\longrightarrow \mathbb{C} \\ f &\mapsto \wedge_g(f) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z), \end{aligned}$$

where  $c_k$  is the positive constant in Lemma 3.3.9. We observe that  $\wedge_g$  is linear and is well defined on  $A_\alpha^p(\mathbb{B}_n, E)$ . Indeed, let  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . We have

$$\begin{aligned} |\wedge_g(f)| &= c_k \left| \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z) \right| \\ &\leq c_k \int_{\mathbb{B}_n} \|f(z)\|_E \|D_k g(z)\|_{E^*} (1 - |z|^2)^k d\nu_\alpha(z) \\ &= c_k \int_{\mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k g(z)\|_{E^*} (1 - |z|^2)^\gamma \|f(z)\|_E d\nu_\alpha(z) \\ &\leq c_k \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k g(z)\|_{E^*} \int_{\mathbb{B}_n} (1 - |z|^2)^\gamma \|f(z)\|_E d\nu_\alpha(z) \\ &\lesssim \|g\|_{\Lambda_\gamma(\mathbb{B}_n, E^*)} \int_{\mathbb{B}_n} (1 - |z|^2)^{\left(\frac{1}{p}-1\right)(n+1+\alpha)} \|f(z)\|_E d\nu_\alpha(z) \\ &\lesssim \|g\|_{\Lambda_\gamma(\mathbb{B}_n, E^*)} \|f\|_{p, \alpha, E}, \end{aligned}$$

where for the last inequality we used Corollary 2.2.12. We conclude that  $\wedge_g$  is bounded on  $A_\alpha^p(\mathbb{B}_n, E)$  and  $\|\wedge_g\| \lesssim \|g\|_{\Lambda_\gamma(\mathbb{B}_n, E^*)}$ .

Conversely, let  $\wedge$  be a bounded linear functional on  $A_\alpha^p(\mathbb{B}_n, E)$ . Let us show that there exists  $g \in \Lambda_\gamma(\mathbb{B}_n, E^*)$ , with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$  such that  $\wedge = \wedge_g$ . Since  $A_\alpha^2(\mathbb{B}_n, E) \subset A_\alpha^p(\mathbb{B}_n, E)$  and  $\wedge$  is bounded on  $A_\alpha^p(\mathbb{B}_n, E)$ ,  $\wedge$  is also bounded on  $A_\alpha^2(\mathbb{B}_n, E)$ . Then by Theorem 2.2.22, there exists  $g \in A_\alpha^2(\mathbb{B}_n, E^*)$  such that

$$\wedge(f) = \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z), \quad (3.23)$$

for all  $f \in A_\alpha^2(\mathbb{B}_n, E)$ . Since  $g \in A_\alpha^2(\mathbb{B}_n, E^*)$ , for any positive integer  $k$ , we have  $D_k g \in A_{\alpha+k}^2(\mathbb{B}_n, E^*)$ . Applying Lemma 3.3.9 in (3.23), we obtain that

$$\wedge(f) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z), \quad (3.24)$$

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for all  $f \in A_\alpha^2(\mathbb{B}_n, E)$ . Now, we fix  $x \in E$ ,  $w \in \mathbb{B}_n$  and an integer  $k > \gamma$ . Let

$$f(z) = \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x, \quad z \in \mathbb{B}_n.$$

By Theorem 2.1.25, we have that  $f \in A_\alpha^2(\mathbb{B}_n, E)$ . Proposition 2.2.2 and (3.24), give us

$$\begin{aligned} \wedge(f) &= c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z) \\ &= c_k \int_{\mathbb{B}_n} \left\langle \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x, D_k g(z) \right\rangle_{E, E^*} (1 - |z|^2)^k d\nu_\alpha(z) \\ &= \frac{c_\alpha c_k}{c_{\alpha+k}} (1 - |w|^2)^{k-\gamma} \left\langle x, \int_{\mathbb{B}_n} \frac{D_k g(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} d\nu_{\alpha+k}(z) \right\rangle_{E, E^*} \\ &= \frac{c_\alpha c_k}{c_{\alpha+k}} (1 - |w|^2)^{k-\gamma} \langle x, D_k g(w) \rangle_{E, E^*}. \end{aligned}$$

By Theorem 2.1.25,  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $\|f\|_{p, \alpha, E} \lesssim \|x\|_E$ . Since  $x$  is arbitrary, by duality, we have that

$$\begin{aligned} \|D_k g(w)\|_{E^*} &= \sup_{\|x\|_E=1} |\langle x, D_k g(w) \rangle_{E, E^*}| \\ &= \frac{c_{\alpha+k}}{c_\alpha c_k} \sup_{\|x\|_E=1} \frac{1}{(1 - |w|^2)^{k-\gamma}} |\wedge(f)| \\ &\lesssim \sup_{\|x\|_E=1} \frac{1}{(1 - |w|^2)^{k-\gamma}} \|\wedge(f)\| \\ &\lesssim \sup_{\|x\|_E=1} \frac{\|\wedge(f)\|}{(1 - |w|^2)^{k-\gamma}} \|x\|_E \\ &\lesssim \frac{\|\wedge(f)\|}{(1 - |w|^2)^{k-\gamma}}. \end{aligned}$$

According to Corollary 3.3.12, we conclude that

$$g \in \Lambda_\gamma(\mathbb{B}_n, E^*) \quad \text{and} \quad \|g\|_{\Lambda_\gamma(\mathbb{B}_n, E^*)} \lesssim \|\wedge(f)\|,$$

$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$ . To finish the proof, it remains to show that (3.23) remains true for functions in  $A_\alpha^p(\mathbb{B}_n, E)$  which is a direct consequence of the density in Corollary 2.2.6.  $\square$

# Little Hankel operators on vector-valued Bergman spaces

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In 1957 Nehari [50] described the bounded Hankel operators on the sequence space  $\ell^2$ . This description turned out to be very important and started the contemporary period of the study of Hankel operators. Hankel operators can be defined in many different ways and they admit different useful realizations. Such a variety of realizations are important in applications, since in a concrete case we can choose a realization that is most suitable for the problem under consideration. The history of the Hankel operators began with the Hardy spaces and the Hankel matrices. Immediately they became very popular and they led to very good applications [51, 55]. Since then many other generalizations were made.

It is also well known that the Hankel operators are closely related to Toeplitz operators, another well known operator. There is only one type of Hankel operators on Hardy spaces. In the Bergman setting, there are two different Hankel operators, the little (or small) Hankel operator and the big Hankel operator.

In this chapter, we are concerned with the question of characterizing the operator-valued holomorphic symbols  $b$  for which the little Hankel operator  $h_b$  extends into a bounded (resp compact) operator from  $A_0^p(\mathbb{B}_1, E)$  into  $A_0^q(\mathbb{B}_1, F)$  where  $0 < p, q < \infty$  (resp where  $1 < p \leq q < \infty$ ). The study of the boundedness of the little Hankel operator with operator-valued symbol began with [1] where Aleman and Constantin solved this problem for the particular case  $n = 1, p = q = 2$  and  $E = F = \mathcal{H}$  where  $\mathcal{H}$  is a separable complex Hilbert space. They showed that the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  into  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  if and only if the symbol  $b$  belongs to the Bloch space  $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\mathcal{H}))$ . For the compactness problem, Constantin proved in [26] that the little Hankel operator  $h_b$  is a compact operator from  $A_0^2(\mathbb{B}_1, \mathcal{H})$  into  $A_0^2(\mathbb{B}_1, \mathcal{H})$  if and only if the symbol  $b$  belongs to the little vector-valued Bloch space  $\mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(\mathcal{H}))$ . The more general case for the boundedness problem was obtained in [52] where Oliver proved that for  $1 < p < \infty$ , and  $E, F$  be two complex Banach spaces, the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if the symbol  $b$  belongs to the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  and this result clearly generalizes the one obtained by Aleman and Constantin in [1]. Moreover, for  $1 < p \leq q < \infty$ , Oliver showed that the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if the symbol  $b$  belongs to the  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  with  $\gamma = 1 + (n + 1 + \alpha) \left( \frac{1}{q} - \frac{1}{p} \right)$ . Also for  $1 < q < p < \infty$ , Oliver showed that if  $F$  has finite cotype (see [52] for the definition of the cotype of a Banach space), then the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if  $b \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , with  $1/t = 1/q - 1/p$ , which generalizes the main result in [53].

In this chapter, we will study the boundedness problem in the remaining cases and we will also study the compactness problem for  $1 < p \leq q < \infty$ . We start by remarking that there is an error in the result by

## 4.1. Basic properties and definitions

Oliver on the case  $1 < p \leq q < \infty$ . Firstly, we will correct the result by Oliver in the case  $1 < p \leq q < \infty$ , we shall continue the work with the study of the boundedness problem in the cases  $0 < p, q \leq 1$ . We finally will solve the boundedness problem in the case  $0 < p \leq q$  and  $1 < q < \infty$ . In the case  $1 < p < \infty$  and  $0 < q < \infty$ , we give some estimates with loss. After the boundedness problem, we will study the compactness problem. Precisely, we will prove that for two reflexive complex Banach spaces  $E, F$  and  $1 < p \leq q < \infty$ , the little Hankel operator  $h_b$  with operator-valued symbol  $b$  extends into a compact operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if its symbol  $b$  is in the little vector-valued Lipschitz space  $\Lambda_{\gamma_0, 0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$ , where  $\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ .

## 4.1 Basic properties and definitions

In this section, we are going to introduce basic concepts of the little Hankel operator. In the scalar case, the little Hankel operator  $h_\phi : A_\alpha^2(\mathbb{B}_n, \mathbb{C}) \rightarrow \overline{A_\alpha^2(\mathbb{B}_n, \mathbb{C})}$  with symbol  $\phi \in L^\infty(\mathbb{B}_n, \mathbb{C})$  is defined (see [53]) by

$$h_\phi(f)(z) := \overline{P_\alpha}(\phi f)(z), \quad f \in A_\alpha^2(\mathbb{B}_n, \mathbb{C}).$$

Since

$$\overline{P_\alpha}(f)(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \quad f \in L_\alpha^2(\mathbb{B}_n, \mathbb{C})$$

is an integral operator we have that  $h_\phi$  is also an integral operator with the form

$$h_\phi(f)(z) = \int_{\mathbb{B}_n} \frac{f(w) \cdot \phi(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \quad f \in A_\alpha^2(\mathbb{B}_n, \mathbb{C}).$$

The characterization of this operator is widely studied and established, see [78] for example. The little Hankel operators has many applications, mostly in physics [55, 51]. In this thesis, we are interested in little Hankel operators with operator-valued symbols and vector-valued functions.

From now on, let  $E$  and  $F$  be two complex Banach spaces, and let  $b : \mathbb{B}_n \rightarrow \mathcal{L}(\overline{E}, F)$  be a holomorphic function, i.e.,  $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , and then the most general little Hankel operator  $h_b$  with operator-valued symbol  $b$  is defined as

$$h_b(f)(z) := \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \quad f \in \mathcal{P}(\mathbb{B}_n, E).$$

It is clear that if  $b$  satisfies

$$\int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{E}, F)}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) < \infty, \quad z \in \mathbb{B}_n, \quad (4.1)$$

then the little Hankel operator  $h_b$  is well-defined for polynomials, which is a dense subspace of  $A_\alpha^p(\mathbb{B}_n, E)$  for  $1 \leq p < \infty$ , by Lemma 2.2.4, meaning  $h_b$  is densely defined on  $A_\alpha^p(\mathbb{B}_n, E)$  for any  $1 \leq p < \infty$ . Since

$$|1 - \langle z, w \rangle| \geq |1 - |\langle z, w \rangle|| \geq (1 - |z|) \quad z, w \in \mathbb{B}_n,$$

it is clear that condition (4.1) is equivalent to  $b \in A_\alpha^1(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . In what will follow, we will assume that the operator-valued holomorphic symbol  $b$  satisfy the following condition:

$$\int_{\mathbb{B}_n} \|b(z)\|_{\mathcal{L}(\overline{E}, F)} \log \left( \frac{1}{1 - |z|^2} \right) d\nu_\alpha(z) < \infty. \quad (4.2)$$

## 4.1. Basic properties and definitions

The following technical lemma is useful for what follows.

**Lemma 4.1.1.** ([52, Lemma 4.1.1]) Let  $f \in \mathcal{P}(\mathbb{B}_n, E)$  and  $g \in \mathcal{P}(\mathbb{B}_n, F^*)$ . If  $b \in A_\alpha^r(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , for some  $1 < r < \infty$ , then

$$\langle h_b(f), g \rangle_{\alpha, F} = \int_{\mathbb{B}_n} \langle b(z)(\overline{f(z)}), g(z) \rangle_{F, F^*} d\nu_\alpha(z).$$

The aim of this chapter is to characterize the general little Hankel operator  $h_b$  in terms of the properties of the symbol  $b$ . The techniques used here are similar to [1]. In order to do that, we need to show the following generalization of Lemma 2.2.7.

**Lemma 4.1.2.** ([52, Lemma 4.1.2]) Let  $t > 0$ . Then

$$\begin{aligned} \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z) &= \int_{\mathbb{B}_n} \langle R^{\alpha, t} f(z), g(z) \rangle_{E, E^*} d\nu_{\alpha+t}(z) \\ &= \int_{\mathbb{B}_n} \langle f(z), R^{\alpha, t} g(z) \rangle_{E, E^*} d\nu_{\alpha+t}(z) \end{aligned}$$

for any  $f \in \mathcal{P}(\mathbb{B}_n, E)$  and  $g \in \mathcal{P}(\mathbb{B}_n, E^*)$ .

**Corollary 4.1.3.** ([52, Lemma 4.1.3]) Suppose  $t > 0$  and  $1 < p < \infty$ . If  $b \in A_\alpha^{p'}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where  $p'$  is the conjugate exponent of  $p$ , then the equality

$$\int_{\mathbb{B}_n} \langle b(z)(\overline{f(z)}), g(z) \rangle_{F, F^*} d\nu_\alpha(z) = \int_{\mathbb{B}_n} \langle R^{\alpha, t} b(z)(\overline{f(z)}), g(z) \rangle_{F, F^*} d\nu_{\alpha+t}(z)$$

holds whenever  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $g \in \mathcal{P}(\mathbb{B}_n, F^*)$ .

The proof of the following lemma can also be found in [52].

**Lemma 4.1.4.** Let  $f \in H^\infty(\mathbb{B}_n, E)$  and  $g \in H^\infty(\mathbb{B}_n, F^*)$ . If  $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  is such that (4.1) and (4.2) hold. Then we have

$$\langle h_b(f), g \rangle_{\alpha, F} = \int_{\mathbb{B}_n} \langle b(z)(\overline{f(z)}), g(z) \rangle_{F, F^*} d\nu_\alpha(z). \quad (4.3)$$

*Proof.* Let  $f \in H^\infty(\mathbb{B}_n, E)$  and  $g \in H^\infty(\mathbb{B}_n, F^*)$ . By the definition of  $\langle \cdot, \cdot \rangle_{\alpha, F}$ , Fubini's theorem, Lemma 1.2.24 and the reproducing kernel property, we have:

$$\begin{aligned} \langle h_b(f), g \rangle_{\alpha, F} &= \int_{\mathbb{B}_n} \langle h_b(f)(z), g(z) \rangle_{F, F^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left\langle \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)}) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, g(z) \right\rangle_{F, F^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} g(z) \left( \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)}) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z) \left( \frac{b(w)(\overline{f(w)})}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right) d\nu_\alpha(w) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{g(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(z) \right) (b(w)(\overline{f(w)})) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} g(w) (b(w)(\overline{f(w)})) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \langle b(w)(\overline{f(w)}), g(w) \rangle_{F, F^*} d\nu_\alpha(w). \end{aligned}$$

## 4.1. Basic properties and definitions

It remains to show that the assumption of Fubini's theorem is fulfilled. Indeed, since  $f \in H^\infty(\mathbb{B}_n, E)$  and  $g \in H^\infty(\mathbb{B}_n, F^*)$ , by Tonelli's theorem, Theorem 2.1.25 and relation (4.2) we have that

$$\begin{aligned} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left| \frac{g(z) \left( b(w) \overline{f(w)} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right| d\nu_\alpha(w) d\nu_\alpha(z) &\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{E}, F)}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) d\nu_\alpha(z) \\ &\lesssim \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{E}, F)} \log \left( \frac{1}{1 - |w|^2} \right) d\nu_\alpha(w) < \infty. \quad \square \end{aligned}$$

**Lemma 4.1.5.** Let  $f \in H^\infty(\mathbb{B}_n, E)$  and  $z \in \mathbb{B}_n$ . For  $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  satisfying (4.1) and (4.2) the function

$$g_z(w) := \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}}, \quad w \in \mathbb{B}_n$$

belongs to  $H^\infty(\mathbb{B}_n, E)$  and the following identity holds:

$$h_b(f)(z) = C_k \int_{\mathbb{B}_n} L_k \left( b(w) \overline{g_z(w)} \right) d\nu_{\alpha+k}(w),$$

where  $k$  is any positive integer and  $C_k$  is a positive constant depending only on  $k$ .

*Proof.* It is clear that  $g_z \in H^\infty(\mathbb{B}_n, E)$ . By the definition of the little Hankel operator and the reproducing kernel property, we have

$$\begin{aligned} h_b(f)(z) &= \int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} b(w) \overline{\left( \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right)} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} b(w) \overline{g_z(w)} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} b(w) \overline{\left( \int_{\mathbb{B}_n} \frac{g_z(\zeta)}{(1 - \langle w, \zeta \rangle)^{n+1+\alpha+k}} d\nu_{\alpha+k}(\zeta) \right)} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{b(w) \overline{g_z(\zeta)}}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) \right) d\nu_{\alpha+k}(\zeta) \\ &= c_k^{-1} \int_{\mathbb{B}_n} L_k \left( \int_{\mathbb{B}_n} \frac{b(w) \overline{g_z(\zeta)}}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right) d\nu_{\alpha+k}(\zeta) \\ &= c_k^{-1} \int_{\mathbb{B}_n} L_k \left( b(\zeta) \overline{g_z(\zeta)} \right) d\nu_{\alpha+k}(\zeta). \end{aligned}$$

The assumption of Fubini's theorem is fulfilled. Indeed by (4.2), we have that

$$\begin{aligned} &\int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} \frac{b(w) \overline{g_z(\zeta)}}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) \right\|_F d\nu_{\alpha+k}(\zeta) \\ &\leq \|g_z\|_{\infty, E} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{E}, F)}}{|1 - \langle w, \zeta \rangle|^{n+1+\alpha+k}} d\nu_\alpha(w) d\nu_{\alpha+k}(\zeta) \\ &= \|g_z\|_{\infty, E} \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{E}, F)} \times \left( \int_{\mathbb{B}_n} \frac{d\nu_{\alpha+k}(\zeta)}{|1 - \langle w, \zeta \rangle|^{n+1+\alpha+k}} \right) d\nu_\alpha(w) \\ &\leq \|g_z\|_{\infty, E} \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{E}, F)} \left( \log \frac{1}{1 - |w|^2} \right) d\nu_\alpha(w) < \infty. \quad \square \end{aligned}$$

**4.2. Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

**4.2 Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

Most of the results presented in this section can be found in [52] and [9].

Here, we present one of the most important results of this chapter, the boundedness of the little Hankel operators  $h_b$  with symbol  $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . In this section we are going to characterize bounded little Hankel operators  $h_b$  from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  for  $1 < p \leq q < \infty$ . For convenience, we recall that the little Hankel operator  $h_b$  is defined as

$$h_b(f)(z) := \int_{\mathbb{B}_n} \frac{b(w)\overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w),$$

for any  $f \in \mathcal{P}(\mathbb{B}_n, E)$ .

In this case, the result obtained by Oliver ([52]) is stated as follow:

**Theorem 4.2.1.** Let  $1 < p \leq q < \infty$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator if and only if  $b \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where

$$\gamma = 1 + (n + 1 + \alpha) \left( \frac{1}{q} - \frac{1}{p} \right).$$

Moreover

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}.$$

**Remark 4.2.2.** Suppose  $1 < p < q < \infty$ , and  $\gamma = 1 + (n + 1 + \alpha) \left( \frac{1}{q} - \frac{1}{p} \right)$ . Then  $\gamma$  is not always positive. Indeed, since  $1/q - 1/p \in (-1, 0)$ , then  $\gamma \in (-n - \alpha, 1)$ . It follows that when  $\gamma \in (-(n + \alpha), 0)$ , the vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  is not interesting and does not make sense since the definition of the vector-valued  $\gamma$ -Bloch space introduced by Oliver only takes into account the case where  $\gamma > 0$ . In the next theorem, we correct the problem by replacing the vector-valued  $\gamma$ -Bloch space with the generalized vector-valued Lipschitz space  $\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where  $\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ . Since  $\gamma = 1 - \gamma_0$ , we see that when  $0 < \gamma_0 < 1$ , we have that

$$\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F)) = \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F)).$$

As one of our modesty contribution, we improved Theorem 4.2.1 by introducing the vector-valued Lipschitz space  $\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ .

**Theorem 4.2.3.** Suppose  $1 < p \leq q < \infty$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator if and only if  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where

$$\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right).$$

Moreover,  $\|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}$ .

*Proof.* Let  $p'$  and  $q'$  such that  $1/p + 1/p' = 1$  and  $1/q + 1/q' = 1$ . We first assume that  $h_b$  is a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  to  $A_\alpha^q(\mathbb{B}_n, F)$  with norm  $\|h_b\| = \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)}$ . Let  $x \in E$  and  $k >$

**4.2. Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

$(n + 1 + \alpha)/p$ . Let  $z \in \mathbb{B}_n$  and put

$$f(w) = \frac{x}{(1 - \langle w, z \rangle)^k}, \quad w \in \mathbb{B}_n.$$

Since  $k > (n + 1 + \alpha)/p$ , by Theorem 2.1.25, we have that  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and

$$\|f\|_{p,\alpha,E} \lesssim \frac{\|x\|_E}{(1 - |z|^2)^{k-(n+1+\alpha)/p}}.$$

By Proposition 2.2.3, we have that

$$\begin{aligned} h_b(f)(z) &= \int_{\mathbb{B}_n} \frac{b(w)\overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{b(w)\overline{x}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) = R^{\alpha,k}b(z)(\overline{x}). \end{aligned}$$

It follows by Theorem 2.2.1 that

$$\begin{aligned} \|R^{\alpha,k}b(z)(\overline{x})\|_F &= \|h_b(f)(z)\|_F \\ &\leq \frac{\|h_b(f)\|_{q,\alpha,F}}{(1 - |z|^2)^{(n+1+\alpha)/q}} \\ &\leq \frac{\|h_b\| \|f\|_{p,\alpha,E}}{(1 - |z|^2)^{(n+1+\alpha)/q}} \\ &\lesssim \frac{\|h_b\| \|x\|_E}{(1 - |z|^2)^{k+(n+1+\alpha)(1/q-1/p)}} \\ &= \frac{\|h_b\| \|x\|_E}{(1 - |z|^2)^{k-\gamma_0}}. \end{aligned}$$

Since  $x \in E$  is arbitrary and  $\|x\|_E = \|\overline{x}\|_{\overline{E}}$  we get that

$$\|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{E},F)} \lesssim \frac{\|h_b\|}{(1 - |z|^2)^{k-\gamma_0}}.$$

Thus

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma_0} \|R^{\alpha,k}b(z)\|_{\mathcal{L}(\overline{E},F)} \lesssim \|h_b\|.$$

By Lemma 3.3.6 this means that  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  and  $\|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \lesssim \|h_b\|$ .

Conversely, assume that  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . Let  $f \in A_\alpha^p(\mathbb{B}_n, E)$ ,  $g \in A_\alpha^q(\mathbb{B}_n, F^*)$  and  $k > \gamma_0$ . By Corollary 3.3.7, we have that

$$b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F)) \subset A_\alpha^{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F)),$$

so by Lemma 4.1.1, Corollary 4.1.3 and Lemma 3.3.6 it follows that

$$\begin{aligned} |\langle h_b(f), g \rangle_{\alpha,F}| &= \left| \int_{\mathbb{B}_n} \langle b(z)\overline{f(z)}, g(z) \rangle_{F,F^*} d\nu_\alpha(z) \right| \\ &= \left| \int_{\mathbb{B}_n} \langle R^{\alpha,k+1}b(z)\overline{f(z)}, g(z) \rangle_{F,F^*} d\nu_{\alpha+k+1}(z) \right| \\ &\lesssim \int_{\mathbb{B}_n} \|R^{\alpha,k+1}b(z)\|_{\mathcal{L}(\overline{E},F)} \|f(z)\|_{\overline{E}} \|g(z)\|_{F^*} (1 - |z|^2)^{k+1+\alpha} d\nu(z) \\ &\lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \int_{\mathbb{B}_n} \|f(z)\|_E \|g(z)\|_{F^*} (1 - |z|^2)^{\alpha+\gamma_0} d\nu(z). \end{aligned}$$

**4.3. Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p, q \leq 1$ .**

By Hölder's inequality the last integral is less than or equal to

$$\left( \int_{\mathbb{B}_n} \|f(z)\|_E^q (1 - |z|^2)^{\alpha + q\gamma_0} d\nu(z) \right)^{1/q} \left( \int_{\mathbb{B}_n} \|g(z)\|_{F^*}^{q'} (1 - |z|^2)^\alpha d\nu(z) \right)^{1/q'}.$$

For  $q = p$ , we have  $\gamma_0 = 0$  and thus

$$|\langle h_b(f), g \rangle_{\alpha, F}| \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \|f\|_{p, \alpha, E} \|g\|_{p', \alpha, F^*}.$$

For  $q - p > 0$ , using Theorem 2.2.1, we have

$$\|f(z)\|_E^q = \|f(z)\|_E^p \|f(z)\|_E^{q-p} \leq \frac{\|f(z)\|_E^p \|f\|_{p, \alpha, E}^{q-p}}{(1 - |z|^2)^{(q-p)(n+1+\alpha)/p}} = \frac{\|f(z)\|_E^p \|f\|_{p, \alpha, E}^{q-p}}{(1 - |z|^2)^{q\gamma_0}}.$$

It follows that

$$\left( \int_{\mathbb{B}_n} \|f(z)\|_E^q (1 - |z|^2)^{\alpha + q\gamma_0} d\nu(z) \right)^{1/q} \lesssim \|f\|_{p, \alpha, E}^{1-p/q} \left( c_\alpha \int_{\mathbb{B}_n} \|f(z)\|_E^p \frac{(1 - |z|^2)^{\alpha + q\gamma_0}}{(1 - |z|^2)^{q\gamma_0}} d\nu(z) \right)^{1/q} = \|f\|_{p, \alpha, E}.$$

Therefore, by duality, we obtain that

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} = \sup_{\|f\|_{p, \alpha, E}=1; \|g\|_{q', \alpha, F^*}=1} |\langle h_b(f), g \rangle_{\alpha, F}| \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}.$$

□

**4.3 Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p, q \leq 1$ .**

In this section, we present some of our contribution which can also be found in [9].

**4.3.1 Boundedness of  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p \leq 1$  and  $0 < q < 1$ .**

**Theorem 4.3.1.** Let  $0 < p \leq 1$  and  $0 < q < 1$ .

- 1) If the little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$ , then the symbol  $b$  is in  $\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$ .
- 2) If  $b$  is in  $\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$ , then the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^{1, \infty}(\mathbb{B}_n, F)$  is a bounded operator.

*Proof.* First assume that  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  to  $A_\alpha^q(\mathbb{B}_n, F)$ , with  $q < 1$ .

**4.3. Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p, q \leq 1$ .**

Let  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)}$ . We want to show that  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . Since  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator, we have by Theorem 3.3.19 that

$$|\langle h_b(f), g \rangle_{\alpha, F}| \lesssim \|h_b\| \|f\|_{p, \alpha, E} \|g\|_{\Lambda_\beta(\mathbb{B}_n, F^*)},$$

for every  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $g \in \Lambda_\beta(\mathbb{B}_n, F^*)$ , with  $\beta = (n + 1 + \alpha) \left(\frac{1}{q} - 1\right)$ . Let  $x \in E$ ,  $y^* \in F^*$ ,  $w \in \mathbb{B}_n$  and an integer  $k$  such that  $k > \gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ . Let  $g(z) = y^*$ , and  $f(z) = \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x$ . It is clear that  $f \in H^\infty(\mathbb{B}_n, E)$  and  $g \in \Lambda_\beta(\mathbb{B}_n, F^*)$ , with  $\|g\|_{\Lambda_\beta(\mathbb{B}_n, F^*)} = \|y^*\|_{F^*}$ . We also have by Theorem 2.1.25 that  $f \in A_\alpha^p(\mathbb{B}_n, E)$ , with  $\|f\|_{p, \alpha, E} \lesssim \|x\|_E$ . Hence

$$|\langle h_b(f), g \rangle_{\alpha, F}| \lesssim \|h_b\| \|x\|_E \|y^*\|_{F^*}, \quad (4.4)$$

Applying Lemma 4.1.4 and the reproducing kernel property, we have that

$$\begin{aligned} |\langle h_b(f), g \rangle_{\alpha, F}| &= \left| \int_{\mathbb{B}_n} \langle b(z) \left( \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x \right), y^* \rangle_{F, F^*} d\nu_\alpha(z) \right| \\ &= (1 - |w|^2)^{k-\gamma} \left| \int_{\mathbb{B}_n} \langle b(z) \left( \frac{\bar{x}}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} \right), y^* \rangle_{F, F^*} d\nu_\alpha(z) \right| \\ &= (1 - |w|^2)^{k-\gamma} \left| \int_{\mathbb{B}_n} \left\langle \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}}, y^* \right\rangle_{F, F^*} d\nu_\alpha(z) \right| \\ &= (1 - |w|^2)^{k-\gamma} \left| \left\langle \int_{\mathbb{B}_n} \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \right| \\ &= \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle \int_{\mathbb{B}_n} L_k \left( \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right) d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \right| \\ &= \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle L_k \left( \int_{\mathbb{B}_n} \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(z) \right), y^* \right\rangle_{F, F^*} \right| \\ &= \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle L_k (b(w) (\bar{x})), y^* \right\rangle_{F, F^*} \right|. \end{aligned}$$

Thus,

$$|\langle h_b(f), g \rangle_{\alpha, F}| = \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle L_k (b(w) (\bar{x})), y^* \right\rangle_{F, F^*} \right|. \quad (4.5)$$

From (4.4), (4.5) and the fact that  $\|x\|_E = \|\bar{x}\|_{\overline{E}}$ , we deduce that

$$(1 - |w|^2)^{k-\gamma} \left| \left\langle L_k (b(w) (\bar{x})), y^* \right\rangle_{F, F^*} \right| \lesssim \|h_b\| \|\bar{x}\|_{\overline{E}} \|y^*\|_{F^*}. \quad (4.6)$$

Since  $x$  and  $y^*$  are arbitrary, we get that

$$\sup_{w \in \mathbb{B}_n} (1 - |w|^2)^{k-\gamma} \|L_k b(w)\|_{\mathcal{L}(\overline{E}, F^*)} \lesssim \|h_b\|.$$

That is,  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F^*))$  with  $\|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F^*))} \lesssim \|h_b\|$ .

Conversely, assume that  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  and let us prove that  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  to  $A_\alpha^{1, \infty}(\mathbb{B}_n, F)$ . Choose a positive integer  $k > \gamma$ , and let  $f \in H^\infty(\mathbb{B}_n, E)$ . Taking

$$g_z(w) = \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}},$$

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with  $w \in \mathbb{B}_n$  and applying Lemma 4.1.5, Corollary 2.2.12 and the assumption we obtain

$$\begin{aligned}
\|h_b(f)(z)\|_F &= \left\| \int_{\mathbb{B}_n} \frac{b(w)\overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right\|_F \\
&= c_k \left\| \int_{\mathbb{B}_n} L_k \left( b(w)\overline{g_z(w)} \right) d\nu_{\alpha+k}(w) \right\|_F \\
&= c_k \left\| \int_{\mathbb{B}_n} \frac{L_k \left( b(w)\overline{f(w)} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha+k}(w) \right\|_F \\
&\leq c_k \int_{\mathbb{B}_n} \left\| \frac{L_k \left( b(w)\overline{f(w)} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\|_F d\nu_{\alpha+k}(w) \\
&\leq \frac{c_k c_{\alpha+k}}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^k \|L_k b(w)\|_{\mathcal{L}(\overline{E}, F)} \|f(w)\|_{\overline{E}}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) \\
&\lesssim \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\gamma \|f(w)\|_E}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) \\
&= \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} P_\alpha^+ g(z),
\end{aligned}$$

where the reproducing kernel is justified by (4.2) and

$$P_\alpha^+ g(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\gamma \|f(w)\|_E}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w)$$

is the positive kernel Bergman operator of the positive function  $g(z) = (1 - |z|^2)^\gamma \|f(z)\|_E$ .

Now, let  $\lambda > 0$ . We have that

$$\nu_\alpha(\{z \in \mathbb{B}_n : \|h_b(f)(z)\|_F > \lambda\}) \leq \nu_\alpha(\{z \in \mathbb{B}_n : c_k \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} P_\alpha^+ g(z) > \lambda\}).$$

Since the positive Bergman operator  $P_\alpha^+ : L_\alpha^1(\mathbb{B}_n) \rightarrow L_\alpha^{1,\infty}(\mathbb{B}_n)$  is bounded (cf. eg. [11]), there exists a constant  $c$  such that

$$\begin{aligned}
\nu_\alpha(\{z \in \mathbb{B}_n : c_k \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} P_\alpha^+ g(z) > \lambda\}) &\leq \frac{c}{\frac{\lambda}{c_k \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}}} \|g\|_{L_\alpha^1(\mathbb{B}_n)} \\
&= \frac{cc_k}{\lambda} \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \|g\|_{L_\alpha^1(\mathbb{B}_n)}.
\end{aligned}$$

Applying Corollary 2.2.12 to the function  $f$ , we get that

$$\begin{aligned}
\|g\|_{L_\alpha^1(\mathbb{B}_n)} &= \int_{\mathbb{B}_n} (1 - |z|^2)^\gamma \|f(z)\|_E d\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} (1 - |z|^2)^{\left(\frac{1}{p}-1\right)(n+1+\alpha)} \|f(z)\|_E d\nu_\alpha(z) \\
&\leq \|f\|_{p,\alpha,E}.
\end{aligned}$$

It follows that

$$\lambda \nu_\alpha(\{z \in \mathbb{B}_n : \|h_b f(z)\|_F > \lambda\}) \lesssim \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \|f\|_{p,\alpha,E}$$

for all  $\lambda > 0$ . Therefore,  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  to  $A_\alpha^{1,\infty}(\mathbb{B}_n, F)$  with

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^{1,\infty}(\mathbb{B}_n, F)} \lesssim \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}.$$

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By density of  $H^\infty(\mathbb{B}_n, E)$  on  $A_\alpha^p(\mathbb{B}_n, E)$ , the proof of the theorem is finished.  $\square$

As a direct consequence, we have the following result:

**Corollary 4.3.2.** Suppose  $0 < p \leq 1$ , and  $0 < q < 1$ . The little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  if and only if its symbol  $b$  belongs to  $\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right).$$

Moreover,  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}$ .

*Proof.* Just apply Lemma 2.2.15 and the second part of Theorem 4.3.1 to conclude.  $\square$

**4.3.2 Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p \leq 1$  and  $q = 1$ .**

**Definition 4.3.3.** Let  $\gamma \geq 0$ . The vector-valued logarithmic Lipschitz space denoted by  $\Lambda_{\gamma, \log}(\mathbb{B}_n, E)$  consists of all  $E$ -valued holomorphic function  $f$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \log \left( \frac{1}{1 - |z|^2} \right) \|\mathcal{N}^k f(z)\|_E < \infty,$$

where  $k$  is any positive integer greater than  $\gamma$ .

**Theorem 4.3.4.** Let  $0 < p \leq 1$ . The little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^1(\mathbb{B}_n, F)$  if and only if  $b$  is in the vector-valued logarithmic Lipschitz space  $\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right).$$

Moreover,  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^1(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}$ .

*Proof.* We first prove the sufficiency of the theorem. We assume that  $b \in \Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . Thus,

$$\|\mathcal{N}^k b(w)\|_{\mathcal{L}(\overline{E}, F)} \leq \frac{\|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1}, \quad w \in \mathbb{B}_n.$$

Likewise by Corollary 3.3.12, we have that

$$\|L_k b(w)\|_{\mathcal{L}(\overline{E}, F)} \lesssim \frac{\|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1}.$$

Applying Lemma 4.1.5 for any  $f \in H^\infty(\mathbb{B}_n, E)$ , we get

$$\int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) = c_k \int_{\mathbb{B}_n} \frac{L_k b(w)(\overline{f(w)})}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha+k}(w).$$

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Thus, by the assumption, Lemma 4.1.5 and Corollary 2.2.12 we have that

$$\begin{aligned}
 \|h_b(f)\|_{A_\alpha^1(\mathbb{B}_n, F)} &= \int_{\mathbb{B}_n} \left\| c_k \int_{\mathbb{B}_n} \frac{L_k b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha+k}(w) \right\|_F d\nu_\alpha(z) \\
 &\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left\| \frac{L_k b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\|_F (1 - |w|^2)^k d\nu_\alpha(w) d\nu_\alpha(z) \\
 &\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|L_k b(w)\|_{\mathcal{L}(\overline{E}, F)}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \|f(w)\|_{\overline{E}} (1 - |w|^2)^k d\nu_\alpha(w) d\nu_\alpha(z) \\
 &= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{1}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(z) \right) \|L_k b(w)\|_{\mathcal{L}(\overline{E}, F)} \|f(w)\|_{\overline{E}} (1 - |w|^2)^k d\nu_\alpha(w) \\
 &\lesssim \|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \int_{\mathbb{B}_n} \left( \log \frac{1}{1 - |w|^2} \right) \|f(w)\|_{\overline{E}} \frac{(1 - |w|^2)^k}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1} d\nu_\alpha(w) \\
 &= \|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \int_{\mathbb{B}_n} \|f(w)\|_E (1 - |w|^2)^\gamma d\nu_\alpha(w) \\
 &= \|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \int_{\mathbb{B}_n} \|f(w)\|_E (1 - |w|^2)^{\left(\frac{1}{p}-1\right)(n+1+\alpha)} d\nu_\alpha(w) \\
 &\lesssim \|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \|f\|_{p, \alpha, E}.
 \end{aligned}$$

Conversely, we assume that  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  to  $A_\alpha^1(\mathbb{B}_n, F)$ . Then for all  $f \in H^\infty(\mathbb{B}_n, E)$  and  $g \in \mathcal{B}(\mathbb{B}_n, F^*)$ , we have

$$|\langle h_b(f), g \rangle_{\alpha, F}| \leq \|h_b\| \|f\|_{p, \alpha, E} \|g\|_{\mathcal{B}(\mathbb{B}_n, F^*)}. \quad (4.7)$$

We choose the particular function  $g(z) = y^*$ , with  $y^* \in F^*$ . Applying Lemma 4.1.4, relation (4.7) becomes

$$\begin{aligned}
 \left| \int_{\mathbb{B}_n} \langle h_b(f)(z), y^* \rangle_{F, F^*} d\nu_\alpha(z) \right| &= \left| \left\langle \int_{\mathbb{B}_n} b(z) \overline{f(z)} d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \right| \\
 &\leq \|h_b\| \|f\|_{p, \alpha, E} \|y^*\|_{F^*}.
 \end{aligned}$$

Thus

$$\left| \int_{\mathbb{B}_n} \langle b(z) \overline{f(z)}, y^* \rangle_{F, F^*} d\nu_\alpha(z) \right| \leq \|h_b\| \|f\|_{p, \alpha, E} \|y^*\|_{F^*} \quad (4.8)$$

for all  $f \in H^\infty(\mathbb{B}_n, E)$  and  $y^* \in F^*$ . Now, take  $x \in E$ ,  $y^* \in F^*$ , and an integer  $k$  such that  $k > \gamma$ . Fix  $w \in \mathbb{B}_n$  and put

$$f(z) = \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x ; \quad g(z) = \log(1 - \langle z, w \rangle) y^*,$$

where  $\log$  is the principal branch of the logarithm. Since  $f \in H^\infty(\mathbb{B}_n, E)$  and  $g \in \mathcal{B}(\mathbb{B}_n, F^*)$ , by relation (4.7), we have that

$$|\langle h_b(f), g \rangle_{\alpha, F}| \leq \|h_b\| \|x\|_E \|y^*\|_{F^*}. \quad (4.9)$$

Applying Lemma 4.1.4 for those particular vector-valued holomorphic functions  $f$  and  $g$  and using the fact that

$$\log(1 - \langle w, z \rangle) = \log(1 - |w|^2) + \log\left(\frac{1 - \langle w, z \rangle}{1 - |w|^2}\right),$$

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we obtain

$$\begin{aligned}
\langle h_b(f), g \rangle_{\alpha, F} &= \int_{\mathbb{B}_n} \langle b(z) \left( \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x \right), \log(1 - \langle z, w \rangle) y^* \rangle_{F, F^*} d\nu_\alpha(z) \\
&= \left\langle \int_{\mathbb{B}_n} b(z) \left[ \frac{(1 - |w|^2)^{k-\gamma} \log(1 - \langle w, z \rangle)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} \bar{x} \right] d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \\
&= \left\langle \int_{\mathbb{B}_n} \frac{b(z)(\bar{x})(1 - |w|^2)^{k-\gamma} \log(1 - |w|^2)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \\
&+ \left\langle \int_{\mathbb{B}_n} b(z) \left[ \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} \log \left( \frac{1 - \langle w, z \rangle}{1 - |w|^2} \right) \bar{x} \right] d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \\
&= \langle (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \int_{\mathbb{B}_n} \frac{b(z)(\bar{x}) d\nu_\alpha(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}}, y^* \rangle_{F, F^*} \\
&+ \left\langle \int_{\mathbb{B}_n} b(z) \left( \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) x \right) d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \\
&= (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \langle L_k \left( \int_{\mathbb{B}_n} \frac{b(z)(\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(z) \right), y^* \rangle_{F, F^*} \\
&+ \left\langle \int_{\mathbb{B}_n} b(z) \left( f(z) \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) \right) d\nu_\alpha(z), y^* \right\rangle_{F, F^*} \\
&= (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*} \\
&+ \left\langle \int_{\mathbb{B}_n} b(z)(\overline{\varphi(z)}) d\nu_\alpha(z), y^* \right\rangle_{F, F^*},
\end{aligned}$$

where  $\varphi(z) = f(z) \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right)$ . Therefore, we can write  $\langle h_b f, g \rangle_{\alpha, F} = I_1 + I_2$ , with

$$I_1 = (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*}$$

and

$$I_2 = \left\langle \int_{\mathbb{B}_n} b(z)(\overline{\varphi(z)}) d\nu_\alpha(z), y^* \right\rangle_{F, F^*}.$$

Applying Lemma 2.2.13 with  $\delta = p$ , and  $\beta = p(k - \gamma)$ , we obtain that

$$\begin{aligned}
\|\varphi\|_{p, \alpha, E} &= \left( \int_{\mathbb{B}_n} \left| \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) \right|^p \frac{(1 - |w|^2)^{p(k-\gamma)}}{|1 - \langle z, w \rangle|^{p(n+1+\alpha+k)}} \|x\|_E^p d\nu_\alpha(z) \right)^{1/p} \\
&= \|x\|_E \left( \int_{\mathbb{B}_n} \left| \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) \right|^p \frac{(1 - |w|^2)^{p(k-\gamma)}}{|1 - \langle z, w \rangle|^{n+1+\alpha+p(k-\gamma)}} d\nu_\alpha(z) \right)^{1/p} \\
&\lesssim \|x\|_E.
\end{aligned}$$

According to the relation (4.8), we obtain the following estimation of  $I_2$

$$|I_2| \leq \|h_b\| \|\varphi\|_{p, \alpha, E} \|y^*\|_{F^*} \lesssim \|h_b\| \|x\|_E \|y^*\|_{F^*}.$$

Since  $I_1 = \langle h_b f, g \rangle_{\alpha, F} - I_2$ , by the relation (4.9) and the previous estimates on  $I_2$ , we have that

$$|I_1| \leq |\langle h_b(f), g \rangle_{\alpha, F}| + |I_2| \lesssim \|h_b\| \|x\|_E \|y^*\|_{F^*}.$$

**4.3. Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p, q \leq 1$ .**

Since  $x \in E$ ,  $y^* \in F^*$  are arbitrary and  $\|x\|_E = \|\bar{x}\|_{\bar{E}}$ , we get that

$$\begin{aligned} |I_1| &= (1 - |w|^2)^{k-\gamma} \log \left( \frac{1}{1 - |w|^2} \right) |\langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*}| \\ &\lesssim \|h_b\| \|\bar{x}\|_{\bar{E}} \|y^*\|_{F^*}. \end{aligned}$$

Since  $\bar{x} \in \bar{E}$  and  $y^* \in F^*$  are arbitrary, we deduce that :

$$\begin{aligned} \|L_k b(w)\|_{\mathcal{L}(\bar{E}, F)} &= \sup_{\|\bar{x}\|_{\bar{E}}=1, \|y^*\|_{F^*}=1} |\langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*}| \\ &\lesssim \frac{\|h_b\|}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1}. \end{aligned}$$

Applying Corollary 3.3.12, it follows that  $b \in \Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))$  and  $\|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))} \lesssim \|h_b\|$ .  $\square$

**4.3.3 Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p \leq q$  and  $1 < q < \infty$ .**

**Theorem 4.3.5.** Let  $0 < p \leq q$ , and  $1 < q < \infty$ . Then the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator if and only if  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))$ , with

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right).$$

Moreover,  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))}$ .

*Proof.* To prove the necessary part, we assume that the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator. It follows that

$$|\langle h_b(f), g \rangle_{\alpha, F}| \leq \|h_b\| \|f\|_{p, \alpha, E} \|g\|_{q', \alpha, F^*}, \quad (4.10)$$

for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$ , and  $g \in A_\alpha^{q'}(\mathbb{B}_n, F^*)$ , where  $\frac{1}{q} + \frac{1}{q'} = 1$ . Fix  $x \in E$ ,  $y^* \in F^*$  and  $w \in \mathbb{B}_n$ . Let

$$f(z) = \frac{1}{(1 - \langle z, w \rangle)^k} x; \quad g(z) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}} y^*,$$

where  $z \in \mathbb{B}_n$  and  $k$  is an integer such that  $kp > n + 1 + \alpha$ . It is clear that  $f \in H^\infty(\mathbb{B}_n, E)$  and  $g \in H^\infty(\mathbb{B}_n, F^*)$ . By using Theorem 2.1.25 it follows that  $\|f\|_{p, \alpha, E} \lesssim (1 - |w|^2)^{-k + \frac{n+1+\alpha}{p}} \|x\|_E$ , and  $\|g\|_{q', \alpha, F^*} \lesssim (1 - |w|^2)^{-\frac{1}{q'}(n+1+\alpha)} \|y^*\|_{F^*}$ . We also have that

$$\begin{aligned} \langle h_b(f), g \rangle_{\alpha, F} &= \int_{\mathbb{B}_n} \langle b(z)(\overline{f(z)}), g(z) \rangle_{F, F^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left\langle \frac{b(z)(\bar{x})}{(1 - \langle w, z \rangle)^k}, \frac{y^*}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\rangle_{F, F^*} d\nu_\alpha(z) \\ &= \left\langle \int_{\mathbb{B}_n} \frac{b(z)(\bar{x}) d\nu_\alpha(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}}, y^* \right\rangle_{F, F^*} \\ &= \left\langle \frac{1}{c_k} L_k \left( \int_{\mathbb{B}_n} \frac{b(z)(\bar{x}) d\nu_\alpha(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right), y^* \right\rangle_{F, F^*} \\ &= \frac{1}{c_k} \langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*}. \end{aligned}$$

**4.3. Boundedness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $0 < p, q \leq 1$ .**

Therefore, the inequality (4.10) becomes

$$|\langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*}| \lesssim \|h_b\| (1 - |w|^2)^{-k + (n+1+\alpha)(\frac{1}{p} - \frac{1}{q})} \|x\|_E \|y^*\|_{F^*}. \quad (4.11)$$

Since  $x \in E$  is arbitrary and  $\|x\|_E = \|\bar{x}\|_{\bar{E}}$ , it follows that

$$|\langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*}| \lesssim \|h_b\| (1 - |w|^2)^{-k+\gamma} \|\bar{x}\|_{\bar{E}} \|y^*\|_{F^*}, \quad (4.12)$$

for every  $\bar{x} \in \bar{E}$  and  $y^* \in F^*$ . Thus,

$$\|L_k(b(w))\|_{\mathcal{L}(\bar{E}, F)} = \sup_{\|x\|_E=1, \|y^*\|_{F^*}=1} |\langle L_k(b(w)(\bar{x})), y^* \rangle_{F, F^*}| \lesssim \|h_b\| (1 - |w|^2)^{-k+\gamma}.$$

Since  $w \in \mathbb{B}_n$  is arbitrary, it follows that

$$\sup_{w \in \mathbb{B}_n} (1 - |w|^2)^{k-\gamma} \|L_k(b(w))\|_{\mathcal{L}(\bar{E}, F)} \lesssim \|h_b\|.$$

Thus,

$$b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{E}, F)), \quad \text{and} \quad \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))} \lesssim \|h_b\|,$$

with  $\gamma = (n+1+\alpha)(\frac{1}{p} - \frac{1}{q})$ . Which is the desired result.

For the sufficiently part, we assume that

$$b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{E}, F)),$$

where  $\gamma = (n+1+\alpha)(\frac{1}{p} - \frac{1}{q})$ . Let us show that the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator. Let an integer  $k > \gamma$ . Applying Lemma 4.1.5, with

$$g_z(w) = \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}},$$

with  $w \in \mathbb{B}_n$ ,  $f \in H^\infty(\mathbb{B}_n, E)$ , and  $z \in \mathbb{B}_n$ , we obtain that

$$\begin{aligned} h_b(f)(z) &= C_k \int_{\mathbb{B}_n} L_k b(w) (\overline{g_z(w)}) d\nu_{\alpha+k}(w) \\ &= \frac{C_k c_{k+\alpha}}{c_\alpha} \int_{\mathbb{B}_n} \frac{L_k b(w) (\overline{f(w)}) (1 - |w|^2)^k}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \frac{C_k c_{k+\alpha}}{c_\alpha} P_\alpha(T_b(f))(z), \end{aligned}$$

where

$$T_b(f)(w) = L_k b(w) (\overline{f(w)}) (1 - |w|^2)^k, \quad w \in \mathbb{B}_n.$$

We have that

$$\begin{aligned} \int_{\mathbb{B}_n} \|T_b(f)(w)\|_F^q d\nu_\alpha(w) &= \int_{\mathbb{B}_n} \|L_k b(w) (\overline{f(w)})\|_F^q (1 - |w|^2)^{qk} d\nu_\alpha(w) \\ &\leq \int_{\mathbb{B}_n} \|L_k b(w)\|_{\mathcal{L}(\bar{E}, F)}^q \|\overline{f(w)}\|_{\bar{E}}^q (1 - |w|^2)^{q(k-\gamma+\gamma)} d\nu_\alpha(w) \\ &\lesssim \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))}^q \int_{\mathbb{B}_n} \|\overline{f(w)}\|_{\bar{E}}^q (1 - |w|^2)^{q\gamma} d\nu_\alpha(w) \end{aligned}$$

#### 4.4. Summary of the results obtained and estimates with loss.

Since  $f \in H^\infty(\mathbb{B}_n, E)$  is arbitrary, we know that  $\|\overline{f(w)}\|_{\overline{E}} = \|f(w)\|_E$ , for every  $w \in \mathbb{B}_n$ . Therefore, by Lemma 2.2.11, it follows that

$$\begin{aligned} \int_{\mathbb{B}_n} \|\overline{f(w)}\|_{\overline{E}}^q (1 - |w|^2)^{q\gamma} d\nu_\alpha(w) &= \int_{\mathbb{B}_n} \|f(w)\|_E^q (1 - |w|^2)^{q(n+1+\alpha)(\frac{1}{p} - \frac{1}{q})} d\nu_\alpha(w) \\ &\leq \|f\|_{p,\alpha,E}^q. \end{aligned}$$

Thus  $T_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $L_\alpha^q(\mathbb{B}_n, F)$  and

$$\|T_b f\|_{q,\alpha,F} \lesssim \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \|f\|_{p,\alpha,E}.$$

Since  $1 < q < \infty$ , by Theorem 2.2.18, we know that the Bergman projection  $P_\alpha$  is bounded on  $L_\alpha^q(\mathbb{B}_n, F)$ . Therefore, it follows that

$$\|h_b(f)\|_{q,\alpha,F} \lesssim \|P_\alpha(T_b(f))\|_{q,\alpha,F} \lesssim \|T_b(f)\|_{q,\alpha,F} \lesssim \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} \|f\|_{p,\alpha,E},$$

for all  $f \in H^\infty(\mathbb{B}_n, E)$ . Since the space  $H^\infty(\mathbb{B}_n, E)$  is dense in  $A_\alpha^p(\mathbb{B}_n, E)$ , the desired result follows at once.  $\square$

## 4.4 Summary of the results obtained and estimates with loss.

In this section we summarize all the important results we have obtained in the previous section and we obtained some estimates with loss of the boundedness of the little Hankel operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, F)$  in the two remaining cases  $1 < p < \infty$ ,  $0 < q < 1$ , and  $1 < p < \infty$ ,  $q = 1$ .

