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DÉPARTEMENT DE MATHÉMATIQUES
DEPARTMENT OF MATHEMATICS

Laboratoire de Mathématiques et Applications Fondamentales
Laboratory of Mathematics and Fundamentals Applications

NON CLASSICAL APPROACHES TO COOPERATIVE GAMES

DISSERTATION

Presented in partial fulfillment of the requirements for the award of

Ph.D. in MATHEMATICS

Specialty: **Mathematics and Fundamentals Applications**
Option: **Applied Mathematics to Social Choice**

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Academic year: 2020 - 2021

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DÉPARTEMENT DE MATHÉMATIQUES

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ATTESTATION DE CORRECTION DE LA THÈSE DE DOCTORAT/PhD

Nous soussignés, Pr. **TCHANTCHO Bertrand,** Pr. **MOYOUWOU Issofa,** Pr. **MBANG Joseph;** membres du jury de la thèse de Doctorat/PhD présenté par **Monsieur NJOYA NGANMEGNI NDOUMBEU Marc Donald**, Matricule 12V0885, Thèse intitulé: **«Non classical approaches to cooperative games»** et soutenu en vue de l'obtention du diplôme de **DOCTORAT/PhD en Mathématiques**, attestons que toutes les corrections demandées par le jury de soutenance en vue de l'amélioration de ce travail, ont été effectuées.

En foi de quoi la présente attestation lui est délivrée pour servir et valoir ce que de droit.

Rapporteur	Président	Examinateur
Pr. **MOYOUWOU Issofa**	Pr. **TCHANTCHO Bertrand**	Pr. **MBANG Joseph**

⋆⋆ Contents ⋆⋆

⋆⋆ Declaration ⋆⋆

I hereby declare that this submission is my own work and to the best of my knowledge, it contains no materials previously published or written by another person and no material which to a substantial extent has been accepted for the award of any other degree or diploma at The University of Yaoundé I or any other educational institution except where due acknowledgement is made in this thesis. Any contribution made to the research by others, with whom I have worked at The University of Yaoundé I or elsewhere, is explicitly cited in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in style, presentation, and linguistic expression is acknowledged.

Donald Njoya Nganmegni Ndoumbeu

⋆⋆ Dedication ⋆⋆

I dedicate this thesis both to:

The memory of my late father ***NDOUMBEU JEAN***;

my mother **TCHOUTOUO MARIE LOUISE**.

⋆⋆ Acknowledgments ⋆⋆

There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted.

- ☞ First of all, I give thanks to the Almighty God for all the graces he has bestowed on me during these years of hard work, for Him honor and glory.
- ☞ A special feeling of gratitude to Pr. Issofa MOYOUWOU and Pr. Nicolas for having supervised me throughout this thesis. Dear professors, I cannot estimate the time and energy you have devoted to supervise this thesis. Your encouragements, advices and expertise in this research field have been a decisive contribution to achieve this work. I'm very grateful to you.
- ☞ I thank all members of the MAP laboratory, in particular the coordinating members of the MASS research team Pr. Bertrand TCHANTCHO, Pr. Lawrence DIFFO LAMBO, Pr. Louis A. FONO and Dr. Hugue TCHANTCHO. Your constructive exchanges during our monthly seminars have helped to improve the quality of this work.
- ☞ I thank Dr. Aurélien MEKUKO and Dr. Monge K. OUAMBO KAMDEM, for the proof-reading of this work and for their expertise in latex. Please accept my sincere gratitude.
- ☞ I express all my gratitude to my wife, D'avila NJOYA. Your words of encouragement during these years of research played a key role. A very special thank for your practical and emotional support.
- ☞ I thank my son Evyd Landry NDOUMBEU. With your smile and the grace you bring in my life, you and your mother have always given me the strength to go through with

this work. May this work be a source of inspiration for you and your brothers and sisters.

- ☞ To my nuclear family, I thank you for the multifaceted supports throughout the process of writing this dissertation.
- ☞ I thank my stepfather Timothée TAGNE and Edith KAPCHE for their permanent encouragements.
- ☞ I would like to thank all my friends, in particular M. Hilaire TOUYEM, M. Ivan DONGMO and M. Adin SAFOKEM, for their comments and permanent encouragements.
- ☞ I would like to thank the choir JERICHO MESSA MOKOLO, for the multifaceted supports throughout the process of writing this dissertation.

⋆⋆ Abstract ⋆⋆

In this thesis, our aim is threefold and consists in exploring three non classical approaches of cooperative games. In a cooperative game with transferable utilities, each coalition that is formed is endowed with a pre-determined worth. This is not always the case as shown in our investigations. Firstly, we assume that the payoff of a coalition depends on the choice of its members between two or more available alternatives. By so doing, we introduce the class of multi-cooperative games with transferable utilities (*MTU-games*) by assuming that, instead of a single game, players are offered two or more TU-games. For MTU-games, we define some core concepts; and then we prove necessary and sufficient conditions of the non-emptiness for the corresponding cores. Secondly, we consider cooperative games with possibly non-monetary sharing issues by assuming that each outcome of cooperation is a raw material each share of which is made profitable by players in their own way. For these games, called cooperative games with local utilities functions (LUF-games), two core concepts are introduced. For core sharing vectors, no coalition exists such that all its members are better off by staying out of the grand coalition; while for strong core sharing vectors, any deviation that is profitable for some members of a coalition also makes some others in that coalition worse off. The results obtained include a generalization of the Shapley-Bondareva theorem to linear utility functions with possibly distinct rates. Thirdly and finally, we follow Charnes and Granot (1973,1976) on cooperative games when payoffs of coalitions are random variables; the so-called chance-constrained games. On this strand of the literature, we encompass the absence of a single- valued solution by defining a two-stage value as an ex-ante agreement among players. In support of the tractability of the newly introduced value called *equal-surplus Shapley value*, a simple and compact formula as well as axiomatic solutions are established.

Keywords: Cooperative games; random payoffs; non-monetary shares; core solutions.

⋆⋆ Resumé ⋆⋆

Dans cette thèse, nos préoccupations se déclinent en trois volets et consistent à explorer trois approches non classiques des jeux coopératifs (JCs). En fait, à chaque coalition qui se forme est affecté classiquement un gain connu à l'avance. Ceci n'est pas toujours le cas dans les modÃ¨les que nous investigons. Premièrement, nous supposons que le gain d'une coalition dépend du choix consensuel d'une option parmi deux ou plusieurs alternatives par ses membres. En fait, nous introduisons la classe des jeux multi-coopératifs à utilités transférables sous l'hypothèse que les membres de chaque coalition peuvent avoir la possibilité de choisir suivant quelle option s'exerce leur coopération entre une ou plusieurs options de jeux coopératifs à utilités transférables. Pour cette classe de jeux, nous définissons une solution de type cœur et établisssons une condition nécessaire et suffisante pour que le cœur correspondant soit non vide. Deuxièmement, nous considérons les JCs avec possibilité de partage non monétaire entre les joueurs en supposant que le produit d'une coopération est un bien dont l'utilité de chaque part dépend du joueur. Pour cette classe de jeux appelés jeux coopératifs à fonctions d'utilités locales, deux concepts de solution de type cœur sont introduits. Pour un partage du cœur, il n'existe pas de coalition dont tous les membres gagneraient à quitter la grande coalition; tandis que pour tout partage du cœur fort, toute déviation par une coalition qui est rentable pour certains l'est au détriment de certains autres. Entre autres résultats, nous généralisons le théorème de Shapley-Bondareva aux jeux avec des fonctions d'utilités linéaires de coefficients distincts. Troisièmement et pour terminer, nous reprenons les travaux de Charnes and Granot (1973,1976) sur les JCs lorsque les gains des coalitions sont des variables aléatoires, jeux très souvent appelés jeux coopératifs à paiements aléatoires. Nous comblons l'absence d'une solution ponctuelle dans la littérature sur cette classe de jeux en définissant une valeur en termes d'un contrat préalable de coopération à deux étapes entre les joueurs. Pour prouver la maniabilité et l'attractivité de la nouvelle valeur que nous avons convenue d'appeler valeur de Shapley à suppléments égaux, une formule simple et compacte ainsi que plusieurs axiomatisations sont établies.

Mots clés: Jeux coopératifs; gains aléatoires; partages non monétaires; cœurs.

⋆⋆ Introduction ⋆⋆

Since Morgenstern and Von Neumann (1953) who defined the cooperative game in characteristic function form of a strategic game, cooperative games with transferable utilities (TU-games) have been used in modeling many other economic interactions where players form coalitions to produce goods, make profits, save costs or enjoy power; for various applications, see Curiel (2013) for optimization issues; Slikker and Van den Nouweland (2012) and Myerson (1977) for network allocation problems; Thomson (2003) for bankruptcy problems; Wang et al. (2008) and Young et al. (1982) for water supply problems; or Peleg and Sudhölter (2007) for a systematic study of some salient solutions to cooperative games. Classically, the payoff of a coalition in all those contributions is deterministic. The aim of this thesis is to study three non classical families of cooperative games in which a coalition may be proposed several opportunities; or the outcome of cooperation may not be monetary (such as an intermediate good differently valued by players) or may simply be a random variable.

The first class of games that we consider differs with the usual TU-games on the way coalitions get their collective payments. There are two or more alternatives for cooperation offered to the members of each coalition. For the coalitional worth, players thus care not only about their partners (as it is the case with TU-games) but also which of the alternative to adopt. To illustrate this, consider two neighbors, X and Y, planning each for a new house to be built by one of the three qualified building firms of their town hall. Each building firm makes a pricing that specifies the cost for bills (if X and Y separately contact the firm) and the cost for a joint bill (if X and Y jointly contact the firm). Clearly, the pricing by each building firm is a TU-game involving X and Y. The question is, when should the two neighbors cooperate? In the case of cooperation, which building firm is chosen and what is the share of each partner? Note that, the situation we just describe is a collection of three TU-games on the same set of players. Any such game that consists in a finite collection of two or more TU-games is called a *multi-cooperative game with transferable utilities* (MTU-game).

In the first chapter, two core concepts for MTU-games are presented. The first core concept for MTU-games we consider consists in assuming that players form an arbitrary coalition structure and partners share their coalitional worth in such a way that no coalition in the game can improve the payoffs of all its members by behaving differently. The set of all payoff vectors that satisfy the latter requirement of stability is called the $\mathcal{B}$-core (core following the coalition structure $\mathcal{B}$) of the MTU-game in consideration. Of course, this is a generalization of the $\mathcal{B}$-core of TU-games; see Gillies (1953) and Aumann and Dreze (1974). Now, for our second core concept, no a priori coalition structure is given. A core payoff is now any payoff vector supported by a coalition structure such that no coalition in the game can improve the payoff of all its members. In both cases, we provide necessary and sufficient conditions for the non-emptiness of the corresponding cores. This is achieved by the introduction of an appropriate notion of balanced families of coalitions. The characterization results that we derive can then be seen as an extension, from TU-games to MTU-games, of the Shapley-Bondareva theorem; see Shapley (1967) and Bondareva (1963).

In the second class of games analyzed here, we reconsider another key assumption in TU-games which amounts to assuming that money (or an infinitely divisible commodity) is available as a means of exchange. Aumann (1960) showed that transferable utilities are met in games with at least three players only if utilities are linear in money with a common rate. In the second chapter, we introduce *cooperative games with local utility functions* (LUF-games) which are cooperative games such that (i) each outcome of cooperation is a raw material each share of which is made profitable by players in their own way; and (ii) the utility of a player depends on their coalition of partners. The presentation of a cooperative game with its characteristic function and a collection of utility functions was also mentioned by Peleg and Sudhölter [2007, page 211, equation (11. 4.1)] to construct a non transferable utilities "pregame" associated with a given TU game. We provide an analysis of LUF-games as a primitive class of games.

More precisely, we provide two core concepts for LUF-games assuming that the grand coalition is formed. The core of a LUF-game is the set of all sharing vectors such that all the members of no coalition are better off by staying out of the grand coalition. This is the usual requirement on core elements. Since we only require utility functions to be non decreasing, it is shown that the core may contain some weakly Pareto dominated sharing vectors. The strong core is then introduced as a refinement of the core that consists of all sharing vectors such that any deviation that is profitable for some members of a coalition also makes some other members of the same coalition worse off. When utilities are transferable, it is shown that the two cores coincide with the well known Gillies core; see Gillies (1959). It is shown that the non emptiness of each of the two cores depends not only on the characteristic function but also on the collection of utility functions.

To characterize core sharing vectors of a LUF-game, we introduce the notion of (lower or upper) compensation share. Intuitively, consider a sharing vector that allocates the outcome of the grand coalition to players. The lower compensation share of a player i from the grand coalition to a proper coalition S is the smallest share player i needs to move from the grand coalition to S; that is the smallest share of the coalitional worth of S such that a larger share of player i in S provides him with at least as much utility as his initial share in the grand coalition. An upper compensation share is defined in a similar way. It is shown that the stability of a sharing vector depends on whether the collection of its compensation shares meets a set of constraints e provide. In particular, when all utility functions are linear, upper compensation shares coincide with lower compensation shares and the two cores coincide. This allows us to derive a Shapley-Bondareva like theorem for the non-emptiness of this LUF-games with linear utility functions.

The last class of games we analyze is the class of *chance-constrained games* (CC-games) introduced by Charnes and Granot (1973,1976). CC-games depart from TU-game on the fact that coalitional payoffs are rather random variables than of being deterministic. We reconsider this class of games mainly for two reasons. Firstly, for CC-games, only set-valued solutions have been defined. More precisely, Charnes and Granot (1977) consider a two-stage core and a two-stage nucleolus for CC-games. No single-valued rule, a value, that assigns to a CC-game with a single payoff vector is not yet defined together with some of its axiomatizations. Secondly, Charnes and Granot (1977) advocate that, in CC-games, the *payoff process* to the players should be composed of two parts: a prior payoff and an adjustment disposition. Such a payoff process can be viewed as a two-stage contract which works as follows: when a given coalition is formed, its members are first promised, taking into the account the expectation of each of its sub-coalitions, their respective prior payoffs; after a realization of the random payoff of the coalition is observed, the second part of the contract is applied to reallocate the surplus to the members of the coalition. This approach for profit sharing takes into account all possible realizations of the profit. Although Timmer et al. (2004) find this approach to be "time-consuming, inefficient and perhaps even impossible", we think these comments are misleading since the alternative approach that consists in assigning to each agent a share of the total profit can also be viewed as a two-stage payoff with null prior payments. Moreover, the value we propose has a two-stage shape, but it still has some desirable properties and a very simple interpretation.

Our aim in the third chapter is to fill this gap on the absence of a value for CC-games by providing a value together with a simple and compact formula as well as some characterization results. In doing so, we opt for the normative approach in which we only care about individual contributions to the collective worth in order to set up some justice and equity norms like efficiency, symmetry, null player property, etc. Moreover, we consider CC-games each equipped with a collection of finite sample spaces from which coalitional payoffs are derived. A similar approach was considered for cooperative games with uncertainty by Habis and Herings (2011) who introduce a single sample space, and then define all coalitional worths as functions on this sample space. Assuming that the grand coalition is formed, one obtains a two-stage payoff by equally re-allocating the surplus when a realization of the collective payoff is observed. We refer to the corresponding rule as the *equal-surplus Shapley value*. In We explore some interesting features of the equal-surplus Shapley value of CC-games from three distinct perspectives as mentioned above. Firstly, from a computational point of view, the equal-surplus Shapley value has a simple formula that linearly depends on the inputs that define the game. Secondly, we show that the equal-surplus Shapley value complies with the Shapley procedure with the only adjustment that the marginal contribution of a player to a given coalition is not known in advance. Finally, axiomatizations are provided to exhibit what are the normative dispositions that completely describe the equal-surplus Shapley value.

The idea of the equal-surplus shares is also known in the case of TU-games; see for example van den Brink (2007) or Béal et al. (2016). Béal et al. (2019) introduce the efficient egalitarian Shapley value of a TU-game as the sum of the Shapley value of the game and the equal shares of the surplus generated for the grand coalition as one moves from the game to its super-additive cover. This allocation rule has the same shape as the equal-surplus Shapley value for CC-games. Another similarity with the TU-game setting is that some of the axioms we use are extensions of known axioms from TU-games to CC-games. This is the case for *efficiency, null player property or additivity*. However, some other axioms are purely designed for CC-games. This is for example the case with *Independence of Local Relabeling* (ILR) or *Independence of Local Duplication* (ILD). Axiom (ILR) requires that any relabeling of events in the sample space associated with a proper coalition should have no effect on individual shares. In the same way, (ILD) requires that no change on individual shares occurs when an event in a sample space is split into two new events which preserve the probability and the payments: each of the two new events yields the same payment as the initial event and the sum of the probabilities of the two new events is equal to the probability of the initial event. Depending on which probability distributions are admissible, we provide distinct axiomatizations of the equal-surplus Shapley value for CC-games. To make our presentation easier, two assumptions are made. It is assumed that the grand coalition is formed; and that all sample spaces are finite; that is, the random payoff of each coalition consists of a finite number of elementary events.

A detailed presentation of what we outline above includes three chapters as follows. In chapter 1, we recall the class of TU-games and introduce the model of MTU-games. We then enlarge two concepts of core solution to the new class of games. The main result of this chapter, gives necessary and sufficient conditions for the non-emptiness of the corresponding cores. The class of LUF-games is introduced and an analysis of stability on this class is presented in chapter 2. More precisely, after modeling the new class of games, we define two core concepts on LUF-games and give a generalization of the Shapley-Bondareva's result like theorem for the non-emptiness of this LUF-games with linear utility functions. Chapter 3 is devoted to the definition and axiomatizations of the *equal-surplus Shapley value* for CC-games. For each axiomatization, we show the independence of the axioms. In the conclusion, after a summary of the work carried out, we highlight some of these lines of future research.

⋆ ⋆ Chapter One ⋆ ⋆

Multi-cooperative games with transferable utilities

In this chapter, we generalize the approach of cooperative games with transferable utilities (TU-games) by including situations in which the members of a coalition have to choose an activity for cooperation out of several available activities. We define two core solutions that are both generalizations of cores presented by Gillies (1953) and Aumann and Dreze (1974) on classical TU-games. Our main results provide necessary and sufficient conditions for the non-emptiness of the newly introduced cores. These results are based on a generalization of the balancedness conditions introduced by Bondareva (1963) and Shapley (1967) to the new class of games.

The present chapter is organized as follows: Section 1.1 is devoted to the presentation of the TU-games, the concept of core solution on those games and the stability of the core solution. Multi-cooperative games with transferable utilities (MTU-games) are introduced in Section 1.2. Two concepts of core solution on this new class of cooperative games are presented together with necessary and sufficient conditions for the non-emptiness of each of the two cores.

1.1 Cooperative games with transferable utilities

In many practical situations, agents (called actors here) interact with potentially conflicting individual interests for a result that depends on individual or coordinated actions. In such situations or games), players care not only about their own choices but also about the decisions of others. There are mainly two forms of games which are:

- non-cooperative games where no binding agreement is allowed between players;

- cooperative games where binding agreements are allowed between players who may coordinate their actions by forming coalitions.

We are especially interested in cooperative games with transferable utilities we recall below. To this purpose, we adopt the following notational dispositions. In what follows, N is a finite non-empty set of players. Any non-empty subset of N is called a coalition. We denote by Π_N the set of partitions of N and by 2^N the set of coalitions of N (including the empty set). The set of all coalitions of N is denoted by $\mathcal{C}_N$.

Hereafter a partition $\mathcal{B}$ of N will be called a coalition structure. Furthermore, two players i and j are called partners with respect to $\mathcal{B}$ if there exists $B \in \mathcal{B}$ such that $\{i, j\} \subseteq B$.

1.1.1 On the cores of TU-games

At the end of each TU-games, each player belongs to exactly one coalition and all coalitions are admissible.

Definition 1.1.1. *A cooperative game with transferable utilities (TU-game) on N, or simply a TU-game, is a mapping $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The set of all TU-games on N is denoted by Γ^N.*

Interpretation Given a TU-game (N, v),

- v indicates the state of the cooperation between players of N;
- for any coalition S, $v(S)$ is the quantity of goods (collective payoff or collective loss) that the coalition S obtains when its members act together by a mutual agreement.

  It is assumed that:
- all coalitions are likely to forme;
- players are free to cooperate;
- players are rational and the aim of everyone, is to maximize his/her individual interest;
- information on the state of cooperation is fully contained in the data of v.

The following example is from Moulin [2003, Example 2.4a, page 28].

Example 1.1.1. *Teresa is a pianist and David is a violinist. They are proposed to work as a full-time duo with a revenue of \$210,000. Before the duo was formed, Teresa was earning \$50, 000 a year as a teacher and solo artist, and David \$100, 000 as the first violinist of a symphony orchestra.*

It is interesting to know:

(Q1) *What is the end agreement on cooperation (should the two artists form the duo or not)?*

(Q2) *What is a fair split of the duo revenue in the case the two artists opt for a joint venture?*

The outcome of a TU-game consists of a partition of the player set together with a sharing vector that informs on the way the worth of each coalition is split among its members. If the players free decide how to cooperate and share the outcome of the cooperation, the main questions raised by a TU-game are *(a) which partition of N will be observed at the end of the game?* and *(a) what will be the share of each player in the selected coalition structure?* Question *(a)* refers to *coalition formation issues* while Question *(b)* is on *sharing issues.*

Sharing vectors and imputations

Definition 1.1.2. *Given (N,v) a TU-game and $\mathcal{B}$ a coalition structure. A* **$\mathcal{B}$-sharing vector** *(sharing vector given $\mathcal{B}$) is any mapping $x : N \to \mathbb{R}$ that assigns to each player i of N a real number denoted by x_i such that:*

$$\forall B \in \mathcal{B}, x(B) = v(B);$$

where $x(B) = \sum_{i \in B} x_i$.

We denote by $\mathbb{R}^N$ the set of all mappings from N to $\mathbb{R}$. Given $x \in \mathbb{R}^N$ a $\mathcal{B}$-sharing vector:

- x_i is the share of player i;
- when $N = \{1, 2, \ldots, n\}$, we identify x with the n-tuple $x = (x_1, x_2, ..., x_n)$;

- the n-tuple x is also denoted by $(x_i)_{i\in N}$.

Definition 1.1.3. *A* **sharing vector** *is any element x of $\mathbb{R}^N$ which is a $\mathcal{B}$-sharing vector with respect to some coalition structure $\mathcal{B}$.*

Definition 1.1.4. *Given a TU-game (N, v) and a coalition structure $\mathcal{B}$,*

- *A $\mathcal{B}$-sharing vector x is* **individually rational** *if, for all $i \in N$, $x_i \geq v(i)$;*

- *A $\mathcal{B}$-sharing vector x is* **collectively rational** *if for all $S \in \mathcal{C}_N$, $x(S) \geq v(S)$.*

An individually rational $\mathcal{B}$-sharing vector is also called a $\mathcal{B}$-imputation or an imputation given $\mathcal{B}$. The set of all imputations with respect to a coalition structure $\mathcal{B}$ is denoted by $\chi(\mathcal{B}, v)$. More specifically, for $\mathcal{B} = \{N\}$, we simply note $\chi(\mathcal{B}, v)$ by $\chi(N, v)$.

Classical core

Most of the existing studies in the literature deal with redistribution issues after assuming that the problem of coalition formation is solved: the players are already grouped together according to a coalition structure. Concretely, according to the core solution (see Gillies (1953)), it is assumed that:

(C_1) players form the grand coalition (that is $\mathcal{B} = \{N\}$);

(C_2) coalitional payoffs are infinitely divisible;

(C_3) players are rational.

Under these conditions, the problem which remains to be solved is therefore that of sharing the payoff of each of the observed coalitions. It is natural to assume that the members of no coalition S will accept a sharing vector in which the total amount they receive is less than what they might win by unilaterally forming S. In this sense, only sharing vectors that are collectively rational are observable and are the only payoff outcomes advocated by the core solution. Formally,

Definition 1.1.5. *Given a TU-game (N, v), the* **classical core** *of (N, v) is the set of all collectively rational $\{N\}-$sharing vector.*

This set is denoted $\mathcal{C}(N,v)$ and in other words:

$$\mathcal{C}(N,v)=\left\{x\in\mathbb{R}^N : x(N)=v(N) \text{ and } \forall S\in\mathcal{C}_N, x(S)\geq v(S)\right\}. \tag{1.1}$$

The TU-game (N,v) is **stable** if $\mathcal{C}(N,v)$ is a non-empty set.

Another definition of the core solution based on the notion of dominance is given by Gillies (1953) as follows:

Definition 1.1.6. *Given a TU-game (N,v), $S\in\mathcal{C}_N$ and $x,y\in\chi(N,v)$,*

1. *y dominates x via S if*

   (i) $\forall i\in S, y_i>x_i$;

   (ii) $y(S)\leq v(S)$.

2. *x is dominated if there exists $z\in\chi(N,v)$ and $T\in\mathcal{C}_N$ such that z dominates x via T.*

In this case we, also say that x is not stable.

For x to be dominated by y via S, there are two requirements. Firstly, item (i) requires that each member of the coalition S expects a better payoff in y than in x; we say that item (i) is the *incentive* condition. Secondly, for item (ii), the total amount of claims should not exceed the worth of the objecting coalition; we say that (ii) is the *means* condition. It appears the following proposition (see Gillies (1953)):

Proposition 1.1.1. *Given a TU-game (N,v), the set of all sharing vector (following $\{N\}$) that are dominated by no sharing vector (or that are stable) coincides with the core solution of the TU-game (N,v).*

In other words

$$\mathcal{C}(N,v)=\{x\in\chi(N,v) : x \text{ is stable}\} \tag{1.2}$$

The core solution can therefore be perceived in its two forms (1.1) and (1.2). The form (1.1) is called static form and give a subset of possible shares. The form (1.2) is called dynamic form. In fact, when the players cooperate following the notion of dominance, the only stable shares are non-dominated shares.

The next definition gives an alternative definition of the notion of domination among sharing vector.

Definition 1.1.7. *Given a TU-game* (N, v), $S \in \mathcal{C}_N$ *and* $x, y \in \chi(N, v)$,

1. y *weakly dominated* x *via* S *if*

   (i) $y_i \geq x_i$ *for all* $i \in S$; *and* $y_i > x_i$ *for some* $i \in S$;

   (ii) $y(S) \leq v(S)$.

2. x *is weakly dominated if there exists* $z \in \chi(N, v)$ *and* $T \in \mathcal{C}_N$ *such that* z *weakly dominated* x *via* T.

The weak domination relation among sharing vector is obtained by weakening only the incentive condition (i) in Definition 1.1.6: the share of each member of the objecting coalition should not be altered from x to y; and y should provide a better share for some of those players as compared to what they receive in x.

Definition 1.1.8. *Let* (N, v) *be a TU-game.*

The strong core of the game (N, v) *is the set* $\mathcal{C}^s(N, v)$ *of all payoff vectors that are weakly dominated by no sharing vector.*

It is obvious that the strong core $\mathcal{C}^s(N, v)$ of a TU-game (N, v) is a subset of the core $\mathcal{C}(N, v)$ of that game. Moreover, it has been proved that the two core coincide as stated below.

Proposition 1.1.2. *The strong core and the core of each TU-game coincide.*

The $\mathcal{B}$-Core

The core solution has been defined above when the players are grouped according to the grand coalition N. Aumann and Dreze (1974) generalize the classical core approach to any coalition structure $\mathcal{B}$. Concretely, the authors suppose that :

(D_1) The players are grouped following a partition $\mathcal{B}$ of N;

(D_2) Goods are infinitely divisible;

(D_3) Players are rational.

Under the three conditions (D_1) to (D_3), the $\mathcal{B}$-core is defined by the following static formulation:

Definition 1.1.9. *Given a TU-game (N,v) and a coalition structure $\mathcal{B}$ on N,*

1. *the $\mathcal{B}$-core of the TU-game (N,v), denoted by $\mathcal{C}(\mathcal{B},v)$, is the set of all $\mathcal{B}$-imputations that are collectively rational. In other words,*

$$\mathcal{C}(\mathcal{B},v) \quad = \quad \{x \in \chi(\mathcal{B},v) : \forall S \in \mathcal{C}_N, x(S) \geq v(S)\}$$

2. *the TU-game (N,v) is **$\mathcal{B}$-stable** (or stable following $\mathcal{B}$) if $\mathcal{C}(\mathcal{B},v)$ is a non-empty set.*

Remark 1.1.1. *It follows by definition that the $\mathcal{B}$-core is the set of sharing vector of $\chi(\mathcal{B},v)$ that are collectively rational. Moreover, for $\mathcal{B} = \{N\}$, we obtain the classical core. In other words:*

$$\mathcal{C}(\{N\},v) = \mathcal{C}(N,v).$$

The above definition of the $\mathcal{B}$-core of a TU-game does not guarantee its stability as shown in the following example:

Example 1.1.2. *Let (N,v) be the TU-game defined as follows:*

S	1	2	3	12	13	23	123
$v(S)$	0	0	0	10	6	10	12

For $\mathcal{B} = \{N\}$, suppose x belongs to $\mathcal{C}(N\ ,v)$. It follows that:

$$(x_1+x_2)+(x_1+x_3)+(x_2+x_3) \geq 10+6+10; \text{ and then } x_1+x_2+x_3 \geq 13$$

Since, x is a $\mathcal{B}$-sharing vector, it appears that:

$$x_1+x_2+x_3 = 12 \geq 13$$

A contradiction arises and contradicts the assumption that the core is not empty. That is $\mathcal{C}(N,v) = \emptyset$. Furthermore, for any other coalition structure $\mathcal{B} \neq \{N\}$, we have:

$$\sum_{B\in\mathcal{B}} v(B) < v(N) = 12.$$

This proves that $\mathcal{C}(\mathcal{B},v) = \emptyset$ for all coalition structures $\mathcal{B}$.

Example 1.1.2 shows that the core of a TU-game following a coalition structure might be empty. In this case, there exists no sharing vector that is collectively rational. In other words, given any sharing vector, there exists some coalition S whose members have incentives to depart from the coalition structure in consideration. This is a clearly a drawback of core solutions on TU-game. In order to identify all TU-games that do not exhibit this shortcoming issue with core solutions, Bondareva (1963) and Shapley (1967) independently presented necessary and sufficient conditions for the non-emptiness of the core of an arbitrary TU-game.

1.1.2 The Shapley-Bondareva theorem

Before we move to Shapley-Bondareva theorem, it is worth recalling a relationship between the collection of $\mathcal{B}$-cores for a given TU-game.

Definition 1.1.10. *Given a TU-game (N, v) and a partition $\mathcal{B}_0$ of N, the coalition structure $\mathcal{B}_0$ is v-efficient if:*

$$\forall \mathcal{B} \in \Pi_N, \quad \sum_{B \in \mathcal{B}} v(B) \leq \sum_{B \in \mathcal{B}_0} v(B).$$

In this case, we pose:

$$v(\Pi_N) = \sum_{B \in \mathcal{B}_0} v(B) = \max_{\mathcal{B} \in \Pi_N} \sum_{B \in \mathcal{B}} v(B).$$

It can be observed that $\mathcal{B}$ is v-efficient when $\mathcal{B}$ maximizes over all partitions of N, the total sum of coalitional payoffs. The following proposition underlines not only the importance of efficient coalition structures; but also the link between those coalitions structure and the TU-game $(N, \overline{v})$ obtained from (N, v) as follows :

$$\forall S \in \mathcal{C}_N, \overline{v}(S) = \begin{cases} v(\Pi_N) & \text{if } S = N \\ \\ v(S) & \text{otherwise} \end{cases}$$

Proposition 1.1.3. *Given a TU-game (N, v) and a partition $\mathcal{B}$ of N,*

1. *$\mathcal{C}(\mathcal{B}, v) = \emptyset$ whenever $\mathcal{B}$ is not v-efficient;*

2. *$\mathcal{C}(\mathcal{B}, v) = \mathcal{C}(N, \overline{v})$ whenever $\mathcal{B}$ is v-efficient.*

According to Proposition 1.1.3, any partition that is not v-efficient generates an empty core. Furthermore two v- efficient coalition structures always generate the same $\mathcal{B}$-core. Proposition 1.1.3 also tells us that the problem of the non-emptiness of the core with respect to a coalition structure is essentially the one of the non-emptiness of the core with respect to the grand coalition of the game $(N, \overline{v})$. For this purpose, the notion of balanced families of coalitions has been introduced. We just recall it below.

Definition 1.1.11. *Given a TU-game (N, v),*

1. *a non-empty set $\mathcal{F}$ of coalitions is* **balanced** *if there exists a family of non negative coefficients $(\lambda_S)_{S\in\mathcal{F}}$ such that for each player i,*

$$\sum_{S\in\mathcal{F}:i\in S} \lambda_S = 1.$$

The collection $\lambda = (\lambda_S)_{S\in\mathcal{F}}$ will be called balancedness coefficients (associated with $\mathcal{F}$).

2. *the TU-game (N, v) is* **balanced** *if for all balanced families $\mathcal{F}$ with balancedness coefficients $(\lambda_S)_{S\in\mathcal{F}}$,*

$$\sum_{S\in\mathcal{F}} \lambda_S v(S) \leq v(N).$$

Before we continue, here below are some examples of balanced family of coalitions.

Example 1.1.3. *Given a TU-game (N, v),*

1. *each possible partition $\mathcal{Q}$ of N is balanced with a unique family of balancedness coefficients $\lambda = (\lambda_S)_{S\in\mathcal{Q}}$ defined by $\lambda_S = 1$ for all $S \in \mathcal{Q}$;*

2. *given $k \in \{1, 2, ..., n\}$, the set $\mathcal{Q}_k$ of all coalitions of size k is balanced with a family of balancedness coefficients $\lambda = (\lambda_S)_{S\in\mathcal{Q}_k}$ defined by $\lambda_S = \frac{1}{\binom{n-1}{k-1}}$ for all $S \in \mathcal{Q}_k$.*

3. *given $i \in \{1, 2, ..., n\}$, the set*
$mathcal P_i$ is balanced with a family of balancedness coefficients $\lambda = (\lambda_S)_{S\in\mathcal{P}_i}$ defined by $\lambda_S = \frac{1}{\mathcal{P}_i}$ for all $S \in \mathcal{P}_i$.

Note that for some balanced families, it is possible to extract a sub-family which is still balanced. But there exists other balanced families such that no non empty sub-family is balanced.

Definition 1.1.12. *A balanced family of coalition is called minimal if it contains no proper sub-family of coalitions that is balanced.*

It can be shown that any minimal balanced family of coalitions admits a unique family of balancedness coefficients. Since we know a classification of these sets we can simplify the condition for the non-emptiness of the core.

Balanced families of coalitions play a key role for the statement of the conditions of the non-emptiness of the core of a TU-game. The following result, established independently by Bondareva (1963) and Shapley (1967), gives a characterization of stable TU-games when players form the grand coalition $\{N\}$.

Theorem 1.1.1 (Shapley (1967) -Bondareva (1963))**.**

Given a TU-game (N, v),
the game (N, v) is **stable following the coalition structure** $\{N\}$ if and only if the game (N, v) is **balanced.** ★

Now, note that for all TU-game (N, v) and for all partition $\mathcal{B}$ v-efficient of N, it follows from Proposition 1.1.3 that

$$\mathcal{C}(\mathcal{B}, v) = \mathcal{C}(N, \overline{v}).$$

This leads to a generalization of Theorem 1.1.1 to the case of the core with respect an arbitrary coalition structure. To achieve this, Aumann and Dreze (1974) introduce the following definition which generalizes the notion of balanced family for any coalition structure.

Definition 1.1.13. *Given a TU-game (N, v) and a coalition structure $\mathcal{B}$ of N, the TU-game (N, v) is **$\mathcal{B}$-balanced** if for all balanced families $\mathcal{F}$ of N with balancedness coefficients $(\gamma_S)_{S\in\mathcal{F}}$,*

$$\sum_{S\in\mathcal{F}} \gamma_S v(S) \leq \sum_{B\in\mathcal{B}} v(B).$$

With $\mathcal{B}$-balancedness, the following result holds.

Theorem 1.1.2 (Aumann and Dreze (1974))**.**

A TU-game (N, v) is **$\mathcal{B}$-stable** if and only if the game (N, v) is **$\mathcal{B}$-balanced**. ★

1.2 Multi-cooperative games with transferable utilities

In this section, we introduce a new class of cooperative games as possible generalization of TU-games. We also extend to this class of games the two core concepts presented above. New stability conditions are then provided.

1.2.1 The model

Generally, in the modeling of cooperative games with transferable utilities, the strategic aspects of the players are temporarily set aside (non-cooperative situations)and the characteristic function is offered. More precisely, in a TU-game (N, v), each player cares about which coalition to join and about what would be his/her share. Now we add another strategy dimension by assuming that there may be two or more options for cooperation instead of a single one with TU-games. These games are called **multi-cooperative games with transferable utilities** and are formalized as follows.

Definition 1.2.1. A **multi-cooperative game with transferable utilities (MTU-game)** *is a couple* $\left(N, v = (v_j)_{1\leq j\leq m}\right)$ *where* N *is the set of players;* m *is a positive integer and for all* $j \in \{1, 2, \ldots, m\}$, (n, v_j) *is a TU-game.*

We denote by $MTU^{N,m}$ the class of MTU-games with the player set N and m activities (opportunities).

Interpretation Given an MTU-game $\left(N, v = (v_j)_{1\leq j\leq m}\right)$,

- v_j (or j) denotes an activity (opportunity) in which a player or group of players (coalition) can be involve;
- given an activity v_j and a coalition S, $v_j(S)$ is the quantity of goods (collective payoff or collective loss) assigned to coalition S when its members decide by mutual agreement to form S and choose to be involve in activity v_j.

  We suppose that:
- all coalitions are likely to form and players have access to all the available activities;
- players are free to cooperate;
- players are rational, the aim of each player is to maximize his/her interest;

- at the end of the game, each player should belong to only one coalition and chooses with his/her partners one option on the available opportunities;
- information on the state of cooperation is fully contained in the data of each v_j.

Remark 1.2.1. *With only one activity ($m = 1$), MTU-games are identified with TU-games. In other words, $MTU^{N,1} = \Gamma^N$.*

To better understand the new class of games that interests us in this thesis, consider the following examples:

Example 1.2.1 (Pipework renewal). *Two neighboring cities X and Y, in order to improve their water supplies, have the choice between two experts. The two experts' evaluations of costs are as follows:*

- *When each city opts for a separate pipework renewal, Expert 1 charges 25 and 20 for city X and city Y respectively; meanwhile Expert 2 charges 23 and 27 for city X and city Y respectively. The difference between the two evaluations is due to equipment constraints and the technology each expert uses.*
- *When the two cities decide for a joint venture, the cost for pipework renewal is 40 by Expert 1 and 39 by Expert 2.*

The situation is summarized by:

	city X	city Y	cities X and Y
Expert 1	25	20	40
Expert 2	23	27	39

where the unit for cost evaluation is a million of FCFA.

Some questions emerge among which the followings:

(Q1) How will the player behave?

(Q2) What advice will you give to?

(Q3) In the case of cooperation, how much should be the contribution of each of the two cities?

Example 1.2.2. *Three countries A, B and C have opted for the implementation of submarine optical fiber in order to improve their digital economy. Two expert companies E_1 and E_2 on the market offer the following pricing: costs are expressed in billions of CFA francs.*

For E_1:

- *Each installation costs 50 when the three countries separately negotiate for their respective networks.*
- *If two countries make a joint venture, they will spend together an amount of 70.*
- *When they form a trio, they will spend together an amount of 125.*

For E_2:

- *When the three countries separately negotiate for their respective networks, the cost of individual network installation is 30 for Country A and 60 for each of the two other countries.*
- *In the case of a joint venture between two of the three countries, the cost is 65 for B and C; and 80 for A and any one of two other countries.*
- *When the three countries form a trio, they will pay together an amount of 130.*

As in the Example 1.2.1, It becomes interesting to know:

(Q1) What is the best way to proceed for the three countries: which coalition structures will emerge?

(Q2) In case of cooperation for a joint venture for a coalition of two or three countries, which of the two experts should the partners choose?

(Q3) In the case of cooperation, how much should be the contribution of each of the three countries?

The problem described in these two examples is an illustration of many situations encountered in economic environments where producers (resp. consumers) meet several opportunities of cooperation that can generate profits. They might form coalitions and choose an opportunity in order to maximize the coalitional worth (resp. minimize the coalitional costs). Addressing questions (Q1), (Q2) and (Q3) in the class of MTU-games is our main concern. To this purpose, we define two concepts of core solution.

1.2.2 Core solutions for MTU-games and characterization

Given an MTU-game, answering questions (Q1), (Q2) and (Q3) consists in giving a partition of the players' set together with the activity chosen by each coalition of the partition and a sharing vector stating how partners share their coalitional worths. Considering this, **which partition will be observed at the end of the game and which activities will be chosen by the coalitions of that partition? And what will be the sharing vector in each of the partition that emerges?**

An analysis with a predefined coalition structure

Definition 1.2.2. *Given an MTU-game $G = \Big(N, (v_j)_{1\leq j\leq m}\Big)$, a coalition structure $\mathcal{P}$ is a collection of ordered pairs $((S_1; v_{k_1}), (S_2; v_{k_2}), ..., (S_p; v_{k_p}))$ such that $(S_j)_{j=1,...,p} \in \Pi_N$ and for $j = 1, 2, \ldots, m$, v_{k_j} is the activity that the members of S_j choose. The set of all coalition structures in the game G is denoted by $\Pi_N(G)$.*

Definition 1.2.3. *Given an MTU-game $\Big(N, (v_j)_{1\leq j\leq m}\Big)$ and a coalition structure $\mathcal{P}$, a* **$\mathcal{P}$-sharing vector** *(sharing vector according to $\mathcal{P}$) is any application $x : N \to \mathbb{R}$ that assigns to each player i of N a real number noted x_i such that:*

$$\forall (T, u) \in \mathcal{P}, x\,(T) = u\,(T)\,;$$

with $x\,(T) = \sum_{i\in T} x_i$.

Given an MTU-game $G = \Big(N, v = (v_j)_{1\leq j\leq m}\Big)$, we denote by $R_{\mathcal{P}}(G)$ the set of all $\mathcal{P}$-sharing vectors. In other words:

$$R_{\mathcal{P}}(G) = \left\{x \in \mathbb{R}^N : \forall j = 1, \ldots, p,\ x(S_j) = v_{k_j}(S_j)\right\}.$$

Definition 1.2.4. *Given an MTU-game $G = \Big(N, v = (v_j)_{1\leq j\leq m}\Big)$ and a coalition structure $\mathcal{P}$,*

- *A* **multi-individually rational** *$\mathcal{P}$-sharing vector is any sharing vector x following $\mathcal{P}$ such that, for all $i \in N$, for all $l \in \{1, 2, ..., m\}$, $x_i \geq v_l\,(i)$;*

- A **multi-collectively rational** $\mathcal{P}$-sharing vector is any sharing vector x following $\mathcal{P}$ such that, for all $S \in \mathcal{C}_N$, for all $l \in \{1, 2, ..., m\}$, $x(S) \geq v_l(S)$.

Definition 1.2.5. Given an MTU-game $G = \left(N, v = (v_j)_{1 \leq j \leq m}\right)$ and a coalition structure $\mathcal{P}$ on the MTU-game G, a $\mathcal{P}$-sharing vector x is unstable if there exist an activity l in $\{1, ..., m\}$, a coalition $S \in \mathcal{C}_N$ and $y \in \mathbb{R}^S$ such that:

$$\begin{cases} y_i > x_i & \forall i \in S \\ y(S) \leq v_l(S) \end{cases}$$

In this case, we say that x is dominated via (S, l, y).

A $\mathcal{P}$-sharing vector x is called stable if x is not dominated in the game.

In other words a $\mathcal{P}$-sharing vector is unstable if there exists an activity where the members of a coalition can improve their respective shares in the game. Recall that we assume that players are grouped with respect to a predefined coalition structure $\mathcal{P}$. The remaining problem is thus the sharing issue within each coalition that is formed. Definition 1.2.5 provides a way to test whether a given $\mathcal{P}$-sharing vector is stable or not. This leads us to the following core concept for MTU-games.

Definition 1.2.6. Given an MTU-game $G = \left(N, v = (v_j)_{1 \leq j \leq m}\right)$ and a coalition structure $\mathcal{P}$, the $\mathcal{P}$-core of G, denoted by $\mathcal{CM}(\mathcal{P}, G)$, is the set of all $\mathcal{P}$-sharing vectors that are multi-collectively rational. In other words,

$$\mathcal{CM}(\mathcal{P}, G) = \{x \in R_{\mathcal{P}}(G) : \ x \text{ is stable } \}. \tag{1.3}$$

The game G is $\mathcal{P}$-stable if $\mathcal{CM}(\mathcal{P}, G)$ is not empty.

The following proposition characterizes all $\mathcal{P}$-sharing vectors that are stable.

Proposition 1.2.1. A $\mathcal{P}$-sharing vector x is stable in an MTU-game G if and only if x is multi-collectively rational.

Proof.

Consider an MTU-game $G = \left(N, v = (v_j)_{1 \leq j \leq m}\right)$ and a coalition structure $\mathcal{P}$. Let $x \in R_{\mathcal{P}}(G)$.

Sufficiency. First suppose that x is unstable. Then there exist an activity l in $\{1, ..., m\}$, a coalition $S \in \mathcal{C}_N$ and $y \in \mathbb{R}^S$ such that

$$\begin{cases} y_i > x_i & \forall i \in S \\ y(S) \leq v_l(S) \end{cases}$$

We deduce that

$$\begin{cases} y(S) = \sum_{i \in S} y_i > \sum_{i \in S} x_i = x(S) \\ y(S) \leq v_l(S) \end{cases}$$

Therefore $x(S) < v_l(S)$ and x is not multi-collectively rational.

Necessity. Conversely, suppose that x is not multi-collectively rational. Then, there exists $l \in \{1, ..., p\}$ and $S \in \mathcal{C}_N$ such that: $x(S) < v_l(S)$. Consider $y \in \mathbb{R}^S$ such that

$$y_i = x_i + \frac{v_l(S) - x(S)}{|S|}$$

for all $i \in S$. Since $x(S) < v_l(S$, it immediately follows that

$$\begin{cases} y_i > x_i & \forall i \in S \\ y(S) \leq v_l(S) \end{cases}$$

Therefore, x is unstable. ■

Proposition 1.2.1 is a characterization of stable $\mathcal{P}$-sharing vectors. It thus provides the following characterization of the $\mathcal{P}$-core.

Corollary 1.2.1. *Given a MTU-game $G = \left(N, v = (v_j)_{1 \leq j \leq m}\right)$ and a coalition structure $\mathcal{P}$ on the MTU-game G. The $\mathcal{P}$-core of MTU-game G is also given by:*

$$\mathcal{CM}(\mathcal{P}, G) = \{x \in R_{\mathcal{P}}(G) : \forall S \in \mathcal{C}_N, \forall l \in \{1, 2, ..., m\}, x(S) \geq v_l(S)\}. \tag{1.4}$$

Proof.

Immediately follows from the definition of a multi-collectively rational sharing vector and Proposition 1.2.1. ■

To check whether a sharing vector is stable or not, we now simply have to check whether some linear constraints are satisfied. This helps us to identify the set of all MTU-games with non-empty cores.

Definition 1.2.7. *Given an MTU-game $G = \Big(N, v = (v_j)_{1\leq j\leq m}\Big)$ and a coalition structure $\mathcal{P}$ on the MTU-game G. The MTU-game G is* **$\mathcal{P}$-stable** *if $\mathcal{CM}(\mathcal{P}, G)$ is a non-empty set.*

For an MTU-game that is $\mathcal{P}$-stable, players can be groups with respect to $\mathcal{P}$ and share in a stable way the outcome of their cooperation. But how to check whether an MTU-game is stable or not. To address this question, we introduce the following definition.

Definition 1.2.8. *Given an MTU-game $G = \Big(N, v = (v_j)_{1\leq j\leq m}\Big)$ and a coalition structure $\mathcal{P}$ on the MTU-game G, the max-game associated with G is the TU-game denoted by v_G and defined on N for $\forall S \subseteq N$ by,*

$$v_G(S) = \max\{v_k(S), k = 1, ..., m\}. \tag{1.5}$$

Given an MTU-game G and a coalition structure $\mathcal{P}$ in G, we slightly abuse notation to identify $\mathcal{P} = (S_j, v_j)_{j\in J}$ with the partition $(S_j)_{j\in J}$. Recall that $\mathcal{C}(\mathcal{P}, v_G)$ is the $\mathcal{P}$-core of the TU-game (N, v_G); that is

$$x \in \mathcal{C}(\mathcal{P}, v_G) \Longleftrightarrow \begin{cases} x(N) = \sum_{l=1}^{p} v_G(S_l) \\ \qquad\qquad x(S) \geq v_G(S) \quad \forall S \in \mathcal{C}_N \end{cases} \tag{1.6}$$

Now, we denote by $\mathcal{C}^*(\mathcal{P}, v_G)$ the subset of $\mathbb{R}^N$ defined by

$$x \in \mathcal{C}^*(\mathcal{P}, v_G) \Longleftrightarrow \begin{cases} x(N) = \sum_{l=1}^{p} v_{k_l}(S_l) \\ \qquad\qquad x(S) \geq v_G(S) \quad \forall S \in \mathcal{C}_N \end{cases} \tag{1.7}$$

The next result equates the $\mathcal{P}$-core of an MTU-game G with the set $\mathcal{C}^*(\mathcal{P}, v_G)$. The advantage is that $\mathcal{C}^*(\mathcal{P}, v_G)$ is completely characterized by a set of linear inequalities.

Proposition 1.2.2. *Given an MTU-game $G = \Big(N, v = (v_j)_{1\leq j\leq m}\Big)$, for all coalition structure $\mathcal{P}$ on the MTU-game G, $\mathcal{CM}(\mathcal{P}, G) = \mathcal{C}^*(\mathcal{P}, v_G)$.*

Proof.

Consider an MTU-game $G = \Big(N, v = (v_j)_{1\leq j\leq m}\Big)$, a coalition structure $\mathcal{P}$ of G and $x \in \mathbb{R}^N$. Then we have:

$$\begin{aligned} x \in \mathcal{CM}(\mathcal{P}, G) &\Longleftrightarrow \begin{cases} \quad x \in R_{\mathcal{P}}(G) \\ \forall l = 1, ..., m, \quad \forall S \in \mathcal{C}_N \quad x(S) \geq v_l(S) \end{cases} \\ &\Longleftrightarrow \begin{cases} x(S_j) = v_{k_j}(S_j), \quad \forall j = 1, ..., p \\ \qquad\qquad \forall S \in \mathcal{C}_N, \quad x(S) \geq v_l(S), \quad \forall l \in \{1, ..., m\} \end{cases} \end{aligned}$$

It then follows by the definition of the max-game that

$$x \in \mathcal{CM}(\mathcal{P}, G) \Longleftrightarrow \begin{cases} x(S_j) = v_{k_j}(S_j) \quad \forall j = 1, ..., p \\ x(S) \geq \max\limits_{l \in \{1,...,m\}} v_l(S) \quad \forall S \in \mathcal{C}_N \end{cases}$$

$$\Longleftrightarrow \begin{cases} x(S_j) = v_{k_j}(S_j) \quad \forall j = 1, ..., p \\ x(S) \geq v_G(S) \quad \forall S \in \mathcal{C}_N \quad \text{By Equation (1.5)} \end{cases}$$

$$\Longleftrightarrow x \in \mathcal{C}^*(\mathcal{P}, v_G)$$

■

Clearly, finding a stable $\mathcal{P}$-sharing vector given an MTU-game G amounts to finding a sharing vector with respect to the partition embedded to $\mathcal{P}$ that is stable in the max-game associated with G.

Example 1.2.3. An MTU-game model of Example 1.2.1 is as follows:

S	$\{1\}$	$\{2\}$	$\{1,2\}$
v_1	-25	-20	-40
v_2	-23	-27	-39
v_G	-23	-20	-39

Costs has been turned into utilities.

For the coalition structure $\mathcal{P} = \{(\{N\}, v_2)\}$, the core $\mathcal{CM}(\mathcal{P}, G)$ is determined as follows: Let x be an element of $\mathbb{R}^N$. Then

$$x \in \mathcal{CM}(\mathcal{P}, G) \Longleftrightarrow \begin{cases} x_1 + x_2 = -39 \\ x_1 \geq -23 \\ x_2 \geq -20 \\ x_1 + x_2 \geq -39 \end{cases}$$

By canceling the redundant constraint $x_1 + x_2 \geq -39$, it follows that

$$x \in \mathcal{CM}(\mathcal{P}, G) \Longleftrightarrow \begin{cases} x_1 + x_2 = -39 \\ x_2 \geq -20 \\ -x_2 - 39 \geq -23 \end{cases} \Longleftrightarrow \begin{cases} x_1 + x_2 = -39 \\ -20 \leq x_2 \leq -16 \end{cases}$$

For example $(-21, -18)$ is a stable $\mathcal{P}$-sharing vector in which City X and City Y are charged 21 and 18 respectively.Now, for $\mathcal{P}' = \{(\{N\}, v_1)\}$, it can be noted that a stable $\mathcal{P}'$-sharing vector x should satisfy :

$$x_1 + x_2 = -40; \text{ and } x_1 + x_2 \geq -39;$$

which lead to a contradiction. Therefore, $\mathcal{CM}(\mathcal{P}', G)$ is a empty set.

We conclude that if the two municipalities agree to cooperate and choose Expert 2, there would be no objection to bear the charges of 21 million and 18 million for City X and City Y respectively. But there exists no sharing vector which encourages in a stable way the two municipalities to cooperate and choose the Expert 1.

The following example gives an application of core solution for three players and two activities:

Example 1.2.4. Example 1.2.2 can be modeled by the following MTU-game:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
v_1	-50	-50	-50	-70	-70	-70	-125
v_2	-30	-60	-60	-80	-80	-65	-130
v_G	-30	-50	-50	-70	-70	-65	-125

Given the coalition structure $\mathcal{P} = \{(\{1\}, v_2), (\{2,3\}, v_2)\}$ and $x \in \mathbb{R}^N$,

$$x \in \mathcal{CM}(\mathcal{P}, G) \Longleftrightarrow \begin{cases} x_2 + x_3 = -65 \text{ and } x_1 = -30 \\ x_1 \geq -30; \ x_2 \geq -50 \text{ et } x_3 \geq -50 \\ x_1 + x_2 \geq -70; \ x_1 + x_3 \geq -70 \text{ and } x_2 + x_3 \geq -65 \\ x_1 + x_2 + x_3 \geq -125 \end{cases}$$

Therefore

$$x \in \mathcal{CM}(\mathcal{P}, G) \Longleftrightarrow \begin{cases} x_2 + x_3 = -65 \text{ and } x_1 = -30 \\ x_2 \geq -50 \text{ and } x_3 \geq -50 \\ x_2 \geq -40 \text{ and } x_3 \geq -40 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_2 + x_3 = -65 \text{ and } x_1 = -30 \\ x_2 \geq -40 \text{ and } x_3 \geq -40 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_3 = -65 - x_2 \text{ and } x_1 = -30 \\ -40 \leq x_2 \leq -25 \end{cases}$$

It appears that $(-30, -32, -33)$ is an example of stable $\mathcal{P}$-sharing vector. We conclude that if Country A stays alone and choose E_2 while Countries B and C agree to cooperate and choose E_2, there would be no objection to bear the charges of 30 billion, 32 billion and 33 billion for A, B and C respectively.

An Analysis when no predefined coalition structure

We start by enlarging the set of sharing vectors players might be offered.

Definition 1.2.9. Given an MTU-game $\left(N, (v_j)_{1\leq j\leq m}\right)$, an element x of $\mathbb{R}^N$ is a **G-sharing vector** if there exists a coalition structure $\mathcal{P}$ such that x is a $\mathcal{P}$-sharing vector.

We denote by $\mathcal{R}_G$ the set of all G-shares; that is

$$\mathcal{R}_G = \bigcup_{P\in\Pi_N} R_{\mathcal{P}}(G).$$

As in Definition 1.2.4, we also extend the notion of being multi-individually rational and multi-collectively rational to G-sharing vectors.

Definition 1.2.10. Consider an MTU-game $G = \left(N, v = (v_j)_{1\leq j\leq m}\right)$ and a coalition structure $\mathcal{P}$ on the MTU-game G,

- A **multi-individually rational** G-sharing vector is any multi-individually $\mathcal{P}$-sharing vector x for some coalition structure $\mathcal{P}$ such that, for all $i \in N$, for all $l \in \{1, 2, ..., m\}$, $x_i \geq v_l\,(i)$;

- A **multi-collectively rational** G-sharing vector is any multi-collectively rational $\mathcal{P}$-sharing vector x for some coalition structure $\mathcal{P}$ such that, for all $S \in \mathcal{C}_N$, for all $l \in \{1, 2, ..., m\}$, $x(S) \geq v_l(S)$.

We assume that when players face a G-sharing vector x, each player simply cares about his/her share by checking whether there exists a possibility of cooperation for a better share.

Definition 1.2.11. Given an MTU-game $G = \left(N, v = (v_j)_{1\leq j\leq m}\right)$. An element x of $\mathcal{R}_G$ is unstable if there exist an activity l in $\{1, ..., m\}$, a coalition $S \in \mathcal{C}_N$ and $y \in \mathbb{R}^S$ such that:

$$\begin{cases} y_i > x_i & \forall i \in S \\ y(S) \leq v_l(S) \end{cases}$$

In this case, we say that x is dominated via (S, l, y).

A G-sharing vector x is called stable if x is not dominated in the game.

The stability of a G-sharing vector has the same interpretation we provide for $\mathcal{P}$-sharing vectors. The core concept associated with the stability of G-sharing vectors is the following:

Definition 1.2.12. The **core** of an MTU-game $G = \left(N, v = (v_j)_{1\leq j\leq m}\right)$ is the set $\mathcal{CM}(G)$ of all stable G-sharing vectors; that is

$$\mathcal{CM}(G) = \{x \in \mathcal{R}_G : \ x \text{ is stable}\}. \tag{1.8}$$

The game G is stable if $\mathcal{CM}(G)$ is not empty.

Proposition 1.2.3. Given an an MTU-game $G = \left(N, v = (v_j)_{1\leq j\leq m}\right)$, a G-sharing vector x is stable if and only if x is multi-collectively rational.

Proof.

Consider an MTU-game $G = \left(N, v = (v_j)_{1\leq j\leq m}\right)$ and a G-sharing vector x.

Necessity. First suppose that x is stable. To prove that x is multi-collectively rational, suppose on the contrary that this is not the case. Then, there exist $l \in \{1, ..., m\}$ and $S \in \mathcal{C}_N$ such that $x(S) < v_l(S)$. Consider $y \in \mathbb{R}^S$ defined by

$$y_i = x_i + \frac{v_l(S) - x(S)}{|S|} \text{ for all } i \in S.$$

It is immediate that x is dominated via (S, l, y). A contradiction.

Sufficiency. Suppose that x is multi-collectively rational. To prove that x is stable, suppose the contrary. Then there exist an activity $l \in \{1, ..., m\}$, a coalition $S \in \mathcal{C}_N$ and $y \in \mathbb{R}^S$ such that

$$\begin{cases} y_i > x_i & \forall i \in S \\ y(S) \leq v_l(S) \end{cases}$$

It follows that

$$\left\{ v_l(S) \geq y(S) = \sum_{i \in S} y_i > \sum_{i \in S} x_i = x(S) \right.$$

Hence $v_l(S) > x(S)$. A contradiction arises since x is multi-collectively rational. ∎

As in Proposition 1.2.1, the stability of a G-sharing vector coincides by Proposition 1.2.3 with the notion of being multi-collectively rational. It then provides the following characterization of core sharing vectors.

Corollary 1.2.2. *Given an MTU-game $G = \Big(N, v = (v_j)_{1 \leq j \leq m}\Big)$; the classical core of MTU-game G is also given by:*

$$\mathcal{CM}(G) = \{x \in \mathcal{R}_G : \forall S \in \mathcal{C}_N, \forall l \in \{1, 2, ..., m\}, x(S) \geq v_l\,(S)\}. \tag{1.9}$$

Proof.

Immediately follows from the definition of a multi-collectively rational sharing vector and Proposition 1.2.3. ∎

The next result provides a relationship between the core of an MTU-game when no coalition structure is provided and the core of an MTU-game with respect to a predefined coalition structure.

Proposition 1.2.4. *Given an MTU-game $G = \Big(N, v = (v_j)_{1 \leq j \leq m}\Big)$, an element x of $\mathbb{R}^N$ is an element of $\mathcal{CM}(G)$ if and only if there exists a coalition structure $\mathcal{P}$ such that x is an element of $\mathcal{CM}(\mathcal{P}, G)$, that is $\mathcal{CM}(G) = \bigcup_{\mathcal{P} \in \Pi_N} \mathcal{CM}(\mathcal{P}, G)$.*

Proof.

Consider an MTU-game $G = \Big(N, v = (v_j)_{1\leq j\leq m}\Big)$, $x \in \mathbb{R}^N$ an element of $\mathcal{CM}(G)$. Since x is an element of $\mathcal{R}_G$, then there exists $\mathcal{P}$ a coalition structure such that x be an element of $R_{\mathcal{P}}(G)$.

It follows that:

$$\begin{aligned}
x \in \mathcal{CM}(G) \Longleftrightarrow & \begin{cases} \exists \mathcal{P} \in \Pi_N(G), \quad x \in R_{\mathcal{P}} \\ \forall l = 1, ..., m, \quad \forall S \in \mathcal{C}_N \quad x(S) \geq v_l(S) \end{cases} \\
\Longleftrightarrow & \exists \mathcal{P} \in \Pi_N(G), \begin{cases} x(S_j) = v_{k_j}(S_j), \quad \forall j = 1, ..., p \\ \qquad \forall S \in \mathcal{C}_N, \quad x(S) \geq v_l(S), \quad \forall l \in \{1, ..., m\} \end{cases} \\
\Longleftrightarrow & \exists \mathcal{P} \in \Pi_N(G), \begin{cases} \qquad x(S_j) = v_{k_j}(S_j), \quad \forall j = 1, ..., p \\ x(S) \geq \max\limits_{l\in\{1,...,m\}} v_l(S), \quad \forall S \in \mathcal{C}_N \end{cases} \\
\Longleftrightarrow & \exists \mathcal{P} \in \Pi_N(G), x \in \mathcal{CM}(\mathcal{P}, G) \\
\Longleftrightarrow & x \in \bigcup_{\mathcal{P}\in\Pi_N} \mathcal{CM}(\mathcal{P}, G)
\end{aligned}$$

This proves that that is $\mathcal{CM}(G) = \bigcup_{\mathcal{P}\in\Pi_N} \mathcal{CM}(\mathcal{P}, G)$. ∎

Example 1.2.5. *Recall that Example 1.2.1 is modeled the following MTU-game:*

S	$\{1\}$	$\{2\}$	$\{1,2\}$
v_1	-25	-20	-40
v_2	-23	-27	-39
v_G	-23	-20	-39

All coalition structures in this game are listed as follows:

$$\mathcal{P}_1 = \{(\{N\}, v_1)\},\ \mathcal{P}_2 = \{(\{N\}, v_2)\},$$

$$\mathcal{P}_3 = \{(\{1\}, v_1); (\{2\}, v_1)\}$$

$$\mathcal{P}_4 = \{(\{1\}, v_2); (\{2\}, v_2)\},\ \mathcal{P}_5 = \{(\{1\}, v_1); (\{2\}, v_2)\}$$

$$\text{and } \mathcal{P}_6 = \{(\{1\}, v_2); (\{2\}, v_1)\}.$$

Now, given $x = (x_1, x_2) \in \mathbb{R}^N$, we have:

$$x \in \mathcal{CM}(G) \Longleftrightarrow \begin{cases} x_1 + x_2 &= -40 \\ x_1 + x_2 &\geq -39 \\ x_1 &\geq -23 \\ x_2 &\geq -20 \end{cases} \text{ or } \begin{cases} x_1 + x_2 &= -39 \\ x_1 + x_2 &\geq -39 \\ x_1 &\geq -23 \\ x_2 &\geq -20 \end{cases}$$

$$\text{or } \begin{cases} x_1 = -25 & \text{and} & x_2 = -20 \\ x_1 + x_2 & \geq & -39 \\ x_1 & \geq & -23 \\ x_2 & \geq & -20 \end{cases} \text{ or } \begin{cases} x_1 = -23 & \text{and} & x_2 = -27 \\ x_1 + x_2 & \geq & -39 \\ x_1 & \geq & -23 \\ x_2 & \geq & -20 \end{cases}$$

$$\text{or } \begin{cases} x_1 = -23 & \text{and} & x_2 = -20 \\ x_1 + x_2 & \geq & -39 \\ x_1 & \geq & -23 \\ x_2 & \geq & -20 \end{cases} \text{ or } \begin{cases} x_1 = -25 & \text{and} & x_2 = -27 \\ x_1 + x_2 & \geq & -39 \\ x_1 & \geq & -23 \\ x_2 & \geq & -20 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 + x_2 &= -39 \\ x_1 + x_2 &\geq -39 \\ x_1 &\geq -23 \\ x_2 &\geq -20 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 + x_2 = -39 \\ -20 \leq x_2 \leq -16 \end{cases}$$

For this game, the sets of constraints for the stability of $\mathcal{P}_t$-sharing vectors for $\mathcal{P}_1$, $\mathcal{P}_3$, P_4, $\mathcal{P}_5$ and $\mathcal{P}_6$ are not feasible. It then appears that $\mathcal{CM}(G) = \mathcal{CM}_{\mathcal{P}_2}(G)$ which is the $\mathcal{P}$-core we found for this game in the Example 1.2.3.

Example 1.2.6. Let $G = (N, u = (u_j)_{1 \leq j \leq 2})$ the MTU-game defined as follows:

S	1	2	3	12	13	23	123
$u_1(S)$	0	0	0	10	6	10	12
$u_2(S)$	0	0	0	8	7	11	11
$u_{G'}(S)$	0	0	0	10	7	11	12

Given a coalition structure $\mathcal{P} = (S_j, u_{l_j})_{1 \leq j \leq p}$, the set of stability conditions for a $\mathcal{P}$-sharing vector x includes:

$$(x_1 + x_2) + (x_1 + x_3) + (x_2 + x_3) \geq 10 + 7 + 11; \text{ and then } x_1 + x_2 + x_3 \geq 14.$$

Since $\mathcal{P}$ is coalition structure, the game is such that:

$$\sum_{j=1}^{p} u_{l_j}(S_j) < u_G(N) = 12.$$

A contradiction arises. We conclude that $\mathcal{CM}(\mathcal{P}, G) = \emptyset$ for all coalition structure $\mathcal{P}$. Therefore $\mathcal{CM}(G) = \emptyset$ by Proposition 1.2.4.

It clearly appears in the core of an MTU-games might be empty. It is therefore interesting to find necessary and sufficient conditions for the non-emptiness of the core of an MTU-game.

1.2.3 Stabilities conditions of MTU-games

We extend the notion of efficient coalition structures from TU-games to MTU-games as follows:

Definition 1.2.13. *A coalition structure $\mathcal{P} = (S'_j, v_{k_j})_{1\leq j\leq p}$ of an MTU-game G is poly-efficient if for all coalition structures $(S_j, v_{l_j})_{1\leq j\leq r}$ we have:*

$$\sum_{j=1}^{r} v_{l_j}(S_j) \leq \sum_{j=1}^{p} v_{k_j}(S'_j). \tag{1.10}$$

In other words, a coalition structure $\mathcal{P}$ is poly-efficient means that there it maximizes, over all possible coalitions structures, the total sum of coalitional payoffs in the game. The coincidence between $\mathcal{CM}(G) = \mathcal{CM}_{\mathcal{P}_2}(G)$ observed in Example 1.2.5 is underlined in the next result in terms of poly-efficient coalition structures.

Proposition 1.2.5. *Given an MTU-game $G = (N, v = (v_j)_{1\leq j\leq m})$ and a coalition structure $\mathcal{P}$ of the game G,*

1. *$\mathcal{CM}(\mathcal{P}, G) = \emptyset$ whenever $\mathcal{P}$ is not poly-efficient in the game G;*

2. *If a coalition structure $\mathcal{P} = (S_j, v_{l_j})_{1\leq j\leq r}$ is poly-efficient in the game G, then $(S_j)_{1\leq j\leq r}$ is efficient in the max-game.*

3. *$\mathcal{CM}(\mathcal{P}, G) = \mathcal{C}^*(\mathcal{P}, v_G) = \mathcal{C}(N, \overline{v_G})$ whenever $\mathcal{P}$ is poly-efficient in the game G.*

Proof.

Consider an MTU-game $G = (N, v = (v_j)_{1\leq j\leq m})$ and a coalition structure $\mathcal{P} = (S_j, v_{l_j})_{1\leq j\leq r}$ of the game G.

1. Suppose that $\mathcal{P}$ is not poly-efficient and suppose that $\mathcal{CM}(\mathcal{P}, G)$ is not empty. Consider $x \in \mathcal{CM}(\mathcal{P}, G)$. Since $\mathcal{P}$ is not poly-efficient in the game G, then there exists a coalition structure $\mathcal{P} = (S'_j, v_{k_j})_{1\leq j\leq p}$ of the game G such that

$$x(N) \geq \sum_{j=1}^{r} v_{l_j}(S_j) > \sum_{j=1}^{p} v_{k_j}(S'_j) = x(N). \tag{1.11}$$

   A contradiction holds. We conclude that $\mathcal{CM}(\mathcal{P}, G)$ is empty.

2. Suppose that a coalition structure $\mathcal{P} = (S_j, v_{l_j})_{1\leq j\leq r}$ is poly-efficient in the game G. Consider a partition $(S'_j)_{1\leq j\leq p}$ of N that is v_G-efficient. By the definition of v_G, there exists for each $j \in \{1, 2, \ldots, p\}$, an activity k_j such that $v_G(S_j) = v_{k_j}(S_j)$. Since $\mathcal{P}$ is poly-efficient and $(S'_j)_{1\leq j\leq p}$ of N that is v_G-efficient, it follows that

$$\sum_{j=1}^{r} v_{l_j}(S_j) = \sum_{j=1}^{p} v_G(S'_j) = \sum_{j=1}^{p} v_{k_j}(S'_j). \tag{1.12}$$

   Therefore $(S_j)_{1\leq j\leq r}$ is efficient in the max-game.

3. Suppose that $\mathcal{P}$ is poly-efficient. Consider a partition $(S'_j)_{1\leq j\leq p}$ of N that is v_G-efficient. As shown at the second item above, it follows that

$$\sum_{j=1}^{r} v_{l_j}(S_j) = \sum_{j=1}^{p} v_G(S'_j). \tag{1.13}$$

   By 1.6 and 1.7, it follows that $\mathcal{C}^*(\mathcal{P}, v_G) = \mathcal{C}^*(\mathcal{P}', v_G)$. Therefore by Proposition 1.1.3, we get $\mathcal{CM}(\mathcal{P}, G) = \mathcal{C}^*(\mathcal{P}', v_G) = \mathcal{C}(N, \overline{v_G})$.

■

Corollary 1.2.3. *Given an MTU-game* $G = (N, v = (v_j)_{1\leq j\leq m})$,

$$\mathcal{CM}(G) = \mathcal{C}(N, \overline{v_G}).$$

Proof.

See Proposition 1.2.4 and Proposition 1.2.5. ■

Proposition 1.2.5 inspires us the following definition.

Definition 1.2.14. Consider an MTU-game $G = (N, v = (v_j)_{1 \le j \le m})$ and a coalition structure $\mathcal{P} = (S_j, v_{l_j})_{1 \le j \le p}$ of G,

1. The game G is **$\mathcal{P}$-balanced** if for all balanced families $\mathcal{F}$ of N with balancedness coefficients $(\gamma_S)_{S \in \mathcal{F}}$,
$$\sum_{S \in \mathcal{F}} \gamma_S v_G(S) \le \sum_{j=1}^{p} v_{l_j}(S_j).$$
2. The game G is **max-balanced** if for all balanced families $\mathcal{F}$ of N with balancedness coefficients $(\gamma_S)_{S \in \mathcal{F}}$,
$$\sum_{S \in \mathcal{F}} \gamma_S v_G(S) \le \max_{\mathcal{B} \in \Pi_N} \sum_{B \in \mathcal{B}} v_G(B).$$

We are now ready to characterize all MTU-games that are stable with respect to a coalition structure.

Theorem 1.2.1.

Given an MTU-game G and a coalition structure $\mathcal{P}$ of G, the game G is $\mathcal{P}$-stable if and only if the game G is $\mathcal{P}$-balanced. ⋆

Proof.
See Theorem 1.1.2 and Proposition 1.2.5 ■

Similarly,

Theorem 1.2.2.

Given an MTU-game G, the game G is stable if and only if the game G is max-balanced. ⋆

Proof.
See Theorem 1.1.1 and Corollary 1.2.3 ■

Theorem 1.2.1 and Theorem 1.2.1 complete the study of the stability of MTU-games. Il appears that the stability of an MTU-game relies on its associated max-game. If the game is not max-balanced, the game is not stable. When the game is max-stable, its core coincides with the core with respect to an arbitrary coalition structure that is poly-efficient. Any coalition structure that is not poly-efficient leads to an empty core: no sharing vector with respect to a non poly-efficient coalition structure is stable.

In MTU-games, utilities are still transferable. In the next chapter, we weaken this condition to introduce a new class of cooperative games.

⋆ ⋆ Chapter Two ⋆ ⋆

Cooperative games with local utilities functions

Cooperative games with transferable utilities (TU-games) have been used in modeling some economic interactions where players form coalitions to produce goods, to make profits, to save costs or to enjoy power. For transferable utilities, it is generally assumed that money (or an infinitely divisible commodity) is available as a means of exchange, but there exist many other economic environments where players form coalitions to produce goods that do not fit this model. In this chapter, we reconsider the transferability assumption in TU-games. More precisely, we consider cooperative games in which (i) every outcome of cooperation is a raw material and each share is made profitable by the player in their own way; and (ii) the utility of a player depends on the current coalition. Due to these two assumptions, those games will be called cooperative games with local utility functions (LUF-games). We define two core concepts for LUF-games that give a generalization of core solutions introduced by Gillies (1953) and Aumann and Dreze (1974) on TU-games. Note that one of our important results in this chapter gives necessary and sufficient conditions of the non-emptiness for the two core concepts using a generalization of balanced family introduced by Bondareva (1963) and Shapley (1967).

The chapter comprises two sections presented as follows. Section 2.1 is devoted to a presentation of the notion of LUF-games. Two core concepts for LUF-games is presented in Section 2.2. The main result in Section 2.2 generalizes the Shapley-Bondareva theorem by providing necessary and sufficient conditions of the non-emptiness for a subclass of LUF-games.

2.1 Cooperative games with local utilities

2.1.1 Illustrative examples

For illustration, consider two providers who separately supply water in two neighboring municipalities. In order to reduce costs, control the sustainability of resources, and increase the profitability of their respective businesses, building some common water treatment facilities is generally a solution. An option for cooperation is to opt for a joint managing authority in charge of maximizing the total profit. This profit is redistributed to the two providers according to a predetermined contract that underpinned the creation of the joint managing authority. This presumes that an outcome of cooperation is an amount of money and that side payments are possible between the most profitable and the least profitable network. In another option for cooperation, each provider contributes to common expenses provided that he gets access to a certain volume of water he makes profitable in his way. The presence of a state agency is sometimes necessary to avoid collusion or side payments that would result in a minimum service offer to the least profitable municipality. Instead of a centralized management based on the common profit as in the first option, any binding agreement between the two suppliers in the second option is directly made on how to share, between the two networks, the volume of water available. The stability of such water quota agreements depends on the unit profitability of the water supply in each of the two networks, and the two characteristics can be of any type, there by departing from Aumann's transferable utility conditions. Still in this example, it is worth mentioning that the profitability level of each of the two networks actually depends on whether the cooperation is implemented or not. To better understand the model we are describing, the following few examples perfectly illustrate the life situations that relate to the model we formalize in this chapter.

Example 2.1.1 (Abroad joint venture). *Two businessmen* 1 *and* 2 *are operating from two distinct countries C_1 and C_2 respectively. The two businessmen are investigating a possible joint venture in a third country C. The possibilities open to the two businessmen is summarized in terms of a cooperative game, where the two players can act separately* *or form a coalition, as follows:*

Country	C_1	C_2	C
Coalition S	$\{1\}$	$\{2\}$	$\{1,2\}$
Worth $v(S)$	100	200	330
Utility rate	0.94	0.88	$0.85; 0.82$

When player 1 *is operating in C_1, the rate of income tax for national investment in C_1 is* 6%*; the corresponding* utility rate *is* 0.94*; that is, a worth of one unit produced in C_1 finally gives* 0.94 *units of utility to player* 1*. In case of a joint venture, the rate of income tax applicable for investment abroad by player* 1 *is* 15% *(this includes taxes in both C_1 and C); the corresponding utility rate is* 0.85*: each benefit unit in country C finally represents to player* 1 *a utility of* 0.85*. A similar reasoning applies to player* 2*. The question is, what would be the share of each player in case of a joint venture?*

To see the contrast with the classical monotonicity condition of utility functions, note that in the case of cooperation in the previous example, a share of 21 for player 2 yields less utility than the worth of 20 realized by player 2 when he acts alone. The reason is that, the utility enjoyed by a player now depends on both the coalition formed and the share received. For example, the utility function u_1 of player 1 is the function defined by

$$u_1(S,x) = 0.94x \text{ if } S = \{1\}\,; \text{ and } u_1(S,x) = 0.85x \text{ if } S = \{1,2\}\,. \tag{2.1}$$

Similarly, the utility function u_2 of Firm 2 when it receives a share x and joint a coalition S is as follows:

$$u_2(S,x) = 0.88x \text{ if } S = \{2\}\,; \text{ and } u_2(S,x) = 0.82x \text{ if } S = \{1,2\}\,. \tag{2.2}$$

It is worth noting that, for a given coalition S, the utility of each player in S is a non decreasing function of his share of the collective worth of S.

The next example now deals with individuals externalities when investment return rates are brought into consideration for water supply in two neighboring municipalities.

Example 2.1.2 (Water providers). *Consider two water providers* 1 *and* 2 *in two neighboring towns. Each provider has some specific geographical constraints that impact on the profit per unit of its water supply offer. A joint water tower will improve on the* *volume of water available for the two municipalities, but will also impact on the profit per unit in both municipalities. The situation is the following:*

Coalition S	$\{1\}$	$\{2\}$	$\{1,2\}$
Volume of water $v(S)$	100	140	300
(profit per unit, fix costs)	$(1.2;10)$	$(1.5;8)$	$(1.1;7);(1.2;5)$

when each provider builds a water tower for his/her own, the profit per unit for providers 1 *and* 2 *are* 1.2 *and* 1.5 *respectively; together with a fix cost of* 10 *for* 1 *and* 8 *for* 2. *With a common water tower located at a boundary area of the two towns, the profit per unit for providers* 1 *and* 2 *are now* 1.1 *and* 1.2 *respectively; together with a fix cost of* 7 *for* 1 *and* 5 *for* 2. *The question is, in the case of a common water tower, which quantity of water would be allocated to each provider?*

It should be noted that, even in the case of cooperation, each water provider wishes to manage its own network and to retain the resulting profit. This is for example the case when one assumes that the two water providers have already agreed on a method of sharing vector the cost of the common water tower; such methods have been investigated in Suzuki and Nakayama (1976) or Young et al. (1982); see also Kruś and Bronisz (2000) for a multicriteria analysis of the case of multiple goods.