### 4.4.1 Summary of the results obtained.

In this section we summarize all the important results we have obtained in Section 4.3.

**Theorem 4.4.1.** Let  $0 < p, q < \infty$ . and  $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ . Then

- (i) for  $1 < p \leq q < \infty$ , then  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is bounded if and only if  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$$

and, moreover,  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}$ .

- (ii) for  $0 < p \leq 1$  and  $0 < q < 1$ , then  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  extends into a bounded operator if and only if  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$$

and moreover,  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}$ .

- (iii) for  $0 < p \leq 1$  and  $q = 1$ , then  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  extends to a bounded operator if and only

#### 4.4. Summary of the results obtained and estimates with loss.

if  $b \in \Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$$

and moreover,  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^1(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}$ .

(iv) for  $0 < p \leq q$  and  $1 < q < \infty$ ,  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator if and only if  $b \in \Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right).$$

Moreover,  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \simeq \|b\|_{\Lambda_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}$ .

Notice that since  $E$  is isometrically equivalent to  $\overline{E}$  we have the same result if we replace  $\overline{E}$  by  $E$ . Now we present some important examples of little Hankel operators. For example, if  $F = \mathbb{C}$  we get the little Hankel operator  $h_\phi$  with symbol  $\phi \in \mathcal{H}(\mathbb{B}_n, E^*)$ . The little Hankel operator  $h_\phi : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n)$  is the linear operator defined as

$$h_\phi(f)(z) = \int_{\mathbb{B}_n} \frac{\langle f(w), \phi(w) \rangle_{E, E^*}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \quad z \in \mathbb{B}_n, f \in \mathcal{P}(\mathbb{B}_n, E).$$

**Theorem 4.4.2.** Let  $0 < p, q < \infty$  and  $\phi \in \mathcal{H}(\mathbb{B}_n, \overline{E}^*)$ .

(i) For  $1 < p \leq q < \infty$ , the little Hankel operator  $h_\phi$  is bounded if and only if  $\phi \in \Lambda_\gamma(\mathbb{B}_n, \overline{E}^*)$  where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$$

and, moreover,  $\|h_\phi\| := \|h_\phi\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n)} \simeq \|\phi\|_{\Lambda_\gamma(\mathbb{B}_n, \overline{E}^*)}$ .

(ii) For  $0 < p \leq 1$  and  $1 < q < \infty$ , the little Hankel operator  $h_\phi$  is bounded if  $\phi \in \Lambda_\gamma(\mathbb{B}_n, \overline{E}^*)$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$$

and moreover,  $\|h_\phi\| := \|h_\phi\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n)} \simeq \|\phi\|_{\Lambda_\gamma(\mathbb{B}_n, \overline{E}^*)}$ .

(iii) For  $0 < p \leq 1$ , and  $0 < q < 1$ , the little Hankel operator  $h_\phi$  is bounded  $\phi \in \Lambda_\gamma(\mathbb{B}_n, \overline{E}^*)$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$$

and moreover,  $\|h_\phi\| := \|h_\phi\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n)} \simeq \|\phi\|_{\Lambda_\gamma(\mathbb{B}_n, \overline{E}^*)}$ .

(iv) For  $0 < p \leq 1$  and  $q = 1$ , the little Hankel operator  $h_b$  extends to a bounded if and only if  $\phi \in \Lambda_{\gamma, \log}(\mathbb{B}_n, \overline{E}^*)$ , where

$$\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$$

and moreover,  $\|h_\phi\| := \|h_\phi\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^1(\mathbb{B}_n)} \simeq \|\phi\|_{\Lambda_{\gamma, \log}(\mathbb{B}_n, \overline{E}^*)}$ .

### 4.4.2 Estimates with loss for the boundedness of the little Hankel operator, case $1 < p < \infty$ and $0 < q \leq 1$ .

In this section, we consider conditions for which the little Hankel operator with operator-valued symbol  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , where  $E, F$  are two complex Banach spaces and  $q < p$ . We give very partial results, which are far from necessary being and sufficient conditions as in the case of improving estimates.

**Proposition 4.4.3.** Let  $1 < p < \infty$ .

- 1) If  $b \in A_\alpha^{p'}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ , then the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^{1,\infty}(\mathbb{B}_n, F)$ .
- 2) If  $F = \mathbb{C}$ , and the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^{1,\infty}(\mathbb{B}_n, \mathbb{C})$ , then  $b \in A_\alpha^{p'}(\mathbb{B}_n, \overline{E}^*)$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

*Proof.* 1) Let  $b \in A_\alpha^{p'}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , with  $\frac{1}{p} + \frac{1}{p'} = 1$ . For every  $f \in A_\alpha^p(\mathbb{B}_n, E)$ , the function  $b\overline{f}$  is in  $L_\alpha^1(\mathbb{B}_n, F)$ . In fact, if  $f \in A_\alpha^p(\mathbb{B}_n, E)$ , then by using Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{B}_n} \|b(w)(\overline{f})(w)\|_F d\nu_\alpha(w) &\leq \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{E}, F)} \|\overline{f}(w)\|_{\overline{E}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{E}, F)} \|f(w)\|_E d\nu_\alpha(w) \\ &\leq \|b\|_{p', \alpha, \mathcal{L}(\overline{E}, F)} \|f\|_{p, \alpha, E}. \end{aligned}$$

Since the Bergman projection  $P_\alpha$  is bounded from  $L_\alpha^1(\mathbb{B}_n, F)$  into  $A_\alpha^{1,\infty}(\mathbb{B}_n, F)$ , it follows that

$$\|h_b(f)\|_{A_\alpha^{1,\infty}(\mathbb{B}_n, F)} = \|P_\alpha(b\overline{f})\|_{A_\alpha^{1,\infty}(\mathbb{B}_n, F)} \leq c\|b\overline{f}\|_{1, \alpha, F} \leq c\|b\|_{p', \alpha, \mathcal{L}(\overline{E}, F)} \|f\|_{p, \alpha, E}.$$

2) Let  $F = \mathbb{C}$ , and  $0 < t < 1$ . Assume that  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^{1,\infty}(\mathbb{B}_n)$  is bounded. Then by the sub-mean inequality (or the pointwise inequality), for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$ ,  $h_b(f) \in A_\alpha^{1,\infty}(\mathbb{B}_n) \subset A_\alpha^t(\mathbb{B}_n)$  and we have that

$$|h_b(f)(0)| \leq \left( \int_{\mathbb{B}_n} |h_b(f)(z)|^t d\nu_\alpha(z) \right)^{1/t} = \|h_b(f)\|_{t, \alpha}.$$

Also, by Lemma 2.2.15, there is a constant  $c > 0$  such that  $\|h_b(f)\|_{t, \alpha} \leq c\|h_b(f)\|_{A_\alpha^{1,\infty}(\mathbb{B}_n)}$

Using the assumption, we obtain a constant  $C > 0$  such that

$$|h_b(f)(0)| \leq c\|h_b f\|_{A_\alpha^{1,\infty}(\mathbb{B}_n)} \leq cC\|f\|_{p, \alpha, E}.$$

Since

$$\begin{aligned} h_b(f)(0) &= \int_{\mathbb{B}_n} b(w)(\overline{f}(w)) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \langle \overline{f}(w), b(w) \rangle_{\overline{E}, \overline{E}^*} d\nu_\alpha(w) \\ &= \langle \overline{f}, b \rangle_{\alpha, \overline{E}}, \end{aligned}$$

it follows that

$$|\langle \overline{f}, b \rangle_{\alpha, \overline{E}}| \leq cC\|f\|_{p, \alpha, \overline{E}}.$$

That is,  $b$  is in the dual space of  $A_\alpha^p(\mathbb{B}_n, \overline{E})$ , that is,  $A_\alpha^{p'}(\mathbb{B}_n, \overline{E}^*)$ . □

**4.5. Compactness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

**Corollary 4.4.4.** Let  $1 < p < \infty$ ,  $0 < q < 1$  and  $F = \mathbb{C}$ . If  $b$  is in  $A_\alpha^{p'}(\mathbb{B}_n, \overline{E}^*)$ , then the little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^q(\mathbb{B}_n, \mathbb{C})$ .

**Proposition 4.4.5.** Suppose  $1 < p < \infty$  and  $q = 1$ .

- 1) If the function  $b(z) \log \frac{1}{1-|z|^2}$  is in  $L_\alpha^{p'}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , with  $\frac{1}{p} + \frac{1}{p'} = 1$ , then the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^1(\mathbb{B}_n, F)$ .
- 2) If  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, E)$  into  $A_\alpha^1(\mathbb{B}_n, F)$ , then there exists a constant  $C > 0$  such that for all  $z \in \mathbb{B}_n$ ,

$$\|b(z)\|_{\mathcal{L}(\overline{E}, F)} \leq C(1 - |z|^2)^{-\frac{n+1+\alpha}{p'}} \left( \log \frac{1}{1 - |z|^2} \right)^{-1}.$$

*Proof.* 1) Let  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . We have

$$\begin{aligned} \|h_b(f)\|_{1, \alpha, F} &= \int_{\mathbb{B}_n} \|h_b(f)(w)\|_F d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} \frac{b(z) \overline{f(z)} dz}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right\|_F d\nu_\alpha(w) \\ &\leq \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(z) \overline{f(z)}\|_F}{|1 - \langle w, z \rangle|^{n+1+\alpha}} d\nu_\alpha(z) d\nu_\alpha(w) \\ &\leq \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(z)\|_{\mathcal{L}(\overline{E}, F)} \|f(z)\|_{\overline{E}}}{|1 - \langle w, z \rangle|^{n+1+\alpha}} d\nu_\alpha(z) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(z)\|_{\mathcal{L}(\overline{E}, F)} \|f(z)\|_E}{|1 - \langle w, z \rangle|^{n+1+\alpha}} d\nu_\alpha(z) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|b(z)\|_{\mathcal{L}(\overline{E}, F)} \|f(z)\|_E \left( \int_{\mathbb{B}_n} \frac{1}{|1 - \langle w, z \rangle|^{n+1+\alpha}} d\nu_\alpha(w) \right) d\nu_\alpha(z) \\ &\lesssim \int_{\mathbb{B}_n} \left[ \|b(z)\|_{\mathcal{L}(\overline{E}, F)} \log \frac{1}{1 - |z|^2} \right] \|f(z)\|_E d\nu_\alpha(z) \\ &\lesssim \left( \int_{\mathbb{B}_n} \left[ \|b(z)\|_{\mathcal{L}(\overline{E}, F)} \log \frac{1}{1 - |z|^2} \right]^{p'} d\nu_\alpha(z) \right)^{1/p'} \left( \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) \right)^{1/p} \\ &= \left\| b(z) \log \frac{1}{1 - |z|^2} \right\|_{p', \alpha, \mathcal{L}(\overline{E}, F)} \|f\|_{p, \alpha, E}. \end{aligned}$$

2) It is enough to follow step by step the second part of the proof of Theorem 4.3.4 by taking this two particular functions: Fix  $w \in \mathbb{B}_n$ ,  $x \in E$  and  $y^* \in F^*$  and choose

$$f_w(z) = \frac{(1 - |w|^2)^{\frac{n+1+\alpha}{p'}}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} x, \quad g_w(z) = \log(1 - \langle z, w \rangle) y^* \quad z \in \mathbb{B}_n.$$

□

**4.5 Compactness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

The compactness properties of the little Hankel operator on the vector-valued Bergman space  $A_\alpha^2(\mathbb{D}, \mathcal{H})$ , (where  $\mathbb{D} = \mathbb{B}_1$ , and  $\mathcal{H}$  is a separable Hilbert space) has been studied by Constantin ([26, Theorem 3.1]). In

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this section, we obtain a similar result (see [9, Theorem 8]) for vector-valued Bergman spaces.

In this section, we characterize those symbols  $b$  for which the little Hankel operator  $h_b$  extends into a compact operator from  $A_\alpha^p(\mathbb{B}_n, E)$  to  $A_\alpha^q(\mathbb{B}_n, F)$ , where  $1 < p \leq q < \infty$  and  $E, F$  are two reflexive complex Banach spaces.

**4.5.1 Preliminaries notions.**

To present the principal result compactness of the little Hankel operator with operator-valued holomorphic symbol, we first need the following results.

**Lemma 4.5.1.** Let  $1 < t < \infty$ , and  $b(z) = \sum_{\beta \in \mathbb{N}^n} \hat{b}(\beta) z^\beta \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))$ . Then

$$\int_{\mathbb{B}_n} \langle b(z) (\overline{f(z)}), y_0^* \rangle_{F, F^*} d\nu_\alpha(z) = \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} z^\beta \langle \hat{b}(\beta) (\overline{f(z)}), y_0^* \rangle_{F, F^*} d\nu_\alpha(z),$$

for every  $f \in H^\infty(\mathbb{B}_n, E)$  and  $y_0^* \in F^*$  with  $\|y_0^*\|_{F^*} = 1$ .

*Proof.* Since  $b(z) = \sum_{\beta \in \mathbb{N}^n} \hat{b}(\beta) z^\beta \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\bar{E}, F))$ , we have that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\bar{E}, F)}^t d\nu_\alpha(z) = 0.$$

We have

$$\begin{aligned} & \left| \int_{\mathbb{B}_n} \left\langle \left( b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right) (\overline{f(z)}), y_0^* \right\rangle_{F, F^*} d\nu_\alpha(z) \right| \\ & \leq \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\bar{E}, F)} \| \overline{f(z)} \|_{\bar{E}} \| y_0^* \|_{F^*} d\nu_\alpha(z) \\ & = \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\bar{E}, F)} \| f(z) \|_E d\nu_\alpha(z) \\ & \lesssim \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\bar{E}, F)}^t d\nu_\alpha(z) \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ , where we have used Hölder's inequality on the last estimate.

Therefore, we have that

$$\begin{aligned} \int_{\mathbb{B}_n} \langle b(z) (\overline{f(z)}), y_0^* \rangle_{F, F^*} d\nu_\alpha(z) &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \left\langle \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta (\overline{f(z)}), y_0^* \right\rangle_{F, F^*} d\nu_\alpha(z) \\ &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \langle \hat{b}(\beta) z^\beta (\overline{f(z)}), y_0^* \rangle_{F, F^*} d\nu_\alpha(z) \\ &= \lim_{N \rightarrow \infty} \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \int_{\mathbb{B}_n} \langle \hat{b}(\beta) z^\beta (\overline{f(z)}), y_0^* \rangle_{F, F^*} d\nu_\alpha(z) \\ &= \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \langle \hat{b}(\beta) z^\beta (\overline{f(z)}), y_0^* \rangle_{F, F^*} d\nu_\alpha(z). \quad \square \end{aligned}$$

**4.5. Compactness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

In the following lemma, we compute the little Hankel operator when the operator-valued symbol is a monomial.

**Lemma 4.5.2.** Let  $1 < p < \infty$ ,  $\gamma \in \mathbb{N}^n$  and  $a_\gamma \in \mathcal{L}(\overline{E}, F)$ . For every  $f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta \in A_\alpha^p(\mathbb{B}_n, E)$ , we have

$$h_{a_\gamma z^\gamma}(f)(z) = \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta}.$$

*Proof.* Since

$$f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta \in A_\alpha^p(\mathbb{B}_n, E),$$

and  $p > 1$  by using [74, Corollary 4], it follows that

$$\int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta z^\beta \right\|_E^p d\nu_\alpha(z) \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (4.13)$$

Firstly, let us prove that

$$\int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) = \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \frac{a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \quad (4.14)$$

Let  $N \in \mathbb{N}$ . We have that

$$\begin{aligned} \int_{\mathbb{B}_n} \left\| \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta} - \sum_{|\beta| \leq N} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} \right\|_F d\nu_\alpha(w) &= \int_{\mathbb{B}_n} \left\| \frac{\sum_{|\beta| \geq N+1} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} \right\|_F d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \left\| \frac{a_\gamma \left( \sum_{|\beta| \geq N+1} (\overline{c_\beta}) \overline{w^\beta} \right)}{(1-\langle z, w \rangle)^{n+1+\alpha}} \right\|_F d\nu_\alpha(w) \\ &\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{E}, F)}}{(1-|z|)^{n+1+\alpha}} \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_E d\nu_\alpha(w) \\ &\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{E}, F)}}{(1-|z|)^{n+1+\alpha}} \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_E d\nu_\alpha(w) \\ &\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{E}, F)}}{(1-|z|)^{n+1+\alpha}} \left( \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_E^p d\nu_\alpha(w) \right)^{1/p}. \end{aligned}$$

Therefore

$$\left\| \int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta} - \sum_{|\beta| \leq N} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right\|_F$$

is less than or equal to

$$\frac{\|a_\gamma\|_{\mathcal{L}(\overline{E}, F)}}{(1-|z|)^{n+1+\alpha}} \left( \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_E^p d\nu_\alpha(w) \right)^{1/p}. \quad (4.15)$$

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By using (4.13) and (4.15), it follows that

$$\left\| \int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\bar{c}_\beta) \overline{w^\beta} - \sum_{|\beta| \leq N} a_\gamma(\bar{c}_\beta) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right\|_F \rightarrow 0$$

as  $N \rightarrow \infty$ , and so

$$\begin{aligned} \int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\bar{c}_\beta) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \frac{\sum_{|\beta| \leq N} a_\gamma(\bar{c}_\beta) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \lim_{N \rightarrow \infty} \sum_{|\beta| \leq N} \int_{\mathbb{B}_n} \frac{a_\gamma(\bar{c}_\beta) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \frac{a_\gamma(\bar{c}_\beta) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \end{aligned}$$

which is the desired result. Secondly, let us prove that

$$\int_{\mathbb{B}_n} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) = \sum_{k=0}^{\infty} \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w). \quad (4.16)$$

Let  $N \in \mathbb{N}$ . We have

$$\begin{aligned} \left| \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k \right| &\leq \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} |z|^k \\ &\leq \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} |z|^k \\ &= \frac{1}{(1 - |z|)^{n+1+\alpha}}. \end{aligned}$$

Since  $\int_{\mathbb{B}_n} \frac{1}{(1 - |z|)^{n+1+\alpha}} d\nu_\alpha(w) = \frac{1}{(1 - |z|)^{n+1+\alpha}}$ , by the dominated convergence theorem, we have that

$$\begin{aligned} \sum_{k=0}^{\infty} \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) &= \lim_{N \rightarrow \infty} \sum_{k=0}^N \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) \\ &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w). \end{aligned}$$

We are now ready to prove our lemma. For  $f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta \in A_\alpha^p(\mathbb{B}_n, E)$ , by using the following multi-nomial formula [77, (1.1)] and the following formula [77, (1.23)] respectively

$$\langle z, w \rangle^k = \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w^m}, \quad \int_{\mathbb{B}_n} |z^m|^2 d\nu_\alpha(z) = \frac{m! \Gamma(n + \alpha + 1)}{\Gamma(n + |m| + \alpha + 1)},$$

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we get that, using (4.14) and (4.16)

$$\begin{aligned}
h_{a_\gamma z^\gamma} f(z) &= \int_{\mathbb{B}_n} \frac{a_\gamma w^\gamma \left( \overline{\sum_{\beta \in \mathbb{N}^n} c_\beta w^\beta} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \frac{w^\gamma \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \frac{w^\gamma a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w^m} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \sum_{|m|=k} \frac{k!}{m!} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} z^m \overline{w^m} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \sum_{|m|=k} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)m!} \int_{\mathbb{B}_n} w^\gamma z^m \overline{w^{m+\beta}} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{m \in \mathbb{N}^n} \frac{\Gamma(n+1+\alpha+|m|)}{\Gamma(n+1+\alpha)m!} z^m \int_{\mathbb{B}_n} w^\gamma \overline{w^{\beta+m}} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\Gamma(n+1+\alpha+|\gamma-\beta|)}{\Gamma(n+1+\alpha)(\gamma-\beta)!} z^{\gamma-\beta} \int_{\mathbb{B}_n} |z^\gamma|^2 d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\Gamma(n+1+\alpha+|\gamma-\beta|)}{\Gamma(n+1+\alpha)(\gamma-\beta)!} \frac{\gamma! \Gamma(n+1+\alpha)}{\Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta} \\
&= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta}.
\end{aligned}$$

□

The goal of the following lemma is to prove that the linear span of the vector-valued Bergman kernel  $\frac{x^*}{(1 - \langle w, z \rangle)^{n+1+\alpha}}$ , where  $x^* \in E^*$  and  $z, w \in \mathbb{B}_n$  form a dense subspace in the vector-valued Bergman space  $A_\alpha^{p'}(\mathbb{B}_n, E^*)$ , with  $1 < p < \infty$  and  $p'$  is the conjugate exponent of  $p$ .

**Lemma 4.5.3.** Let  $1 < p < \infty$ . For each  $x^* \in E^*$  and  $z \in \mathbb{B}_n$ , let

$$e_{z, x^*}(w) = \frac{x^*}{(1 - \langle w, z \rangle)^{n+1+\alpha}}; \quad w \in \mathbb{B}_n.$$

Then  $e_{z, x^*} \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$  and the subspace generated by  $e_{z, x^*}$  is dense in  $A_\alpha^{p'}(\mathbb{B}_n, E^*)$ .

*Proof.* Let  $\phi \in A_\alpha^p(\mathbb{B}_n, E)$  such that  $\langle \phi, e_{z, x^*} \rangle_{\alpha, E} = 0$  for all  $z \in \mathbb{B}_n$  and  $x^* \in E^*$ . Let  $f^* \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ . According to the Hahn-Banach theorem, it suffices to prove that  $\langle \phi, f^* \rangle_{\alpha, E} = 0$ . For all  $z \in \mathbb{B}_n$  and  $x^* \in E^*$ ,

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using Proposition 1.2.24 and the reproducing kernel formula, it follows that

$$\begin{aligned}
 0 &= \langle \phi, e_{z, x^*} \rangle_{\alpha, E} \\
 &= \int_{\mathbb{B}_n} \langle \phi(w), e_{z, x^*}(w) \rangle_{E, E^*} d\nu_\alpha(w) \\
 &= \int_{\mathbb{B}_n} \langle \phi(w), \frac{x^*}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \rangle_{E, E^*} d\nu_\alpha(w) \\
 &= \int_{\mathbb{B}_n} \langle \frac{\phi(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, x^* \rangle_{E, E^*} d\nu_\alpha(w) \\
 &= \langle \phi(z), x^* \rangle_{E, E^*}.
 \end{aligned}$$

Therefore, for all  $x^* \in E^*$ , we have

$$\langle \phi(z), x^* \rangle_{E, E^*} = 0.$$

Thus  $\phi(z) = 0$  for every  $z \in \mathbb{B}_n$ . It follows that for each  $f^* \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ , we have

$$\langle \phi, f^* \rangle_{\alpha, E} = \int_{\mathbb{B}_n} \langle \phi(z), f^*(z) \rangle_{E, E^*} d\nu_\alpha(z) = 0.$$

□

**Lemma 4.5.4.** Suppose that  $1 < p < \infty$ , and  $E$  is a reflexive complex Banach space. Let  $\{f_j\} \subset A_\alpha^p(\mathbb{B}_n, E)$  such that  $f_j \rightarrow 0$  weakly in  $A_\alpha^p(\mathbb{B}_n, E)$  as  $j \rightarrow \infty$ . Then for each  $\beta \in \mathbb{N}^n$ , we have that  $\partial^\beta f_j(0) \rightarrow 0$  weakly in  $E$  as  $j \rightarrow \infty$ , where  $\partial^\beta = \frac{\partial^{|\beta|}}{\partial z^\beta}$ .

*Proof.* Since for each  $j \in \mathbb{N}$ ,  $f_j \in A_\alpha^p(\mathbb{B}_n, E)$ , using the reproducing kernel formula in Proposition 2.2.2, we have that

$$f_j(z) = \int_{\mathbb{B}_n} \frac{f_j(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \quad z \in \mathbb{B}_n.$$

Differentiating both sides of the previous relation with respect to  $z$ , we obtain

$$\partial^\beta f_j(z) = C(n, \alpha, |\beta|) \int_{\mathbb{B}_n} \frac{f_j(w) \bar{w}^\beta}{(1 - \langle z, w \rangle)^{n+1+\alpha+|\beta|}} d\nu_\alpha(w).$$

Therefore, we have

$$\partial^\beta f_j(0) = C(n, \alpha, |\beta|) \int_{\mathbb{B}_n} f_j(w) \bar{w}^\beta d\nu_\alpha(w).$$

Now, let  $x^* \in E^*$  and let us show that  $\langle \partial^\beta f_j(0), x^* \rangle_{E, E^*} \rightarrow 0$  as  $j \rightarrow \infty$ . But we have that

$$\begin{aligned}
 \langle \partial^\beta f_j(0), x^* \rangle_{E, E^*} &= C(n, \alpha, |\beta|) \left\langle \int_{\mathbb{B}_n} f_j(w) \bar{w}^\beta d\nu_\alpha(w), x^* \right\rangle_{E, E^*} \\
 &= \int_{\mathbb{B}_n} \langle f_j(w), x^* w^\beta \rangle_{E, E^*} d\nu_\alpha(w) \\
 &= \langle f_j, g \rangle_{\alpha, E} \rightarrow 0 \quad \text{as } j \rightarrow \infty,
 \end{aligned}$$

with  $g(z) = x^* z^\beta \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ . Thus,  $\langle \partial^\beta f_j(0), x^* \rangle_{E, E^*} \rightarrow 0$  as  $j \rightarrow \infty$ . □

We recall that the symbol  $b$  used in the following lemma satisfies (4.1) and (4.2). We also recall that the set  $\mathcal{K}(E, F)$  is the space of all compacts operators from  $E$  to  $F$ .

**Lemma 4.5.5.** Suppose that  $E$  is a reflexive complex Banach space and  $k$  is a nonnegative integer. If the holomorphic mapping  $z \mapsto b(z)$  maps  $\mathbb{B}_n$  into  $\mathcal{K}(\bar{E}, F)$ , then the holomorphic mapping  $z \mapsto R^{\alpha, k} b(z)$  also

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maps  $\mathbb{B}_n$  into  $\mathcal{K}(\overline{E}, F)$ .

*Proof.* Let  $z \in \mathbb{B}_n$ . Let  $\{f_j\}$  a sequence of elements of  $E$  which converges weakly to 0 in  $E$  as  $j$  tends to infinity. Let us prove that  $\lim_{j \rightarrow \infty} \|R^{\alpha,k}b(z)(\overline{f_j})\|_F = 0$ . We know that the sequence  $\{f_j\}$  is strongly bounded in  $E$ . Let  $j \in \mathbb{N}$ , by using (4.1) for  $z = 0$ , we get that the function  $z \mapsto b(z)(\overline{f_j}) \in A_\alpha^1(\mathbb{B}_n, F)$ . By the reproducing kernel formula, it follows that

$$b(z)(\overline{f_j}) = \int_{\mathbb{B}_n} \frac{b(w)(\overline{f_j})}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w). \quad (4.17)$$

Applying the partial differential operator  $R^{\alpha,k}$  to (4.17), we have

$$R^{\alpha,k}b(z)(\overline{f_j}) = \int_{\mathbb{B}_n} \frac{b(w)(\overline{f_j})}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w).$$

We also have

$$\begin{aligned} \frac{\|b(w)(\overline{f_j})\|_F}{|1 - \langle z, w \rangle|^{n+1+\alpha+k}} &\leq \frac{\|b(w)\|_{\mathcal{L}(\overline{E}, F)} \|f_j\|_E}{(1 - |z|)^{n+1+\alpha+k}} \\ &\leq \frac{C(n+1+\alpha)}{(1 - |z|)^{n+1+\alpha+k}} \|b(w)\|_{\mathcal{L}(\overline{E}, F)}, \end{aligned}$$

and

$$\int_{\mathbb{B}_n} \frac{C(n+1+\alpha)}{(1 - |z|)^{n+1+\alpha+k}} \|b(w)\|_{\mathcal{L}(\overline{E}, F)} d\nu_\alpha(w) < \infty.$$

Therefore, by applying the dominated convergence theorem, we have that

$$\begin{aligned} \limsup_{j \rightarrow \infty} \|R^{\alpha,k}b(z)(\overline{f_j})\|_F &\leq \limsup_{j \rightarrow \infty} \int_{\mathbb{B}_n} \frac{\|b(w)(\overline{f_j})\|_F}{|1 - \langle z, w \rangle|^{n+1+\alpha+k}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{\lim_{j \rightarrow \infty} \|b(w)(\overline{f_j})\|_F}{|1 - \langle z, w \rangle|^{n+1+\alpha+k}} d\nu_\alpha(w) = 0. \end{aligned}$$

Thus for each  $z \in \mathbb{B}_n$

$$\lim_{j \rightarrow \infty} \|R^{\alpha,k}b(z)(\overline{f_j})\|_F = 0.$$

□

The following result will be also important in the sequel.

**Lemma 4.5.6.** Suppose  $\beta_0 \in \mathbb{N}^n$ ,  $\{f_j\}$  a sequence of elements of  $E$  which converges weakly to 0 as  $j$  tends to infinity. For  $z \in \mathbb{B}_n$ , let  $x_j(z) = z^{\beta_0} f_j$ . Then  $\{x_j\} \subset A_\alpha^p(\mathbb{B}_n, E)$  and  $\{x_j\}$  converges weakly to 0 in  $A_\alpha^p(\mathbb{B}_n, E)$ .

*Proof.* Let  $j \in \mathbb{N}$ . Since  $f_j \rightarrow 0$  weakly in  $E$  as  $j \rightarrow \infty$ , it follows that  $\{f_j\}$  is strongly bounded in  $E$  (see [22]). Let  $\beta_0 \in \mathbb{N}^n$  and  $x_j(z) = z^{\beta_0} f_j$ . It is clear that  $\{x_j\} \subset A_\alpha^p(\mathbb{B}_n, E)$ . For every  $g \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ , we have

$$\begin{aligned} \langle x_j, g \rangle_{\alpha, E} &= \int_{\mathbb{B}_n} \langle x_j(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle z^{\beta_0} f_j, g(z) \rangle_{E, E^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} z^{\beta_0} \langle f_j, g(z) \rangle_{E, E^*} d\nu_\alpha(z). \end{aligned}$$

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Since

$$\begin{aligned} |z^{\beta_0} \langle f_j, g(z) \rangle_{E, E^*}| &\leq |z^{\beta_0} \langle f_j, g(z) \rangle_{E, E^*}| \\ &\leq \|f_j\|_E \|g(z)\|_{X^*} \\ &\leq C \|g(z)\|_{E^*}, \end{aligned}$$

and

$$\int_{\mathbb{B}_n} \|g(z)\|_{E^*} d\nu_\alpha(z) \leq \left( \int_{\mathbb{B}_n} \|g(z)\|_{E^*}^{p'} d\nu_\alpha(z) \right)^{1/p'} < \infty.$$

By using the dominated convergence theorem and the assumption, it follows that

$$\limsup_{j \rightarrow \infty} \langle x_j, g \rangle_{\alpha, E} = \int_{\mathbb{B}_n} z^{\beta_0} \lim_{j \rightarrow \infty} \langle f_j, g(z) \rangle_{E, E^*} d\nu_\alpha(z) = 0. \quad \square$$

**4.5.2 Compactness of  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

From now on, we choose  $\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ , with  $1 < p \leq q < \infty$ , and we consider the vector-valued Lipschitz space  $\Lambda_{\gamma_0}(\mathbb{B}_n, E)$ . In what follows, we assume that  $E, F$  are reflexive complex Banach spaces. We first introduce the following proposition which will be useful in the proof of Theorem 4.5.8.

**Proposition 4.5.7.** Suppose  $1 < p \leq q < \infty$ ,  $0 \leq r < 1$  and  $\gamma \in \mathbb{N}^n$ . If  $a_\gamma \in \mathcal{K}(\overline{E}, F)$ , then the little Hankel operator  $h_{g_r^\gamma} : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a compact operator, where  $g_r^\gamma(z) = a_\gamma(rz)^\gamma$  for every  $z \in \mathbb{B}_n$ .

*Proof.* Let  $\{f_j\}$  be a sequence in  $A_\alpha^p(\mathbb{B}_n, E)$  such that  $f_j \rightarrow 0$  weakly in  $A_\alpha^p(\mathbb{B}_n, E)$  as  $j$  tends to infinity. We want to prove that  $\lim_{j \rightarrow \infty} \|h_{g_r^\gamma}(f_j)\|_{q, \alpha, F} = 0$ . Let the Taylor expansion of  $f_j$  given by  $f_j(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta^j z^\beta \in A_\alpha^p(\mathbb{B}_n, E)$ . Since  $f_j \rightarrow 0$  weakly in  $A_\alpha^p(\mathbb{B}_n, E)$ , applying Lemma 4.5.4, using the fact that  $c_\beta^j = \partial^\beta f_j(0)/\beta!$ , we have that for all  $\beta \in \mathbb{N}^n$ ,  $c_\beta^j \rightarrow 0$  weakly in  $E$  as  $j \rightarrow \infty$ . By Lemma 4.5.2, for every  $z \in \mathbb{B}_n$ , we have

$$h_{g_r^\gamma}(f_j)(z) = \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta^j}) \frac{\gamma! \Gamma(n + 1 + \alpha + |\gamma - \beta|)}{(\gamma - \beta)! \Gamma(n + 1 + \alpha + |\gamma|)} r^{|\gamma - \beta|} z^{\gamma - \beta}.$$

Therefore,

$$\begin{aligned} \|h_{g_r^\gamma}(f_j)\|_{q, \alpha, F} &= \left( \int_{\mathbb{B}_n} \left\| \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta^j}) \frac{\gamma! \Gamma(n + 1 + \alpha + |\gamma - \beta|)}{(\gamma - \beta)! \Gamma(n + 1 + \alpha + |\gamma|)} (rz)^{\gamma - \beta} \right\|_F^q d\nu_\alpha(z) \right)^{1/q} \\ &\leq \left( \int_{\mathbb{B}_n} \left( \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n + 1 + \alpha + |\gamma - \beta|) \|a_\gamma(\overline{c_\beta^j})\|_F |r|^{|\gamma - \beta|}}{(\gamma - \beta)! \Gamma(n + 1 + \alpha + |\gamma|)} \right)^q d\nu_\alpha(z) \right)^{1/q} \\ &\leq \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \left( \int_{\mathbb{B}_n} \left( \|a_\gamma(\overline{c_\beta^j})\|_F \frac{\gamma! \Gamma(n + 1 + \alpha + |\gamma - \beta|)}{(\gamma - \beta)! \Gamma(n + 1 + \alpha + |\gamma|)} (r|z|)^{|\gamma - \beta|} \right)^q d\nu_\alpha(z) \right)^{1/q} \\ &= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \|a_\gamma(\overline{c_\beta^j})\|_F \frac{\gamma! \Gamma(n + 1 + \alpha + |\gamma - \beta|)}{(\gamma - \beta)! \Gamma(n + 1 + \alpha + |\gamma|)} \left( \int_{\mathbb{B}_n} (r|z|)^{|\gamma - \beta| q} d\nu_\alpha(z) \right)^{1/q} \\ &\lesssim \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n + 1 + \alpha + |\gamma - \beta|)}{(\gamma - \beta)! \Gamma(n + 1 + \alpha + |\gamma|)} \|a_\gamma(\overline{c_\beta^j})\|_F, \end{aligned}$$

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where the third line above is justified by the Minkowski's inequality for integrals. Thus,

$$\|h_{g_r^\gamma}(f_j)\|_{q,\alpha,F} \lesssim \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} \|a_\gamma(\overline{c_\beta^j})\|_F. \quad (4.18)$$

Now, since  $c_\beta^j \rightarrow 0$  weakly in  $E$  as  $j \rightarrow \infty$ , it is clear that  $\overline{c_\beta^j} \rightarrow 0$  weakly in  $\overline{E}$  as  $j \rightarrow \infty$ . By the assumption, we know that  $a_\gamma \in \mathcal{K}(\overline{E}, F)$ . Since  $\overline{c_\beta^j} \rightarrow 0$  weakly in  $\overline{E}$  as  $j \rightarrow \infty$ , we have that  $\|a_\gamma(\overline{c_\beta^j})\|_F \rightarrow 0$  as  $j \rightarrow \infty$ . It follows that

$$\limsup_{j \rightarrow \infty} \|h_{g_r^\gamma}(f_j)\|_{q,\alpha,F} \lesssim \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} \lim_{j \rightarrow \infty} \|a_\gamma(\overline{c_\beta^j})\|_F = 0.$$

□

We are now ready to give the proof of the main result of this section.

**Theorem 4.5.8.** Let  $E$  and  $F$  be two reflexive complex Banach spaces. Suppose that  $1 < p \leq q < \infty$ , and  $\alpha > -1$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a compact operator if and only if

$$b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F)),$$

where  $\Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$  denotes the little vector-valued Lipschitz space and  $\gamma_0 = (n+1+\alpha) \left(\frac{1}{p} - \frac{1}{q}\right)$ , see (3.15).

*Proof.* First assume that  $b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$  and denote by  $b_r(z) := b(rz)$  with  $z \in \mathbb{B}_n$  and  $0 < r < 1$ . Since  $b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{E}, F))$ , by Theorem 4.2.3, we have that

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}.$$

Therefore, we have

$$\|h_b - h_{b_r}\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \lesssim \|b - b_r\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}.$$

By using Proposition 3.3.17, we have that

$$\lim_{r \rightarrow 1^-} \|b - b_r\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} = 0,$$

so to prove that  $h_b$  is a compact operator, it suffices to prove that  $h_{b_r}$  is a compact operator. Since  $b_r$  is analytic on a neighbourhood of  $\overline{\mathbb{B}_n}$ , it can be approximated by its Taylor polynomial in the vector-valued Lipschitz norm. Thus,

$$\lim_{N \rightarrow \infty} \|b_r - P_{N,r}\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))} = 0, \quad (4.19)$$

with  $P_{N,r}(z) = \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) r^{|\beta|} z^\beta$ , where  $\hat{b}(\beta) \in \mathcal{K}(\overline{E}, F)$  are the Taylor coefficients of  $b$ . We also have by Theorem 4.2.3 that

$$\|h_{b_r} - h_{P_{N,r}}\|_{A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)} \lesssim \|b_r - P_{N,r}\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))}.$$

So by (4.19), to prove that  $h_{b_r}$  is a compact operator, it is enough to prove that  $h_{P_{N,r}}$  is a compact operator. Since  $P_{N,r}$  is a polynomial, it is enough to do the proof for monomials of the form  $\hat{b}(\beta) r^{|\beta|} z^\beta$ , with  $\beta \in \mathbb{N}^n$ ,  $z \in \mathbb{B}_n$  and  $\hat{b}(\beta) \in \mathcal{K}(\overline{E}, F)$ . Thus, according to Proposition 4.5.7, the proof of this part is complete.

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Conversely, for the "only if part", let us assume that

$$h_b : A_\alpha^p(\mathbb{B}_n, E) \longrightarrow A_\alpha^q(\mathbb{B}_n, F)$$

is a compact operator. Since  $h_b$  is compact,  $h_b$  is then bounded and Theorem 4.2.3 yields

$$b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F)).$$

We shall first prove that the Taylor coefficients  $\hat{b}(\beta)$ ,  $\beta \in \mathbb{N}^n$  of  $b$  belongs to  $\mathcal{K}(\overline{E}, F)$ . Let  $\{f_j\} \subset E$  such that  $f_j \rightarrow 0$  weakly in  $E$  as  $j \rightarrow \infty$ , fix  $\beta_0 \in \mathbb{N}^n$ , and let  $x_j(z) = z^{\beta_0} f_j$ . By Lemma 4.5.6, we have  $\{x_j\} \subset A_\alpha^p(\mathbb{B}_n, E)$  and  $\{x_j\}$  converges weakly to 0 in  $A_\alpha^p(\mathbb{B}_n, E)$ . Since

$$\|\hat{b}(\beta_0)(\overline{f_j})\|_F = \sup_{\|y^*\|_{F^*}=1} |\langle \hat{b}(\beta_0)(\overline{f_j}), y^* \rangle_{F, F^*}|$$

and  $F$  is reflexive, by the Kakutani's theorem [22, Theorem 3.17] there exists  $y_j^* \in F^*$  with  $\|y_j^*\|_{F^*} = 1$  such that

$$\|\hat{b}(\beta_0)(\overline{f_j})\|_F = |\langle \hat{b}(\beta_0)(\overline{f_j}), y_j^* \rangle_{F, F^*}|.$$

But  $y_j^* \in A_\alpha^{p'}(\mathbb{B}_n, F^*)$ . By Lemma 4.1.4, we have

$$\begin{aligned} |\langle h_b(x_j), y_j^* \rangle_{\alpha, F}| &= \left| \int_{\mathbb{B}_n} \langle b(z)(\overline{x_k})(z), y_j^* \rangle_{F, F^*} d\nu_\alpha(z) \right| \\ &= \left| \int_{\mathbb{B}_n} \overline{z}^{\beta_0} \langle \sum_{\beta \in \mathbb{N}^n} z^\beta \hat{b}(\beta)(\overline{f_j}), y_j^* \rangle_{F, F^*} d\nu_\alpha(z) \right| \\ &= \left| \sum_{\beta \in \mathbb{N}^n} \langle \hat{b}(\beta)(\overline{f_j}), y_j^* \rangle_{F, F^*} \int_{\mathbb{B}_n} z^\beta \overline{z}^{\beta_0} d\nu_\alpha(z) \right| \\ &= |\langle \hat{b}(\beta_0)(\overline{f_j}), y_j^* \rangle_{F, F^*}| \int_{\mathbb{B}_n} |z^{\beta_0}|^2 d\nu_\alpha(z) \\ &= \frac{\beta_0! \Gamma(n + \alpha + 1)}{\Gamma(n + |\beta_0| + \alpha + 1)} |\langle \hat{b}(\beta_0)(\overline{f_j}), y_j^* \rangle_{F, F^*}| \\ &= \frac{\beta_0! \Gamma(n + \alpha + 1)}{\Gamma(n + |\beta_0| + \alpha + 1)} \|\hat{b}(\beta_0)(\overline{f_j})\|_F, \end{aligned}$$

where Fubini's theorem is justified by Lemma 4.5.1 with  $\{x_j\} \subset H^\infty(\mathbb{B}_n, E)$ . Since  $h_b$  is compact and  $\{x_j\}$  converges weakly to 0 as  $j$  tends to infinity, we have that  $\{h_b(x_j)\}$  converges strongly to 0 as  $j$  tends to infinity, therefore one gets that

$$\lim_{j \rightarrow \infty} \langle h_b(x_j), y_j^* \rangle_{\alpha, F} = 0.$$

Thus

$$\lim_{j \rightarrow \infty} \frac{\beta_0! \Gamma(n + \alpha + 1)}{\Gamma(n + |\beta_0| + \alpha + 1)} \|\hat{b}(\beta_0)(\overline{f_j})\|_F = 0.$$

We then obtain

$$\lim_{j \rightarrow \infty} \|\hat{b}(\beta_0)(\overline{f_j})\|_F = 0.$$

In fact, we have shown that  $\hat{b}(\beta_0)$  belongs to  $\mathcal{K}(\overline{E}, F)$  and as  $\beta_0$  is arbitrary, this holds for all  $\beta \in \mathbb{N}^n$ . Let

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$1 < t < \infty$ . Since  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , we have that  $b \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  and

$$\lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \|b(w) - \sum_{|\beta| \leq N} \hat{b}(\beta) w^\beta\|_{\mathcal{L}(\overline{E}, F)}^t d\nu_\alpha(w) = 0.$$

Let  $z \in \mathbb{B}_n$ . There exists a constant  $C_z > 0$  such that

$$\|b(z) - \sum_{|\beta| \leq N} \hat{b}(\beta) z^\beta\|_{\mathcal{L}(\overline{E}, F)}^t \leq C_z \int_{\mathbb{B}_n} \|b(w) - \sum_{|\beta| \leq N} \hat{b}(\beta) w^\beta\|_{\mathcal{L}(\overline{E}, F)}^t d\nu_\alpha(w).$$

Thus,

$$\lim_{N \rightarrow \infty} \|b(z) - \sum_{|\beta| \leq N} \hat{b}(\beta) z^\beta\|_{\mathcal{L}(\overline{E}, F)} = 0.$$

Since  $z \in \mathbb{B}_n$  is arbitrary, we deduce that  $b(z) \in \mathcal{K}(\overline{E}, F)$ , for each  $z \in \mathbb{B}_n$ . It remains to show that  $b$  satisfy the "little  $\gamma_0$ - Lipschitz" condition. Let  $x \in E$  and  $y^* \in F^*$ . Since  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$ , then the mapping  $z \mapsto \langle b(z)\overline{x}, y^* \rangle_{F, F^*}$  belongs to  $A_\alpha^1(\mathbb{B}_n)$ . By using the reproducing kernel formula, it follows that

$$\langle b(z)\overline{x}, y^* \rangle_{F, F^*} = \int_{\mathbb{B}_n} \frac{\langle b(w)\overline{x}, y^* \rangle_{F, F^*}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w). \quad (4.20)$$

Let  $k > \gamma_0$ . Applying the operator  $R^{\alpha, k}$  in (4.20), we obtain that

$$\langle R^{\alpha, k} b(z)\overline{x}, y^* \rangle_{F, F^*} = \int_{\mathbb{B}_n} \frac{\langle b(w)\overline{x}, y^* \rangle_{F, F^*}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w). \quad (4.21)$$

Let  $z \in \mathbb{B}_n$ . Since  $\|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{E}, F)} = \sup_{\|x\|_E=1} \|R^{\alpha, k} b(z)\overline{x}\|_F$ , by Lemma 4.5.5, the operator  $R^{\alpha, k} b(z)$  is compact. So there exists  $x_0(z) \in E$  with  $\|x_0(z)\|_E = 1$  and

$$\|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{E}, F)} = \|R^{\alpha, k} b(z)\overline{x_0(z)}\|_F.$$

Also

$$\|R^{\alpha, k} b(z)\overline{x_0(z)}\|_F = \sup_{\|y^*\|_{F^*}=1} |\langle R^{\alpha, k} b(z)\overline{x_0(z)}, y^* \rangle_{F, F^*}|.$$

Since  $F$  is reflexive, it follows by the Kakutani's theorem [22, Theorem 3.17] that there exists  $y_0^*(z) \in F^*$  with  $\|y_0^*(z)\|_{F^*} = 1$  such that

$$\|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{E}, F)} = \|R^{\alpha, k} b(z)\overline{x_0(z)}\|_F = |\langle R^{\alpha, k} b(z)\overline{x_0(z)}, y_0^*(z) \rangle_{F, F^*}|. \quad (4.22)$$

By (4.21) and (4.22) we get

$$(1 - |z|^2)^{k-\gamma_0} \|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{E}, F)} = \left| \int_{\mathbb{B}_n} \langle b(w)\overline{x_0(z)}, y_0^*(z) \rangle_{F, F^*} \frac{(1 - |z|^2)^{k-\gamma_0}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) \right| = |\langle h_b(x_z), y_z^* \rangle_{\alpha, F}|,$$

with

$$x_z(w) = \frac{x_0(z)(1 - |z|^2)^{\beta-(n+1+\alpha)/p}}{(1 - \langle w, z \rangle)^\beta}, \quad w \in \mathbb{B}_n$$

and

$$y_z^*(w) = \frac{y_0^*(z)(1 - |z|^2)^{k+(n+1+\alpha)/q-\beta}}{(1 - \langle w, z \rangle)^{n+1+\alpha+k-\beta}}, \quad w \in \mathbb{B}_n,$$

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where  $\beta$  is chosen such that

$$(n + 1 + \alpha)/p < \beta < k + (n + 1 + \alpha)/q.$$

By Theorem 2.1.25 we have  $x_z \in A_\alpha^p(\mathbb{B}_n, E)$ ,  $y_z^* \in A_\alpha^{q'}(\mathbb{B}_n, F^*)$ , and

$$\sup_{z \in \mathbb{B}_n} \|x_z\|_{p,\alpha,E} < \infty, \quad \sup_{z \in \mathbb{B}_n} \|y_z^*\|_{q',\alpha,F^*} < \infty.$$

Let us prove that

$$x_z \longrightarrow 0 \text{ weakly in } A_\alpha^p(\mathbb{B}_n, E) \text{ as } |z| \longrightarrow 1^-. \quad (4.23)$$

Since

$$\sup_{z \in \mathbb{B}_n} \|x_z\|_{p,\alpha,E} < \infty,$$

to prove (4.23), by Lemma 4.5.3, it suffices to prove that

$$\langle x_z, e_{w,a^*} \rangle_{\alpha,E} \longrightarrow 0 \text{ as } |z| \longrightarrow 1^-,$$

where for each  $a^* \in E^*$  and  $w \in \mathbb{B}_n$ , we have

$$e_{w,a^*}(\zeta) = \frac{1}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha}} a^*, \quad \zeta \in \mathbb{B}_n.$$

By using the definition of  $e_{w,a^*}$  and the reproducing kernel formula, it follows that

$$\begin{aligned} \langle x_z, e_{w,a^*} \rangle_{p,\alpha,E} &= \int_{\mathbb{B}_n} \langle x_z(\zeta), e_{w,a^*}(\zeta) \rangle_{E,E^*} d\nu_\alpha(\zeta) \\ &= \int_{\mathbb{B}_n} \langle x_z(\zeta), \frac{1}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha}} a^* \rangle_{E,E^*} d\nu_\alpha(\zeta) \\ &= \left\langle \int_{\mathbb{B}_n} \frac{x_z(\zeta)}{(1 - \langle w, \zeta \rangle)^{n+1+\alpha}} d\nu_\alpha(\zeta), a^* \right\rangle_{E,E^*} \\ &= \langle x_z(w), a^* \rangle_{E,E^*}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |\langle x_z, e_{w,a^*} \rangle_{p,\alpha,E}| &= |\langle x_z(w), a^* \rangle_{E,E^*}| \\ &= \left| \frac{(1 - |z|^2)^{\beta - (n+1+\alpha)/p}}{(1 - \langle w, z \rangle)^\beta} \langle x_0(z), a^* \rangle_{E,E^*} \right| \\ &\leq \frac{(1 - |z|^2)^{\beta - (n+1+\alpha)/p}}{(1 - |w|)^\beta} \|a^*\|_{E^*} \longrightarrow 0 \end{aligned}$$

as  $|z| \longrightarrow 1^-$ . By using (4.23), the compactness of  $h_b$  and the fact that

$$\sup_{z \in \mathbb{B}_n} \|y_z^*\|_{q',\alpha,F^*} < \infty,$$

it follows that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma_0} \|R^{\alpha,k} b(z)\|_{\mathcal{L}(\bar{E}, F)} = \lim_{|z| \rightarrow 1^-} |\langle h_b(x_z), y_z^* \rangle_{\alpha,F}| = 0,$$

**4.5. Compactness of the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$ , case  $1 < p \leq q < \infty$ .**

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which completes the proof of the theorem. □

# Toeplitz operators on vector-valued Bergman spaces

Toeplitz operators are natural classes of specific operators on spaces of holomorphic functions. Exploring their properties on a variety of spaces brings deeper understanding not only on the operators, but also on the spaces in question. The study of their behaviour on the Hardy and Bergman spaces has generated an extensive list of results in operator theory and in theory of functions spaces. One of the useful approaches in this area is the use of the Berezin transform as a determining factor of the behaviour of the Toeplitz operator. This method is motivated by its connections with quantum physics and noncommutative geometry [51]. In this chapter, we study the boundedness and the compactness of Toeplitz operator  $T_b$  with operator-valued symbol  $b$  acting on the weighted vector-valued Bergman spaces  $A_\alpha^2(\mathbb{B}_n, E)$ , where  $E$  is a Banach space. The first result on the compactness of the Toeplitz operator  $T_b$  on the vector-valued Bergman space  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  with matrix-valued symbol  $b$  appeared in [60]. More precisely, for an integer  $d \geq 1$ , let  $L_{\mathcal{M}^d}^\infty$  denote the set of  $d \times d$  matrix-valued functions  $f : \mathbb{B}_n \rightarrow \mathcal{M}^d$  such that  $z \mapsto \|f(z)\|$  is in  $L^\infty(\mathbb{B}_n)$ .