Note that in Example 2.1.2, the utility function u_i of provider $i=1,2$ when he receives a share x, is as follows:

$$u_1(S,x) = 1.2x - 10 \text{ if } S = \{1\}; \text{ and } u_1(S,x) = 1.1x - 7 \text{ if } S = \{1,2\} \tag{2.3}$$

$$u_2(S,x) = 1.5x - 8 \text{ if } S = \{2\}; \text{ and } u_2(S,x) = 1.2x - 5 \text{ if } S = \{1,2\}. \tag{2.4}$$

In case of cooperation, an amount x of the collective worth represents $x' = 1.1x - 7$ amount of utility for provider 1 and $x'' = 1.2x - 5$ amount of utility for provider 2. It appears that the two providers value differently the same share of the collective worth.

Example 2.1.3 (One sales unit - n production units). *Consider* n *production units* 1, 2, ..., n *that supply a certain sales unit. Each production unit has the equipment that enables it to produce certain goods, each in a limited quantity and with a certain level of profitability. A coalition* S *of production units can be formed to supply up to a quantity* $v(S)$ *of the goods of type* T_S *that can be absorbed by the sales unit. In this case, the* *members of S simply have to agree on the maximum quantity of the goods of type T_S that each partner in S should produce. Furthermore, a production unit i in S can produce at most a quantity $q_{S,i}$ of T_S type of goods with a profitability of $p_{i,S}$ per unit. Clearly, what each production unit gains depends on its coalition and its share of the maximum quantity of goods to be supplied. Due to this production limitation, the utility $u_i(S,q)$ associated with a production unit i when it is allowed a maximum quantity q of T_S type of goods as a member of a coalition S is given by:*

$$u_i(S,q) = min(q, q_{S,i})p_{S,i}. \tag{2.5}$$

Note that, a utility function of a player in this game is no longer a linear function of his share.

In each of the examples above, analyzing cooperation in the corresponding games is no longer possible under the TU-game model; see Aumann (1960) on utility transferability: with at least three players, only linear utility functions with the same slope guarantee the transferability of utilities. The utility function of a player, say i, is now a function with several arguments that possibly include the coalition that i joins. Even in the case of linear utility functions, the slope of the function on the same item may change from one player to another. A formal presentation of such games is the subject of the next section.

2.1.2 The model

To give a formal presentation of the games considered here, we denote by $\mathcal{P}_i(N)$ the set of all coalitions that contain a given player i; that is

$$\mathcal{P}_i(N) = \{S \subseteq N : i \in S\}.$$

Definition 2.1.1. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is:

- nondecreasing if for all $q, q' \in \mathbb{R}$, $q' \geq q \Longrightarrow f(q') \geq f(q)$;
- increasing if for all $q, q' \in \mathbb{R}$, $q' > q \Longleftrightarrow f(q') > f(q)$.

Definition 2.1.2. A cooperative game with local utilities functions(LUF-game) is a triple $\left(N, v, u = (u_i)_{i\in N}\right)$ such that $v : \mathcal{P}(N) \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$; and for each $i \in N$, $u_i : \mathcal{P}_i(N) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing function with respect to its second argument; that is for all $S \in \mathcal{P}_i(N)$, $u_{i,S} : q \longmapsto u_i(S,q)$ is nondecreasing.

Given a LUF-game (N, v, u), the characteristic function v can be seen as the (coalitional) production function; and given $i \in N$, u_i is the utility function of player i. Note that the worth of a coalition or the share of a player in a LUF-game may be an amount of money as in Example 2.1.1; or an amount of a certain good as in Example 2.1.2 or Example 2.1.3. Furthermore, distinct coalitions may produce distinct goods. This is the case in Example 2.1.3 where the members of a coalition S produce T_S type of goods. The possibility of several commodities was also considered by Aumann (1960) who assumes that there exists a set of possible coalitional outcomes of the game before side payments are made. It appears from the conclusion by the author that, a transferable utility game (TU-game) with at least three players and a single commodity (when a coalition is formed, its worth is an amount of a given commodity) can be seen as a LUF-game (N, v, u) where for some real number $a > 0$ and for some collection $b = (b_i)_{i\in N}$ of real numbers,

$$\text{for all } q \in \mathbb{R}, u_i(S, q) = aq + b_i \text{ for all } S \in \mathcal{P}_i(N) \text{ and for all } i \in N. \tag{2.6}$$

In this case, players all have linear utility functions that all have the same rate and do not depend on the coalition in consideration. It clearly appears that the class of LUF-games constitutes a generalization of the class of TU-games. From now on,

Definition 2.1.3. *A TU-game is a LUF-game (N, v, u) such that for some real number $a > 0$ and for some collection $b = (b_i)_{i\in N}$ of real numbers, $u_i(S, q) = aq + b_i$ for all $i \in N$, for all $S \in \mathcal{P}_i(N)$ and for all $q \in \mathbb{R}$.*

In a LUF-game, the utility of a player depends on the coalition he finally belongs to; that is $u_i(S, q)$ is the utility of player i provided that i joins S and is given a share q of the collective worth $v(S)$ achieved by the members of S. Clearly, the same share (even with only one commodity) may now provides to i distinct utilities from a coalition to another. To allow utility comparability by each player on distinct commodities or from distinct coalitions, we assume that each player is able to measure his utility on any of his share using monetary units. In Example 2.1.2, the utility function of a water provider, his benefit, changes from a coalition to another; but the benefit is measured in each case using the same monetary unit.

As already described above, one can obviously observe that Examples 2.1.1 and 2.1.2 are supported by LUF-games rather by the classical TU-games. In Example 2.1.1, the set of players is $N = \{1, 2\}$, the coalitional production function is defined by

$$v(\emptyset) = 0, \quad v(\{1\}) = 10, \quad v(\{2\}) = 20 \text{ and } v(\{1,2\}) = 35. \tag{2.7}$$

and the utility functions of the two players are defined by Equations (2.1) and (2.2). This can be summarized as follows

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$v(S)$	100	200	330
$(u_i(S,q))_{i\in S}$	$0.94q$	$0.88q$	$0.85q; 0.82q$

(2.8)

Similarly, the game of Example 2.1.2 can be represented as follows:

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$v(S)$	100	140	300
$(u_i(S,q))_{i\in S}$	$1.2q - 10$	$1.5q - 8$	$1.1q - 7; 1.2q - 5$

(2.9)

Finally the situation in Example 2.1.3 is formalized as a LUF-game (N, v, u) such that $N = \{1, 2, ..., n\}$; for all coalition S and given $i \in S$,

$$v(S) = \sum_{i\in S} q_{S,i} \text{ and } u_i(S,q) = min(q, q_{S,i}) \times p_{S,i} \text{ for all } q \in \mathbb{R}$$

Remark 2.1.1. *In a LUF-game, note that the share received by a player while joining a coalition may be an amount of money (see Example 2.1.1) or not (see Example 2.1.2).*

Now depending on the type of utility functions, one may defined subclasses of LUF-games.

Definition 2.1.4. *A LUF-game with local fixed-term externalities is any LUF-game (N, v, u) such that for all $i \in N$,*

$$u_i(S,q) = q + b_i(S)$$

for some local constants (depending of the cooperation of player i) $b_i(S) \in \mathbb{R}$.

In this case, $b_i(S)$ is the utility fixed-term of i in S.

In a LUF-game with local fixed term-externalities, the utility of a player in a coalition is obtained by shifting his/her share by a given local constant.

Definition 2.1.5. *A LUF-game with local fixed-rate externalities is any LUF-game* (N, v, u) *such that for all* $i \in N$,

$$u_i\left(S, q\right) = a_i\left(S\right) q$$

for some local non negative constants (depending of the cooperation of player i*)* $a_i\left(S\right)$.

In this case, $a_i\left(S\right)$ is the utility fixed-rate of i in S.

In a LUF-game with local fixed-rate externalities, the utility of a player in a coalition is the product of his/her share by a local constant rate.

Definition 2.1.6. *A LUF-game* (N, v, u) *is linear (in utilities) if for all coalitions* S *and for all players* $i \in S$ *there exists some real numbers* $a_{i,S} > 0$ *and* $b_{i,S}$ *such that*

$$u_i(S, q) = a_{i,S}q + b_{i,S}. \tag{2.10}$$

In this case, $a = \left(a_{i,S}\right)_{S \in 2^N, i \in S}$ *and* $b = \left(b_{i,S}\right)_{S \in 2^N, i \in S}$ *will be called the gradient collection and the fixed-term collection respectively.*

We also say that (N, v, u) is a LUF-game with local uniform externalities, the utility of a player in a coalition is an affine transformation of his/her share by combining an utility fixed-rate with an utility fixed-term. Therefore when the collection gradient are all positives reals numbers we call it the positive gradient collection.

Definition 2.1.7. *A* regular *LUF-game is any LUF-game* (N, v, u) *such that for all* $i \in N$, *there exists an increasing function* μ_i *from* $\mathbb{R}$ *to* $\mathbb{R}$ *and*

$$u_i\left(S, q\right) = \mu_i(q) \text{ for all } S \in \mathcal{P}_i\left(N\right).$$

Note that the word "regular" only refers to the fact that each player has a unique utility function which is increasing and does not change from one coalition to another.

Definition 2.1.8. *A LUF-game* (N, v, u) *is consistent if for all* $i \in N$, u_i *is increasing function with respect to its second argument.*

When a LUF-game is consistent, a larger share of the worth of a coalition to a player provides him/her a better utility.

As in TU-games, two questions arise given a LUF-game: which coalition would emerge? In case of cooperation, what would be the share of each player? In this paper, we assume that players form the grand coalition and propose two notions of core concepts. Each of the two cores corresponds to a set of sharing vector that satisfy the stand-alone test: no coalition exists such that its members have incentive to form their own coalition.

2.2 Core solution for LUF-games and characterization

As with TU-games, we provide here a core concept for LUF-games. The main idea consists in redefining a stand alone test that captures players preferences as modeled by the collection of individual utility functions.

2.2.1 Classical core on LUF-games

From now on, we assume that:

(CL_1) players agree to form the grand coalition (the coalition structure is $\{N\}$);

(CL_2) goods are infinitely divisible;

(CL_3) players are rational.

Under these hypothesis, the problem which remains to be solved is therefore that of sharing the payoff of the grand coalition. It is natural to think that the members of each coalition S cannot accept a sharing vector if they have an incentive to stand alone once S is formed. This is the main idea of the following notion of dominance between sharing vector in LUF-games.

Definition 2.2.1. *Given a LUF-game (N, v, u) and a coalition S, an S-sharing vector is any collection $x = (x_i)_{i\in S}$ of real numbers that sum to $v(S)$; that is $x(S) = v(S)$.*

The set of all S-sharing vectors in the game (N, v, u) will be denoted by $\chi(S, v)$. In particular, an N-sharing vector will be called a sharing vector for short.

Definition 2.2.2. Let (N, v, u) be a LUF-game and $x \in \chi(N, v)$.

The payoff vector x is dominated if there exists a coalition S and a s-tuple $y \in \chi(S, v)$ such that:

(a) $u_i(S, y_i) > u_i(N, x_i)$ for all $i \in S$;

(b) $y(S) \leq v(S)$.

In this case, we say that x is dominated by y via S; or that (S, y) is an objection on x via S or say simply that x is unstable.

Firstly, condition (a) in Definition 2.2.2 is the requirement that each player i in the objecting coalition S expects in S a share y_i that provides much more utility than what is available in x_i: this is the *incentive condition.* Secondly, condition (b) expresses the fact that, the total amount $y(S)$ of claims should not exceed the worth $v(S)$ of the objecting coalition S: this is the *feasibility condition.* Being dominated dismisses the corresponding sharing vector agreement from being a (*one-shot*) stable outcome of a LUF-game when collective rationality is the sole basis of stability: in case of domination, the members of an objecting coalition would reject the proposal since they may find their respective utilities improved when they appropriately reorganize themselves.

Example 2.2.1. *Let (N, v, u) be the LUF-game in Equation (2.8). Consider the proportional sharing vector vector $x = (110, 220)$. The share $x_1 = 110$ of player 1 represents a total utility of $u_1(N, 110) = 110 * 0.88 = 93.5$. However, player 1 can stay alone to produce 100 with a total utility of $u_1(\{1\}, 100) = 100 * 0.94 = 94$. Thus player 1 is better off by operating alone. The proportional sharing vector x is therefore dominated via $(\{1\}, 100)$.*

Note that the existence of an objection to a sharing vector implies that some players are better off by disrupting the grand coalition to form a coalition and share the corresponding worth in an appropriate way. When the stand alone test is the minimal requirement, such a sharing vector is discarded as a possible outcome of the a LUF-game.

Definition 2.2.3. Let (N, v, u) be a LUF-game.

The core of the game (N, v, u) is the set $\mathcal{CL}(N, v, u)$ of all stable (undominated) sharing vectors.

in other words:

$$\mathcal{CL}(N,v) = \{x \in \chi(N,v) : \ x \text{ stable } \}. \tag{2.11}$$

The LUF-game (N,v,u) is **stable** if $\mathcal{CL}(N,v,u)$ is non-empty.

Example 2.2.2. *Let (N,v,u) be the LUF-game at (2.8). Consider a sharing vector x. An objection $(\{1\}, y_1)$ by player 1 on x holds if $y_1 = 100$ and $u_1(\{1\}, y_1) = u_1(\{1\}, 100) = 100 \times 0.94 > u_1(\{1,2\}, x_1) = 0.85x_1$. Therefore, no objection on x by player 1 exists if and only if $0.85x_1 \geq 94$. Similarly, no objection on x by player 2 exists if and only if $0.82x_2 \geq 176$. Thus*

$$\mathcal{CL}(N,v,u) = \left\{x \in \mathbb{R}^2 : x_1 + x_2 = 330, x_1 \geq \frac{94}{0.85}, x_2 \geq \frac{176}{0.82}\right\} = \left\{(q, 330-q) : \frac{94}{0.85} \leq q \leq \frac{94.6}{0.82}\right\}.$$

Besides, when we replace the collection of utility functions u by any other collection u' of utility functions that satisfy Equation (2.6), the corresponding Gillies core $\mathcal{C}(N,v)$ is given by

$$\mathcal{C}(N,v) = \left\{x \in \mathbb{R}^2 : x_1 + x_2 = 330, x_1 \geq 100, x_2 \geq 200\right\} = \{(q, 330-q) : 100 \leq q \leq 130\}$$

Note that for the current game, $\mathcal{CL}(N,v,u)$ is a proper subset of $\mathcal{C}(N,v)$. Both $\mathcal{C}(N,v)$ and $\mathcal{CL}(N,v,u)$ are sketched in Figure 2.2.1 to highlight the impact of utility functions on the shape of the core:

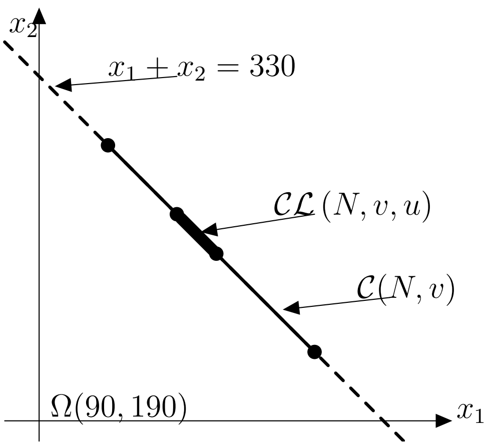

Figure 2.1: **Core of the LUF-game at Equation (2.8) and the corresponding Gillies core**

Remember that we assume in this section that the grand coalition is formed. A set-valued solution for LUF-games is any mapping $\mathcal{S}$ that associates each LUF-game (N,v,u)

with a subset $\mathcal{S}(N, v, u)$ of the $\chi(N, v)$. The set $\mathcal{S}(N, v, u)$ is the set of possible outcomes in the game (N, v, u) with respect to the solution concept $\mathcal{S}$. A minimum requirement for a desirable solution concept is the Pareto principle: if from a sharing vector x to another sharing vector y, all players are better off, then the Pareto principle stipulates that x should not be selected. We rephrase the Pareto principle in the current setting of LUF-games as follows:

Definition 2.2.4. *Given a LUF-game (N, v, u), a sharing vector x is Pareto dominated if for some sharing vector y, $u_i(N, y_i) > u_i(N, x_i)$ for all $i \in N$.*

A solution concept $\mathcal{S}$ for LUF-games is Pareto efficient if $\mathcal{S}(N, v, u)$ contains a Pareto dominated sharing vector for no LUF-game (N, v, u).

Clearly, the core of LUF-game is Pareto efficient. To go beyond the Pareto principle, note that in a LUF-game, a player may be indifferent on two distinct shares. To see this, consider the following example:

Example 2.2.3. *Let (N, v, u) be the two-player game defined as follows:*

S	$\{1\}$	$\{2\}$	$\{1, 2\}$
$v(S)$	8	15	11
$(u_i(S, q))_{i \in S}$	$\min(q, 5)$	$\min(q, 20)$	$(2\min(q, 4)\,; 3\min(q, 10))$

The intuition behind this LUF-game comes from Example 2.1.3 with one sales unit and two production units 1 and 2, the players. We rephrase it as follows: player i masters a certain technology to produce goods of type T_i. The market can absorb up to 8 units of T_1 type of goods; but player 1 can produce only up to 5 units of this good with a benefit per unit of 1. The market can absorb only 15 units of T_2 type of goods although player 2 can produce up to 20 units of this good. By cooperating, the two players benefit from the experience of each other to produce each an advanced design good of type T_{12}. The market can absorb up to 11 units of T_{12} goods. For the good of type T_{12}, the production ability of player 1 is 4 units at most with a benefit per unit of 2 while player 2 is able to produce up to 7 units with a benefit per unit of 3. In case of cooperation, player 1 can produce at most 4 units of goods of type T_{12}. Thus, any share of more than 4 units provides as much utility as a share of 4 units. Thus instead of a share of $4 + \varepsilon$ with $\varepsilon > 0$ to player 1, it is optimal to give player 1 only 4 units and reallocate the surplus of ε to player 2.

The previous illustration leads us to the following refinements of the core. The next definition gives a notion of a weak domination relation for LUF-games.

Definition 2.2.5. Let (N, v, u) be a LUF-game and $x \in \chi(N, v)$.

The sharing vector vector x is weakly dominated if there exists a coalition S and a s-tuple $y \in \chi(S, v)$ such that:

(i) $u_i(S, y_i) \geq u_i(N, x_i)$ *for all* $i \in S$*; and* $u_i(S, y_i) > u_i(N, x_i)$ *for some* $i \in S$*;*

(ii) $y(S) \leq v(S)$.

In this case, we say that x is weakly dominated by y via S, or that (S, y) is a weak objection on x via S or simply that x is weakly unstable.

The weak dominance relation is obtained by weakening only the incentive condition (i) in Definition 2.2.2: the utility of each member of the objecting coalition should not be altered from x to y; and y should provide a better utility for some of those players as compared to what they receive in x.

Definition 2.2.6. Let (N, v, u) be a LUF-game.

The strong core of the game (N, v, u) is the set $\mathcal{CL}^s(N, v, u)$ of all sharing vector that are weakly dominated by no sharing vector vector,

that is:

$$\mathcal{CL}^s(N, v) = \{x \in \chi(N, v) : \ x \text{ strongly stable } \}. \tag{2.12}$$

The LUF-game (N, v, u) is **strongly stable** if $\mathcal{CL}^s(N, v, u)$ is non-empty.

A sharing vector that is weakly Pareto-dominated is sub-optimal since some players can be better off without any adverse effect on other players.

Clearly, a sharing vector that is weakly Pareto dominated is suboptimal since some players can be better off without any adverse effect on other players. The strong core is thus a natural refinement of the core. It is obvious that the strong core $\mathcal{CL}^s(N, v, u)$ of a LUF-game (N, v, u) is a subset of the core $\mathcal{C}(N, v, u)$; but $\mathcal{CL}^s(N, v, u)$ may differ from $\mathcal{CL}(N, v, u)$ as shown below.

Proposition 2.2.1. *For all LUF-game (N, v, u), $\mathcal{CL}^s(N, v, u) \subseteq \mathcal{CL}(N, v, u)$; and the inclusion is strict for some LUF-games.*

Proof.

Firstly, consider $x \in \chi(N, v)$ if x is dominated, then there exists a coalition S and a s-tuple $y \in \chi(S, v)$ satisfy items (a) and (b) in Definition 2.2.2. Thus there exists then there exists a coalition S and a s-tuple $y \in \chi(S, v)$ satisfy items (a) and (b) in Definition 2.2.5. By contraposition, it follows that: if x is strongly stable, then x is stable. Secondly, consider the sharing vector $x = (5, 6)$ in the LUF-game given in Example 2.2.3. The utility of player 1 is $u_1(\{1, 2\}, x_1) = 2 \min(x_1, 4) = 8$. Since the maximum amount that player 1 can achieve by opting out is at must equal to 5, there is no objection of player 1 on x. Similarly, the maximum amount that player 2 can achieve by opting out is 15. Since $u_2(\{1, 2\}, x_2) = 3 \min(x_2, 9) = 18$, there is no objection of player 2 on x. Hence x is a core sharing vector.

Now, let $S = \{1, 2\}$ and $y = (4, 7)$. It can be checked that y weakly dominates x via N. Thus x is not a strong core selection. This proofs that the strong core of this game is a proper subset of its core. ■

Taking into account the possibility of indifference in comparing distinct shares by some players, we consider the following refinement of the Pareto principle:

Definition 2.2.7. *Given a LUF-game (N, v, u):*

- *A sharing vector x is weakly Pareto dominated if for some sharing vector y, $u_i(N, y_i) \geq u_i(N, x_i)$ for all $i \in N$ and $u_i(N, y_i) > u_i(N, x_i)$ for some $i \in N$.*
- *A solution concept $\mathcal{S}$ for LUF-game is strongly Pareto efficient if $\mathcal{S}(N, v, u)$ never contains a weakly Pareto dominated sharing vector for every LUF-game (N, v, u).*

By definition, any strongly Pareto efficient solution concept for LUF-games is also Pareto efficient.

Proposition 2.2.2. The core of any LUF-game is Pareto efficient; but not strongly Pareto efficient.

Proof.

The core of LUF-game is Pareto efficient by its definition, see Definitions 2.2.2 and 2.2.3. Now, consider the LUF-game presented in Example 2.2.3. In the proof of Proposition 2.2.1, it has been shown that $x = (5, 6)$ is a core sharing vector and is weakly dominated by $y = (4, 7)$. Therefore the core of LUF-games is not Paretro efficient. ■

Proposition 2.2.3. *The strong core of LUF-games is strongly Pareto efficient.*

The proof is obvious and is therefore omitted. For LUF-games in general, the core and the strong core differ on some LUF-games. However, it is shown in the next proposition that the two core concepts coincide on each *regular* LUF-game (N, v, u).

Proposition 2.2.4. *If a LUF-game (N, v, u) is regular, then $\mathcal{CL}^s(N, v, u) = \mathcal{CL}(N, v, u) = \mathcal{C}(N, v)$.*

Proof.

Consider a constant LUF-game (N, v, u). We show that $\mathcal{C}(N, v) \subseteq \mathcal{CL}^s(N, v, u)$ and $\mathcal{CL}(N, v, u) \subseteq \mathcal{C}(N, v)$. By definition of a regular LUF-game, there exists a collection $(\mu_i)_{i\in N}$ of increasing utility functions which satisfy (2.1.7). Consider a sharing vector $x \notin \mathcal{C}(N, v)$. Then by the definition of $\mathcal{C}(N, v)$, x is not collectively rational. Thus there exists a coalition S such that $x(S) < v(S)$. Let $\varepsilon = \frac{v(S)-x(S)}{|S|} > 0$ and pose $y_i = x_i + \varepsilon$ for all $i \in S$. It follows that $y(S) = v(S)$ and $u_i(S, y_i) = \mu_i(y_i) > \mu_i(x_i) = u_i(N, x_i)$. Therefore (S, y) is an objection on x via S. Thus $x \notin \mathcal{CL}(N, v, u)$. This proves that $\mathcal{CL}(N, v, u) \subseteq \mathcal{C}(N, v)$.

Now, consider a sharing vector $x \notin \mathcal{CL}^s(N, v, u)$. Then by the definition of $\mathcal{CL}^s(N, v, u)$, there exists a coalition S and S-sharing vector y such that $u_i(S, y_i) \geq u_i(N, x_i)$ for all $i \in S$ and $u_i(S, y_i) = u_i(N, x_i)$ for some $i \in S$. By item (a) in Definition 2.2.5, let j be a player in S such that $u_j(S, y_j) = \mu_j(y_j) > u_i(N, x_j) = \mu_j(x_j)$. Since, μ_j is an increasing function, we deduce that $y_j > x_j$. Pose $\varepsilon = \frac{y_j - x_j}{|S|} > 0$ and pose $z_i = y_i + \varepsilon$ for all $i \in S$. It follows that $z(S) = v(S)$ and $z_i > x_i$ for all $i \in S$. Therefore (S, z) is an objection on x via S. Thus $x \notin \mathcal{C}(N, v)$. This proves that $\mathcal{C}(N, v) \subseteq \mathcal{CL}^s(N, v, u)$. The results follows since the strong core is a subset of the core. ■

Corollary 2.2.1. *For any TU-game (N, v, u) in the sense of (2.6), $\mathcal{CL}^s(N, v, u) = \mathcal{CL}(N, v, u) = \mathcal{CL}^s(N, v)$.*

Proof.

The result holds from Proposition 2.2.4 since any TU-game (N, v, u) in the sense of Equation 2.6 is a regular LUF-game. ■

Corollary 2.2.1 shows that the core and the strong core of LUF-games are both extensions of the core of TU-games. Furthermore, the core (or the strong core) of a LUF-game may differ from the core of the associated TU-game only if the utility of some players on a given share depends on the coalition in consideration; or the preference relation of some player admits indifference (distinct shares provide the same amount of utility).

The coincidence between the core and the strong core is mainly due to the properties of utility functions. In the case of regular LUF-games, utility functions depend only on the share of each player; and not on the coalition a player joins. This can be seen as an inter-coalition regularity. One can also think about other types on regularity such as topological regularities. For example, one may require each utility function to be continuous: each player compares his shares in a smooth way, without any jump[1]

In the proposition below, it is shown that the continuity is sufficient condition for the coincidence between the core and the strong core of LUF-games when utility functions are increasing.

Proposition 2.2.5. *Let (N, v, u) be any LUF-game. Assume that for all coalitions S and for each player $i \in S$, the function $u_i(S, \cdot)$ is continuous and increasing on $\mathbb{R}$.*

Then a sharing vector x is dominated if and only if x is weakly dominated; that is $\mathcal{CL}^s(N, v, u) = \mathcal{CL}(N, v, u)$.

Proof.

Consider a LUF-game (N, v, u) and a sharing vector x. Assume that for all coalitions S and for each player $i \in S$, the function $u_i(S, \cdot)$ is continuous on $\mathbb{R}$.

Sufficiency. Assume that x is dominated. Then x is, by definition, weakly dominated via S.

Necessity. Assume that x is weakly dominated. Then there exists an objection (S, y) on x such that:

(i) $u_i(S, y_i) \geq u_i(N, x_i)$ for all $i \in S$; and $u_i(S, y_i) > u_i(N, x_i)$ for some $i \in S$;

(ii) $y(S) \leq v(S)$.

First note that if S is a singleton, then (S, y) is an objection on x; and x is dominated. Now suppose that S contains at least two players. Let j be a player in S such

[1] More formally, a function f from $\mathbb{R} \to \mathbb{R}$ is continuous if for each $a \in \mathbb{R}$ and for any variation $\varepsilon > 0$ of $f(a)$, there exists a small variation $\alpha > 0$ of a such that for all $q \in \mathbb{R}$, $|q - a| < \alpha \Longrightarrow |f(q) - f(a)| < \varepsilon$.

that $u_j(S, y_j) > u_j(N, x_j)$. Since $u_j(S, y_j) - u_j(N, x_j) > 0$ and $u_j(S, \cdot)$ is continuous, then for the variation $\varepsilon = u_j(S, y_j) - u_j(N, x_j) > 0$ of $u_j(S, y_j)$, there exists a small variation $\alpha > 0$ of y_j such that for all $q \in \mathbb{R}$:

$$|q - y_j| < \alpha \implies |u_j(S, q) - u_j(S, y_j)| < \varepsilon.$$

Let q be any share of $v(S)$ such that $|q - y_j| < \alpha$. We have $|u_j(S, q) - u_j(S, y_j)| < \varepsilon$. Since $\varepsilon = u_j(S, y_j) - u_j(N, x_j)$, we deduce that $-\varepsilon = u_j(N, x_j) - u_j(S, y_j) < u_j(S, q) - u_j(S, y_j)$. Hence $u_j(N, x_j) < u_j(S, q)$. Define the S-sharing vector z by

$$z_i = \begin{cases} y_i + \frac{\alpha}{2|S\setminus\{j\}|} & \text{if } i \in S\setminus\{j\} \\ \\ y_j - \frac{\alpha}{2} & \text{if } i = j \end{cases}$$

Note that $|z_j - y_j| < \alpha$. Thus $u_j(N, z_j) < u_j(S, x_j)$ as shown above. Moreover, for each player $i \in S\setminus\{j\}$, $z_i > y_i$. Since utility functions are increasing, $u_i(S, z_i) > u_i(S, y_i) \geq u_i(S, x_i)$. Hence $u_i(S, z_i) > u_i(S, x_i)$. Finally and by construction, z satisfies $z(S) = y(S) \leq v(S)$. In summary, (S, z) is an objection on x and x is dominated. ■

Beside continuity, it is also required in Proposition 2.2.5 that individual utility functions should also be increasing. For example, utility functions in the LUF-game presented in Example 2.2.3 are continuous; but not increasing. Therefore one can not drop in Proposition 2.2.5 the monotonicity condition.

2.2.2 Characterization of core elements for LUF-games

The lower transition correspondence and strong core

Consider a LUF-game (N, v, u) and a sharing vector x on (N, v, u). Given a coalition S, the question is whether the members of S can be better off by opting out from the grand coalition. To this end, each member of S examines what would be his minimum share of the worth $v(S)$ that guarantees at least as much utility as the current share x_i of $v(N)$. The answer is given by a set-valued map denoted by $\mathrm{LT}^{i,S}$ and defined as follows:

$$LT_{i,S}(q) = \{Q \in \mathbb{R} : u_i(S, Q) \geq u_i(N, q)\}. \tag{2.13}$$

The set $LT_{i,S}(q)$ will be called the *lower transition set* of q as the share q of $v(N)$ provides to player i at most as much utility as any share $Q \in LT_{i,S}(q)$ received by i when S is formed. Note that for the sharing vector x, any share Q of $v(S)$ which is out of $LT_{i,S}(x_i)$ provides to player i less utility than the share x_i of $v(N)$. Since each player is assumed to be rational, player i will never accept to leave the grand coalition when he is offered a share Q of $v(S)$ which is out of $LT_{i,S}(x_i)$. For the stability of the sharing vector x, the collection $LT_{i,S}(x_i), i \in S$ plays the role of a control test to check the feasibility of any weak objection via S in a sense which will be made clearer later. But before, we introduce the following definition:

Definition 2.2.8. *Let (N, v, u) be a LUF-game, S a coalition and i a player in S.*

- *The lower transition correspondence of player i from N to S is the map $LT_{i,S}$ defined by (2.13).*
- *The lower transition function of player i from N to S is the map $u_{i,S}$ defined for all $q \in \mathbb{R}$ by*

$$u_{i,S}(q) = \begin{cases} \inf\left(LT_{i,S}(q)\right) & \textit{if } LT_{i,S}(q) \neq \emptyset \\ \\ +\infty & \textit{otherwise} \end{cases}$$

In case $u_{i,S}(q)$ is finite, it is called the lower compensation share *of q when player i opts out from N to S.*

Note that $LT_{i,S}(q)$ is the set of all shares Q of the worth of S that provides to player i as much utility as the share q of $v(N)$. Moreover, when $u_{i,S}(q) = +\infty$, $LT_{i,S}(q) = \emptyset$. In this case, no share received by i in S compensates the utility provided to i by the share q of $v(N)$. This occurs when the line of $q \longmapsto u_i(N, q)$ is above the curve of $Q \longmapsto u_i(S, Q)$. When $u_{i,S}(q)$ is finite, $LT_{i,S}(q) \neq \emptyset$; and $u_{i,S}(q)$ is the smallest share of $v(S)$ such that any greater share of $v(S)$ provides at least as much utility to player i as the share q of $v(N)$. Note that it is still possible that $u_i(S, u_{i,S}(q)) < u_i(N, q)$ when the utility function $u_i(S, \cdot)$ is discontinuous for example. One may also have $u_{i,S}(q) = -\infty$. In this later case, the curve of $Q \longmapsto u_i(S, Q)$ is above the line $q \longmapsto u_i(N, q)$: the share q of $v(N)$ provides to player i at most as much utility as any share of the worth $v(S)$ of coalition S.

When $u_{i,S}(q)$ is finite, it is worth mentioning that $u_i(S, u_{i,S}(q) + \varepsilon) \geq u_i(N, q)$ for any small variation $\varepsilon > 0$. The word "compensation" in Definition 2.2.8 only refers to the fact that $u_{i,S}(q)$ is a threshold up to which a greater share of $v(S)$ provides to player i at least at most utility as the share q of $v(N)$. We say that the lower compensation share $u_{i,S}(q)$ is *exact* if $u_{i,S}(q)$ is finite and $u_i(S, u_{i,S}(q)) = u_i(N, q)$.

Example 2.2.4. In the game of Example 2.1.2 formalized by Equation (2.9), the utility function u_1 of player 1 is defined by

$$u_1(S, x) = 1.2x - 10 \text{ if } S = \{1\};\ \text{and } u_1(S, x) = 1.1x - 7 \text{ if } S = \{1, 2\}.$$

Suppose that the grand coalition is formed and that 1 receives $x_1 = q$. Thus the lower transition correspondence of 1 from N to coalition $S = \{1\}$ is such that

$$UT_{1,S}(q) = \{q' \in \mathbb{R} : 1.2q' - 10 \geq 1.1q - 7\} = \left\{q' \in \mathbb{R} : q' \geq \frac{11}{12}q + \frac{10}{4}\right\}$$

Therefore the associated lower transition function is given by $u_{1,S}(q) = \frac{11}{12}q + \frac{25}{6}$. In other words, from $\{1,2\}$ to $\{1\}$, the $\{1\}$-correspondent of q is $\frac{11}{12}q + \frac{10}{4}$: in terms of utility for player 1, a share q of $v(N)$ corresponds to the share $\frac{11}{12}q + \frac{10}{4}$ of $v(\{1\})$.

Example 2.2.5. *In the game of Example 2.2.3, the utility function u_1 of player 1 is defined by*

$$u_1(S, q) = \min(q, 5) \text{ if } S = \{1\};\ \text{and } u_1(S, q) = 2\min(q, 4) \text{ if } S = \{1, 2\}.$$

The lower transition correspondence of 1 from N to coalition $S = \{1\}$ is such that

$$LT_{1,S}(q) = \{Q \in \mathbb{R} : \min(Q, 5) \geq 2\min(q, 4)\}$$

Therefore the associated strong transition function is such that $u_{1,S}(q) = 2q$ if $q \leq \frac{5}{2}$; and $u_{1,S}(q) = +\infty$ otherwise. As illustrated in Figure 2.2.2, for $q \leq \frac{5}{2}$, the utility of player 1 is $2q$ and the horizontal line $u = 2\min(q, 4) = 2q$ intercepts the curve of $u = \min(Q, 5)$ at a point which locates, for player 1, the compensation share of q. However, for $q > \frac{5}{2}$, $2\min(q, 4) > 5$. The horizontal line $u = 2\min(q, 4)$ is now above the curve of $u = \min(Q, 5)$ and no longer intercept it. In this later case, $UT_{i,\{1\}}(q) = \emptyset$ and $u_{i,\{1\}}(q) = +\infty$: player 1 has no incentive to opt out from the grand coalition.

Recall that in a TU-game seen as a LUF-game, the utility function u_i of player i in each coalition $S \in \mathcal{P}_i(N)$ is such that $u_i(q) = aq + b$ for some constant $a > 0$ and b. Therefore the strong transition function of i is defined by $u_{i,S}(q) = q$. Thus, the utility function and the strong transition function coincide for TU-games.

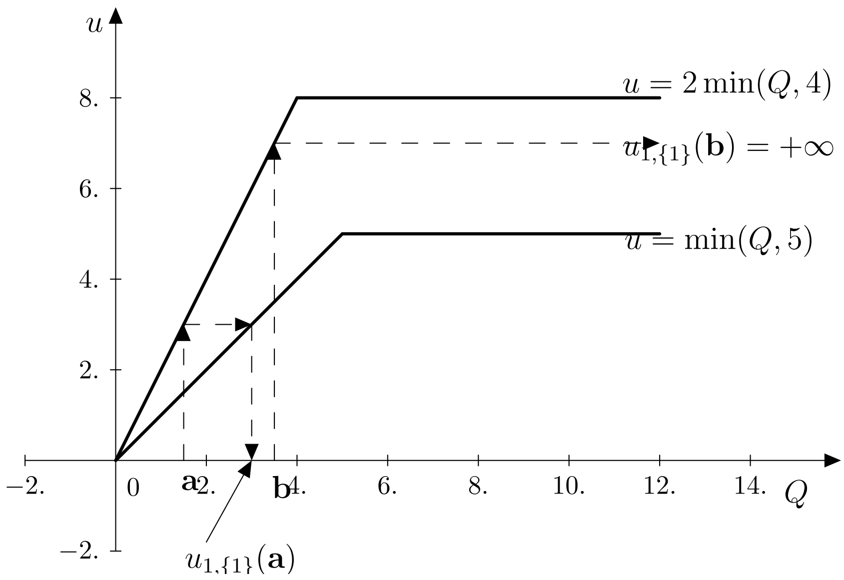

Figure 2.2: **Lower compensation shares when player 1 opts out from $\{1,2\}$ in the game of Example 2.2.3**

PROPOSITION 2.2.6. *Let (N, v, u) be a LUF-game and S a coalition.*

1. *If x is dominated via S, then $\sum_{i\in S} u_{i,S}(x_i) \leq v(S)$.*
2. *If x is weakly dominated via S, then $\sum_{i\in S} u_{i,S}(x_i) \leq v(S)$.*

Proof. Consider a LUF-game (N, v, u), a sharing vector x and a coalition S.

Assume that x is dominated via S. Then there exists an S-sharing vector y such that (S, y) is an objection on x. Let i be a player in S. Then $u_i(S, y_i) > u_i(N, x_i)$. It follows by definition of $u_{i,S}$ that $y_i \in LT_{i,S}$. Therefore $u_{i,S}(x_i) \leq y_i$. This implies that

$$\sum_{i\in S} u_{i,S}(x_i) \leq y(S) \leq v(S).$$

The later inequality holds from the fact that (S, y) is an objection on x.

The second item of proposition is proved by using similar arguments to those presented above. ∎

The previous result shows that when a sharing vector is dominated via a coalition S, the sum of all compensation shares of the members of S is less than or equal to the worth of S. Put another way, if a sharing vector x is such that for all coalitions S, the sum of all compensation shares of players in S is greater than the worth of S in the game, then x admits no objection and is therefore a core selection. The next result is a general characterization of all sharing vectors that are belong to the strong core for a LUF-game.

Lemma 2.2.1. *Let (N, v, u) be any LUF-game. Then the two assertions below are equivalent:*

1. *A sharing vector x belongs to the strong core of (N, v, u).*
2. *For all $S \in 2^N$, x satisfies*

   (2-a) for some $i \in S$, $u_{i,S}(x_i) = +\infty$; or else

   (2-b) for all $i \in S$, $u_i(N, x_i) \geq u_i\left(S, v(S) - \sum_{j \in S\setminus\{i\}} u_{j,S}(x_j)\right)$ [2].

Proof.