For a function  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , the Toeplitz operator  $T_b$  on  $A_\alpha^2(\mathbb{B}_n, E)$  is densely defined by

$$T_b(f)(z) := \int_{\mathbb{B}_n} \frac{b(w)(f(w))}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \tag{5.1}$$

where  $f$  is in  $H^\infty(\mathbb{B}_n, E)$ . The following theorem was proved by Rahm [60].

**Theorem 5.0.9.** Let  $d \geq 1$  be an integer and  $T \in \mathcal{L}(A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d))$  which can be written as

$$T := \sum_{j=1}^m \prod_{k=1}^{m_j} T_{b_{jk}}, \tag{5.2}$$

where  $b = (b_{jk}) \in L_{\mathcal{M}^d}^\infty$ . Then the following are equivalent:

- (1)  $T$  is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ .
- (2)  $\langle \tilde{T}(z)(e), h \rangle_{\mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$  for all  $e, h \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = \|h\|_{\mathbb{C}^d} = 1$ .
- (3)  $T^z e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ , for all  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .
- (4)  $\|T^z e\|_{L_\alpha^p(\mathbb{B}_n, \mathbb{C}^d)} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for any  $p > 1$  and  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .

## 5.1. Some reminders and definitions.

Moreover, Rahm shows that if the condition (5.2) can be replaced by a bounded operator  $T$  with

$$\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_z e\|_{p,\alpha,\mathbb{C}^d} < \infty, \quad p \geq 1. \quad (5.3)$$

the same result holds.

From Theorem 5.0.9, we have a complete description of the compactness of the Toeplitz operator  $T_b$  on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  with  $b \in L^\infty_{\mathcal{M}^d}$ . In this chapter, we study the compactness of the Toeplitz operator  $T_b$  on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  with bounded mean oscillation symbol ( $b \in BMO^1(\mathbb{B}_n, \mathcal{M}^d)$ ).

The study of Toeplitz operators with  $BMO$  symbols began in [79] where Zorboska proved that the Toeplitz operator  $T_b$  with a bounded mean oscillation ( $BMO^1$ ) symbol  $b$  is bounded (resp. compact) on the scalar-valued Bergman space  $A^2(\mathbb{D})$  (where  $\mathbb{D} = \mathbb{B}_1$  is the unit disk) if and only if its Berezin transform  $\tilde{b}$  is bounded in  $\mathbb{D}$  (resp.  $\tilde{b}$  vanishes at the boundary of  $\mathbb{D}$ ). Later, these results were extended to the scalar-valued weighted Bergman space of the unit ball  $A_\alpha^2(\mathbb{B}_n)$  [70].

In this chapter, we extend main results in [79] to vector-valued settings. Precisely, we prove the two following theorems.

**Theorem 5.0.10.** Let  $E, F$  be two complex Banach spaces and  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . The Toeplitz operator  $T_b : A_\alpha^2(\mathbb{B}_n, E) \rightarrow A_\alpha^2(\mathbb{B}_n, F)$  is a bounded linear operator if and only if its Berezin transform  $\tilde{b}$  is bounded on  $\mathbb{B}_n$ .

**Theorem 5.0.11.** Let  $d \geq 1$  be an integer and  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$  such that  $\tilde{b} \in L^\infty(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ . The following are equivalent.

- (1)  $T_b$  is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ .
- (2)  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .
- (3)  $T_{b \circ \varphi_z} e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .
- (4)  $\|T_{b \circ \varphi_z} e\|_{p,\alpha,\mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and every  $p > 1$ .
- (5)  $\|T_{b \circ \varphi_z} e\|_{p,\alpha,\mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and some  $0 < p - 1 < \frac{\alpha+1}{n+1+\alpha}$ .

The proof of Theorem 5.0.10 follows the same idea of the proof of [70, Theorem 3.1] and the proof of Theorem 5.0.11 used the same approach in the proof of [60, Theorem 4.1].

## 5.1 Some reminders and definitions.

We recall that the reproducing kernel  $K_w^\alpha$  and the normalized reproducing kernel  $k_w^\alpha$  in the classical Bergman space  $A_\alpha^2(\mathbb{B}_n)$ , are given respectively by

$$K_w^\alpha(z) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad (5.4)$$

and

$$k_w^\alpha(z) = \frac{(1 - |w|^2)^{\frac{n+1+\alpha}{2}}}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad (5.5)$$

## 5.2. Carleson type measures for vector-valued Bergman spaces.

for  $z, w \in \mathbb{B}_n$ . We recall that for  $r > 0$  and  $z \in \mathbb{B}_n$ , (see Lemma 2.1.27 and Lemma 2.1.9) we have

$$|1 - \langle z, u \rangle| \simeq 1 - |z|^2 \quad u \in D(z, r), \quad (5.6)$$

$$\nu_\alpha(D(z, r)) \simeq (1 - |z|^2)^{n+1+\alpha}. \quad (5.7)$$

The following identities can be found in [70, 79].

$$\begin{aligned} |k_w^\alpha(\varphi_z(v))|^2 |k_z^\alpha(v)|^2 &= |k_{\varphi_z(w)}^\alpha(v)|^2 & z, w, v \in \mathbb{B}_n, \\ \overline{K_{\varphi_z(u)}^\alpha(w)} k_z^\alpha(u) &= \overline{k_z^\alpha(w)} K_u^\alpha(\varphi_z(w)) & z, w, u \in \mathbb{B}_n, \\ \overline{K_u^\alpha(\varphi_z(v))} k_z^\alpha(v) &= k_z^\alpha(u) \overline{K_{\varphi_z(u)}^\alpha(v)} & z, u, v \in \mathbb{B}_n, \\ k_z^\alpha(a) K^\alpha(\varphi_z(a), \varphi_z(b)) \overline{k_z^\alpha(b)} &= K^\alpha(a, b) & z, a, b \in \mathbb{B}_n, \\ k_z^\alpha(w) k_z^\alpha \circ \varphi_z(w) &= 1 & z, w \in \mathbb{B}_n, \\ \varphi_z \circ \varphi_w &= \varphi_{\varphi_z(w)} \circ U & z, w \in \mathbb{B}_n, \quad \text{where } U \text{ is a unitary transformation of } \mathbb{C}^n. \end{aligned} \quad (5.8)$$

$$(5.9)$$

The Berezin transform of  $f$  defined by

$$\tilde{f}(z) := \int_{\mathbb{B}_n} f(w) |k_z^\alpha(w)|^2 d\nu_\alpha(w). \quad (5.10)$$

For  $f \in L_\alpha^1(\mathbb{B}_n, E)$ , the average of  $f$  over  $D(z, r)$  is defined by

$$\hat{f}(z) := \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} f(w) d\nu_\alpha(w). \quad (5.11)$$

Given two complex Banach spaces  $E$  and  $F$ , we recall that  $\mathcal{L}(E, F)$  is the space of all bounded linear operators  $T : E \rightarrow F$  with the norm

$$\|T\|_{\mathcal{L}(E, F)} = \sup_{\|x\|_E=1} \|T(x)\|_F = \sup_{\|x\|_E=1, \|y^*\|_{F^*}=1} |\langle T(x), y^* \rangle_{F, F^*}|.$$

## 5.2 Carleson type measures for vector-valued Bergman spaces.

The notion of Carleson measures was introduced by Carleson [23] for the unit disk. Carleson's original definition works well in the theory of Hardy spaces, and this can be seen in such classics as Duren [33] and Garnett [36]. The characterization of Carleson measures for the Hardy spaces on the unit ball can be found in Hörmander [41] and Power [56]. In this section, we introduce the notion of Carleson measures for weighted vector-valued Bergman spaces on the unit ball and relate the notion to Toeplitz operators whose symbol is a positive Borel measure. The corresponding notion for scalar-valued functions can be found in [78].

**Definition 5.2.1.** Let  $\mu$  be a positive finite Borel measure on  $\mathbb{B}_n$  and  $p > 0$ . We say that  $\mu$  is a *Carleson measure* for the vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, E)$  if there exists a finite constant  $c_\mu > 0$  such that

$$\int_{\mathbb{B}_n} \|f(z)\|_E^p d\mu(z) \leq c_\mu \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z), \quad (5.12)$$

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for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . Moreover, the Carleson norm  $\|\mu\|_{Carl}$  of  $\mu$  is defined by:

$$\|\mu\|_{Carl} := \inf \left\{ c_\mu > 0 : \int_{\mathbb{B}_n} \|f(z)\|_E^p d\mu(z) \leq c_\mu \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z) \quad \forall f \in A_\alpha^p(\mathbb{B}_n, E) \right\}.$$

We will show later that the property of being a Carleson measure for  $A_\alpha^p(\mathbb{B}_n, E)$  is actually independent of  $p$ , but only depends on  $\alpha$ .

**Theorem 5.2.2.** Let  $\mu$  be a positive finite Borel measure on  $\mathbb{B}_n$ . Let  $p > 0$  and  $r > 0$ . Then  $\mu$  is a Carleson measure for  $A_\alpha^p(\mathbb{B}_n, E)$  if and only if

$$M := \sup_{z \in \mathbb{B}_n} \left\{ \frac{\mu(D(z, r))}{(1 - |z|^2)^{n+1+\alpha}} \right\} < \infty.$$

Moreover, we have that  $M \simeq \|\mu\|_{Carl}$ .

*Proof.* Assume that  $\mu$  is a Carleson measure for  $A_\alpha^p(\mathbb{B}_n, E)$ . It follows that

$$\int_{\mathbb{B}_n} \|f(w)\|_E^p d\mu(w) \leq \|\mu\|_{Carl} \int_{\mathbb{B}_n} \|f(w)\|_E^p d\nu_\alpha(w), \quad (5.13)$$

for all  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . For a fixed  $a \in \mathbb{B}_n$ , and a unit vector  $x \in E$ , let

$$f(w) = \left( \frac{(1 - |a|^2)^{n+1+\alpha}}{(1 - \langle w, a \rangle)^{2(n+1+\alpha)}} \right)^{1/p} x, \quad w \in \mathbb{B}_n.$$

From Theorem 2.1.25, we have that  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $\|f\|_{A_\alpha^p(\mathbb{B}_n, E)} \simeq 1$ . Therefore, by (5.13) we have that

$$\int_{D(a, r)} \frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - \langle w, a \rangle|^{2(n+1+\alpha)}} d\mu(w) \leq \|\mu\|_{Carl} \int_{\mathbb{B}_n} \frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - \langle w, a \rangle|^{2(n+1+\alpha)}} d\nu_\alpha(w) = \|\mu\|_{Carl}.$$

Since for  $w \in D(a, r)$ , Lemma 2.1.27 gives

$$\frac{(1 - |a|^2)^{n+1+\alpha}}{|1 - \langle w, a \rangle|^{2(n+1+\alpha)}} \simeq \frac{1}{(1 - |a|^2)^{n+1+\alpha}}.$$

It follows that for every  $a \in \mathbb{B}_n$ , we have

$$\frac{1}{(1 - |a|^2)^{n+1+\alpha}} \mu(D(a, r)) \lesssim \|\mu\|_{Carl}.$$

Thus,

$$M := \sup_{a \in \mathbb{B}_n} \left\{ \frac{\mu(D(a, r))}{(1 - |a|^2)^{n+1+\alpha}} \right\} \lesssim \|\mu\|_{Carl} < \infty.$$

Conversely, assume that

$$M := \sup_{a \in \mathbb{B}_n} \left\{ \frac{\mu(D(a, r))}{(1 - |a|^2)^{n+1+\alpha}} \right\} < \infty.$$

Let  $0 < r < 1$ , and  $\{a_k\}_k$  a  $r$ -lattice in the Bergman metric (see Definition 2.2.31). We have that

$$\begin{aligned} \int_{\mathbb{B}_n} \|f(z)\|_E^p d\mu(z) &\leq \sum_{k=1}^{\infty} \int_{D(a_k, r)} \|f(z)\|_E^p d\mu(z) \\ &\leq \sum_{k=1}^{\infty} \mu(D(a_k, r)) \sup_{z \in D(a_k, r)} \|f(z)\|_E^p. \end{aligned}$$

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Let  $k \geq 1$ , and  $z \in D(a_k, r)$ . By Proposition 2.1.24 there is a constant  $C > 0$  such that

$$\|f(z)\|_E^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{D(z,r)} \|f(w)\|_E^p d\nu_\alpha(w).$$

Since  $1 - |z|^2 \simeq 1 - |a_k|^2$  and  $D(z, r) \subset D(a_k, 2r)$  we get that

$$\sup_{z \in D(a_k, r)} \|f(z)\|_E^p \lesssim \frac{C}{(1 - |a_k|^2)^{n+1+\alpha}} \int_{D(a_k, 2r)} \|f(w)\|_E^p d\nu_\alpha(w).$$

It follows that

$$\begin{aligned} \int_{\mathbb{B}_n} \|f(w)\|_E^p d\mu(w) &\lesssim C \sum_{k=1}^{\infty} \frac{\mu(D(a_k, r))}{(1 - |a_k|^2)^{n+1+\alpha}} \int_{D(a_k, 2r)} \|f(w)\|_E^p d\nu_\alpha(w) \\ &\lesssim CM \sum_{k=1}^{\infty} \int_{D(a_k, 2r)} \|f(w)\|_E^p d\nu_\alpha(w) \\ &\lesssim CMN \int_{\mathbb{B}_n} \|f(w)\|_E^p d\nu_\alpha(w), \end{aligned}$$

where  $N$  is a positive integer such that each point of  $\mathbb{B}_n$  belongs to at most  $N$  of the sets  $D(a_k, 2r)$ . Thus,  $\mu$  is a Carleson measure for  $A_\alpha^p(\mathbb{B}_n, E)$  and  $\|\mu\|_{Carl} \lesssim M$ . The equivalence follows at once, thus ends the proof.  $\square$

**Remark 5.2.3.** From Theorem 5.2.2, we remark that  $\mu$  is a Carleson measure for  $A_\alpha^p(\mathbb{B}_n, E)$ , independently of  $p$  and  $E$ .

**Definition 5.2.4.** Let  $\mu$  be a finite measure on  $\mathbb{B}_n$ . The Berezin transform  $\tilde{\mu}$  of  $\mu$  is defined by

$$\tilde{\mu}(z) := \int_{\mathbb{B}_n} |k_z^\alpha(w)|^2 d\mu(w), \quad z \in \mathbb{B}_n.$$

The Toeplitz operator  $T_\mu$ , with symbol  $\mu$  is defined by

$$T_\mu(g)(z) := \int_{\mathbb{B}_n} \frac{g(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\mu(w),$$

where  $g \in H^\infty(\mathbb{B}_n, E)$  and  $z \in \mathbb{B}_n$ .

**Lemma 5.2.5.** Let  $\mu$  a finite positive Borel measure on  $\mathbb{B}_n$ . We have :

$$\langle T_\mu(f), g \rangle_{\alpha, E} = \int_{\mathbb{B}_n} \langle f(w), g(w) \rangle_{E, E^*} d\mu(w),$$

for all  $f \in \mathcal{H}^\infty(\mathbb{B}_n, E)$  and  $g \in \mathcal{H}^\infty(\mathbb{B}_n, E^*)$ .

*Proof.* let  $f \in \mathcal{H}^\infty(\mathbb{B}_n, E)$  and  $g \in \mathcal{H}^\infty(\mathbb{B}_n, E^*)$ . By Fubini's theorem, the Hille's theorem (Lemma 1.2.24)

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and the reproducing kernel formula (see Proposition 2.2.2), we have that

$$\begin{aligned}
 \langle T_\mu(f), g \rangle_{\alpha, E} &= \int_{\mathbb{B}_n} \langle T_\mu(f)(z), g(z) \rangle_{E, E^*} d\nu_\alpha(z) \\
 &= \int_{\mathbb{B}_n} \left\langle \int_{\mathbb{B}_n} \frac{f(w) d\mu(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, g(z) \right\rangle_{E, E^*} d\nu_\alpha(z) \\
 &= \int_{\mathbb{B}_n} \left\langle f(w), \int_{\mathbb{B}_n} \frac{g(z) d\nu_\alpha(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right\rangle_{E, E^*} d\mu(w) \\
 &= \int_{\mathbb{B}_n} \langle f(w), g(w) \rangle_{E, E^*} d\mu(w).
 \end{aligned}$$

□

The following theorem generalizes [78, Theorem 7.5].

**Theorem 5.2.6.** Let  $1 < p < \infty$  and  $\mu$  a finite positive Borel measure on  $\mathbb{B}_n$ . Then the following are equivalent.

- (a)  $T_\mu$  is bounded on  $A_\alpha^p(\mathbb{B}_n, E)$ .
- (b)  $\tilde{\mu}$  is a bounded function on  $\mathbb{B}_n$ .
- (c)  $\mu$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$ .

Moreover,

$$\|T_\mu\| \simeq \|\tilde{\mu}\|_{L^\infty(\mathbb{B}_n, \mathbb{C})} \simeq \|\mu\|_{Carl}.$$

*Proof.* Assume that (a) holds. Then we have

$$|\langle T_\mu f, g \rangle_{\alpha, E}| \leq \|T_\mu\| \|f\|_{A_\alpha^p(\mathbb{B}_n, E)} \|g\|_{A_\alpha^{p'}(\mathbb{B}_n, E^*)} \quad (5.14)$$

for every  $f \in A_\alpha^p(\mathbb{B}_n, E)$  and  $g \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$ . Now,

Let  $z \in \mathbb{B}_n$ ,  $x_0 \in E$  with  $\|x_0\|_E = 1$ . Let  $x^* \in E^*$  with  $\|x^*\|_{E^*} = 1$  such that  $|\langle x_0, x^* \rangle_{E, E^*}| = 1$ . Let

$$f_z(w) = \frac{(1 - |z|^2)^{(n+1+\alpha)/p'}}{(1 - \langle w, z \rangle)^{n+1+\alpha}} x_0 \quad \text{and} \quad g_z(w) = \frac{(1 - |z|^2)^{(n+1+\alpha)/p}}{(1 - \langle w, z \rangle)^{n+1+\alpha}} x^*,$$

where  $p'$  is the conjugate exponent of  $p$ . We have, using Theorem 2.1.25,  $f_z \in A_\alpha^p(\mathbb{B}_n, E)$ ,  $g_z \in A_\alpha^{p'}(\mathbb{B}_n, E^*)$  and

$$\|f_z\|_{A_\alpha^p(\mathbb{B}_n, E)} \simeq 1 \simeq \|g_z\|_{A_\alpha^{p'}(\mathbb{B}_n, E^*)}.$$

Applying (5.14) for these particular functions and using Lemma 5.2.5, we have

$$\left| \int_{\mathbb{B}_n} \left\langle \frac{(1 - |z|^2)^{(n+1+\alpha)/p'}}{(1 - \langle w, z \rangle)^{n+1+\alpha}} x_0, \frac{(1 - |z|^2)^{(n+1+\alpha)/p}}{(1 - \langle w, z \rangle)^{n+1+\alpha}} x^* \right\rangle_{E, E^*} d\mu(w) \right| \lesssim \|T_\mu\|.$$

That is for all  $z \in \mathbb{B}_n$ ,

$$|\langle x_0, x^* \rangle_{E, E^*}| \int_{\mathbb{B}_n} |k_z^\alpha(w)|^2 d\mu(w) = \tilde{\mu}(z) \lesssim \|T_\mu\|.$$

Thus,  $\tilde{\mu}$  is bounded in  $\mathbb{B}_n$  and

$$\|\tilde{\mu}\|_{L^\infty(\mathbb{B}_n, \mathbb{C})} := \sup_{z \in \mathbb{B}_n} \tilde{\mu}(z) \lesssim \|T_\mu\|.$$

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This completes the proof of "(a) implies (b)".

(b)  $\Rightarrow$  (c). Assume that  $\tilde{\mu}$  is a bounded function on  $\mathbb{B}_n$ . Let  $z \in \mathbb{B}_n$  and  $r > 0$ . By the assumption, we have that

$$\int_{D(z,r)} |k_z^\alpha(w)|^2 d\mu(w) \leq \int_{\mathbb{B}_n} |k_z^\alpha(w)|^2 d\mu(w) \leq \|\tilde{\mu}\|_{L^\infty(\mathbb{B}_n, \mathbb{C})} < \infty. \quad (5.15)$$

Since for  $w \in D(z, r)$ , by Lemma 2.1.27, we have

$$|k_z^\alpha(w)|^2 = \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle w, z \rangle|^{2(n+1+\alpha)}} \simeq \frac{1}{(1 - |z|^2)^{n+1+\alpha}}. \quad (5.16)$$

From (5.16) and (5.15), it follows that

$$\frac{\mu(D(z, r))}{(1 - |z|^2)^{n+1+\alpha}} \lesssim \|\tilde{\mu}\|_{L^\infty(\mathbb{B}_n, \mathbb{C})},$$

for all  $z \in \mathbb{B}_n$ , and  $r > 0$ . Using Theorem 5.2.2, we obtain that  $\mu$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$  and

$$\|\mu\|_{Carl} \simeq \sup_{z \in \mathbb{B}_n} \left\{ \frac{\mu(D(z, r))}{(1 - |z|^2)^{n+1+\alpha}} \right\} \lesssim \|\tilde{\mu}\|_{L^\infty(\mathbb{B}_n, \mathbb{C})}$$

This completes the proof "(b) implies (c)".

(c)  $\Rightarrow$  (a). Assume that  $\mu$  is a Carleson measure for the vector-valued Bergman space  $A_\alpha^1(\mathbb{B}_n, E)$ . Let us prove that  $T_\mu : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^p(\mathbb{B}_n, E)$  is a bounded linear operator. Fix  $z \in \mathbb{B}_n$ , and  $f \in A_\alpha^p(\mathbb{B}_n, E)$ . The function

$$F_z(w) = \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}}, \quad w \in \mathbb{B}_n,$$

belongs to  $A_\alpha^1(\mathbb{B}_n, E)$ . Indeed, by Hölder's inequality, we have

$$\begin{aligned} \int_{\mathbb{B}_n} \|F_z(w)\|_E d\nu_\alpha(w) &\leq \int_{\mathbb{B}_n} \frac{\|f(w)\|_E}{(1 - |z|)^{n+1+\alpha}} d\nu_\alpha(w) \\ &\leq \frac{1}{(1 - |z|)^{n+1+\alpha}} \|f\|_{A_\alpha^p(\mathbb{B}_n, E)}. \end{aligned}$$

Since  $\mu$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$ , there holds

$$\int_{\mathbb{B}_n} \|F_z(w)\|_E d\mu(w) \leq \|\mu\|_{Carl} \int_{\mathbb{B}_n} \|F_z(w)\|_E d\nu_\alpha(w). \quad (5.17)$$

By using (5.17) and the fact that for  $p > 1$ , the positive Bergman operator  $P_\alpha^+$  defined by

$$P_\alpha^+(\phi)(z) := \int_{\mathbb{B}_n} \frac{\phi(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w),$$

### 5.3. Introduction to vector-valued $BMO^p$ spaces for $p \geq 1$ .

is a bounded operator on  $L_\alpha^p(\mathbb{B}_n, \mathbb{C})$  (see [77]), we then obtain

$$\begin{aligned}
 \int_{\mathbb{B}_n} \|T_\mu f(z)\|_E^p d\nu_\alpha(z) &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\mu(w) \right\|_E^p d\nu_\alpha(z) \\
 &\leq \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{\|f(w)\|_E}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\mu(w) \right)^p d\nu_\alpha(z) \\
 &= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \|F_z(w)\|_E d\mu(w) \right)^p d\nu_\alpha(z) \\
 &\leq \|\mu\|_{Carl}^p \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \|F_z(w)\|_E d\nu_\alpha(w) \right)^p d\nu_\alpha(z) \\
 &= \|\mu\|_{Carl}^p \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{\|f(w)\|_E}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) \right)^p d\nu_\alpha(z) \\
 &= \|\mu\|_{Carl}^p \int_{\mathbb{B}_n} [P_\alpha^+(\|f\|_E)]^p d\nu_\alpha(z) \\
 &\lesssim \|\mu\|_{Carl}^p \int_{\mathbb{B}_n} \|f(z)\|_E^p d\nu_\alpha(z).
 \end{aligned}$$

We have proved that  $T_\mu : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^p(\mathbb{B}_n, E)$  is bounded and  $\|T_\mu\| \lesssim \|\mu\|_{Carl}$ . This completes the proof of the theorem.  $\square$

### 5.3 Introduction to vector-valued $BMO^p$ spaces for $p \geq 1$ .

It is well-known that  $BMO$  spaces play an important role in many domains of Mathematics. The  $BMO$  space have been extensively studied (see [12, 13, 10, 54, 73, 78, 45]) for scalar-valued functions. In this section, we introduce the vector-valued  $BMO$  spaces.

Let  $z \in \mathbb{B}_n$  and  $r > 0$ . For a  $\nu_\alpha$ -integrable vector-valued function  $f : \mathbb{B}_n \rightarrow E$ , we define a function  $\hat{f}_r : \mathbb{B}_n \rightarrow E$  as follows:

$$\hat{f}_r(z) = \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} f(w) d\nu_\alpha(w), \quad z \in \mathbb{B}_n.$$

If  $f$  is a vector-valued integrable function, we define the mean oscillation of  $f$  at  $z$  in the Bergman metric as

$$MO_{r, \alpha, E}^1(f)(z) := \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \hat{f}_r(z)\|_E d\nu_\alpha(w).$$

We say that  $f$  has a bounded mean oscillation in the Bergman metric if

$$\|f\|_{BMO_{r, \alpha}^1(\mathbb{B}_n, E)} = \sup_{z \in \mathbb{B}_n} MO_{r, \alpha, E}^1(f)(z) < \infty.$$

We denote by  $BMO_{r, \alpha}^1(\mathbb{B}_n, E)$  the space of  $E$ -valued integrable functions on  $\mathbb{B}_n$  with bounded mean oscillation. Let  $p \geq 1$ , and  $BMO_{r, \alpha}^p(\mathbb{B}_n, E)$  be the space of all  $f$  in  $L_\alpha^p(\mathbb{B}_n, E)$  such that

$$\|f\|_{BMO_{r, \alpha}^p(\mathbb{B}_n, E)} := \sup_{z \in \mathbb{B}_n} MO_{r, \alpha, E}^p(f)(z) < \infty,$$

with

$$MO_{r, \alpha, E}^p(f)(z) = \left( \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \hat{f}_r(z)\|_E^p d\nu_\alpha(w) \right)^{\frac{1}{p}}.$$

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We observe that

$$L^\infty(\mathbb{B}_n, E) \subset BMO_{r,\alpha}^p(\mathbb{B}_n, E) \subset L_\alpha^p(\mathbb{B}_n, E), \quad p \geq 1,$$

$$BMO_{r,\alpha}^q(\mathbb{B}_n, E) \subset BMO_{r,\alpha}^p(\mathbb{B}_n, E) \subset BMO_{r,\alpha}^1(\mathbb{B}_n, E), \quad 1 \leq p \leq q.$$

**Lemma 5.3.1.** Let  $1 \leq p < \infty$ . A function  $f$  is in  $BMO_{r,\alpha}^p(\mathbb{B}_n, E)$  if and only if there exists a constant  $C > 0$  such that for any  $z \in \mathbb{B}_n$  there is a vector  $\lambda_z \in E$  with

$$\frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \lambda_z\|_E^p d\nu_\alpha(w) \leq C.$$

*Proof.* The "only if" part follows by taking  $\lambda_z = \widehat{f}_r(z)$ . For the "if" part, we assume that the above inequality holds for all  $z \in \mathbb{B}_n$ . By the triangle inequality for  $L_\alpha^p(\mathbb{B}_n, E)$  integral, we have

$$\left[ \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \widehat{f}_r(z)\|_E^p d\nu_\alpha(w) \right]^{1/p}$$

$$\leq \left[ \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \lambda_z\|_E^p d\nu_\alpha(w) \right]^{1/p} + \|\widehat{f}_r(z) - \lambda_z\|_E.$$

By Hölder's inequality,

$$\|\widehat{f}_r(z) - \lambda_z\|_E = \left\| \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} (f(w) - \lambda_z) d\nu_\alpha(w) \right\|_E$$

$$\leq \left[ \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \lambda_z\|_E^p d\nu_\alpha(w) \right]^{1/p}.$$

It follows that

$$\left[ \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \widehat{f}_r(z)\|_E^p d\nu_\alpha(w) \right]^{1/p}$$

$$\leq 2 \left[ \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \lambda_z\|_E^p d\nu_\alpha(w) \right]^{1/p}.$$

□

**Definition 5.3.2.** Let  $r > 0$  and  $f : \mathbb{B}_n \rightarrow E$ . We define a function  $\omega_r(f)$  on  $\mathbb{B}_n$  by

$$\omega_r(f)(z) := \sup\{\|f(z) - f(w)\|_E : w \in D(z, r)\}.$$

$\omega_r(f)(z)$  is called the oscillation of  $f$  at  $z$  in the Bergman metric. We say that a vector-valued continuous function  $f : \mathbb{B}_n \rightarrow E$  has bounded oscillation in the Bergman metric if

$$\sup_{z \in \mathbb{B}_n} \omega_r(f)(z) < \infty.$$

We denote  $BO_r(E)$  the space of all vector-valued continuous functions on  $\mathbb{B}_n$  with bounded oscillation.

**Lemma 5.3.3.** A vector-valued function  $f : \mathbb{B}_n \rightarrow E$  belongs to  $BO_r(E)$  if and only if there exists a constant  $C > 0$  such that

$$\|f(z) - f(w)\|_E \leq C(\beta(z, w) + 1),$$

for all  $z, w \in \mathbb{B}_n$ .

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*Proof.* We first suppose that  $f \in BO_r(E)$ . If  $\beta(z, w) \leq 1$ , then the desired result is obvious. Now, fix two points  $z$  and  $w$  in  $\mathbb{B}_n$  with  $\beta(z, w) > 1$ . Let  $\gamma(t)$ ,  $0 \leq t \leq 1$ , be the geodesic in the Bergman metric from  $z$  to  $w$ . Let  $N = [\beta(z, w)] + 1$  and  $t_i = i/N$ ,  $0 \leq i \leq N$ , where  $[x]$  denotes the largest integer less than or equal to  $x$ . Then

$$\beta(\gamma(t_i), \gamma(t_{i+1})) = \frac{\beta(z, w)}{N} \leq 1$$

and hence

$$\begin{aligned} \|f(z) - f(w)\|_E &\leq \sum_{i=1}^N \|f(\gamma(t_{i-1})) - f(\gamma(t_i))\|_E \\ &\leq N \|f\|_{BO(E)} \leq \|f\|_{BO(E)} (\beta(z, w) + 1). \end{aligned}$$

The converse implication is obvious. □

A consequence of the above lemma is that the space  $BO_r(E)$  is independent of the choice of  $r$ . Thus, we simply write  $BO(E) = BO_r(E)$ . We equipped  $BO(E)$  with the following semi-norm

$$\|f\|_{BO(E)} = \sup\{\omega_r(f)(z) : z \in \mathbb{B}_n\}.$$

**Definition 5.3.4.** Let  $0 < p < \infty$  and  $r > 0$ . We say that  $f \in BA_r^p(E)$  if  $f \in L_\alpha^p(\mathbb{B}_n, E)$  and

$$\|f\|_{BA_r^p(E)}^p := \sup_{z \in \mathbb{B}_n} (\widehat{\|f\|_E^p})_r(z) = \sup_{z \in \mathbb{B}_n} \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(u)\|_E^p d\nu_\alpha(u) < \infty.$$

The initials  $BA$  stand for "bounded average".

Functions in  $BA_r^p(E)$  are described as follows.

**Lemma 5.3.5.** Let  $1 \leq p < \infty$  and  $f \in L_\alpha^p(\mathbb{B}_n, E)$ . Set  $d\mu_{f, \alpha} = \|f\|_E^p d\nu_\alpha$ . The following conditions are equivalent:

- (i) The measure  $\mu_{f, \alpha}$  is a Carleson measure for  $A_\alpha^p(\mathbb{B}_n, E)$ .
- (ii)  $f \in BA_r^p(E)$  for some (for all)  $r > 0$ .
- (iii)  $\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w)\|_E^p d\nu_\alpha(w) < \infty$ .

Moreover,

$$\|f\|_{BA_r^p(E)}^p \simeq \|\mu_{f, \alpha}\|_{Carl} \simeq \sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z\|_{L_\alpha^p(\mathbb{B}_n, E)}^p.$$

*Proof.* By Theorem 5.2.2, condition (i) is equivalent to

$$\|\mu_{f, \alpha}\|_{Carl} \simeq \sup_{z \in \mathbb{B}_n} \frac{\mu_{f, \alpha}(D(z, r))}{(1 - |z|^2)^{n+1+\alpha}}, \tag{5.18}$$

for some  $r > 0$ . Since

$$\sup_{z \in \mathbb{B}_n} (\widehat{\|f\|_E^p})_r(z) \simeq \sup_{z \in \mathbb{B}_n} \frac{\mu_{f, \alpha}(D(z, r))}{(1 - |z|^2)^{n+1+\alpha}},$$

it follows that (i) and (ii) are equivalent with  $\|f\|_{BA_r^p(E)}^p \simeq \|\mu_{f, \alpha}\|_{Carl}$ .

We recall that  $|1 - \langle z, u \rangle| \simeq (1 - |z|^2)$  for  $u \in D(z, r)$ . Using Proposition 2.1.12 and Lemma 2.1.9, for every

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$z \in \mathbb{B}_n$ , we have

$$\begin{aligned} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w)\|_E^p d\nu_\alpha(w) &= \int_{\mathbb{B}_n} \|f(u)\|_E^p |k_z^\alpha(u)|^2 d\nu_\alpha(u) \\ &\geq \int_{D(z,r)} \|f(u)\|_E^p |k_z^\alpha(u)|^2 d\nu_\alpha(u) \\ &\simeq \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} \|f(u)\|_E^p d\nu_\alpha(u) = (\widehat{\|f\|_E^p})_r(z). \end{aligned}$$

This proves that (iii) implies (ii) and  $\|f\|_{BA_r^p(E)}^p \lesssim \sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z\|_{L_\alpha^p(\mathbb{B}_n, E)}^p$ .

To complete the proof, we will prove that (i) implies (iii). Let  $\{a_j\}$  be an  $r$ -lattice in the Bergman metric. Then,

$$\begin{aligned} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w)\|_E^p d\nu_\alpha(w) &= \int_{\mathbb{B}_n} \|f(u)\|_E^p |k_z^\alpha(u)|^2 d\nu_\alpha(u) \\ &= (1 - |z|^2)^{n+1+\alpha} \int_{\mathbb{B}_n} \frac{\|f(u)\|_E^p}{|1 - \langle z, u \rangle|^{2(n+1+\alpha)}} d\nu_\alpha(u) \\ &\leq (1 - |z|^2)^{n+1+\alpha} \sum_{j=1}^{\infty} \int_{D(a_j, r)} \frac{\|f(u)\|_E^p}{|1 - \langle z, u \rangle|^{2(n+1+\alpha)}} d\nu_\alpha(u). \end{aligned}$$

By Lemma 2.1.28, we have

$$|1 - \langle z, u \rangle| \simeq |1 - \langle z, a_j \rangle|,$$

for  $u \in D(a_j, r)$ . It follows by (5.18) that

$$\begin{aligned} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w)\|_E^p d\nu_\alpha(w) &\lesssim (1 - |z|^2)^{n+1+\alpha} \sum_{j=1}^{\infty} \frac{\mu_{f, \alpha}(D(a_j, r))}{|1 - \langle z, a_j \rangle|^{2(n+1+\alpha)}} \\ &\lesssim (1 - |z|^2)^{n+1+\alpha} \|\mu_{f, \alpha}\|_{Carl} \sum_{j=1}^{\infty} \frac{(1 - |a_j|^2)^{n+1+\alpha}}{|1 - \langle z, a_j \rangle|^{2(n+1+\alpha)}}. \end{aligned}$$

By using Lemma 2.1.31 with  $s = 2(n + 1 + \alpha)$  and  $t = (n + 1 + \alpha)$ , we obtain that

$$\sum_{j=1}^{\infty} \frac{(1 - |a_j|^2)^{n+1+\alpha}}{|1 - \langle z, a_j \rangle|^{2(n+1+\alpha)}} \lesssim (1 - |z|^2)^{-(n+1+\alpha)}.$$

Therefore, we obtain that:

$$\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w)\|_E^p d\nu_\alpha(w) \lesssim \|\mu_{f, \alpha}\|_{Carl}.$$

Thus, (i) implies (iii) and  $\sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z\|_{L_\alpha^p(\mathbb{B}_n, E)}^p \lesssim \|\mu_{f, \alpha}\|_{Carl}$ .  $\square$

**Remark 5.3.6.** As a consequence of Lemma 5.3.5, we see that the space  $BA_r^p(E)$  is independent of the choice of  $r$ . Thus, we simply write  $BA^p(E)$ .

The scalar-valued version of the following theorem can be found in [77].

**Theorem 5.3.7.** Let  $r > 0$  and  $p \geq 1$ . Then the following are equivalent :

- (1)  $f \in BMO_{r, \alpha}^p(\mathbb{B}_n, E)$ .
- (2)  $f = f_1 + f_2$ , with  $f_1 \in BO(E)$  and  $f_2 \in BA^p(E)$ ;
- (3)  $\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w) - \tilde{f}(z)\|_E^p d\nu_\alpha(w) < \infty$ .

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(4) There exists a constant  $C > 0$  such that for any  $z \in \mathbb{B}_n$  there is a vector  $\lambda_z \in E$  with

$$\int_{\mathbb{B}_n} \|f \circ \varphi_z(w) - \lambda_z\|_E^p d\nu_\alpha(w) \leq C.$$

Moreover,

$$\|f\|_{BMO_{r,\alpha}^p(\mathbb{B}_n, E)} \simeq \|f_1\|_{BA^p(E)} + \|f_2\|_{BO(E)} \simeq \sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z - \tilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}.$$

*Proof.* (1)  $\Rightarrow$  (2): Since  $r$  is arbitrary, it suffices to show that  $BMO_{2r,\alpha}^p(\mathbb{B}_n, E) \subset BO(E) + BA^p(E)$ . Given  $f \in BMO_{2r,\alpha}^p(\mathbb{B}_n, E)$  and  $\beta(z, w) \leq r$ . Put  $f_1 = \widehat{f}_r$  and  $f_2 = f - \widehat{f}_r$ . Let us show that  $f_1 \in BO(E)$  and  $f_2 \in BA^p(E)$ . We have that

$$\begin{aligned} \|\widehat{f}_r(z) - \widehat{f}_r(w)\|_E &\leq \|\widehat{f}_r(z) - \widehat{f}_{2r}(z)\|_E + \|\widehat{f}_{2r}(z) - \widehat{f}_r(w)\|_E \\ &\leq \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(u) - \widehat{f}_{2r}(z)\|_E d\nu_\alpha(u) \end{aligned} \quad (5.19)$$

$$+ \frac{1}{\nu_\alpha(D(w, r))} \int_{D(w, r)} \|f(u) - \widehat{f}_{2r}(z)\|_E d\nu_\alpha(u). \quad (5.20)$$

By Hölder's inequality and the fact that  $D(z, r) \subset D(z, 2r)$  and  $f \in BMO_{2r,\alpha}^p(\mathbb{B}_n, E)$ , (5.19) becomes :

$$\begin{aligned} \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(u) - \widehat{f}_{2r}(z)\|_E d\nu_\alpha(u) \\ \lesssim \frac{1}{\nu_\alpha(D(z, 2r))} \left( \int_{D(z, r)} \|f(u) - \widehat{f}_{2r}(z)\|_E^p d\nu_\alpha(u) \right)^{1/p} \\ \lesssim \|f\|_{BMO_{2r,\alpha}^p(\mathbb{B}_n, E)}. \end{aligned}$$

For the inequality (5.20), since  $D(w, r) \subset D(z, 2r)$  and  $f \in BMO_{2r,\alpha}^p(\mathbb{B}_n, E)$ , it follows from Hölder's inequality that:

$$\begin{aligned} \frac{1}{\nu_\alpha(D(w, r))} \int_{D(w, r)} \|f(u) - \widehat{f}_{2r}(z)\|_E d\nu_\alpha(u) &\lesssim \frac{1}{\nu_\alpha(D(z, 2r))} \times \\ &\left( \int_{D(z, 2r)} \|f(u) - \widehat{f}_{2r}(z)\|_E^p d\nu_\alpha(u) \right)^{1/p} \lesssim \|f\|_{BMO_{2r,\alpha}^p(\mathbb{B}_n, E)}. \end{aligned}$$

It follows that the function  $f_1 = \widehat{f}_r$  belongs to  $BO(E)$  and

$$\|f_1\|_{BO(E)} \lesssim \|f\|_{BMO_{2r,\alpha}^p(\mathbb{B}_n, E)}.$$

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For  $f_2 = f - \widehat{f}_r$  with  $f \in BMO_{2r,\alpha}^p(\mathbb{B}_n, E)$ , we next show that  $f_2 \in BAP(E)$ . By the Minkowski inequality,

$$\begin{aligned} [\widehat{\|f_2\|_{r,E}^p}]^{1/p} &= \left[ \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} \|f(u) - \widehat{f}_r(u)\|_E^p d\nu_\alpha(u) \right]^{1/p} \\ &\leq \left[ \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} \|f(u) - \widehat{f}_r(z)\|_E^p d\nu_\alpha(u) \right]^{1/p} \\ &\quad + \left[ \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} \|\widehat{f}_r(u) - \widehat{f}_r(z)\|_E^p d\nu_\alpha(u) \right]^{1/p} \\ &\leq \|f\|_{BMO_{2r,\alpha}^p(\mathbb{B}_n, E)} + \omega_r(\widehat{f}_r)(z). \end{aligned}$$

Since  $\widehat{f}_r = f_1$  is in  $BO(E)$ , and  $f \in BMO_{2r,\alpha}^p(\mathbb{B}_n, E)$ , it follows that the function  $f_2$  is in  $BAP(E)$  and  $\|f_2\|_{BAP(E)} \lesssim \|f\|_{BMO_{2r,\alpha}^p(\mathbb{B}_n, E)} + \|f_1\|_{BO(E)}$ . We have proved that  $BMO_{2r,\alpha}^p(\mathbb{B}_n, E) \subset BAP(E) + BO(E)$  and

$$\|f_1\|_{BO(E)} + \|f_2\|_{BAP(E)} \lesssim \|f\|_{BMO_{2r,\alpha}^p(\mathbb{B}_n, E)}.$$

(2)  $\Rightarrow$  (3): First note that by Minkowski's inequality, Proposition 2.1.12 and Hölder's inequality, we have

$$\begin{aligned} \|f \circ \varphi_z - \widetilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)} &\leq \|f \circ \varphi_z\|_{L_\alpha^p(\mathbb{B}_n, E)} + \|\widetilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)} \\ &= \|f \circ \varphi_z\|_{L_\alpha^p(\mathbb{B}_n, E)} + \|\widetilde{f}(z)\|_E \leq 2\|f \circ \varphi_z\|_{L_\alpha^p(\mathbb{B}_n, E)}. \end{aligned}$$

If  $f$  is in  $BAP(E)$ , it follows by Lemma 5.3.5, that  $\|f \circ \varphi_z - \widetilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}$  is bounded and

$$\sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z - \widetilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)} \lesssim \sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z\|_{L_\alpha^p(\mathbb{B}_n, E)} \simeq \|f\|_{BAP(E)}. \quad (5.21)$$

On the other hand,

$$\begin{aligned} \|f \circ \varphi_z - \widetilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}^p &= \int_{\mathbb{B}_n} \|f \circ \varphi_z(w) - \widetilde{f}(z)\|_E^p d\nu_\alpha(w) \\ &\leq \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \|f \circ \varphi_z(w) - f \circ \varphi_z(u)\|_E^p d\nu_\alpha(w) d\nu_\alpha(u). \end{aligned}$$

If  $f \in BO(E)$ , then Lemma 5.3.3 shows that

$$\|f(z) - f(w)\|_E \leq (\beta(z, w) + 1)\|f\|_{BO(E)}$$

for all  $z, w \in \mathbb{B}_n$ . This along with the Möbius invariance of the Bergman metric, implies that

$$\begin{aligned} \|f \circ \varphi_z - \widetilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}^p &\leq \|f\|_{BO(E)}^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (\beta(w, u) + 1)^p d\nu_\alpha(w) d\nu_\alpha(u) \\ &\leq \|f\|_{BO(E)}^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (\beta(0, w) + \beta(0, u) + 1)^p d\nu_\alpha(w) d\nu_\alpha(u) \\ &= \|f\|_{BO(E)}^p \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} (h(w) + h(u) + 1)^p d\nu_\alpha(w) d\nu_\alpha(u), \end{aligned}$$

where  $h(w) = \frac{1}{2} \log \left( \frac{1+|w|}{1-|w|} \right)$ . Since

$$\int_{\mathbb{B}_n} \left[ \log \left( \frac{1+|w|}{1-|w|} \right) \right]^p d\nu_\alpha(w) = 2nc_\alpha \int_0^1 r^{2n-1} \left[ \log \left( \frac{1+r}{1-r} \right) \right]^p (1-r^2)^\alpha dr < \infty.$$

In fact,

$$\begin{aligned}
 \int_0^1 r^{2n-1} \left[ \log \left( \frac{1+r}{1-r} \right) \right]^p (1-r^2)^\alpha dr &\simeq \int_0^1 r^{2n-1} \left[ \log \left( \frac{1+r}{1-r} \right) \right]^p (1-r)^\alpha dr \\
 &\leq \int_0^1 \left[ \log \left( \frac{2}{1-r} \right) \right]^p (1-r)^\alpha dr \\
 &= \int_0^1 \left[ \log \left( \frac{2}{\rho} \right) \right]^p \rho^\alpha d\rho \\
 &\leq 2^p \int_0^1 [\log^p(2) - \log^p(\rho)] \rho^\alpha d\rho \\
 &= 2^p \log^p(2) \int_0^1 \rho^\alpha dr - 2^p \int_0^1 \log^p(\rho) \rho^\alpha dr \\
 &= \frac{2^p \log^p(2)}{\alpha+1} - 2^p I_{p,\alpha},
 \end{aligned}$$

with

$$I_{p,\alpha} = \int_0^1 \log^p(\rho) \rho^\alpha d\rho.$$

To conclude, we need to show that  $I_{p,\alpha} < \infty$ . Let  $0 < \lambda < 1$  be a real such that  $0 < \alpha + \lambda$ . Since  $\lim_{\rho \rightarrow 0} \rho^{\alpha+\lambda} \log^p(\rho) = 0$ , there exists a real  $\eta > 0$  such that for  $\rho < \eta$ , we have

$$\rho^\alpha \log^p(\rho) < \frac{1}{\rho^\lambda}.$$

It follows that

$$I_{p,\alpha} \leq \int_0^\eta \frac{1}{\rho^\lambda} d\rho + \int_\eta^1 \log^p(\rho) \rho^\alpha d\rho < \infty.$$

It follows that

$$\sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z - \tilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}^p \lesssim \|f\|_{BO(E)}^p.$$

Thus, we have shown that  $f \in BO(E)$  implies that  $\|f \circ \varphi_z - \tilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}$  is bounded and

$$\sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z - \tilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)} \lesssim \|f\|_{BO(E)}. \quad (5.22)$$

Now, for  $f = f_1 + f_2$ , with  $f_1 \in BA^p(E)$  and  $f_2 \in BO(E)$ , we have by Minkowski's inequality, (5.21) and (5.22), we have

$$\begin{aligned}
 \|f \circ \varphi_z - \tilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}^p &\leq \|f_1 \circ \varphi_z - \tilde{f}_1(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}^p + \|f_2 \circ \varphi_z - \tilde{f}_2(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}^p \\
 &\lesssim \|f_1\|_{BA^p(E)} + \|f_2\|_{BO(E)}.
 \end{aligned}$$

(3)  $\Leftrightarrow$  (4): Since the proof of the equivalence of (3) and (4) is similar to that of Lemma 5.3.1, we omit the details.

(3)  $\Rightarrow$  (1): By Lemma 2.1.9 and Lemma 2.1.27, we have that

$$1 \simeq \nu_\alpha(D(z, r)) |k_z^\alpha(w)|^2,$$

### 5.3. Introduction to vector-valued $BMO^p$ spaces for $p \geq 1$ .

for all  $z \in \mathbb{B}_n$  and  $w \in D(z, r)$ . It follows that

$$\begin{aligned} \frac{1}{\nu_\alpha(D(z, r))} \int_{D(z, r)} \|f(w) - \tilde{f}(z)\|_E^p d\nu_\alpha(w) &\lesssim \int_{D(z, r)} \|f(w) - \tilde{f}(z)\|_E^p |k_z^\alpha(w)|^2 d\nu_\alpha(w) \\ &\lesssim \int_{\mathbb{B}_n} \|f \circ \varphi_z(w) - \tilde{f}(z)\|_E^p d\nu_\alpha(w). \end{aligned}$$

The desired result then follows at once.  $\square$

**Remark 5.3.8.** One of the consequences of the above result is that the space  $BMO_{r, \alpha}^p(\mathbb{B}_n, E)$  is also independent of  $r$ . Thus, from now on it will be denoted by  $BMO_\alpha^p(\mathbb{B}_n, E)$ . A canonical semi-norm on  $BMO_\alpha^p(\mathbb{B}_n, E)$  is given by

$$\|f\|_{BMO_\alpha^p(\mathbb{B}_n, E)} := \sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z - \tilde{f}(z)\|_{L_\alpha^p(\mathbb{B}_n, E)}.$$

A norm on  $BMO_\alpha^p(\mathbb{B}_n, E)$  is defined by

$$\|f\|_{BMO_\alpha^p(\mathbb{B}_n, E)} = \|\tilde{f}(0)\|_E + \|f\|_{BMO_\alpha^p(\mathbb{B}_n, E)}.$$

**Proposition 5.3.9.** Let  $f \in L_\alpha^1(\mathbb{B}_n, E)$ .

- (1) If  $f \in BMO_\alpha^1(\mathbb{B}_n, E)$ , then  $\sup_{z \in \mathbb{B}_n} (\widetilde{\|f\|_E}(z) - \|\tilde{f}(z)\|_E) < \infty$ .
- (2) If  $\tilde{f}$  is bounded in  $\mathbb{B}_n$  and  $\sup_{z \in \mathbb{B}_n} (\widetilde{\|f\|_E}(z) - \|\tilde{f}(z)\|_E) < \infty$ , then  $f \in BMO_\alpha^1(\mathbb{B}_n, E)$ .