Sufficiency. Suppose that x satisfies condition (2) for all coalitions S. To prove that x is in the strong core, suppose the contrary that x is weakly dominated. Then there exists a weak objection (S, y) on x. By definition, $u_i(S, y_i) \geq u_i(N, x_i)$ for all $i \in S$ and $u_i(S, y_i) > u_i(N, x_i)$ for some $i \in S$. It follows that $LT_{i,S}(x_i) \neq \emptyset$ and $u_{i,S}(x_i) \leq y_i$ for all $i \in S$. Note that in this case, $u_{i,S}(x_i) < +\infty$ for all $i \in S$. Consider $j \in S$ such that $u_j(S, y_j) > u_j(N, x_j)$. Since $y(S) \leq v(S)$, then

$$y_j \leq v(S) - \sum_{i \in S\setminus\{j\}} y_i \leq v(S) - \sum_{i \in S\setminus\{j\}} u_{i,S}(x_i).$$

The utility function $u_j(S, \cdot)$ is a nondecreasing function. Therefore

$$u_j(S, y_j) \leq u_j\left(S, v(S) - \sum_{i \in S\setminus\{j\}} u_{i,S}(x_i)\right) \leq u_j(N, x_j) < u_j(S, y_j).$$

A contradiction holds. This proves that x belongs to the strong core.

Necessity. Assume that x is in the strong core. Consider a coalition S. To prove that x satisfies *(2-a)*; or else *(2-b)*, suppose on the contrary that $u_{i,S}(x_i) < +\infty$ for all $i \in S$, and that the set S^* that consists of all $j \in S$ such that $u_j(N, x_j) < u_j\left(S, v(S) - \sum_{i \in S\setminus\{j\}} u_{i,S}(x_i)\right)$ is a non-empty subset of S. Note that for each player

[2]In case $u_{j,S}(x_j) = -\infty$ for some $j \in S\setminus\{i\}$, the quantity $v(S) - \sum_{j \in S\setminus\{i\}} u_{j,S}(x_j)$ is set to $+\infty$ and the condition $u_i(N, x_i) \geq u_i(S, +\infty)$ simply means that $u_i(N, x_i) \geq u_i(S, q)$ for all $q \in \mathbb{R}$.

$j \in S^*$, there exists some $q_j \in \mathbb{R}$ such that $u_j(N, x_j) < u_j(S, q_j)$. Denote by S_0 the set that consists of all $k \in S$ such that $u^{k,S}(x_k) = -\infty$.

First suppose that S_0 is empty. Then for each player $i \in S$, the compensation share $u_{i,S}(x_i)$ is finite. Choose a player j in S^* and define the S-sharing vector y by $y_i = u_{i,S}(x_i)$ if $i \in S\backslash\{j\}$; and $y_j = v(S) - \sum_{i\in S\backslash\{j\}} u_{i,S}(x_i)$. It follows that $y(S) = v(S)$, $u_j(S, y_j) > u_i(N, x_i)$ and $u_i(S, y_i) \geq u_i(N, x_i)$ for all $i \in S$. Thus, (S, y) is a weak objection on x. A contradiction arises since x belongs to the strong core by assumption.

Now suppose that S_0 is a singleton and pose $S_0 = \{k\}$. Note that for each player $i \in S\backslash\{k\}$, the compensation share $u_{i,S}(x_i)$ is finite and $u_k(S, q) \geq u_k(N, x_k)$ for all $q \in \mathbb{R}$. Two cases are possible. Firstly, suppose that $u_k(N, x_k) < u_k\left(S, v(S) - \sum_{i\in S\backslash\{k\}} u_{i,S}(x_i)\right)$. In this case, define the S-sharing vector y by $y_k = v(S) - \sum_{i\in S\backslash\{k\}} u_{i,S}(x_i)$ and $y_i = u_{i,S}(x_i)$ if $i \in S\backslash\{k\}$. Clearly, $y(S) = v(S)$, $u_k(S, y_k) > u_k(N, x_k)$ and $u_i(S, y_i) \geq u_i(N, x_i)$ for all $i \in S\backslash\{k\}$. Thus, (S, y) is a weak objection on x. A contradiction. Secondly, suppose that $u_k(N, x_k) = u_k\left(S, v(S) - \sum_{i\in S\backslash\{k\}} u_{i,S}(x_i)\right)$. This implies that $k \notin S^*$. Choose $j \in S^*$ and $q_j \in \mathbb{R}$ such that $u_j(S, q_j) > u_j(N, x_j)$. Now define the S-sharing vector y by $y_j = q_j$, $y_i = u_{i,S}(x_i)$ if $i \in S\backslash\{j, k\}$ and $y_k = v(S) - q_j - \sum_{i\in S\backslash\{k,j\}} u_{i,S}(x_i)$. The vector y satisfies $y(S) = v(S)$, $u_j(S, y_j) > u_i(N, x_i)$ and $u_i(S, y_i) \geq u_i(N, x_i)$ for all $i \in S\backslash\{j\}$. Thus, (S, y) is a weak objection on x. A contradiction also arises.

Finally, suppose that S_0 contains a pair $\{k, l\}$ of players. Choose $j \in S^*$ and $q_j \in \mathbb{R}$ such that $u_j(S, q_j) > u_j(N, x_j)$. Suppose without lost of generality that $j \neq k$. Define the S-sharing vector y by $y_j = q_j$, $y_i = u_{i,S}(x_i)$ if $i \in S\backslash(S_0 \cup \{j\})$, $y_i = 0$ if $i \in S_0\backslash\{j, k\}$ and $y_k = v(S) - q_j - \sum_{i\in S\backslash\{k\}} u_{i,S}(x_i)$. It follows that $y(S) = v(S)$, $u_j(S, y_j) > u_i(N, x_i)$ and $u_i(S, y_i) \geq u_i(N, x_i)$ for all $i \in S\backslash\{j\}$. Thus, (S, y) is a weak objection on x. A contradiction holds.

In each of the three possible cases, a contradiction arises. This proves that x is the element of strong core. ■

Theorem 2.2.1.

Let (N, v, u) be a LUF-game in which all players have increasing utility functions and only exact lower compensation shares from the grand coalition to any other coalition $S \in 2^N$. Then the two assertions below are equivalent:

1. A sharing vector x belongs to the strong core.

2. For all $S \in 2^N$, $\sum_{i \in S} u_{i,S}(x_i) \geq v(S)$.

Proof.

Consider a LUF-game (N, v, u) in which all players have increasing utility functions and only exact lower compensation shares from the grand coalition to any other coalition $S \in 2^N$.

Sufficiency. Assume that $\sum_{i \in S} u_{i,S}(x_i) \geq v(S)$ for all $S \in 2^N$. Consider a coalition S and some $i_0 \in S$. By assumption on the game, $u_{j,S}(x_j)$ is finite for all $j \in S$. Note that $\sum_{i \in S} u_{i,S}(x_i) \geq v(S)$ implies $u_{i,S}(x_{i_0}) \geq v(S) - \sum_{j \in S \setminus \{i_0\}} u_{j,S}(x_j)$. The utility function of player i is an increasing function. Therefore

$$u_{i_0}\left(S, v(S) - \sum_{j \in S \setminus \{i_0\}} u_{j,S}(x_j)\right) \leq u_{i_0}(S, u_{i_0,S}(x_{i_0})) = u_{i_0}(N, x_{i_0}).$$

The last equality holds thanks to the fact that $u_{i_0,S}(x_{i_0})$ is an exact lower compensation share to x_{i_0}. Therefore, x satisfies condition (2) in Lemma 2.2.1. It follows that x is in the strong core.

Necessity. Assume that x is a strong core selection. Consider a coalition S and $i \in S$. It follows that x satisfies condition (2) in Lemma 2.2.1. Since for all coalitions $S \in 2^N$ and for all $j \in S$, $u_{j,S}(x_j)$ is finite, it follows from Lemma 2.2.1 that

$$u_i\left(S, v(S) - \sum_{j \in S \setminus \{i\}} u_{j,S}(x_j)\right) \leq u_i(N, x_i) = u_i(S, u_{i,S}(x_i)).$$

Recalling that u_i is an increasing utility function, it holds that $v(S) - \sum_{j \in S \setminus \{i\}} u_{j,S}(x_j) \leq u_i(S, u_{i,S}(x_i))$. Therefore $\sum_{i \in S} u_{i,S}(x_i) \geq v(S)$. ∎

Condition *(2-b)* in Lemma 2.2.1, which is better explicit in the Theorem 2.2.1 represents a kind of coalition rationality. It says that, for any sharing vector in the strong core of a LUF-game, each player obtains at most as much utility in the grand coalition as with any other coalition when each of his partners in the new coalition is compensated even minimally. Furthermore, under assumptions in Theorem 2.2.1, the strong core is completely determined by a set of $2^n - 1$ constraints as the classical core of TU-games; that is, provided that all players have increasing utility functions and only exact compensation shares from the grand coalition to any other coalition $S \in 2^N$,

$$\mathcal{CL}^s(N,v,u) = \left\{ x \in \chi(N,v) : \sum_{i \in S} u_{i,S}(x_i) \geq v(S), \forall S \in 2^N \right\}. \tag{2.14}$$

As shown in Proposition 2.2.5, the core and the strong core coincide for all LUF-games in which for all coalitions S and for each player $i \in S$, the function $u_i(S, \cdot)$ is continuous and increasing. In addition, if all compensation shares are exact, then the core is also determined by (2.14). However, the characterization of all sharing vectors that are in the core is more involving for an arbitrary LUF-game. This is addressed in the next section using further notation.

The upper transition correspondence and core sharing vectors

Given a LUF-game (N,v,u), a coalition S and $i \in S$, we pose

$$UT^{i,S}(q) = \{Q \in \mathbb{R} : u_i(S,Q) \leq u_i(N,q)\}. \tag{2.15}$$

The set $UT^{i,S}(q)$ will be called the *upper transition set* of q, as the share q of $v(N)$ provides to player i at least as much utility as any $Q \in UT^{i,S}(q)$ received by i when S is formed. Thus each player has no proper insensitive (or no profit for his own) to leave the grand coalition and receive a share q of $v(S)$ which is in $UT^{i,S}(x_i)$. For the stability of the sharing vector x, the collection $UT^{i,S}(x_i), i \in S$ also plays the role of a control test since no player will not accept less than upper compensation share while opting out from N to any other coalition. This leads us to the following definition:

Definition 2.2.9. *Let (N,v,u) be a LUF-game, S a coalition and i a player in S.*

- *The upper transition correspondence of player i from N to S is the map $UT^{i,S}$ defined by Equation 2.15.*
- *The upper transition function of player i from N to S is the map $u^{i,S}$ defined for*

all $q \in \mathbb{R}$ *by*

$$u^{i,S}(q) = \begin{cases} \sup\left(UT^{i,S}(q)\right) & \textit{if } UT_{i,S}(q) \neq \emptyset \\ \\ -\infty & \textit{otherwise} \end{cases}$$

In case $u^{i,S}(q)$ is finite, it is called the *upper compensation share* of q when player i opts out from N to S. Furthermore, $u^{i,S}(q)$ is exact if $u^{i,S}(q)$ is finite and $u_i(S, u^{i,S}(q)) = u_i(N, q)$.

Similarly to the case of the lower transition set, note that $UT^{i,S}(q)$ is the set of all shares Q of the worth of S that provides to player i at most as much utility as the share q of $v(N)$. Moreover, when $u^{i,S}(q) = -\infty$, it holds that $UT^{i,S}(q) = \emptyset$. In this case, no share received by i in S provides to player i less utility than the share q of $v(N)$. This occurs when the line of $q \longmapsto u_i(N, q)$ is below the curve of $Q \longmapsto u_i(S, Q)$. When $u^{i,S}(q)$ is finite, $UT^{i,S}(q) \neq \emptyset$; and $u^{i,S}(q)$ is the greatest share of $v(S)$ such that any smaller share of $v(S)$ provides at most as much utility to player i as the share q of $v(N)$. It is also possible that $u^{i,S}(q) = +\infty$. In this later case, the curve of $Q \longmapsto u_i(S, Q)$ is below the line $q \longmapsto u_i(N, q)$: the share q of $v(N)$ provides to player i at least as much utility as any share of $v(S)$.

Before we continue, the next proposition gives a relationship between the notions of compensation shares we introduce.

Proposition 2.2.7. *Let (N, v, u) be any LUF-game, S a coalition and $i \in S$. Then,*

1. *for all $q \in \mathbb{R}$, $u_{i,S}(q) \leq u^{i,S}(q)$;*
2. *if the utility function $u_i(S, \cdot)$ is increasing, $u_{i,S}(q) = u^{i,S}(q)$ for all $q \in \mathbb{R}$.*

Proof.

Let (N, v, u) be any LUF-game. Consider a coalition S, a player $i \in S$ and $q \in \mathbb{R}$.

1. To prove that $u_{i,S}(q) \leq u^{i,S}(q)$, first suppose that $u_{i,S}(q) = +\infty$. Then by definition, $LT_{i,S}(q) = \emptyset$; that is for all $Q \in \mathbb{R}$, $u_i(S, Q) < u_i(N, q)$. Therefore $UT^{i,S}(q) = \mathbb{R}$ and $u^{i,S}(q) = +\infty$. Now suppose that $u_{i,S}(q)$ is finite. To prove that $u_{i,S}(q) \leq u^{i,S}(q)$, suppose on the contrary that $u_{i,S}(q) > u^{i,S}(q)$. Then choose a real number Q such that $u_{i,S}(q) > Q > u^{i,S}(q)$. By the definition of $u^{i,S}(q)$ and

since $Q > u^{i,S}(q)$, it holds that $Q \notin UT^{i,S}(q)$. Thus $u_i(S,Q) > u_i(N,q)$. This implies that $Q \in LT_{i,S}(q)$. Therefore $Q \geq u_{i,S}(q)$. A contradiction arises due to the fact that $u_{i,S}(q) > Q$. This proves that $u_{i,S}(q) \leq u^{i,S}(q)$. Finally, if $u_{i,S}(q) = -\infty$, then it is obvious that $u_{i,S}(q) \leq u^{i,S}(q)$.

2. Assume that the utility function $u_i(S,\cdot)$ is increasing. To prove that $u_{i,S}(q) = u^{i,S}(q)$, suppose on the contrary that $u_{i,S}(q) \neq u^{i,S}(q)$. Then it follows from the item part of the present proof that $u_{i,S}(q) < u^{i,S}(q)$. Choose a real number Q_0 such that $u_{i,S}(q) < Q_0 < u^{i,S}(q)$. Two cases arise. First suppose that $u_i(S,Q_0) \leq u_i(N,q)$. Then for all $Q \in LT_{i,S}(q)$, $u_i(S,Q_0) \leq u_i(N,q) \leq u_i(S,Q)$. Since $u_i(S,\cdot)$ is increasing, $Q_0 \leq Q$ for all $Q \in LT_{i,S}(q)$. Thus, by the definition of $u_{i,S}(q)$, it appears that $Q \leq u_{i,S}(q)$. A contradiction arises. Now, suppose that $u_i(S,Q_0) > u_i(N,q)$. Then for all $Q \in UT^{i,S}(q)$, $u_i(S,Q_0) > u_i(N,q) \geq u_i(S,Q)$. Since $u_i(S,\cdot)$ is increasing, $Q_0 > Q$ for all $Q \in UT^{i,S}(q)$. Thus, by the definition of $u^{i,S}(q)$, it follows that $Q \geq u^{i,S}(q)$. A contradiction holds. This proves that $u_{i,S}(q) = u^{i,S}(q)$.

■

The next result provides a characterization of all sharing vectors that are in the core of a given LUF-game.

Theorem 2.2.2.

Let (N,v,u) be any LUF-game. Then the two assertions below are equivalent:

a. A sharing vectors x belongs to the core.

b. For all $S \in 2^N$, x satisfies

(b-1) for some $i \in S$, $u^{i,S}(x_i) = +\infty$; or else

(b-2) $\sum_{i \in S} u^{i,S}(x_i) > v(S)$; or else

(b-3) $\sum_{i \in S} u^{i,S}(x_i) = v(S)$ and $u_i(S, u^{j,S}(x_j)) \leq u_j(N,x_j)$ for some $j \in S$.

★

Proof.

Sufficiency. Suppose that x satisfies the current condition (b) for all coalitions S. To prove that x is in the core, suppose on the contrary that x is dominated. Then there exists an objection (S,y) on x. By definition, $u_i(S,y_i) > u_i(N,x_i)$ for all $i \in S$. Given $i \in S$, it follows that for all $q \in UT^{i,S}(x_i)$, $u_i(S,q) \leq u_i(N,x_i) < u_i(S,y_i)$. Since the utility function $u(S,\cdot)$ is nondecreasing, it holds that for all $q \in UT^{i,S}(x_i)$, $q < y_i$. Therefore $u^{i,S}(x_i) \leq y_i < +\infty$ for all $i \in S$ and thus $\sum_{i\in S} u^{i,S}(x_i) \leq y(S) = v(S)$. This implies that x does not satisfy *(b-1)*; nor *(b-2)*. Therefore x necessarily satisfies *(b-3)*. Consider $j \in S$ such that $u_i(S, u^{j,S}(x_j)) \leq u_j(N,x_j)$. Recalling that (S,y) is an objection on x, it follows that $u_i(S,u^{j,S}(x_j)) \leq u_j(N,x_j) < u_j(S,y_j)$ and thus $u^{j,S}(x_j) < y_j$. In summary, $u^{i,S}(x_i) \leq y_i$ for all $i \in S$ and $u^{j,S}(x_j) < y_j$. This implies that $\sum_{i\in S} u^{i,S}(x_i) < y(S) = v(S)$. A contradiction arises since by *(b-3)*, $\sum_{i\in S} u^{i,S}(x_i) = v(S)$.

Necessity. Assume that $x \in \mathcal{CL}(N,v,u)$ and consider a coalition S. To prove that x satisfies condition *(b)*, suppose the contrary that this is not the case. Then x simultaneously satisfies

($c1$) for all $i \in S$, $u^{i,S}(x_i) < +\infty$; and

($c2$) $\sum_{i\in S} u^{i,S}(x_i) \leq v(S)$; and

($c3$) $\sum_{i\in S} u^{i,S}(x_i) \neq v(S)$ or $u_i(S,u^{i,S}(x_i)) > u_i(N,x_i)$ for all $i \in S$.

Denote by S' the set of all $i \in S$ such that $u^{i,S}(x_i) = -\infty$. Then for all $i \in S'$, $u_i(S,q) > u_i(N,x_i)$. Moreover, by ($c1$), $u^{i,S}(x_i)$ is finite for all $i \in S\backslash S'$. Thus, for each $i \in S\backslash S'$, there exists some real number q_i such that $u_i(S,q_i) > u_i(N,x_i)$. Two possible cases arise:

Case 1: The set S' is not empty. Choose $j \in S'$ and define the S-sharing vector y by

$$y_i = q_i \text{ if } i \in S\backslash S',\ y_j = v(S) - \sum_{i\in S\backslash(S'\cup\{j\})} q_i \text{ and } y_i = 0 \text{ if } i \in S'\backslash\{j\}. \tag{2.16}$$

By definition of S' and q_i for $i \in S\backslash S'$, y is such that $y(S) = v(S)$ and $u_i(S,y_i) > u_i(N,x_i)$ for all $i \in S$. Therefore (S,y) is an objection on x. A contradiction holds.

Case 2: The set S' is empty. Then, by $(c1)$, $u^{i,S}(x_i)$ is finite for all $i \in S$. First suppose that $\sum_{i\in S} u^{i,S}(x_i) \notin v(S)$. The by $(c2)$, it follows that $\sum_{i\in S} u^{i,S}(x_i) < v(S)$. We define the S-sharing vector y by

$$y_i = u^{i,S}(x_i) + \varepsilon \text{ with } \varepsilon = \frac{1}{|S|}\left(v(S) - \sum_{i\in S} u^{i,S}(x_i)\right) > 0. \tag{2.17}$$

Let $i \in S$. Then by definition of $u^{i,S}(x_i)$ and since $y_i > u^{i,S}(x_i)$, it comes that $y_i \notin UT^{i,S}(x_i)$ and thus $u_i(S, y_i) > u_i(N, x_i)$. Moreover $y(S) = v(S)$. Therefore (S, y) is an objection on x. A contradiction arises since x belong to the core. Now suppose that $\sum_{i\in S} u^{i,S}(x_i) = v(S)$. Then by $(c3)$, $u_i(S, u^{i,S}(x_i)) > u_i(N, x_i)$ for all $i \in S$. Define the S-sharing vector y by $y_i = u^{i,S}(x_i)$ for all $i \in S$. Clearly, (S, y) is an objection on x. A contradiction holds.

In both cases, a contradiction arises. This proves that x necessarily satisfies condition (b). ∎

In Theorem 2.2.2, the stability of a sharing vector x is guaranteed by condition (b) which consists for each possible coalition S in three steps. At the first step, one should check whether *(b-1)* is satisfied or not. If x meets *(b-1)*, the shares for some players in x provide at least as much utility as any shares they may be offered when S is formed. Since those players have no incentive to opt out from N to S, no objection via S exists. If x does not satisfies *(b-1)*, the stability test leads to the second step; that is *(b-2)*. If this condition is satisfied, the worth $v(S)$ is not enough to simultaneously compensate all members of S. This makes any objection via S impossible. If condition *(b-2)* also fails, then the third and final step to be checked is *(b-3)*. When x satisfies *(b-3)*, the worth of S is just enough to minimally serve each member of S; even in these conditions, some players in S are not better off. This discards any possibility for an objection via S. Instead of a three-stage test, conditions *(b1)-(b-3)*, reduce to a single inequality as soon as all utility functions in a LUF-game have some minimal properties such as monotonicity.

Corollary 2.2.2. *Let (N, v, u) be a LUF-game in which all players have only exact upper compensation shares from the grand coalition to any other coalition $S \in 2^N$. Then the two following assertions below are equivalent:*

1. *A sharing vector x belongs to the core $\mathcal{CL}(N, v, u)$.*

2. *For all $S \in 2^N$, $\sum_{i\in S} u^{i,S}(x_i) \geq v(S)$.*

Proof.

Consider a LUF-game (N, v, u) in which all players only have exact upper compensation shares from the grand coalition to any other coalition $S \in 2^N$.

Sufficiency. Assume that $\sum_{i\in S} u^{i,S}(x_i) \geq v(S)$ for all $S \in 2^N$ and consider a coalition S. By assumption on the game, x does not meet condition *(b-1)* in Theorem 2.2.2 since each upper compensation share is finite. It is then sufficient to prove that x meets *(b-2)*; or else *(b-3)* in Theorem 2.2.2. Clearly, if $\sum_{i\in S} u^{i,S}(x_i) > v(S)$, then x satisfies *(b-2)*. Otherwise $\sum_{i\in S} u^{i,S}(x_i) = v(S)$ and x satisfies *(b-3)* since $u_i(S, u^{i,S}(x_j)) = u_i(N, x_i)$ for all $i \in S$ by assumption on the game. In both cases, x satisfies condition *(b)* in Theorem 2.2.2. Therefore, $x \in \mathcal{CL}(N, v, u)$.

Necessity. Assume that $x \in \mathcal{CL}(N, v, u)$ and consider a coalition S. Then x satisfies condition (b) in Theorem 2.2.2. Since each upper compensation share $u_{i,S}(x_i)$ is finite for all $i \in S$. Therefore, x does not meet condition *(b-1)* in Theorem 2.2.2. This implies that x meets *(b-2)*; or else *(b-3)*. In both cases, $\sum_{i\in S} u^{i,S}(x_i) \geq v(S)$. ■

Provided that all players only have exact upper compensation shares from the grand coalition to any other coalition $S \in 2^N$ as stated in Corollary 2.2.2, the core is also determined by a set of $2^n - 1$ constraints as the classical core of TU-games; that is,

$$\mathcal{CL}(N, v, u) = \left\{ x \in \chi(N, v) : \sum_{i\in S} u^{i,S}(x_i) \geq v(S), \forall S \in 2^N \right\}. \tag{2.18}$$

It is worth mentioning that in both Equations (2.14) and (2.18), the two characterizations may involve nonlinear constraints since some compensation shares may be derived from utility functions of any shape. Whether the core (or the strong core) of a given LUF-game is empty or not surely depends on the collection of utility functions embedded. This dependence is illustrated in the next proposition.

Proposition 2.2.8. *Consider a coalitional production function v. Then depending on the collection $u = (u_i)_{i\in N}$ of utility functions, the core (or the strong core) may be empty or not.*

Proof.

Consider a coalitional production function v.

To construct a LUF-game (N, v, u) with a non-empty strong core, pose

$$M = \max\{v(S) : S \in 2^N\} \text{ and } m = \min\{v(S) : S \in 2^N\}. \tag{2.19}$$

and define the utility of player i for all coalitions $S \in 2^N$ and for all $q \in \mathbb{R}$ by

$$u_i(S, q) = q - \frac{m}{|S|} \text{ if } S \neq N \text{ and } u_i(N, q) = \max(M - m, q). \tag{2.20}$$

Consider the sharing vector x such that $x_i = \frac{V(N)}{n}$. It is clear that for all $i \in N$, the lower compensation share of player i from N to a coalition S is $u_{i,S}(x_i) = \max(M-m, x_i)+\frac{m}{|S|}$. Therefore, it follows that for all coalitions S and for all $i \in S$, the lower compensation share $u_{i,S}(x_i)$ is finite and

$$\begin{aligned}
v(S) - \textstyle\sum_{j\in S\setminus\{i\}} u_{i,S}(x_i) &\leq v(S) - \textstyle\sum_{j\in S\setminus\{i\}} \left(M - m + \frac{m}{|S|}\right) \\
&= v(S) - m - (|S| - 1)(M - m) + \tfrac{m}{|S|} \\
&\leq v(S) - m + \tfrac{m}{|S|} \text{ since } M - m \geq 0
\end{aligned} \tag{2.21}$$

Since $u_i(S, \cdot)$ is increasing,

$$u_i\left(S, v(S) - \sum_{j\in S\setminus\{i\}} u_{i,S}(x_i)\right) \leq u_i\left(S, v(S) - m + \frac{m}{|S|}\right) = v(S) - m \leq u_i(N, x_i) \tag{2.22}$$

By Theorem 2.2.1, the strong core of the game is not empty; and so is also the core by Proposition 2.2.1.

Now, to construct a LUF-game (N, v, u') with an empty core, define the utility of player i for all coalitions $S \in 2^N$ and for all $q \in \mathbb{R}$ by

$$u_i(S, q) = q \text{ if } S \neq N \text{ and } u_i(N, q) = q + v(\{i\}) - 1 - \frac{v(N)}{n}. \tag{2.23}$$

Let x be any sharing vector. By definition of a sharing vector $x(S) = v(N)$. Therefore, there exists some player $j \in N$ such that $x_j \leq \frac{v(N)}{n}$. For player j, $u_j(N, x_j) = v(\{j\})-1$ and $u_j(\{j\}, v(\{j\})) = v(\{j\})$. Therefore player j is better off by standing alone. Thus, x is dominated via $S = \{i\}$. This proves that the core of the game (N, v, u') is empty; and so is also the strong core by Proposition 2.2.1. ∎

For a given coalitional production function, the non emptiness of the core (or the strong core) clearly depends on the collection of the utility functions. The well known Shapley-Bondareva theorem, by Shapley (1967) and Bondareva (1963), provides necessary and sufficient conditions for the non-emptiness of the core of a TU-game. In the next section, we extend this theorem to a class of LUF-games.

2.2.3 Stability conditions of LUF-games

In this section, we consider the class of LUF-games in which utility functions are linear (See, Definition 2.1.6).

Assume for example that in a LUF-game (N, v, u), players form coalition to produce a given good. Then in Definition 2.1.6, $a_{i,S}$ can be interpreted as the investment return rate of player $i \in S$ when S is formed; in this case, the constant term $b_{i,S}$ may be seen as a fixed cost term or a fixed profit term. It can be easily checked that under (2.10), utility functions are increasing. Moreover, given $i \in S$, upper compensations shares and lower compensations shares of player i from N to a coalition S coincide, are exact and are given by:

$$u_{i,S}(q) = \frac{a_{i,N}}{a_{i,S}}q + \frac{b_{i,N} - b_{i,S}}{a_{i,S}} = u^{i,S}(q). \tag{2.24}$$

This leads us to the following results:

Proposition 2.2.9. *The core and the strong core of any linear LUF-game coincide.*

Proof.

For a linear LUF-game, all utility functions are increasing and continuous. Thus, the results follows from Proposition 2.2.5. ∎

Proposition 2.2.10. *If (N, v, u) is a linear LUF-game with gradient collection a and fixed-term collection b, then $\mathcal{CL}(N, v, u)$ and $\mathcal{CL}^s(N, v, u)$ coincide with the set of all solutions x to the following set of $2^n - 1$ linear constraints:*

$$\sum_{i \in S} \frac{a_{i,N}}{a_{i,S}} x_i \geq v(S) - \sum_{i \in S} \frac{b_{i,N} - b_{i,S}}{a_{i,S}} \text{ for all } S \in 2^N. \tag{2.25}$$

Proof.

For any linear LUF-game, all utility functions are increasing and continuous. Furthermore, players have only exact upper compensation shares and exact lower compensation shares from N to any other coalition. Thus, the result follows from Proposition 2.2.5, Theorem 2.2.1 and Corollary 2.2.2. ■

The notion of balanced collections of coalitions is the sole basis of the Shapley-Bondareva theorem. We provide here a result using characteristic vectors as in Peleg and Sudhölter (2007). We give a definition of balanced family to a LUF-game below:

Definition 2.2.10. *Given a linear LUF-game (N, v, u) with positive gradient collection a, a family F of subsets of N and a family of nonnegative reals numbers $(\delta_S)_{S\in 2^N}$*

A couple $(F, (\delta_S)_{S\in} 2^N)$ of subsets of N is an a-balanced family if each player j satisfies the following condition:

$$\sum_{\substack{S\in F \\ j\in S}} \delta_S \frac{a_{j,N}}{a_{j,S}} = 1.$$

The family $(\delta_S)_{S\in 2^N}$ is so called a-balancing weights.

Given a linear LUF-game (N, v, u) with positive gradient collection a and fixed-term collection b, we associate the following linear programming problem which will be called the *primal problem* of the game (N, v, u).

$$(P): \begin{cases} \min x(N) \\ \\ \text{subject to } \displaystyle\sum_{i\in S} \frac{a_{i,N}}{a_{i,S}} x_i \geq v_a(S) \text{ for all coalitions } S \in 2^N \end{cases} \tag{2.26}$$

where

$$v_a(S) = v(S) - \sum_{i\in S} \frac{b_{i,N} - b_{i,S}}{a_{i,S}} \text{ for all } S \in 2^N. \tag{2.27}$$

The dual program of (P) is

$$(D): \begin{cases} \max \sum_{S\in 2^N} \delta_S v_a(S) \\ \\ \text{subject to } \displaystyle\sum_{\substack{S\in 2^N \\ i\in S}} \delta_S \frac{a_{i,N}}{a_{i,S}} \mathbb{1}_S = \mathbb{1}_N \text{ and } \delta_S \geq 0 \text{ for all coalitions } S \in 2^N \end{cases} \tag{2.28}$$

Where for a given coalition S, $\mathbb{1}_S$ is the characteristic vector of S denoted by $\mathbb{1}_S = (\mathbb{1}_S^i)_{i\in N}$ and defined such that for all $i \in N$, $\mathbb{1}_S^i = 1$ if $i \in S$; and $\mathbb{1}_S^i = 0$ otherwise.

A feasible solution to a linear programming problem is any solution to the set of linear inequalities of that problem. A feasible solution x^* of (P) is optimal if $p^* = x(N)$ takes the minimum value among all feasible solutions of (P); while a feasible solution δ^* of (D) is optimal if $d^* = \sum_{S\in 2^N} \delta_S v_a(S)$ takes the maximum value among all feasible solutions of (D); in this case p^* and d^* are the optimal value of (P) and the optimal value of D respectively.

Lemma 2.2.2. *The primal problem of any linear LUF-game with positive gradient collection admits an optimal value $p^* = d^*$.*

Proof.
Consider a linear LUF-game (N, v, u) with gradient collection a and fixed-term collection b. By the duality theorem, we only need to prove that the primal problem of the game admits an optimal value. Note that the collections $\left\{\frac{v_a(S)}{|S|} : S \in 2^N\right\}$ and $\left\{\frac{a_{i,N}}{a_{i,S}} : i \in S \in 2^N\right\}$ are finite. Thus, there exist some coalitions $K, L \in 2^N$ and some player $j \in N$ such that

$$\frac{v_a(K)}{|K|} \geq \frac{v_a(S)}{|S|} \text{ and } \frac{a_{j,N}}{a_{j,L}} \leq \frac{a_{i,N}}{a_{i,S}} \text{ for all } S \in 2^N \text{ and for all } i \in S. \tag{2.29}$$

Define the n-tuple y by

$$y_i = \frac{a_{j,L}}{a_{j,N}} \frac{v_a(K)}{|K|} \text{ and for all } i \in N. \tag{2.30}$$

Let S be any coalition. Then

$$\sum_{i\in S} \frac{a_{i,N}}{a_{i,S}} y_i \geq \sum_{i\in S} \frac{a_{i,N}}{a_{i,L}} x_i = |S| \frac{v_a(K)}{|K|} \geq |S| \frac{v_a(S)}{|S|} = v_a(S). \tag{2.31}$$

Therefore, y is a feasible solution the dual problem (P) of the game. To prove that (P) admits an optimal solution, we only need to prove that for any feasible solution x of (P), $x(N)$ greater than or equal to some number. For this purpose, consider a feasible solution x of (P). Since x satisfies all constraints of (P), it follows that for all $i \in N$, $\frac{a_{i,N}}{a_{i,\{i\}}} x_i \geq v_a(\{i\})$. Therefore $x(N) \geq \sum_{i\in N} \frac{a_{i,\{i\}}}{a_{i,N}} v_a(\{i\})$. ■

Lemma 2.2.3. *A linear LUF-game is* **stable** *if and only if, the optimal value p^* of its primal problem satisfies $p^* \leq v(N)$.*

Proof.

Consider a linear LUF-game (N, v, u) with gradient collection a and fixed-term collection b.

Necessity. Suppose that $C(N, v, u) \neq \emptyset$. Consider $x \in C(N, v, u)$. Then by Proposition 2.2.10, x is a feasible solution to the primal problem (P) of the game. Thus, $p^* \leq x(N) = v(N)$.

Sufficiency. Suppose that the optimal value p^* of the primal problem of the game is such that $p^* \leq v_a(N)$. Consider an optimal solution x^* of (P) and define the sharing vector x for all $i \in N$ by $x_i = x_i^* + \frac{v(N)-p^*}{n}$. By assumption $v(N) - p^* = v(N) - x^*(N) \geq 0$. Therefore, $x_i \geq x_i^*$ for all $i \in N$. Since the gradient collection contains only positive numbers, it follows that x satisfies (2.25). We conclude by Proposition 2.2.10 that $x \in C(N, v, u)$. ■

Lemmas 2.2.2 and 2.2.3 lead to the following result:

Theorem 2.2.3.

A linear LUF-game (N, v, u) with gradient collection a and fixed-term collection b is **stable** if, and only if, for all a-balanced family $\mathcal{B}$ of a-balancing weights $(\delta_S)_{S\in\mathcal{B}}$,

$$\sum_{S\in\mathcal{B}} \delta_S v_a(S) \leq v(N). \tag{2.32}$$

Proof.

Consider a linear LUF-game (N, v, u) with gradient collection a and fixed-term collection b.

Necessity. Suppose that $C(N, v, u) \neq \emptyset$. Consider an a-balanced collection $\mathcal{B}$ and a collection of a-balancing weights $(\delta_S)_{S\in\mathcal{B}}$ for $\mathcal{B}$. Extend δ from coalitions in $\mathcal{B}$ to any coalition in 2^N by considering the collection δ' defined by $\delta'_S = \delta_S$ if $S \in \mathcal{B}$ and $\delta'_S = 0$ otherwise. Since $\mathcal{B}$ is an a-balanced collection and $(\delta_S)_{S\in\mathcal{B}}$ is a collection of a-balancing weights for $\mathcal{B}$, it follows that δ' is a feasible solution to the dual program (D) of the game. Therefore,

$$\sum_{S\in\mathcal{B}} \delta_S v_a(S) = \sum_{S\in 2^N} \delta'_S v_a(S) \leq d^* = p^* \leq v(N).$$

Note that $d^* = p^*$ comes from Lemma 2.2.2; and the last inequality holds from Lemma 2.2.3 since $C(N, v, u) \neq \emptyset$ by assumption.

Sufficiency. Suppose that (2.32) holds for all a-balanced collections $\mathcal{B}$ and for all collections of a-balancing weights $(\delta_S)_{S\in\mathcal{B}}$ for $\mathcal{B}$. By Lemma 2.2.2, the primal problem and the dual problem of the game admit, each, some feasible solutions and the same optimal value. Consider a feasible solution $\delta = (\delta_S)_{S\in 2^N}$ of the dual problem (D). It follows that the set 2^N of all coalitions is a balanced and that δ is a collection of a-balancing weights for 2^N. Therefore,

$$\sum_{S\in 2^N} \delta'_S v_a(S) \leq v(N).$$

This proves that $p^* = d^* \leq v(N)$. Hence by Lemma 2.2.3, $C(N, v, u) \neq \emptyset$. ■

Before moving to the next chapter, we recall that cooperative games with transferable utility (TU-games), multi-cooperative games with transferable utility (MTU-games) or cooperative games with local utilities functions (LUF-games) model interactions in which agents form coalitions and gain some payments. The payoff of a coalition is classically deterministic. However, in many situations, payoffs are not known in advance but are random variables. This is the case in the next chapter where the outcome of a cooperation is a random variable.

⋆ ⋆ Chapter Three ⋆ ⋆

Chance-constrained cooperative games: a value solution

In this chapter, we focus our attention on chance constrained games (CC-games) introduced by Charnes and Granot (1973). Up to now, only set-valued solutions of CC-games have been defined. More precisely, Charnes and Granot (1977) consider a two-stage core and a two-stage nucleolus for this class of games. No single-valued rule, a value, that assigns a CC-game with a single payoff vector is not yet defined together with some of its axiomatizations. Our aim is to fill this gap by providing a value for CC-games together with a simple and compact formula as well as some characterization results[1]. More precisely, we present here a two-stage value for chance-constrained games (on discrete sample spaces) as an ex-ante agreement among players.

The chapter is organized into three sections. Section 3.1 is devoted to the presentation of CC-games. We mainly introduce the model and adjust it by adding sample spaces and give some algebraic properties of CC-games. In Section 3.2, we define a value for CC-games and introduce some intuitive axioms. In Section 3.3, we define the Equal-surplus Shapley value and give a simple and compact formula of this value. Axiomatizations of the newly introduced value are presented.

3.1 On chance-constrained games

Hereafter, the cardinality of a finite set A is denoted by $|A|$, a permutation of A is a one-to-one function from A onto itself and a transposition of A is a permutation π of A such that for some $\{k, l\} \subseteq A$, $\pi(k) = l$, $\pi(l) = k$ and $\pi(t) = t$ for all $t \in A \setminus \{k, l\}$. in this case, π is denoted by $\pi = (k, l)$.

[1] Njoya et al. (2021) present the essentials of the achievements in this chapter

3.1.1 Models of cooperative games with random payments

The theory of cooperative games in characteristic function form was extended by Charnes and Granot (1973), so as to encompass situations in which the values of the various coalitions are not deterministic but are rather random variables with given distribution functions.

Definition 3.1.1. A ***chance-constrained game (CC-game)*** *is simply a couple* $(N, (X_S)_{S\in 2^N})$ *with* X_S *a random variable representing a coalitional payoff of* S*. These games are also called* n*-person cooperative game in stochastic characteristic function form.*

Richer models of cooperative games with random payments exist. When some coalitions have several actions and random payoffs. one obtains cooperative game with stochastic payoffs which are each, a collection $(N, (A_S)_{S\in\mathcal{C}_N}, (X_S)_{S\in\mathcal{C}_N}, (\succeq_i)_{i\in N})$ where for each $a \in A_S$, $X_S(a)$ is a finite expectation random variable each realization of which is a coalitional payoff of S when its members jointly choose action a; and $(\succeq_i)_{i\in N}$ is the collection of individual preferences over random payoff vectors. Preferences of players are needed to analyze the desirability or the stability of random payoff vectors; see Suijs et al. (1999) for more details. Another model in Habis and Herings (2011) deals with cooperative games with uncertainty (TUU-games): players are involved in a TU-game depending on some given states of the nature. Precisely, a TUU-game can be viewed as a five-ingredient collection (N, S, v, T, u) in which N is a finite set of players, S is a finite set of the states of nature, $v = (v_s)_{s\in S}$ is a collection of TU-games such that v_s is the TU-game associated with the state of nature s, T is a finite set of periods, and $u = (u_i)_{i\in N}$ is a collection of individual utility functions. Over each period in T, a state of nature s is (randomly) observed from S and all players play the game (N, v_s). Our model of games departs from all these settings and is presented below.