*Proof.* (1) For  $z \in \mathbb{B}_n$ , we have that

$$\begin{aligned} (\widetilde{\|f\|_E}(z) - \|\tilde{f}(z)\|_E) &= \int_{\mathbb{B}_n} (\|f(w)\|_E - \|\tilde{f}(z)\|_E) |k_z^\alpha(w)|^2 d\nu_\alpha(w) \\ &\leq \int_{\mathbb{B}_n} \|f(w) - \tilde{f}(z)\|_E |k_z^\alpha(w)|^2 d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|f \circ \varphi_z(w) - \tilde{f}(z)\|_E d\nu_\alpha(w) \\ &= \|f \circ \varphi_z - \tilde{f}(z)\|_{1, \alpha, E}, \end{aligned}$$

where we have used the change of variables  $w = \varphi_z(v)$  in the third line of the previous equation.

Since  $f \in BMO_\alpha^1(\mathbb{B}_n, E)$ ,

$$\sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z - \tilde{f}(z)\|_{1, \alpha, E} < \infty$$

and so

$$\sup_{z \in \mathbb{B}_n} (\widetilde{\|f\|_E}(z) - \|\tilde{f}(z)\|_E) < \infty.$$

(2) Assume that  $\tilde{f}$  is bounded in  $\mathbb{B}_n$ , and  $c := \sup_{z \in \mathbb{B}_n} (\widetilde{\|f\|_E}(z) - \|\tilde{f}(z)\|_E) < \infty$ . For any  $z \in \mathbb{B}_n$ , we have that

$$\widetilde{\|f\|_E}(z) = (\widetilde{\|f\|_E}(z) - \|\tilde{f}(z)\|_E) + \|\tilde{f}(z)\|_E \leq c + \sup_{z \in \mathbb{B}_n} \|\tilde{f}(z)\|_E < \infty.$$

But since

$$\|f \circ \varphi_z - \tilde{f}(z)\|_{1, \alpha, E} \leq \|f \circ \varphi_z\|_{1, \alpha, E} + \|\tilde{f}(z)\|_E = \widetilde{\|f\|_E}(z) + \|\tilde{f}(z)\|_E,$$

we get that

$$\sup_{z \in \mathbb{B}_n} \|f \circ \varphi_z - \tilde{f}(z)\|_{1,\alpha,E} < \infty$$

and thus  $f$  is in  $BMO_\alpha^1(\mathbb{B}_n, E)$ . □

The scalar-valued version of the following theorem has been proved by Li and Luecking in [45].

**Theorem 5.3.10.** The Bergman projection

$$P_\alpha : BMO_\alpha^1(\mathbb{B}_n, E) \rightarrow \mathfrak{B}(\mathbb{B}_n, E)$$

is a bounded operator. Moreover, we have

$$\|P_\alpha f\|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \|f\|_{BMO_\alpha^1(\mathbb{B}_n, E)}.$$

*Proof.* Let  $f \in BMO_\alpha^1(\mathbb{B}_n, E)$ . By Theorem 5.3.7, we have  $f = f_1 + f_2$ , with  $f_1 \in BO(E)$ ,  $f_2 \in BA^1(E)$  and  $\|f\|_{BMO_\alpha^1(\mathbb{B}_n, E)} \simeq \|f_1\|_{BO(E)} + \|f_2\|_{BA^1(E)}$ . Since  $P_\alpha f = P_\alpha(f_1) + P_\alpha(f_2)$ , we have

$$\|P_\alpha f\|_{\mathfrak{B}(\mathbb{B}_n, E)} \leq \|P_\alpha f_1\|_{\mathfrak{B}(\mathbb{B}_n, E)} + \|P_\alpha f_2\|_{\mathfrak{B}(\mathbb{B}_n, E)}.$$

So, to complete the proof of the theorem, we will prove the following two inequalities:

$$\|P_\alpha f_1\|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \|f_1\|_{BO(E)},$$

and

$$\|P_\alpha f_2\|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \|f_2\|_{BA^1(E)}.$$

For the first inequality, we first show that  $P_\alpha(f_1) \in \mathfrak{B}(\mathbb{B}_n, E)$ . For this, it suffices to show that the function  $(1 - |z|^2)\mathcal{N}P_\alpha(f_1)(z)$  is bounded. Let  $z \in \mathbb{B}_n$ . We have

$$P_\alpha(f_1)(z) = \int_{\mathbb{B}_n} K_w^\alpha(z) f_1(w) d\nu_\alpha(w).$$

Therefore,

$$\mathcal{N}P_\alpha(f_1)(z) = \int_{\mathbb{B}_n} \mathcal{N}[K_w^\alpha(z)] f_1(w) d\nu_\alpha(w),$$

with

$$\mathcal{N}K_w^\alpha(z) = \frac{(n+1+\alpha)\langle z, w \rangle}{(1 - \langle z, w \rangle)^{n+2+\alpha}}.$$

Now, fix  $z \in \mathbb{B}_n$  and consider the holomorphic function

$$g_z(w) = \frac{(n+1+\alpha)\langle w, z \rangle}{(1 - \langle w, z \rangle)^{n+2+\alpha}}.$$

By using the reproducing kernel formula, we get

$$\begin{aligned} 0 = g_z(0) &= \int_{\mathbb{B}_n} \frac{g_z(w)}{(1 - \langle 0, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} g_z(w) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{(n+1+\alpha)\langle w, z \rangle}{(1 - \langle w, z \rangle)^{n+2+\alpha}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \mathcal{N}K_w^\alpha(z) d\nu_\alpha(w). \end{aligned}$$

Therefore,  $\int_{\mathbb{B}_n} \mathcal{N}K_w^\alpha(z) d\nu_\alpha(w) = 0$ . It follows that, using Lemma 5.3.3

$$\begin{aligned} \|\mathcal{N}P_\alpha f_1(z)\|_E &= \left\| \int_{\mathbb{B}_n} \mathcal{N}[K_w^\alpha(z)]f_1(w) d\nu_\alpha(w) \right\|_E \\ &= \left\| \int_{\mathbb{B}_n} \mathcal{N}[K_w^\alpha(z)][f_1(w) - f_1(z)] d\nu_\alpha(w) \right\|_E \\ &\leq (n+1+\alpha) \int_{\mathbb{B}_n} \frac{\|f_1(w) - f_1(z)\|_E}{|1 - \langle z, w \rangle|^{n+2+\alpha}} d\nu_\alpha(w) \\ &\leq (n+1+\alpha) \|f_1\|_{BO(E)} \int_{\mathbb{B}_n} \frac{\beta(z, w) + 1}{|1 - \langle z, w \rangle|^{n+2+\alpha}} d\nu_\alpha(w) \\ &\lesssim \|f_1\|_{BO(E)} \int_{\mathbb{B}_n} \frac{\beta(z, w)}{|1 - \langle z, w \rangle|^{n+2+\alpha}} d\nu_\alpha(w) \\ &\quad + (1 - |z|^2)^{-1} \|f_1\|_{BO(E)}. \end{aligned}$$

By Lemma 2.1.26, we know that

$$\int_{\mathbb{B}_n} \frac{\beta(z, w) d\nu_\alpha(w)}{|1 - \langle z, w \rangle|^{n+2+\alpha}} \leq C''(1 - |z|^2)^{-1}.$$

It follows that

$$\|\mathcal{N}P_\alpha f_1(z)\|_E \lesssim (1 - |z|^2)^{-1} \|f_1\|_{BO(E)}.$$

Thus,

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|\mathcal{N}P_\alpha(f_1)(z)\|_E \lesssim \|f_1\|_{BO(E)} < \infty.$$

That is  $P_\alpha(f_1)$  is in  $\mathfrak{B}(\mathbb{B}_n, E)$  and  $\|P_\alpha f_1\|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \|f_1\|_{BO(E)}$ .

It remains to prove that  $P_\alpha(f_2) \in \mathfrak{B}(\mathbb{B}_n, E)$ . Since  $f_2 \in BA^1(E)$ , by condition (i) of Lemma 5.3.5 we have that the measure  $d\mu_{f_2, \alpha} = \|f_2\|_E d\nu_\alpha$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$  and  $\|f_2\|_{BA^1(E)} \simeq \|\mu_{f_2, \alpha}\|_{Carl}$ . Consider the function

$$g_2(z) = P_\alpha(f_2)(z) = \int_{\mathbb{B}_n} \frac{f_2(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

We have

$$\mathcal{N}g_2(z) = (n+1+\alpha) \int_{\mathbb{B}_n} \frac{\langle z, w \rangle f_2(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+2+\alpha}}.$$

Let  $x \in E$  with  $\|x\|_E = 1$ . Using Lemma 5.3.5 and Theorem 2.1.25, we obtain a constant  $c$  independent of

$z$  such that

$$\begin{aligned} (1 - |z|^2) \|\mathcal{N}g_2(z)\|_E &\leq (n + 1 + \alpha) \int_{\mathbb{B}_n} \|f_2(w)\|_E \frac{1 - |z|^2}{|1 - \langle z, w \rangle|^{n+2+\alpha}} d\nu_\alpha(w) \\ &= (n + 1 + \alpha) \int_{\mathbb{B}_n} \frac{1 - |z|^2}{|1 - \langle z, w \rangle|^{n+2+\alpha}} d\mu_{f_2, \alpha}(w) \\ &\leq (n + 1 + \alpha) \|\mu_{f_2, \alpha}\|_{Carl} \int_{\mathbb{B}_n} \frac{1 - |z|^2}{|1 - \langle z, w \rangle|^{n+2+\alpha}} d\nu_\alpha(w) \\ &\lesssim \|f_2\|_{BA^1(E)}. \end{aligned}$$

This shows that  $g_2 = P_\alpha(f_2)$  is in  $\mathfrak{B}(\mathbb{B}_n, E)$  and

$$\|P_\alpha f_2\|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \|f_2\|_{BA^1(E)}.$$

Therefore,  $P_\alpha f \in \mathfrak{B}(\mathbb{B}_n, E)$  and

$$\|P_\alpha f\|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \|f_1\|_{BO(E)} + \|f_2\|_{BA^1(E)} \simeq \|f\|_{BMO_\alpha^1(E)}.$$

Since  $P_\alpha f(0) = \tilde{f}(0)$ , it follows that

$$\| \|P_\alpha f\| \|_{\mathfrak{B}(\mathbb{B}_n, E)} = \|P_\alpha f(0)\|_E + \|P_\alpha f\|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \| \|f\| \|_{BMO_\alpha^1(E)} = \| \tilde{f}(0) \|_E + \|f\|_{BMO_\alpha^1(E)}.$$

□

Since  $BMO_\alpha^p(\mathbb{B}_n, E) \subset BMO_\alpha^1(\mathbb{B}_n, E)$ , for all  $p \geq 1$ , then we have the following result:

**Corollary 5.3.11.** Let  $1 \leq p < \infty$ . The Bergman projection

$$P_\alpha : BMO_\alpha^p(\mathbb{B}_n, E) \rightarrow \mathfrak{B}(\mathbb{B}_n, E)$$

is a bounded operator. Moreover, we have

$$\| \|P_\alpha f\| \|_{\mathfrak{B}(\mathbb{B}_n, E)} \lesssim \| \|f\| \|_{BMO_\alpha^p(\mathbb{B}_n, E)}.$$

## 5.4 Boundedness of Toeplitz operators with operator-valued $BMO^1$ symbols.

In this section, for  $E, F$  two complex Banach spaces and  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , we show that the Toeplitz operator  $T_b$  is bounded from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$  if and only if its Berezin transform  $\tilde{b}$  is bounded on  $\mathbb{B}_n$ . We recall that the Toeplitz operator  $T_b$ , with operator-valued symbol  $b : \mathbb{B}_n \rightarrow \mathcal{L}(E, F)$  is defined by

$$T_b(g)(z) := \int_{\mathbb{B}_n} \frac{b(w)(g(w))}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w),$$

and that the Toeplitz operator  $T_{\|b\|_{\mathcal{L}(E, F)}}$ , is defined by

$$T_{\|b\|_{\mathcal{L}(E, F)}}(g)(z) = \int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(E, F)} g(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) = \int_{\mathbb{B}_n} \frac{g(w) \|b(w)\|_{\mathcal{L}(E, F)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) = T_{\mu_{b, \alpha}} g(z),$$

where  $d\mu_{b, \alpha}(w) = \|b(w)\|_{\mathcal{L}(E, F)} d\nu_\alpha(w)$ , and  $g \in H^\infty(\mathbb{B}_n, E)$ . Since  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , the positive measure  $d\mu_{b, \alpha}(w)$  is a finite Borel measure on  $\mathbb{B}_n$ . Let  $E, F$  be two complex Banach spaces. We recall that the Berezin

## 5.4. Boundedness of Toeplitz operators with operator-valued $BMO^1$ symbols.

transform  $\tilde{b}$  of an operator-valued function  $b : \mathbb{B}_n \rightarrow \mathcal{L}(E, F)$  is the operator-valued function defined by

$$\tilde{b}(z) := \int_{\mathbb{B}_n} |k_z^\alpha(w)|^2 b(w) d\nu_\alpha(w),$$

such that for each  $x \in E$  we have

$$\tilde{b}(z)(x) = \int_{\mathbb{B}_n} |k_z^\alpha(w)|^2 b(w)(x) d\nu_\alpha(w). \quad (5.23)$$

We also note that the Berezin transform of the scalar-valued function  $\|b\|_{\mathcal{L}(E, F)}$  is given by

$$\|\tilde{b}\|_{\mathcal{L}(E, F)}(z) := \int_{\mathbb{B}_n} |k_z^\alpha(w)|^2 \|b(w)\|_{\mathcal{L}(E, F)} d\nu_\alpha(w).$$

The following definition can also be found in [59].

**Definition 5.4.1.** The Berezin transform of the Toeplitz operator  $T_b$  is the operator-valued function defined by

$$\langle \widetilde{T}_b(z)(x), y^* \rangle_{F, F^*} = \langle T_b(k_z^\alpha x), k_z^\alpha y^* \rangle_{\alpha, F}, \quad (5.24)$$

for all  $z \in \mathbb{B}_n$ ,  $x \in E$  and  $y^* \in F^*$ .

**Lemma 5.4.2.** For  $x \in E$  and  $y^* \in F^*$ , the following identity holds

$$\langle \widetilde{T}_b(z)(x), y^* \rangle_{F, F^*} = \langle \tilde{b}(z)(x), y^* \rangle_{F, F^*}. \quad (5.25)$$

*Proof.* Let  $x \in E$  and  $y^* \in F^*$ . By Fubini's theorem, for all  $z \in \mathbb{B}_n$ , we have

$$\begin{aligned} \langle T_b(k_z^\alpha x), k_z^\alpha y^* \rangle_{\alpha, F} &= \int_{\mathbb{B}_n} \langle T_b(k_z^\alpha(w)x), k_z^\alpha(w)y^* \rangle_{F, F^*} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \left\langle \int_{\mathbb{B}_n} \frac{b(u)(k_z^\alpha(u)x)}{(1 - \langle w, u \rangle)^{n+1+\alpha}} d\nu_\alpha(u), k_z^\alpha(w)y^* \right\rangle_{F, F^*} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \left\langle b(u)(k_z^\alpha(u)x), \int_{\mathbb{B}_n} \frac{[k_z^\alpha(w)y^*] d\nu_\alpha(w)}{(1 - \langle u, w \rangle)^{n+1+\alpha}} \right\rangle_{F, F^*} d\nu_\alpha(u) \\ &= \int_{\mathbb{B}_n} \langle b(u)(k_z^\alpha(u)x), k_z^\alpha(u)y^* \rangle_{F, F^*} d\nu_\alpha(u) \\ &= \int_{\mathbb{B}_n} \langle b(u)(x) |k_z^\alpha(u)|^2, y^* \rangle_{F, F^*} d\nu_\alpha(u) \\ &= \left\langle \int_{\mathbb{B}_n} b(u)(x) |k_z^\alpha(u)|^2 d\nu_\alpha(u), y^* \right\rangle_{F, F^*} \\ &= \langle \tilde{b}(z)(x), y^* \rangle_{F, F^*}. \end{aligned}$$

□

**Remark 5.4.3.** For every  $z \in \mathbb{B}_n$ , we have  $\widetilde{T}_b(z) = \tilde{b}(z)$ .

**Proposition 5.4.4.** Let  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . If the Toeplitz operator  $T_b : A_\alpha^2(\mathbb{B}_n, E) \rightarrow A_\alpha^2(\mathbb{B}_n, F)$  is bounded, then its Berezin transform  $\tilde{b}$  is bounded on  $\mathbb{B}_n$ .

*Proof.* Assume that the Toeplitz operator  $T_b$  is bounded from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$ . Let us prove that

$$\sup_{z \in \mathbb{B}_n} \|\tilde{b}(z)\|_{\mathcal{L}(E, F)} < \infty.$$

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Let  $x \in E$  and  $y^* \in F^*$ . Since  $T_b : A_\alpha^2(\mathbb{B}_n, E) \rightarrow A_\alpha^2(\mathbb{B}_n, F)$  is bounded, by Lemma 5.4.2, for all  $z \in \mathbb{B}_n$ , we have

$$\begin{aligned} |\langle \tilde{b}(z)(x), y^* \rangle_{F, F^*}| &= |\langle T_b(k_z^\alpha x), k_z^\alpha y^* \rangle_{\alpha, F}| \\ &\leq \|T_b\| \|k_z^\alpha x\|_{2, \alpha, E} \|k_z^\alpha y^*\|_{2, \alpha, F^*} \\ &= \|T_b\| \|x\|_E \|y^*\|_{F^*}. \end{aligned}$$

Since  $x \in E$  and  $y^* \in F^*$  are arbitrary, it follows that

$$\|\tilde{b}(z)\|_{\mathcal{L}(E, F)} = \sup_{\|x\|_E=1, \|y^*\|_{F^*}=1} |\langle \tilde{b}(z)(x), y^* \rangle_{F, F^*}| \leq \|T_b\| < \infty.$$

That is,  $\tilde{b}$  is bounded in  $\mathbb{B}_n$ . □

**Proposition 5.4.5.** Let  $E, F$  be two complex Banach spaces and  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . If  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$  and  $\tilde{b}$  is bounded in  $\mathbb{B}_n$ , then  $d\mu_{b, \alpha} = \|b\|_{\mathcal{L}(E, F)} d\nu_\alpha$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$ .

*Proof.* We assume that  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$  and  $\tilde{b}$  is bounded in  $\mathbb{B}_n$ . Let us prove that  $\|b\|_{\mathcal{L}(E, F)} d\nu_\alpha$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$ . Since  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , by (1) of Proposition 5.3.9 we have that

$$\sup_{z \in \mathbb{B}_n} \left( \widetilde{\|b\|_{\mathcal{L}(E, F)}}(z) - \|\tilde{b}(z)\|_{\mathcal{L}(E, F)} \right) < \infty$$

and thus  $\widetilde{\|b\|_{\mathcal{L}(E, F)}}$  is bounded in  $\mathbb{B}_n$ . Since  $\|b\|_{\mathcal{L}(E, F)} \geq 0$ , and  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , then the measure  $d\mu_{b, \alpha} = \|b\|_{\mathcal{L}(E, F)} d\nu_\alpha$  is a finite positive Borel measure on  $\mathbb{B}_n$ . By Theorem 5.2.6, the Berezin transform  $\widetilde{\mu_{b, \alpha}} = \widetilde{\|b\|_{\mathcal{L}(E, F)}}$  is bounded in  $\mathbb{B}_n$  implies that  $d\mu_{b, \alpha} = \|b\|_{\mathcal{L}(E, F)} d\nu_\alpha$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$ . □

**Proposition 5.4.6.** Let  $E, F$  be two complex Banach spaces and  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . If  $\|b\|_{\mathcal{L}(E, F)} d\nu_\alpha$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$  (for  $A_\alpha^1(\mathbb{B}_n, F^*)$ ), then  $T_b$  is bounded from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$ .

*Proof.* Assume that  $\|b\|_{\mathcal{L}(E, F)} d\nu_\alpha$  is a Carleson measure for  $A_\alpha^1(\mathbb{B}_n, E)$ . Let us show that  $T_b$  is bounded from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$ . In fact, let  $g \in H^\infty(\mathbb{B}_n, E)$  and  $h \in H^\infty(\mathbb{B}_n, F^*)$ . By using the Fubini's theorem, and the reproducing kernel formula, we have

$$\begin{aligned} \langle T_b(g), h \rangle_{\alpha, F} &= \int_{\mathbb{B}_n} \langle T_b(g)(z), h(z) \rangle_{F, F^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left\langle \int_{\mathbb{B}_n} \frac{b(w)(g(w))}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), h(z) \right\rangle_{F, F^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left\langle b(w)(g(w)), \frac{h(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right\rangle_{F, F^*} d\nu_\alpha(z) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \left\langle b(w)(g(w)), \int_{\mathbb{B}_n} \frac{h(z) d\nu_\alpha(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right\rangle_{F, F^*} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \langle b(w)(g(w)), h(w) \rangle_{F, F^*} d\nu_\alpha(w). \end{aligned}$$

Let  $g \in H^\infty(\mathbb{B}_n, E)$ , and  $h \in H^\infty(\mathbb{B}_n, F^*)$ . Using Hölder's inequality and the fact that  $\|b\|_{\mathcal{L}(E, F)} d\nu_\alpha$  is a

Carleson measure respectively for  $A_\alpha^2(\mathbb{B}_n, E)$  and for  $A_\alpha^2(\mathbb{B}_n, F^*)$ , we get

$$\begin{aligned}
 \left| \int_{\mathbb{B}_n} \langle b(w)(g(w)), h(w) \rangle_{F, F^*} d\nu_\alpha(w) \right| &\leq \int_{\mathbb{B}_n} |\langle b(w)(g(w)), h(w) \rangle_{F, F^*}| d\nu_\alpha(w) \\
 &\leq \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(E, F)} \|g(w)\|_E \|h(w)\|_{F^*} d\nu_\alpha(w) \\
 &= \int_{\mathbb{B}_n} \left[ \|b(w)\|_{\mathcal{L}(E, F)}^{1/2} \|g(w)\|_E \right] \left[ \|b(w)\|_{\mathcal{L}(E, F)}^{1/2} \|h(w)\|_{F^*} \right] d\nu_\alpha(w) \\
 &\leq \left( \int_{\mathbb{B}_n} \|g(w)\|_E^2 \|b(w)\|_{\mathcal{L}(E, F)} d\nu_\alpha(w) \right)^{1/2} \\
 &\quad \times \left( \int_{\mathbb{B}_n} \|h(w)\|_{F^*}^2 \|b(w)\|_{\mathcal{L}(E, F)} d\nu_\alpha(w) \right)^{1/2} \\
 &\lesssim \|g\|_{2, \alpha, E} \|h\|_{2, \alpha, F^*}.
 \end{aligned}$$

So, we have shown that

$$|\langle T_b(g), h \rangle_{\alpha, F}| \lesssim \|g\|_{2, \alpha, E} \|h\|_{2, \alpha, F^*},$$

for all  $g \in H^\infty(\mathbb{B}_n, E)$  and  $h \in H^\infty(\mathbb{B}_n, F^*)$ . Thus,

$$\|T_b(g)\|_{\alpha, F} \lesssim \|g\|_{2, \alpha, E},$$

for every  $g \in H^\infty(\mathbb{B}_n, E)$ . Since  $H^\infty(\mathbb{B}_n, E)$  is dense in  $A_\alpha^2(\mathbb{B}_n, E)$ , we conclude that the  $T_b$  is bounded from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$ .  $\square$

The following theorem is the main result of this section.

**Theorem 5.4.7.** Let  $E, F$  be two complex Banach spaces and  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . The Toeplitz operator  $T_b : A_\alpha^2(\mathbb{B}_n, E) \rightarrow A_\alpha^2(\mathbb{B}_n, F)$  is a bounded linear operator if and only if its Berezin transform  $\widetilde{b}$  is bounded in  $\mathbb{B}_n$ .

*Proof.* The proof is a direct consequence of Proposition 5.4.4, Proposition 5.4.5 and Proposition 5.4.6.  $\square$

In what will follow, we let  $E, F$  to be two Banach spaces.

**Lemma 5.4.8.** Let  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ . Then  $b \circ \varphi_z \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$  for all  $z \in \mathbb{B}_n$ .

*Proof.* Let  $z \in \mathbb{B}_n$ . By using the change of variables  $v = \varphi_z(u)$  ( see Proposition 2.1.12) and the identity

$$|k_w^\alpha(\varphi_z(v))|^2 |k_z^\alpha(v)|^2 = |k_{\varphi_z(w)}^\alpha(v)|^2,$$

we have

$$\begin{aligned}
 \widetilde{b \circ \varphi_z}(w) &= \int_{\mathbb{B}_n} b \circ \varphi_z(u) |k_w^\alpha(u)|^2 d\nu_\alpha(u) \\
 &= \int_{\mathbb{B}_n} b(v) |k_w^\alpha(\varphi_z(v))|^2 |k_z^\alpha(v)|^2 d\nu_\alpha(v) \\
 &= \int_{\mathbb{B}_n} b(v) |k_{\varphi_z(w)}^\alpha(v)|^2 d\nu_\alpha(v) \\
 &= \widetilde{b}(\varphi_z(w)).
 \end{aligned}$$

## 5.4. Boundedness of Toeplitz operators with operator-valued $BMO^1$ symbols.

Now, let  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$  and  $z \in \mathbb{B}_n$ . By using the identity

$$\varphi_z \circ \varphi_w = \varphi_{\varphi_z(w)} \circ U, \quad \text{where } U \text{ is a unitary transformation of } \mathbb{C}^n,$$

and the invariance of  $\nu_\alpha$  under  $U$ , we obtain that

$$\begin{aligned} \|b \circ \varphi_z\|_{BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))} &= \sup_{w \in \mathbb{B}_n} \|b \circ \varphi_z \circ \varphi_w - \widetilde{b \circ \varphi_z}(w)\|_{1, \alpha, \mathcal{L}(E, F)} \\ &= \sup_{w \in \mathbb{B}_n} \|b \circ \varphi_{\varphi_z(w)} - \widetilde{b}(\varphi_z(w))\|_{1, \alpha, \mathcal{L}(E, F)} \\ &= \sup_{\psi_z(w) \in \mathbb{B}_n} \|b \circ \varphi_{\psi_z(w)} - \widetilde{b}(\psi_z(w))\|_{1, \alpha, \mathcal{L}(E, F)} \\ &= \sup_{u \in \mathbb{B}_n} \|b \circ \varphi_u - \widetilde{b}(u)\|_{1, \alpha, \mathcal{L}(E, F)} \\ &= \|b\|_{BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))} < \infty. \end{aligned}$$

□

**Proposition 5.4.9.** Let  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$  and  $\widetilde{b}$  be bounded in  $\mathbb{B}_n$ . Then

$$\sup_{\|e\|_E=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{p, \alpha, F} < \infty, \quad \text{for } p \geq 1. \quad (5.26)$$

*Proof.* We first recall that if  $g \in \mathcal{H}(\mathbb{B}_n, F)$ , then its  $L_\alpha^p(\mathbb{B}_n, F)$  norm  $\|g\|_{p, \alpha, F}$  is equivalent to  $\|g(0)\|_F + \|(1 - |z|^2)\mathcal{N}g(z)\|_{p, \alpha, F}$  (see Theorem 2.2.26). For  $g \in \mathcal{B}(\mathbb{B}_n, F)$ , we have that

$$\begin{aligned} \|g\|_{p, \alpha, F} &\simeq \|g(0)\|_F + \|(1 - |z|^2)\mathcal{N}g(z)\|_{p, \alpha, F} \\ &\lesssim \|g(0)\|_F + \|(1 - |z|^2)\mathcal{N}g(z)\|_{\infty, F} = \|g\|. \end{aligned} \quad (5.27)$$

Let  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$  and  $e \in E$ , with  $\|e\|_E = 1$ . By Lemma 5.4.8, we have that  $(b \circ \varphi_z)e$  is in  $BMO_\alpha^1(\mathbb{B}_n, F)$  and

$$\|(b \circ \varphi_z)e\|_{BMO_\alpha^1(\mathbb{B}_n, F)} \leq \|b \circ \varphi_z\|_{BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))}. \quad (5.28)$$

Using (5.27), (5.28), Theorem 5.3.10, Remark 5.3.8 and (5.9), we get

$$\begin{aligned} \|P_\alpha(b \circ \varphi_z)e\|_{p, \alpha, F} &\lesssim \|P_\alpha(b \circ \varphi_z e)\|_{\mathcal{B}(\mathbb{B}_n, F)} \\ &\lesssim \|(b \circ \varphi_z)e\|_{BMO_\alpha^1(\mathbb{B}_n, F)} \\ &\lesssim \|b \circ \varphi_z\|_{BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))} \\ &\lesssim \|b \circ \varphi_z\| \\ &= \|(\widetilde{b \circ \varphi_z})(0)\|_{\mathcal{L}(E, F)} + \sup_{w \in \mathbb{B}_n} \|b \circ \varphi_z \circ \varphi_w - \widetilde{b \circ \varphi_z}(w)\|_{1, \alpha, \mathcal{L}(E, F)} \\ &= \|\widetilde{b}(z)\|_{\mathcal{L}(E, F)} + \sup_{\varphi_z(w) \in \mathbb{B}_n} \|b \circ \varphi_{\varphi_z(w)} - \widetilde{b}(\varphi_z(w))\|_{1, \alpha, \mathcal{L}(E, F)} \\ &= \|\widetilde{b}(z)\|_{\mathcal{L}(E, F)} + \sup_{u \in \mathbb{B}_n} \|b \circ \varphi_u - \widetilde{b}(u)\|_{1, \alpha, \mathcal{L}(E, F)}. \end{aligned}$$

Thus, for  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$  and  $\widetilde{b}$  being bounded in  $\mathbb{B}_n$  we get

$$\sup_{\|e\|_E=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{p, \alpha, F} < \infty, \quad \text{for } p \geq 1.$$

□

## 5.4. Boundedness of Toeplitz operators with operator-valued $BMO^1$ symbols.

The following lemma is an extension of Lemma 3.2 in [79, 70] to vector-valued settings.

**Lemma 5.4.10.** Let  $E, F$  be two complex Banach spaces and  $b \in L^1_\alpha(\mathbb{B}_n, \mathcal{L}(E, F))$ . Let  $T_b$  be bounded from  $A^2_\alpha(\mathbb{B}_n, E)$  into  $A^2_\alpha(\mathbb{B}_n, F)$ . Then for every  $z \in \mathbb{B}_n$ , and every unit vector  $e \in E$  the following holds:

- (a)  $(T_b(K_z^\alpha e))(u) = K_z^\alpha(u)P_\alpha(b \circ \varphi_z e)(\varphi_z(u))$ .
- (b)  $\|T_b(k_z^\alpha e)\|_{2,\alpha,F} = \|T_{b \circ \varphi_z} e\|_{2,\alpha,F}$ .
- (c) For  $b \in BMO^1_\alpha(\mathbb{B}_n, \mathcal{L}(E, F))$ , each  $T_{b \circ \varphi_z}$  is bounded from  $A^2_\alpha(\mathbb{B}_n, E)$  into  $A^2_\alpha(\mathbb{B}_n, F)$ .

*Proof.* (a) Let  $z \in \mathbb{B}_n$ , and  $e \in E$  with  $\|e\|_E = 1$ . By using the change of variable formula (Proposition 2.1.12), we have

$$\begin{aligned} T_b(K_z^\alpha e)(u) &= P_\alpha(b K_z^\alpha e)(u) = \int_{\mathbb{B}_n} b(w)(K_z^\alpha(w)e)\overline{K_u^\alpha(w)}d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} b(\varphi_z(v))(K_z^\alpha(\varphi_z(v))e)\overline{K_u^\alpha(\varphi_z(v))}|k_z^\alpha(v)|^2d\nu_\alpha(v). \end{aligned}$$

Since

$$K_z^\alpha(\varphi_z(v))k_z^\alpha(v) = \frac{1}{(1 - |z|^2)^{\frac{n+1+\alpha}{2}}},$$

and

$$\overline{K_u^\alpha(\varphi_z(v))k_z^\alpha(v)} = k_z^\alpha(u)\overline{K_{\varphi_z(u)}^\alpha(v)},$$

it follows that

$$\begin{aligned} (T_b(K_z^\alpha e))(u) &= \int_{\mathbb{B}_n} [b \circ \varphi_z](v)(e) [K_z^\alpha(\varphi_z(v))k_z^\alpha(v)] [\overline{K_u^\alpha(\varphi_z(v))k_z^\alpha(v)}] d\nu_\alpha(v) \\ &= \int_{\mathbb{B}_n} [b \circ \varphi_z](v)(e) \frac{1}{(1 - |z|^2)^{\frac{n+1+\alpha}{2}}} k_z^\alpha(u)\overline{K_{\varphi_z(u)}^\alpha(v)} d\nu_\alpha(v) \\ &= \int_{\mathbb{B}_n} [b \circ \varphi_z](v)(e) K_z^\alpha(u)\overline{K_{\varphi_z(u)}^\alpha(v)} d\nu_\alpha(v) \\ &= K_z^\alpha(u) \int_{\mathbb{B}_n} b(\varphi_z(v))(e) K_{\varphi_z(u)}^\alpha(v) d\nu_\alpha(v) \\ &= K_z^\alpha(u)P_\alpha(b \circ \varphi_z e)(\varphi_z(u)). \end{aligned}$$

(b) By using the change of variables  $w = \varphi_z(u)$ , we have

$$\begin{aligned} \|T_b(k_z^\alpha e)\|_{2,\alpha,F}^2 &= \int_{\mathbb{B}_n} \|P_\alpha(b(\cdot)(k_z^\alpha(\cdot)e))(w)\|_F^2 d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|P_\alpha(b(\cdot)(k_z^\alpha(\cdot)e))(\varphi_z(u))\|_F^2 |k_z^\alpha(u)|^2 d\nu_\alpha(u) \\ &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} b(w)(k_z^\alpha(w)e)\overline{K_{\varphi_z(u)}^\alpha(w)}k_z^\alpha(u) d\nu_\alpha(w) \right\|_F^2 d\nu_\alpha(u). \end{aligned}$$

By the change of variables  $w = \varphi_z(v)$ , and using the fact that

$$\overline{K_{\varphi_z(u)}^\alpha(w)}k_z^\alpha(u) = \overline{k_z^\alpha(w)K_u^\alpha(\varphi_z(w))},$$

and

$$k_z^\alpha(\varphi_z(v))k_z^\alpha(v) = 1,$$

it follows that

$$\begin{aligned}
 \|T_b(k_z^\alpha e)\|_{2,\alpha,F}^2 &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} b(w)(k_z^\alpha(w)e)\overline{k_z^\alpha(w)K_u^\alpha(\varphi_z(w))}d\nu_\alpha(w) \right\|_F^2 d\nu_\alpha(u) \\
 &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} b(w)(e)|k_z^\alpha(w)|^2\overline{K_u^\alpha(\varphi_z(w))}d\nu_\alpha(w) \right\|_F^2 d\nu_\alpha(u) \\
 &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} b(\varphi_z(v))(e)|k_z^\alpha(\varphi_z(v))|^2\overline{K_u^\alpha(\varphi_z(\varphi_z(v)))}|k_z^\alpha(v)|^2d\nu_\alpha(v) \right\|_F^2 d\nu_\alpha(u) \\
 &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} b(\varphi_z(v))(e)|k_z^\alpha(\varphi_z(v))|^2|k_z^\alpha(v)|^2\overline{K_u^\alpha(v)}d\nu_\alpha(v) \right\|_F^2 d\nu_\alpha(u) \\
 &= \int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} b(\varphi_z(v))(e)\overline{K_u^\alpha(v)}d\nu_\alpha(v) \right\|_F^2 d\nu_\alpha(u) \\
 &= \int_{\mathbb{B}_n} \|P_\alpha(b \circ \varphi_z e)(u)\|_F^2 d\nu_\alpha(u) \\
 &= \|P_\alpha(b \circ \varphi_z e)\|_{2,\alpha,F}^2 = \|T_{b \circ \varphi_z} e\|_{2,\alpha,F}^2.
 \end{aligned}$$

(c) If  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , then  $b \circ \varphi_z \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , for all  $z \in \mathbb{B}_n$  (see Lemma 5.4.8). Since for  $z, w \in \mathbb{B}_n$ ,  $\widetilde{b \circ \varphi_z}(w) = \widetilde{b}(\varphi_z(w))$  (see the proof of Lemma 5.4.8), then  $\widetilde{b}$  being bounded (Theorem 5.4.7) implies that  $\widetilde{b \circ \varphi_z}$  is bounded independently of  $z$ . It follows by Theorem 5.4.7 that  $T_{b \circ \varphi_z}$  is a bounded operator from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$ .

□

## 5.5 Compactness of Toeplitz operators with matrix-valued $BMO^1$ symbol.

In this section, we will study an extension of Theorem 3.1 in [79] and Theorem 3.4 in [70] to the vector-valued settings. In particular, we will show that when the target space is of finite dimension, we obtain a similar result. From now on, we consider an integer  $d \in \mathbb{N}$  with  $d > 1$ . We recall that  $\mathcal{L}(\mathbb{C}^d)$  is identified with the space  $\mathcal{M}^d$  of  $d \times d$  matrix with coefficients in  $\mathbb{C}$ . Since all norms of matrices are equivalent in finite dimension, for a  $d \times d$  matrix  $M$ , we will denote  $\|M\|$  any convenient norm of  $M$ . Let  $L^\infty_{\mathcal{M}^d} := L^\infty(\mathbb{B}_n, \mathcal{M}^d)$  denote the set of  $d \times d$  matrices-valued functions,  $b$ , such that the function  $z \mapsto \|b(z)\|_{\mathcal{L}(\mathbb{C}^d)}$  is in  $L^\infty(\mathbb{B}_n)$ . We denote by  $\{e_i\}_{i=1}^d$  the standard orthonormal basis in  $\mathbb{C}^d$ . For  $x = (x_1, \dots, x_d)$  and  $y = (y_1, \dots, y_d)$  in  $\mathbb{C}^d$ , the inner product on  $\mathbb{C}^d$  is given by

$$\langle x, y \rangle_{\mathbb{C}^d} = \sum_{k=1}^d x_k \overline{y_k}.$$

Given  $x = (x_1, \dots, x_d)$  in  $\mathbb{C}^d$ , we denote by  $\|x\|_{\mathbb{C}^d}^2 = \sum_{i=1}^d |x_i|^2$  the norm on  $\mathbb{C}^d$ . If  $M$  is a  $d \times d$  matrix,  $\|M\|$  will denote any convenient matrix norm.

**Definition 5.5.1.** Let  $\mathcal{H}$  be a complex Hilbert space and  $z \in \mathbb{B}_n$ . We consider the linear operator

$$U_z : A_\alpha^2(\mathbb{B}_n, \mathcal{H}) \rightarrow A_\alpha^2(\mathbb{B}_n, \mathcal{H})$$

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defined by

$$U_z f := f \circ \varphi_z k_z^\alpha \quad f \in A_\alpha^2(\mathbb{B}_n, \mathcal{H}), \quad (5.29)$$

where we recall that  $k_z^\alpha$  is defined in (5.5). Given a complex Hilbert space  $\mathcal{H}$ , we have the following lemma.

**Lemma 5.5.2.** For any  $z \in \mathbb{B}_n$ , the operator  $U_z$  is self-adjoint, idempotent, and isometric on  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ .

*Proof.* We first show idempotency. Given  $f \in A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ , we have using (5.8)

$$\begin{aligned} U_z^2 f &= U_z(U_z f) \\ &= ((U_z f) \circ \varphi_z) k_z^\alpha \\ &= (f \circ \varphi_z \circ \varphi_z)(k_z^\alpha \circ \varphi_z) k_z^\alpha \\ &= f. \end{aligned}$$

Next, we show the isometry. Using the change of variables  $u = \varphi_z(w)$  and the identity (5.8), we have

$$\begin{aligned} \|U_z f\|_{2,\alpha,\mathcal{H}}^2 &= \int_{\mathbb{B}_n} \|U_z f(w)\|_{\mathcal{H}}^2 d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|f \circ \varphi_z(w) k_z^\alpha(w)\|_{\mathcal{H}}^2 d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|f \circ \varphi_z(w)\|_{\mathcal{H}}^2 |k_z^\alpha(w)|^2 d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|f(u)\|_{\mathcal{H}}^2 |k_z^\alpha(\varphi_z(u))|^2 |k_z^\alpha(u)|^2 d\nu_\alpha(u) \\ &= \int_{\mathbb{B}_n} \|f(u)\|_{\mathcal{H}}^2 |k_z^\alpha(\varphi_z(u)) k_z^\alpha(u)|^2 d\nu_\alpha(u) \\ &= \int_{\mathbb{B}_n} \|f(u)\|_{\mathcal{H}}^2 d\nu_\alpha(u) \\ &= \|f\|_{2,\alpha,\mathcal{H}}^2. \end{aligned}$$

We finally show that  $U_z$  is self-adjoint. Since  $U_z$  is an isometry, for every  $f, g \in A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ , we have  $\langle f, g \rangle_{\alpha,\mathcal{H}} = \langle U_z f, U_z g \rangle_{\alpha,\mathcal{H}}$ . It follows that

$$\langle U_z f, g \rangle_{\alpha,\mathcal{H}} = \langle U_z U_z f, U_z g \rangle_{\alpha,\mathcal{H}} = \langle f, U_z g \rangle_{\alpha,\mathcal{H}}.$$

□

Let  $\mathcal{H}$  be a complex Hilbert space.

**Definition 5.5.3.** Let  $A \in \mathcal{L}(A_\alpha^2(\mathbb{B}_n, \mathcal{H}))$ . We define the operator  $A_z$  on  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  by

$$A_z := U_z A U_z = U_z^* A U_z = U_z A U_z^*. \quad (5.30)$$

**Proposition 5.5.4.** Let  $\psi \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathcal{H}))$ . If  $A = T_\psi$  is a bounded Toeplitz operator on  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ , then  $A_z = T_{\psi \circ \varphi_z}$  for any  $z \in \mathbb{B}_n$ . In particular, each  $A_z$  is still a bounded Toeplitz operator on  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ .

*Proof.* Let  $f \in A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  and  $z \in \mathbb{B}_n$ . We have, using the change of variables  $v = \varphi_z(u)$  and the identity

(5.8)

$$\begin{aligned}
 (T_\psi(U_z f))(w) &= \int_{\mathbb{B}_n} K^\alpha(w, u) \psi(u) (f \circ \varphi_z(u) k_z^\alpha(u)) d\nu_\alpha(u) \\
 &= \int_{\mathbb{B}_n} K^\alpha(w, u) \psi(u) (f \circ \varphi_z(u)) \frac{|k_z^\alpha(u)|^2 d\nu_\alpha(u)}{\overline{k_z^\alpha(u)}} \\
 &= \int_{\mathbb{B}_n} K^\alpha(w, \varphi_z(v)) \psi \circ \varphi_z(v) (f(v)) \frac{d\nu_\alpha(v)}{\overline{k_z^\alpha(\varphi_z(v))}} \\
 &= \int_{\mathbb{B}_n} K^\alpha(w, \varphi_z(v)) \overline{k_z^\alpha(v)} (\psi \circ \varphi_z(v)) (f(v)) d\nu_\alpha(v) \\
 &= k_z^\alpha(w) \int_{\mathbb{B}_n} k_z^\alpha(\varphi_z(w)) K^\alpha(\varphi_z(\varphi_z(w)), \varphi_z(v)) \overline{k_z^\alpha(v)} \psi \circ \varphi_z(v) f(v) d\nu_\alpha(v).
 \end{aligned}$$

Since

$$k_z^\alpha(a) K^\alpha(\varphi_z(a), \varphi_z(b)) \overline{k_z^\alpha(b)} = K^\alpha(a, b),$$

 for all  $a, b \in \mathbb{B}_n$ , it follows that

$$\begin{aligned}
 (T_\psi(U_z f))(w) &= k_z^\alpha(w) \int_{\mathbb{B}_n} K^\alpha(\varphi_z(w), v) \psi \circ \varphi_z(v) f(v) d\nu_\alpha(v) \\
 &= k_z^\alpha(w) (T_{\psi \circ \varphi_z}(f))(\varphi_z(w)) \\
 &= (U_z T_{\psi \circ \varphi_z}(f))(w).
 \end{aligned}$$

Hence

$$U_z T_\psi U_z(f) = T_{\psi \circ \varphi_z}(f).$$

 This proves the desired identity. □

We will use the following extension of Berezin transform of vector-valued operator introduced in [58].

**Definition 5.5.5.** Let  $\mathcal{H}$  be a complex Hilbert space and  $A \in \mathcal{L}(A_\alpha^2(\mathbb{B}_n, \mathcal{H}))$ . We define the Berezin transform  $\tilde{A}$  of  $A$  by

$$\langle \tilde{A}(z)e, h \rangle_{\mathcal{H}} := \langle A(k_z^\alpha e), k_z^\alpha h \rangle_{\alpha, \mathcal{H}},$$

 where  $z \in \mathbb{B}_n$  and  $e, h \in \mathcal{H}$ .

**Proposition 5.5.6.** Let  $A \in \mathcal{L}(A_\alpha^2(\mathbb{B}_n, \mathcal{H}))$  and  $z \in \mathbb{B}_n$ . We have  $\tilde{A} \circ \varphi_z = \tilde{A}_z$ .

*Proof.* Let  $A \in \mathcal{L}(A_\alpha^2(\mathbb{B}_n, \mathcal{H}))$  and  $z, w \in \mathbb{B}_n$ . For any  $e \in \mathcal{H}$  we claim that

$$U_z(K_w^\alpha e) = \overline{k_z^\alpha(w)} K_{\varphi_z(w)}^\alpha e. \quad (5.31)$$

 Indeed, for  $f \in A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ , and  $e \in \mathcal{H}$  we have, using the definition of  $U_z$  and the change of variables

$$\zeta = \varphi_z(w)$$

$$\begin{aligned} \langle f, U_z(K_w^\alpha e) \rangle_{\alpha, \mathcal{H}} &= \langle U_z f, (K_w^\alpha e) \rangle_{\alpha, \mathcal{H}} \\ &= \langle U_z f(w), e \rangle_{\mathcal{H}} \\ &= \langle (f \circ \varphi_z)(w) k_z^\alpha(w), e \rangle_{\mathcal{H}} \\ &= \langle (f \circ \varphi_z)(w), \overline{k_z^\alpha(w)} e \rangle_{\mathcal{H}} \\ &= \langle f(\varphi_z(w)), \overline{k_z^\alpha(w)} e \rangle_{\mathcal{H}} \\ &= \langle f, \overline{k_z^\alpha(w)} K_{\varphi_z(w)}^\alpha e \rangle_{\alpha, \mathcal{H}}, \end{aligned}$$

which implies the claim.

By writing (5.31) as

$$\begin{aligned} U_z(K_w^\alpha e) &= \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha}^{-1} (U_z K_w^\alpha e) \\ &= \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} \overline{k_z^\alpha(w)} \left( \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha}^{-1} K_{\varphi_z(w)}^\alpha \right) e \\ &= \left( \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} \overline{k_z^\alpha(w)} \right) k_{\varphi_z(w)}^\alpha e. \end{aligned}$$

It follows that

$$k_{\varphi_z(w)}^\alpha e = \left( \|K_w^\alpha\|_{2, \alpha}^{-1} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} \overline{k_z^\alpha(w)} \right)^{-1} U_z(K_w^\alpha e). \quad (5.32)$$

By (2.40) and (2.11), it follows that

$$\begin{aligned} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} &= \frac{1}{(1 - |\varphi_z(w)|^2)^{(n+1+\alpha)/2}} \\ &= \frac{|1 - \langle z, w \rangle|^{n+1+\alpha}}{(1 - |z|^2)^{(n+1+\alpha)/2} (1 - |w|^2)^{(n+1+\alpha)/2}} = \frac{\|K_w^\alpha\|_{2, \alpha} \|K_z^\alpha\|_{2, \alpha}}{|\overline{K_z^\alpha(w)}|}. \end{aligned}$$

It follows that

$$\|K_w^\alpha\|_{2, \alpha}^{-1} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} |\overline{K_z^\alpha(w)}| = \|K_z^\alpha\|_{2, \alpha}.$$

Therefore, there holds

$$\|K_w^\alpha\|_{2, \alpha}^{-1} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} |\overline{k_z^\alpha(w)}| = 1. \quad (5.33)$$

Using (5.32) and (5.33), for every  $e_1, e_2 \in \mathcal{H}$ , we have

$$\begin{aligned} \langle \widetilde{A} \circ \varphi_z(w) e_1, e_2 \rangle_{\mathcal{H}} &= \langle \widetilde{A}(\varphi_z(w) e_1), e_2 \rangle_{\mathcal{H}} \\ &= \langle A(k_{\varphi_z(w)}^\alpha e_1), k_{\varphi_z(w)}^\alpha e_2 \rangle_{\alpha, \mathcal{H}} \\ &= \left( \|K_w^\alpha\|_{2, \alpha}^{-1} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} |\overline{K_z^\alpha(w)}| \right)^{-2} \langle AU_z(K_w^\alpha e_1), U_z(K_w^\alpha e_2) \rangle_{\alpha, \mathcal{H}} \\ &= \left( \|K_w^\alpha\|_{2, \alpha}^{-2} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} |\overline{K_z^\alpha(w)}| \right)^{-2} \langle AU_z(k_w^\alpha e_1), U_z(k_w^\alpha e_2) \rangle_{\alpha, \mathcal{H}} \\ &= \left( \|K_w^\alpha\|_{2, \alpha}^{-1} \|K_{\varphi_z(w)}^\alpha\|_{2, \alpha} |\overline{k_z^\alpha(w)}| \right)^{-2} \langle AU_z(k_w^\alpha e_1), U_z(k_w^\alpha e_2) \rangle_{\alpha, \mathcal{H}} \\ &= \langle AU_z(k_w^\alpha e_1), U_z(k_w^\alpha e_2) \rangle_{\alpha, \mathcal{H}} \\ &= \langle U_z AU_z(k_w^\alpha e_1), k_w^\alpha e_2 \rangle_{\alpha, \mathcal{H}} \\ &= \langle A_z(k_w^\alpha e_1), k_w^\alpha e_2 \rangle_{\alpha, \mathcal{H}} \\ &= \langle \widetilde{A}_z(w) e_1, e_2 \rangle_{\mathcal{H}}. \end{aligned}$$

This shows that  $\widetilde{A} \circ \varphi_z = \widetilde{A}_z$  as maps on  $\mathcal{H}$ . □

## 5.5. Compactness of Toeplitz operators with matrix-valued $BMO^1$ symbol.

The following result already appears in [60] for  $\mathcal{H} = \mathbb{C}^d$ ,  $d \geq 1$ .

**Theorem 5.5.7.** Let  $\mathcal{H}$  be a separable Hilbert space and  $A$  be a bounded linear operator on  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  and let  $e \in \mathcal{H}$  with  $\|e\|_{\mathcal{H}} = 1$  such that  $\tilde{A}(z)e \rightarrow 0$  weakly in  $\mathcal{H}$  as  $|z| \rightarrow 1^-$ . Then  $A_z e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  as  $|z| \rightarrow 1^-$ .