3.1.2 The new model of game and basic definitions

A random variable is derived from a random experiment, that is a process with many uncertain issues. The set of all possible issues or all possible random events for the experiment forms the *sample space.* CC-games (or games with stochastic payoffs general) and TUU-games with uncertainty are usually defined without any information on the sample spaces. In this chapter, we embed sample spaces with a probability distribution from which coalitional payoffs are derived.

Definition 3.1.2. *Given a finite set N of $n \geq 2$ players, a chance-constrained game with discrete sample spaces on N is a tuple (Ω, v, ϖ) such that for some collection $\Omega = (\Omega_S)_{S \in 2^N}$ of coalitional sample spaces; the mapping v and ϖ respectively give, for all $S \in 2^N$ and for all $k \in \Omega_S$, the coalitional payoff $v(S,k)$ of the members of S and the probability $\varpi(S,k) > 0$ of observing event k; with $\sum_{k \in \Omega_S} \varpi(S,k) = 1$ for all coalitions S.*

The set of all chance-constrained games with discrete sample spaces on N is denoted by $\mathcal{CC}(N)$, $\mathcal{CC}(N, \Omega, \varpi)$ is the subset of $\mathcal{CC}(N)$ that consists of all CC-games on N with the same probability distribution mapping ϖ on Ω and $\mathcal{CC}^r(N)$ is the subset of $\mathcal{CC}(N)$ on the full class of rational probability distributions. Provided that ϖ and Ω are known, the game (Ω, v, ϖ) will be identified with its coalitional payoff function v. Note that given a coalition S, the mapping $v_S : k \in \Omega_S \longmapsto v(S,k)$ is a random variable with probability distribution $\varpi_S : k \in \Omega_S \longmapsto \varpi(S,k)$. In other words, $(N, (v_S)_{S \in 2^N})$ is a CC-game in Charnes and Granot sense. The little change in our setting is that we have embedded the collection of sample spaces from which the random payoff of each coalition is derived.

Example 3.1.1. *Consider two business units BU1 and BU2 who may purchase each a basic printer. Each business unit may experience, over a given period, a mechanic breakdown (M), an electronic breakdown (E), or none of them (Z). Event M occurs with probability 0.1 for BU1 and 0.05 for BU2. Event E occurs with probability 0.02 for BU1 and 0.1 for $BU2$. In case of a joint professional printer, M and E are observed with the same probability of 0.05. The assistance charges for BU1 are 5 for M and 10 for E; BU2 pays 8 for M and 4 for E; and for the joint printer, M and E cost 10 and 12 respectively. It is assumed that the functioning of printers are independent. The question is, in case of a joint printer, what would each business unit pay for M? for E? Here the sample space for each coalition S is $\Omega_S = \{M, E, Z\}$ where Z stands for no breakdown (observed with no charge).*

This situation can be formalised by a chance-constrained game with discrete sample spaces $(\Omega, v, \varpi) \in \mathcal{CC}(N)$ and be rewritten as follows:

Example 3.1.2. In Example 3.1.1, $N = \{1, 2\}$ and $\Omega_S = \{M, E, Z\}$ for all $S \in 2^N$. Here, player 1 stands for BU1 and player 2 for BU2. The coalitional payoff function v and the probability distribution function ϖ are summarized below with respect to each of the three possible coalitions:

$S = \{1\}$			$S = \{2\}$			$S = \{1, 2\}$		
k	$v(S, k)$	$\varpi(S, k)$	k	$v(S, k)$	$\varpi(S, k)$	k	$v(S, k)$	$\varpi(S, k)$
M	5	0.1	M	8	0.05	M	10	0.05
E	10	0.02	E	4	0.1	E	12	0.05
Z	0	0.88	Z	0	0.85	Z	0	0.9

The second following example described a situation with more than 2 players and where the samples spaces are not the same for all coalitions:

Example 3.1.3. During a festival, three types of tombola are organized for single tickets, two-person tickets and group tickets respectively. Buying any ticket gives rights to a Wheel of Fortune trial depending on the nature of the ticket. The Wheel of Fortune for a one-person ticket may return a golden (G) band with probability 0.02 for a win of 10 euros; or a red (R) band for a zero win. For a two-person ticket, one may win 25 euros for a yellow (Y) band with probability 0.04; 250 euros for a golden (G) band with probability 0.02; or nothing for a red (R) band. A group ticket for at least $k \geq 3$ visitors is offered an initial discount of k euros and may further win $20k$ euros with probability 0.02 for a yellow (Y) band; $200k$ euros with probability 0.001 for a golden (G) band; and nothing for a red (R) band on the Wheel of Fortune. John (player 1) has to purchase the entrance rights for three persons (himself, his wife and his young daughter); Penny (player 2) would like to pay access for two persons (herself and Jenny); and Andrew (player 3) intends to attain the festival. They may separately purchase their tickets or form coalition for group ticket.

It can be noted that the sample space for each coalition S is $\Omega_S = \{Y, G, R\}$ if $1 \in S$ or $2 \in S$; and $\Omega_S = \{Y, R\}$ if $S = \{3\}$. It is important to note that the sample space depends on the coalition considered.

Definition 3.1.3. Given a CC-game $v \in \mathcal{CC}(N, \Omega, \varpi)$, a coalition S and a permutation π of Ω_S. $(\pi\Omega, \pi v_S, \pi\varpi_S)$ is the chance-constrained game with discrete sample spaces defined for all $T \in 2^N \setminus \{S\}$ and for all $k \in \Omega_T$ by $\pi v_S(T,k) = v(T,k)$ and $\pi\varpi_S(T,k) = \varpi(T,k)$; together with $\pi v_S(S,\pi(k)) = v(S,k)$ and $\pi\varpi_S(S,\pi(k)) = \varpi(S,k)$.

We will simply denote this CC-game by πv_S. For an illustration, consider Example 3.1.2; let $S = \{1\}$ and π be the transposition of the mechanic breakdown (M) and the electronic breakdown (E). Then the game πv_S is obtained from the representation of v in Example 3.1.2 by simply interchanging M and E in the first column of the first table. Equivalently, this simply amounts to interchanging for the business unit BU1 the costs and the probabilities of observing M and E. Globally, the risk incurred by BU1 remains unchanged; only a relabeling of the possible events occurred.

Definition 3.1.4. *Given a game $u \in \Gamma^N$. The chance-constrained game with discrete sample spaces associates to u is the game $\tilde{u}$ defined for all $S \in \mathcal{C}_N$ by $\tilde{u}(S,k) = u(S)$ for all $k \in \Omega_S$ and ϖ is the uniform distribution probability on Ω_S.*

Remark 3.1.1. *Given a c.c game $v \in \mathcal{CC}(N,\Omega,\varpi)$. The expectation game noted E_v is the TU-game that assigns to each coalition S, its expected worth $E_v(S)$ defined by $E_v(S) := \sum_{k\in\Omega_S} \varpi(S,k)v(S,k)$.*

3.1.3 Algebraic definitions and properties on CC-games

The following definitions give some algebraic operations for chance-constrained games with discrete sample spaces.

Definition 3.1.5. *Given a probability distribution function ϖ on Ω, $\mathcal{CC}(N,\Omega,\varpi)$ and $u, v \in \mathcal{CC}(N,\Omega,\varpi)$. $u+v$ is the c.c game in $\mathcal{CC}(N,\Omega,\varpi)$ such that for all $S \in 2^N$ and for all $k \in \Omega$ $(u+v)(S,k) = u(S,k) + v(S,k)$.*

For example, in the game of Example 3.1.1, if u models the charges for possible breakdowns and v the transport cost, then $u + v$ gives the total cost for each possible breakdown.

Definition 3.1.6. *Given a probability distribution function ϖ on Ω, $\mathcal{CC}(N,\Omega,\varpi)$ and a real number λ. λv is the c.c game in $\mathcal{CC}(N,\Omega,\varpi)$ such that $(\lambda v)(S,k) = \lambda v(S,k)$.*

Similarly, in the game of Example 3.1.1, if u models the charges for possible breakdowns. Then λv corresponds to the game obtained when the transport cost is updated by a constant rate λ.

It is well-known that the set Γ^N of all TU-games on N is a space vector of dimension $2^n - 1$. It can be also checked that $\mathcal{CC}(N, \Omega, \varpi)$ is also a space vector of dimension $\sum_{S\in 2^N} |\Omega_S|$. This assertion is proven below together with some other nice properties of $\mathcal{CC}(N, \Omega, \varpi)$ that will be useful in characterizing Ψ.

Given a coalition S, $k \in \Omega_S$ and a collection $c = (c_{k,S})_{S\in\mathcal{C}_N, k\in\Omega_S}$ we define:

- $\gamma_S \in \Gamma^N$ and $\gamma_S^* \in \Gamma^N$ for all $T \in \mathcal{C}_N$ by

$$\gamma_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \gamma_S^*(T) = \begin{cases} 1 & \text{if } S \subsetneq T \\ 0 & \text{otherwise} \end{cases}$$

- $g^{k,S}$, $\Upsilon^{k,S}$, $\Upsilon^{*,S} \in \mathcal{CC}(N, \Omega, \varpi)$ for all $T \in \mathcal{C}_N$ and $l \in \Omega_T$ by $\Upsilon^{*,S}(T, l) = \gamma_S^*(T)$,

$$g^{k,S}(T, l) = \begin{cases} 1 & \text{if } l = k \text{ and } T = S \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \Upsilon^{k,S}(T, l) = \begin{cases} \gamma_S(T) & \text{if } l = k \\ \gamma_S^*(T) & \text{otherwise} \end{cases}$$

- $\Upsilon^{c,k,S} \in \mathcal{CC}(N, \Omega, \varpi)$ by $\Upsilon^{c,k,S} = c_{k,S}g^{k,S} + \Upsilon^{*,S}$.

Note that γ_S and γ_S^* are TU-games; whereas $g^{k,S}$, $\Upsilon^{k,S}$ and $\Upsilon^{*,S}$ are CC-games in $\mathcal{CC}(N, \Omega, \varpi)$. Furthermore,

$$\Upsilon^{k,S} = g^{k,S} + \Upsilon^{*,S} = \Upsilon^{c,k,S} \text{ provided that } c_{k,S} = 1. \tag{3.1}$$

Proposition 3.1.1. *Given N, Ω and ϖ.*

1. *$\mathcal{CC}(N, \Omega, \varpi)$ is a real space vector.*

2. *Any collection $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is a basis of $\mathcal{CC}(N, \Omega, \varpi)$ assuming that that $c_{k,S} \neq 0$ for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$.*

Proof.

To prove that $\mathcal{CC}(N, \Omega, \varpi)$ is a real space vector, we have just to prove that it is a subspace vector of space vector of application from 2^N to $\mathbb{R}$. It is clear that $\mathcal{CC}(N, \Omega, \varpi) \neq \emptyset$ and the proof is completed by Definitions 3.1.5 and 3.1.6.

Now to prove that $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is a basis of $\mathcal{CC}(N, \Omega, \varpi)$, consider $c = (c_{k,S})_{S\in\mathcal{C}_N, k\in\Omega_S}$ a collection of real numbers such that $c_{k,S} \neq 0$ for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$. Note that for all games $v \in \mathcal{CC}(N, \Omega, \varpi)$,

$$v = \sum_{S\in\mathcal{C}_N, k\in\Omega_S} v(S, k)g^{k,S}.$$

Therefore, $\{g^{k,S} : S \in \mathcal{C}_N, k \in \Omega_S\}$ is a generating set for the vector space $\mathcal{CC}(N,\Omega,\varpi)$. It follows that the dimension of $\mathcal{CC}(N,\Omega,\varpi)$ is at most $\sum_{S\in\mathcal{C}_N} |\Omega_S|$. Now, the collection $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N,k\in\Omega_S}$ contains exactly $\sum_{S\in\mathcal{C}_N} |\Omega_S|$ distinct games. To prove that $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N,k\in\Omega_S}$ is a basis of $\mathcal{CC}(N,\Omega,\varpi)$, it is sufficient to prove that the games in $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N,k\in\Omega_S}$ are linearly independent. To see this, let $(\alpha_{k,S})_{S\in\mathcal{C}_N,k\in\Omega_S}$ be some real numbers such that

$$\sum_{S\in\mathcal{C}_N,k\in\Omega_S} \alpha_{k,S}\Upsilon^{c,k,S} = \widetilde{0}_\Omega. \tag{3.2}$$

where $\widetilde{0}_\Omega(T,l) = 0$ for all $T \in \mathcal{C}_N$ and for all $l \in \Omega_S$. We prove by induction on the cardinality of S that $\alpha_{k,S} = 0$ for all $S \in \mathcal{C}_N$ and $k \in \Omega_S$. First assume that $|S| = 1$; that is $S = \{i\}$ for some $i \in N$. Consider $k \in \Omega_S$. Then by the definition of $\Upsilon^{c,l,T}$,

$$\widetilde{0}_\Omega(\{i\},k) = 0 = \sum_{T\in\mathcal{C}_N,l\in\Omega_T} \alpha_{l,T}\Upsilon^{c,l,T}(\{i\},k) = c_{k,\{i\}}\alpha_{k,\{i\}}.$$

Therefore $\alpha_{k,\{i\}} = 0$ since $c_{k,\{i\}} \neq 0$. Assume that for some s such that $1 \leq s < n$, it holds that for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$, $\alpha_{k,S} = 0$ whenever $1 \leq |S| \leq s$. Consider a coalition S of cardinality $s+1$ and $k \in \Omega_S$. The definition of $\Upsilon^{c,l,T}$ together with the induction assumption imply

$$\widetilde{0}_\varpi(S,k) = 0 = \sum_{T\in\mathcal{C}_N,l\in\Omega_T} \alpha_{l,Tk}\Upsilon^{c,l,T}(S,k) = c_{k,S}\alpha_{k,S} + \sum_{T\subsetneq S,l\in\Omega_T} c_{l,T}\alpha_{l,T} = c_{k,S}\alpha_{k,S}.$$

Therefore $\alpha_{k,S} = 0$ since $c_{k,S} \neq 0$. This proves that $\alpha_{k,S} = 0$ for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$. ∎

In particular, the collection $\left(\Upsilon^{k,S}\right)_{S\in\mathcal{C}_N,k\in\Omega_S}$ is, by (3.1), a basis of $\mathcal{CC}(N,\Omega,\varpi)$. In the following propositions show that, the sum of all $\Upsilon^{k,S}$ over Ω_S can be also rewritten as a linear combination of some specific games.

Proposition 3.1.2. *Given a coalition S,*

$$\sum_{k\in\Omega_S} \Upsilon^{k,S} = (\widetilde{\gamma}_S)_\Omega + (|\Omega_S| - 1)\Upsilon^{*,S}. \tag{3.3}$$

Proof.

Let $S \in \mathcal{C}_N$. Proving (3.3) amounts to showing that for all $T \in \mathcal{C}_N$ and for all $l \in \Omega_T$,

$$\sum_{k\in\Omega_S} \Upsilon^{k,S}(T,l) = \gamma_S(T) + (|\Omega_S| - 1)\Upsilon^{*,S}(T,l). \tag{3.4}$$

Consider $T \in \mathcal{C}_N$ and $l \in \Omega_T$. There are three possible cases we distinguish.

(a) First suppose that $S \nsubseteq T$. By definition, $\Upsilon^{k,S}(T,l) = \gamma_S(T) = \Upsilon^{*,S}(T,l) = 0$ for all $k \in \Omega_S$. Hence,

$$\sum_{k\in\Omega_S} \Upsilon^{k,S}(T,l) = 0 \text{ and } \gamma_S(T) + (|\Omega_S| - 1)\,\Upsilon^{*,S}(T,l) = 0.$$

(b) Now, for $S = T$. Then $\Upsilon^{k,S}(T,l) = 0$ for $k \neq l$; $\Upsilon^{l,S}(T,l) = 1 = \gamma_S(T)$; and $\Upsilon^{*,S}(T,l) = 0$. Therefore

$$\sum_{k\in\Omega_S} \Upsilon^{k,S}(T,l) = 1 \text{ and } \gamma_S(T) + (|\Omega_S| - 1)\,\Upsilon^{*,S}(T,l) = 1.$$

(c) Finally, assume that $S \subsetneq T$, then $\gamma_S(T) = \Upsilon^{k,S}(T,l) = \Upsilon^{*,S}(T,l) = 1$ for all $k \in \Omega_S$. Thus,

$$\sum_{k\in\Omega_S} \Upsilon^{k,S}(T,l) = |\Omega_S| \text{ and } \gamma_S(T) + (|\Omega_S| - 1)\,\Upsilon^{*,S}(T,l) = |\Omega_S|.$$

For each of the three possible cases, (3.3) holds. ∎

The game $\Upsilon^{k,S}$ is a kind of unanimity game in which a win of one unit is guaranteed provided that either the members of S cooperate and event k is observed; or the members of S cooperate with some players out of S. Thus, (3.3) simply tells us that by summing over all events in Ω_S, the members of S secure, independently of the event that is observed, a win of one unit by forming S, or a win of $|\Omega_S|$ by cooperating with some players out of S. More importantly, Proposition 3.1.2 together with the next result help in linking unanimity CC-games with the known unanimity TU-games.

Proposition 3.1.3. *Given a coalition S, when the collection c is such that for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$, $c_{k,S} = \frac{1}{\varpi(S,k)}$:*

$$\sum_{k\in\Omega_S} \frac{1}{c_{k,S}} \Upsilon^{c,k,S} = (\widetilde{\gamma}_S)_\Omega\,. \tag{3.5}$$

Proof.
The collection $c = (c_{k,S})_{S\in\mathcal{C}_N, k\in\Omega_S}$ is well-defined since only events with positive probabilities are considered. Let $S \in \mathcal{C}_N$, since for all $k \in \Omega_S$, $\Upsilon^{c,k,S} = \frac{1}{\varpi(S,k)} g^{k,S} + \Upsilon^{*,S}$ we have:

$$\begin{aligned}
\sum_{k\in\Omega_S}\frac{1}{c_{k,S}}\Upsilon^{c,k,S} &= \sum_{k\in\Omega_S} g^{k,S} + \varpi(S,k)\Upsilon^{*,S} \\
&= \sum_{k\in\Omega_S}\left[\Upsilon^{k,S} - \Upsilon^{*,S}\right] + \Upsilon^{*,S}\sum_{k\in\Omega_S}\varpi(S,k) \\
&= \sum_{k\in\Omega_S}\Upsilon^{k,S} - |S|\,\Upsilon^{*,S} + \Upsilon^{*,S} \\
&= (\widetilde{\gamma}_S)_\Omega + (|\Omega_S|-1)\,\Upsilon^{*,S} - |\Omega_S|\,\Upsilon^{*,S} + \Upsilon^{*,S} \text{ by equation (3.3)} \\
&= (\widetilde{\gamma}_S)_\Omega\,.
\end{aligned}$$

■

PROPOSITION 3.1.4. *Given a coalition S,*

$$\Upsilon^{*,S} = \sum_{T\in 2^N : S\subsetneqq T}(-1)^{|T|-|S|+1}\,\tilde{\gamma}_T. \tag{3.6}$$

Proof.

Consider a coalition S. Proving (3.6) amounts to showing that for all $K\in 2^N$ and for all $l\in\Omega_K$

$$\Upsilon^{*,S}(K,l) = \sum_{T\in 2^N : S\subsetneqq T}(-1)^{|T|-|S|+1}\gamma_T(K)\,. \tag{3.7}$$

Consider $K\in 2^N$ and $l\in\Omega_K$. First suppose that $S\nsubseteq K$ or $K=S$. Then each coalition T such that $S\subsetneqq T$ satisfies $T\nsubseteq K$; otherwise one would have $S\subsetneqq K$. Thus $\Upsilon^{*,S}(K,l)=\gamma^*_S(K)=0$ and $\gamma_T(K)=0$. Therefore

$$\sum_{T\in 2^N : S\subsetneqq T}(-1)^{|T|-|S|+1}\gamma_T(K) = 0 = \Upsilon^{*,S}(K,l)\,.$$

Now suppose that $S\subsetneqq K$, then $\Upsilon^{*,S}(K,l)=1$ and $\gamma_T(K)=0$ for all coalitions T such that $T\nsubseteq K$. Thus

$$\begin{aligned}
\sum_{T\in 2^N : S\subsetneqq T}(-1)^{|T|-|S|+1}\gamma_T(K) &= \sum_{T/S\subsetneqq T\subseteq K}(-1)^{|T|-|S|+1} \\
&= (-1)^{|S|-1}\sum_{t=1}^{|K|-|S|}\binom{|K|-|S|}{t}(-1)^{|S|+t} \\
&\qquad \text{where } t=|T\backslash S| \\
&= -\left((1-1)^{|K|-|S|}-1\right) = 1 = \Upsilon^{*,S}_l(K)
\end{aligned}$$

In both possible cases, (3.6) holds. ■

3.2 A value for CC-games with discrete sample spaces

A CC-game is played as follows: players have to form coalitions. When a coalition S is formed, a realization from Ω_S of a random event is observed and the members of S obtain the corresponding payoff. Furthermore, when a coalition structure $\mathcal{P} = \{S_1, S_2, \ldots, S_p\}$ is formed, outcomes of coalitions in $\mathcal{P}$ have independent realizations. If $\mathcal{P}$ is disrupted, new coalitions are formed and again new events with possibly new payoff realizations are randomly observed.

3.2.1 Basic definitions

Our concern is as follows. The grand coalition N is formed; but an *a priori sharing vector rule* F is to be designed such that when an event $k \in \Omega_N$ is observed, the coalitional payoff $v(N, k)$ is shared accordingly. In a two stage process, an allocation usually has the shape (d, r) such that given $i \in N$, player i who is first promised d_i finally gets $d_i + f_i(P)$ with $f_i(P) = r_i \left(P - \sum_{i \in N} d_i\right)$ when P is the actual worth of the grand coalition; see Suijs et al. (1999). We are interested in such an operational solution. But before, we have the following general definition:

Definition 3.2.1. *Assume that the grand coalition N is formed. An allocation rule (a value) on $\mathcal{F} \subseteq \mathcal{CC}(N)$ is a mapping F that associates each CC-game $G = (\Omega, v, \varpi) \in \mathcal{F}$ with a list $F(G) = (F(G, k))_{k \in \Omega_N}$ of payoff vectors.*

$F_i(G, k)$ is the share of player i with respect to F when the grand coalition is formed and event $k \in \Omega_N$ is observed.

Given a game $G = (\Omega, v, \varpi)$, a value F assigns to G a collection of deterministic payoff vectors $F(G, k)$ for $k \in \Omega_N$. This can be viewed as a *post ante* contract that states the share of each player for each specific event $k \in \Omega_N$ that may be encountered as the grand coalition is formed. But observing an event in Ω_N is a stochastic event. Thus, the pair $((F(G, k))_{k \in \Omega_N}, (\varpi(N, k))_{k \in \Omega_N})$ can also be seen as an n-tuple of random variables $F_i(G) = ((F_i(G, k))_{k \in \Omega_N}, (\varpi(N, k))_{k \in \Omega_N})$ for $i \in N$, that sum to the random variable $v_N = (v(N, k), \varpi(N, k))_{k \in \Omega_N}$. Roughly, before an event $k \in \Omega_N$ is observed for the grand coalition, the share of player i is the random variable $F_i(G)$; and provided that $k \in \Omega_N$ is observed, the share of player i is equal to $F_i(G, k)$ for sure.

3.2.2 Axioms for a value on CC-games with discrete sample spaces

The desirability of a value depends on how appealing are the shares it generates. Below are some properties of a value given a family $\mathcal{F}$ of CC-games.

Definition 3.2.2. *An allocation rule (a value) F on $\mathcal{F} \subseteq \mathcal{CC}(N)$ satisfies* **Efficiency (E)** *if for all $v \in \mathcal{F}$ and for all $k \in \Omega_N$, $\sum\limits_{i \in N} F_i(v,k) = v(N,k)$.*

Efficiency requires that when an event $k \in \Omega_N$ is observed for the grand coalition, individual deterministic payments are summed to the collective worth $v(N,k)$.

Definition 3.2.3. *An allocation rule (a value) F on $\mathcal{F} \subseteq \mathcal{CC}(N)$ satisfies* **Additivity (A)** *if for all $u, v \in \mathcal{F} \cap \mathcal{CC}(N, \Omega, \varpi)$, for some Ω and ϖ, $F(u+v) = F(u) + F(v)$.*

Additivity states that when an event $k \in \Omega_N$ occurred for the grand coalition, the share of a player in the game $u + v$ should be the sum of his/her shares in the games u and v.

Definition 3.2.4. *A player $i \in N$ is a* **null player** *in a CC-game $v \in \mathcal{F} \cap \mathcal{CC}(N, \Omega, \varpi)$ if for all $S \subseteq N \setminus \{i\}$, $\Omega_S = \Omega_S \cup \{i\}$ and for all $k, l \in \Omega_S$, $v(S \cup \{i\}, k) = v(S, l)$.*

In other words, a null player is simply a player who can not changes the possibles actions of a coalition when he/she joints this coalition. Therefore he/she could not change the coalition's payoffs by joining a coalition. 4

Definition 3.2.5. *An allocation rule (a value) F on $\mathcal{F} \subseteq \mathcal{CC}(N)$ satisfies* **Null Player Property (NP*)** *if for all* **null player** *in a CC-game $v \in \mathcal{F} \cap \mathcal{CC}(N, \Omega, \varpi)$, $F_i(v,k) = 0$ for all $k \in \Omega_N$*

Axiom (NP*) simply requires that any null player in a CC-game should receive a zero share.

Definition 3.2.6. *Given a CC-game $v \in \mathcal{CC}(N, \Omega, \varpi)$. Two players i and j are* stochastically symmetric *in the CC-game v if for all 444 $S \subseteq N \setminus \{i, j\}$ and for all $k \in \Omega_{S \cup \{i\}}$, $\sum_{l \in \Omega_{S \cup \{i\},k}} \varpi(S \cup \{i\}, l) = \sum_{l \in \Omega_{S \cup \{j\},k}} \varpi(S \cup \{j\}, l)$;*

Where $\Omega_{T,k} = \{l \in \Omega_T : v(T,l) = v(T,k)\}$

We simply say that, interchanging (transposing) i and j does not affect the chance of any coalition to realize each of its feasible worth.

Moreover in the game $v \in \mathcal{CC}(N, \Omega, \varpi)$, given a coalition T and an elementary k in Ω_T. $\Omega_{T,k}$ is the set of all elementary events l in Ω_T such that both l and k produce the same payoff for T; l is called a duplication of k in Ω_T.

Definition 3.2.7. *An allocation rule (a value) F on $\mathcal{F} \subseteq \mathcal{CC}(N)$ satisfies* **Stochastic Symmetry (SS)** *if for two players i and j who are stochastically symmetric in a game $v \in \mathcal{F}$, then $F_i(v,k) = F_j(v,k)$ for all $k \in \Omega_N$.*

Following (SS), two stochastically symmetric players in a CC-game always receive equal shares.

Definition 3.2.8. *An allocation rule (a value) F on $\mathcal{F} \subseteq \mathcal{CC}(N)$ satisfies* **Independence of Local Relabeling (ILR)** *if for all coalitions $S \neq N$ and for all permutations π of Ω_S, $F(\pi v) = F(v)$ whenever $v, \pi v \in \mathcal{F}$.*

Axiom (ILR) requires that any *local relabeling* of events in a sample space associated with a proper coalition of players should have no effect on individual shares.

One would expect any conceivable value to return in the new game the same shares as in the initial game.

Another change we consider is the duplication of an event.

Definition 3.2.9. *Given a c.c game $v' \in \mathcal{CC}(N, \Omega', \varpi')$. A game $v \in \mathcal{CC}(N, \Omega, \varpi)$ is a local duplication of v' if there exists a coalition $S \neq N$ and $k, k' \in \Omega_S$ satisfies both three items:*

(i) $k' \in \Omega_S \setminus \Omega'_S$, $\Omega_S = \Omega'_S \cup \{k'\}$ and $\Omega_T = \Omega'_T$ for all $T \neq S$;

(ii) $v(S,k) = v(S,k') = v'(S,k)$ and $\varpi(S,k') + \varpi(S,k) = \varpi'(S,k)$;

(iii) $v(T,l) = v'(T,l)$ and $\varpi(T,l) = \varpi'(T,l)$ whenever $(l \neq k$ or $T \neq S)$.

This is denoted by $v' = v^{S,k,k'}$. We also say that v' is obtained from v by canceling the duplicated event k' of k. We also say that $v' = v^{S,k,k'}$ is a *local duplication game.*

Assume for illustration that in Example 3.1.3, the organizer modifies the Wheel of Fortune for one-person tickets by only splitting the golden band into a new golden band which provides a win of 10 euros with probability p_g and a yellow (Y) band which also provides 10 euros with probability p_y in such a way that $p_g + p_y = 0.02$. Then a one-person ticket still wins 10 euros with probability 0.02 although such a win now comes from two distinct events. Such a fake change should normally have no effect on individual shares in a game.

Definition 3.2.10. *An allocation rule (a value) F on $\mathcal{F} \subseteq \mathcal{CC}(N)$ satisfies* **Independence of Local Duplication (ILD)** *if for all local duplication game $v^{S,k,k'}$ of any CC-game v, $F\left(v^{S,k,k'}\right) = F(v)$ whenever $v^{S,k,k'} \in \mathcal{F}$.*

Condition (ILD) is the requirement that any local duplication has no change on individual shares.

Remark 3.2.1. *One can iterate (ILD), if feasible, to cancel any subset of events that are duplications of a given event $k \in \Omega_S$. To see this, consider a nonempty subset $K = \{k_1, k_2, \ldots, k_t\}$ of $\Omega_{S,k} \setminus \{k\}$ and denote by $v^{S,k,K}$ the game obtained from v by successively canceling $k_1, k_2, \ldots, k_t$. Condition (ILD) is equivalent to say that $F(v^{S,k,K}) = F(v)$. In words, by canceling any finite number of duplications of a given event k in a game and by updating the probability of observing k to the sum of the probabilities of observing k or some of its duplications, the shares of players are not affected.*

Remark 3.2.2. *By duplicating some event for a given game in $\mathcal{F}$, it is not guaranteed that the new game is still in $\mathcal{F}$. Similarly, by canceling a duplication of an event, the new game is not necessarily in $\mathcal{F}$. Some family may be rich enough to allow these two operations. In this case, $\mathcal{F}$ will be simply called rich. This is formally stated in the next definition.*

Definition 3.2.11. *A family $\mathcal{F}$ of CC-games is rich if $\mathcal{F}$ meets the following two conditions:*

($\mathbf{c_1}$) *for all $v \in \mathcal{F}$ such that $v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, for all coalitions $S \neq N$, for all events $k \in \Omega_S$ and for all events $k' \notin \Omega_S$, there exists $u \in \mathcal{F}$ such that $v = u^{S,k,k'}$;*

($\mathbf{c_2}$) *for all $v \in \mathcal{F}$ such that $v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, for all coalitions $S \neq N$ and for all events $k, k' \in \Omega_S$, if k' is a duplication of k in v then $v^{S,k,k'} \in \mathcal{F}$.*

Condition ($\mathbf{c_1}$) means that for each game v in $\mathcal{F}$ and for each event k in the universe of a proper subset of the player set, there exists a game in $\mathcal{F}$ obtained from v by a duplication of k that brings into the game v a new event k'. Condition ($\mathbf{c_2}$) is the requirement that by merging an event and some of its duplications from a game in $\mathcal{F}$, the new game should stay in $\mathcal{F}$. For example, the set of all CC-games on N is rich. Furthermore, one also obtains a rich family of CC-games by considering the set of all CC-games such that all probabilities are rational numbers. The following result holds on rich families of CC-games.

Proposition 3.2.1. (ILD) *implies* (ILR) *on any rich domain.*

Proof.

Consider a rich family $\mathcal{F}$ of CC-games and a value F on $\mathcal{F}$ that meets (ILD). Since any permutation of a finite set is a finite product of transpositions, to prove that F necessarily meets (ILR), it is sufficient to prove that for all $v \in \mathcal{F}$ such that $v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, for all coalitions $S \neq N$ and for all transpositions π of Ω_S, $F(\pi v) = F(v)$. Suppose that π is a transposition of Ω_S. That is $\pi = (k, l)$ for some $\{k, l\} \subseteq \Omega_S$. Consider two events k' and l' such that $k', l' \notin \Omega_S$. Since $\mathcal{F}$ is rich and $k \in \Omega_S$, then there exists $u_1 \in \mathcal{F}$ such that $v = u_1^{S,k,k'}$ (u_1 is obtained from v by a duplication of event k). By (ILD),

$$F(v) = F(u_1). \tag{3.8}$$

Similarly, $l \in \Omega_S \cup \{k'\}$ and $l' \notin \Omega_S \cup \{k'\}$. Since $\mathcal{F}$ is rich, there exists a game u_2 in $\mathcal{F}$ such that $u_1 = u_2^{S,l,l'}$. By (ILD),

$$F(u_1) = F(u_2). \tag{3.9}$$

Note that in the game u_2, k' and l' are duplications of k and l respectively. Since $\mathcal{F}$ is rich, the game u_3 such that $u_3 = u_2^{S,k',k}$ (obtained from u_2 by merging k' and k into k') belongs to $\mathcal{F}$. By (ILD),

$$F(u_2) = F(u_3). \tag{3.10}$$

Similarly, the game u_4 such that $u_4 = u_3^{S,l',l}$ belongs to $\mathcal{F}$. By (ILD),

$$F(u_4) = F(u_3). \tag{3.11}$$

By construction, the game u_4 is obtained from v by only renaming k to k' and l to l'. It appears from Equations (3.8) – (3.11) that $F(u_4) = F(v)$. In words, renaming k to k' and l to l' with $k', l' \notin \Omega_S$ does not affect individual shares in the game v. Note that $u_4 \in \mathcal{CC}(N, \Omega', \varpi')$ where

$$\Omega'_S = (\Omega_S \backslash \{k, l\}) \cup \{k', l'\},$$

$$\varpi'(S, k') = \varpi(S, k), \quad \varpi'(S, l') = \varpi(S, l),$$

$$u_4(S, k') = v(S, k), \quad u_4(S, l') = v(S, l);$$

and for all coalitions $T \neq S$,

$$\Omega'_T = \Omega_T \text{ and } \varpi'(T, t) = \varpi(T, t) \text{ for all } t \in \Omega'_T.$$

Since $k, l \notin \Omega'_S$, one obtains from u_4 a new game u_5 by only renaming k' to l and l' to k without altering individual shares. That is $F(u_5) = F(u_4)$. Hence $F(u_5) = F(v)$. This complete the proof since $u_5 = \pi v$. ∎

3.3 The equal-surplus Shapley value for CC-games

Is there any value that meets all the properties mentioned in Section 3.2? The answer is yes and a solution is constructed below in two different ways.

3.3.1 Two ways of constructing a value

We construct two values Ψ and Φ for CC-games using to distinct approaches.

Via an analytic approach

We associate each CC-game $v \in \mathcal{CC}(N, \Omega, \varpi)$ with its expectation game E_v that assigns to each coalition S, its expected worth $E_v(S)$. More formally,

$$E_v(S) := \sum_{k \in \Omega_S} \varpi(S, k) v(S, k).$$

The game E_v is a TU-game and its Shapley value is called the *prior Shapley value* of the CC-game v. We then define the two-stage value Ψ for CC-games as follows. Players in N are first promised their prior Shapley shares of the game v and then proceed to observe a random event from Ω_N. When $k \in \Omega_N$ is observed at the second stage, the actual worth of the grand coalition is $v(N,k)$ and the surplus to be re-allocated is $v(N,k) - E_v(N)$. This surplus is equally split among players in such a way that the final share of a each player $i \in N$ is

$$\Psi_i(v,k) = Shap_i(E_v) + \frac{1}{n}(v(N,k) - E_v(N)). \tag{3.12}$$

The value Ψ will be called the *equal-surplus Shapley value* for CC-games.

Via a bargaining model

Given a game $v \in \mathcal{CC}(N)$, the intuition on how the payoff vector $\Phi(v,k)$ is derived can be obtained as in Shapley (1953) using a bargaining procedure. Once the grand coalition is formed and an event $k \in \Omega_N$ is observed, the question is how to share $v(N,k)$ among the n players? The attributes in favor of a player, say i, are measured only by all possible marginal contributions that may be observed when i joins a coalition $S \subseteq N \setminus \{i\}$ as in the following threefold procedure:

(A_1) Once an event $k \in \Omega_N$ is observed, players row up in a line to join the coalition one at a time; N_s denotes the coalition of the first s players to get in.

(A_2) While the s^{th} player, say e^s, is joining the coalition, a trial from Ω_{N_s} is made for $s < n$ to observe a random event e^s with probability $\varpi(N_s, e^s)$ and player e_s is promised his/her marginal contribution $v(N_s, e^s) - v(N_{s-1}, e^{s-1})$ with the convention $N_0 = \emptyset$, $k_0 = 0$, $v(\emptyset, k_0) = 0$ and $e^n = k$, k is the final issue.

(A_3) All the $n!$ orderings of the players are equally probable; for each ordering of players, the $n-1$ first trials have independent realizations; and the event $e^n = k \in \Omega_N$ observed for the grand coalition remains unchanged.

$\Phi_i(v,k)$ is the expectation of player i in the procedure $(A_1) - (A_3)$.

Here, an entry-trial scenario is a pair $e = (e_1 e_2 ... e_n, e^1 e^2 ... e^n)$ such that player e_s gets in at the s^{th} position and e^s is the event that is observed as the coalition $\{e_1, e_2, \dots, e_s\}$ of the first s players is formed. A k-entry-trial scenario is any entry-trial scenario e such that $e^n = k$. The set of all k-entry-trial scenarios will be denoted by $\mathcal{E}_k$. The probability of observing $e \in \mathcal{E}_k$ is $\varpi^e = \varpi(N_1, e^1) \times \varpi(N_2, e^2) \times \cdots \times \varpi(N_n - 1, e^n - 1)$ with $N_s = \{e_1, e_2, \dots, e_s\}$.

Interestingly, $\Psi = \Phi$ and the value Ψ meets all the six axioms above as shown in the following propositions.

Proposition 3.3.1. *The value Ψ satisfies (E), (A), (NP^*), (SS), (ILR) and (ILD).*

Proof.

Note that the operator that associates each CC-game v with the TU-game E_v is linear. Moreover, the Shapley value is efficient and additive. It then follows from (3.12) that Ψ is efficient and additive. Also note that if two players are stochastically symmetric in the game v, they are symmetric in the TU-game E_v. Recalling that the Shapley value is symmetric, it follows from (3.12) that Ψ satisfies (SS). Given a TU-game u, note that the expectation game of $\widetilde{u}_\Omega$ coincides with u and that by definition, $\widetilde{u}(N,k) = u(N) = E_{\widetilde{u}_\Omega}(N)$. Again, if follows from (3.12) that Ψ satisfies (NP^*). By the definition of the game πv_S given a coalition S and a permutation π of Ω_S, note that $E_{\pi v_S}(T) = E_v(T)$ for all coalitions T. Since $\pi v_S(N,k) = v(N,k)$ for all $k \in \Omega_N$, it follows from (3.12) that Ψ satisfies (ILR). In the same way, a local duplication does not affect the expectation game; nor the random coalitional worth of the grand coalition. Thus by (3.12), a local duplication does not affect individual shares by Ψ; and Ψ then meets (ILD). ∎

Remark 3.3.1. *Given a player $i \in N$, it immediately follows from (3.12) that, the payoff of player i in a CC-game is a random variable the mean of which is the Shapley payoff of player i in the expectation game and its standard deviation is $\frac{1}{n}$ the standard deviation of the random payoff of the grand coalition. Therefore, the smaller of the standard deviation of the payoff of the grand coalition is, the better is the chance of player i to obtain his/her prior Shapley share.*

Proposition 3.3.2. *For all $v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all $k \in \Omega_N$, $\Phi(v,k) = \Psi(v,k)$.*

Proof.