*Proof.* We recall that if  $\{e_k\}$  is a countable orthonormal basis for  $\mathcal{H}$ , then

$$\left( \sqrt{\frac{\Gamma(n+|m|+\alpha+1)}{m!\Gamma(n+\alpha+1)}} z^m e_k \right)_{m \in \mathbb{N}^n, k \in \mathbb{N}}$$

is a countable orthonormal basis for the vector-valued Bergman space  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$ . So, (up to a normalization constant which does not matter us) it is enough to show that  $\langle A_z e, h w^s \rangle_{\alpha, \mathcal{H}} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for every  $s \in \mathbb{N}^n$  and for every  $h \in \mathcal{H}$  with  $\|h\|_{\mathcal{H}} = 1$ . From Proposition 5.5.6 and Definition 5.5.5, we have that

$$\langle \tilde{A}(\varphi_z(\lambda))e, h \rangle_{\mathcal{H}} = \langle \tilde{A}_z(\lambda)e, h \rangle_{\mathcal{H}} = \langle A_z(k_\lambda^\alpha e), k_\lambda^\alpha h \rangle_{\alpha, \mathcal{H}}.$$

Since  $k_\lambda^\alpha(w) = (1-|\lambda|^2)^{(n+1+\alpha)/2} \sum_{\beta \in \mathbb{N}^n} \frac{w^\beta \bar{\lambda}^\beta}{\gamma_\beta}$ , where

$$\gamma_\beta = \frac{\beta! \Gamma(n+1+\alpha)}{\Gamma(n+|\beta|+1+\alpha)},$$

we obtain

$$\begin{aligned} \langle \tilde{A}(\varphi_z(\lambda))e, h \rangle_{\mathcal{H}} &= \langle A_z(k_\lambda^\alpha e), k_\lambda^\alpha h \rangle_{\alpha, \mathcal{H}} \\ &= (1-|\lambda|^2)^{n+1+\alpha} \sum_{\beta, m \in \mathbb{N}^n} \langle A_z e w^\beta, h w^m \rangle_{\alpha, \mathcal{H}} \frac{\lambda^m \bar{\lambda}^\beta}{\gamma_\beta \gamma_m}. \end{aligned} \quad (5.34)$$

Now, we fix  $r \in (0, 1)$ . We multiply both sides of (5.34) by  $\bar{\lambda}^s / (1-|\lambda|^2)^{n+1+\alpha}$  and we then integrate over  $r\mathbb{B}_n$  to obtain

$$\begin{aligned} \int_{r\mathbb{B}_n} \frac{\langle \tilde{A}(\varphi_z(\lambda))e, h \rangle_{\mathcal{H}} \bar{\lambda}^s}{(1-|\lambda|^2)^{n+1+\alpha}} d\nu_\alpha(\lambda) &= \sum_{\beta, m \in \mathbb{N}^n} \frac{\langle A_z e w^\beta, h w^m \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta \gamma_m} \int_{r\mathbb{B}_n} \lambda^m \bar{\lambda}^{\beta+s} d\nu_\alpha(\lambda) \\ &= \sum_{\beta \in \mathbb{N}^n} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta \gamma_{\beta+s}} \int_{r\mathbb{B}_n} |\lambda^{\beta+s}|^2 d\nu_\alpha(\lambda). \end{aligned}$$

We recall that

$$\begin{aligned} \int_{r\mathbb{B}_n} |\lambda^{\beta+s}|^2 d\nu_\alpha(\lambda) &= \int_{\mathbb{B}_n} |(ru)^{\beta+s}|^2 r^{2n} d\nu_\alpha(u) \\ &= r^{2(n+|\beta+s|)} \frac{(\beta+s)! \Gamma(n+1+\alpha)}{\Gamma(n+|\beta+s|+\alpha+1)} = r^{2(n+|\beta+s|)} \gamma_{\beta+s}. \end{aligned}$$

It follows that

$$\int_{r\mathbb{B}_n} \frac{\langle \tilde{A}(\varphi_z(\lambda))e, h \rangle_{\mathcal{H}} \bar{\lambda}^s}{(1-|\lambda|^2)^{n+1+\alpha}} d\nu_\alpha(\lambda) = r^{2|s|+2n} \sum_{|\beta| \geq 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r^{2|\beta|}.$$

## 5.5. Compactness of Toeplitz operators with matrix-valued $BMO^1$ symbol.

Since  $1 - |\varphi_z(\lambda)|^2 = \frac{(1 - |z|^2)(1 - |\lambda|^2)}{|1 - \langle \lambda, z \rangle|^2}$ , then for each  $\lambda \in \mathbb{B}_n$ , we get  $|\varphi_z(\lambda)| \rightarrow 1$  as  $|z| \rightarrow 1^-$ . Thus, by the assumption, we have that  $\tilde{A}(\varphi_z(\lambda))e \rightarrow 0$  weakly in  $\mathcal{H}$  as  $|z| \rightarrow 1^-$ . On the other hand, for each  $\lambda \in r\mathbb{B}_n$ , we have

$$\left| \frac{\langle \tilde{A}(\varphi_z(\lambda))e, h \rangle_{\mathcal{H}} \bar{\lambda}^s}{(1 - |\lambda|^2)^{n+1+\alpha}} \right| \leq \frac{\|A\|}{(1 - r^2)^{n+1+\alpha}}.$$

So, by the dominated convergence theorem, we have that

$$\lim_{|z| \rightarrow 1^-} \int_{r\mathbb{B}_n} \frac{\langle \tilde{A}(\varphi_z(\lambda))e, h \rangle_{\mathcal{H}} \bar{\lambda}^s}{(1 - |\lambda|^2)^{n+1+\alpha}} d\nu_\alpha(\lambda) = 0.$$

Therefore, we have that

$$\frac{\langle A_z e, h w^s \rangle_{\alpha, \mathcal{H}}}{\gamma_0} + \sum_{|\beta| > 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r^{2|\beta|} \rightarrow 0, \quad (5.35)$$

as  $|z| \rightarrow 1^-$ . Since  $\|w^\beta\|_{2, \alpha} = \gamma_\beta^{\frac{1}{2}}$  and  $\|w^{\beta+s}\|_{2, \alpha} \leq \|w^\beta\|_{2, \alpha}$ , we obtain that

$$\begin{aligned} \left| \sum_{|\beta| > 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r^{2|\beta|} \right| &\lesssim \|A\| \left( \sum_{|\beta|=1}^{\infty} r^{2|\beta|} \right) \\ &= \|A\| \left( \sum_{k=1}^{\infty} \sum_{|\beta|=k} r^{2k} \right) \\ &= \|A\| \left( \sum_{k=1}^{\infty} a_k(n) r^{2k} \right) \\ &= \|A\| \left( \sum_{k=1}^{\infty} C_{n+k-1}^k r^{2k} \right), \end{aligned}$$

where  $a_k(n) = C_{n+k-1}^k$  is the cardinal of the set  $\{(\beta_1, \dots, \beta_n) \in \mathbb{N}^n : \beta_1 + \dots + \beta_n = k\}$ . Using (2.22), we get

$$\sum_{k=1}^{\infty} C_{n+k-1}^k r^{2k} = \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k! \Gamma(n)} r^{2k} - 1 = \frac{1}{(1-r^2)^n} - 1.$$

It follows that

$$\left| \sum_{|\beta| > 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r^{2|\beta|} \right| \lesssim \|A\| \left( \frac{1}{(1-r^2)^n} - 1 \right). \quad (5.36)$$

Let  $\epsilon > 0$ . By (5.36) and the fact that  $\|A\| \left( \frac{1}{(1-r^2)^n} - 1 \right) \rightarrow 0$  as  $r \rightarrow 0$ , there exists  $r_0 < 1$  small enough such that

$$\left| \sum_{|\beta| > 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r_0^{2|\beta|} \right| \lesssim \|A\| \left( \frac{1}{(1-r_0^2)^n} - 1 \right) < \frac{\epsilon}{2}. \quad (5.37)$$

Since  $\gamma_0 = 1$ , by using (5.35) there exists  $\delta > 0$  such that  $1 - |z| < \delta$  implies

$$\left| \langle A_z e, h w^s \rangle_{\alpha, \mathcal{H}} + \sum_{|\beta| > 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r_0^{2|\beta|} \right| < \frac{\epsilon}{2}. \quad (5.38)$$

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Now, let  $z \in \mathbb{B}_n$  be such that  $1 - |z| < \delta$ . We have, using 5.37 and 5.38

$$\begin{aligned} |\langle A_z e, h w^s \rangle_{\alpha, \mathcal{H}}| &\leq \left| \langle A_z e, h w^s \rangle_{\alpha, \mathcal{H}} + \sum_{|\beta| > 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r_0^{2|\beta|} \right| \\ &+ \left| \sum_{|\beta| > 0} \frac{\langle A_z e w^\beta, h w^{\beta+s} \rangle_{\alpha, \mathcal{H}}}{\gamma_\beta} r_0^{2|\beta|} \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, we have

$$\lim_{|z| \rightarrow 1^-} \langle A_z e, h w^s \rangle_{\alpha, \mathcal{H}} = 0.$$

□

From now on, we fix  $\mathcal{H} = \mathbb{C}^d$ , where  $d \geq 1$  is an integer. A useful tool in the study the problem of the compactness of operators between Bergman spaces is the following Schur's test.

**Lemma 5.5.8.** Let  $(\Omega, \mu)$  be a measure space,  $1 < p < \infty$  and  $1/p + 1/q = 1$ . For a  $d \times d$ -matrix-valued kernel  $M(x, y)$ , consider the integral operator

$$Tf(x) = \int_{\Omega} M(x, y) f(y) d\mu(y).$$

If there is a  $C_1 > 0$ ,  $C_2 > 0$  and a positive function  $h$  on  $\Omega$  such that

$$\int_{\Omega} \|M(x, y)\| h(y)^q d\mu(y) \leq C_1 h(x)^q \quad (5.39)$$

for almost every  $x \in \Omega$ , and

$$\int_{\Omega} \|M(x, y)\| h(x)^p d\mu(x) \leq C_2 h(y)^p \quad (5.40)$$

for almost every  $y \in \Omega$ , the operator  $T : L^p(\Omega, \mathbb{C}^d, \mu) \rightarrow L^p(\Omega, \mathbb{C}^d, \mu)$  is bounded and

$$\|T\| \leq C_1^{1/q} C_2^{1/p}.$$

*Proof.* Given a function  $f \in L^p(\Omega, \mathbb{C}^d, \mu)$ , Hölder's inequality gives

$$\|Tf(x)\|_{\mathbb{C}^d} \leq \left[ \int_{\Omega} \|M(z, w)\| h(y)^q d\mu(y) \right]^{1/q} \left[ \int_{\Omega} \|M(z, w)\| h(y)^{-p} \|f(y)\|_{\mathbb{C}^d}^p d\mu(y) \right]^{1/p}$$

for almost all  $x \in \Omega$ . By using (5.39), we get that

$$\|Tf(x)\|_{\mathbb{C}^d}^p \leq C_1^{p/q} h(x)^p \int_{\Omega} \|M(z, w)\| h(y)^{-p} \|f(y)\|_{\mathbb{C}^d}^p d\mu(y)$$

for almost all  $x \in \Omega$ .

By the Fubini theorem and (5.40), we obtain

$$\begin{aligned} \int_{\Omega} \|Tf(x)\|_{\mathbb{C}^d}^p d\mu(x) &\leq C_1^{p/q} \int_{\Omega} h(x)^p \int_{\Omega} \|M(z, w)\| h(y)^{-p} \|f(y)\|_{\mathbb{C}^d}^p d\mu(y) d\mu(x) \\ &= C_1^{p/q} \int_{\Omega} \|f(y)\|_{\mathbb{C}^d}^p h(y)^{-p} \left[ \int_{\Omega} \|M(z, w)\| h(x)^p d\mu(x) \right] d\mu(y) \\ &\leq C_1^{p/q} C_2 \int_{\Omega} \|f(y)\|_{\mathbb{C}^d}^p h(y)^{-p} h(y)^p d\mu(y) = C_1^{p/q} C_2 \int_{\Omega} \|f(y)\|_{\mathbb{C}^d}^p d\mu(y). \end{aligned}$$

Thus,  $T$  is bounded on  $L^p(\Omega, \mathbb{C}^d, \mu)$  and

$$\|T\| \leq C_1^{1/q} C_2^{1/p}.$$

□

There is also a vector-valued test to determine an operator's membership in the Hilbert-Schmidt class.

**Definition 5.5.9.** The matrix-valued function  $M(z, w)$  is in  $L^2(\mathbb{B}_n \times \mathbb{B}_n, \mathcal{L}(\mathbb{C}^d), d\nu_\alpha \otimes d\nu_\alpha)$  if

$$\|M\|_{L^2(\mathbb{B}_n \times \mathbb{B}_n, \mathcal{L}(\mathbb{C}^d), d\nu_\alpha \otimes d\nu_\alpha)}^2 := \iint_{\mathbb{B}_n} \|M(z, w)\|^2 d\nu_\alpha(w) d\nu_\alpha(z) < \infty.$$

**Definition 5.5.10.** Let  $\mathcal{H}$  be a separable Hilbert space, and  $\{\psi_k\}$  be an Hilbertian basis of  $\mathcal{H}$ . We say that the operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  is a *Hilbert-Schmidt operator* if

$$\sum_{k=1}^{\infty} \|T\psi_k\|^2 < \infty.$$

The Hilbert-Schmidt norm of  $T$  is given by

$$\|T\|_{HS} := \left( \sum_{k=1}^{\infty} \|T\psi_k\|^2 \right)^{1/2}.$$

The definition of the Hilbert-Schmidt operator is independent on the choice of the Hilbertian basis (see [78, 22]).

**Proposition 5.5.11.** ([22, p.169]) Each Hilbert-Schmidt operator  $T$  is a compact operator.

**Lemma 5.5.12** (Vector-valued Hilbert Schmidt test). A linear operator  $T \in \mathcal{L}(L_\alpha^2(\mathbb{B}_n, \mathbb{C}^d))$  is a Hilbert-Schmidt operator if there is a matrix-valued function  $M$  in  $L^2(\mathbb{B}_n \times \mathbb{B}_n, \mathcal{L}(\mathbb{C}^d), d\nu_\alpha \otimes d\nu_\alpha)$  such that

$$T(f)(z) := \int_{\mathbb{B}_n} M(z, w) f(w) d\nu_\alpha(w).$$

*Proof.* Let  $M^*(z, w)$  be the adjoint of the matrix  $M(z, w)$ . Let  $\{\psi_n\}$  be an orthonormal basis for  $L_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ , let  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and let  $M_z(w) := M(z, w)$ , then there holds

$$\begin{aligned} \|M_z^* e\|_{2, \alpha, \mathbb{C}^d}^2 &= \sum_{k=1}^{\infty} \left| \langle M_z^* e, \psi_k \rangle_{\alpha, \mathbb{C}^d} \right|^2 \\ &= \sum_{k=1}^{\infty} \left| \int_{\mathbb{B}_n} \langle M_z^*(w) e, \psi_k(w) \rangle_{\mathbb{C}^d} d\nu_\alpha(w) \right|^2 \\ &= \sum_{k=1}^{\infty} \left| \int_{\mathbb{B}_n} \langle e, M_z(w) \psi_k(w) \rangle_{\mathbb{C}^d} d\nu_\alpha(w) \right|^2 \\ &= \sum_{k=1}^{\infty} \left| \left\langle e, \int_{\mathbb{B}_n} M_z(w) \psi_k(w) d\nu_\alpha(w) \right\rangle_{\mathbb{C}^d} \right|^2 \\ &= \sum_{k=1}^{\infty} |\langle e, T\psi_k(z) \rangle_{\mathbb{C}^d}|^2. \end{aligned} \tag{5.41}$$

Using Fubini theorem and (5.41), we get that

$$\begin{aligned}
 \sum_{k=1}^{\infty} \|T\psi_k\|_{2,\alpha,\mathbb{C}^d}^2 &= \sum_{k=1}^{\infty} \int_{\mathbb{B}_n} \|T\psi_k(z)\|_{\mathbb{C}^d}^2 d\nu_{\alpha}(z) \\
 &= \sum_{k=1}^{\infty} \int_{\mathbb{B}_n} \sum_{i=1}^d |\langle e_i, T\psi_k(z) \rangle_{\mathbb{C}^d}|^2 d\nu_{\alpha}(z) \\
 &= \sum_{i=1}^d \int_{\mathbb{B}_n} \sum_{k=1}^{\infty} |\langle e_i, T\psi_k(z) \rangle_{\mathbb{C}^d}|^2 d\nu_{\alpha}(z) \\
 &= \sum_{i=1}^d \int_{\mathbb{B}_n} \|M_z^* e_i\|_{2,\alpha,\mathbb{C}^d}^2 d\nu_{\alpha}(z) \\
 &= \sum_{i=1}^d \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \|M^*(z, w) e_i\|_{\mathbb{C}^d}^2 d\nu_{\alpha}(w) d\nu_{\alpha}(z) \\
 &\leq d \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \|M(z, w)\|^2 d\nu_{\alpha}(w) d\nu_{\alpha}(z) < \infty.
 \end{aligned}$$

Therefore,  $T$  is a Hilbert-Schmidt operator. □

We recall that (see Theorem 2.1.25) the integral

$$\mathcal{J}_{c,t}(z) := \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+t+c}} \tag{5.42}$$

is bounded on  $\mathbb{B}_n$  if  $c < 0$  and  $t > -1$ . Before we state the next result, we need to introduce a new operator.

**Definition 5.5.13.** Let  $f \in A_{\alpha}^2(\mathbb{B}_n, \mathbb{C}^d)$ . For  $w \in \mathbb{B}_n$ , we denote by  $\mathcal{R}$  the operator defined on  $\mathbb{C}^d$  by

$$\mathcal{R}(w) = -w.$$

We define the linear operator  $U_{\mathcal{R}} : A_{\alpha}^2(\mathbb{B}_n, \mathbb{C}^d) \rightarrow A_{\alpha}^2(\mathbb{B}_n, \mathbb{C}^d)$  by

$$U_{\mathcal{R}} f = f \circ \mathcal{R},$$

for all  $f \in A_{\alpha}^2(\mathbb{B}_n, \mathbb{C}^d)$ . If  $T : A_{\alpha}^2(\mathbb{B}_n, \mathbb{C}^d) \rightarrow A_{\alpha}^2(\mathbb{B}_n, \mathbb{C}^d)$  is a bounded linear operator, we define the following operator

$$T_{\mathcal{R}} := U_{\mathcal{R}} T U_{\mathcal{R}}.$$

We observe that the operator  $U_{\mathcal{R}}$  is self-adjoint, idempotent and isometric. For  $z \in \mathbb{B}_n$ , we recall that

$$T_{-z} = U_{-z} T U_{-z},$$

where  $U_z$  is defined in (5.29).

We now state the following lemma that is useful to obtain our Schur's test.

**Lemma 5.5.14.** Let  $p \in \mathbb{R}$ ,  $0 < p - 1 < \frac{\alpha+1}{n+1+\alpha}$ ,  $q \in \mathbb{R}$  such that  $1/p + 1/q = 1$  and  $T \in \mathcal{L}(A_{\alpha}^2(\mathbb{B}_n, \mathbb{C}^d))$ . For  $i \in \{1, \dots, d\}$ , we have

$$(a) \int_{\mathbb{B}_n} \|T_{\mathcal{R}}(K_z^{\alpha} e_i)(w)\|_{\mathbb{C}^d} \|K_w^{\alpha}\|_{2,\alpha}^{\epsilon} d\nu_{\alpha}(w) \leq \|K_z^{\alpha}\|_{2,\alpha}^{\epsilon} \|T_{-z} e_i\|_{q,\alpha,\mathbb{C}^d} \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p},$$

$$(b) \int_{\mathbb{B}_n} \|T_{\mathcal{R}}(K_z^\alpha e_i)(w)\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \leq \|K_w^\alpha\|_{2,\alpha}^\epsilon \left( \sum_{j=1}^d \|T_{-w}^* e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p},$$

where  $\frac{2(p-1)}{p} < \epsilon < \frac{2(\alpha+1)}{(n+1+\alpha)p}$ ,  $a = (n+1+\alpha)(p-1-\frac{\epsilon p}{2})$ ,  $b = \alpha - \frac{(n+1+\alpha)\epsilon p}{2}$ , and  $\mathcal{J}_{a,b}$  is given by (5.42). Moreover, the quantity,

$$\sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p} < \infty.$$

*Proof.* We prove the first inequality and we show that the second one is a consequence of the first. We begin by observing that for every  $i \in \{1, \dots, d\}$ , and  $z \in \mathbb{B}_n$ , we have, using (5.29), the fact that  $U_{-z}$  is idempotent and (5.30)

$$\begin{aligned} T_{\mathcal{R}}(K_z^\alpha e_i) &= U_{\mathcal{R}} T U_{\mathcal{R}}(K_z^\alpha e_i) \\ &= \|K_z^\alpha\|_{2,\alpha} U_{\mathcal{R}} T U_{\mathcal{R}}(k_z^\alpha e_i) \\ &= \|K_z^\alpha\|_{2,\alpha} U_{\mathcal{R}} T(U_{-z} e_i) \\ &= \|K_z^\alpha\|_{2,\alpha} U_{\mathcal{R}} U_{-z} U_{-z} T U_{-z} e_i \\ &= \|K_z^\alpha\|_{2,\alpha} U_{\mathcal{R}} U_{-z} T_{-z} e_i \\ &= \|K_z^\alpha\|_{2,\alpha} U_{\mathcal{R}} ((T_{-z} e_i) \circ \varphi_{-z} \cdot k_{-z}^\alpha) \\ &= \|K_z^\alpha\|_{2,\alpha} ((T_{-z} e_i) \circ \varphi_{-z} \circ \mathcal{R}) \cdot k_z^\alpha \\ &= ((T_{-z} e_i) \circ \varphi_{-z} \circ \mathcal{R}) \cdot K_z^\alpha \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\mathbb{B}_n} \|T_{\mathcal{R}}(K_z^\alpha e_i)(w)\|_{\mathbb{C}^d} \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|((T_{-z} e_i) \circ \varphi_{-z}(-w))\|_{\mathbb{C}^d} |k_z^\alpha(w)| \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) \\ &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|((T_{-z} e_i) \circ \varphi_{-z}(w))\|_{\mathbb{C}^d} |k_z^\alpha(-w)| \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) \\ &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|((T_{-z} e_i) \circ \varphi_{-z}(w))\|_{\mathbb{C}^d} |k_{-z}^\alpha(w)| \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) \\ &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|(T_{-z} e_i)(u)\|_{\mathbb{C}^d} |k_{-z}^\alpha(\varphi_{-z}(u))| \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha}^\epsilon |k_{-z}^\alpha(u)|^2 d\nu_\alpha(u), \end{aligned} \tag{5.43}$$

where we used the fact that  $\|K_w^\alpha\|_{2,\alpha} = \|K_{-w}^\alpha\|_{2,\alpha}$  and the change of variables  $u = \varphi_{-z}(w)$ . If we write

$$\|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha}^\epsilon = \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha}^{\epsilon-1} \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha}, \tag{5.44}$$

we observe that

$$\begin{aligned} \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha}^2 &= K^\alpha(\varphi_{-z}(u), \varphi_{-z}(u)) \\ &= K^\alpha(u, u) |k_{-z}^\alpha(u)|^{-2} \\ &= \|K_u^\alpha\|_{2,\alpha}^2 |k_{-z}^\alpha(u)|^{-2}. \end{aligned}$$

Thus,

$$\|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha} = \|K_u^\alpha\|_{2,\alpha} |k_{-z}^\alpha(u)|^{-1}. \tag{5.45}$$

It then follows from (5.45) and the identity (5.8) that

$$\begin{aligned} \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha} |k_{-z}^\alpha(\varphi_{-z}(u))| |k_{-z}^\alpha(u)|^2 &= \|K_u^\alpha\|_{2,\alpha} |k_{-z}^\alpha(u)|^{-1} |k_{-z}^\alpha(\varphi_{-z}(u))| |k_{-z}^\alpha(u)|^2 \\ &= \|K_u^\alpha\|_{2,\alpha} |k_{-z}^\alpha(\varphi_{-z}(u)) k_{-z}^\alpha(u)| \\ &= \|K_u^\alpha\|_{2,\alpha}. \end{aligned}$$

This implies that, using (5.43) and (5.45)

$$\begin{aligned} &\int_{\mathbb{B}_n} \|T_{\mathcal{R}}(K_z^\alpha e_i)(w)\|_{\mathbb{C}^d} \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) \\ &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|(T_{-z} e_i)(u)\|_{\mathbb{C}^d} \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha} |k_{-z}^\alpha(\varphi_{-z}(u))| |k_{-z}^\alpha(u)|^2 \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha}^{\epsilon-1} d\nu_\alpha(u) \\ &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|(T_{-z} e_i)(u)\|_{\mathbb{C}^d} \|K_u^\alpha\|_{2,\alpha} \|K_{\varphi_{-z}(u)}^\alpha\|_{2,\alpha}^{\epsilon-1} d\nu_\alpha(u) \\ &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|(T_{-z} e_i)(u)\|_{\mathbb{C}^d} \|K_u^\alpha\|_{2,\alpha} \left(\|K_u^\alpha\|_{2,\alpha} |k_{-z}^\alpha(u)|^{-1}\right)^{\epsilon-1} d\nu_\alpha(u) \\ &= \|K_z^\alpha\|_{2,\alpha} \int_{\mathbb{B}_n} \|(T_{-z} e_i)(u)\|_{\mathbb{C}^d} \|K_u^\alpha\|_{2,\alpha}^\epsilon |k_{-z}^\alpha(u)|^{1-\epsilon} d\nu_\alpha(u) \\ &= \|K_z^\alpha\|_{2,\alpha}^\epsilon \int_{\mathbb{B}_n} \|(T_{-z} e_i)(u)\|_{\mathbb{C}^d} \|K_u^\alpha\|_{2,\alpha}^\epsilon |K_{-z}^\alpha(u)|^{1-\epsilon} d\nu_\alpha(u). \end{aligned}$$

Let

$$\mathcal{I}(z) = \|K_z^\alpha\|_{2,\alpha}^\epsilon \int_{\mathbb{B}_n} \|(T_{-z} e_i)(u)\|_{\mathbb{C}^d} \|K_u^\alpha\|_{2,\alpha}^\epsilon |K_{-z}^\alpha(u)|^{1-\epsilon} d\nu_\alpha(u).$$

Applying Hölder's inequality with  $p^{-1} + q^{-1} = 1$ , we have

$$\mathcal{I}(z) \leq \|K_z^\alpha\|_{2,\alpha}^\epsilon \|T_{-z} e_i\|_{q,\alpha,\mathbb{C}^d} \left( \int_{\mathbb{B}_n} \|K_u^\alpha\|_{2,\alpha}^{p\epsilon} |K_{-z}^\alpha(u)|^{p(1-\epsilon)} d\nu_\alpha(u) \right)^{1/p}.$$

In order to complete the proof, we need to study the integral in the right hand side of the previous inequality.

Since we know explicitly the kernel, we have

$$\begin{aligned} \int_{\mathbb{B}_n} \|K_u^\alpha\|_{2,\alpha}^{p\epsilon} |K_{-z}^\alpha(u)|^{p(1-\epsilon)} d\nu_\alpha(u) &= \int_{\mathbb{B}_n} \frac{(1 - |u|^2)^{\frac{-(n+1+\alpha)p\epsilon}{2}} d\nu_\alpha(u)}{|1 - \langle -z, u \rangle|^{(n+1+\alpha)p(1-\epsilon)}} \\ &= c_\alpha \int_{\mathbb{B}_n} \frac{(1 - |u|^2)^{\alpha - \frac{(n+1+\alpha)p\epsilon}{2}} d\nu(u)}{|1 - \langle -z, u \rangle|^{n+1 + [\alpha - \frac{(n+1+\alpha)p\epsilon}{2}] + (n+1+\alpha)(p-1 - \frac{\epsilon p}{2})}} \\ &= c_\alpha \int_{\mathbb{B}_n} \frac{(1 - |u|^2)^{\alpha - \frac{(n+1+\alpha)p\epsilon}{2}} d\nu(u)}{|1 - \langle -z, u \rangle|^{n+1+a+b}} = c_\alpha \mathcal{J}_{a,b}(-z), \end{aligned}$$

where  $a = (n + 1 + \alpha)(p - 1 - \frac{\epsilon p}{2})$ ,  $b = \alpha - \frac{(n+1+\alpha)p\epsilon}{2}$ , and  $\mathcal{J}_{a,b}$  is given by (5.42). Because we need  $a > -1$  and  $b < 0$ , we obtain the following double inequality  $\frac{2(p-1)}{p} < \epsilon < \frac{2(\alpha+1)}{(n+1+\alpha)p}$ . It is clear that we can find such an  $\epsilon$  if and only if  $p - 1 < \frac{\alpha+1}{n+1+\alpha}$ . That is,  $p < 1 + \frac{\alpha+1}{n+1+\alpha}$ . So we have proved the first inequality. Therefore,

$$\int_{\mathbb{B}_n} \|T_{\mathcal{R}}(K_z^\alpha e)(w)\|_{\mathbb{C}^d} \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) \leq \|K_z^\alpha\|_{2,\alpha}^\epsilon \|T_{-z} e_i\|_{q,\alpha,\mathbb{C}^d} \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p}. \quad (5.46)$$

To prove the second inequality, we start by remarking that

$$\begin{aligned} T_{\mathcal{R}}(K_z^\alpha e_i)(w) &= \sum_{j=1}^d \langle T_{\mathcal{R}}(K_z^\alpha e_i)(w), e_j \rangle_{\mathbb{C}^d} e_j \\ &= \sum_{j=1}^d \langle T_{\mathcal{R}}(K_z^\alpha e_i), K_w^\alpha e_j \rangle_{\alpha, \mathbb{C}^d} e_j \\ &= \sum_{j=1}^d \langle K_z^\alpha e_i, T_{\mathcal{R}}^*(K_w^\alpha e_j) \rangle_{\alpha, \mathbb{C}^d} e_j \\ &= \sum_{j=1}^d \langle e_i, T_{\mathcal{R}}^*(K_w^\alpha e_j)(z) \rangle_{\mathbb{C}^d} e_j. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\mathbb{B}_n} \|T_{\mathcal{R}}(K_z^\alpha e_i)(w)\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) &= \int_{\mathbb{B}_n} \left\| \sum_{j=1}^d \langle e_i, T_{\mathcal{R}}^*(K_w^\alpha e_j)(z) \rangle_{\mathbb{C}^d} e_j \right\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \\ &\leq \sum_{j=1}^d \int_{\mathbb{B}_n} \|T_{\mathcal{R}}^*(K_w^\alpha e_j)(z)\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z). \end{aligned}$$

Using (5.46) replacing  $T$  by  $T^*$  and  $w$  by  $z$ , we obtain for each  $j \in \{1, \dots, d\}$  that

$$\int_{\mathbb{B}_n} \|T_{\mathcal{R}}^*(K_w^\alpha e_j)(z)\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \leq \|K_w^\alpha\|_{2,\alpha}^\epsilon \|T_{-w}^* e_j\|_{q,\alpha,\mathbb{C}^d} |\mathcal{J}_{a,b}(-w)|^{1/p}.$$

Thus,

$$\int_{\mathbb{B}_n} \|T_{\mathcal{R}}(K_z^\alpha e_i)(w)\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \leq \|K_w^\alpha\|_{2,\alpha}^\epsilon \left( \sum_{j=1}^d \|T_{-w}^* e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{w \in \mathbb{B}_n} |\mathcal{J}_{a,b}(w)|^{1/p}.$$

□

We are now ready to prove the main result on compactness of  $T_b$  on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ .

**Theorem 5.5.15.** Let  $d \geq 1$  be an integer and  $\tilde{b} \in L^\infty(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ . For  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , the following are equivalent.

- (1)  $T_b$  is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ .
- (2)  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .
- (3)  $T_{b \circ \varphi_z} e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ .
- (4)  $\|T_{b \circ \varphi_z} e\|_{p,\alpha,\mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and every  $p > 1$ .
- (5)  $\|T_{b \circ \varphi_z} e\|_{p,\alpha,\mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and some  $0 < p - 1 < \frac{\alpha+1}{n+1+\alpha}$ .

*Proof.* (1)  $\Rightarrow$  (2). We assume that  $T_b$  is a compact operator on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ . Let  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ . We want to show that  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$ . Let  $h \in \mathbb{C}^d$ . We have that

$$|\langle \tilde{b}(z)e, h \rangle_{\mathbb{C}^d}| = |\langle T_b(k_z^\alpha e), k_z^\alpha h \rangle_{\alpha, \mathbb{C}^d}| \leq \|h\|_{\mathbb{C}^d} \|T_b(k_z^\alpha e)\|_{2,\alpha,\mathbb{C}^d}. \quad (5.47)$$

## 5.5. Compactness of Toeplitz operators with matrix-valued $BMO^1$ symbol.

Since  $T_b$  is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ , by (5.47), it is enough to show that  $k_z^\alpha e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ . By Lemma 4.5.3, it is enough to prove that  $\langle k_z^\alpha e, K_w^\alpha h' \rangle_{\alpha, \mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$  for each  $h' \in \mathbb{C}^d$  and  $w \in \mathbb{B}_n$ . By the reproducing kernel formula, we have  $\langle k_z^\alpha e, K_w^\alpha h' \rangle_{\alpha, \mathbb{C}^d} = \langle k_z^\alpha(w)e, h' \rangle_{\mathbb{C}^d}$ . It follows that

$$|\langle k_z^\alpha e, K_w^\alpha h' \rangle_{\alpha, \mathbb{C}^d}| = |\langle k_z^\alpha(w)e, h' \rangle_{\mathbb{C}^d}| \leq \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{2}}}{(1 - |w|)^{n+1+\alpha}} \|h'\|_{\mathbb{C}^d} \rightarrow 0$$

as  $|z| \rightarrow 1^-$ .

(2)  $\Rightarrow$  (3). Assume that  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$ , for  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ . Since  $\tilde{b}$  is bounded and  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , it follows that  $T_b$  is bounded on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ . Let  $h \in \mathbb{C}^d$ , and  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ . Since  $T_b$  is bounded on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ , by using Proposition 5.5.4,  $(T_b)_z = T_{b \circ \varphi_z}$ . By Lemma 5.4.2 and the fact that  $\langle \tilde{T}_b(z)e, h \rangle_{\mathbb{C}^d} = \langle \tilde{b}(z)e, h \rangle_{\mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , it follows by Theorem 5.5.7 that  $(T_b)_z e = T_{b \circ \varphi_z} e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ .

(3)  $\Rightarrow$  (4). Suppose that  $T_{b \circ \varphi_z} e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ , for  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ . Let  $0 < r < 1$ . We first show (4) for  $p = 2$ .

$$\begin{aligned} \|T_{b \circ \varphi_z} e\|_{2, \alpha, \mathbb{C}^d}^2 &= \int_{\mathbb{B}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w) + \int_{r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w) = I_1(z) + I_2(z), \end{aligned}$$

where

$$I_1(z) = \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w),$$

and

$$I_2(z) = \int_{r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w).$$

On the first hand, using the Cauchy-Schwarz inequality and Proposition 5.4.9, we have that

$$\begin{aligned} I_1(z) &\leq \left( \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^4 d\nu_\alpha(w) \right)^{1/2} \left( \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} 1 d\nu_\alpha(w) \right)^{1/2} \\ &\leq \|T_{b \circ \varphi_z} e\|_{4, \alpha, \mathbb{C}^d}^2 \left( \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} 1 d\nu_\alpha(w) \right)^{1/2} \\ &\leq C(n, \alpha, r)(1 - r)^{\frac{\alpha+1}{2}}, \end{aligned}$$

where

$$C(n, \alpha, r) = \left( \frac{2nc_\alpha \max\{(1+r)^\alpha, 2^\alpha\}}{\alpha+1} \right)^{1/2} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{4, \alpha, \mathbb{C}^d}^2.$$

On the second hand, we have that

$$I_2(z) = \int_{r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w) \leq \sup_{w \in r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2.$$

It follows that

$$\|T_{b \circ \varphi_z} e\|_{2, \alpha, \mathbb{C}^d}^2 \leq C(n, \alpha, r)(1 - r)^{\alpha+1} + \sup_{w \in r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2. \quad (5.48)$$

Let  $w \in r\overline{\mathbb{B}}_n$ . Since  $T_{b \circ \varphi_z} e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ , for  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ , we have,

using the reproducing kernel formula

$$\begin{aligned} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 &= \sum_{i=1}^d |\langle T_{b \circ \varphi_z} e(w), e_i \rangle_{\mathbb{C}^d}|^2 \\ &= \sum_{i=1}^d |\langle T_{b \circ \varphi_z} e, K_w^\alpha e_i \rangle_{\alpha, \mathbb{C}^d}|^2 \rightarrow 0, \end{aligned}$$

as  $|z| \rightarrow 1^-$ . Therefore, we get that

$$\lim_{|z| \rightarrow 1^-} \sup_{w \in r\bar{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 = 0. \tag{5.49}$$

Let  $\epsilon > 0$ . Choose  $r_0 < 1$  such that

$$C(n, \alpha, r_0)(1 - r_0)^{\alpha+1} < \frac{\epsilon}{2}.$$

By (5.49), there exists  $\delta > 0$  such that for all  $1 - |z| < \delta$  we have

$$\sup_{w \in r_0\bar{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 < \frac{\epsilon}{2}.$$

For  $1 - |z| < \delta$ , (5.48) becomes

$$\|T_{b \circ \varphi_z} e\|_{2, \alpha, \mathbb{C}^d}^2 \leq C(n, \alpha, r_0)(1 - r_0)^{\alpha+1} + \sup_{w \in r_0\bar{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

That is,

$$\lim_{|z| \rightarrow 1^-} \|T_{b \circ \varphi_z} e\|_{2, \alpha, \mathbb{C}^d}^2 = 0.$$

This proves the claim for the case  $p = 2$ . For  $1 \leq p < 2$ , we have that

$$\|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d} \leq \|T_{b \circ \varphi_z} e\|_{2, \alpha, \mathbb{C}^d},$$

and the desired result follows from the case  $p = 2$ . Finally, for  $2 < p < \infty$ , we have using the Cauchy-Schwarz inequality

$$\begin{aligned} \|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d}^p &= \int_{\mathbb{B}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^p d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^{p-1} d\nu_\alpha(w) \\ &\leq \left( \int_{\mathbb{B}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w) \right)^{1/2} \left( \int_{\mathbb{B}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^{2(p-1)} d\nu_\alpha(w) \right)^{1/2} \\ &= \|T_{b \circ \varphi_z} e\|_{2, \alpha, \mathbb{C}^d} \|T_{b \circ \varphi_z} e\|_{2p-2, \alpha, \mathbb{C}^d}^{p-1}. \end{aligned}$$

Since  $2p - 2 > 2 \geq 1$ , the desired result follows from Proposition 5.4.9 and the the case  $p = 2$ .

The implication (4)  $\Rightarrow$  (5) is obvious.

(5)  $\Rightarrow$  (1). Assume that  $\|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for every  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$  and  $p \geq 1$ . Let us prove that  $T_b$  is a compact operator on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ . Considering the operator  $U_{\mathcal{R}}$  defined in Definition 5.5.13, since  $U_{\mathcal{R}}$  is bounded and invertible, we have that  $T_b$  is compact if and only if  $U_{\mathcal{R}} T_b U_{\mathcal{R}} := (T_b)_{\mathcal{R}}$  is compact. So, we will show that  $(T_b)_{\mathcal{R}}$  is compact. To do this, we are going to prove that  $(T_b)_{\mathcal{R}}$  is an integral

operator with matrix-valued kernel  $M_b(z, w)$  given by the following relation

$$\langle M_b(z, w)e_i, e_j \rangle_{\mathbb{C}^d} := \langle (T_b)_{\mathcal{R}}(K_z^\alpha e_j)(w), e_i \rangle_{\mathbb{C}^d}. \quad (5.50)$$

Indeed, for  $f \in A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ , we have, using the reproducing kernel formula

$$\begin{aligned} \langle (T_b)_{\mathcal{R}}(f)(w), e_i \rangle_{\mathbb{C}^d} &= \langle (T_b)_{\mathcal{R}}f, K_w^\alpha e_i \rangle_{\alpha, \mathbb{C}^d} \\ &= \langle f, [(T_b)_{\mathcal{R}}]^*(K_w^\alpha e_i) \rangle_{\alpha, \mathbb{C}^d} \\ &= \int_{\mathbb{B}_n} \langle f(z), [(T_b)_{\mathcal{R}}]^*(K_w^\alpha e_i)(z) \rangle_{\mathbb{C}^d} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle e_j, [(T_b)_{\mathcal{R}}]^*(K_w^\alpha e_i)(z) \rangle_{\mathbb{C}^d} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle K_z^\alpha e_j, [(T_b)_{\mathcal{R}}]^*(K_w^\alpha e_i) \rangle_{\alpha, \mathbb{C}^d} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle (T_b)_{\mathcal{R}}(K_z^\alpha e_j), K_w^\alpha e_i \rangle_{\alpha, \mathbb{C}^d} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle (T_b)_{\mathcal{R}}(K_z^\alpha e_j)(w), e_i \rangle_{\mathbb{C}^d} d\nu_\alpha(z). \end{aligned}$$

Hence,

$$\begin{aligned} (T_b)_{\mathcal{R}}f(w) &= \sum_{i=1}^d \langle (T_b)_{\mathcal{R}}f(w), e_i \rangle_{\mathbb{C}^d} e_i \\ &= \sum_{i=1}^d \int_{\mathbb{B}_n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle (T_b)_{\mathcal{R}}(K_z^\alpha e_j)(w), e_i \rangle_{\mathbb{C}^d} d\nu_\alpha(z) e_i \\ &= \sum_{i=1}^d \int_{\mathbb{B}_n} \sum_{j=1}^d \langle f(z), e_j \rangle_{\mathbb{C}^d} \langle M_b(z, w)e_i, e_j \rangle_{\mathbb{C}^d} d\nu_\alpha(z) e_i \\ &= \int_{\mathbb{B}_n} \sum_{i=1}^d \left( \sum_{j=1}^d M_b(z, w)e_i, e_j \rangle_{\mathbb{C}^d} \langle f(z), e_j \rangle_{\mathbb{C}^d} \right) e_i d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} M_b(z, w)f(z) d\nu_\alpha(z). \end{aligned}$$

We conclude that  $(T_b)_{\mathcal{R}}$  is an integral operator with matrix-valued kernel  $M_b(z, w)$ . As in the scalar-valued case, we introduce now the truncations of this operator. Set now on  $T^b := T_b$ ,  $(T_b)_{\mathcal{R}} := T_{\mathcal{R}}^b$  and  $t \in (0, 1)$ . We define the truncated operator  $\left[ T_{\mathcal{R}}^b \right]_{[t]}$  of  $T_{\mathcal{R}}^b$  by:

$$\left[ T_{\mathcal{R}}^b \right]_{[t]}(f)(w) := \int_{t\mathbb{B}_n} M_b(z, w)f(z) d\nu_\alpha(z). \quad (5.51)$$

Thus,  $\left[ T_{\mathcal{R}}^b \right]_{[t]}$  is an integral operator with kernel given by  $M_b^{[t]}(z, w) := \chi_{t\mathbb{B}_n}(z)M_b(z, w)$ . Let us prove that

$\left[ T_{\mathcal{R}}^b \right]_{[t]}$  is a Hilbert-Schmidt operator. To this, let  $\|(M_{i,j})_{i,j=1}^d\|^2 = \sum_{i=1}^d \sum_{j=1}^d |M_{i,j}|^2$  be a norm on the space of

$d \times d$  matrices. Using (5.50), we have

$$\begin{aligned} \|M_b(z, w)\|^2 &= \sum_{i=1}^d \sum_{j=1}^d |\langle M_b(z, w)e_i, e_j \rangle_{\mathbb{C}^d}|^2 \\ &= \sum_{i=1}^d \sum_{j=1}^d |\langle T_{\mathcal{R}}^b(K_z^\alpha e_j)(w), e_i \rangle_{\mathbb{C}^d}|^2 \\ &= \sum_{j=1}^d \|T_{\mathcal{R}}^b(K_z^\alpha e_j)(w)\|_{\mathbb{C}^d}^2. \end{aligned} \tag{5.52}$$

It follows that for any  $t \in (0, 1)$ , we have, using (5.52)

$$\begin{aligned} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \|M_b^{[t]}(z, w)\|^2 d\nu_\alpha(z) d\nu_\alpha(w) &= \int_{\mathbb{B}_n} \int_{t\mathbb{B}_n} \|M_b(z, w)\|^2 d\nu_\alpha(z) d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \int_{t\mathbb{B}_n} \sum_{j=1}^d \|T_{\mathcal{R}}^b(K_z^\alpha e_j)(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(z) d\nu_\alpha(w) \\ &= \sum_{j=1}^d \int_{t\mathbb{B}_n} \int_{\mathbb{B}_n} \|T_{\mathcal{R}}^b(K_z^\alpha e_j)(w)\|_{\mathbb{C}^d}^2 d\nu_\alpha(w) d\nu_\alpha(z) \\ &= \sum_{j=1}^d \int_{t\mathbb{B}_n} \|T_{\mathcal{R}}^b(K_z^\alpha e_j)\|_{2, \alpha, \mathbb{C}^d}^2 d\nu_\alpha(z) \\ &\leq \sum_{j=1}^d \int_{t\mathbb{B}_n} \|T_{\mathcal{R}}^b\|^2 \|K_z^\alpha e_j\|_{2, \alpha, \mathbb{C}^d}^2 d\nu_\alpha(z) \\ &= d \|T_{\mathcal{R}}^b\|^2 \int_{t\mathbb{B}_n} \|K_z^\alpha\|_{2, \alpha}^2 d\nu_\alpha(z) \\ &= d \|T_{\mathcal{R}}^b\|^2 \int_{t\mathbb{B}_n} \frac{1}{(1 - |z|^2)^{n+1+\alpha}} d\nu_\alpha(z) \leq \frac{d \|T_{\mathcal{R}}^b\|^2}{(1 - t^2)^{n+1+\alpha}} < \infty. \end{aligned}$$

According to Lemma 5.5.12, the operator  $[T_{\mathcal{R}}^b]_{[t]}$  is Hilbert-Schmidt, and therefore is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ .

So, to prove that  $T_{\mathcal{R}}^b$  is a compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ , it is enough to prove that

$$\lim_{t \rightarrow 1^-} \left\| T_{\mathcal{R}}^b - [T_{\mathcal{R}}^b]_{[t]} \right\| = 0.$$

First, we note that  $T_{\mathcal{R}}^b - [T_{\mathcal{R}}^b]_{[t]}$  is an integral operator with kernel given by  $H_{[1-t]}(z, w) = \chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) M_b(z, w)$ .

We are going to use Schur's test and Lemma 5.5.14 to estimate the norm of this operator. For Schur's test, choose as test function  $h(z) = (K^\alpha(z, z))^{\epsilon/2} = \|K_z^\alpha\|_{2, \alpha}^\epsilon$ . If we choose  $p$  such that  $0 < p - 1 < \frac{\alpha+1}{n+1+\alpha}$ , and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $\epsilon \in \left(\frac{2(p-1)}{p}, \frac{2(\alpha+1)}{(n+1+\alpha)p}\right)$  then we can apply

Lemma 5.5.14. Thus,

$$\begin{aligned} \int_{\mathbb{B}_n} \|H_{[1-t]}(z, w)\| \|K_w^\alpha\|_{2, \alpha}^\epsilon d\nu_\alpha(w) &= \int_{\mathbb{B}_n} \|\chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) M_b(z, w)\| \|K_w^\alpha\|_{2, \alpha}^\epsilon d\nu_\alpha(w) \\ &\simeq \chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) \int_{\mathbb{B}_n} \sum_{i=1}^d \sum_{j=1}^d |\langle M_b(z, w)e_i, e_j \rangle_{\mathbb{C}^d}| \|K_w^\alpha\|_{2, \alpha}^\epsilon d\nu_\alpha(w) \\ &= \chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) \int_{\mathbb{B}_n} \sum_{i=1}^d \sum_{j=1}^d |\langle T_{\mathcal{R}}^b(K_z^\alpha e_j)(w), e_i \rangle_{\mathbb{C}^d}| \|K_w^\alpha\|_{2, \alpha}^\epsilon d\nu_\alpha(w) \\ &\leq d \chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) \sum_{j=1}^d \int_{\mathbb{B}_n} \|T_{\mathcal{R}}^b(K_z^\alpha e_j)(w)\|_{\mathbb{C}^d} \|K_w^\alpha\|_{2, \alpha}^\epsilon d\nu_\alpha(w). \end{aligned}$$

## 5.5. Compactness of Toeplitz operators with matrix-valued $BMO^1$ symbol.

By Lemma 5.5.14, for each  $j \in \{1, \dots, d\}$ , we have

$$\int_{\mathbb{B}_n} \|T_{\mathcal{R}}^b(K_z^\alpha e_j)(w)\|_{\mathbb{C}^d} \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) \leq \|K_z^\alpha\|_{2,\alpha}^\epsilon \|T_{-z}^b e_j\|_{q,\alpha,\mathbb{C}^d} \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p},$$

where  $a$  and  $b$  are given in Lemma 5.5.14. It follows that, using (a) of Lemma 5.5.14

$$\begin{aligned} \int_{\mathbb{B}_n} \|H_{[1-t]}(z, w)\| \|K_w^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(w) &\leq d \chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) \|K_z^\alpha\|_{2,\alpha}^\epsilon \sum_{j=1}^d \|T_{-z}^b e_j\|_{q,\alpha,\mathbb{C}^d} \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(-z)|^{1/p} \\ &= d \chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) \|K_z^\alpha\|_{2,\alpha}^\epsilon \left( \sum_{j=1}^d \|T_{-z}^b e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p} \\ &\leq d \|K_z^\alpha\|_{2,\alpha}^\epsilon \left( \sum_{j=1}^d \sup_{|z| \geq t} \|T_{-z}^b e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p}, \end{aligned}$$

where  $a$  and  $b$  are given in Lemma 5.5.14. Similarly, we have that, using the norm  $\|(M_{i,j})_{i,j=1}^d\| = \sum_{i=1}^d \sum_{j=1}^d |M_{i,j}|$  and (b) of Lemma 5.5.14

$$\begin{aligned} \int_{\mathbb{B}_n} \|H_{[1-t]}(z, w)\| \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) &= \int_{\mathbb{B}_n} \chi_{\mathbb{B}_n \setminus t\mathbb{B}_n}(z) \sum_{i=1}^d \sum_{j=1}^d |\langle M_b(z, w) e_i, e_j \rangle_{\mathbb{C}^d}| \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \\ &\leq \int_{\mathbb{B}_n} \sum_{i=1}^d \sum_{j=1}^d |\langle T_{\mathcal{R}}^b(K_z^\alpha e_j)(w), e_i \rangle_{\mathbb{C}^d}| \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \\ &\leq d \sum_{j=1}^d \int_{\mathbb{B}_n} \|T_{\mathcal{R}}^b(K_z^\alpha e_j)(w)\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \\ &\leq d \sum_{j=1}^d \int_{\mathbb{B}_n} \|(T_{\mathcal{R}}^b)^*(K_w^\alpha e_j)(z)\|_{\mathbb{C}^d} \|K_z^\alpha\|_{2,\alpha}^\epsilon d\nu_\alpha(z) \\ &\leq d \|K_w^\alpha\|_{2,\alpha}^\epsilon \left( \sum_{j=1}^d \|(T^b)^*_{-w} e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{w \in \mathbb{B}_n} |\mathcal{J}_{a,b}(w)|^{1/p} \\ &\leq d \|K_w^\alpha\|_{2,\alpha}^\epsilon \left( \sum_{j=1}^d \sup_{w \in \mathbb{B}_n} \|(T^b)^*_{-w} e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{w \in \mathbb{B}_n} |\mathcal{J}_{a,b}(w)|^{1/p}. \end{aligned}$$

Since  $a$  and  $b$  are as in Lemma 5.5.14, we have, using Theorem 2.1.25

$$\sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p} < \infty.$$

Also, using the condition  $0 < p - 1 < \frac{\alpha+1}{n+1+\alpha}$  and the fact that  $\frac{1}{p} + \frac{1}{q} = 1$ , we obtain  $q > 2 + \frac{n}{1+\alpha}$ . It follows by Proposition 5.4.9 that for each  $j \in \{1, \dots, d\}$ ,

$$\sum_{j=1}^d \sup_{|z| \geq t} \|T_{-z}^b e_j\|_{q,\alpha,\mathbb{C}^d} \leq d \sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|(T_b)_{-z} e\|_{q,\alpha,\mathbb{C}^d} < \infty,$$

and

$$\sum_{j=1}^d \sup_{w \in \mathbb{B}_n} \|(T^b)^*_{-w} e_j\|_{q,\alpha,\mathbb{C}^d} \leq d \sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{w \in \mathbb{B}_n} \|(T_b^*)_{-w} e\|_{q,\alpha,\mathbb{C}^d} < \infty.$$

By using the Schur's test (Lemma 5.5.8), we get

$$\left\| T_{\mathcal{R}}^b - [T_{\mathcal{R}}^b]_{[t]} \right\| \leq \left[ d \left( \sum_{j=1}^d \sup_{|z| \geq t} \|T_{-z}^b e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{z \in \mathbb{B}_n} |\mathcal{J}_{a,b}(z)|^{1/p} \right] \times \left[ d \left( \sum_{j=1}^d \sup_{w \in \mathbb{B}_n} \|(T^b)^*_{-w} e_j\|_{q,\alpha,\mathbb{C}^d} \right) \sup_{w \in \mathbb{B}_n} |\mathcal{J}_{a,b}(w)|^{1/p} \right]. \quad (5.53)$$

Using the assumption (5), we have  $\|T_{-z}^b e_j\|_{q,\alpha,\mathbb{C}^d} = \|T_{b \circ \varphi_{-z}} e_j\|_{q,\alpha,\mathbb{C}^d} \rightarrow 0$ , as  $|z| \rightarrow 1^-$ . It follows by (5.53) that

$$\lim_{t \rightarrow 1^-} \left\| T_{\mathcal{R}}^b - [T_{\mathcal{R}}^b]_{[t]} \right\| = 0.$$

Therefore we have proved that  $U_{\mathcal{R}} T_b U_{\mathcal{R}} := T_{\mathcal{R}}^b$  is compact, hence  $T_b$  is compact.  $\square$

According to the method of the above proof, we also get the following results.