Given a k-entry-trial scenario e, denote by $e\,[i] \in \{1, 2, \ldots, n\}$ the position of player i. By definition,

$$\Phi_i\,(v,k) = \sum_{r=1}^{n} \sum_{e \in \mathcal{E}_k : e[i]=r} \frac{\varpi^e}{n!} v\,(N_r, e^r)\,.$$

This sum can be split into three distinct sums $\Phi_i(v,k) = K_1 + K_2 + K_3$ where the first sum K_1 is the weighted sum of contributions of player i for all entry-trial scenarios where he/she gets in first.

$$\begin{aligned}
K_1 &= \sum_{e\in\mathcal{E}_k:e_1=i} \frac{\varpi^e}{n!} v\left(N_1, e^1\right) \\
&= \sum_{e\in\mathcal{E}_k:e_1=i} \frac{\varpi^e}{n!} v\left(\{i\}, e^1\right) \\
&= \frac{(n-1)!}{n!} \sum_{a\in\Omega_{\{i\}}} \varpi\left(\{i\}, a\right) v\left(\{i\}, a\right) \\
&= \frac{(n-1)!}{n!} E_v\left(\{i\}\right) = \frac{0!\,(n-1)!}{n!}\left(E_v\left(\{i\}\right) - E_v\left(\emptyset\right)\right) \text{ since } E_v\left(\emptyset\right) = 0
\end{aligned}$$

The sum K_2 is the weighted sum of contributions of player i for all k-entry-trial scenarios where i gets in last.

$$\begin{aligned}
K_2 &= \sum_{e\in\mathcal{E}_k:e_n=i} \frac{\varpi^e}{n!} \left(v\left(N_n, e^n\right) - v\left(N_{n-1}, e^{n-1}\right)\right) \\
&= \sum_{e\in\mathcal{E}_k:e_n=i} \frac{\varpi^e}{n!} \left(v\left(N, k\right) - v\left(N\backslash\{i\}, e^{n-1}\right)\right) \\
&= \frac{(n-1)!0!}{n!} \sum_{a\in\Omega_{N\backslash\{i\}}} \varpi\left(N\backslash\{i\}, a\right)\left(v\left(N,k\right) - v\left(N\backslash\{i\}, a\right)\right) \\
&= \frac{(n-1)!0!}{n!}\left(v\left(N,k\right) - E_v\left(N\backslash\{i\}\right)\right) \\
&= \frac{(n-1)!0!}{n!}\left(E_v\left(N\right) - E_v\left(N\backslash\{i\}\right)\right) + \frac{1}{n}\left(v\left(N,k\right) - E_v\left(N\right)\right)
\end{aligned}$$

The sum K_3 is the weighted sum of all contributions of player i for all entry-trial scenarios where he/she gets in at a position s such that $1 < s < n$. Let $P_{s,n} = \frac{(s-1)!(n-s)!}{n!}$ for $1 \leq s \leq n$.

$$\begin{aligned}
K_3 &= \sum_{e\in\mathcal{E}_k:e_s=i\wedge 1<s<n} \frac{\varpi^e}{n!} \left(v\left(N_s, e^s\right) - v\left(N_{s-1}, e^{s-1}\right)\right) \\
&= \sum_{S:i\in S\wedge 1<|S|=s<n} \sum_{e\in\mathcal{E}_k:e_s=i} \wedge N_s = S \frac{\varpi^e}{n!} \left(v\left(S, e^s\right) - v\left(S\backslash\{i\}, e^{s-1}\right)\right) \\
&= \sum_{S:i\in S\wedge 1<|S|=s<n} \sum_{b\in\Omega_S} \sum_{a\in\Omega_{S\backslash\{i\}}} \left(v\left(S,b\right) - v\left(S\backslash\{i\}, a\right)\right) \sum_{e\in\mathcal{E}_k:e_s=i\wedge N_s=S\wedge e^s=b\wedge e^{s-1}=a} \frac{\varpi^e}{n!} \\
&= \sum_{S:i\in S\wedge 1<|S|=s<n} P_{s,n} \sum_{b\in\Omega_S} \sum_{a\in\Omega_{S\backslash\{i\}}} \varpi\left(S,b\right)\varpi\left(S\backslash\{i\}, a\right)\left(v\left(S,b\right) - v\left(S\backslash\{i\}, a\right)\right) \\
&= \sum_{S:i\in S\wedge 1<|S|=s<n} P_{s,n}\left[E_v\left(S\right) - E_v\left(S\backslash\{i\}\right)\right]
\end{aligned}$$

Therefore, combining the three sums gives

$$\begin{aligned}\Phi_i(v,k) &= \sum_{S:i\in S}\frac{(s-1)!\,(n-s)!}{n!}\left(E_v(S)-E_v(S\backslash\{i\})\right)+\frac{1}{n}\left(v(N,k)-E_v(N)\right)\\ &= Shap_i(E_v)+\frac{1}{n}\left(v(N,k)-E_v(N)\right)=\Psi_i(v,k)\end{aligned}$$

Clearly, the two solutions Φ and Ψ coincide. ■

Procedure $(A_1)-(A_3)$ can then be viewed as a bargaining model of the value Ψ. It provides an intuitive way of deriving the shares of all players as their expected shares from a dynamic process. The following example give in detail an application of the equal-surplus Shapley value on the game v in Example 3.1.2.

Example 3.3.1. *Consider the game v in Example 3.1.2. When the event M for the joint printer is observed, all scenarii in sharing vector $v(N,M)$ are as follows:*

Scenarii	1's contribution	ϖ	2's contribution	ϖ
$(21,MM)$	2	0.05	8	0.05
$(21,EM)$	6	0.1	4	0.1
$(21,ZM)$	10	0.85	0	0.85
$(12,MM)$	5	0.1	5	0.1
$(12,EM)$	10	0.02	0	0.02
$(12,ZM)$	0	0.88	10	0.88
Expectations	$\Phi_1(v,M)=\frac{9.9}{2}=4.95$		$\Phi_2(v,M)=\frac{10.1}{2}=5.05$	

For illustration, note that when the entry-trial is $(21,MM)$, player 1's contribution is $v(\{1,2\},M)-v(\{2\},M)=2$ while player 2's contribution is $v(\{2\},M)-0=8$. Similarly, when the entry-trial is $(12,ZM)$, player 1's contribution is $v(\{1\},Z)-0=0$ while player 2's contribution is $v(\{1,2\},M)-v(\{1\},Z)=10$.

We easily compute that the expectation game E_v is given by $E_v(\{1\})=0.7$, $E_v(\{2\})=0.8$ and $E_v(\{1,2\})=1.1$. The prior shares is then $Shap_1(E_v)=0.5$ for player 1 and $Shap_2(E_v)=0.6$ for player 2. When a mechanic breakdown (M) for the joint printer is observed, the surplus cost is $v(\{1,2\},M)-0.7-0.6=8.9$. Each of the two players gets the same extra cost of 4.45. Finally $\Psi_1(v,M)=0.5+4.45=4.95$

and $\Psi_2(v, M) = 0.6 + 4.45 = 5.05$. Thus $\Psi(v, M)$ coincides with the expected payoffs from procedure $(A_1) - (A_3)$. Similarly, it can be also checked that $\Phi(v, k) = \Psi(v, k)$ for $k \in \{E, Z\}$. Thus $\Phi(v) = \Psi(v)$ for the game of Example 3.1.2.

The coincidence $\Phi(v) = \Psi(v)$ observed in Example 3.3.1 turns out to be true for an arbitrary CC-game.

3.3.2 Characterization of equal-surplus Shapley value

Characterization with a uniform probability distribution function

The probability distribution function ϖ is a collection of *uniform probability distributions* if for all coalitions $S \neq N$, all events in Ω_S occur with the same probability. Since all probabilities over Ω_S sum to 1, we have,

Uniform probability distributions ϖ : for all coalitions $S \neq N$ and for all $k \in \Omega_S$, $\varpi(S, k) = \frac{1}{|\Omega_S|}$. In this case, we simply say that ϖ is uniform.

Note that the collection ϖ is uniform if for all coalitions $S \neq N$, $\varpi(S, .) := (\varpi(S, k))_{k \in \Omega_S}$ is a uniform probability distribution on Ω_S. Proposition 3.3.1 holds on $\mathcal{CC}(N, \Omega, \varpi)$ for any probability distribution function ϖ. When ϖ is uniform, we now show that the first five properties in Proposition 3.3.1 completely characterize Ψ.

Lemma 3.3.1. *Consider an arbitrary probability distribution ϖ on Ω.*

Then for any two values F and F' that satisfy (E), (NP^*) and (SS) on $\mathcal{CC}(N, \Omega, \varpi)$,

$$F\left(\alpha\left(\widetilde{\gamma}_S\right)_\Omega\right) = F'\left(\alpha\left(\widetilde{\gamma}_S\right)_\Omega\right)$$

for all $S \in \mathcal{C}_N$ and for all $\alpha \in \mathbb{R}$.

Proof.

Members of S are symmetric players in $\alpha\gamma_S$ as well as in $\alpha\left(\widetilde{\gamma}_S\right)_\Omega$. Thus by (SS), players in S all have the same shares with respect to both F and F'. Furthermore, members of $N \setminus S$ are null players in γ_S. Therefore by (NP*), players in $N \setminus S$ have each a zero share in $\alpha\gamma_S$ as well as in $\alpha\left(\widetilde{\gamma}_S\right)_\Omega$ with respect to both F and F'. The result then follows by efficiency. ■

Lemma 3.3.2. *Assume that* ϖ *is uniform on* Ω*. Then for any two values* F *and* F' *that satisfy* (E), (A), (NP^*), (SS) *and* (ILR) *on* $\mathcal{CC}(N,\Omega,\varpi)$,

$$F\left(\alpha\Upsilon^{k,S}\right) = F'\left(\alpha\Upsilon^{k,S}\right)$$

for all $S\in\mathcal{C}_N$*, for all* $k\in\Omega_S$ *and for all* $\alpha\in\mathbb{R}$.

Proof. First suppose that $S=N$. Then any two players are stochastically symmetric in the game $\alpha\Upsilon^{k,S}$. Thus by (SS) and (E), $F\left(\alpha\Upsilon^{k,S}\right) = F'\left(\alpha\Upsilon^{k,S}\right)$. Now suppose that $S\neq N$. Then by Propositions 3.1.2 and 3.1.4, it follows that,

$$\begin{aligned}
& F\left(\sum_{k\in\Omega_S}\alpha\Upsilon^{k,S}\right) \\
= \; & F\left(\alpha\left(\widetilde{\gamma}_S\right)_\Omega + \alpha(|\Omega_S|-1)\sum_{T\in\mathcal{C}_N:S\subsetneq T}(-1)^{|T|-|S|+1}\left(\widetilde{\gamma}_T\right)_\Omega\right) \\
= \; & F\left(\alpha\left(\widetilde{\gamma}_S\right)_\Omega\right) + \alpha(|\Omega_S|-1)\sum_{T\in\mathcal{C}_N:S\subsetneq T}(-1)^{|T|-|S|+1}F\left(\left(\widetilde{\gamma}_T\right)_\Omega\right) \text{ by } (A) \\
= \; & F'\left(\alpha\left(\widetilde{\gamma}_S\right)_\Omega\right) + \alpha(|\Omega_S|-1)\sum_{T\in\mathcal{C}_N:S\subsetneq T}(-1)^{|T|-|S|+1}F'\left(\left(\widetilde{\gamma}_T\right)_\Omega\right) \text{ by Lemma 3.3.1} \\
= \; & F'\left(\sum_{l\in\Omega_S}\alpha\Upsilon^{l,S}\right) \text{ by } (A)
\end{aligned}$$

Moreover, for all $l\in\Omega_S\setminus\{k\}$, $\pi\Upsilon^{k,S}=\Upsilon^{l,S}$ where π is the transposition of k and l in Ω_S. Therefore by (ILR), $F\left(\alpha\Upsilon^{l,S}\right) = F\left(\alpha\Upsilon^{k,S}\right)$. Thus, by (A), $|\Omega_S|F\left(\alpha\Upsilon^{k,S}\right) = |\Omega_S|F'\left(\alpha\Upsilon^{k,S}\right)$. Finally, $F\left(\alpha\Upsilon^{k,S}\right) = F'\left(\alpha\Upsilon^{k,S}\right)$. ∎

The following result is an axiomatization of the equal-surplus Shapley value on CC-games with uniform probability distributions:

Theorem 3.3.1 (Njoya et al. (2021))**.**

Assume that ϖ is uniform on Ω. Then, the equal-surplus Shapley value Ψ is the unique value on $\mathcal{CC}(N,\Omega,\varpi)$ that simultaneously satisfies (E), (A), (NP^*), (SS) and (ILR). ★

Proof.

Assume that ϖ is uniform on Ω.

Necessity. See Proposition 3.3.1.

Sufficiency. Suppose that F is a value on $\mathcal{CC}(N,\Omega,\varpi)$ that satisfies (E), (A), (NP^*), (SS) and (ILR). To see that $F=\Psi$, consider a game $v\in\mathcal{CC}(N,\Omega,\varpi)$. By Proposition 3.1.1, the family $\left(\Upsilon^{k,S}\right)_{S\in\mathcal{C}_N,k\in\Omega_S}$ is a basis of the space vector $\mathcal{CC}(N,\Omega,\varpi)$. Therefore there exists a family of reals $(\alpha_{k,S})_{S\in\mathcal{C}_N,k\in\Omega_S}$ such that

$$v=\sum_{S\in\mathcal{C}_N}\sum_{k\in\Omega_S}\alpha_{k,S}\Upsilon^{k,S}.$$

By Lemma 3.3.2, $F\left(\alpha_{k,S}\Upsilon^{k,S}\right)=\Psi\left(\alpha_{k,S}\Upsilon^{k,S}\right)$ for all $k\in\Omega_S$, since F and Ψ both satisfy (E), (A), (NP^*), (SS) and (ILR) on $\mathcal{CC}(N,\Omega,\varpi)$. Therefore, $F\left(v\right)=\Psi\left(v\right)$ by additivity. We then conclude that, $F=\Psi$. ■

In Theorem 3.3.1, condition (ILR) may be omitted for some specific Ω. To illustrate this, we assume that all sample spaces with proper coalitions are of the same cardinality. The following counterpart of Lemma 3.3.2 holds for this specific configuration.

Lemma 3.3.3. *Assume that ϖ is uniform on Ω and that $|\Omega_S|=|\Omega_T|$ for all coalitions $S,T\in\mathcal{C}_N\setminus\{N\}$. Then for any two values F and F' that satisfy (E), (A), (NP^*) and (SS) on $\mathcal{CC}(N,\Omega,\varpi)$,*

$$F\left(\alpha\Upsilon^{k,S}\right)=F'\left(\alpha\Upsilon^{k,S}\right)$$

for all $S\in\mathcal{C}_N$, for all $k\in\Omega_S$ and for all $\alpha\in\mathbb{R}$.

Proof.

Assume that ϖ is uniform on Ω and that $|\Omega_S|=|\Omega_T|$ for all coalitions $S,T\in\mathcal{C}_N\setminus\{N\}$. Consider two values F and F' that satisfy (E), (A), (NP^*) and (SS). As above, $F\left(\alpha\Upsilon^{k,S}\right)=F'\left(\alpha\Upsilon^{k,S}\right)$ for $S=N$ by (SS) and (E). Now, let S be a coalition other than N; and $k,l\in\Omega_S$. Choose an arbitrary event k_T for each possible coalition T such that $|T|=|S|$ and define the CC-game v by

$$v=\sum_{T\neq S:|T|=|S|}\alpha\Upsilon^{k_T,T}.$$

Taking into the account that the probability distribution is uniform, it follows that for both $\alpha\Upsilon^{k,S}+v$ and $\alpha\Upsilon^{l,S}+v$, any pair of players are stochastically symmetric. Therefore by efficiency, $F(\alpha\Upsilon^{k,S}+v)=F'(\alpha\Upsilon^{l,S}+v)$. Thus by additivity, one gets $F(\alpha\Upsilon^{k,S})=F'(\alpha\Upsilon^{l,S})$. ■

Lemma 3.3.3 leads us to the following result:

Theorem 3.3.2 (Njoya et al. (2021))**.**

Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \setminus \{N\}$. Then

The equal-surplus Shapley value Ψ is the unique value on $\mathcal{CC}(N, \Omega, \varpi)$ that satisfies axioms (E), (A), (NP^*) and (SS).

Proof.

Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \setminus \{N\}$. *Necessity.* See Proposition 3.3.1.

Sufficiency. Very similar to the proof of Theorem 3.3.1 using Lemma 3.3.3 instead of Lemma 3.3.2. ∎

As shown in the next proposition, axioms in Theorem 3.3.2 are independent, none of them can dropped.

Proposition 3.3.3. *Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \setminus \{N\}$. Axioms (E), (A), (NP^*) and (SS) are independent on $\mathcal{CC}(N, \Omega, \varpi)$.*

Proof.

Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \setminus \{N\}$. For any four axioms out of (E), (A), (NP^*) and (SS), we show that there exists a value that meets all these three axioms, but which does not satisfy the fourth.

1. $F_i^1(v, k) = \frac{1}{n} v(N, k)$ for all $v \in \mathcal{CC}(N, \Omega, \varpi)$, for all coalitions S, for all $k \in \Omega_N$ and for all $i \in N$.

   (a) It is clear that given a CC-game $v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ: $\sum_{i \in N} F_i^1(v, k) = v(N, k)$. Thus F^1 is (E).

   (b) Given two CC-games $u, v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, we have for any player i: $F_i^1(u + v) = \frac{1}{n}[u + v](N, k) = F_i^1(u) + F_i^1(v)$, thus F^1 is (A).

(c) Now consider two players i and j who are stochastically symmetric in a game $v \in \mathcal{CC}(N, \Omega, \varpi)$, by definition of F^1 we have $F_i^1(v,k) = F_j^1(v,k) = \frac{1}{n} v(N,k)$. Thus F^1 is (SS).

(d) Although for some TU-game $u \in \Gamma^N$ and for any null player i in $u \in \Gamma^N$ and in $\widetilde{u}_\varpi \in \mathcal{CC}(N, \Omega, \varpi)$, we have $F_i^1(v,k) = \frac{1}{n} v(N,k)$. Thus F^1 is not (NP^*).

2. Consider the value F^2 defined by: $F^2(v) = 2\Psi(v)$ for all $v \in \mathcal{CC}(N, \Omega, \varpi)$. Given a player i in N, consider the CC-game $v = \left(\widetilde{\gamma}_{\{i\}}\right)_\Omega \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ. Then all player $N \setminus \{i\}$ are null players in $\left(\widetilde{\gamma}_{\{i\}}\right)_\Omega$ and By definition of Ψ for an event $k \in \Omega_N$, $\sum_{j \in N} F_j^2(v,k) = 2\Psi_i(v,k) = 2v(N,k)$. Thus, F^2 is not (E). Since Ψ is (NP^*), (A) and (SS); it is clear that F^2 is also (NP^*), (A) and (SS).

3. Given $v \in \mathcal{CC}(N, \Omega, \varpi)$, denote by $N^*(v)$ the set of all null players in the expectation game E_v and for all $S \in \mathcal{C}_N$, let $V(S) = \{v(S,l) : l \in \Omega_S\}$ be the set of all possible worths of coalition S. Define the value F^3 for all $v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all $i \in N$ by

$$F_i^3(v,k) = 0 \text{ if } i \in N^*(v); \text{ and } F_i^3(v,k) = \frac{v(N,k)}{|N \backslash N^*(v)|} \text{ otherwise.}$$

(a) It is clear that F^3 satisfies (E) and (NP^*).

(b) Suppose that i and j are two stochastically symmetric players in a CC-game $v \in \mathcal{CC}(N, \Omega, \varpi)$. To see that i and j are symmetric players in the expectation game, consider $S \subseteq N \backslash \{i,j\}$.

$$\begin{aligned}
E_v(S \cup \{i\}) &= \sum_{l \in \Omega_{S \cup \{i\}}} \varpi((S \cup \{i\}, l) v((S \cup \{i\}, l) \\
&= \sum_{x \in V(S \cup \{i\})} x \sum_{l \in \Omega_{S \cup \{i\}} : v(S \cup \{i\}, l) = x} \varpi(S \cup \{i\}, l) \\
&= \sum_{x \in V(S \cup \{j\})} x \sum_{l \in \Omega_{S \cup \{j\}} : v(S \cup \{j\}, l) = x} \varpi(S \cup \{j\}, l) \\
&\quad \text{since } i \text{ and } j \text{ are stochastically symmetric in } v \\
&= \sum_{l \in \Omega_{S \cup \{j\}}} \varpi((S \cup \{j\}, l) v((S \cup \{j\}, l) \\
&= E_v(S \cup \{j\})
\end{aligned}$$

Thus, i and j are symmetric players in E_v. By the definition of F^3, $F_i^3(v,k) = F_j^3(v,k)$ for all $k \in \Omega_N$. Therefore F^3 satisfies (SS).

(c) Consider $\{i,j\} \in \mathcal{C}_N$. Pose $u = \left(\widetilde{\gamma}_{\{i\}}\right)_\Omega$ and $v = \left(\widetilde{\gamma}_{\{i,j\}}\right)_\Omega$. Let $k \in \Omega_S$. We have $N^*(u) = N \setminus \{i\}$ and $N^*(v) = N^*(u+v) = N \setminus \{i,j\}$. By the definition of F^3,

$$F_i^3(u,k) + F_i^3(v,k) = 1 + \frac{1}{2} = \frac{3}{2} \text{ and } F_i^3(u+v,k) = 1.$$

Therefore $F^3(v+u) \neq F^3(u) + F^3(v)$ since $F_i^3(u+v) \neq F_i^3(u) + F_i^3(v,k)$. This proves that F^3 does not satisfy (A).

4. Given two distinct players i and j in N, denote by a the n-tuple defined by $a_i = 1$, $a_j = -1$ and $a_h = 0$ for all $h \in N\backslash\{i,j\}$. Let the value F^4 be defined for all $v \in \mathcal{CC}(N,\Omega,\varpi)$ and for all $k \in \Omega_N$ by

$$F^4(v,k) = \Psi(v,k) + \left[\sum_{l\in\Omega_{\{i,j\}}} \varpi(\{i,j\},l)v(\{i,j\},l) - \sum_{l\in\Omega_{\{i\}}} \varpi(\{i\},l)v(\{i\},l) - \sum_{l\in\Omega_{\{j\}}} \varpi(\{j\},l)v(\{j\},l)\right]$$

(a) Since the terms of $\Psi(v,k)$ sum to $v(N,k)$ and the terms of a sum to zero, it follows that the terms of $F^4(,k)$ sum to $v(N,k)$. Therefore F^4 satisfies (E).

(b) Suppose that u is a TU-game on N and $h \in N$ is a null player in u. If $h \in N\backslash\{i,j\}$, then by the definition of a, $a_h = 0$ and $F_h^4\left(\widetilde{u}_\varpi, k\right) = \Psi_h(\widetilde{u}_\varpi, k) = 0$ since Ψ is (NP^*). Now, without lost generality, suppose that $h = i$. Since Ψ is (NP^*), $\Psi_i(\widetilde{u}_\varpi, k) = 0$. Moreover, $\widetilde{u}_\varpi(S,l) = u(S)$ for all coalitions S and for all $l \in \Omega_S$. Thus, by the definition of F^4, we have:

$$\begin{aligned} F_i^4\left(\widetilde{u}_\varpi,k\right) &= \left[\sum_{l\in\Omega_{\{i,j\}}} \varpi(\{i,j\},l)u(\{i,j\}) - \sum_{l\in\Omega_{\{i\}}} \varpi(\{i\},l)u(\{i\}) - \sum_{l\in\Omega_{\{j\}}} \varpi(\{j\},l)u(\{j\})\right] a \\ &= \left[\underbrace{u(\{i,j\}) - u(\{j\})}_{0} - \underbrace{u(\{i\})}_{0}\right] a_i \text{ since } \varpi \text{ is a probability distribution function} \\ &= 0. \end{aligned}$$

We conclude that F^4 satisfies (NP^*).

(c) F^4 verifies (A) since Ψ verifies (A) and the coefficient of vector a in the definition of F^4 is linear.

(d) Pose $v = \Upsilon^{l,\{i,j\}}$ for some $k \in \Omega_{\{i,j\}}$. Players i and j are stochastically symmetric in v. Moreover,

$$F_i^4\left(v,k\right) = \Psi_i(v,k) + 1 \text{ and } F_j^4\left(v,k\right) = \Psi_j(v,k) - 1.$$

> Since $\Psi_i(v,k) = \Psi_j(v,k)$ by symmetry of Ψ, we deduce that $F^4(_i(v,k) \neq F^4_j(v,k)$. Therefore F^4 does not satisfy (SS).
>
> In summary, the four axioms are independent. ■

In Theorem 3.3.1, (ILR) is a necessary condition for some sample spaces. An illustration is provided in the following example.

Example 3.3.2. Let $N = \{1,2\}$ be a set of two players. Consider the coalitional sample spaces $\Omega_{\{1\}} = \{a,b\}$, $\Omega_{\{2\}} = \{c\}$ and $\Omega_{\{1,2\}} = \{x,y\}$. A probability distribution function ϖ that is uniform on Ω is such that $\varpi(\{1\},a) = \varpi(\{1\},b) = \frac{1}{2}$ and $\varpi(\{2\},c) = 1$. Define the value F on $\mathcal{CC}(N,\Omega,\varpi)$ as follows:

$$F(v) = \begin{pmatrix} \begin{array}{|c|c|} \hline F_1(v,x) & F_2(v,x) \\ \hline F_1(v,y) & F_2(v,y) \\ \hline \end{array} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{array}{|c|c|} \hline v_{1,b} - \frac{1}{2}v_{1,a} - \frac{1}{2}v_{2,c} + \frac{1}{2}v_{12,x} & \frac{1}{2}v_{1,a} - v_{1,b} + \frac{1}{2}v_{2,c} + \frac{1}{2}v_{12,x} \\ \hline \frac{1}{2}v_{1,a} - \frac{1}{2}v_{2,c} + \frac{1}{2}v_{12,y} & \frac{1}{2}v_{2,c} - \frac{1}{2}v_{1,a} + \frac{1}{2}v_{12,y} \\ \hline \end{array} \end{pmatrix}$$

where for simplicity $v_{1,k} = v(\{1\},k)$ for $k \in \{a,b\}$, $v_{2,c} = v(\{2\},c)$ and $v_{12,k} = v(\{1,2\},k)$ for $k \in \{x,y\}$. Note that F satisfies (E), (A), (NP*) and (SS); but not (ILR). This shows that for Theorem 3.3.1 to hold with this specific sample space, one can no more discard condition (ILR).

Characterization over the full class of rational probability distributions

A uniform probability distribution is completely described by the collection of its sample spaces. This is no longer the case for non uniform probability distributions. Also note that conditions (E), (A) and (NP*) are entirely normative considerations on individual shares with respect to coalitional payoffs in a game. Thus, apart from (SS), axioms in Theorem 3.3.1 do not fully capture the full strength of non uniform probability distributions. Condition (ILR) is some type of neutral treatment of elementary events in a sample space. Thus none of the five properties in Theorem 3.3.1 tells about how changes on the collection of probability distributions impact on individual shares. Further properties are needed to characterize the equal-surplus Shapley value on $\mathcal{CC}^r(N)$.

Remark 3.3.2. *Consider a pair $\{a,b\}$ of integers and Ω^0 such that $\Omega^0_S = \{a,b\}$ for all $S \in \mathcal{C}_N$. Let $p \geq 3$ be a prime number. Define the probability distribution function ϖ_p for all coalitions S, by $\varpi_p(S,a) = 2/p$ and $\varpi_p(S,b) = 1 - 2/p$. Now, define the value F^5 for all $v \in \mathcal{CC}(N,\Omega,\varpi) \subseteq \mathcal{CC}^r(N)$ by*

$$F^5(v) = \begin{cases} \Psi(\widehat{v}) & \text{if } \Omega = \Omega^0 \text{ and } \varpi = \varpi_p \\ \Psi(v) & \text{otherwise} \end{cases}$$

where the game $\widehat{v}$ is obtained from v by substituting to ϖ_p the uniform probability distribution function ϖ^0 on Ω^0. It can be checked that both F and Ψ meet (E), (A), (NP^*), (SS) and (ILR). Since $F^5 \neq \Psi$, one needs further requirements to characterize Ψ on $\mathcal{CC}^r(N)$.

Recall that condition (ILD) allows to reshape the sample space as well as the probability distribution function. Over the full class of rational probability distributions, we obtain the following:

Theorem 3.3.3 (Njoya et al. (2021)).

A value F on $\mathcal{CC}^r(N)$ satisfies (E), (A), (NP^*), (SS) and (ILD) if and only if $F = \Psi$.

Proof.

Sufficiency. See Proposition 3.3.1.

Necessity. Suppose that a value F on $\mathcal{CC}^r(N)$ satisfies(E), (A), (NP^*), (SS) and (ILD). Then by Theorem 3.3.2, it follows that $F(v) = \Psi(v)$ whenever v is a game with a uniform probability distribution function on a sample space Ω such that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \setminus \{N\}$. Now, consider an arbitrary game $v \in \mathcal{CC}(N,\Omega,\varpi) \subseteq \mathcal{CC}^r(N)$. Given a coalition $S \neq N$, pose $\varpi(S,k) = \frac{a_{S,k}}{b_{S,k}}$ with $b_{S,k} \geq 1$. Denote by q, the least common multiple of the collection $\{b_{S,k}\colon S \in \mathcal{C}_N \setminus \{N\}$ and $k \in \Omega_S\}$. Then given a coalition $S \neq N$, $q = q_{S,k} b_{S,k}$ for some integer $q_{S,k} \geq 1$. By duplicating $q_{S,k} - 1$ times each event $k \in \Omega_S$ for all coalitions $S \neq N$ and by applying (ILD) as indicated in

Remark 3.2.1, one constructs a new game $v^d \in \mathcal{CC}\left(N, \Omega^d, \varpi^d\right)$ such that ϖ^d is uniform on Ω^d, $F(v) = F(v^d)$ and for all coalitions $S, T \neq N$, $|\Omega^d_S| = |\Omega^d_T| = q$. Since ϖ^d is uniform, then by Theorem 3.3.2, $F(v^d) = \Psi(v^d)$. Hence $F(v) = \Psi(v^d) = \Psi(v)$. ■

Proposition 3.3.4. *The axioms (E), (A), (NP^*), (SS) and (ILD) are independent on $\mathcal{CC}^r(N)$.*

Proof.
Each of the four values presented in Proposition 3.3.3 satisfies (ILD) but fails to satisfy exactly one axiom among (E), (A), (NP^*) and (SS). Therefore, we only have to prove that (ILD) can not be deduced from the other four axioms in consideration on $\mathcal{CC}^r(N)$. To see this, we prove that the value introduced in Remark 3.3.2 is (E), (A), (NP^*) and (SS), but not (ILD).

1. The value F^5 satisfies (E) since Ψ verifies (E).

2. To prove that F^5 satisfies (NP^*), suppose that u is a TU-game on N and $i \in N$ is a null player in u. We have to prove that for all $k \in \Omega_N$, $F^5_i(v, k) = 0$ where $v = \widetilde{u}_\varpi$ for an arbitrary probability distribution function ϖ on a collection Ω of sample spaces. First suppose that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$; that is $\Omega = \Omega^0$ and $\varpi = \varpi_p$. Then $\widehat{v} = \widetilde{u}_{\varpi^0}$ and $F^5_i(v, k) = \Psi_i(\widetilde{u}_{\varpi^0}, k) = 0$ since Ψ satisfies (NP^*). Now, suppose that $v \notin \mathcal{CC}(N, \Omega^0, \varpi_p)$. Then $F^5_i(v, k) = \Psi_i(\widetilde{u}_\varpi, k) = 0$ since Ψ satisfies (NP^*). Thus, we conclude that F^5 satisfies (NP^*).

3. By noting that $\widehat{u+v} = \widehat{u} + \widehat{v}$ for all $u, v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$, it follows that F^5 verifies (A) since Ψ verifies (A).

4. Suppose that i and j are two stochastically symmetric players in a CC-game $v \in \mathcal{CC}(N, \Omega, \varpi)$. Let $k \in \Omega_N$. First suppose that $v \notin \mathcal{CC}(N, \Omega^0, \varpi_p)$. By the definition of F^5, $F^5(v) = \Psi(v)$. Since Ψ verifies (SS), it follows that $F^5_i(v, k) = F^5_j(v, k)$ for all $k \in \Omega_N$. Now, suppose that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$. By the definition of Ω^0, i and j are such that $v(S \cup \{i\}, l) = v(S \cup \{j\}, l)$ for all $S \subseteq N \backslash \{i, j\}$ and for all $l \in \Omega_{S\cup\{i\}} = \{a, b\} = \Omega_{S\cup\{j\}}$. From v to $\widehat{v}$, only the probability distribution function changes. Therefore, $\widehat{v}(S \cup \{i\}, l) = \widehat{v}(S \cup \{j\}, l)$ for all $S \subseteq N \backslash \{i, j\}$ and for all $l = \{a, b\}$. This proves that i and j are stochastically symmetric in $\widehat{v}$ and

that $F^5(v) = \Psi(\widehat{v})$. Since Ψ verifies (SS), it follows that $F_i^5(v,k) = F_j^5(v,k)$ for all $k \in \Omega_N$. This prove that F^5 satisfies (SS).

5. Consider $i \in N$. Pose $S = N\backslash\{i\}$, $v = \Upsilon^{a,S} \in \mathcal{CC}(N, \Omega^0, \varpi_p)$ and $u = v^{S,b,b'}$ where u is obtained from v by only duplicating, in Ω_S, b into b and b'. Note that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$, $\widehat{v} \in \mathcal{CC}(N, \Omega^0, \varpi^0)$ and $u \notin \mathcal{CC}(N, \Omega^0, \varpi_p)$. Also note that,
$$E_u = E_v = \frac{2}{p}\gamma_S + \left(1 - \frac{2}{p}\right)\gamma_N \text{ and } E_{\widehat{v}} = \frac{1}{p}\gamma_S + \left(1 - \frac{1}{p}\right)\gamma_N. \tag{3.13}$$
Therefore,
$$F_i^5(v,a) = \Psi_i(\widehat{v},a) = Shap_i\left(\frac{1}{p}\gamma_S + \left(1 - \frac{1}{p}\right)\gamma_N, a\right) = \frac{1}{n} - \frac{1}{np}$$
and
$$F_i^5(u,a) = \Psi_i(u,a) = Shap_i\left(\frac{2}{p}\gamma_S + \left(1 - \frac{2}{p}\right)\gamma_N, a\right) = \frac{1}{n} - \frac{2}{np}$$
It follows that $F_i^5(u,a) \neq F_i^5(v,a)$. Since u is obtained from v by a duplication of b in Ω_S, we conclude that F^5 does not satisfy (ILD).

The proof is thus completed. ■

The arguments in the proof of Theorem 3.3.3 are strongly related to the fact that the probability of each coalitional event is a rational number. The main step of the proof consists in moving from any game with rational probability distributions to a game with a uniform probability distribution.

Characterization in the whole set of all CC-games

In this section, the whole set $\mathcal{CC}(N)$ of all CC-games on N is considered (there is no restriction on the probability distribution function). New axioms are introduced to capture how a value behaves when some specific changes on the payoffs or on the probability distribution occur. But before, we extend the scope of Theorem 3.3.3 from $\mathcal{CC}^r(N)$ to $\mathcal{CC}(N)$. To this end, the following lemma is introduced.

Lemma 3.3.4. *If a value F satisfies (E), (A), (NP^*), (SS) and (ILD) on $\mathcal{CC}(N)$, then*
$$F\left(\alpha\Upsilon^{k,S}\right) = \Psi\left(\alpha\Upsilon^{k,S}\right)$$

for all collections $\Omega = (\Omega_S)_{S\in\mathcal{C}_N}$ of sample spaces, for all probability distribution functions ϖ on Ω, for all $S \in \mathcal{C}_N$, for all $k \in \Omega_S$ and for all $\alpha \in \mathbb{R}$.

Proof.

Consider a value F that satisfies (E), (A), (NP^*), (SS) and (ILD) on $\mathcal{CC}(N)$; a collection Ω of sample spaces, a probability distribution function ϖ on Ω, a coalition $S \in \mathcal{C}_N$, an event $k \in \Omega_S$ and a real number $\alpha \in \mathbb{R}$. First note that by Theorem 3.3.3, the result holds when ϖ is rational-valued. Now, suppose that the probability distribution function ϖ is not rational-valued and cannot be reduced to a rational-valued function by merging only some duplicate events. We prove that, even in this case, $F\left(\alpha\Upsilon^{k,S}\right) = \Psi\left(\alpha\Upsilon^{k,S}\right)$. By the definition of $\Upsilon^{k,S}$, for all coalitions $T \neq S$, all events in Ω_T lead to the same coalitional payment of 1 when $S \subsetneqq T$, and 0 when $S \nsubseteq T$. Thus by (ILD), $F\left(\alpha\Upsilon^{k,S}\right) = F(u_0)$ where the game u_0 is obtained from $\Upsilon^{k,S}$ by merging all duplicated events in each sample space Ω_T, with $T \notin \{S, N\}$, into a single event e_T. By so doing, we move from (Ω, ϖ) to (Ω_1, ϖ_1) such that for $T \in \{S, N\}$, $(\Omega_1)_T = \Omega_T$, $\varpi_1(T,k) = \varpi(T,k)$ for all $k \in \Omega_T$; and for all $T \neq S$, $(\Omega_1)_T = \{e_T\}$ and $\varpi_1(T, e_T) = 1$. Since ϖ cannot be reduced to a rational-valued probability distribution, Ω_S necessarily contains at least two events.

First suppose that $\Omega_S = \{k, l\}$ for some events k and l with the associated probability distribution $(p_1; p_2)$. To continue, we introduce three new games. To this end, we consider an arbitrary rational number q such that $p_1 - \frac{1}{2} < q < \min\{p_1, \frac{1}{2}\}$. Such a rational number q necessarily exists since $p_1 - \frac{1}{2} < p_1$ and $p_1 - \frac{1}{2} < \frac{1}{2}$. The new games are:

- $u_1 \in \mathcal{CC}(N, \Omega_1, \varpi_1)$ is defined such that
$$u_1(S,k) = 0,\ u_1(S,l) = 1,\ \varpi'(S,k) = \varpi'(S,l) = \frac{1}{2}$$
and
$$u_1(T,t) = 1 \text{ for all } T \neq S \text{ and } t \in (\Omega_1)_T.$$
- $u_2 \in \mathcal{CC}(N, \Omega_2, \varpi_2)$ is obtained from u_0 by only duplicating k into k and k', and l into l and l' in such a way that $(\Omega_2)_S = \{k_1, k_2, l_1, l_2\}$ and $(\Omega_2)_T = (\Omega_1)_T$ for all $T \neq S$, with
$$\varpi_2(S,k) = q,\ \varpi_2(S,k') = p_1 - q;\ \varpi_2(S,l') = \frac{1}{2} - q;\ \varpi_2(S,l) = q + \frac{1}{2} - p_1 = p_2 - \left(\frac{1}{2} - q\right).$$

- $u_3 \in \mathcal{CC}(N, \Omega_2, \varpi_3)$ is obtained from u_1 by only duplicating k into k and l', and l into l and k' in such a way that

  $$\varpi_3(S,k) = q,\ \varpi_3(S,l') = \frac{1}{2} - q;\ \varpi_3(S,k') = p_1 - q;\ \varpi_3(S,l) = q + \frac{1}{2} - p_1 = \frac{1}{2} - (p_1 - q)$$

Note that u_2 and u_3 are defined on the same sample space Ω_2. Moreover, from ϖ_2 to ϖ_3, only the probabilities of k' and l' are permuted. Let π be the transposition of Ω_2 that interchanges k' and l'; that is $\pi = (k', l')$. Then $\pi\varpi_3 = \varpi_2$. We deduce that $\pi u_3 \in \mathcal{CC}(N, \Omega_2, \varpi_2)$. Furthermore,

$$(u_2 + \pi u_3)(S,t) = 1 \text{ for all } t \in (\Omega_2)_S\,.$$

In words, all events in $(\Omega_2)_S$ lead to the same coalitional payment of 1. Therefore, by (ILD), k, k', l and l' can be merged from $u_2 + \pi u_3$ into a single event e_S that occurs with probability 1. This leads us to the game $u_4 = u_2 + \pi u_3 \in \mathcal{CC}(N, (\Omega_3, \varpi_4)$ such that

$$(\Omega_2)_T = \{e_T\} \text{ and } \varpi_4(T, e_T) = 1$$

for all coalitions $T \neq N$. Thus, u_1 and u_4 belongs $\mathcal{CC}^r(N)$. Therefore

$$\begin{aligned}
F\left(\alpha\Upsilon^{k,S}\right) &= F\left(u_0\right) \text{ by } (ILD) \\
&= F\left(u_2\right) \text{ by } (ILD) \\
&= F\left(u_2 + \pi u_3\right) - F(\pi u_3) \text{ by additivity of } F \\
&= \Psi\left(u_2 + \pi u_3\right) - F(\pi u_3) \text{ by Theorem 3.3.3 since } u_4 = u_2 + \pi u_3 \in \mathcal{CC}^r(N) \\
&= \Psi(u_2) + \Psi\left(\pi u_3\right) - F(\pi u_3) \text{ by additivity of } \Psi \\
&= \Psi(u_2) + \Psi\left(u_3\right) - F(u_3) \text{ by } (ILD) \text{ and Proposition 3.2.1} \\
&= \Psi(u_0) + \Psi\left(u_1\right) - F(u_1) \text{ by } (ILD) \\
&= \Psi(u_0) \text{ by Theorem 3.3.3 since } u_1 \in \mathcal{CC}^r(N) \\
&= \Psi\left(\alpha\Upsilon^{k,S}\right) \text{ by } (ILD)
\end{aligned}$$

Now, suppose that Ω_S contains more than two events. By definition, $\Upsilon^{k,S}(S,k) = 1$ and $\Upsilon^{k,S}(S,t) = 0$ for all $t \in \Omega_S \backslash \{k\}$. Thus by merging all events $t \in \Omega_S \backslash \{k\}$ into a single event l, one returns to the previous case by applying (ILD).

■

Thanks to the previous lemma, we have the following result :

Theorem 3.3.4 (Njoya et al. (2021)).

A value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), (SS) and (ILD) if and only if $F = \Psi$.

Proof.

Necessity. See Proposition 3.3.1.

Sufficiency. For a given collection Ω of sample spaces and a probability distribution function on ϖ on Ω, $\left(\Upsilon^{k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is, by Proposition 3.1.1, a basis of $\mathcal{CC}(N,\Omega,\varpi)$. Thus, the result follows from Lemma 3.3.4 and additivity. ■

Proposition 3.3.5. *Axioms (E), (A), (NP^*), (SS) and (ILD) are independent on $\mathcal{CC}(N)$.*

Proof.

Since $\mathcal{CC}^r(N) \subseteq \mathcal{CC}(N)$, each of the five values invoked in the proof of Proposition 3.3.4 also permit to prove that none of the five axioms in consideration on $\mathcal{CC}(N)$ can not be deduced from the four others. ■

In Theorem 3.3.4 , axioms (E), (A) and (NP^*) are related to how, independently of the probability distribution, the information on coalitional payoffs in a game impacts on the shares of players. Axioms (SS) and (ILD) describe some specific patterns or changes that may be observed either on the coalitional payoffs; or on the probability distribution function separately. Some more mixture changes that combine changes on the probability distribution and the coalitional payoffs are explored below. The first change consists in merging two events as follows: given $v \in \mathcal{CC}(N)$, a coalition $S \neq N$ and $\{k,l\} \subseteq \Omega_S$, the game $\left(\Omega^{S,k\sim l}, v^{S,k\sim l}, \varpi^{S,k\sim l}\right) \in \mathcal{CC}(N)$ is defined by

(i) $\Omega_T^{S,k\sim l} = \Omega_T \setminus \{l\}$ if $T = S$ and $\Omega_T^{S,k\sim l} = \Omega_T$ if $T \neq S$;

(ii) $\varpi^{S,k\sim l}(S,k) = \varpi(S,k) + \varpi(S,l)$ and

$$v^{S,k\sim l}(S,k) = \frac{\varpi(S,k)}{\varpi(S,k)+\varpi(S,l)}v(S,k) + \frac{\varpi(S,l)}{\varpi(S,k)+\varpi(S,l)}v(S,l);$$

(iii) $\varpi^{S,k\sim l}(T,j)=\varpi(T,j)$ and $v^{S,k\sim l}(S,j)=v(S,j)$ if $T\neq S$ or $(T=S$ and $j\in\Omega_S\setminus\{k,l\})$.

To obtain the game $v^{S,k\sim l}$, event l is deleted, the probability of event k is updated to the sum of the probability of events k and l while the corresponding payoff is obtained by aggregating the payoffs associated with k and l to their weighted mean. Hereafter, we say that moving from v to $v^{S,k\sim l}$ is an merging and cancelling operation (MC-merging).

Definition 3.3.1. *An allocation rule (a value) F on $\mathcal{F}\subseteq\mathcal{CC}(N)$ satisfies* **Merge-and-Cancel Invariance (MCI)** *if for all $v\in\mathcal{CC}(N)$, for all coalitions $S\neq N$ and for all $k,l\in\Omega_S$, $F\left(v^{S,k\sim l}\right)=F(v)$.*

In contrast with (ILD), mergeability invariance captures changes that combine probabilities with payoffs of coalitions. Although, (MCI) seems to be almost a mathematical disposition, it turns out in the following result that (MCI) is equivalent to (ILD) when one assumes (E), (A), (NP^*) and (SS).

Considering (MCI), we have the following:

Theorem 3.3.5 (Njoya et al. (2021))**.**

A value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), (SS) and (MCI) if and only if $F=\Psi$.

Proof.

Sufficiency Due to Proposition 3.3.1, one only needs to prove that Ψ satisfies (MCI). By merging two events $k,l\in\Omega_S$ into k in the CC-game v, it can be checked that the expectation game in the new game $v^{S,k\sim l}$ coincides with that of v. Therefore, $F\left(v^{S,k\sim l}\right)=F(v)$ by Equation (3.12).

Necessity. Suppose that a value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), (SS) and (MCI). Given a coalition $S\neq N$, all events in Ω_S can be merged into a single event, say k_S, by a successive use of the merging operation. By MCI, this operation leaves unchanged all individual shares with respect to F and Ψ from v to the new game. By iterating this procedure for all coalitions $S\neq N$, one moves from v to the game

$$\Upsilon=\left(\widetilde{E_v}\right)_{\varpi'}+\sum_{k\in\Omega_N}\left(v(N,k)-E_v(N)\right)\Upsilon^{k,N}$$

such that for all $k \in \Omega_N$, the payoff and the probability of event k are the same in Υ as in v; for all coalitions $S \neq N$, k_S is the unique event for S in Υ while the payoff of the members of S is the expectation $E_v(S)$ of S in v. By (MCI), $F(v) = F(\Upsilon)$ and $\Psi(\Upsilon) = \Psi(v)$. Moreover, the probability distribution in Υ is uniform on Ω'. Thus by Theorem 3.3.2, $F(\Upsilon) = \Psi(\Upsilon)$. Therefore $F(v) = \Psi(v)$. ■

Proposition 3.3.6. *Axioms (E), (A), (NP^*), (SS) and (MCI) are independent on $\mathcal{CC}\,(N)$.*

Proof.

Each of the four values presented in Proposition 3.3.3 fails to satisfy exactly one axiom among (E), (A), (NP^*) and (SS). Each of those four values obviously satisfies (MCI). Now, we have proved that the value F^5 in the proof of Proposition 3.3.4 satisfies (E), (A), (NP^*) and (SS). To prove that F^5 fails to meet (MCI), consider $i \in N$. Pose $S = N\backslash\{i\}$, $v = \Upsilon^{a,S} \in \mathcal{CC}(N, \Omega^0, \varpi_p)$ and $u = v^{S,a\sim b}$ where u is obtained from v by merging, in Ω_S, a and b into a. Note that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$, $\widehat{v} \in \mathcal{CC}(N, \Omega^0, \varpi^0)$ and $u \notin \mathcal{CC}(N, \Omega^0, \varpi_p)$. Since the expectation game does not change by applying an MC-merging operation, (3.13) still holds. Therefore,

$$F_i^5(v,a) = \Psi_i(\widehat{v},a) = Shap_i\left(\frac{1}{p}\gamma_S + \left(1 - \frac{1}{p}\right)\gamma_N, a\right) = \frac{1}{n} - \frac{1}{np}$$

and

$$F_i^5(u,a) = \Psi_i(u,a) = Shap_i\left(\frac{2}{p}\gamma_S + \left(1 - \frac{2}{p}\right)\gamma_N, a\right) = \frac{1}{n} - \frac{2}{np}$$

It follows that $F_i^5(u,a) \neq F_i^5(v,a)$. Since u is obtained from v by an MC-merging operation, we conclude that F^5 does not satisfy (MCI). The proof is thus completed. ■

By an MC-merging operation, the size of a sample space is reduced by canceling some events. An alternative would consist in maintaining all events. To achieve such an operation, associate to a game $v \in \mathcal{CC}(N, \Omega, \varpi)$ the game $v^{S,k\wedge l} \in \mathcal{CC}\,(N, \Omega, \varpi)$ defined by $v^{S,k\wedge l}(T,j) = v(T,j)$ if $T \neq S$ or ($T = S$ and $j \in \Omega_S \setminus \{k,l\}$); together with

$$v^{S,k\wedge l}(S,k) = v^{S,k\wedge l}(S,l) = \frac{\varpi(S,k)}{\varpi(S,k) + \varpi(S,l)}v(S,k) + \frac{\varpi(S,l)}{\varpi(S,k) + \varpi(S,l)}v(S,l).$$

For the game $v^{S,k\wedge l}(S,k)$, one merges the payoffs of events k and l to the their weighted mean; but keeps all events with their respective probabilities. However, for the game $v^{k\sim l}$, one merges the payoffs of the two events and delete l from the sample space of S.

Definition 3.3.2. *An allocation rule (a value) F on $\mathcal{F} \subseteq \mathcal{CC}(N)$ satisfies* **Merge-and-Keep Invariance (MKI)** *if for all $v \in \mathcal{CC}(N)$, for all coalitions $S \neq N$ and for all $k, l \in \Omega_S$, $F\left(v^{S,\wedge l}\right) = F(v)$.*

Both (MKI) and (MCI) are based on the same intuition that merging some events in a consistent way should not affect individual shares in a game. Hereafter, we say that moving from v to $v^{S,k,l}$ is an MK-merging operation.

To prove the next result, we consider the basis $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ of $\mathcal{CC}(N,\Omega,\varpi)$ where the collection c is such that for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$,

$$c_{k,S} = \frac{1}{\varpi(S,k)}. \tag{3.14}$$

It can be checked that all games $\Upsilon^{c,k,S}$ for $k \in \Omega_S$ yield the same expectation of one unit to coalition S. Moreover, the following result holds:

Lemma 3.3.5. *Let $c = (c_{k,S})_{S\in\mathcal{C}_N, k\in\Omega_S}$ be a collection defined by (3.14).*

Then, all values F that satisfy (MKI) on $\mathcal{CC}(N,\Omega,\varpi)$ are such that

$$F\left(\alpha\Upsilon^{c,k,S}\right) = F\left(\alpha\Upsilon^{c,l,S}\right) \tag{3.15}$$

for all coalitions S, for all events $k, l \in \Omega_S$ and for all real numbers α.

Proof.

Suppose that $c = (c_{k,S})_{S\in\mathcal{C}_N, k\in\Omega_S}$ is defined by (3.14). Consider a value F on $\mathcal{CC}(N,\Omega,\varpi)$ that satisfies (MKI), a coalition S, two events $k, l \in \Omega_S$ and a real number α. Pose $u = \alpha\Upsilon^{k,S}$ and $v = \alpha\Upsilon^{l,S}$. By the definition of $\alpha\Upsilon^{k,S}$ and $\alpha\Upsilon^{l,S}$, $u(T,t) = v(T,t)$ for all coalitions T and for all $t \in \Omega_T$ such that $T \neq S$ or ($T = S$ and $t \in \Omega_S\backslash\{k,l\}$). Therefore, $u^{S,k\wedge l}(T,t) = v^{S,k\wedge l}(T,t)$ for all coalitions T and for all $t \in \Omega_T$ such that $T \neq S$ or ($T = S$ and $t \in \Omega_S\backslash\{k,l\}$). Furthermore, for $t \in \{k,l\}$,

$$u^{S,k\wedge l}(S,t) = v^{S,k\wedge l}(S,t) = \frac{1}{\varpi(S,k) + \varpi(S,l)}.$$

This proves that $u^{S,k\wedge l} = v^{S,k\wedge l}$. It appears from (MKI) that,

$$F(\alpha\Upsilon^{k,S}) = F(u^{S,k\wedge l}) = F(v^{S,k\wedge l}) = F(\alpha\Upsilon^{l,S}).$$

■

As with (MCI), axiom (MKI) leads us to the following result:

Theorem 3.3.6 (Njoya et al. (2021)).

A value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), (SS) and (MKI) if and only if $F = \Psi$.

Proof.

Sufficiency. Due to Proposition 3.3.1, we only needs to prove that Ψ satisfies (MKI). Consider $v \in \mathcal{CC}(N)$. By the definition of an (MKI)-merging operation, the expectation game of $v^{S,k\wedge l}$ coincides with that of v. Therefore, by Equation (3.12), $\Psi\left(v^{S,k\wedge l}\right) = \Psi(v)$.

Necessity. Suppose that a value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), (SS) and (MKI). Consider $v \in \mathcal{CC}(N, \Omega, \varpi)$. Since F satisfies (A) and the collection $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ with c defined by (3.14) is a basis of $\mathcal{CC}(N, \Omega, \varpi)$, to prove that $F(v) = \Psi(v)$, we only have to prove that $F\left(\alpha\Upsilon^{c,k,S}\right) = \Psi\left(\alpha\Upsilon^{c,k,S}\right)$ for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$.

For this purpose, consider a coalition S and $k \in \Omega_S$. On the one hand, we have

$$F\left(\sum_{l\in\Omega_S} \varpi(S,l)\alpha\Upsilon^{c,l,S}\right) = F\left(\alpha\left(\widetilde{\gamma}_S\right)_{\varpi}\right) \text{ by Proposition 3.1.3}$$

On the other hand,

$$\begin{aligned} F\left(\sum_{l\in\Omega_S} \varpi(S,l)\alpha\Upsilon^{c,l,S}\right) &= \sum_{l\in\Omega_S} F\left(\varpi(S,l)\alpha\Upsilon^{c,l,S}\right) \text{ by additivity} \\ &= \sum_{l\in\Omega_S} F\left(\varpi(S,l)\alpha\Upsilon^{c,k,S}\right) \text{ by Lemma 3.3.5} \\ &= F\left(\sum_{l\in\Omega_S} \varpi(S,l)\alpha\Upsilon^{c,k,S}\right) \text{ by additivity} \\ &= F\left(\alpha\Upsilon^{k,S}\right) \text{ since } \sum_{l\in\Omega_S} \varpi(S,l) = 1 \end{aligned}$$

This proves that $F\left(\alpha\Upsilon^{k,S}\right) = F\left(\alpha\left(\widetilde{\gamma}_S\right)_{\Omega}\right)$. Since Ψ also satisfies (E), (A), (NP^*), (SS) and (MKI), we deduce that $\Psi\left(\alpha\Upsilon^{k,S}\right) = \Psi\left(\alpha\left(\widetilde{\gamma}_S\right)_{\Omega}\right)$. In the CC-game $\alpha\left(\widetilde{\gamma}_S\right)_{\Omega}$, all players in S are stochastically symmetric while all players out of S are null players in the game. The result follows by applying (E), (NP^*) and (SS). ∎

PROPOSITION 3.3.7. *Axioms* (E), (A), (NP^*), (SS) *and* (MKI) *are independent on* $\mathcal{CC}(N)$.

Proof. Each of the four values presented in Proposition 3.3.3 fails to satisfy exactly one axiom among (E), (A), (NP^*) and (SS). Each of those four values obviously satisfies (MKI). Now, we have proved that the value F^5 in the proof of Proposition 3.3.4 satisfies (E), (A), (NP^*) and (SS). To prove that F^5 fails to meet (MKI), consider $i \in N$. Pose $S = N\backslash\{i\}$, $v = \Upsilon^{a,S} \in \mathcal{CC}(N, \Omega^0, \varpi_p)$ and $u = v^{S,a\wedge b}$ where u is obtained from v by an MK-merging operation, in Ω_S. Note that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$, $\widehat{v} \in \mathcal{CC}(N, \Omega^0, \varpi^0)$ and $u \in \mathcal{CC}(N, \Omega^0, \varpi_p)$. Since the expectation game does not change by applying an MK-merging operation, (3.13) still holds. Therefore,

$$F_i^5(v,a) = \Psi_i(\widehat{v},a) = Shap_i\left(\frac{1}{p}\gamma_S + \left(1-\frac{1}{p}\right)\gamma_N, a\right) = \frac{1}{n} - \frac{1}{np}$$

and

$$F_i^5(u,a) = \Psi_i(u,a) = Shap_i\left(\frac{2}{p}\gamma_S + \left(1-\frac{2}{p}\right)\gamma_N, a\right) = \frac{1}{n} - \frac{2}{np}$$

It follows that $F_i^5(u,a) \neq F_i^5(v,a)$. Since u is obtained from v by an MK-merging operation, we conclude that F^5 does not satisfy (MCI). The proof is thus completed. ∎

REMARK 3.3.3. *It appears from Theorems 3.3.5 and 3.3.6 that* (MCI) *and* (MKI) *are equivalent axioms for values that satisfy* (E), (A), (NP^*) *and* (SS). *The noticeable advantage of Theorem 3.3.6 is that it is still valid for a given sample space* Ω *while Theorem 3.3.5 stands for a framework with a variable sample space. Furthermore, the intuition behind both* (MCI) *or* (MKI) *seems quite to be the preservation of the expectations of proper coalitions in a game. Each of the two axioms is equivalent to* (ILD) *for values that satisfy* (E), (A), (NP^*) *and* (SS). *Hence* (ILD) *tells a more perceivable story behind* (MCI) *or* (MKI).

⋆⋆ Conclusion ⋆⋆

At the end of this thesis, it is worth noticing that our aim was the study of some non classical approaches to cooperative games. In this sense, we have explored three new classes of cooperative games namely MTU-games, LUF-games and CC-games (with sample spaces). For each of these three classes, our main concerns consist in analyzing how the worth of a cooperation can be shared amount partners.

With MTU-games presented in Chapter 1, the novelty is that the payoff of a coalition depends on the choice of its members between two or more available alternatives. We have defined two core concepts and provided necessary and sufficient conditions for the non-emptiness of each of the two cores (see Theorem 1.2.1 and Theorem 1.2.2). These results can be viewed as two extensions of the Shapley-Bondareva theorem from TU-games to MTU-games. To achieve this, we prove in Proposition 1.2.5 that the core of an MTU-game with respect to a given coalition structure is empty whenever the coalition structure is not poly-efficient; and coincides, for all poly-efficient coalition structures, with the classical core of the corresponding max-game.

In chapter 2, we introduce the class of LUF-games by weakening the utility transferability assumption in TU-games. Now, each outcome of cooperation is a raw material the value depends on the player and the coalition in consideration. Two core concepts for LUF-games are defined and the characterization of sharing vectors in each of the two cores are presented. Necessary and sufficient non-emptiness conditions are stated and proved in Theorem 2.2.3 for linear LUF-games. In the general case, we provide, in Theorem 2.2.1 and Theorem 2.2.2, the characterizations of core sharing vectors in LUF-games. The notions of lower compensation shares and upper compensation shares of players, two concepts we introduce, were very useful tools in proving each of the characterizations provided.

In chapter 3, we enrich the model of CC-games by embedding sample spaces to coalitional random payoffs. This allows us to define a value for CC-games called equal-surplus Shapley value. By so doing, we encompass the absence of single-valued solutions for CC-games in the literature. The newly introduced value has been characterized in several ways depending on the class of admissible probability distributions; see Theorems 3.3.1 to 3.3.6. These axiomatizations scrutinize the equal-surplus Shapley value on its ability to fulfill some desirable properties of value solutions defined on CC-games. For example, when only uniform probability distributions are observable, the equal-surplus Shapley value is the unique value on CC-games that simultaneously satisfies efficiency, additivity, null player property, stochastic symmetry and independence of local relabeling.

Still in our framework, some issues are left open. For example, it is of interesting to find necessary and sufficient conditions for the non-emptiness of the cores of LUF-games. Possible directions to explore include the study of possible links between LUF-games and the general model of non transferable utility cooperative games. It is worth noticing that an interpretation of a LUF-game as a non transferable utility cooperative game will surely emerge to a loss of information about individual utility functions. Another pending issue is the extension and the characterization of the equal-surplus Shapley value for CC-games with possibly infinite sample spaces. An immediate difficulty in doing that is the fact that some coalitional random payoffs might lead to infinite means (that is when the means of some coalitional payoffs do not exist as real numbers on which algebraic operations might be defined).

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⋆⋆ Published Articles ⋆⋆

- Njoya Nganmegni, D., Moyouwou, I. & Andjiga, Nicolas, G. The equal-surplus Shapley value for chance-constrained games on finite sample spaces. Mathematical Methods of Operations Research (2021). https://doi.org/10.1007/s00186-021-00738-w

Mathematical Methods of Operations Research (2021) 93:463–499
https://doi.org/10.1007/s00186-021-00738-w

ORIGINAL ARTICLE

The equal-surplus Shapley value for chance-constrained games on finite sample spaces

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Received: 17 January 2020 / Revised: 8 January 2021 / Accepted: 11 January 2021 /
Published online: 4 February 2021

Abstract

Many interactions from linear production problems, financial markets, or sequencing problems are modeled by cooperative games where payoffs to a coalition of players is a random variable. For this class of cooperative games, we introduce a two-stage value as an ex-ante agreement among players. Players are first promised their prior Shapley shares which are exactly their respective shares by the Shapley value of the expectation game. The final payoff vector is obtained by equally re-allocating the surplus when a realization of the random payoff of the grand coalition is observed. In support of the tractability of the newly introduced value called equal-surplus Shapley value, we provide a simple and compact formula. Depending on which probability distributions over the sample spaces are admissible, we present several characterization results of the equal-surplus Shapley value. This is achieved by using some classical axioms together with some other appealing axioms such as the independence of local duplication which simply requires that individual shares in a game remain unchanged when only certain events are duplicated in the sample space of a coalition without altering the probability of observing the others.

Keywords Game theory · Random coalitional payoffs · Equal surplus · Finite sample spaces

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1 Introduction

Cooperative games with transferable utility (TU-games) model interactions in which agents form coalitions and gain some payments. The payoff of a coalition is classically deterministic. However, in many situations, payoffs are not known in advance but are random variables. A variety of stochastic environments for cooperative decision-making is nicely reported in Suijs (2012) and Suijs et al. (1998) who explore applications in linear production tasks, queuing models, financial markets as well as insurance deals. In this strand of the literature, Charnes and Granot (1973; 1976) were the first to consider cooperative games when payoffs of coalitions are random variables; the so-called chance-constrained games *(c.c. games)*. Suijs et al. (1999), in order to define stochastic cooperative games, enrich this model by embedding a set of possible actions for each coalition and a profile of individual preferences on payoff vectors. We are essentially interested in this paper on Charnes–Granot model mainly for two reasons:

On the one hand, Charnes and Granot (1977) advocate that, in c.c. games, the *payoff process* to the players should be composed of two parts: a prior payoff and an adjustment disposition. Such a payoff process can be viewed as a two-stage contract which works as follows: when a given coalition is formed, its members are first promised, taking into the account the expectation of each of its sub-coalitions, their respective prior payoffs; after a realization of the random payoff of the coalition is observed, the second part of the contract is applied to reallocate the surplus to the members of the coalition. This approach for profit sharing takes into account all possible realizations of the profit. Although Timmer et al. (2004) find this approach to be “time-consuming, inefficient and perhaps even impossible”, we think these comments are misleading since the alternative approach that consists in assigning to each agent a share of the total profit can also be viewed as a two-stage payoff with null prior payments. Moreover, the value we propose in this paper has a two-stage shape, but it still has some desirable properties and a very simple interpretation.

On the other hand, it can be checked that for c.c. games, only set-valued solutions have been defined. More precisely, Charnes and Granot (1977) consider a two-stage core and a two-stage nucleolus for this class of games; further, they present some conditions, on the distribution functions associated with the random payoffs of coalitions, that describe some situations in which the two-stage nucleolus contains a unique payoff vector. No single-valued rule, a value, that assigns a c.c. game with a single payoff vector is not yet defined together with some of its axiomatizations. Our aim is to fill this gap by providing a value for c.c. games together with a simple and compact formula as well as some characterization results.

To define core concepts which lead to set-valued solutions, preferences are used in Suijs et al. (1999, 1998) or Charnes and Granot (1977). Timmer (2006) and Timmer et al. (2004) provide, for cooperative games with random payoffs, the only solutions that are single-valued. In both contributions, individual preferences play a central role in designing the allocation rules. Individual preferences are omitted here. In doing so, we opt for the normative approach in which we only care about individual contributions to the collective worth in order to set up some justice and equity norms like efficiency, symmetry, null player property, etc. Moreover, we consider chance-constrained games each equipped with a collection of finite sample spaces from which coalitional payoffs are derived. A similar approach was considered for cooperative games with uncertainty by Habis and Herings (2011) who introduce a single sample space, and then define all coalitional worths as functions on this sample space. In this latter framework, coalitional worths, which may be correlated or not, are simultaneously generated as a state of nature is observed. This is the case, for example, of the worths of coalitions of agents on the same market with distinct volumes of the same good, the price of which depends on each state of nature. However, chance-constrained games that we consider deal with situations in which only the worths of coalitions formed by players are observed at each state of nature. However, all the parameters of the game are common knowledge (the sample spaces, the coalitional function and all the probabilities are known in advance). Illustrations of this class of games are provided later in the paper.

Note that the expectation game of a given c.c. game associates each coalition with the expectation of its random payoff. The prior Shapley value of a c.c. game (see Charnes and Granot (1973),) is simply the Shapley value of its expectation game. Assuming that the grand coalition is formed, one obtains a two-stage payoff by equally re-allocating the surplus when a realization of the collective payoff is observed. We refer to the corresponding rule as the *equal-surplus Shapley value*. We explore some interesting features of the equal-surplus Shapley value of c.c. games from three distinct perspectives as mentioned above. Firstly, from a computational point of view, the equal-surplus Shapley value has a simple formula that linearly depends on the inputs that define the game. Secondly, we show that the equal-surplus Shapley value complies with the Shapley procedure with the only adjustment that the marginal contribution of a player to a given coalition is not known in advance. Finally, axiomatizations are provided to exhibit what are the normative dispositions that completely describe the equal-surplus Shapley value.

The idea of the equal-surplus shares is also known in the case of TU-games; see for example van den Brink (2007) or Béal et al. (2016). Béal et al. (2019) introduce the efficient egalitarian Shapley value of a TU-game as the sum of the Shapley value of the game and the equal shares of the surplus generated for the grand coalition as one moves from the game to its superadditive cover. This allocation rule has the same shape as the equal-surplus Shapley value for c.c. games in this paper. Another similarity with the TU-game setting is that some of the axioms we use are extensions of known axioms from TU-games to c.c. games. This is the case for *efficiency, null player property or additivity*. However, some other axioms are purely designed for c.c. games. This is for example the case with *Independence of Local Relabeling* (ILR) or *Independence of Local Duplication* (ILD). Axiom (ILR) requires that any relabeling of events in the sample space associated with a proper coalition should have no effect on individual shares. In the same way, (ILD) requires that no change on individual shares occurs when an event in a sample space is split into two new events which preserve the probability and the payments: each of the two new events yields the same payment as the initial event and the sum of the probabilities of the two new events is equal to the probability of the initial event. All these new axioms are illustrated through examples.

Depending on which probability distributions are admissible, we provide distinct axiomatizations of the equal-surplus Shapley value for c.c. games. We first consider the uniform probability distribution on sample spaces. We then move to the class of probability distributions that are rational-valued. In the general case, no restriction on the probability distribution on sample spaces is imposed. To make our presentation easier, two assumptions are made. It is assumed that the grand coalition is formed; and that all sample spaces are finite; that is, the random payoff of each coalition consists of a finite number of elementary events.

The rest of the paper is organized into three main sections. Section 2 is devoted to some basic definitions and differences between TU-games, c.c. games and cooperative games with stochastic payoffs. We essentially present some illustrations of c.c. games with sample spaces on which we structure our analysis. Still in this section, the notion of value for c.c. games is defined and the equal-surplus Shapley value is introduced. In Sect. 3, an interpretation of the equal-surplus Shapley value is provided. Some algebraic properties on c.c. games are then presented to make the proofs of some subsequent results in the paper more understandable. In this section, we mainly present some axiomatizations of the equal-surplus Shapley value depending on which probability distributions are considered; detailed proofs of the independence between axioms we use in each result are relegated to the appendices. Section 4 concludes the paper.

2 Basic definitions and some preliminary results

Denote by $N = \{1, 2, \ldots, n\}$ a nonempty finite set of $n \geq 2$ players; and by 2^N the set of all subsets of N. A non empty subset of N is called coalition and the set of all coalitions of N is denoted by $\mathcal{C}_N$. Given a finite set X, the cardinality of X is denoted by $|X|$, a permutation of X is a one-to-one function from X onto itself. A transposition of X is any permutation π of X such that for some $\{k, l\} \subseteq X$, $\pi(k) = l$, $\pi(l) = k$ and $\pi(t) = t$ for all $t \in X\backslash\{k, l\}$; in this case, π is denoted by $\pi = (k, l)$.

2.1 TU-games, c.c. games and cooperative games with stochastic payoffs

A cooperative game with transferable utilities on N, or simply a TU-game, is a mapping $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$; $v(S)$ is the payoff that the members of S can jointly achieve. The set of all TU-games on N is denoted by Γ^N. Given a game $v \in \Gamma^N$, a player i is a *null player* if for all $S \subseteq N\backslash\{i\}$, $v(S \cup \{i\}) - v(S) = 0$; two players i and j are *symmetric* if for all $S \subseteq N\backslash\{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$. The sum of $u, v \in \Gamma^N$ is the TU-game $u + v \in \Gamma^N$ defined for all $S \subseteq N$ by $(u + v)(S) = u(S) + v(S)$.

It is assumed that the grand coalition N is formed. An *allocation rule*, or *a value*, on Γ^N is a mapping f assigning each TU-game v with a payoff vector $f(v) = (f_i(v))_{i \in N}$; $f_i(v)$ is the share of player i in the game v with respect to f. A value f satisfies *Symmetry (Sym)* if for all $v \in \Gamma^N$, $f_i(v) = f_j(v)$ whenever i and j are symmetric players in v; *Null Player (NP)* property if for all $v \in \Gamma^N$, $f_i(v) = 0$ whenever i is a null player in v; *Efficiency (E)* if for all $v \in \Gamma^N$, $\sum_{i \in N} f_i(v) = v(N)$; and *Additivity (A)* if for all $u, v \in \Gamma^N$, $f(u + v) = f(u) + f(v)$. It is known from

Shapley (1953) that the unique value on Γ^N that simultaneously meets (NP), (Sym), (E) and (A) is the *Shapley value* defined for all $v \in \Gamma^N$ by

$$\forall i \in N, Shap_i(v) = \sum_{S \ni i} \frac{(|S|-1)!\,(n-|S|)!}{n!}\left(v(S) - v(S\backslash\{i\})\right).$$

When some coalitions have several actions and random payoffs; one obtains cooperative game with stochastic payoffs which are each, a collection $(N, (A_S)_{S\in\mathcal{C}_N}, (X_S)_{S\in\mathcal{C}_N}, (\succeq_i)_{i\in N})$ where for each $a \in A_S$, $X_S(a)$ is a finite expectation random variable each realization of which is a coalitional payoff of S when its members jointly choose action a; and $(\succeq_i)_{i\in N}$ is the collection of individual preferences over random payoff vectors. Preferences of players are needed to analyze the desirability or the stability of random payoff vectors; see Suijs et al. (1999) for more details. These attributes can be omitted when one is mainly interested in value theory from a point of view of a social planner. However, individual preferences are inescapable when one cares about the stability of a payoff vector as it is the case with core concepts; see Timmer et al. (2005) and Suijs et al. (1998) for cooperative games with stochastic payoffs; or Habis and Herings (2011) for cooperative games with uncertainty (players are involved in a TU-game depending on some given states of nature). Moreover, when each strategy set A_S is a singleton and strategies are omitted, the corresponding game, which is simply identified to $(N, (X_S)_{S\in\mathcal{C}_N})$, is hereafter called a *chance-constrained game (c.c. game)* following Charnes and Granot (1973). It is worth noticing that values for cooperative games with random payoffs that take into account individual preferences on random payoffs are presented by Timmer (2006) and Timmer et al. (2004).

Note that a random variable is derived from a random experiment, that is a process with many uncertain issues. The set of all possible issues or all possible random events for the experiment forms the *sample space*. Chance-constrained games and games with stochastic payoffs are usually defined without any information on the sample spaces. In this paper, we embed sample spaces from which coalitional payoffs are derived. Before we introduce chance-constrained games with sample spaces, let us give a short view of cooperative games with uncertainty (TUU-games) by Habis and Herings (2011). A TUU-game can be summarized as a five-ingredient collection (N, S, v, T, u) in which N is a finite set of players, S is a finite set of the states of nature, $v = (v_s)_{s\in S}$ is a collection of TU-games such that v_s is the TU-game associated with the state of nature s, T is a finite set of periods, and $u = (u_i)_{i\in N}$ is a collection of individual utility functions. Over each period in T, a state of nature s is (randomly) observed from S and all players play the game (N, v_s).In TUU-games, the worths of all coalitions are known for each state of nature. In the following illustrative examples, the worths of coalitions are independently drawn from distinct sample spaces.

Example 1 Consider two business units BU1 and BU2 who may purchase each a basic printer. Each business unit may experience, over a given period, a mechanic breakdown (M), an electronic breakdown (E), or none of them (Z). Event M occurs with probability 0.1 for BU1 and 0.05 for BU2. Event E occurs with probability 0.02 for BU1 and 0.1 for $BU2$. In the case of a joint professional printer, M and E are observed with the same probability of 0.05. The assistance charges for BU1 are 5 for

M and 10 for E; BU2 pays 8 for M and 4 for E; and for the joint printer, M and E cost 10 and 12 respectively. It is assumed that the printers operate independently. The question is, in case of a joint printer, what would each business unit pay for M? for E? Here the sample space for each coalition S is $\Omega_S = \{M, E, Z\}$ where Z stands for no breakdown (observed with no charge).

Remark 1 Note that in Example 1, there are only two coalition structures, namely P_1 when BU1 and BU2 opt for individual printers; and P_2 when they purchase a joint printer. For P_1 the states of nature are (M, M), (M, E), (M, Z), (E, M), (E, E), (E, Z), (Z, M), (Z, E) and (Z, Z) where, for example, (M, E) means that BU1 and BU2 experience M and E respectively. For each of these nine states of nature, note that nothing is known about what would have happened for a joint printer since P_1 and P_2 are conjointly not possible here. Similarly, for P_2, the states of nature are M, E and Z. Now, for each of these three states of nature associated with P_2, nothing is said about what would have happened for individual printers.

Example 2 During a festival, three types of tombola are organized for single tickets, two-person tickets and group tickets respectively. Buying any ticket gives rights to a Wheel of Fortune trial depending on the nature of the ticket. The Wheel of Fortune for a one-person ticket may return a golden (G) band with probability 0.02 for a win of 10 euros; or a red (R) band for a zero win. For a two-person ticket, one may win 25 euros for a yellow (Y) band with probability 0.04; 250 euros for a golden (G) band with probability 0.02; or nothing for a red (R) band. A group ticket for at least $k \geq 3$ visitors is offered an initial discount of k euros and may further win $20k$ euros with probability 0.02 for a yellow (Y) band; $200k$ euros with probability 0.001 for a golden (G) band; and nothing for a red (R) band on the Wheel of Fortune. John (player 1) has to purchase the entrance rights for three persons (himself, his wife and his young daughter); Penny (player 2) would like to pay access for two persons (herself and Jenny); and Andrew (player 3) intends to attain the festival. They may separately purchase their tickets or form a coalition for a group ticket. The question is, how would they share a joint win from the Wheel of Fortune in case of a group ticket (for six visitors)? Note that the sample space for each coalition S is $\Omega_S = \{Y, G, R\}$ if $1 \in S$ or $2 \in S$; and $\Omega_S = \{Y, R\}$ if $S = \{3\}$. Note that the sample space depends on the coalition considered.

Definition 1 Given a finite set N of $n \geq 2$ players, a chance-constrained game on N (with sample spaces) is a triple (Ω, v, ϖ) such that $\Omega = (\Omega_S)_{S \in \mathcal{C}_N}$ is a collection of coalitional sample spaces; the mapping v and ϖ respectively give, for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$, the coalitional payoff $v(S, k)$ of the members of S and the probability $\varpi(S, k) > 0$ of observing event k; with $\sum_{k \in \Omega_S} \varpi(S, k) = 1$ for all coalitions S.

The set of all chance-constrained games on N is denoted by $\mathcal{CC}(N)$ and $\mathcal{CC}(N, \Omega, \varpi)$ is the subset of $\mathcal{CC}(N)$ that consists of all c.c. games on N with the same probability distribution function ϖ on Ω. Provided that ϖ and Ω are known, the game (Ω, v, ϖ) will be identified with its coalitional payoff function v. Note that given a coalition S, the mapping $v_S : k \in \Omega_S \longmapsto v(S, k)$ is a random variable with probability distribution $\varpi_S : k \in \Omega_S \longmapsto \varpi(S, k)$. In other words, $\big(N, (v_S)_{S \in \mathcal{C}_N}\big)$ is a c.c. game in Charnes and Granot sense. The little change in our setting is that we have embedded the collection of sample spaces from which the random payoff of each coalition is derived.

Example 3 In Example 1, the corresponding game can be formalized as a c.c. game $v \in \mathcal{CC}(N, \Omega, \varpi)$ with $N = \{1, 2\}$ and $\Omega_S = \{M, E, Z\}$ for all $S \in \mathcal{C}_N$. Here, player 1 stands for BU1 and player 2 for BU2. The coalitional payoff function v and the probability distribution function ϖ are summarized below with respect to each of the three possible coalitions:

$S = \{1\}$			$S = \{2\}$			$S = \{1, 2\}$		
k	$v(S, k)$	$\varpi(S, k)$	k	$v(S, k)$	$\varpi(S, k)$	k	$v(S, k)$	$\varpi(S, k)$
M	5	0.1	M	8	0.05	M	10	0.05
E	10	0.02	E	4	0.1	E	12	0.05
Z	0	0.88	Z	0	0.85	Z	0	0.9

Given a TU-game u on N, a collection $\Omega = (\Omega_S)_{S \in \mathcal{C}_N}$ of coalitional sample spaces and a probability distribution function ϖ on Ω, the game $\widetilde{u}_\varpi$ is the c.c. game defined for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$ by $\widetilde{u}_\varpi(S, k) = u(S)$. In the c.c. game $\widetilde{u}_\varpi$, the coalitional payoff of a coalition S is, for all events $k \in \Omega_S$, identical to the coalitional payoff of S in the TU-game u. Although several events can be observed when S is formed, each of them exactly produces $u(S)$. The game $\widetilde{u}_\varpi$ can then be seen as the randomized version of u in $\mathcal{CC}(N, \Omega, \varpi)$.