**Lemma 5.5.16.** Let  $b \in L^1_{\alpha}(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$  and  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ , such that  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$ . Let  $T_b$  be bounded on  $A^2_{\alpha}(\mathbb{B}_n, \mathbb{C}^d)$ . If there exists  $p > 2 + \frac{n}{1+\alpha}$  such that

$$\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{p,\alpha,\mathbb{C}^d} < \infty,$$

then we have  $\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{q,\alpha,\mathbb{C}^d} < \infty$  and  $\|T_{b \circ \varphi_z} e\|_{q,\alpha,\mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ , for every  $q < p$ .

*Proof.* Let  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ . Since  $q < p$ , and  $\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{p,\alpha,\mathbb{C}^d} < \infty$ , by Hölder's inequality we get

$$\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{q,\alpha,\mathbb{C}^d} < \infty.$$

Let  $s = \frac{p}{q} > 1$  and  $t$  such that  $\frac{1}{t} + \frac{1}{s} = 1$ . Let  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ . Then

$$\|T_{b \circ \varphi_z} e\|_{q,\alpha,\mathbb{C}^d}^q = \int_{r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q d\nu_{\alpha}(w) + \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q d\nu_{\alpha}(w) = I_1(z) + I_2(z),$$

where

$$I_1(z) = \int_{r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q d\nu_{\alpha}(w)$$

and

$$I_2(z) = \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q d\nu_{\alpha}(w).$$

Since  $\tilde{b}(z)e \rightarrow 0$  weakly as  $|z| \rightarrow 1^-$  for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ , by the implication of (2)  $\Rightarrow$  (3) in Theorem 5.5.15, we get that  $T_{b \circ \varphi_z} e \rightarrow 0$  weakly as  $|z| \rightarrow 1^-$ . We have that

$$I_1(z) \leq \sup_{w \in r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q.$$

Using Hölder's inequality, we get

$$\begin{aligned} I_2(z) &\leq \left( \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^{qs} d\nu_{\alpha}(w) \right)^{\frac{1}{s}} \left( \int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}}_n} d\nu_{\alpha}(w) \right)^{\frac{1}{t}} \\ &\leq C(n, \alpha, r)(1-r)^{\frac{1+\alpha}{t}}, \end{aligned}$$

## 5.5. Compactness of Toeplitz operators with matrix-valued $BMO^1$ symbol.

where

$$C(n, \alpha, r) = \left( \frac{2nc_\alpha \max\{(1+r)^\alpha, 2^\alpha\}}{\alpha+1} \right)^{1/t} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d}^{\frac{p}{s}}$$

(we recall that by Proposition 5.4.9,  $\sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d}^{\frac{p}{s}} < \infty$ ). It follows that

$$\|T_{b \circ \varphi_z} e\|_{q, \alpha, \mathbb{C}^d}^q \leq \sup_{w \in r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q + C(n, \alpha, r)(1-r)^{\frac{1+\alpha}{t}}. \quad (5.54)$$

Let  $w \in r\overline{\mathbb{B}}_n$ . Since  $T_{b \circ \varphi_z} e \rightarrow 0$  weakly in  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  as  $|z| \rightarrow 1^-$ , for  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ , we have, using the reproducing kernel formula

$$\begin{aligned} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q &= \left( \sum_{i=1}^d |\langle T_{b \circ \varphi_z} e(w), e_i \rangle_{\mathbb{C}^d}|^2 \right)^{q/2} \\ &= \left( \sum_{i=1}^d |\langle T_{b \circ \varphi_z} e, K_w^\alpha e_i \rangle_{\alpha, \mathbb{C}^d}|^2 \right)^{q/2} \rightarrow 0, \end{aligned}$$

as  $|z| \rightarrow 1^-$ . Thus,

$$\lim_{|z| \rightarrow 1^-} \sup_{w \in r\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q = 0. \quad (5.55)$$

Let  $\epsilon > 0$  and  $0 < r_0 < 1$  such that

$$C(n, \alpha, r_0)(1-r_0)^{\frac{1+\alpha}{t}} < \frac{\epsilon}{2}.$$

Using (5.54) and (5.55) we obtain  $\delta > 0$  such that for all  $|z| < \delta$ , we have that

$$\|T_{b \circ \varphi_z} e\|_{q, \alpha, \mathbb{C}^d}^q \leq \sup_{w \in r_0\overline{\mathbb{B}}_n} \|T_{b \circ \varphi_z} e(w)\|_{\mathbb{C}^d}^q + C(n, \alpha, r_0)(1-r_0)^{\frac{1+\alpha}{t}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

**Theorem 5.5.17.** Let  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$  such that  $T_b$  is bounded on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  and suppose that there exists  $p > 2 + \frac{n}{\alpha+1}$  such that

$$\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d} < \infty$$

and

$$\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_{b^* \circ \varphi_z} e\|_{p, \alpha, \mathbb{C}^d} < \infty,$$

where for every  $z \in \mathbb{B}_n$ ,  $b^*(z)$  is the adjoint of the operator  $b(z)$ . Then  $\tilde{b}(z)e \rightarrow 0$  weakly in  $\mathbb{C}^d$  as  $|z| \rightarrow 1^-$  for each  $e \in \mathbb{C}^d$  with  $\|e\|_{\mathbb{C}^d} = 1$ , implies that  $T_b$  is a compact operator on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ .

*Proof.* By Lemma 5.5.16, there exists  $q > 2 + \frac{n}{\alpha+1}$  such that

$$\sup_{\|e\|_{\mathbb{C}^d}=1} \sup_{z \in \mathbb{B}_n} \|T_{b \circ \varphi_z} e\|_{q, \alpha, \mathbb{C}^d} < \infty$$

and  $\|T_{b \circ \varphi_z} e\|_{q, \alpha, \mathbb{C}^d} \rightarrow 0$  as  $|z| \rightarrow 1^-$ . The same holds for  $\|T_{b^* \circ \varphi_z} e\|_{q, \alpha, \mathbb{C}^d}$ . The proof of (4)  $\Rightarrow$  (1) in Theorem 5.5.15 shows that  $T_b$  is compact on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$ . □

**Discussion of the Theorem 5.5.15.** We first observe that in Theorem 5.5.15 the target space is a finite dimension space. So, the natural question we can ask is: "It is possible to replace the set  $\mathbb{C}^d$  by any

infinite dimensional Hilbert space  $\mathcal{H}$  ?” We notice that the very important idea in the proof of Theorem 5.5.15 is the fact that the truncated operator defined by (5.51) is a Hilbert-Schmidt operator, and therefore is a compact operator. Since each constant in the proof of the compactness of the truncated operator in (5.51) depend on the dimension  $d$  of the space  $\mathbb{C}^d$ , we cannot expect that this method could work for the infinite dimensional case. In fact, if  $\mathcal{H}$  is an infinite dimension separable Hilbert space with the orthonormal basis  $\{e_k\}_{k \geq 1}$ , we prove that the multiplication operator  $M_{\chi_{r\mathbb{B}_n}}$  defined in [79, (3.19)] is not a compact operator from  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  into  $L_\alpha^2(\mathbb{B}_n, \mathcal{H})$  ( we show that the sequence  $\{e_k\}_{k \geq 1}$  converges weakly to 0 on  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  and that  $\|M_{\chi_{r\mathbb{B}_n}} e_k\|_{2,\alpha,\mathcal{H}} > 0$  ). Thus, to handle this problem, it will be necessary to change our approach.

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## ♣ Conclusion and future work ♣

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This thesis aimed to study Hankel and Toeplitz operators between vector-valued Bergman spaces. More precisely, the main goal of this thesis was to extend some known results [20, 79] on the boundedness and the compactness of Hankel and Toeplitz operators acting on Bergman spaces of scalar-valued functions to Bergman spaces of vector-valued functions.

We started this thesis with some useful recalls on functional analysis and the integration of vector-valued functions. The definition and the study of vector-valued Bergman spaces allowed us to build our framework. After defining and studying vector-valued Bloch and Lipschitz spaces, we have defined little Hankel operators. Namely, given two complex Banach spaces  $E, F$  and  $\alpha > -1$ , we defined the little Hankel operator  $h_b$  with operator-valued symbol  $b \in A_\alpha^1(\mathbb{B}_n, \mathcal{L}(\overline{E}, F))$  by

$$h_b(f)(z) := \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}},$$

where  $f \in H^\infty(\mathbb{B}_n, E)$  and  $z \in \mathbb{B}_n$ . Let  $0 < p, q < \infty$ . We were first interested in the problem of characterizing those operator-valued holomorphic symbols  $b$  for which the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a bounded operator. This problem was solved by Oliver [52] in the cases  $1 < p, q < \infty$ . Our contribution to this problem was to correct the Oliver's result in the case  $1 < p < q < \infty$  and to solve the boundedness problem of the little Hankel operator on vector-valued Bergman spaces in the cases  $0 < p, q \leq 1$ . We also obtained a complete characterization in the case  $0 < p < 1$  and  $1 < q < \infty$ . Some estimates with loss are given for  $1 < p < \infty$  and  $0 < q \leq 1$ . To solve this problem, we have adapted the methods in [20] and used those in [52].

Secondly, we were interested in the problem of characterizing those operator-valued holomorphic symbols  $b$  for which the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, E) \rightarrow A_\alpha^q(\mathbb{B}_n, F)$  is a compact operator, where  $1 < p \leq q < \infty$ . We have completely solved this problem when  $E, F$  are reflexive complex Banach spaces. Our approach to solve this problem is similar to that in [26]. Notice that in the proof of Proposition 4.5.7, we gave a new method to prove the compactness of the little Hankel operator with monomial symbol by using directly the characterization of the compactness of an operator.

The last part of this thesis is devoted to the study of the boundedness and the compactness properties of Toeplitz operators with operator-valued symbols on vector-valued Bergman spaces. Our work is to extend those results in [79] to vector-valued settings. For  $b \in L_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , the Toeplitz operator is defined by

$$T_b(f)(z) := \int_{\mathbb{B}_n} \frac{b(w)(f(w))d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}},$$

## 5.5. Compactness of Toeplitz operators with matrix-valued $BMO^1$ symbol.

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where  $f \in H^\infty(\mathbb{B}_n, E)$  and  $z \in \mathbb{B}_n$ . Unlike the Hankel operators, we do not have a complete characterization of the boundedness and the compactness of the Toeplitz operator in terms of its symbol. Many tools are introduced in the literature to study Toeplitz operators. In this thesis, we introduced Carleson measures for vector-valued Bergman spaces and we studied some of its properties. Using Berezin transforms introduced in [60, 59], we have defined and studied vector-valued  $BMO$  spaces. Using Carleson measures, Berezin transforms and  $BMO$  spaces, we proved that for  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(E, F))$ , the Toeplitz operator is a bounded operator from  $A_\alpha^2(\mathbb{B}_n, E)$  into  $A_\alpha^2(\mathbb{B}_n, F)$  if and only if  $\tilde{b}$  is bounded on  $\mathbb{B}_n$ . In the case  $E = F = \mathbb{C}^d$ , where  $d \geq 1$ , we proved that for  $b \in BMO_\alpha^1(\mathbb{B}_n, \mathcal{L}(\mathbb{C}^d))$ , the Toeplitz operator  $T_b$  is a compact operator on  $A_\alpha^2(\mathbb{B}_n, \mathbb{C}^d)$  if and only if for each  $e \in \mathbb{C}^d$ , with  $\|e\|_{\mathbb{C}} = 1$ , we have  $\tilde{b}(z)e \rightarrow 0$  weakly as  $|z| \rightarrow 1^-$ . The approach used to prove these results is adapted from the scalar-valued cases.

Despite the work done in this thesis, many problems remain open (see for example [52, Chapter 7]). Some of the following problems can be interesting to look in future work.

The first problem is to look for a complete characterization of the boundedness of the little Hankel operator with operator-valued symbol acting on vector-valued Bergman spaces for  $1 < p < \infty$  and  $0 < q < 1$ . The second problem is the characterization of the compactness of the little Hankel operator from  $A_\alpha^1(\mathbb{B}_n, E)$  into  $A_\alpha^1(\mathbb{B}_n, F)$ . The compactness properties of the big Hankel [52] operator on vector-valued Bergman spaces is still an open problem. For the Toeplitz operator, we hope that the results obtained on the boundedness of Toeplitz operators with  $BMO$  symbols can be extended on the vector-valued Bergman spaces  $A_\alpha^p(\mathbb{B}_n, E)$ , for  $1 < p < \infty$  and  $\alpha > -1$ . We are also interested on the characterization of compact Toeplitz operators on the vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, \mathcal{H})$ , where  $1 < p < \infty$  and  $\mathcal{H}$  is infinite dimension complex Hilbert space.

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# Little Hankel Operators Between Vector-Valued Bergman Spaces on the Unit Ball

David Békollé, Hugues Olivier Defo, Edgar L. Tchoundja   
and Brett D. Wick

**Abstract.** In this paper, we study the boundedness and the compactness of the little Hankel operators  $h_b$  with operator-valued symbols  $b$  between different weighted vector-valued Bergman spaces on the open unit ball  $\mathbb{B}_n$  in  $\mathbb{C}^n$ . More precisely, given two complex Banach spaces  $X, Y$ , and  $0 < p, q \leq 1$ , we characterize those operator-valued symbols  $b : \mathbb{B}_n \rightarrow \mathcal{L}(\overline{X}, Y)$  for which the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$ , is a bounded operator. Also, given two reflexive complex Banach spaces  $X, Y$  and  $1 < p \leq q < \infty$ , we characterize those operator-valued symbols  $b : \mathbb{B}_n \rightarrow \mathcal{L}(\overline{X}, Y)$  for which the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$ , is a compact operator.

**Mathematics Subject Classification.** 32A10, 32A36, 46E40, 47B35.

**Keywords.** Little Hankel operator, Operator-valued symbol, Vector-valued Bergman spaces.

## 1. Introduction

It is well known that Hankel operators constitute a very important class of operators in spaces of analytic functions. The study of these operators on different analytic spaces is not only motivated by the mathematical challenges it raises, but also by many applications on applied mathematics and in physics (see for example [13] for more information). In this paper, we are interested on the boundedness and the compactness problem of the little Hankel operator with operator-valued symbols on weighted vector-valued Bergman spaces on the unit ball.

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Throughout this paper, we fix a nonnegative integer  $n$  and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the  $n$ -dimensional Euclidean space. For

$$z = (z_1, \dots, z_n), \quad w = (w_1, \dots, w_n),$$

in  $\mathbb{C}^n$ , we define the inner product of  $z$  and  $w$  by

$$\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n},$$

where  $\overline{w_k}$  is the complex conjugate of  $w_k$ . The resulting norm is then

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Endowed with the above inner product,  $\mathbb{C}^n$  become a Hilbert space whose canonical basis consists of the following vectors

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad e_n = (0, \dots, 0, 1).$$

The open unit ball in  $\mathbb{C}^n$  is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

When  $\alpha > -1$ , the weighted Lebesgue measure  $d\nu_\alpha$  in  $\mathbb{B}_n$  is defined by

$$d\nu_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha d\nu(z), \quad z \in \mathbb{B}_n$$

where  $d\nu$  is the Lebesgue measure in  $\mathbb{C}^n$  and

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}$$

is the normalizing constant so that  $d\nu_\alpha$  becomes a probability measure on  $\mathbb{B}_n$ . A function defined on the unit ball  $\mathbb{B}_n$  will be called a vector-valued function when it takes its values in some vector space. If  $X$  is a complex Banach space, a vector-valued function  $f : \mathbb{B}_n \rightarrow X$  (a  $X$ -valued function) is said to be strongly holomorphic in  $\mathbb{B}_n$  if for every  $z \in \mathbb{B}_n$  and for every  $k \in \{1, \dots, n\}$ , the limit

$$\lim_{\lambda \rightarrow 0} \frac{f(z + \lambda e_k) - f(z)}{\lambda}$$

exists in  $X$ , where  $\lambda \in \mathbb{C} - \{0\}$ . The space of all  $X$ -valued strongly holomorphic functions on  $\mathbb{B}_n$  will be denoted by  $\mathcal{H}(\mathbb{B}_n, X)$ . We will also denote by  $H^\infty(\mathbb{B}_n, X)$  the space of all bounded  $X$ -valued holomorphic functions. Let  $X^*$  denotes the space of all bounded linear functionals  $x^* : X \rightarrow \mathbb{C}$  (the topological dual space of  $X$ ). We say that a vector-valued function  $f : \mathbb{B}_n \rightarrow X$  is weakly holomorphic if for every  $x^* \in X^*$ , the scalar-valued function  $x^*(f) : \mathbb{B}_n \rightarrow \mathbb{C}$  is holomorphic in the usual sense. An important result by Dunford [7] shows that a vector-valued function is strongly holomorphic if and only if it is weakly holomorphic.

### 1.1. The Conjugate $\overline{X}$ of the Complex Banach Space $X$

In the sequel, we will need the notion of “conjugate” of a complex Banach space [11].

We will use the following definition and notation which can be found in [11]. Let  $x \in X$ ,  $x^* \in X^*$  and  $\lambda \in \mathbb{C}$ . We define

$$(\lambda x^*)(x) := \overline{\lambda} x^*(x).$$

We also use the notation

$$\langle x, x^* \rangle_{X, X^*} = x^*(x)$$

to represent the ‘inner product’ in the complex Banach space  $X$ . We have the following identities

$$\langle \lambda x, x^* \rangle_{X, X^*} = \lambda \langle x, x^* \rangle_{X, X^*} = \langle x, \overline{\lambda} x^* \rangle_{X, X^*},$$

so that we have a regular rule of an inner product. The complex conjugate  $\overline{x}$  of  $x \in X$ , is the linear functional on  $X^*$  defined by

$$\overline{x}(x^*) = \overline{\langle x, x^* \rangle_{X, X^*}},$$

for every  $x^* \in X^*$ . Therefore,

$$\overline{X} = \{\overline{x} : x \in X\}$$

is called the complex conjugate of the Banach space  $X$ . With the norm defined by

$$\|\overline{x}\| := \sup_{\|x^*\|_{X^*}=1} |\overline{x}(x^*)|,$$

$\overline{X}$  becomes a Banach space. Moreover, we have that  $\|x\|_X = \|\overline{x}\|_{\overline{X}}$  for any  $x \in X$ , so that  $X$  and  $\overline{X}$  are isometrically anti-isomorphic.

### 1.2. Vector-Valued Bergman Space

In the sequel, we will integrate vector-valued measurable functions in the sense of Bochner (see [7] for more information). Let  $X$  be a complex Banach space. A measurable function  $f : \mathbb{B}_n \rightarrow X$  is Bochner-integrable with respect to the measure  $\nu_\alpha$  in the unit ball  $\mathbb{B}_n$  if and only if the Lebesgue integral

$$\|f\|_{1, \alpha, X} = \int_{\mathbb{B}_n} \|f(z)\|_X d\nu_\alpha(z)$$

is finite. For  $0 < p < \infty$ , the Bochner-Lebesgue space  $L^p_{\nu_\alpha}(\mathbb{B}_n, X)$  consists of all vector-valued measurable functions  $f : \mathbb{B}_n \rightarrow X$  such that

$$\|f\|_{p, \alpha, X}^p = \int_{\mathbb{B}_n} \|f(z)\|_X^p d\nu_\alpha(z) < \infty.$$

The vector-valued Bergman space  $A^p_\alpha(\mathbb{B}_n, X)$  is defined by

$$A^p_\alpha(\mathbb{B}_n, X) = L^p_{\nu_\alpha}(\mathbb{B}_n, X) \cap \mathcal{H}(\mathbb{B}_n, X).$$

The weak Bochner-Lebesgue space  $L_\alpha^{p,\infty}(\mathbb{B}_n, X)$  consists of all vector-valued measurable functions  $f : \mathbb{B}_n \rightarrow X$  for which

$$\|f\|_{L_\alpha^{p,\infty}(\mathbb{B}_n, X)} = \left( \sup_{\lambda > 0} \lambda^p \nu_\alpha(\{z \in \mathbb{B}_n : \|f(z)\|_X > \lambda\}) \right)^{1/p} < \infty.$$

The weak vector-valued Bergman space  $A_\alpha^{p,\infty}(\mathbb{B}_n, X)$  is defined by

$$A_\alpha^{p,\infty}(\mathbb{B}_n, X) = \mathcal{H}(\mathbb{B}_n, X) \cap L_\alpha^{p,\infty}(\mathbb{B}_n, X).$$

Let  $X, Y$  be two complex Banach spaces and  $\alpha > -1$ . We have the following two lemmas whose proofs can be found in [11].

**Lemma 1.** *Let  $T : X \rightarrow Y$  be a bounded linear operator. If  $f : \mathbb{B}_n \rightarrow X$  is  $\nu_\alpha$ -Bochner integrable in the unit ball, then  $Tf : \mathbb{B}_n \rightarrow Y$  is  $\nu_\alpha$ -Bochner integrable in the unit ball and we have*

$$\int_{\mathbb{B}_n} Tf(z) d\nu_\alpha(z) = T \left( \int_{\mathbb{B}_n} f(z) d\nu_\alpha(z) \right).$$

**Lemma 2.** *If  $f : \mathbb{B}_n \rightarrow X$  is a  $\nu_\alpha$ -Bochner integrable vector-valued function in the unit ball, then the following inequality holds*

$$\left\| \int_{\mathbb{B}_n} f(z) d\nu_\alpha(z) \right\|_X \leq \int_{\mathbb{B}_n} \|f(z)\|_X d\nu_\alpha(z).$$

### 1.3. Vector-Valued Lipschitz Spaces and Vector-Valued $\gamma$ -Bloch Spaces

The radial derivative of a vector-valued holomorphic function  $f : \mathbb{B}_n \rightarrow X$  denoted  $Nf$  is defined for  $z \in \mathbb{B}_n$  by

$$Nf(z) := \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z). \tag{1.1}$$

Let  $f \in \mathcal{H}(\mathbb{B}_n, X)$  and

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \quad z \in \mathbb{B}_n$$

the homogeneous expansion of the function  $f$  where  $f_k$  are homogeneous holomorphic polynomials of degree  $k$  with coefficients in  $X$ . For any two real parameters  $\alpha$  and  $t$  such that neither  $n + \alpha$  nor  $n + \alpha + t$  is a negative integer, we define an invertible operator  $R^{\alpha,t} : \mathcal{H}(\mathbb{B}_n, X) \rightarrow \mathcal{H}(\mathbb{B}_n, X)$  as

$$R^{\alpha,t} f(z) := \sum_{k=0}^{\infty} \frac{\Gamma(n + 1 + \alpha)\Gamma(n + 1 + k + \alpha + t)}{\Gamma(n + 1 + \alpha + t)\Gamma(n + 1 + k + \alpha)} f_k(z), \tag{1.2}$$

where  $z \in \mathbb{B}_n$  and  $\Gamma$  is the classical Euler Gamma function. For  $\gamma \geq 0$ , we denote by  $\Gamma_\gamma(\mathbb{B}_n, X)$  the space of vector-valued holomorphic functions  $f : \mathbb{B}_n \rightarrow X$  for which there exists an integer  $k > \gamma$  such that

$$\|f\|_{\gamma, X} = \|f(0)\|_X + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|N^k f(z)\|_X < \infty,$$

where  $N^k = N \circ N \circ \dots \circ N$   $k$ -times. The definition of the space  $\Gamma_\gamma(\mathbb{B}_n, X)$  is independent of the integer  $k$  used. The space  $\Gamma_\gamma(\mathbb{B}_n, X)$  will be called the vector-valued holomorphic Lipschitz space and for  $\gamma = 0$ , we write

$\mathcal{B}(\mathbb{B}_n, X) = \Gamma_0(\mathbb{B}_n, X)$ . It is clear that  $f \in \mathcal{B}(\mathbb{B}_n, X)$  if and only if  $f$  is a vector-valued holomorphic function and

$$\|f\|_{\mathcal{B}(\mathbb{B}_n, X)} = \|f(0)\|_X + \sup_{z \in \mathbb{B}_n} (1 - |z|^2) \|Nf(z)\|_X < \infty.$$

That is,  $\mathcal{B}(\mathbb{B}_n, X) = \Gamma_0(\mathbb{B}_n, X)$  is the vector-valued Bloch space. The vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, X)$  for  $\gamma > 0$ , is defined as the space of vector-valued holomorphic functions  $f \in \mathcal{H}(\mathbb{B}_n, X)$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^\gamma \|Nf(z)\|_X < \infty.$$

The little vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_{\gamma,0}(\mathbb{B}_n, X)$  for  $\gamma > 0$ , is the subspace of  $\mathcal{B}_\gamma(\mathbb{B}_n, X)$  consisting of functions  $f$  such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\gamma \|Nf(z)\|_X = 0.$$

It is easy to see that  $\mathcal{B}_1(\mathbb{B}_n, X) = \mathcal{B}(\mathbb{B}_n, X)$ . Therefore, the vector-valued  $\gamma$ -Bloch spaces with  $\gamma > 0$  generalize the vector-valued Bloch space. Let  $\gamma \geq 0$ . The generalized vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, X)$  consists of vector-valued holomorphic functions  $f$  in  $\mathbb{B}_n$  such that for some nonnegative integer  $k > \gamma$ , we have

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n, X)} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X < \infty.$$

We consider the following norm on the generalized vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, X)$  by

$$\|f\|_{\Lambda_\gamma(\mathbb{B}_n, X)} = \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X,$$

where  $k > \gamma$  is a nonnegative integer. Equipped with this norm, the generalized vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, X)$  becomes a Banach space. The generalized little vector-valued Lipschitz space  $\Lambda_{\gamma,0}(\mathbb{B}_n, X)$  is the subspace of  $\Lambda_\gamma(\mathbb{B}_n, X)$ , which consists of functions  $f \in \Lambda_\gamma(\mathbb{B}_n, X)$  such that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X = 0. \quad (1.3)$$

When  $\gamma = 0$  and  $k = 1$ , then  $\Lambda_0(\mathbb{B}_n, X) = \mathcal{B}(\mathbb{B}_n, X)$ . It is also important to note that as in the classical case, when  $0 < \gamma < 1$ , we have  $\Lambda_\gamma(\mathbb{B}_n, X) = \mathcal{B}_{1-\gamma}(\mathbb{B}_n, X)$ .

#### 1.4. Little Hankel Operator with Operator-Valued Symbol

Given two complex Banach spaces  $X$  and  $Y$ , we denote by  $\mathcal{L}(X, Y)$  the space of all bounded linear operators  $T : X \rightarrow Y$  endowed with the following norm

$$\|T\|_{\mathcal{L}(X, Y)} = \sup_{\|x\|_X=1} \|Tx\|_Y = \sup_{\|x\|_X=1, \|y^*\|_{Y^*}=1} |\langle Tx, y^* \rangle_{Y, Y^*}|,$$

where  $T \in \mathcal{L}(X, Y)$ . Then  $\mathcal{L}(X, Y)$  is a Banach space. We consider an operator-valued function  $b : \mathbb{B}_n \rightarrow \mathcal{L}(\overline{X}, Y)$  and we suppose that  $b \in$

$\mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ . The little Hankel operator with operator-valued symbol  $b$ , denoted  $h_b$  is defined for  $z \in \mathbb{B}_n$  by

$$h_b f(z) := \int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \quad f \in H^\infty(\mathbb{B}_n, X).$$

In the sequel, we will assume that the symbol  $b$  satisfies the following condition

$$\int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{X}, Y)}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) < \infty, \quad \text{for every } z \in \mathbb{B}_n. \quad (1.4)$$

It is easy to check that if  $b$  satisfies (1.4), then the little Hankel operator  $h_b$  is well defined on  $H^\infty(\mathbb{B}_n, X)$ .

### 1.5. Problems and Known Results

The boundedness properties of the little Hankel operator in the classical case (that is, when  $X = Y = \mathbb{C}$ ) have been extensively studied and many results are now well known. For the case  $n = 1$ , important references are [6, 15]. For  $n > 1$ , a complete characterization has been obtained by Aline Bonami and Luo Luo in [4] when  $p \leq q$ . In 2015, Pau and Zhao [12] solved the case  $1 < q < p < \infty$ . Indeed, they showed that if  $b$  is a holomorphic symbol, the little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, \mathbb{C})$  into  $A_\alpha^q(\mathbb{B}_n, \mathbb{C})$ , with  $1 < q < p < \infty$ , if and only if the symbol  $b$  belongs to the weighted Bergman space  $A_\alpha^t(\mathbb{B}_n, \mathbb{C})$  where  $1/t = 1/q - 1/p$ . We are here concerned with the question of characterizing the operator-valued holomorphic symbols  $b$  for which the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$  where  $0 < p, q < \infty$ . In [1] Aleman and Constantin solved this problem for the particular case  $n = 1$ ,  $p = q = 2$  and  $X = Y = \mathcal{H}$  where  $\mathcal{H}$  is a separable Hilbert space. They showed that the little Hankel operator  $h_b$  extends into a bounded operator from  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  into  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  if and only if the symbol  $b$  belongs to the Bloch space  $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\mathcal{H}))$ . Constantin also obtained in [5] that the little Hankel operator  $h_b$  is a compact operator from  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  into  $A_\alpha^2(\mathbb{B}_n, \mathcal{H})$  if and only if the symbol  $b$  belongs to the little vector-valued Bloch space  $\mathcal{B}_0(\mathbb{B}_n, \mathcal{K}(\mathcal{H}))$ . Their results extend clearly the one known in the classical case (when  $\mathcal{H} = \mathbb{C}$ ). In [11], Oliver solved this problem in the case  $1 < p, q < \infty$ . Mainly, he showed that for  $1 < p < \infty$ , the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$  if and only if the symbol  $b$  belongs to the vector-valued Bloch space  $\mathcal{B}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  and this result clearly generalizes the one obtained by Aleman and Constantin in [1]. Moreover, for  $1 < p \leq q < \infty$ , Oliver showed that the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$  if and only if the symbol  $b$  belongs to the  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  with  $\gamma = 1 + (n + 1 + \alpha) \left( \frac{1}{q} - \frac{1}{p} \right)$ . Also for  $1 < q < p < \infty$ , Oliver showed that the little Hankel operator  $h_b$  is bounded from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$  if and only if  $b \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , with  $1/t = 1/q - 1/p$ , which generalizes the main result in [12]. We are also concerned here with the question of characterizing the operator-valued holomorphic symbols for which  $h_b$  extends into a compact operator from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$  where  $1 < p \leq q < \infty$ .

### 1.6. Statement of Results

Let  $X$  be a complex Banach space and  $0 < p \leq 1$ . The topological dual of the Bergman space  $A_\alpha^p(\mathbb{B}_n, X)$  can be identified with the Lipschitz space  $\Gamma_\gamma(\mathbb{B}_n, X^*)$  as follows:

**Theorem 3.** *Let  $0 < p \leq 1$ . The space  $(A_\alpha^p(\mathbb{B}_n, X))^*$  can be identified with  $\Gamma_\gamma(\mathbb{B}_n, X^*)$  with  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$  under the pairing*

$$\langle f, g \rangle_{\alpha, X} = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{X, X^*} (1 - |z|^2)^k d\nu_\alpha(z), \quad (1.5)$$

where  $D_k$  is defined by (2.3),  $k > \gamma$ , is an integer,  $g \in \Gamma_\gamma(\mathbb{B}_n, X^*)$  and  $f \in A_\alpha^p(\mathbb{B}_n, X)$ . Moreover,

$$\|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^*)} \simeq \sup_{\|f\|_{A_\alpha^p(\mathbb{B}_n, X)}=1} |\langle f, g \rangle_{\alpha, X}|.$$

Before stating the next results, we need to make another assumption on the operator-valued symbol  $b$ . More precisely, we assume that the operator-valued holomorphic symbol  $b$  satisfies the following condition:

$$\int_{\mathbb{B}_n} \|b(z)\|_{\mathcal{L}(\overline{X}, Y)} \log \left( \frac{1}{1 - |z|^2} \right) d\nu_\alpha(z) < \infty. \quad (1.6)$$

Let  $X$  and  $Y$  be two complex Banach spaces. Our contributions to the boundedness problem of the little Hankel operator with operator-valued symbol for  $0 < p, q \leq 1$  are the following :

**Theorem 4.** *Suppose  $0 < p \leq 1$ , and  $\alpha > -1$ . If the little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$  for some positive  $q < 1$ , then the symbol  $b$  is in  $\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  with  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ . Conversely, if  $b$  is in  $\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  with  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ , then the little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, X) \longrightarrow A_\alpha^{1, \infty}(\mathbb{B}_n, Y)$  is a bounded operator.*

As a direct consequence, we have the following result:

**Corollary 5.** *Suppose  $0 < p \leq 1$ , and  $\alpha > -1$ . The little Hankel operator  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$  for some positive  $q < 1$  if and only if its symbol  $b$  belongs to  $\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , where  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ .*

**Theorem 6.** *Let  $0 < p \leq 1$ ,  $\alpha > -1$  and  $\gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ . The little Hankel operator extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^1(\mathbb{B}_n, Y)$  if and only if for some integer  $k > \gamma$ ,*

$$\|N^k b(w)\|_{\mathcal{L}(\overline{X}, Y)} \leq \frac{C}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1} \quad w \in \mathbb{B}_n. \quad (1.7)$$

**Theorem 7.** *Suppose  $1 < p \leq q < \infty$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$  is a bounded operator if and only if  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , where  $\gamma_0 = (n + 1 + \alpha) \left(\frac{1}{p} - \frac{1}{q}\right)$ . Moreover,*

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)} \simeq \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}.$$

If  $X, Y$  are reflexive complex Banach spaces, then we have the following theorem

**Theorem 8.** *Suppose that  $1 < p \leq q < \infty$ , and  $\alpha > -1$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$  is a compact operator if and only if*

$$b \in \Lambda_{\gamma_0, 0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y)),$$

where  $\Lambda_{\gamma_0, 0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$  denotes the generalized little vector-valued Lipschitz space and  $\gamma_0 = (n + 1 + \alpha) \left(\frac{1}{p} - \frac{1}{q}\right)$ , see (1.3).

### 1.7. Plan of the Paper

The paper is divided into six sections. In Sect. 2, we recall some preliminary notions on vector-valued holomorphic functions and we also give the proofs of some important results. Sect. 3 contains the proof of Theorem 3 on the dual of the vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, X)$  for  $0 < p \leq 1$ . In Sect. 4, we give the proof of Theorem 4 and Corollary 5. In Sect. 5, we give the proof of Theorem 6. In Sect. 6, We first give some preliminaries results to prepare the proof of Theorem 8. We recall the result by Oliver [11] of the boundedness of the little Hankel operator with operator-valued symbol  $h_b$  from  $A_\alpha^p(\mathbb{B}_n, X)$  into  $A_\alpha^q(\mathbb{B}_n, Y)$ , with  $1 < p \leq q < \infty$  and we generalize it. In the same section, we give the proof of Theorem 8.

Throughout this paper, when there is no additional condition,  $X$  and  $Y$  will denotes two complex Banach spaces, the real parameter  $\alpha$  will be chosen such that  $\alpha > -1$  and  $c$  will be a positive constant whose value may change from one occurrence to the next. We will also adopt the following notation: we will write  $A \lesssim B$  whenever there exists a positive constant  $c$  such that  $A \leq cB$ . We also write  $A \simeq B$  when  $A \lesssim B$  and  $B \lesssim A$ .

## 2. Preliminaries

### 2.1. Vector-Valued Bergman Projection and Integral Estimates

Here we give some definitions and notations which will be used later and can be found in [4, 11].

For  $f \in L_\alpha^1(\mathbb{B}_n, X)$  and  $z \in \mathbb{B}_n$ , the Bergman projection  $P_\alpha f$  of  $f$  is the integral operator defined by

$$P_\alpha f(z) := \int_{\mathbb{B}_n} K_\alpha(z, w) f(w) d\nu_\alpha(w),$$

where  $K_\alpha(z, w) := \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}$  is the Bergman reproducing kernel of  $\mathbb{B}_n$ . In this situation,  $P_\alpha f$  is also a  $X$ -valued holomorphic function.

**Lemma 9.** (Density) *Suppose that  $0 < p < \infty$ . Then the space of all bounded vector-valued holomorphic functions  $H^\infty(\mathbb{B}_n, X)$  is dense in  $A_\alpha^p(\mathbb{B}_n, X)$ .*

*Proof.* We are going to give the proof for  $0 < p < 1$ , since the case  $1 \leq p < \infty$  is [11, Lemma 2.1.4]. Given a function  $f \in A_\alpha^p(\mathbb{B}_n, X)$ , let  $f_\rho$  defined for  $z \in \mathbb{B}_n$  by  $f_\rho(z) := f(\rho z)$ , where  $0 < \rho < 1$ . The function  $f_\rho$  is holomorphic in the set  $\{z \in \mathbb{B}_n : |z| < 1/\rho\}$  hence is bounded on  $\mathbb{B}_n$ . We first recall that the integral means

$$M_p(r, f) := \int_{\mathbb{S}_n} \|f(r\zeta)\|_X^p d\sigma(\zeta), \quad 0 \leq r < 1$$

are increasing with  $r$ , see [14, Corollary 4.21]. Since  $M_p(r, f_\rho) = M_p(\rho r, f)$ , we have by Minkowski's inequality that

$$M_p^p(r, f_\rho - f) \leq M_p^p(r, f) + M_p^p(r, f_\rho) \leq 2M_p^p(r, f).$$

By the formula of [11, (1.1.1)], (integration in polar coordinates formula) we get

$$\|f - f_\rho\|_{p, \alpha, X}^p = 2nc_\alpha \int_0^1 M_p^p(r, f_\rho - f)(1 - r^2)^\alpha r^{2n-1} dr. \quad (2.1)$$

Since  $f \in A_\alpha^p(\mathbb{B}_n, X)$ , we have that the function  $M_p^p(r, f)$  is integrable over the interval  $[0, 1)$  with respect to the measure  $2n(1 - r^2)^\alpha r^{2n-1} dr$ . It is also clear that  $f_\rho \rightarrow f$  on any compact subsets of  $\mathbb{B}_n$  which implies that  $M_p^p(r, f_\rho - f) \rightarrow 0$  for each  $r \in [0, 1)$  as  $\rho \rightarrow 1$ . Applying the dominated convergence theorem in (2.1), we obtain that  $\|f - f_\rho\|_{p, \alpha, X}^p \rightarrow 0$ , as  $\rho \rightarrow 1$ .  $\square$

**Corollary 10.** *For  $0 < p \leq 1$ , the following inclusion is dense*

$$A_\alpha^2(\mathbb{B}_n, X) \subset A_\alpha^p(\mathbb{B}_n, X).$$

*Proof.* The proof follows directly from Lemma 9.  $\square$

In [3], Oscar Blasco obtained the duality theorem for the vector-valued Bergman spaces in the unit disc  $\mathbb{B}_1$  without any restriction on the Banach space. The proof also works for the unit ball  $\mathbb{B}_n$ . The result is stated as follows:

**Theorem 11. (Duality).** *Suppose  $1 < p < \infty$ . The dual space  $(A_\alpha^p(\mathbb{B}_n, X))^*$  can be identified with  $A_\alpha^{p'}(\mathbb{B}_n, X^*)$ , where  $p'$  is the conjugate exponent of  $p$  given by  $\frac{1}{p} + \frac{1}{p'} = 1$ , under the integral pairing defined by*

$$\langle f, g \rangle_{\alpha, X} := \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{X, X^*} d\nu_\alpha(z), \quad (2.2)$$

for any  $f \in A_\alpha^p(\mathbb{B}_n, X)$ ,  $g \in A_\alpha^{p'}(\mathbb{B}_n, X^*)$ .

*Remark 12.* Suppose  $1 < p < \infty$ . If  $X$  is a reflexive complex Banach space, then the vector-valued Bergman space  $A_\alpha^p(\mathbb{B}_n, X)$  is a reflexive Banach space.

The following reproducing kernel formula also holds for vector-valued Bergman spaces. The proof can be found in [11, Proposition 2.1.2].

**Proposition 13.** *Let  $f \in A_\alpha^1(\mathbb{B}_n, X)$ . We have*

$$f(z) := \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w),$$

for any  $z \in \mathbb{B}_n$ .

We have the following pointwise estimate on the vector-valued Bergman spaces. The proof can be found in [11].

**Theorem 14.** *Let  $0 < p < \infty$ . Then*

$$\|f(z)\|_X \leq \frac{\|f\|_{p,\alpha,X}}{(1 - |z|^2)^{(n+1+\alpha)/p}},$$

for any  $f \in A_\alpha^p(\mathbb{B}_n, X)$  and  $z \in \mathbb{B}_n$ .

The following lemma is critical for many problems concerning the weighted vector-valued Bergman spaces  $A_\alpha^p(\mathbb{B}_n, X)$  whenever  $0 < p \leq 1$  and will be extensively used.

**Lemma 15.** *Let  $0 < p \leq 1$ . Then*

$$\int_{\mathbb{B}_n} \|f(z)\|_X (1 - |z|^2)^{(\frac{1}{p}-1)(n+1+\alpha)} d\nu_\alpha(z) \leq \|f\|_{p,\alpha,X},$$

for all  $f \in A_\alpha^p(\mathbb{B}_n, X)$ .

*Proof.* Write

$$\|f(z)\|_X = \|f(z)\|_X^p \|f(z)\|_X^{1-p},$$

and estimate the second factor using Theorem 14. The desired result follows.  $\square$

The following technical result is proved in [4, Lemma 3.1]

**Lemma 16.** *Let  $\beta, \delta > 0$ . For all  $w \in \mathbb{B}_n$ , we have*

$$I_\alpha(w) := \int_{\mathbb{B}_n} \left| \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) \right|^\delta \frac{(1 - |w|^2)^\beta}{|1 - \langle z, w \rangle|^{n+1+\alpha+\beta}} d\nu_\alpha(z) \leq C,$$

where  $C$  is independent of  $w$  and  $\log$  is the principal branch of the logarithm.

In the sequel, we will also need the following lemma which the scalar version can be found in [8].

**Lemma 17.** *If  $0 < q < 1$ , then the identity  $i : L_\alpha^{1,\infty}(\mathbb{B}_n, X) \hookrightarrow L_\alpha^q(\mathbb{B}_n, X)$  is continuous in the sense that there exists a constant  $C(q) > 0$  such that for every  $f \in L_\alpha^{1,\infty}(\mathbb{B}_n, X)$ , we have*

$$\|f\|_{q,\alpha,X} \leq C(q) \|f\|_{L_\alpha^{1,\infty}(\mathbb{B}_n, X)}.$$

The following result will be very useful in many situations. A proof can be found in [14].

**Theorem 18.** *For  $\beta \in \mathbb{R}$ , let*

$$I_{\alpha,\beta}(z) := \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\alpha d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+\beta}}, \quad z \in \mathbb{B}_n.$$

(i) If  $\beta = 0$ , there exists a constant  $C > 0$  such that

$$I_{\alpha,\beta}(z) \leq C \log \frac{1}{1 - |z|^2}, \quad z \in \mathbb{B}_n.$$

(ii) If  $\beta > 0$ , there exists a constant  $C > 0$  such that

$$I_{\alpha,\beta}(z) \leq C \frac{1}{(1 - |z|^2)^\beta}, \quad z \in \mathbb{B}_n.$$

(iii) If  $\beta < 0$ , there exists a constant  $C > 0$  such that

$$I_{\alpha,\beta}(z) \leq C.$$

## 2.2. Differential Operators and Equivalent Norms for $\Gamma_\gamma$

Given a positive integer  $k$ , we define the differential operator  $D_k$  by

$$D_k := (2I + N) \circ (3I + N) \circ \dots \circ ((k + 1)I + N), \quad (2.3)$$

where  $I$  is the identity operator and  $N$  is the differential operator given in (1.1).

In the sequel, we denote by  $\mathcal{P}(\mathbb{B}_n, X)$  the space of all vector-valued holomorphic polynomials. The proof of the following lemma is similar as in the scalar case in [10].

**Lemma 19.** *For all  $f \in \mathcal{P}(\mathbb{B}_n, X)$  and  $g \in \mathcal{P}(\mathbb{B}_n, X^*)$ , we have the following identity*

$$\int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{X, X^*} d\nu_\alpha(z) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{X, X^*} (1 - |z|^2)^k d\nu_\alpha(z),$$

where  $c_k$  is a positive constant depending only on the integer  $k$ . The above identities are valid for vector-valued holomorphic functions when both sides make sense.

The following lemma will be very useful in the sequel.

**Lemma 20.** *Let  $\{a_k\}$  a sequence of positive numbers. For any positive integer  $k$ , let  $M_k$  the differential operator of order  $k$  defined by*

$$M_k := (a_0 I + N) \circ (a_1 I + N) \circ \dots \circ (a_{k-1} I + N).$$

*Then a vector-valued holomorphic function  $f$  belongs to  $\Gamma_\gamma(\mathbb{B}_n, X)$  if and only if there exists an integer  $k > \gamma$  such that*

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|M_k f(z)\|_X < \infty.$$

*Proof.* Let us assume first that  $f \in \Gamma_\gamma(\mathbb{B}_n, X)$ , and we prove the desired estimate on  $M_k$ . By assumption, there exists an integer  $k > \gamma$  and a positive constant  $C$  such that

$$\|N^k f(z)\|_X \leq C(1 - |z|^2)^{\gamma-k},$$

for any  $z \in \mathbb{B}_n$ . It is enough to prove that the following inequality

$$\|N^j f(z)\|_X < C(1 - |z|^2)^{\gamma-k},$$

holds for  $0 \leq j < k$ , since the assumption give the case  $j = k$ . For  $g \in \mathcal{H}(\mathbb{B}_n, X)$  and  $z = rz'$ , where  $r = |z|$ , and  $z'$  is in the unit sphere. We have

$$Ng(rz') = r\partial_r g(rz').$$

Thus,

$$g(rz') - g(z'/2) = \int_{\frac{1}{2}}^r Ng(sz') \frac{ds}{s}.$$

Now, for  $g \in \mathcal{H}(\mathbb{B}_n, X)$  such that  $\|Ng(z)\|_X \leq C(1 - |z|^2)^{\gamma-k}$ . We have that

$$\begin{aligned} \|g(rz') - g(z'/2)\|_X &\leq 2 \int_{\frac{1}{2}}^r \|Ng(sz')\|_X ds \\ &\leq 4C \int_{\frac{1}{2}}^r (1 - s^2)^{\gamma-k} s ds \\ &= -2C \int_{\frac{1}{2}}^r -2s(1 - s^2)^{\gamma-k} ds \\ &= \left[ \frac{-2C}{\gamma - k + 1} (1 - s^2)^{\gamma-k+1} \right]_{\frac{1}{2}}^r \\ &= \frac{-2C}{\gamma - k + 1} \left\{ (1 - r^2)^{\gamma-k+1} - \left(1 - \frac{1}{4}\right)^{\gamma-k+1} \right\}. \end{aligned}$$

Now, if  $\gamma - k + 1 < 0$ , then

$$\|g(rz') - g(z'/2)\|_X \leq \frac{-2C}{\gamma - k + 1} (1 - r^2)^{\gamma-k} = C_{k,\gamma} (1 - |z|^2)^{\gamma-k}.$$

If  $\gamma - k + 1 > 0$ , then

$$\begin{aligned} \|g(rz') - g(z'/2)\|_X &\leq \frac{2C}{\gamma - k + 1} \left\{ \left(1 - \frac{1}{4}\right)^{\gamma-k+1} - (1 - r^2)^{\gamma-k+1} \right\} \\ &\leq \frac{2C}{\gamma - k + 1} \left(1 - \frac{1}{4}\right)^{\gamma-k+1} = C'_{k,\gamma} \\ &\leq C'_{k,\gamma} (1 - |z|^2)^{\gamma-k}, \end{aligned}$$

where the last inequality is justified using the fact that  $(1 - |z|^2)^{\gamma-k} > 1$ . It then follows that

$$\|g(z)\|_X \leq C(1 - |z|^2)^{\gamma-k}.$$

Now, we use this fact inductively for  $g = N^k f$ , then  $g = N^{k-1} f, \dots$  to conclude. Conversely, assume that there exists an integer  $k > \gamma$  and a positive constant  $C$  such that

$$\|M_k f(z)\|_X \leq C(1 - |z|^2)^{\gamma-k},$$

for any  $z \in \mathbb{B}_n$ . To conclude, it is sufficient to prove that for a fixed positive real  $a$ , the inequality

$$\|ag(z) + Ng(z)\|_X \leq C(1 - |z|^2)^{\gamma-k} \tag{2.4}$$

implies the inequality

$$\|Ng(z)\|_X \leq C(1 - |z|^2)^{\gamma-k},$$

for any function  $g \in \mathcal{H}(\mathbb{B}_n, X)$ . Choose a real  $\beta$  such that  $\beta + \gamma - k > -1$ . By the assumption (2.4), we have that

$$\int_{\mathbb{B}_n} \|ag(z) + Ng(z)\|_X (1 - |z|^2)^\beta d\nu(z) < \infty.$$

Thus, for any  $z \in \mathbb{B}_n$ , we have

$$ag(z) + Ng(z) = c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + Ng(w)]}{(1 - \langle z, w \rangle)^{n+1+\beta}} (1 - |w|^2)^\beta d\nu(w).$$

Then, differentiating under the integral sign, we obtain that for all  $1 \leq i \leq n$ , we get

$$\begin{aligned} \partial_{z_i} [ag(z) + Ng(z)] &= (n + 1 + \beta)c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + Ng(w)]\bar{w}_i}{(1 - \langle z, w \rangle)^{n+2+\beta}} (1 - |w|^2)^\beta d\nu(w). \end{aligned}$$

Therefore,

$$\begin{aligned} N(ag(z) + Ng(z)) &= (n + 1 + \beta)c_\beta \int_{\mathbb{B}_n} \frac{[ag(w) + Ng(w)]\langle z, w \rangle}{(1 - \langle z, w \rangle)^{n+2+\beta}} (1 - |w|^2)^\beta d\nu(w). \end{aligned}$$

Applying (2.4), and Theorem 18, we get that for all  $1 \leq i \leq n$ ,

$$\begin{aligned} \|N(ag(z) + Ng(z))\|_X &\leq Cc_\beta \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\gamma-k+\beta}}{|1 - \langle z, w \rangle|^{n+1+\gamma-k+\beta+(k-\gamma+1)}} d\nu(w) \\ &\leq C(1 - |z|^2)^{\gamma-k-1}. \end{aligned}$$

Thus, the derivative of  $ag(z) + Ng(z)$  is bounded by  $(1 - |z|^2)^{\gamma-k-1}$ . So, to prove the inequality above, we are reduced to consider smooth functions  $\phi$  of one variable  $r \in [0, 1)$ , and to prove that the inequality

$$\|\psi'(r)\|_X \leq C(1 - r)^{\gamma-k-1},$$

with  $\psi(r) = a\phi(r) + r\phi'(r)$ , implies that

$$\|r\phi'(r)\|_X \leq C(1 - r)^{\gamma-k}$$

(here,  $\phi(r) = g(rz')$ ). Now, differentiating  $\psi$ , we obtain  $\psi'(s) = (a+1)\phi'(s) + s\phi''(s)$ . Multiplying both sides of the previous inequality by  $s^a$ , we obtain that  $s^a\psi'(s) = (a+1)s^a\phi'(s) + s^{a+1}\phi''(s) = [s^{a+1}\phi'(s)]'$ . Then integrating the equality above on  $[0, r]$ , we obtain that

$$\phi'(r) = \frac{1}{r^{a+1}} \int_0^r s^a \psi'(s) ds.$$

Therefore, the desired estimate follows at once, since  $k > \gamma$ . □

*Remark 21.* We shall use extensively this lemma for two particular classes of differential operators: first the class  $D_k$ , then the class  $L_k$ , corresponding to the choice  $a_j = n + \alpha + j + 1$ . For this choice, we have

$$(a_j I + N)(1 - \langle z, w \rangle)^{-n-\alpha-j-1} = \frac{n + \alpha + j + 1}{(1 - \langle z, w \rangle)^{n+\alpha+j+2}},$$

and inductively,

$$L_k(1 - \langle z, w \rangle)^{-n-\alpha-1} = \frac{c_k}{(1 - \langle z, w \rangle)^{n+\alpha+k+1}}.$$

The proof of Lemma 20 allows us to define an equivalent norm of  $f$  in terms of  $M_k f$ . Particularly, we will write the equivalent norms of  $f$  in terms of  $D_k f$  and  $L_k f$ . More precisely, we have the following result:

**Corollary 22.** *Let  $D_k$  a differential operator of order  $k$  defined in (2.3) and  $L_k$  a differential operator of order  $k$  defined in Remark 21. For vector-valued holomorphic functions, the following assertions are equivalent:*

- (1)  $f \in \Gamma_\gamma(\mathbb{B}_n, X)$ .
- (2) There exists an integer  $k > \gamma$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k f(z)\|_X < \infty.$$

- (3) There exists an integer  $k > \gamma$  such that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|L_k f(z)\|_X < \infty.$$

Moreover, the following are equivalent

$$\begin{aligned} \|f\|_{\Gamma_\gamma(\mathbb{B}_n, X)} &\simeq \|f(0)\|_X + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k f(z)\|_X \\ &\simeq \|f(0)\|_X + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|L_k f(z)\|_X. \end{aligned}$$

The proof of some of the results obtained in this paper will be based on the following lemma. A proof is in [11], but for the sake of completeness, we will recall the proof.

**Lemma 23.** *Let  $f \in H^\infty(\mathbb{B}_n, X)$  and  $g \in H^\infty(\mathbb{B}_n, Y^*)$ . If  $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  is such that (1.4) and (1.6) hold. Then we have*

$$\langle h_b f, g \rangle_{\alpha, Y} = \int_{\mathbb{B}_n} \langle b(z) \overline{f(z)}, g(z) \rangle_{Y, Y^*} d\nu_\alpha(z). \tag{2.5}$$

*Proof.* Let  $f \in H^\infty(\mathbb{B}_n, X)$  and  $g \in H^\infty(\mathbb{B}_n, Y^*)$ . By the definition of  $\langle \cdot, \cdot \rangle_{\alpha, Y}$ , Fubini's theorem, Lemma 1 and the reproducing kernel property, we have:

$$\begin{aligned} \langle h_b(f), g \rangle_{\alpha, Y} &= \int_{\mathbb{B}_n} \langle h_b(f)(z), g(z) \rangle_{Y, Y^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left\langle \int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), g(z) \right\rangle_{Y, Y^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} g(z) \left( \int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} g(z) \left( \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right) d\nu_\alpha(w) d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{g(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(z) \right) \left( b(w) \overline{f(w)} \right) d\nu_\alpha(w) \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{B}_n} g(w) \left( b(w) \overline{f(w)} \right) d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \langle b(w) \overline{f(w)}, g(w) \rangle_{Y, Y^*} d\nu_\alpha(w).
\end{aligned}$$

It remains to show that the assumption of Fubini's theorem is fulfilled. Indeed, since  $f \in H^\infty(\mathbb{B}_n, X)$  and  $g \in H^\infty(\mathbb{B}_n, Y^*)$ , by Tonelli's theorem, Theorem 18 and relation (1.6) we have that

$$\begin{aligned}
&\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left| \frac{g(z) \left( b(w) \overline{f(w)} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right| d\nu_\alpha(w) d\nu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{X}, Y)}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) d\nu_\alpha(z) \\
&\lesssim \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{X}, Y)} \log \left( \frac{1}{1 - |w|^2} \right) d\nu_\alpha(w) < \infty.
\end{aligned}$$

□

**Lemma 24.** *Let  $f \in H^\infty(\mathbb{B}_n, X)$  and  $z \in \mathbb{B}_n$ . For  $b \in \mathcal{H}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  satisfying (1.4) and (1.6), the function*

$$g_z(w) := \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}}, \quad w \in \mathbb{B}_n$$

*belongs to  $H^\infty(\mathbb{B}_n, X)$  and the following identity holds:*

$$h_b(f)(z) = C_k \int_{\mathbb{B}_n} L_k \left( b(w) \overline{g_z(w)} \right) d\nu_{\alpha+k}(w),$$

*where  $k$  is any positive integer and  $C_k$  is a positive constant depending only on  $k$ .*

*Proof.* It is clear that  $g_z \in H^\infty(\mathbb{B}_n, X)$ . By the definition of the little Hankel operator and the reproducing kernel property, we have

$$\begin{aligned}
h_b(f)(z) &= \int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} b(w) \left( \overline{\frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}}} \right) d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} b(w) \overline{g_z(w)} d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} b(w) \left( \overline{\int_{\mathbb{B}_n} \frac{g_z(\zeta)}{(1 - \langle w, \zeta \rangle)^{n+1+\alpha+k}} d\nu_{\alpha+k}(\zeta)} \right) d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{b(w) \overline{g_z(\zeta)}}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) \right) d\nu_{\alpha+k}(\zeta)
\end{aligned}$$

$$\begin{aligned} &= c_k^{-1} \int_{\mathbb{B}_n} L_k \left( \int_{\mathbb{B}_n} \frac{b(w)(\overline{g_z(\zeta)})}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right) d\nu_{\alpha+k}(\zeta) \\ &= c_k^{-1} \int_{\mathbb{B}_n} L_k \left( b(\zeta)(\overline{g_z(\zeta)}) \right) d\nu_{\alpha+k}(\zeta). \end{aligned}$$

The assumption of Fubini’s theorem is fulfilled. Indeed by (1.6), we have that

$$\begin{aligned} &\int_{\mathbb{B}_n} \left\| \int_{\mathbb{B}_n} \frac{b(w)(\overline{g_z(\zeta)})}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) \right\|_Y d\nu_{\alpha+k}(\zeta) \\ &\leq \|g_z\|_{\infty, X} \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|b(w)\|_{\mathcal{L}(\overline{X}, Y)}}{|1 - \langle w, \zeta \rangle|^{n+1+\alpha+k}} d\nu_\alpha(w) d\nu_{\alpha+k}(\zeta) \\ &= \|g_z\|_{\infty, X} \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{X}, Y)} \times \left( \int_{\mathbb{B}_n} \frac{d\nu_{\alpha+k}(\zeta)}{|1 - \langle w, \zeta \rangle|^{n+1+\alpha+k}} \right) d\nu_\alpha(w) \\ &\leq \|g_z\|_{\infty, X} \int_{\mathbb{B}_n} \|b(w)\|_{\mathcal{L}(\overline{X}, Y)} \left( \log \frac{1}{1 - |w|^2} \right) d\nu_\alpha(w) < \infty. \end{aligned}$$

□

### 3. The Proof of Theorem 3

*Proof.* We first suppose that  $g \in \Gamma_\gamma(\mathbb{B}_n, X^*)$ , with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$ . Given a positive integer  $k > \gamma$ , we define the functional

$$\begin{aligned} \wedge_g &: A_\alpha^p(\mathbb{B}_n, X) \longrightarrow \mathbb{C} \\ f &\mapsto \wedge_g(f) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{X, X^*} (1 - |z|^2)^k d\nu_\alpha(z), \end{aligned}$$

where  $c_k$  is the positive constant in Lemma 19. It is clear that  $\wedge_g$  is linear and is well defined on  $A_\alpha^p(\mathbb{B}_n, X)$ . Indeed, let  $f \in A_\alpha^p(\mathbb{B}_n, X)$ . By Lemma 15, we have

$$\begin{aligned} |\wedge_g(f)| &= c_k \left| \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{X, X^*} (1 - |z|^2)^k d\nu_\alpha(z) \right| \\ &\leq c_k \int_{\mathbb{B}_n} \|f(z)\|_X \|D_k g(z)\|_{X^*} (1 - |z|^2)^k d\nu_\alpha(z) \\ &= c_k \int_{\mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k g(z)\|_{X^*} (1 - |z|^2)^\gamma \|f(z)\|_X d\nu_\alpha(z) \\ &\leq c_k \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|D_k g(z)\|_{X^*} \int_{\mathbb{B}_n} (1 - |z|^2)^\gamma \|f(z)\|_X d\nu_\alpha(z) \\ &\lesssim \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^*)} \int_{\mathbb{B}_n} (1 - |z|^2)^{\left(\frac{1}{p}-1\right)(n+1+\alpha)} \|f(z)\|_X d\nu_\alpha(z) \\ &\lesssim \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^*)} \|f\|_{p, \alpha, X}. \end{aligned}$$

We conclude that  $\wedge_g$  is bounded on  $A_\alpha^p(\mathbb{B}_n, X)$  and  $\|\wedge_g\| \lesssim \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^*)}$ .

Conversely, let  $\wedge$  be a bounded linear functional on  $A_\alpha^p(\mathbb{B}_n, X)$ . Let us show that there exists  $g \in \Gamma_\gamma(\mathbb{B}_n, X^*)$ , with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$

such that  $\wedge = \wedge_g$ . Since  $A_\alpha^2(\mathbb{B}_n, X) \subset A_\alpha^p(\mathbb{B}_n, X)$  and  $\wedge$  is bounded on  $A_\alpha^p(\mathbb{B}_n, X)$ ,  $\wedge$  is also bounded on  $A_\alpha^2(\mathbb{B}_n, X)$ . Then by Theorem 11, there exists  $g \in A_\alpha^2(\mathbb{B}_n, X^*)$  such that

$$\wedge(f) = \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{X, X^*} d\nu_\alpha(z), \quad (3.1)$$

for all  $f \in A_\alpha^2(\mathbb{B}_n, X)$ . Since  $g \in A_\alpha^2(\mathbb{B}_n, X^*)$ , for any positive integer  $k$ , we have  $D_k g \in A_{\alpha+k}^2(\mathbb{B}_n, X^*)$ . Applying Lemma 19 in (3.1), we obtain that

$$\wedge(f) = c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{X, X^*} (1 - |z|^2)^k d\nu_\alpha(z), \quad (3.2)$$

for all  $f \in A_\alpha^2(\mathbb{B}_n, X)$ . Now, we fix  $x \in X$ ,  $w \in \mathbb{B}_n$  and an integer  $k > \gamma$ . Let

$$f(z) = \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x, \quad z \in \mathbb{B}_n.$$

By Theorem 18, we have that  $f \in A_\alpha^2(\mathbb{B}_n, X)$ . Proposition 13 and (3.2), give us

$$\begin{aligned} \wedge(f) &= c_k \int_{\mathbb{B}_n} \langle f(z), D_k g(z) \rangle_{X, X^*} (1 - |z|^2)^k d\nu_\alpha(z) \\ &= c_k \int_{\mathbb{B}_n} \left\langle \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x, D_k g(z) \right\rangle_{X, X^*} (1 - |z|^2)^k d\nu_\alpha(z) \\ &= \frac{c_\alpha c_k}{c_{\alpha+k}} (1 - |w|^2)^{k-\gamma} \left\langle x, \int_{\mathbb{B}_n} \frac{D_k g(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} d\nu_{\alpha+k}(z) \right\rangle_{X, X^*} \\ &= \frac{c_\alpha c_k}{c_{\alpha+k}} (1 - |w|^2)^{k-\gamma} \langle x, D_k g(w) \rangle_{X, X^*}. \end{aligned}$$

By Theorem 18,  $f \in A_\alpha^p(\mathbb{B}_n, X)$  and  $\|f\|_{p, \alpha, X} \lesssim \|x\|_X$ . Since  $x$  is arbitrary, by duality, we have that

$$\begin{aligned} \|D_k g(w)\|_{X^*} &= \sup_{\|x\|_X=1} |\langle x, D_k g(w) \rangle_{X, X^*}| \\ &= \frac{c_{\alpha+k}}{c_\alpha c_k} \sup_{\|x\|_X=1} \frac{1}{(1 - |w|^2)^{k-\gamma}} |\wedge(f)| \\ &\lesssim \sup_{\|x\|_X=1} \frac{1}{(1 - |w|^2)^{k-\gamma}} \|\wedge\| \|f\|_{p, \alpha, X} \\ &\lesssim \sup_{\|x\|_X=1} \frac{\|\wedge\|}{(1 - |w|^2)^{k-\gamma}} \|x\|_X \\ &\lesssim \frac{\|\wedge\|}{(1 - |w|^2)^{k-\gamma}}. \end{aligned}$$

According to Corollary 22, we conclude that

$$g \in \Gamma_\gamma(\mathbb{B}_n, X^*) \text{ and } \|g\|_{\Gamma_\gamma(\mathbb{B}_n, X^*)} \lesssim \|\wedge\|,$$

with  $\gamma = (n + 1 + \alpha) \left( \frac{1}{p} - 1 \right)$ . To finish the proof, it remains to show that (3.1) remains true for functions in  $A_\alpha^p(\mathbb{B}_n, X)$  which is a direct consequence of the density in Corollary 10.  $\square$

### 4. The Proofs of Theorem 4 and Corollary 5

In this section, we will give the proofs of Theorem 4 and Corollary 5.

#### 4.1. Proof of Theorem 4

*Proof.* First assume that  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  to  $A_\alpha^q(\mathbb{B}_n, Y)$ , with  $q < 1$ . Let  $\|h_b\| := \|h_b\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)}$ . We want to show that  $b \in \Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ . Since  $h_b : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$  is a bounded operator, we have by Theorem 3 that

$$|\langle h_b(f), g \rangle_{\alpha, Y}| \lesssim \|h_b\| \|f\|_{p, \alpha, X} \|g\|_{\Gamma_\beta(\mathbb{B}_n, Y^*)},$$

for every  $f \in A_\alpha^p(\mathbb{B}_n, X)$  and  $g \in \Gamma_\beta(\mathbb{B}_n, Y^*)$ , with  $\beta = (n + 1 + \alpha) \left(\frac{1}{q} - 1\right)$ . Let  $x \in X$ ,  $y^* \in Y^*$ ,  $w \in \mathbb{B}_n$  and an integer  $k$  such that  $k > \gamma = (n + 1 + \alpha) \left(\frac{1}{p} - 1\right)$ . Let  $g(z) = y^*$ , and  $f(z) = \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x$ . It is clear that  $f \in H^\infty(\mathbb{B}_n, X)$  and  $g \in \Gamma_\beta(\mathbb{B}_n, Y^*)$ , with  $\|g\|_{\Gamma_\beta(\mathbb{B}_n, Y^*)} = \|y^*\|_{Y^*}$ . We also have by Theorem 18 that  $f \in A_\alpha^p(\mathbb{B}_n, X)$ , with  $\|f\|_{p, \alpha, X} \lesssim \|x\|_X$ . Hence

$$|\langle h_b(f), g \rangle_{\alpha, Y}| \lesssim \|h_b\| \|x\|_X \|y^*\|_{Y^*}, \tag{4.1}$$

Applying Lemma 23 and the reproducing kernel property, we have that

$$\begin{aligned} & |\langle h_b(f), g \rangle_{\alpha, Y}| \\ &= \left| \int_{\mathbb{B}_n} \langle b(z) \left( \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x \right), y^* \rangle_{Y, Y^*} d\nu_\alpha(z) \right| \\ &= (1 - |w|^2)^{k-\gamma} \left| \int_{\mathbb{B}_n} \langle b(z) \left( \frac{\bar{x}}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} \right), y^* \rangle_{Y, Y^*} d\nu_\alpha(z) \right| \\ &= (1 - |w|^2)^{k-\gamma} \left| \int_{\mathbb{B}_n} \langle \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}}, y^* \rangle_{Y, Y^*} d\nu_\alpha(z) \right| \\ &= (1 - |w|^2)^{k-\gamma} \left| \left\langle \int_{\mathbb{B}_n} \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \right| \\ &= \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle \int_{\mathbb{B}_n} L_k \left( \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \right) d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \right| \\ &= \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle L_k \left( \int_{\mathbb{B}_n} \frac{b(z) (\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(z) \right), y^* \right\rangle_{Y, Y^*} \right| \\ &= \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle L_k (b(w) (\bar{x})), y^* \right\rangle_{Y, Y^*} \right|. \end{aligned}$$

Thus,

$$|\langle h_b(f), g \rangle_{\alpha, Y}| = \frac{(1 - |w|^2)^{k-\gamma}}{c_k} \left| \left\langle L_k (b(w) (\bar{x})), y^* \right\rangle_{Y, Y^*} \right|. \tag{4.2}$$

From (4.1), (4.2) and the fact that  $\|x\|_X = \|\bar{x}\|_{\overline{X}}$ , we deduce that

$$(1 - |w|^2)^{k-\gamma} \left| \left\langle L_k (b(w) (\bar{x})), y^* \right\rangle_{Y, Y^*} \right| \lesssim \|h_b\| \|\bar{x}\|_{\overline{X}} \|y^*\|_{Y^*}. \tag{4.3}$$

Since  $x$  and  $y^*$  are arbitrary, we get that

$$\sup_{w \in \mathbb{B}_n} (1 - |w|^2)^{k-\gamma} \|L_k b(w)\|_{\mathcal{L}(\overline{X}, Y^*)} \lesssim \|h_b\|.$$

That is,  $b \in \Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y^*))$  with  $\|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \lesssim \|h_b\|$ .

Conversely, assume that  $b \in \Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  and let us prove that  $h_b$  extends to a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  to  $A_\alpha^{1,\infty}(\mathbb{B}_n, Y)$ . Choose a positive integer  $k > \gamma$ , and let  $f \in H^\infty(\mathbb{B}_n, X)$ . Taking

$$g_z(w) = \frac{f(w)}{(1 - \langle w, z \rangle)^{n+1+\alpha}},$$

with  $w \in \mathbb{B}_n$  and applying Lemma 24, Lemma 2 and the assumption we obtain

$$\begin{aligned} \|h_b f(z)\|_Y &= \left\| \int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right\|_Y \\ &= c_k \left\| \int_{\mathbb{B}_n} L_k \left( b(w) \overline{g_z(w)} \right) d\nu_{\alpha+k}(w) \right\|_Y \\ &= c_k \left\| \int_{\mathbb{B}_n} \frac{L_k \left( b(w) \overline{f(w)} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha+k}(w) \right\|_Y \\ &\leq c_k \int_{\mathbb{B}_n} \left\| \frac{L_k \left( b(w) \overline{f(w)} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\|_Y d\nu_{\alpha+k}(w) \\ &\leq \frac{c_k c_{\alpha+k}}{c_\alpha} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^k \|L_k b(w)\|_{\mathcal{L}(\overline{X}, Y)} \|f(w)\|_{\overline{X}}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) \\ &\lesssim \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^\gamma \|f(w)\|_X}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} P_\alpha^+ g(z), \end{aligned}$$

where the reproducing kernel is justified by (1.4) and

$$P_\alpha^+ g(z) = \int_{\mathbb{B}_n} \frac{g(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(w)$$

is the positive Bergman operator of the positive function  $g(z) = (1 - |z|^2)^\gamma \|f(z)\|_X$ .

Now, let  $\lambda > 0$ . We have that

$$\nu_\alpha(\{z \in \mathbb{B}_n : \|h_b f(z)\|_Y > \lambda\}) \leq \nu_\alpha(\{z \in \mathbb{B}_n : c_k \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} P_\alpha^+ g(z) > \lambda\}).$$

Since the positive Bergman operator  $P_\alpha^+ : L_\alpha^1(\mathbb{B}_n) \rightarrow L_\alpha^{1,\infty}(\mathbb{B}_n)$  is bounded (cf. e.g [2]), there exists a constant  $c$  such that

$$\begin{aligned} \nu_\alpha(\{z \in \mathbb{B}_n : c_k \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} P_\alpha^+ g(z) > \lambda\}) &\leq \frac{c}{\frac{\lambda}{c_k \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}}} \|g\|_{L_\alpha^1(\mathbb{B}_n)} \\ &= \frac{c c_k}{\lambda} \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \|g\|_{L_\alpha^1(\mathbb{B}_n)}. \end{aligned}$$

Applying Lemma 15 to the function  $f$ , we get that

$$\begin{aligned} \|g\|_{L^1_\alpha(\mathbb{B}_n)} &= \int_{\mathbb{B}_n} (1 - |z|^2)^\gamma \|f(z)\|_X d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} (1 - |z|^2)^{\left(\frac{1}{p}-1\right)(n+1+\alpha)} \|f(z)\|_X d\nu_\alpha(z) \\ &\leq \|f\|_{p,\alpha,X}. \end{aligned}$$

It follows that

$$\lambda \nu_\alpha(\{z \in \mathbb{B}_n : \|h_b f(z)\|_Y > \lambda\}) \lesssim \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{X}, Y))} \|f\|_{p,\alpha,X}$$

for all  $\lambda > 0$ . Therefore,  $h_b$  extends into a bounded operator from  $A^p_\alpha(\mathbb{B}_n, X)$  to  $A^{1,\infty}_\alpha(\mathbb{B}_n, Y)$  with

$$\|h_b\|_{A^p_\alpha(\mathbb{B}_n, X) \rightarrow A^{1,\infty}_\alpha(\mathbb{B}_n, Y)} \lesssim \|b\|_{\Gamma_\gamma(\mathbb{B}_n, \mathcal{L}(\bar{X}, Y))}.$$

By density of  $H^\infty(\mathbb{B}_n, X)$  on  $A^p_\alpha(\mathbb{B}_n, X)$ , the proof of the theorem is finished.  $\square$

### 4.2. Proof of Corollary 5

*Proof.* Just apply Lemma 17 and the second part of Theorem 4 to conclude.  $\square$

## 5. The Proof of Theorem 6

This section is devoted to the proof of Theorem 6.

*Proof.* We first prove the sufficiency of the theorem. We assume that there exists a constant  $C' > 0$  such that

$$\|N^k b(w)\|_{\mathcal{L}(\bar{X}, Y)} \leq \frac{C'}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1}.$$

Likewise by Corollary 22, we have that, there exists a constant  $C > 0$  such that

$$\|L_k b(w)\|_{\mathcal{L}(\bar{X}, Y)} \leq \frac{C}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1}.$$

Applying Lemma 24 for any  $f \in H^\infty(\mathbb{B}_n, X)$ , we get

$$\int_{\mathbb{B}_n} \frac{b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) = c_k \int_{\mathbb{B}_n} \frac{L_k b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha+k}(w).$$

Thus, by the assumption, Lemma 24 and Lemma 15 we have that

$$\begin{aligned} &\|h_b f\|_{A^1_\alpha(\mathbb{B}_n, Y)} \\ &= \int_{\mathbb{B}_n} \left\| c_k \int_{\mathbb{B}_n} \frac{L_k b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha+k}(w) \right\|_Y d\nu_\alpha(z) \\ &\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left\| \frac{L_k b(w) \overline{f(w)}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\|_Y (1 - |w|^2)^k d\nu_\alpha(w) d\nu_\alpha(z) \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{\|L_k b(w)\|_{\mathcal{L}(\overline{X}, Y)}}{|1 - \langle z, w \rangle|^{n+1+\alpha}} \|f(w)\|_{\overline{X}} (1 - |w|^2)^k d\nu_\alpha(w) d\nu_\alpha(z) \\
&= \int_{\mathbb{B}_n} \left( \int_{\mathbb{B}_n} \frac{1}{|1 - \langle z, w \rangle|^{n+1+\alpha}} d\nu_\alpha(z) \right) \\
&\quad \|L_k b(w)\|_{\mathcal{L}(\overline{X}, Y)} \|f(w)\|_{\overline{X}} (1 - |w|^2)^k d\nu_\alpha(w) \\
&\lesssim \int_{\mathbb{B}_n} \left( \log \frac{1}{1 - |w|^2} \right) \|f(w)\|_{\overline{X}} \frac{(1 - |w|^2)^k}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1} d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \|f(w)\|_X (1 - |w|^2)^\gamma d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \|f(w)\|_X (1 - |w|^2)^{(\frac{1}{p}-1)(n+1+\alpha)} d\nu_\alpha(w) \\
&\lesssim \|f\|_{p, \alpha, X}.
\end{aligned}$$

Conversely, we assume that  $h_b$  extends into a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  to  $A_\alpha^1(\mathbb{B}_n, Y)$ . Then for all  $f \in H^\infty(\mathbb{B}_n, X)$  and  $g \in \mathcal{B}(\mathbb{B}_n, Y^*)$ , we have

$$|\langle h_b(f), g \rangle_{\alpha, Y}| \leq \|h_b\| \|f\|_{p, \alpha, X} \|g\|_{\mathcal{B}(\mathbb{B}_n, Y^*)}. \quad (5.1)$$

We choose the particular function  $g(z) = y^*$ , with  $y^* \in Y^*$ . Applying Lemma 23, relation (5.1) becomes

$$\begin{aligned}
\left| \int_{\mathbb{B}_n} \langle h_b f(z), y^* \rangle_{Y, Y^*} d\nu_\alpha(z) \right| &= \left| \left\langle \int_{\mathbb{B}_n} b(z) \overline{f(z)} d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \right| \\
&\leq \|h_b\| \|f\|_{p, \alpha, X} \|y^*\|_{Y^*}.
\end{aligned}$$

Thus

$$\left| \int_{\mathbb{B}_n} \langle b(z) \overline{f(z)}, y^* \rangle_{Y, Y^*} d\nu_\alpha(z) \right| \leq \|h_b\| \|f\|_{p, \alpha, X} \|y^*\|_{Y^*} \quad (5.2)$$

for all  $f \in H^\infty(\mathbb{B}_n, X)$  and  $y^* \in Y^*$ . Now, take  $x \in X$ ,  $y^* \in Y^*$ , and an integer  $k$  such that  $k > \gamma$ . Fix  $w \in \mathbb{B}_n$  and put

$$f(z) = \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x; \quad g(z) = \log(1 - \langle z, w \rangle) y^*,$$

where  $\log$  is the principal branch of the logarithm. Since  $f \in H^\infty(\mathbb{B}_n, X)$  and  $g \in \mathcal{B}(\mathbb{B}_n, Y^*)$ , by relation (5.1), we have that

$$|\langle h_b f, g \rangle_{\alpha, Y}| \leq \|h_b\| \|x\|_X \|y^*\|_{Y^*}. \quad (5.3)$$

Applying Lemma 23 for those particular vector-valued holomorphic functions  $f$  and  $g$  and using the fact that

$$\log(1 - \langle w, z \rangle) = \log(1 - |w|^2) + \log \left( \frac{1 - \langle w, z \rangle}{1 - |w|^2} \right),$$

we obtain

$$\begin{aligned}
&\langle h_b f, g \rangle_{\alpha, Y} \\
&= \int_{\mathbb{B}_n} \langle b(z) \left( \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} x \right), \log(1 - \langle z, w \rangle) y^* \rangle_{Y, Y^*} d\nu_\alpha(z)
\end{aligned}$$

$$\begin{aligned}
 &= \left\langle \int_{\mathbb{B}_n} b(z) \left[ \frac{(1 - |w|^2)^{k-\gamma} \log(1 - \langle w, z \rangle)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} \bar{x} \right] d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \\
 &= \left\langle \int_{\mathbb{B}_n} \frac{b(z)(\bar{x})(1 - |w|^2)^{k-\gamma} \log(1 - |w|^2)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \\
 &+ \left\langle \int_{\mathbb{B}_n} b(z) \left[ \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}} \log \left( \frac{1 - \langle w, z \rangle}{1 - |w|^2} \right) \bar{x} \right] d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \\
 &= \left\langle (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \int_{\mathbb{B}_n} \frac{b(z)(\bar{x}) d\nu_\alpha(z)}{(1 - \langle w, z \rangle)^{n+1+\alpha+k}}, y^* \right\rangle_{Y, Y^*} \\
 &+ \left\langle \int_{\mathbb{B}_n} b(z) \left( \frac{(1 - |w|^2)^{k-\gamma}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) x \right) d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \\
 &= (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \left\langle L_k \left( \int_{\mathbb{B}_n} \frac{b(z)(\bar{x})}{(1 - \langle w, z \rangle)^{n+1+\alpha}} d\nu_\alpha(z) \right), y^* \right\rangle_{Y, Y^*} \\
 &+ \left\langle \int_{\mathbb{B}_n} b(z) \left( \overline{f(z) \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right)} \right) d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*} \\
 &= (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \left\langle L_k(b(w)(\bar{x})), y^* \right\rangle_{Y, Y^*} \\
 &+ \left\langle \int_{\mathbb{B}_n} b(z)(\overline{\varphi(z)}) d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*},
 \end{aligned}$$

where  $\varphi(z) = f(z) \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right)$ . Therefore, we can write  $\langle h_b f, g \rangle_{\alpha, Y} = I_1 + I_2$ , with

$$I_1 = (1 - |w|^2)^{k-\gamma} \log(1 - |w|^2) \langle L_k(b(w)(\bar{x})), y^* \rangle_{Y, Y^*}$$

and

$$I_2 = \left\langle \int_{\mathbb{B}_n} b(z)(\overline{\varphi(z)}) d\nu_\alpha(z), y^* \right\rangle_{Y, Y^*}.$$

Applying Lemma 16 with  $\delta = p$ , and  $\beta = p(k - \gamma)$ , we obtain that

$$\begin{aligned}
 &\|\varphi\|_{p, \alpha, X} \\
 &= \left( \int_{\mathbb{B}_n} \left| \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) \right|^p \frac{(1 - |w|^2)^{p(k-\gamma)}}{|1 - \langle z, w \rangle|^{p(n+1+\alpha+k)}} \|x\|_X^p d\nu_\alpha(z) \right)^{1/p} \\
 &= \|x\|_X \left( \int_{\mathbb{B}_n} \left| \log \left( \frac{1 - \langle z, w \rangle}{1 - |w|^2} \right) \right|^p \frac{(1 - |w|^2)^{p(k-\gamma)}}{|1 - \langle z, w \rangle|^{n+1+\alpha+p(k-\gamma)}} d\nu_\alpha(z) \right)^{1/p} \\
 &\lesssim \|x\|_X.
 \end{aligned}$$

According to the relation (5.2), we obtain the following estimation of  $I_2$

$$|I_2| \leq \|h_b\| \|\varphi\|_{p, \alpha, X} \|y^*\|_{Y^*} \lesssim \|h_b\| \|x\|_X \|y^*\|_{Y^*}.$$

Since  $I_1 = \langle h_b f, g \rangle_{\alpha, Y} - I_2$ , by the relation (5.3) and the previous estimates on  $I_2$ , we have that

$$|I_1| \leq |\langle h_b f, g \rangle_{\alpha, Y}| + |I_2| \lesssim \|h_b\| \|x\|_X \|y^*\|_{Y^*}.$$

Since  $x \in X$ ,  $y^* \in Y^*$  are arbitrary and  $\|x\|_X = \|\bar{x}\|_{\bar{X}}$ , we get that

$$\begin{aligned} |I_1| &= (1 - |w|^2)^{k-\gamma} \log \left( \frac{1}{1 - |w|^2} \right) |\langle L_k(b(w)(\bar{x})), y^* \rangle_{Y, Y^*}| \\ &\leq C \|h_b\| \|\bar{x}\|_{\bar{X}} \|y^*\|_{Y^*}. \end{aligned}$$

Since  $\bar{x} \in \bar{X}$  and  $y^* \in Y^*$  are arbitrary, we deduce that :

$$\begin{aligned} \|L_k b(w)\|_{\mathcal{L}(\bar{X}, Y)} &= \sup_{\|\bar{x}\|_{\bar{X}}=1, \|y^*\|_{Y^*}=1} |\langle L_k(b(w)(\bar{x})), y^* \rangle_{Y, Y^*}| \\ &\leq \frac{C}{(1 - |w|^2)^{k-\gamma}} \left( \log \frac{1}{1 - |w|^2} \right)^{-1}. \end{aligned}$$

The desired result follows at once using Corollary 22. □

### 6. Compactness of the Little Hankel Operator, $h_b$ , with Operator-Valued Symbols $b$ From $A^p_\alpha(\mathbb{B}_n, X)$ to $A^q_\alpha(\mathbb{B}_n, Y)$ , With $1 < p \leq q < \infty$

In this section, we are going to characterize those symbols  $b$  for which the little Hankel operator extends into a bounded compact operator from  $A^p_\alpha(\mathbb{B}_n, X)$  to  $A^q_\alpha(\mathbb{B}_n, Y)$ , where  $1 < p \leq q < \infty$  and  $X, Y$  are two reflexive complex Banach spaces.

#### 6.1. Preliminaries Notions

The proof of the following remark can be found in [11, Proposition 1.6.1]

*Remark 25.* Let  $t \geq 0$ . Then the operator  $R^{\alpha, t}$  is the unique continuous linear operator on  $\mathcal{H}(\mathbb{B}_n, X)$  satisfying

$$R^{\alpha, t} \left( \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right) = \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}},$$

for every  $z \in \mathbb{B}_n$  and  $x \in X$ .

We will use the operator  $R^{\alpha, t}$ , for  $t > 0$ , in the vector-valued Bergman space  $A^1_\alpha(\mathbb{B}_n, X)$  as follows:

**Proposition 26.** *Let  $t > 0$  and  $f \in A^1_\alpha(\mathbb{B}_n, X)$ . Then*

$$R^{\alpha, t} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+t}} d\nu_\alpha(w),$$

for each  $z \in \mathbb{B}_n$ .

The proof of the following proposition is not quite different to the proof in [14, Proposition 1.15], but for the sake of completeness, we will recall the proof.

**Proposition 27.** *Suppose  $N$  is a positive integer and  $\alpha$  is a real such that  $n + \alpha$  is not a negative integer. Then  $R^{\alpha, N}$  as an operator acting on  $\mathcal{H}(\mathbb{B}_n, X)$  is*

a linear partial differential operator of order  $N$  with polynomial coefficients, that is

$$R^{\alpha,N} f(z) = \sum_{m \in \mathbb{N}^n, |m| \leq N} p_m(z) \frac{\partial^{|m|} f}{\partial z^m}(z),$$

where each  $p_m$  is a polynomial.

*Proof.* Let  $x \in X$  and  $w \in \mathbb{B}_n$ . By using the multi-nomial formula

$$\langle z, w \rangle^k = \sum_{|m|=k} \frac{k!}{m!} z^m \bar{w}^m,$$

it follows that

$$\begin{aligned} & \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha+N}} \\ &= \frac{x(1 - \langle z, w \rangle + \langle z, w \rangle)^N}{(1 - \langle z, w \rangle)^{n+1+\alpha+N}} \\ &= \sum_{k=0}^N \frac{N!}{k!(N-k)!} \frac{\langle z, w \rangle^k x (1 - \langle z, w \rangle)^{N-k}}{(1 - \langle z, w \rangle)^{n+1+\alpha+N}} \\ &= \sum_{k=0}^N \frac{N!}{k!(N-k)!} \sum_{|m|=k} \frac{k!}{m!} z^m \frac{\bar{w}^m x}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} \\ &= \sum_{k=0}^N \sum_{|m|=k} \frac{N!}{m!(N-k)!} z^m \frac{\bar{w}^m x}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} \\ &= \sum_{k=0}^N \sum_{|m|=k} \frac{N!}{\prod_{j=0}^k (n+1+\alpha+j)m!(N-k)!} z^m \frac{\partial^k}{\partial z^m} \left( \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right). \end{aligned}$$

Therefore, there exists a constant  $c_{mk}$  such that

$$R^{\alpha,N} \left( \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right) = \sum_{k=0}^N \sum_{|m|=k} c_{mk} z^m \frac{\partial^k}{\partial z^m} \left( \frac{x}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right).$$

Thus

$$R^{\alpha,N} = \sum_{k=0}^N \sum_{|m|=k} c_{mk} z^m \frac{\partial^k}{\partial z^m}.$$

□

We will also need the following results whose proofs can be found in [11].

**Lemma 28.** *Let  $t > 0$ . Then*

$$\int_{\mathbb{B}_n} f(z) \overline{g(z)} d\nu_{\alpha}(z) = \int_{\mathbb{B}_n} R^{\alpha,t} f(z) \overline{g(z)} d\nu_{\alpha+t}(z),$$

for all  $f \in A^1_{\alpha}(\mathbb{B}_n, X)$  and  $g \in H^{\infty}(\mathbb{B}_n, \mathbb{C})$ .

**Lemma 29.** *Let  $t > 0$  and  $X$  a complex Banach space. Then*

$$\begin{aligned} \int_{\mathbb{B}_n} \langle f(z), g(z) \rangle_{X, X^*} d\nu_\alpha(z) &= \int_{\mathbb{B}_n} \langle R^{\alpha, t} f(z), g(z) \rangle_{X, X^*} d\nu_{\alpha+t}(z) \\ &= \int_{\mathbb{B}_n} \langle f(z), R^{\alpha, t} g(z) \rangle_{X, X^*} d\nu_{\alpha+t}(z), \end{aligned}$$

for every  $f \in A_\alpha^1(\mathbb{B}_n, X)$  and  $g \in H^\infty(\mathbb{B}_n, X^*)$ .

**Corollary 30.** *Suppose  $t > 0$  and  $1 < p < \infty$ . If  $b \in A_\alpha^{p'}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , where  $p'$  is the conjugate exponent of  $p$ , then the following equality holds*

$$\int_{\mathbb{B}_n} \langle b(z) \overline{f(z)}, g(z) \rangle_{Y, Y^*} d\nu_\alpha(z) = \int_{\mathbb{B}_n} \langle R^{\alpha, t} b(z) \overline{f(z)}, g(z) \rangle_{Y, Y^*} d\nu_{\alpha+t}(z)$$

for  $f \in H^\infty(\mathbb{B}_n, X)$  and  $g \in H^\infty(\mathbb{B}_n, Y^*)$ .

In the sequel, we will need to interchange the position of the summation symbol and the integral symbol in a particular situation. That is why we introduce this lemma.

**Lemma 31.** *Assume  $1 < t < \infty$ . Let  $b(z) = \sum_{\beta \in \mathbb{N}^n} \hat{b}(\beta) z^\beta \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ . Then*

$$\int_{\mathbb{B}_n} \langle b(z) \left( \overline{f(z)} \right), y_0^* \rangle_{Y, Y^*} d\nu_\alpha(z) = \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} z^\beta \langle \hat{b}(\beta) \left( \overline{f(z)} \right), y_0^* \rangle_{Y, Y^*} d\nu_\alpha(z),$$

for every  $f \in H^\infty(\mathbb{B}_n, X)$  and  $y_0^* \in Y^*$  with  $\|y_0^*\|_{Y^*} = 1$ .

*Proof.* Since  $b(z) = \sum_{\beta \in \mathbb{N}^n} \hat{b}(\beta) z^\beta \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , we have that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\overline{X}, Y)}^t d\nu_\alpha(z) = 0.$$

We have

$$\begin{aligned} & \left| \int_{\mathbb{B}_n} \left\langle \left( b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right) \overline{f(z)}, y_0^* \right\rangle_{Y, Y^*} d\nu_\alpha(z) \right| \leq \\ & \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\overline{X}, Y)} \| \overline{f(z)} \|_{\overline{X}} \| y_0^* \|_{Y^*} d\nu_\alpha(z) = \\ & \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\overline{X}, Y)} \| f(z) \|_X d\nu_\alpha(z) \lesssim \\ & \int_{\mathbb{B}_n} \left\| b(z) - \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) z^\beta \right\|_{\mathcal{L}(\overline{X}, Y)}^t d\nu_\alpha(z) \longrightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ . Therefore, we have that

$$\begin{aligned}
 & \int_{\mathbb{B}_n} \langle b(z) \left( \overline{f(z)} \right), y_0^* \rangle_{Y, Y^*} d\nu_\alpha(z) \\
 &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \left\langle \sum_{\beta \in \mathbb{N}^n: |\beta| \leq N} \hat{b}(\beta) z^\beta \left( \overline{f(z)} \right), y_0^* \right\rangle_{Y, Y^*} d\nu_\alpha(z) \\
 &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \sum_{\beta \in \mathbb{N}^n: |\beta| \leq N} \left\langle \hat{b}(\beta) z^\beta \left( \overline{f(z)} \right), y_0^* \right\rangle_{Y, Y^*} d\nu_\alpha(z) \\
 &= \lim_{N \rightarrow \infty} \sum_{\beta \in \mathbb{N}^n: |\beta| \leq N} \int_{\mathbb{B}_n} \left\langle \hat{b}(\beta) z^\beta \left( \overline{f(z)} \right), y_0^* \right\rangle_{Y, Y^*} d\nu_\alpha(z) \\
 &= \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \left\langle \hat{b}(\beta) z^\beta \left( \overline{f(z)} \right), y_0^* \right\rangle_{Y, Y^*} d\nu_\alpha(z).
 \end{aligned}$$

□

In the following lemma, we compute the little Hankel operator when the operator-valued symbol is a monomial.

**Lemma 32.** *Suppose  $1 < p < \infty$  and  $\gamma \in \mathbb{N}^n$ . If  $a_\gamma \in \mathcal{L}(\overline{X}, Y)$ , then for every  $f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta \in A_\alpha^p(\mathbb{B}_n, X)$ , we have*

$$h_{a_\gamma, z^\gamma} f(z) = \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta}.$$

*Proof.* Since

$$f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta z^\beta \in A_\alpha^p(\mathbb{B}_n, X),$$

and  $p > 1$  by using [16, Corollary 4], it follows that

$$\int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta z^\beta \right\|_X^p d\nu_\alpha(z) \rightarrow 0 \text{ as } N \rightarrow \infty. \tag{6.1}$$

Firstly, let us prove that

$$\int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) = \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \frac{a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \tag{6.2}$$

Let  $N \in \mathbb{N}$ . We have that

$$\begin{aligned}
 & \int_{\mathbb{B}_n} \left\| \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta} - \sum_{|\beta| \leq N} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\|_Y d\nu_\alpha(w) \\
 &= \int_{\mathbb{B}_n} \left\| \frac{\sum_{|\beta| \geq N+1} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\|_Y d\nu_\alpha(w) \\
 &= \int_{\mathbb{B}_n} \left\| \frac{a_\gamma \left( \sum_{|\beta| \geq N+1} (\overline{c_\beta}) \overline{w^\beta} \right)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} \right\|_Y d\nu_\alpha(w)
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{X}, Y)}}{(1-|z|)^{n+1+\alpha}} \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_X d\nu_\alpha(w) \\
&\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{X}, Y)}}{(1-|z|)^{n+1+\alpha}} \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_X d\nu_\alpha(w) \\
&\leq \frac{\|a_\gamma\|_{\mathcal{L}(\overline{X}, Y)}}{(1-|z|)^{n+1+\alpha}} \left( \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_X^p d\nu_\alpha(w) \right)^{1/p}.
\end{aligned}$$

Therefore

$$\left\| \int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta} - \sum_{|\beta| \leq N} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right\|_Y$$

is less than or equal to

$$\frac{\|a_\gamma\|_{\mathcal{L}(\overline{X}, Y)}}{(1-|z|)^{n+1+\alpha}} \left( \int_{\mathbb{B}_n} \left\| \sum_{|\beta| \geq N+1} c_\beta w^\beta \right\|_X^p d\nu_\alpha(w) \right)^{1/p}. \quad (6.3)$$

By using (6.1) and (6.3), it follows that

$$\left\| \int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta} - \sum_{|\beta| \leq N} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \right\|_Y \rightarrow 0$$

as  $N \rightarrow \infty$ , and so

$$\begin{aligned}
\int_{\mathbb{B}_n} \frac{\sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \frac{\sum_{|\beta| \leq N} a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\
&= \lim_{N \rightarrow \infty} \sum_{|\beta| \leq N} \int_{\mathbb{B}_n} \frac{a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \frac{a_\gamma(\overline{c_\beta}) \overline{w^\beta}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w),
\end{aligned}$$

which is the desired result. Secondly, let us prove that

$$\begin{aligned}
\int_{\mathbb{B}_n} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) &= \\
\sum_{k=0}^{\infty} \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w). & \quad (6.4)
\end{aligned}$$

Let  $N \in \mathbb{N}$ . We have

$$\begin{aligned} \left| \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k \right| &\leq \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} |z|^k \\ &\leq \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} |z|^k \\ &= \frac{1}{(1-|z|)^{n+1+\alpha}}. \end{aligned}$$

Since  $\int_{\mathbb{B}_n} \frac{1}{(1-|z|)^{n+1+\alpha}} d\nu_{\alpha}(w) = \frac{1}{(1-|z|)^{n+1+\alpha}}$ , by the dominated convergence theorem, we have that

$$\begin{aligned} &\sum_{k=0}^{\infty} \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_{\alpha}(w) \\ &= \lim_{N \rightarrow \infty} \sum_{k=0}^N \int_{\mathbb{B}_n} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_{\alpha}(w) \\ &= \lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_{\alpha}(w) \\ &= \int_{\mathbb{B}_n} \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_{\alpha}(w) \\ &= \int_{\mathbb{B}_n} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_{\alpha}(w). \end{aligned}$$

We are now ready to prove our lemma. For  $f(z) = \sum_{\beta \in \mathbb{N}^n} c_{\beta} z^{\beta} \in A_{\alpha}^p(\mathbb{B}_n, X)$ , by using the following multi-nomial formula [14, (1.1)] and the following formula [14, (1.23)] respectively

$$\langle z, w \rangle^k = \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w^m}, \quad \int_{\mathbb{B}_n} |z^m|^2 d\nu_{\alpha}(z) = \frac{m! \Gamma(n+\alpha+1)}{\Gamma(n+|m|+\alpha+1)},$$

we get that, using (6.2) and (6.4)

$$\begin{aligned} &h_{a_{\gamma} z^{\gamma}} f(z) \\ &= \int_{\mathbb{B}_n} \frac{a_{\gamma} w^{\gamma} \left( \overline{\sum_{\beta \in \mathbb{N}^n} c_{\beta} w^{\beta}} \right)}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha}(w) \\ &= \int_{\mathbb{B}_n} \frac{w^{\gamma} \sum_{\beta \in \mathbb{N}^n} a_{\gamma}(\overline{c_{\beta}}) \overline{w^{\beta}}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha}(w) \\ &= \sum_{\beta \in \mathbb{N}^n} \int_{\mathbb{B}_n} \frac{w^{\gamma} a_{\gamma}(\overline{c_{\beta}}) \overline{w^{\beta}}}{(1-\langle z, w \rangle)^{n+1+\alpha}} d\nu_{\alpha}(w) \\ &= \sum_{\beta \in \mathbb{N}^n} a_{\gamma}(\overline{c_{\beta}}) \int_{\mathbb{B}_n} w^{\gamma} \overline{w^{\beta}} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_{\alpha}(w) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \langle z, w \rangle^k d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} \sum_{|m|=k} \frac{k!}{m!} z^m \overline{w^m} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)k!} \sum_{|m|=k} \frac{k!}{m!} \int_{\mathbb{B}_n} w^\gamma \overline{w^\beta} z^m \overline{w^m} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{k=0}^{\infty} \sum_{|m|=k} \frac{\Gamma(n+1+\alpha+k)}{\Gamma(n+1+\alpha)m!} \int_{\mathbb{B}_n} w^\gamma z^m \overline{w^{m+\beta}} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n} a_\gamma(\overline{c_\beta}) \sum_{m \in \mathbb{N}^n} \frac{\Gamma(n+1+\alpha+|m|)}{\Gamma(n+1+\alpha)m!} z^m \int_{\mathbb{B}_n} w^\gamma \overline{w^{\beta+m}} d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\Gamma(n+1+\alpha+|\gamma-\beta|)}{\Gamma(n+1+\alpha)(\gamma-\beta)!} z^{\gamma-\beta} \int_{\mathbb{B}_n} |z^\gamma|^2 d\nu_\alpha(w) \\
&= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\Gamma(n+1+\alpha+|\gamma-\beta|)}{\Gamma(n+1+\alpha)(\gamma-\beta)!} \frac{\gamma! \Gamma(n+1+\alpha)}{\Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta} \\
&= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta}) \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} z^{\gamma-\beta}. \quad \square
\end{aligned}$$

The goal of the following lemma is to prove that the linear span of the vector-valued Bergman kernel  $\frac{x^*}{(1 - \langle w, z \rangle)^{n+1+\alpha}}$ , where  $x^* \in X^*$  and  $z, w \in \mathbb{B}_n$  form a dense subspace in the vector-valued Bergman space  $A_\alpha^{p'}(\mathbb{B}_n, X^*)$ , with  $1 < p < \infty$  and  $p'$  is the conjugate exponent of  $p$ .

**Lemma 33.** *Suppose that  $1 < p < \infty$ . For each  $x^* \in X^*$  and  $z \in \mathbb{B}_n$ , let*

$$e_{z, x^*}(w) = \frac{x^*}{(1 - \langle w, z \rangle)^{n+1+\alpha}}; \quad w \in \mathbb{B}_n.$$

*Then  $e_{z, x^*} \in A_\alpha^{p'}(\mathbb{B}_n, X^*)$  and the subspace generated by  $e_{z, x^*}$  is dense in  $A_\alpha^{p'}(\mathbb{B}_n, X^*)$ .*

*Proof.* Let  $\phi \in A_\alpha^p(\mathbb{B}_n, X)$  such that  $\langle \phi, e_{z, x^*} \rangle_{\alpha, X} = 0$  for all  $z \in \mathbb{B}_n$  and  $x^* \in X^*$ . Let  $f^* \in A_\alpha^{p'}(\mathbb{B}_n, X^*)$ . According to the Hahn-Banach theorem, it suffices to prove that  $\langle \phi, f^* \rangle_{\alpha, X} = 0$ . For all  $z \in \mathbb{B}_n$  and  $x^* \in X^*$ , using Lemma 1 and the reproducing kernel formula, it follows that

$$\begin{aligned}
0 &= \langle \phi, e_{z, x^*} \rangle_{\alpha, X} \\
&= \int_{\mathbb{B}_n} \langle \phi(w), e_{z, x^*}(w) \rangle_{X, X^*} d\nu_\alpha(w) \\
&= \int_{\mathbb{B}_n} \langle \phi(w), \frac{x^*}{(1 - \langle w, z \rangle)^{n+1+\alpha}} \rangle_{X, X^*} d\nu_\alpha(w)
\end{aligned}$$

$$\begin{aligned} &= \int_{\mathbb{B}_n} \left\langle \frac{\phi(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, x^* \right\rangle_{X, X^*} d\nu_\alpha(w) \\ &= \langle \phi(z), x^* \rangle_{X, X^*}. \end{aligned}$$

Therefore, for all  $x^* \in X^*$ , we have

$$\langle \phi(z), x^* \rangle_{X, X^*} = 0.$$

Thus  $\phi(z) = 0$  for every  $z \in \mathbb{B}_n$ . It follows that for each  $f^* \in A_\alpha^{p'}(\mathbb{B}_n, X^*)$ , we have that

$$\langle \phi, f^* \rangle_{\alpha, X} = \int_{\mathbb{B}_n} \langle \phi(z), f^*(z) \rangle_{X, X^*} d\nu_\alpha(z) = 0. \quad \square$$

In the proof of the following lemma, we use the fact that when  $X$  is a reflexive complex Banach space and  $1 < p < \infty$ , the dual of the vector-valued Bergman space  $A_\alpha^{p'}(\mathbb{B}_n, X^*)$  can be identified with  $A_\alpha^p(\mathbb{B}_n, X)$ , where  $p'$  is the conjugate exponent of  $p$ .

**Lemma 34.** *Suppose that  $1 < p < \infty$ , and  $X$  is a reflexive complex Banach space. Let  $\{f_j\} \subset A_\alpha^p(\mathbb{B}_n, X)$  such that  $f_j \rightarrow 0$  weakly in  $A_\alpha^p(\mathbb{B}_n, X)$  as  $j \rightarrow \infty$ . Then for each  $\beta \in \mathbb{N}^n$ , we have that  $\partial^\beta f_j(0) \rightarrow 0$  weakly in  $X$  as  $j \rightarrow \infty$ , where  $\partial^\beta = \frac{\partial^{|\beta|}}{\partial z^\beta}$ .*

*Proof.* Since for each  $j \in \mathbb{N}$ ,  $f_j \in A_\alpha^p(\mathbb{B}_n, X)$ , using the reproducing kernel formula we have that

$$f_j(z) = \int_{\mathbb{B}_n} \frac{f_j(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w), \quad z \in \mathbb{B}_n.$$

Differentiating both sides of the previous relation with respect to  $z$ , we obtain

$$\partial^\beta f_j(z) = C(n, \alpha, |\beta|) \int_{\mathbb{B}_n} \frac{f_j(w) \bar{w}^\beta}{(1 - \langle z, w \rangle)^{n+1+\alpha+|\beta|}} d\nu_\alpha(w).$$

Therefore, we have

$$\partial^\beta f_j(0) = C(n, \alpha, |\beta|) \int_{\mathbb{B}_n} f_j(w) \bar{w}^\beta d\nu_\alpha(w).$$

Now, let  $x^* \in X^*$  and let us show that  $\langle \partial^\beta f_j(0), x^* \rangle_{X, X^*} \rightarrow 0$  as  $j \rightarrow \infty$ . But we have that

$$\begin{aligned} \langle \partial^\beta f_j(0), x^* \rangle_{X, X^*} &= C(n, \alpha, |\beta|) \left\langle \int_{\mathbb{B}_n} f_j(w) \bar{w}^\beta d\nu_\alpha(w), x^* \right\rangle_{X, X^*} \\ &= \int_{\mathbb{B}_n} \langle f_j(w), x^* w^\beta \rangle_{X, X^*} d\nu_\alpha(w) \\ &= \langle f_j, g \rangle_{\alpha, X} \rightarrow 0 \text{ as } j \rightarrow \infty, \end{aligned}$$

with  $g(z) = x^* z^\beta \in A_\alpha^{p'}(\mathbb{B}_n, X^*)$ . Thus,  $\langle \partial^\beta f_j(0), x^* \rangle_{X, X^*} \rightarrow 0$  as  $j \rightarrow \infty$ .  $\square$

We recall that the symbol  $b$  used in the following lemma satisfies (1.4) and (1.6).

**Lemma 35.** *Suppose that  $X$  is a reflexive complex Banach space and  $k$  is a nonnegative integer. If the holomorphic mapping  $z \mapsto b(z)$  maps  $\mathbb{B}_n$  into  $\mathcal{K}(\overline{X}, Y)$ , then the holomorphic mapping  $z \mapsto R^{\alpha, k}b(z)$  also maps  $\mathbb{B}_n$  into  $\mathcal{K}(\overline{X}, Y)$ .*

*Proof.* Let  $z \in \mathbb{B}_n$ . Let  $\{f_j\}$  a sequence of elements of  $X$  which converges weakly to 0 in  $X$  as  $j$  tends to infinity. Let us prove that  $\lim_{j \rightarrow \infty} \|R^{\alpha, k}b(z)\overline{f_j}\|_Y = 0$ . We know that the sequence  $\{f_j\}$  is strongly bounded in  $X$ . Let  $j \in \mathbb{N}$ , by using (1.4) for  $z = 0$ , we get that the function  $z \mapsto b(z)\overline{f_j} \in A^1_\alpha(\mathbb{B}_n, Y)$ . By the reproducing kernel formula, it follows that

$$b(z)\overline{f_j} = \int_{\mathbb{B}_n} \frac{b(w)\overline{f_j}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w). \quad (6.5)$$

Applying the partial differential operator  $R^{\alpha, k}$  to (6.5), we have

$$R^{\alpha, k}b(z)\overline{f_j} = \int_{\mathbb{B}_n} \frac{b(w)\overline{f_j}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w).$$

We also have

$$\begin{aligned} \frac{\|b(w)\overline{f_j}\|_Y}{|1 - \langle z, w \rangle|^{n+1+\alpha+k}} &\leq \frac{\|b(w)\|_{\mathcal{L}(\overline{X}, Y)} \|f_j\|_X}{(1 - |z|)^{n+1+\alpha+k}} \\ &\leq \frac{C(n+1+\alpha)}{(1 - |z|)^{n+1+\alpha+k}} \|b(w)\|_{\mathcal{L}(\overline{X}, Y)}, \end{aligned}$$

and

$$\int_{\mathbb{B}_n} \frac{C(n+1+\alpha)}{(1 - |z|)^{n+1+\alpha+k}} \|b(w)\|_{\mathcal{L}(\overline{X}, Y)} d\nu_\alpha(w) < \infty.$$

Therefore, by applying the dominated convergence theorem, we have that

$$\begin{aligned} \limsup_{j \rightarrow \infty} \|R^{\alpha, k}b(z)\overline{f_j}\|_Y &\leq \limsup_{j \rightarrow \infty} \int_{\mathbb{B}_n} \frac{\|b(w)\overline{f_j}\|_Y}{|1 - \langle z, w \rangle|^{n+1+\alpha+k}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{\lim_{j \rightarrow \infty} \|b(w)\overline{f_j}\|_Y}{|1 - \langle z, w \rangle|^{n+1+\alpha+k}} d\nu_\alpha(w) = 0. \end{aligned}$$

Thus for each  $z \in \mathbb{B}_n$

$$\lim_{j \rightarrow \infty} \|R^{\alpha, k}b(z)\overline{f_j}\|_Y = 0. \quad \square$$

The following result will be also important in the sequel.

**Lemma 36.** *Suppose  $\beta_0 \in \mathbb{N}^n$ ,  $\{f_j\}$  a sequence of elements of  $X$  which converges weakly to 0 as  $j$  tends to infinity. For  $z \in \mathbb{B}_n$ , let  $x_j(z) = z^{\beta_0} f_j$ . Then  $\{x_j\} \subset A^p_\alpha(\mathbb{B}_n, X)$  and  $\{x_j\}$  converges weakly to 0 in  $A^p_\alpha(\mathbb{B}_n, X)$ .*

*Proof.* Let  $j \in \mathbb{N}$ . Since  $f_j \rightarrow 0$  weakly in  $X$  as  $j \rightarrow \infty$ , it follows that  $\{f_j\}$  is strongly bounded in  $X$  (see [9]). Let  $\beta_0 \in \mathbb{N}^n$  and  $x_j(z) = z^{\beta_0} f_j$ . It is clear that  $\{x_j\} \subset A^p_\alpha(\mathbb{B}_n, X)$ . For every  $g \in A^{p'}_\alpha(\mathbb{B}_n, X^*)$ , we have

$$\langle x_j, g \rangle_{\alpha, X} = \int_{\mathbb{B}_n} \langle x_j(z), g(z) \rangle_{X, X^*} d\nu_\alpha(z)$$

$$\begin{aligned} &= \int_{\mathbb{B}_n} \langle z^{\beta_0} f_j, g(z) \rangle_{X, X^*} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} z^{\beta_0} \langle f_j, g(z) \rangle_{X, X^*} d\nu_\alpha(z). \end{aligned}$$

Since

$$\begin{aligned} |z^{\beta_0} \langle f_j, g(z) \rangle_{X, X^*}| &\leq |z^{\beta_0} \langle f_j, g(z) \rangle_{X, X^*}| \\ &\leq \|f_j\|_X \|g(z)\|_{X^*} \\ &\leq C \|g(z)\|_{X^*}, \end{aligned}$$

and

$$\int_{\mathbb{B}_n} \|g(z)\|_{X^*} d\nu_\alpha(z) \leq \left( \int_{\mathbb{B}_n} \|g(z)\|_{X^*}^{p'} d\nu_\alpha(z) \right)^{1/p'} < \infty.$$

By using the dominated convergence theorem and the assumption, it follows that

$$\limsup_{j \rightarrow \infty} \langle x_j, g \rangle_{\alpha, X} = \int_{\mathbb{B}_n} z^{\beta_0} \lim_{j \rightarrow \infty} \langle f_j, g(z) \rangle_{X, X^*} d\nu_\alpha(z) = 0. \quad \square$$

### 6.2. Boundedness of the Little Hankel Operator with Operator-Valued Symbol on Vector-Valued Bergman Spaces

The principal result here is that, the little Hankel operator with operator-valued symbol  $h_b$  is a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  to  $A_\alpha^q(\mathbb{B}_n, Y)$  with  $1 < p \leq q < \infty$  if and only if the symbol  $b$  belongs to the generalized vector-valued Lipschitz space  $\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , where

$$\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right).$$

The result obtained generalize the Oliver’s result [11, Theorem 4.2.2]. In the following lemma, we first prove that the definition of the generalized vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, X)$ , with  $\gamma \geq 0$  is independent of the integer  $k$  used.

**Lemma 37.** *Let  $f \in \mathcal{H}(\mathbb{B}_n, X)$ . The following conditions are equivalent:*

(a) *There exists a nonnegative integer  $k > \gamma$  such that*

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_X < \infty.$$

(b) *For every nonnegative integer  $k > \gamma$  we have*

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_X < \infty.$$

*Proof.* It is clear that (b)  $\Rightarrow$  (a). So to complete the proof, we will prove that (a)  $\Rightarrow$  (b). Suppose that there exists an integer  $k > \gamma$  such that

$$c := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha, k} f(z)\|_X < \infty.$$

We want to prove that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha, k+1} f(z)\|_X < \infty.$$

Since  $c < \infty$ , then  $f \in A_\alpha^1(\mathbb{B}_n, X)$ . Indeed, by [11, Theorem 3.1.2], we have that

$$\begin{aligned} \|f\|_{1,\alpha,X} &\simeq \int_{\mathbb{B}_n} (1 - |z|^2)^k \|R^{\alpha,k} f(z)\|_X d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} [(1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X] (1 - |z|^2)^\gamma d\nu_\alpha(z) \\ &\lesssim c \int_{\mathbb{B}_n} (1 - |z|^2)^{\alpha+\gamma} d\nu(z) \\ &< \infty. \end{aligned}$$

By using Proposition 26, we have that

$$R^{\alpha,k+1} f(z) = \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+k+1}} d\nu_\alpha(w).$$

Applying Lemma 28, it follows that

$$R^{\alpha,k+1} f(z) = \int_{\mathbb{B}_n} \frac{R^{\alpha,k} f(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+k+1}} d\nu_{\alpha+k}(w).$$

Thus,

$$\begin{aligned} \|R^{\alpha,k+1} f(z)\|_X &\lesssim \int_{\mathbb{B}_n} \frac{[(1 - |w|^2)^{k-\gamma} \|R^{\alpha,k} f(w)\|_X] (1 - |w|^2)^{\alpha+\gamma}}{|1 - \langle z, w \rangle|^{n+1+\alpha+\gamma+(k+1-\gamma)}} d\nu(w) \\ &\lesssim \frac{c}{(1 - |z|^2)^{k+1-\gamma}}. \end{aligned}$$

Therefore, we have that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha,k+1} f(z)\|_X \lesssim c < \infty.$$

Also, if  $k$  is a nonnegative integer with  $k > \gamma$  such that

$$c' := \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k+1-\gamma} \|R^{\alpha,k+1} f(z)\|_X < \infty,$$

then

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X < \infty.$$

Applying Proposition 26 and Lemma 28 we have that

$$R^{\alpha,k} f(z) = \int_{\mathbb{B}_n} \frac{f(w) d\nu_\alpha(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} = \int_{\mathbb{B}_n} \frac{R^{\alpha,k+1} f(w) d\nu_{\alpha+k+1}(w)}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}},$$

where  $z \in \mathbb{B}_n$ . By using Theorem 18, it follows that

$$\begin{aligned} \|R^{\alpha,k} f(z)\|_X &\lesssim \int_{\mathbb{B}_n} \frac{[(1 - |w|^2)^{k+1-\gamma} \|R^{\alpha,k+1} f(w)\|_X] (1 - |w|^2)^{\alpha+\gamma} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+k}} \\ &= c' \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^{\alpha+\gamma} d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+\alpha+\gamma+(k-\gamma)}} \\ &\lesssim \frac{c'}{(1 - |z|^2)^{k-\gamma}}. \end{aligned}$$

Since  $z \in \mathbb{B}_n$  is arbitrary, we obtain that

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X \lesssim c' < \infty. \quad \square$$

**Proposition 38.** *Let  $\gamma \geq 0$  and  $f \in \Lambda_\gamma(\mathbb{B}_n, X)$ . The following conditions are equivalent:*

- (i)  $f \in \Lambda_{\gamma,0}(\mathbb{B}_n, X)$ .
- (ii)  $\lim_{s \rightarrow 1^-} \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, X)} = 0$ , where  $f_s$  is the dilation function defined for  $z \in \mathbb{B}_n$  by  $f_s(z) := f(sz)$ .
- (iii)  $f$  belongs to the closure of  $\mathcal{P}(\mathbb{B}_n, X)$ , where  $\mathcal{P}(\mathbb{B}_n, X)$  is the space of vector-valued holomorphic polynomials.

*Proof.* (i)  $\Rightarrow$  (ii). Suppose that  $\frac{1}{2} < r < s < 1$ , and let  $f_s(z) = f(sz)$ ,  $z \in \mathbb{B}_n$ . By the definition, we have:

$$\begin{aligned} \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, X)} &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}(f - f_s)(z)\|_X \\ &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}f_s)(z)\|_X \\ &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_X \\ &= \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z) \\ &\quad + \chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_X \\ &\leq \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_X \\ &\quad + \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_X, \end{aligned}$$

where  $\chi_r$  is the characteristic function of the set  $\{|z| \leq r\}$ . We first have the following estimate:

$$\begin{aligned} &\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_X \\ &\leq \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_X \\ &\quad + \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z) - \chi_r(z)(R^{\alpha,k}f)(z)\|_X \\ &= \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z)\|_X \\ &\leq \sup_{r^2 < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k}f)(z)\|_X. \end{aligned}$$

We secondly have the following estimate:

$$\begin{aligned} &\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_X \\ &\leq \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_X \\ &\quad + \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|\chi_r(z)(R^{\alpha,k}f)(z) - (R^{\alpha,k}f)(sz)\|_X \end{aligned}$$

$$\begin{aligned}
&= \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_X \\
&+ \sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(sz)\|_X.
\end{aligned}$$

Using the change of variables  $w = sz$ , we then obtain

$$\begin{aligned}
&\sup_{r < |z| < 1} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(sz)\|_X \\
&= \sup_{rs < |w| < s} \left(1 - \frac{|w|^2}{s^2}\right)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_X \\
&= \sup_{rs < |w| < s} \frac{1}{s^{2(k-\gamma)}} (s^2 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_X \\
&\leq 2^{2(k-\gamma)} \sup_{r^2 < |w| < 1} (1 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_X.
\end{aligned}$$

It follows by using the assumption that

$$\begin{aligned}
\|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, X)} &\leq C_\gamma \sup_{r^2 < |w| < 1} (1 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_X \\
&+ \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} f)(sz)\|_X,
\end{aligned}$$

with  $C_\gamma = 1 + 2^{2(k-\gamma)}$ . Since  $(R^{\alpha,k} f)(sz) \rightarrow (R^{\alpha,k} f)(z)$  in  $X$  uniformly on the compact set  $\{|z| \leq r\}$  as  $s \rightarrow 1^-$ , we have

$$\lim_{s \rightarrow 1^-} \sup_{|z| \leq r} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - R^{\alpha,k} f(sz)\|_X = 0.$$

It follows that

$$\lim_{s \rightarrow 1^-} \|f - f_s\|_{\Lambda_\gamma(\mathbb{B}_n, X)} \leq C_\gamma \limsup_{|w| \rightarrow 1^-} (1 - |w|^2)^{k-\gamma} \|(R^{\alpha,k} f)(w)\|_X = 0.$$

(ii)  $\Rightarrow$  (iii). Given  $\epsilon > 0$ , by the assumption, there exists  $s_0 \in (0, 1)$  such that

$$\|f - f_{s_0}\|_{\Lambda_\gamma(\mathbb{B}_n, X)} < \epsilon. \tag{6.6}$$

Further note that  $f_{s_0} \in \mathcal{H}(\frac{1}{s_0}\mathbb{B}_n, X)$  and  $1 < \frac{2}{1+s_0} < \frac{1}{s_0}$ . From this, and by using Taylor's formula, it follows that for each  $m \in \mathbb{N}$ , there exists a  $X$ -valued polynomial  $p_m$  such that

$$\lim_{m \rightarrow \infty} \sup_{z \in \frac{2}{1+s_0}\mathbb{B}_n} \|f_{s_0}(z) - p_m(z)\|_X = 0.$$

Therefore, there exists  $m_0 \in \mathbb{N}$  such that

$$\sup_{z \in \frac{2}{1+s_0}\mathbb{B}_n} \|f_{s_0}(z) - p_m(z)\|_X < \epsilon, \tag{6.7}$$

for  $m \geq m_0$ . By the Cauchy's inequality, there exists a constant  $c_{s_0} > 0$  such that for each  $i = 1, \dots, n$  we have

$$\sup_{z \in \mathbb{B}_n} \left\| \frac{\partial f_{s_0}}{\partial z_i} - \frac{\partial p_m}{\partial z_i} \right\|_X \leq c_{s_0} \sup_{z \in \frac{2}{1+s_0}\mathbb{B}_n} \|f_{s_0}(z) - p_m(z)\|_X. \tag{6.8}$$

Suppose  $k$  is a nonnegative integer with  $k > \gamma$ . By using (6.8) and Theorem 27, there is a constant  $c = c(s_0, n, \alpha, k)$  such that

$$\sup_{z \in \mathbb{B}_n} \|(R^{\alpha,k} f_{s_0})(z) - (R^{\alpha,k} p_{m_0})(z)\|_X \leq c \sup_{z \in \frac{2}{1+s_0} \overline{\mathbb{B}_n}} \|f_{s_0}(z) - P_{m_0}(z)\|_X. \tag{6.9}$$

It follows by (6.9) and (6.7) that

$$\begin{aligned} & \sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k}(f_{s_0} - p_{m_0})(z)\|_X \\ & \leq \sup_{z \in \mathbb{B}_n} \|(R^{\alpha,k} f_{s_0})(z) - (R^{\alpha,k} p_{m_0})(z)\|_X \\ & \leq c \sup_{z \in \frac{2}{1+s_0} \overline{\mathbb{B}_n}} \|f_{s_0}(z) - p_{m_0}(z)\|_X \\ & < c\epsilon. \end{aligned}$$

Thus

$$\|f_{s_0} - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n, X)} < c\epsilon. \tag{6.10}$$

Using (6.6) and (6.10), it follows that

$$\begin{aligned} \|f - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n, X)} & \leq \|f - f_{s_0}\|_{\Lambda_\gamma(\mathbb{B}_n, X)} + \|f_{s_0} - p_{m_0}\|_{\Lambda_\gamma(\mathbb{B}_n, X)} \\ & < \epsilon + c\epsilon = (1 + c)\epsilon. \end{aligned}$$

(iii)  $\Rightarrow$  (i). Let  $f$  in the closure of the set of vector-valued polynomial  $\mathcal{P}(\mathbb{B}_n, X)$ , in  $\Lambda_\gamma(\mathbb{B}_n, X)$ . There exists a sequence of vector-valued polynomials  $\{p_m\}$  in  $\mathcal{P}(\mathbb{B}_n, X)$  such that

$$\lim_{m \rightarrow \infty} \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, X)} = 0. \tag{6.11}$$

Let us prove that for each  $k > \gamma$ ,

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z)\|_X = 0.$$

Let  $k > \gamma$ . We have that

$$\begin{aligned} \|(R^{\alpha,k} f)(z)\|_X & \leq \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} p_m)(z)\|_X + \|(R^{\alpha,k} p_m)(z)\|_X \\ & \leq \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} p_m)(z)\|_X + \|R^{\alpha,k} p_m\|_{\infty, X}, \end{aligned}$$

where  $\|R^{\alpha,k} p_m\|_{\infty, X} = \max_{z \in \mathbb{B}_n} \|(R^{\alpha,k} p_m)(z)\|_X$ . It follows that for each  $m \in \mathbb{N}$ , we have

$$\begin{aligned} & (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z)\|_X \\ & \leq (1 - |z|^2)^{k-\gamma} \|(R^{\alpha,k} f)(z) - (R^{\alpha,k} p_m)(z)\|_X + (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} p_m\|_{\infty, X} \\ & \leq \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, X)} + (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} p_m\|_{\infty, X}. \end{aligned}$$

Letting  $|z| \rightarrow 1^-$ , we obtain that

$$\limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X \leq \|f - p_m\|_{\Lambda_\gamma(\mathbb{B}_n, X)},$$

for each  $m \in \mathbb{N}$ . Now, letting  $m \rightarrow \infty$  on both sides of the previous inequality, it follows by (6.11) that

$$\limsup_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma} \|R^{\alpha,k} f(z)\|_X = 0. \quad \square$$

*Remark 39.* One of the consequences of the previous result is that, given  $\gamma \geq 0$ , the generalized little vector-valued Lipschitz space  $\Lambda_{\gamma,0}(\mathbb{B}_n, X)$  is a closed subspace of the generalized vector-valued Lipschitz space  $\Lambda_\gamma(\mathbb{B}_n, X)$ .

From now on, we choose  $\gamma_0 = (n+1+\alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ , with  $1 < p \leq q < \infty$ , and we consider the generalized vector-valued Lipschitz space  $\Lambda_{\gamma_0}(\mathbb{B}_n, X)$ .

**Corollary 40.** *Suppose  $1 \leq t < \infty$ . Then  $\Lambda_{\gamma_0}(\mathbb{B}_n, X) \subset A_\alpha^t(\mathbb{B}_n, X)$ .*

*Proof.* Let  $k > \gamma_0$ . Applying [11, Theorem 3.1.2], for  $f \in \Lambda_{\gamma_0}(\mathbb{B}_n, X)$ , we have that

$$\begin{aligned} \|f\|_{t,\alpha,X}^t &\simeq \int_{\mathbb{B}_n} [(1-|z|^2)^k \|R^{\alpha,k} f(z)\|_X]^t d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} [(1-|z|^2)^{k-\gamma_0} \|R^{\alpha,k} f(z)\|_X]^t (1-|z|^2)^{\gamma_0 t} d\nu_\alpha(z) \\ &\lesssim \|f\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, X)} \int_{\mathbb{B}_n} (1-|z|^2)^{\alpha+\gamma_0 t} d\nu(z) < \infty. \quad \square \end{aligned}$$

In what follows, we assume that  $X, Y$  are reflexives complex Banach spaces. We first introduce the following proposition which will be used in the proof of Theorem 8.

**Proposition 41.** *Suppose  $1 < p \leq q < \infty$ ,  $0 \leq r < 1$  and  $\gamma \in \mathbb{N}^n$ . If  $a_\gamma \in \mathcal{K}(\overline{X}, Y)$ , then the little Hankel operator  $h_{g_r^\gamma} : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$  is a compact operator, where  $g_r^\gamma(z) = a_\gamma(rz)^\gamma$  for every  $z \in \mathbb{B}_n$ .*

*Proof.* Let  $\{f_j\}$  be a sequence in  $A_\alpha^p(\mathbb{B}_n, X)$  such that  $f_j \rightarrow 0$  weakly in  $A_\alpha^p(\mathbb{B}_n, X)$  as  $j$  tends to infinity. We want to prove that  $\lim_{j \rightarrow \infty} \|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} = 0$ . Let the Taylor expansion of  $f_j$  given by  $f_j(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta^j z^\beta \in A_\alpha^p(\mathbb{B}_n, X)$ .

Since  $f_j \rightarrow 0$  weakly in  $A_\alpha^p(\mathbb{B}_n, X)$ , applying Lemma 34, using the fact that  $c_\beta^j = \partial^\beta f_j(0)/\beta!$ , we have that for all  $\beta \in \mathbb{N}^n$ ,  $c_\beta^j \rightarrow 0$  weakly in  $X$  as  $j \rightarrow \infty$ . By Lemma 32, for every  $z \in \mathbb{B}_n$ , we have

$$h_{g_r^\gamma} f_j(z) = \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta^j}) \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} r^{|\gamma-\beta|} z^{\gamma-\beta}.$$

Therefore,

$$\begin{aligned} &\|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} \\ &= \left( \int_{\mathbb{B}_n} \left\| \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} a_\gamma(\overline{c_\beta^j}) \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} (rz)^{\gamma-\beta} \right\|_Y^q d\nu_\alpha(z) \right)^{1/q} \\ &\leq \left( \int_{\mathbb{B}_n} \left( \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|) \|a_\gamma(\overline{c_\beta^j})\|_Y (r|z|)^{|\gamma-\beta|}}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} \right)^q d\nu_\alpha(z) \right)^{1/q} \\ &\leq \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \left( \int_{\mathbb{B}_n} \left( \|a_\gamma(\overline{c_\beta^j})\|_Y \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} (r|z|)^{|\gamma-\beta|} \right)^q d\nu_\alpha(z) \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \|a_\gamma(\overline{c}_\beta^j)\|_Y \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} \left( \int_{\mathbb{B}_n} (r|z|)^{|\gamma-\beta|q} d\nu_\alpha(z) \right)^{\frac{1}{q}} \\
 &\lesssim \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} \|a_\gamma(\overline{c}_\beta^j)\|_Y,
 \end{aligned}$$

where the third line above is justified by the Minkowsky’s inequality for integrals. Thus,

$$\|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} \lesssim \sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} \|a_\gamma(\overline{c}_\beta^j)\|_Y. \tag{6.12}$$

Now, since  $c_\beta^j \rightarrow 0$  weakly in  $X$  as  $j \rightarrow \infty$ , it is clear that  $\overline{c}_\beta^j \rightarrow 0$  weakly in  $\overline{X}$  as  $j \rightarrow \infty$ . By the assumption, we know that  $a_\gamma \in \mathcal{K}(\overline{X}, Y)$ . Since  $\overline{c}_\beta^j \rightarrow 0$  weakly in  $\overline{X}$  as  $j \rightarrow \infty$ , we have that  $\|a_\gamma(\overline{c}_\beta^j)\|_Y \rightarrow 0$  as  $j \rightarrow \infty$ . It follows that

$$\begin{aligned}
 &\limsup_{j \rightarrow \infty} \|h_{g_r^\gamma} f_j\|_{q,\alpha,Y} \lesssim \\
 &\sum_{\beta \in \mathbb{N}^n, \beta \leq \gamma} \frac{\gamma! \Gamma(n+1+\alpha+|\gamma-\beta|)}{(\gamma-\beta)! \Gamma(n+1+\alpha+|\gamma|)} \lim_{j \rightarrow \infty} \|a_\gamma(\overline{c}_\beta^j)\|_Y = 0. \quad \square
 \end{aligned}$$

Let us state Oliver’s result on the boundedness of the little Hankel operator with operator-valued symbol between vector-valued Bergman spaces.

**Theorem 42.** *Let  $1 < p \leq q < \infty$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$  is a bounded operator if and only if  $b \in \mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , where*

$$\gamma = 1 + (n + 1 + \alpha) \left( \frac{1}{q} - \frac{1}{p} \right).$$

Moreover

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)} \simeq \|b\|_{\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}.$$

*Remark 43.* Suppose  $1 < p < q < \infty$ , and  $\gamma = 1 + (n + 1 + \alpha) \left( \frac{1}{q} - \frac{1}{p} \right)$ . Then  $\gamma$  is not always positive. Indeed, since  $1/q - 1/p \in (-1, 0)$ , then  $\gamma \in (-n - \alpha, 1)$ . It follows that when  $\gamma \in (-(n + \alpha), 0)$ , the vector-valued  $\gamma$ -Bloch space  $\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  is not interesting and does not make sense since the definition of the vector-valued  $\gamma$ -Bloch space introduced by Oliver only takes into account the case where  $\gamma > 0$ . In Theorem 7, we correct the problem by replacing the vector-valued  $\gamma$ -Bloch space with the generalized vector-valued Lipschitz space  $\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , where  $\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ . Since  $\gamma = 1 - \gamma_0$ , we see that when  $0 < \gamma_0 < 1$ , we have that

$$\mathcal{B}_\gamma(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)) = \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)).$$

In what follows, we give the proof of Theorem 7 which generalize the Theorem 42 and correct the mistake mentioned in Remark 43.

### 6.3. Proof of Theorem 7

Let us recall the statement of Theorem 7.

**Theorem 44.** *Suppose  $1 < p \leq q < \infty$ . The little Hankel operator  $h_b : A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)$  is a bounded operator if and only if  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , where  $\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ . Moreover,*

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)} \simeq \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}.$$

*Proof.* Let  $p'$  and  $q'$  such that  $1/p + 1/p' = 1$  and  $1/q + 1/q' = 1$ . We first assume that  $h_b$  is a bounded operator from  $A_\alpha^p(\mathbb{B}_n, X)$  to  $A_\alpha^q(\mathbb{B}_n, Y)$  with norm  $\|h_b\| = \|h_b\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)}$ . Let  $x \in X$  and  $k > (n + 1 + \alpha)/p$ . Let  $z \in \mathbb{B}_n$  and put

$$f(w) = \frac{x}{(1 - \langle w, z \rangle)^k}, \quad w \in \mathbb{B}_n.$$

Since  $k > (n + 1 + \alpha)/p$ , by Theorem 18, we have that  $f \in A_\alpha^p(\mathbb{B}_n, X)$  and

$$\|f\|_{p, \alpha, X} \lesssim \frac{\|x\|_X}{(1 - |z|^2)^{k - (n+1+\alpha)/p}}.$$

By [11, Proposition 2.1.3 ], we have that

$$\begin{aligned} h_b f(z) &= \int_{\mathbb{B}_n} \frac{b(w)(\overline{f(w)})}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{b(w)(\overline{x})}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) \\ &= R^{\alpha, k} b(z)(\overline{x}). \end{aligned}$$

It follows by Theorem 14 that

$$\begin{aligned} \|R^{\alpha, k} b(z)(\overline{x})\|_Y &= \|h_b f(z)\|_Y \\ &\leq \frac{\|h_b f\|_{q, \alpha, Y}}{(1 - |z|^2)^{(n+1+\alpha)/q}} \\ &\leq \frac{\|h_b\| \|f\|_{p, \alpha, X}}{(1 - |z|^2)^{(n+1+\alpha)/q}} \\ &\lesssim \frac{\|h_b\| \|x\|_X}{(1 - |z|^2)^{k + (n+1+\alpha)(1/q - 1/p)}} \\ &= \frac{\|h_b\| \|x\|_X}{(1 - |z|^2)^{k - \gamma_0}}. \end{aligned}$$

Since  $x \in X$  is arbitrary and  $\|x\|_X = \|\overline{x}\|_{\overline{X}}$  we get that

$$\|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{X}, Y)} \lesssim \frac{\|h_b\|}{(1 - |z|^2)^{k - \gamma_0}}.$$

Thus

$$\sup_{z \in \mathbb{B}_n} (1 - |z|^2)^{k - \gamma_0} \|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{X}, Y)} \lesssim \|h_b\|.$$

By Lemma 37 this means that  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  and  $\|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \lesssim \|h_b\|$ .

Conversely, assume that  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ . Let  $f \in A_{\alpha}^p(\mathbb{B}_n, X)$ ,  $g \in A_{\alpha}^{q'}(\mathbb{B}_n, Y^*)$  and  $k > \gamma_0$ . By Corollary 40, we have that

$$b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)) \subset A_{\alpha}^{p'}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)),$$

so by [11, Lemma 4.1.1], Corollary 30, and Lemma 37 it follows that

$$\begin{aligned} |\langle h_b f, g \rangle_{\alpha, Y}| &= \left| \int_{\mathbb{B}_n} \langle b(z) \overline{f(z)}, g(z) \rangle_Y d\nu_{\alpha}(z) \right| \\ &= \left| \int_{\mathbb{B}_n} \langle R^{\alpha, k+1} b(z) \overline{f(z)}, g(z) \rangle_Y d\nu_{\alpha+k+1}(z) \right| \\ &\lesssim \int_{\mathbb{B}_n} \|R^{\alpha, k+1} b(z)\|_{\mathcal{L}(\overline{X}, Y)} \|f(z)\|_{\overline{X}} \|g(z)\|_{Y^*} (1 - |z|^2)^{k+1+\alpha} d\nu(z) \\ &\lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \int_{\mathbb{B}_n} \|f(z)\|_X \|g(z)\|_{Y^*} (1 - |z|^2)^{\alpha+\gamma_0} d\nu(z). \end{aligned}$$

By Hölder’s inequality the last integral is less than or equal to

$$\left( \int_{\mathbb{B}_n} \|f(z)\|_X^q (1 - |z|^2)^{\alpha+q\gamma_0} d\nu(z) \right)^{1/q} \left( \int_{\mathbb{B}_n} \|g(z)\|_{Y^*}^{q'} (1 - |z|^2)^{\alpha} d\nu(z) \right)^{1/q'}.$$

For  $q = p$ , we have  $\gamma_0 = 0$  and thus

$$|\langle h_b f, g \rangle_{\alpha, Y}| \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} \|f\|_{p, \alpha, X} \|g\|_{p', \alpha, Y}.$$

For  $q - p > 0$ , using Theorem 14, we have

$$\begin{aligned} \|f(z)\|_X^q &= \|f(z)\|_X^p \|f(z)\|_X^{q-p} \leq \frac{\|f(z)\|_X^p \|f\|_{p, \alpha, X}^{q-p}}{(1 - |z|^2)^{(q-p)(n+1+\alpha)/p}} = \\ &= \frac{\|f(z)\|_X^p \|f\|_{p, \alpha, X}^{q-p}}{(1 - |z|^2)^{q\gamma_0}}. \end{aligned}$$

It follows that

$$\begin{aligned} \left( \int_{\mathbb{B}_n} \|f(z)\|_X^q (1 - |z|^2)^{\alpha+q\gamma_0} d\nu(z) \right)^{1/q} &\leq \\ \|f\|_{p, \alpha, X}^{1-p/q} \left( \int_{\mathbb{B}_n} \|f(z)\|_X^p \frac{(1 - |z|^2)^{\alpha+q\gamma_0}}{(1 - |z|^2)^{q\gamma_0}} d\nu(z) \right)^{1/q} &= \|f\|_{p, \alpha, X}. \end{aligned}$$

Therefore, by duality, we obtain that

$$\|h_b\|_{A_{\alpha}^p(\mathbb{B}_n, X) \rightarrow A_{\alpha}^q(\mathbb{B}_n, Y)} = \sup_{\|f\|_{p, \alpha, X}=1; \|g\|_{q', \alpha, Y^*}=1} |\langle h_b f, g \rangle_{\alpha, Y}| \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}. \quad \square$$

### 6.4. Proof of Theorem 8

We are now ready to give the proof of the main result in this section that is Theorem 8 that we recall here.

**Theorem 45.** *Let  $X$  and  $Y$  be two reflexive complex Banach spaces. Suppose that  $1 < p \leq q < \infty$ , and  $\alpha > -1$ . The little Hankel operator  $h_b : A_{\alpha}^p(\mathbb{B}_n, X) \rightarrow A_{\alpha}^q(\mathbb{B}_n, Y)$  is a compact operator if and only if*

$$b \in \Lambda_{\gamma_0, 0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y)),$$

where  $\Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$  denotes the generalized little vector-valued Lipschitz space and  $\gamma_0 = (n + 1 + \alpha) \left( \frac{1}{p} - \frac{1}{q} \right)$ , see (1.3).

*Proof.* First assume that  $b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$  and denote by  $b_r(z) := b(rz)$  with  $z \in \mathbb{B}_n$  and  $0 < r < 1$ . Since  $b \in \Lambda_{\gamma_0,0}(\mathbb{B}_n, \mathcal{K}(\overline{X}, Y))$ , by Theorem 7, we have that

$$\|h_b\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)} \lesssim \|b\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}.$$

Therefore, we have

$$\|h_b - h_{b_r}\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)} \lesssim \|b - b_r\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}.$$

By using Proposition 38, we have that

$$\lim_{r \rightarrow 1^-} \|b - b_r\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} = 0,$$

so to prove that  $h_b$  is a compact operator, it suffices to prove that  $h_{b_r}$  is a compact operator. Since  $b_r$  is analytic on a neighbourhood of  $\overline{\mathbb{B}_n}$ , it can be approximated by its Taylor polynomial in the generalized vector-valued Lipschitz norm. Thus,

$$\lim_{N \rightarrow \infty} \|b_r - P_{N,r}\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))} = 0, \quad (6.13)$$

with  $P_{N,r}(z) = \sum_{\beta \in \mathbb{N}^n, |\beta| \leq N} \hat{b}(\beta) r^{|\beta|} z^\beta$ , where  $\hat{b}(\beta) \in \mathcal{K}(\overline{X}, Y)$  are the Taylor coefficients of  $b$ . We also have by Theorem 7 that

$$\|h_{b_r} - h_{P_{N,r}}\|_{A_\alpha^p(\mathbb{B}_n, X) \rightarrow A_\alpha^q(\mathbb{B}_n, Y)} \lesssim \|b_r - P_{N,r}\|_{\Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))}.$$

So by (6.13), to prove that  $h_{b_r}$  is a compact operator, it is enough to prove that  $h_{P_{N,r}}$  is a compact operator. Since  $P_{N,r}$  is a polynomial, it is enough to do the proof for monomials of the form  $\hat{b}(\beta) r^{|\beta|} z^\beta$ , with  $\beta \in \mathbb{N}^n$ ,  $z \in \mathbb{B}_n$  and  $\hat{b}(\beta) \in \mathcal{K}(\overline{X}, Y)$ . Thus, according to Proposition 41, the proof of this part is complete.

Conversely, for the “only if part”, let us assume that

$$h_b : A_\alpha^p(\mathbb{B}_n, X) \longrightarrow A_\alpha^q(\mathbb{B}_n, Y)$$

is a compact operator. Since  $h_b$  is compact,  $h_b$  is then bounded and Theorem 7 yields

$$b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y)).$$

We shall first prove that the Taylor coefficients  $\hat{b}(\beta)$ ,  $\beta \in \mathbb{N}^n$  of  $b$  belongs to  $\mathcal{K}(\overline{X}, Y)$ . Let  $\{f_j\} \subset X$  such that  $f_j \longrightarrow 0$  weakly in  $X$  as  $j \longrightarrow \infty$ , fix  $\beta_0 \in \mathbb{N}^n$ , and let  $x_j(z) = z^{\beta_0} f_j$ . By Lemma 36, we have  $\{x_j\} \subset A_\alpha^p(\mathbb{B}_n, X)$  and  $\{x_j\}$  converges weakly to 0 in  $A_\alpha^p(\mathbb{B}_n, X)$ . Since

$$\|\hat{b}(\beta_0)\overline{f_j}\|_Y = \sup_{\|y^*\|_{Y^*}=1} |\langle \hat{b}(\beta_0)\overline{f_j}, y^* \rangle_{Y, Y^*}|$$

and  $Y$  is reflexive, by the Kakutani's theorem [9, Theorem 3.17] there exists  $y_j^* \in Y^*$  with  $\|y_j^*\|_{Y^*} = 1$  such that

$$\|\hat{b}(\beta_0)\overline{f_j}\|_Y = |\langle \hat{b}(\beta_0)\overline{f_j}, y_j^* \rangle_{Y, Y^*}|.$$

But  $y_j^* \in A_\alpha^{p'}(\mathbb{B}_n, Y^*)$ . By Lemma 23, we have

$$\begin{aligned} |\langle h_b x_j, y_j^* \rangle_{\alpha, Y}| &= \left| \int_{\mathbb{B}_n} \langle b(z)\overline{x_k}(z), y_j^* \rangle_{Y, Y^*} d\nu_\alpha(z) \right| \\ &= \left| \int_{\mathbb{B}_n} \overline{z}^{\beta_0} \left\langle \sum_{\beta \in \mathbb{N}^n} z^\beta \hat{b}(\beta)\overline{f_j}, y_j^* \right\rangle_{Y, Y^*} d\nu_\alpha(z) \right| \\ &= \left| \sum_{\beta \in \mathbb{N}^n} \langle \hat{b}(\beta)\overline{f_j}, y_j^* \rangle_{Y, Y^*} \int_{\mathbb{B}_n} z^\beta \overline{z}^{\beta_0} d\nu_\alpha(z) \right| \\ &= |\langle \hat{b}(\beta_0)\overline{f_j}, y_j^* \rangle_{Y, Y^*}| \int_{\mathbb{B}_n} |z^{\beta_0}|^2 d\nu_\alpha(z) \\ &= \frac{\beta_0! \Gamma(n + \alpha + 1)}{\Gamma(n + |\beta_0| + \alpha + 1)} |\langle \hat{b}(\beta_0)\overline{f_j}, y_j^* \rangle_{Y, Y^*}| \\ &= \frac{\beta_0! \Gamma(n + \alpha + 1)}{\Gamma(n + |\beta_0| + \alpha + 1)} \|\hat{b}(\beta_0)\overline{f_j}\|_Y, \end{aligned}$$

where Fubini's theorem is justified by Lemma 31 with  $\{x_j\} \subset H^\infty(\mathbb{B}_n, X)$ . Since  $h_b$  is compact and  $\{x_j\}$  converges weakly to 0 as  $j$  tends to infinity, we have that  $\{h_b x_j\}$  converges strongly to 0 as  $j$  tends to infinity, therefore one gets that

$$\lim_{j \rightarrow \infty} \langle h_b x_j, y_j^* \rangle_{\alpha, Y} = 0.$$

Thus

$$\lim_{j \rightarrow \infty} \frac{\beta_0! \Gamma(n + \alpha + 1)}{\Gamma(n + |\beta_0| + \alpha + 1)} \|\hat{b}(\beta_0)\overline{f_j}\|_Y = 0.$$

We then obtain

$$\lim_{j \rightarrow \infty} \|\hat{b}(\beta_0)\overline{f_j}\|_Y = 0.$$

In fact, we have shown that  $\hat{b}(\beta_0)$  belongs to  $\mathcal{K}(\overline{X}, Y)$  and as  $\beta_0$  is arbitrary, this holds for all  $\beta \in \mathbb{N}^n$ . Let  $1 < t < \infty$ . Since  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , we have that  $b \in A_\alpha^t(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$  and

$$\lim_{N \rightarrow \infty} \int_{\mathbb{B}_n} \|b(w) - \sum_{|\beta| \leq N} \hat{b}(\beta)w^\beta\|_{\mathcal{L}(\overline{X}, Y)}^t d\nu_\alpha(w) = 0.$$

Let  $z \in \mathbb{B}_n$ . There exists a constant  $C_z > 0$  such that

$$\|b(z) - \sum_{|\beta| \leq N} \hat{b}(\beta) z^\beta\|_{\mathcal{L}(\overline{X}, Y)}^t \leq C_z \int_{\mathbb{B}_n} \|b(w) - \sum_{|\beta| \leq N} \hat{b}(\beta) w^\beta\|_{\mathcal{L}(\overline{X}, Y)}^t d\nu_\alpha(w).$$

Thus,

$$\lim_{N \rightarrow \infty} \|b(z) - \sum_{|\beta| \leq N} \hat{b}(\beta) z^\beta\|_{\mathcal{L}(\overline{X}, Y)} = 0.$$

Since  $z \in \mathbb{B}_n$  is arbitrary, we deduce that  $b(z) \in \mathcal{K}(\overline{X}, Y)$ , for each  $z \in \mathbb{B}_n$ . It remains to show that  $b$  satisfy the “little  $\gamma_0$ - Lipschitz” condition. Let  $x \in X$  and  $y^* \in Y^*$ . Since  $b \in \Lambda_{\gamma_0}(\mathbb{B}_n, \mathcal{L}(\overline{X}, Y))$ , then the mapping  $z \mapsto \langle b(z)\overline{x}, y^* \rangle_{Y, Y^*}$  belongs to  $A_\alpha^1(\mathbb{B}_n, \mathbb{C})$ . By using the reproducing kernel formula, it follows that

$$\langle b(z)\overline{x}, y^* \rangle_{Y, Y^*} = \int_{\mathbb{B}_n} \frac{\langle b(w)\overline{x}, y^* \rangle_{Y, Y^*}}{(1 - \langle z, w \rangle)^{n+1+\alpha}} d\nu_\alpha(w). \quad (6.14)$$

Let  $k > \gamma_0$ . Applying the operator  $R^{\alpha, k}$  in (6.14), we obtain that

$$\langle R^{\alpha, k} b(z)\overline{x}, y^* \rangle_{Y, Y^*} = \int_{\mathbb{B}_n} \frac{\langle b(w)\overline{x}, y^* \rangle_{Y, Y^*}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w). \quad (6.15)$$

Let  $z \in \mathbb{B}_n$ . Since  $\|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{X}, Y)} = \sup_{\|x\|_X=1} \|R^{\alpha, k} b(z)(\overline{x})\|_Y$ , and by Lemma 35, the operator  $R^{\alpha, k} b(z)$  is compact. So there exists  $x_0(z) \in X$  with  $\|x_0(z)\|_X = 1$  and

$$\|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{X}, Y)} = \|R^{\alpha, k} b(z)\overline{x_0(z)}\|_Y.$$

Also

$$\|R^{\alpha, k} b(z)\overline{x_0(z)}\|_Y = \sup_{\|y^*\|_{Y^*}=1} |\langle R^{\alpha, k} b(z)\overline{x_0(z)}, y^* \rangle_{Y, Y^*}|.$$

Since  $Y$  is reflexive, it follows by the Kakutani’s theorem [9, Theorem 3.17] that there exists  $y_0^*(z) \in Y^*$  with  $\|y_0^*(z)\|_{Y^*} = 1$  such that

$$\|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{X}, Y)} = \|R^{\alpha, k} b(z)\overline{x_0(z)}\|_Y = |\langle R^{\alpha, k} b(z)\overline{x_0(z)}, y_0^*(z) \rangle_{Y, Y^*}|. \quad (6.16)$$

By (6.15) and (6.16) we get

$$(1 - |z|^2)^{k-\gamma_0} \|R^{\alpha, k} b(z)\|_{\mathcal{L}(\overline{X}, Y)} = \left| \int_{\mathbb{B}_n} \langle b(w)\overline{x_0(z)}, y_0^*(z) \rangle_{Y, Y^*} \frac{(1 - |z|^2)^{k-\gamma_0}}{(1 - \langle z, w \rangle)^{n+1+\alpha+k}} d\nu_\alpha(w) \right| = |\langle h_b x_z, y_z^* \rangle_{\alpha, Y}|,$$

with

$$x_z(w) = \frac{x_0(z)(1 - |z|^2)^{\beta-(n+1+\alpha)/p}}{(1 - \langle w, z \rangle)^\beta}, \quad w \in \mathbb{B}_n$$

and

$$y_z^*(w) = \frac{y_0^*(z)(1 - |z|^2)^{k+(n+1+\alpha)/q-\beta}}{(1 - \langle w, z \rangle)^{n+1+\alpha+k-\beta}}, \quad w \in \mathbb{B}_n,$$

where  $\beta$  is chosen such that

$$(n + 1 + \alpha)/p < \beta < k + (n + 1 + \alpha)/q.$$

By Theorem 18, we have  $x_z \in A_\alpha^p(\mathbb{B}_n, X)$ ,  $y_z^* \in A_\alpha^{q'}(\mathbb{B}_n, Y^*)$ , and

$$\sup_{z \in \mathbb{B}_n} \|x_z\|_{p,\alpha,X} < \infty, \quad \sup_{z \in \mathbb{B}_n} \|y_z^*\|_{q',\alpha,Y^*} < \infty.$$

Let us prove that

$$x_z \longrightarrow 0 \text{ weakly in } A_\alpha^p(\mathbb{B}_n, X) \text{ as } |z| \longrightarrow 1^-. \tag{6.17}$$

Since

$$\sup_{z \in \mathbb{B}_n} \|x_z\|_{p,\alpha,X} < \infty,$$

to prove (6.17), by Lemma 33, it suffices to prove that

$$\langle x_z, e_{w,a^*} \rangle_{\alpha,X} \longrightarrow 0 \text{ as } |z| \longrightarrow 1^-,$$

where for each  $a^* \in X^*$  and  $w \in \mathbb{B}_n$ , we have

$$e_{w,a^*}(\zeta) = \frac{1}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha}} a^*, \quad \zeta \in \mathbb{B}_n.$$

By using the definition of  $e_{w,a^*}$  and the reproducing kernel formula, it follows that

$$\begin{aligned} \langle x_z, e_{w,a^*} \rangle_{p,\alpha,X} &= \int_{\mathbb{B}_n} \langle x_z(\zeta), e_{w,a^*}(\zeta) \rangle_{X,X^*} d\nu_\alpha(\zeta) \\ &= \int_{\mathbb{B}_n} \langle x_z(\zeta), \frac{1}{(1 - \langle \zeta, w \rangle)^{n+1+\alpha}} a^* \rangle_{X,X^*} d\nu_\alpha(\zeta) \\ &= \left\langle \int_{\mathbb{B}_n} \frac{x_z(\zeta)}{(1 - \langle w, \zeta \rangle)^{n+1+\alpha}} d\nu_\alpha(\zeta), a^* \right\rangle_{X,X^*} \\ &= \langle x_z(w), a^* \rangle_{X,X^*}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |\langle x_z, e_{w,a^*} \rangle_{p,\alpha,X}| &= |\langle x_z(w), a^* \rangle_{X,X^*}| \\ &= \left| \frac{(1 - |z|^2)^{\beta-(n+1+\alpha)/p}}{(1 - \langle w, z \rangle)^\beta} \langle x_0(z), a^* \rangle_{X,X^*} \right| \\ &\leq \frac{(1 - |z|^2)^{\beta-(n+1+\alpha)/p}}{(1 - |w|)^\beta} \|a^*\|_{X^*} \longrightarrow 0 \end{aligned}$$

as  $|z| \longrightarrow 1^-$ . By using (6.17), the compactness of  $h_b$  and the fact that

$$\sup_{z \in \mathbb{B}_n} \|y_z^*\|_{q',\alpha,Y^*} < \infty,$$

it follows that

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{k-\gamma_0} \|R^{\alpha,k} b(z)\|_{\mathcal{L}(\overline{X}, Y)} = \lim_{|z| \rightarrow 1^-} |\langle h_b x_z, y_z^* \rangle_{\alpha, Y}| = 0,$$

which completes the proof of the theorem.  $\square$

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