Given a probability distribution function ϖ on Ω, $\mathcal{CC}(N, \Omega, \varpi)$ is equipped with two algebraic operations as follows: for all $u, v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all real numbers λ, the games $u + v \in \mathcal{CC}(N, \Omega, \varpi)$ and $\lambda v \in \mathcal{CC}(N, \Omega, \varpi)$ are defined for all $S \in \mathcal{C}_N$ and for all $k \in \Omega$ by $(u + v)(S, k) = u(S, k) + v(S, k)$ and $(\lambda v)(S, k) = \lambda v(S, k)$. The games $u + v$ and λv are defined for a given probability distribution function ϖ by re-evaluating the worth of coalitions. For example, in the game of Example 1, if u models the charges for possible breakdowns and v the transport cost, then $u + v$ gives the total cost for each possible breakdown and λv corresponds to the game obtained when the transport cost is updated by a constant rate λ. It is worth noticing that a distinct sum of u and v may consist in summing the random variables u_S and v_S for all $S \in \mathcal{C}_N$. This is the case if, for example, u is the cost function for a given period and v the cost function for another period. This alternative summation of random variables is used by Cheng-Guo et al. (2014) and Ma et al. (2008).

2.2 Values for c.c. games and the equal-surplus Shapley value

A c.c. game is played as follows: players have to form coalitions. When a coalition S is formed, a realization from Ω_S of a random event is observed and the members of S obtain the corresponding payoff. Furthermore, when a coalition structure $\mathcal{P} = \{S_1, S_2, \ldots, S_p\}$ is formed, outcomes of coalitions in $\mathcal{P}$ have independent realizations.

If $\mathcal{P}$ is disrupted, new coalitions are formed and again new events with possibly new payoff realizations are randomly observed.

Our concern is as follows. The grand coalition N is formed; but an *a priori sharing rule* F is to be designed such that when an event $k \in \Omega_N$ is observed, the coalitional payoff $v(N, k)$ is shared accordingly. In a two stage process, an allocation usually has the shape (d, r) such that given $i \in N$, player i who is first promised d_i finally gets $d_i + f_i(P)$ with $f_i(P) = r_i\left(P - \sum_{i\in N} d_i\right)$ when P is the actual worth of the grand coalition; see Suijs et al. (1999). We are interested in such an operational solution. But before, we have the following general definition:

Definition 2 An allocation rule (a value) on $\mathcal{F} \subseteq \mathcal{CC}(N)$ is a mapping F that associates each c.c. game $G = (\Omega, v, \varpi) \in \mathcal{F}$ with a list $F(G) = (F(G, k))_{k\in\Omega_N}$ of payoff vectors such that $F_i(G, k)$ is the player i's share of the collective worth $v(N, k)$.

Given a game $G = (\Omega, v, \varpi)$, a value F assigns to G a collection of deterministic payoff vectors $F(G, k)$ for $k \in \Omega_N$. This can be viewed as a *post ante* contract that states the share of each player for each specific event $k \in \Omega_N$ that may be encountered as the grand coalition is formed. But observing an event in Ω_N is a stochastic event. Thus, the pair $\left((F(G, k))_{k\in\Omega_N}, (\varpi(N, k))_{k\in\Omega_N}\right)$ can also be seen, as an n-tuple of random variables $F_i(G) = (F_i(G, k), \varpi(N, k))_{k\in\Omega_N}$, $i \in N$, that sum to the random variable $v_N = (v(N, k), \varpi(N, k))_{k\in\Omega_N}$. Roughly, before an event $k \in \Omega_N$ is observed for the grand coalition, the share of player i is the random variable $F_i(G)$; and provided that $k \in \Omega_N$ is observed, the share of player i is equal to $F_i(G, k)$ for sure. This approach consists in making "plans for profit sharing based on all possible realizations of the profit" as observed by Timmer et al. (2004) in the case of cooperative games with random payoffs. The desirability of a value depends on how appealing are the shares it generates. Here are some properties of a value given a family $\mathcal{F}$ of c.c. games.

Efficiency (E): If $v \in \mathcal{F} \cap \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, then for all $k \in \Omega_N$,

$$\sum_{i\in N} F_i(v, k) = v(N, k).$$

Additivity (A). If $u, v \in \mathcal{F} \cap \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, then

$$F(u + v) = F(u) + F(v).$$

Null Player Property (NP*). If a player $i \in N$ is a null player in a TU-game $u \in \Gamma^N$ and ϖ is a probability distribution function on a collection Ω of sample spaces such that $\widetilde{u}_\varpi \in \mathcal{F}$, then $F_i(\widetilde{u}_\varpi, k) = 0$ for all $k \in \Omega_N$.

Efficiency requires that when an event $k \in \Omega_N$ is observed for the grand coalition, individual deterministic payments are summed to the collective worth $v(N, k)$. Additivity states that when an event $k \in \Omega_N$ occurred for the grand coalition, the share of a player in the game $u + v$ should be the sum of his/her shares in the games u and v. Axiom (NP*) simply requires that any null player in a TU-game u should receive a zero share in the game $\widetilde{u}_{\varpi}$ when u is seen as a c.c. game. Thus (NP^{*}) simply says that, any null-player in a TU-game u has a zero share in the randomized version of u.

In a c.c. game $v \in \mathcal{CC}(N, \Omega, \varpi)$, two players i and j are *stochastically symmetric* if interchanging i and j does not affect the chance of any coalition to realize each of its feasible worth; that is for all $S \subseteq N\backslash\{i, j\}$ and for all $k \in \Omega_{S\cup\{i\}}$, there exists some $k' \in \Omega_{S\cup\{j\}}$ such that $v(S \cup \{i\}, k) = v(S \cup \{j\}, k')$ and

$$\sum_{l\in\Omega_{S\cup\{i\},k}} \varpi(S \cup \{i\}, l) = \sum_{l\in\Omega_{S\cup\{j\},k'}} \varpi(S \cup \{j\}, l)$$

where in the game $v \in \mathcal{CC}(N, \Omega, \varpi)$, $\Omega_{T,t} = \{l \in \Omega_T : v(T, l) = v(T, t)\}$ is the set of all elementary events l in Ω_T such that both l and t produce the same payoff for T; any event $l \in \Omega_{T,t}$ is called *a duplication* of t in Ω_T.

Stochastic Symmetry (SSym). If two players i and j are stochastically symmetric in a c.c. game $v \in \mathcal{F}$, then $F_i(v, k) = F_j(v, k)$ for all $k \in \Omega_N$.

Following (SSym), two stochastically symmetric players in a c.c. game always receive equal shares. Now, consider a coalition S and a permutation π of Ω_S; and update, in the game v, only the payoff of the members of S and the probability that the members of S observed each event $k \in \Omega_S$. The corresponding game denoted by $(\pi v_S, \pi\varpi_S)$ or simply by πv_S, is formally defined for all $T \in \mathcal{C}_N\backslash\{S\}$ and for all $k \in \Omega_T$ by $\pi v_S(T, k) = v(T, k)$ and $\pi\varpi_S(T, k) = \varpi(T, k)$; together with $\pi v_S(S, \pi(k)) = v(S, k)$ and $\pi\varpi_S(S, \pi(k)) = \varpi(S, k)$. For an illustration, consider Example 3; let $S = \{1\}$ and π be the transposition of the mechanic breakdown (M) and the electronic breakdown (E). Then the game πv_S is obtained from the representation of v in Example 3 by simply interchanging M and E in the first column of the first table. Equivalently, this simply amounts to interchanging for the business unit BU1 the costs and the probabilities of observing M and E. Globally, the risk incurred by BU1 remains unchanged; only a relabeling of the possible events occurred. One would expect any conceivable value to return in the new game the same shares as in the initial game.

Independence of Local Relabeling (ILR). For all coalitions $S \neq N$ and for all permutations π of Ω_S, $F(\pi v) = F(v)$ whenever $v, \pi v \in \mathcal{F}$.

Condition (ILR) requires that any *local relabeling* of events in a sample space associated with a proper coalition of players should have no effect on individual shares.

Another change we consider is the duplication of an event. Assume, for an illustration, that in Example 2, the organizer modifies the Wheel of Fortune for one-person tickets by only splitting the golden band into a new golden band which provides a win of 10 euros with probability p_g and a yellow (Y) band which also provides 10 euros with probability p_y in such a way that $p_g + p_y = 0.02$. Then a one-person ticket still wins 10 euros with probability 0.02 although such a win now comes from two distinct events. Such a fake change should normally have no effect on individual shares in a game. More formally, a game $v \in \mathcal{CC}(N, \Omega, \varpi)$ is obtained from a game $v' \in \mathcal{CC}(N, \Omega', \varpi')$ by a *local duplication* if there exists a coalition $S \neq N$ and $k, k' \in \Omega_S$ such that *(i)* $k' \in \Omega_S\backslash\Omega'_S$, $\Omega_S = \Omega'_S \cup \{k'\}$ and $\Omega_T = \Omega'_T$ for all $T \neq S$;

(ii) $v(S,k) = v(S,k') = v'(S,k)$ and $\varpi(S,k') + \varpi(S,k) = \varpi'(S,k)$; and *(iii)* $v(T,l) = v'(T,l)$ and $\varpi(T,l) = \varpi'(T,l)$ whenever ($l \notin \{k,k'\}$ or $T \neq S$.) In this case, we also say that v' is obtained from v by *canceling the duplicated* event k' of k. This is denoted by $v' = v^{S,k,k'}$.

Independence of Local Duplication (ILD). If $v(S,k) = v(S,k')$ for some coalition $S \neq N$ and for some events $k, k' \in \Omega_S$, then $F\left(v^{S,k,k'}\right) = F(v)$ whenever $v, v^{S,k,k'} \in \mathcal{F}$.

Condition (ILD) is the requirement that in a c.c. game, any local duplication has no change on individual shares.

Remark 2 One can iterate (ILD), if feasible, to cancel any subset of events that are duplications of a given event $k \in \Omega_S$ wile preserving all individual shares. To see this, consider a nonempty subset $K = \{k_1, k_2, \ldots, k_t\}$ of $\Omega_{S,k}\backslash\{k\}$ and denote by $v^{S,k,K}$ the game obtained from v by successively canceling $k_1, k_2, \ldots, k_t$. Condition (ILD) is equivalent to say that $F(v^{S,k,K}) = F(v)$. In words, by canceling any finite number of duplications of a given event k in a c.c. game and by updating the probability of observing k to the sum of the probabilities of observing k or some of its duplications, the shares of players are not affected.

Remark 3 By duplicating some event for a given game in $\mathcal{F}$, it is not guaranteed that the new game is still in $\mathcal{F}$. Similarly, by canceling a duplication of an event, the new game is not necessarily in $\mathcal{F}$. Some family may be rich enough to allow these two operations. In this case, $\mathcal{F}$ will be simply called rich. This is formally stated in the next definition.

Definition 3 A family $\mathcal{F}$ of c.c. games is rich if $\mathcal{F}$ meets the following two conditions:

- **($\mathbf{c_1}$)** for all $v \in \mathcal{F}$ such that $v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, for all coalitions $S \neq N$, for all events $k \in \Omega_S$ and for all events $k' \notin \Omega_S$, there exists $u \in \mathcal{F}$ such that $v = u^{S,k,k'}$;
- **($\mathbf{c_2}$)** for all $v \in \mathcal{F}$ such that $v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, for all coalitions $S \neq N$ and for all events $k, k' \in \Omega_S$, if k' is a duplication of k in v then $v^{S,k,k'} \in \mathcal{F}$.

Condition ($\mathbf{c_1}$) means that for each game v in $\mathcal{F}$ and for each event k in a sample space Ω_S for $S \neq N$, there exists a game in $\mathcal{F}$ obtained from v by a duplication of k that brings into the game v a new event $k' \notin \Omega_S$. Condition ($\mathbf{c_2}$) is the requirement that by merging an event and some of its duplications from a game in $\mathcal{F}$, the new game should stay in $\mathcal{F}$. For example, the set of all c.c. games on N is rich. Furthermore, one also obtains a rich family of c.c. games by considering the set of all c.c. games such that all probabilities in all sample spaces Ω_S, for $S \neq N$, are rational numbers. The following result holds on rich families of c.c. games.

Proposition 1 *(ILD) implies (ILR) on any rich family $\mathcal{F}$ of c.c. games.*

Proof Consider a rich family $\mathcal{F}$ of c.c. games and a value F on $\mathcal{F}$ that meets (ILD). Since any permutation of a finite set is a finite product of transpositions, to prove that

F necessarily meets (ILR), it is sufficient to prove that for all $v \in \mathcal{F}$ such that $v \in \mathcal{CC}(N, \Omega, \varpi)$ for some Ω and ϖ, for all coalitions $S \neq N$ and for all transpositions π of Ω_S, $F(\pi v) = F(v)$. Suppose that π is a transposition of Ω_S. That is $\pi = (k, l)$ for some $\{k, l\} \subseteq \Omega_S$. Consider two events k' and l' such that $k', l' \notin \Omega_S$. Since $\mathcal{F}$ is rich and $k \in \Omega_S$, then there exists $u_1 \in \mathcal{F}$ such that $v = u_1^{S,k,k'}$ (u_1 is obtained from v by a duplication of event k). By (ILD),

$$F(v) = F(u_1). \tag{1}$$

Similarly, $l \in \Omega_S \cup \{k'\}$ and $l' \notin \Omega_S \cup \{k'\}$. Since $\mathcal{F}$ is rich, there exists a game u_2 in $\mathcal{F}$ such that $u_1 = u_2^{S,l,l'}$. By (ILD),

$$F(u_1) = F(u_2). \tag{2}$$

Note that in the game u_2, k' and l' are duplications of k and l respectively. Since $\mathcal{F}$ is rich, the game u_3 such that $u_3 = u_2^{S,k',k}$ (obtained from u_2 by merging k' and k into k') belongs to $\mathcal{F}$. By (ILD),

$$F(u_2) = F(u_3). \tag{3}$$

Similarly, the game u_4 such that $u_4 = u_3^{S,l',l}$ belongs to $\mathcal{F}$. By (ILD),

$$F(u_4) = F(u_3). \tag{4}$$

By construction, the game u_4 is obtained from v by only renaming k to k' and l to l'. It appears from Eqs. (1)–(4) that $F(u_4) = F(v)$. In words, renaming k to k' and l to l' with $k', l' \notin \Omega_S$ does not affect individual shares in the game v. Note that $u_4 \in \mathcal{CC}(N, \Omega', \varpi')$ where $\Omega'_S = (\Omega_S \backslash \{k, l\}) \cup \{k', l'\}$, $\varpi'(S, k') = \varpi(S, k)$, $\varpi'(S, l') = \varpi(S, l)$, $u_4(S, k') = v(S, k)$, $u_4(S, l') = v(S, l)$; and for all coalitions $T \neq S$, $\Omega'_T = \Omega_T$ and $\varpi'(T, t) = \varpi(T, t)$ for all $t \in \Omega'_T$. Since $k, l \notin \Omega'_S$, one obtains from u_4 a new game u_5 by only renaming k' to l and l' to k without altering individual shares. That is $F(u_5) = F(u_4)$. Hence $F(u_5) = F(v)$. This completes the proof since $u_5 = \pi v$. □

Is there any value that meets all the six above mentioned properties? The answer is yes as shown in the sequel. To any c.c. game $v \in \mathcal{CC}(N, \Omega, \varpi)$, we associate its expectation game E_v that assigns to each coalition S, its expected worth $E_v(S)$ defined by:

$$E_v(S) := \sum_{k \in \Omega_S} \varpi(S, k) v(S, k).$$

The game E_v is a TU-game and its Shapley value is called the *prior Shapley value* of the c.c. game v. We define the two-stage value Ψ for c.c. games as follows. Players in N are first promised their prior Shapley shares of the game v and then proceed to observe a random event from Ω_N. When $k \in \Omega_N$ is observed at the second stage, the actual worth of the grand coalition is $v(N, k)$ and the surplus to be re-allocated is

$v(N, k) - E_v(N)$. This surplus is equally split among players in such a way that the final share of a each player $i \in N$ is

$$\Psi_i(v, k) = Shap_i(E_v) + \frac{1}{n}(v(N, k) - E_v(N)). \tag{5}$$

The value Ψ will be called the *equal-surplus Shapley value* for c.c. games. Interestingly, the value Ψ meets all the six axioms above as shown in the following proposition.

Proposition 2 *The equal-surplus Shapley value* Ψ *satisfies* (E), (A), (NP^*), $(SSym)$, (ILR) *and* (ILD).

Proof Note that the operator that associates each c.c. game v with the TU-game E_v is linear. Moreover, the Shapley value is efficient and additive. It then follows from (5) that Ψ is efficient and additive. Also note that if two players are stochastically symmetric in the game v, they are symmetric in the TU-game E_v. Recalling that the Shapley value is symmetric, it follows from (5) that Ψ satisfies $(SSym)$. Given a TU-game u, note that the expectation game of $\widetilde{u}_{\varpi}$ coincides with u and that by definition, $\widetilde{u}_{\varpi}(N, k) = u(N) = E_{\widetilde{u}_{\varpi}}(N)$. Again, if follows from (5) that Ψ satisfies (NP^*) . By the definition of the game πv_S given a coalition $S \neq N$ and a permutation π of Ω_S, note that $E_{\pi v_S}(T) = E_v(T)$ for all coalitions T. Since $\pi v_S(N, k) = v(N, k)$ for all $k \in \Omega_N$, it follows from (5) that Ψ satisfies (ILR). In the same way, a local duplication does not affect the expectation game; nor the random coalitional worth of the grand coalition. Thus by (5), a local duplication does not affect individual shares by Ψ; and Ψ then meets (ILD). □

Remark 4 Given a player $i \in N$, it immediately follows from (5) that, the payoff of player i in a c.c. game is a random variable the mean of which is the Shapley payoff of player i in the expectation game and its standard deviation is $\frac{1}{n}$ the standard deviation of the random payoff of the grand coalition. Therefore, the smaller the standard deviation of the payoff of the grand coalition, the closer is the worth of player i to his/her prior Shapley share.

In what follows, the value Ψ is further scrutinized based on two characteristics. A procedure we provide tells about the intuition behind Ψ. We also get axiomatization results that exhibit certain properties that uniquely identify the equal-surplus Shapley value among many other solutions for c.c. games.

3 Characterizations of the equal-surplus Shapley value

We provide here some characterization results of the value Ψ depending on the shape of the sample space Ω or on whether all probability distribution mappings ϖ are admissible or not.

3.1 An interpretation of the equal-surplus Shapley value

In this section, we show that the equal-surplus Shapley value of a c.c. game associates each player with the average of all his/her possible marginal contributions under a procedure. Such a procedure which tells about the intuition on how the payoff vector $\Psi(v, k)$ is derived can be obtained as in Shapley (1953). Once the grand coalition is formed and an event $k \in \Omega_N$ is observed, the question is how to share $v(N, k)$ among the n players. The attributes in favor of a player, say i, are measured only by all possible marginal contributions that may be observed when i joins a coalition $S \subseteq N \backslash \{i\}$ as in the following threefold procedure:

(A_1) Once an event $k \in \Omega_N$ is observed, players row up in a line to join the coalition one at a time; N_s denotes the coalition of the first s players to get in.
(A_2) While the s^{th} player, say e_s, is joining the coalition, a trial from Ω_{N_s} is made for $s < n$ to observe a random event e^s with probability $\varpi(N_s, e^s)$. Player e_s is promised his/her marginal contribution $v(N_s, e^s) - v(N_{s-1}, e^{s-1})$ with the convention $N_0 = \emptyset$, $k_0 = 0$, $v(\emptyset, k_0) = 0$ and $e^n = k$, k is the final issue.
(A_3) All the $n!$ orderings of the players are equally probable; for each ordering of players, the $n - 1$ first trials have independent realizations; and the event $e^n = k \in \Omega_N$ observed for the grand coalition remains unchanged.
$\Phi_i(v, k)$ is the expectation of player i in the procedure $(A_1) - (A_3)$.

Here, an entry-trial scenario is a pair $e = \left(e_1 e_2 ... e_n, e^1 e^2 ... e^n\right)$ such that player e_s gets in at the s^{th} position and e^s is the event that is observed as the coalition $\{e_1, e_2, \ldots, e_s\}$ of the first s players is formed. A k-entry-trial scenario is any entry-trial scenario e such that $e^n = k$. The set of all k-entry-trial scenarios will be denoted by $\mathcal{E}_k$. The probability of observing $e \in \mathcal{E}_k$ is $\varpi^e = \varpi(N_1, e^1) \times \varpi(N_2, e^2) \times \cdots \times \varpi(N_n, e^n)$ with $N_s = \{e_1, e_2, \ldots, e_s\}$.

Example 4 Consider the game v in Example 3. When the event M for the joint printer is observed, all scenarii in sharing $v(N, M)$ are as follows:

Scenarii	1's contribution	ϖ	2's contribution	ϖ
$(21, MM)$	2	0.05	8	0.05
$(21, EM)$	6	0.1	4	0.1
$(21, ZM)$	10	0.85	0	0.85
$(12, MM)$	5	0.1	5	0.1
$(12, EM)$	10	0.02	0	0.02
$(12, ZM)$	0	0.88	10	0.88
Expectations	$\Phi_1(v, M) = \frac{9.9}{2} = 4.95$		$\Phi_2(v, M) = \frac{10.1}{2} = 5.05$	

For illustration, note that when the entry-trial is $(21, MM)$, player 1's marginal contribution is $v(\{1, 2\}, M) - v(\{2\}, M) = 2$ while player 2's marginal contribution is $v(\{2\}, M) - 0 = 8$. Similarly, when the entry-trial is $(12, ZM)$, player 1's marginal contribution is $v(\{1\}, Z) - 0 = 0$ while player 2's marginal contribution is $v(\{1, 2\}, M) - v(\{1\}, Z) = 10$.

The reader can check that the expectation game E_v is given by $E_v(\{1\}) = 0.7$, $E_v(\{2\}) = 0.8$ and $E_v(\{1,2\}) = 1.1$. The prior shares is then $Shap_1(E_v) = 0.5$ for player 1 and $Shap_2(E_v) = 0.6$ for player 2. When a mechanic breakdown (M) for the joint printer is observed, the surplus cost is $v(\{1,2\}, M) - 0.7 - 0.6 = 8.9$. Each of the two players gets the same extra cost of 4.45. Finally $\Psi_1(v, M) = 0.5 + 4.45 = 4.95$ and $\Psi_2(v, M) = 0.6 + 4.45 = 5.05$. Thus $\Psi(v, M)$ coincides with the expected payoffs from procedure $(A_1) - (A_3)$. Similarly, it can be also checked that $\Phi(v, k) = \Psi(v, k)$ for $k \in \{E, Z\}$. Thus $\Phi(v) = \Psi(v)$ for the game of Example 3.

The coincidence $\Phi(v) = \Psi(v)$ observed in Example 4 turns out to be true for an arbitrary c.c. game.

Proposition 3 *For all $v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all $k \in \Omega_N$, $\Phi(v, k) = \Psi(v, k)$.*

Proof Given a k-entry-trial scenario e, denote by $e[i] \in \{1, 2, \ldots, n\}$ the position of player i. By definition,

$$\Phi_i(v, k) = \sum_{r=1}^{n} \sum_{e \in \mathcal{E}_k : e[i] = r} \frac{\varpi^e}{n!} [v\left(N_r, e^r\right) - v\left(N_{r-1}, e^{r-1}\right)].$$

This sum can be split into three distinct sums as $\Phi_i(v, k) = K_1 + K_2 + K_3$ where the first sum K_1 is the weighted sum of marginal contributions of player i for all entry-trial scenarios where he/she gets in first.

$$\begin{aligned}
K_1 &= \sum_{e \in \mathcal{E}_k : e_1 = i} \frac{\varpi^e}{n!} v\left(N_1, e^1\right) \\
&= \sum_{e \in \mathcal{E}_k : e_1 = i} \frac{\varpi^e}{n!} v\left(\{i\}, e^1\right) \\
&= \frac{(n-1)!}{n!} \sum_{a \in \Omega_{\{i\}}} \varpi(\{i\}, a)\, v(\{i\}, a) \\
&= \frac{(n-1)!}{n!} E_v(\{i\}) = \frac{0!\,(n-1)!}{n!} \left(E_v(\{i\}) - E_v(\emptyset)\right) \text{ since } E_v(\emptyset) = 0
\end{aligned}$$

The sum K_2 is the weighted sum of marginal contributions of player i for all k-entry-trial scenarios where i gets in last.

$$\begin{aligned}
K_2 &= \sum_{e \in \mathcal{E}_k : e_n = i} \frac{\varpi^e}{n!} \left[v\left(N_n, e^n\right) - v\left(N_{n-1}, e^{n-1}\right)\right] \\
&= \sum_{e \in \mathcal{E}_k : e_n = i} \frac{\varpi^e}{n!} \left[v(N, k) - v\left(N \setminus \{i\}, e^{n-1}\right)\right] \\
&= \frac{(n-1)!0!}{n!} \sum_{a \in \Omega_{N \setminus \{i\}}} \varpi(N \setminus \{i\}, a) \left[v(N, k) - v(N \setminus \{i\}, a)\right]
\end{aligned}$$

$$= \frac{(n-1)!0!}{n!} \left[v(N,k) - E_v(N \setminus \{i\}) \right]$$
$$= \frac{(n-1)!0!}{n!} \left[E_v(N) - E_v(N \setminus \{i\}) \right] + \frac{1}{n} \left[v(N,k) - E_v(N) \right]$$

The sum K_3 is the weighted sum of all marginal contributions of player i for all entry-trial scenarios where he/she gets in at a position s such that $1 < s < n$. Let $P_{s,n} = \frac{(s-1)!(n-s)!}{n!}$ for $1 \leq s \leq n$.

$$\begin{aligned}
K_3 &= \sum_{e \in \mathcal{E}_k : e_s = i \wedge 1 < s < n} \frac{\varpi^e}{n!} \left[v\left(N_s, e^s\right) - v\left(N_{s-1}, e^{s-1}\right) \right] \\
&= \sum_{S : i \in S \wedge 1 < |S| = s < n} \sum_{e \in \mathcal{E}_k : e_s = i \wedge N_s = S} \frac{\varpi^e}{n!} \left[v\left(S, e^s\right) - v\left(S \setminus \{i\}, e^{s-1}\right) \right] \\
&= \sum_{S : i \in S \wedge 1 < |S| = s < n} \sum_{b \in \Omega_S} \sum_{a \in \Omega_{S \setminus \{i\}}} \left[v(S,b) - v(S \setminus \{i\}, a) \right] \\
&\qquad \sum_{e \in \mathcal{E}_k : e_s = i \wedge N_s = S \wedge e^s = b \wedge e^{s-1} = a} \frac{\varpi^e}{n!} \\
&= \sum_{S : i \in S \wedge 1 < |S| = s < n} P_{s,n} \sum_{b \in \Omega_S} \sum_{a \in \Omega_{S \setminus \{i\}}} \varpi(S,b)\, \varpi(S \setminus \{i\}, a) \\
&\quad \times \left[v(S,b) - v(S \setminus \{i\}, a) \right] \\
&= \sum_{S : i \in S \wedge 1 < |S| = s < n} P_{s,n} \left(E_v(S) - E_v(S \setminus \{i\}) \right)
\end{aligned}$$

Therefore, combining the three sums gives

$$\begin{aligned}
\Phi_i(v,k) &= \sum_{S : i \in S} \frac{(s-1)!\,(n-s)!}{n!} \left[E_v(S) - E_v(S \setminus \{i\}) \right] + \frac{1}{n} \left(v(N,k) - E_v(N) \right) \\
&= Shap_i(E_v) + \frac{1}{n} \left(v(N,k) - E_v(N) \right) = \Psi_i(v,k)
\end{aligned}$$

Clearly, the two values Φ and Ψ coincide. □

Procedure (A_1)–(A_3) provides an interpretation of the value Ψ. It provides an intuitive way of deriving the shares of all players for all possible worths of the grand coalition.

3.2 Some algebraic properties on c.c. games

It is well-known that the set Γ^N of all TU-games on N is a space vector of dimension $2^n - 1$. It can be also checked that $\mathcal{CC}(N, \Omega, \varpi)$ is also a space vector of dimension $\sum_{S \in \mathcal{C}_N} |\Omega_S|$. This assertion is proved below together with some other nice properties of $\mathcal{CC}(N, \Omega, \varpi)$ that will be useful in characterizing Ψ.

Given $S \in \mathcal{C}_N$, $k \in \Omega_S$ and a collection $c = \left(c_{l,T}\right)_{T \in \mathcal{C}_N, l \in \Omega_T}$ of real numbers, we define:

– $\gamma_S \in \Gamma^N$ and $\gamma_S^* \in \Gamma^N$ for all $T \in \mathcal{C}_N$ by

$$\gamma_S(T) = \begin{cases} 1 \; if S \subseteq T \\ 0 \text{ otherwise} \end{cases} \text{ and } \gamma_S^*(T) = \begin{cases} 1 \; if S \subsetneq T \\ 0 \text{ otherwise} \end{cases}$$

– $g^{k,S}$, $\Upsilon^{k,S}$, $\Upsilon^{*,S} \in \mathcal{CC}(N, \Omega, \varpi)$ for all $T \in \mathcal{C}_N$ and $l \in \Omega_T$ by $\Upsilon^{*,S}(T, l) = \gamma_S^*(T)$,

$$g^{k,S}(T, l) = \begin{cases} 1 \; if l = k \text{ and } T = S \\ 0 \text{ otherwise} \end{cases} \text{ and } \Upsilon^{k,S}(T, l) = \begin{cases} \gamma_S(T) \; if l = k \\ \gamma_S^*(T) \text{ otherwise} \end{cases}$$

– $\Upsilon^{c,k,S} \in \mathcal{CC}(N, \Omega, \varpi)$ by $\Upsilon^{c,k,S} = c_{k,S} g^{k,S} + \Upsilon^{*,S}$.

Note that γ_S and γ_S^* are TU-games; whereas $g^{k,S}$, $\Upsilon^{k,S}$ and $\Upsilon^{*,S}$ are c.c. games in $\mathcal{CC}(N, \Omega, \varpi)$. Furthermore,

$$\Upsilon^{k,S} = g^{k,S} + \Upsilon^{*,S} = \Upsilon^{c,k,S} \text{ provided that } c_{k,S} = 1. \tag{6}$$

The TU-game γ_S is sometimes called unanimity game; see for example Béal et al. (2016). The game $\Upsilon^{k,S}$ can be seen as a kind of *unanimity c.c. game* in which a win of one unit is guaranteed provided that either the members of S cooperate and event k is observed; or the members of S cooperate with some players out of S.

Proposition 4 *Any collection* $\left(\Upsilon^{c,k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ *is a basis of* $\mathcal{CC}(N, \Omega, \varpi)$ *assuming that* $c_{k,S} \neq 0$ *for all* $S \in \mathcal{C}_N$ *and for all* $k \in \Omega_S$.

Proof Let $c = \left(c_{k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ be a collection of real numbers such that $c_{k,S} \neq 0$ for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$. Note that for all games $v \in \mathcal{CC}(N, \Omega, \varpi)$,

$$v = \sum_{S \in \mathcal{C}_N, k \in \Omega_S} v(S, k) g^{k,S}.$$

Therefore, $\{g^{k,S} : S \in \mathcal{C}_N, k \in \Omega_S\}$ is a generating set for the vector space $\mathcal{CC}(N, \Omega, \varpi)$. It follows that the dimension of $\mathcal{CC}(N, \Omega, \varpi)$ is at most $\sum_{S \in \mathcal{C}_N} |\Omega_S|$. Now, the collection $\left(\Upsilon^{c,k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ contains exactly $\sum_{S \in \mathcal{C}_N} |\Omega_S|$ distinct games. To prove that $\left(\Upsilon^{c,k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ is a basis of $\mathcal{CC}(N, \Omega, \varpi)$, it is sufficient to prove that the games in $\left(\Upsilon^{c,k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ are linearly independent. To see this, let $\left(\alpha_{k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ be some real numbers such that

$$\sum_{S \in \mathcal{C}_N, k \in \Omega_S} \alpha_{k,S} \Upsilon^{c,k,S} = \widetilde{0}_{\varpi}. \tag{7}$$

where $\widetilde{0}_{\varpi}(T,l) = 0$ for all $T \in \mathcal{C}_N$ and for all $l \in \Omega_S$. We prove by induction on the cardinality of S that $\alpha_{k,S} = 0$ for all $S \in \mathcal{C}_N$ and $k \in \Omega_S$. First assume that $|S| = 1$; that is $S = \{i\}$ for some $i \in N$. Consider $k \in \Omega_S$. Then by the definition of $\Upsilon^{c,l,T}$,

$$\widetilde{0}_{\varpi}(\{i\},k) = 0 = \sum_{T\in\mathcal{C}_N, l\in\Omega_T} \alpha_{l,T}\Upsilon^{c,l,T}(\{i\},k) = c_{k,\{i\}}\alpha_{k,\{i\}}.$$

Therefore $\alpha_{k,\{i\}} = 0$ since $c_{k,\{i\}} \neq 0$. Assume that for some s such that $1 \leq s < n$, it holds that for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$, $\alpha_{k,S} = 0$ whenever $1 \leq |S| \leq s$. Consider a coalition S of cardinality $s+1$ and $k \in \Omega_S$. The definition of $\Upsilon^{c,l,T}$ together with the induction assumption imply

$$\begin{aligned}\widetilde{0}_{\varpi}(S,k) = 0 = \sum_{T\in\mathcal{C}_N, l\in\Omega_T} \alpha_{l,Tk}\Upsilon^{c,l,T}(S,k) &= c_{k,S}\alpha_{k,S} \\ + \sum_{T\subsetneq S, l\in\Omega_T} c_{l,T}\alpha_{l,T} &= c_{k,S}\alpha_{k,S}.\end{aligned}$$

Therefore $\alpha_{k,S} = 0$ since $c_{k,S} \neq 0$. This proves that $\alpha_{k,S} = 0$ for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$. □

In particular, the collection $\left(\Upsilon^{k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is, by (6), a basis of $\mathcal{CC}(N,\Omega,\varpi)$. The following propositions show that, the sum of all $\Upsilon^{k,S}$ over Ω_S can also be rewritten as a linear combination of some specific games.

Proposition 5 *In the space vector $\mathcal{CC}(N,\Omega,\varpi)$,*

$$\sum_{k\in\Omega_S} \Upsilon^{k,S} = (\widetilde{\gamma}_S)_{\varpi} + (|\Omega_S| - 1)\,\Upsilon^{*,S}. \tag{8}$$

for all coalitions $S \in \mathcal{C}_N$.

Proof Let $S \in \mathcal{C}_N$. Proving (8) amounts to showing that for all $T \in \mathcal{C}_N$ and for all $l \in \Omega_T$,

$$\sum_{k\in\Omega_S} \Upsilon^{k,S}(T,l) = \gamma_S(T) + (|\Omega_S| - 1)\,\Upsilon^{*,S}(T,l). \tag{9}$$

Consider $T \in \mathcal{C}_N$ and $l \in \Omega_T$. There are three possible cases we distinguish. *(a)* First suppose that $S \nsubseteq T$. By definition, $\Upsilon^{k,S}(T,l) = \gamma_S(T) = \Upsilon^{*,S}(T,l) = 0$ for all $k \in \Omega_S$. Therefore

$$\sum_{k\in\Omega_S} \Upsilon^{k,S}(T,l) = 0 \text{ and } \gamma_S(T) + (|\Omega_S| - 1)\,\Upsilon^{*,S}(T,l) = 0.$$

(b) Now, suppose that $S = T$. Then $\Upsilon^{k,S}(T,l) = 0$ for $k \neq l$; $\Upsilon^{l,S}(T,l) = 1 = \gamma_S(T)$; and $\Upsilon^{*,S}(T,l) = 0$. Therefore

$$\sum_{k\in\Omega_S} \Upsilon^{k,S}(T,l) = 1 \text{ and } \gamma_S(T) + (|\Omega_S| - 1)\,\Upsilon^{*,S}(T,l) = 1.$$

(c) Finally, suppose that $S \subsetneqq T$. Then $\gamma_S(T) = \Upsilon^{k,S}(T,l) = \Upsilon^{*,S}(T,l) = 1$ for all $k \in \Omega_S$. Thus

$$\sum_{k \in \Omega_S} \Upsilon^{k,S}(T,l) = |\Omega_S| \text{ and } \gamma_S(T) + (|\Omega_S| - 1)\,\Upsilon^{*,S}(T,l) = |\Omega_S|.$$

For each of the three possible cases, (8) holds. □

The equality in (8) simply tells us that by summing over all events in Ω_S, the members of S secure, independently of the event that is observed, a win of one unit by forming S, or a win of $|\Omega_S|$ by cooperating with some players out of S. More importantly, Proposition 5 together with the next result help in linking unanimity c.c. games with the known unanimity TU-games.

Proposition 6 *In the space vector $\mathcal{CC}(N, \Omega, \varpi)$,*

$$\Upsilon^{*,S} = \sum_{T \in \mathcal{C}_N : S \subsetneqq T} (-1)^{|T|-|S|+1} \left(\widetilde{\gamma}_T\right)_{\varpi}. \tag{10}$$

for all coalitions $S \in \mathcal{C}_N$.

Proof Consider a coalition S. Proving (10) amounts to showing that for all $K \in \mathcal{C}_N$ and for all $l \in \Omega_K$

$$\Upsilon^{*,S}(K,l) = \sum_{T \in \mathcal{C}_N : S \subsetneqq T} (-1)^{|T|-|S|+1} \gamma_T(K). \tag{11}$$

Consider $K \in \mathcal{C}_N$ and $l \in \Omega_K$. First suppose that $S \nsubseteq K$ or $K = S$. Then each coalition T such that $S \subsetneqq T$ satisfies $T \nsubseteq K$; otherwise one would have $S \subsetneqq K$. Thus $\Upsilon^{*,S}(K,l) = \gamma_S^*(K) = 0$ and $\gamma_T(K) = 0$. Therefore

$$\sum_{T \in \mathcal{C}_N : S \subsetneqq T} (-1)^{|T|-|S|+1} \gamma_T(K) = 0 = \Upsilon^{*,S}(K,l).$$

Now suppose that $S \subsetneqq K$. Then $\Upsilon^{*,S}(K,l) = 1$ and $\gamma_T(K) = 0$ for all coalitions T such that $T \nsubseteq K$. Thus

$$\begin{aligned}
\sum_{T \in \mathcal{C}_N : S \subsetneqq T} (-1)^{|T|-|S|+1} \gamma_T(K) &= \sum_{T / S \subsetneqq T \subseteq K} (-1)^{|T|-|S|+1} \\
&= (-1)^{|S|-1} \sum_{t=1}^{|K|-|S|} \binom{|K|-|S|}{t} (-1)^{|S|+t} \\
&\quad \text{where } t = |T \backslash S| \\
&= -\left((1-1)^{|K|-|S|} - 1\right) = 1 = \Upsilon^{*,S}(K,l)
\end{aligned}$$

In both possible cases, (10) holds. □

The c.c. game $(\widetilde{\gamma}_S)_\varpi$ inherits the intuition behind the TU-game γ_S. Thus, Propositions 5 and 6 help in highlighting the relationship between the unanimity c.c. games $\Upsilon^{k,S}$ for $S \in \mathcal{C}_N$ and $k \in \Omega_S$. The decompositions (8) and (10) will be useful in studying values for c.c. games that meet some interesting properties such as symmetry or additivity.

3.3 The case of rational probability distributions

We denote by $\mathcal{CC}^r(N)$ the set of all c.c. games v such that for some collection Ω of finite sample spaces and for some probability distribution function ϖ on Ω, $v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all coalitions $S \neq N$ and for all $k \in \Omega_S$, $\varpi(S, k)$ is rational; that is $\varpi(S, k) = \frac{a_{S,k}}{b_{S,k}}$ for some integers $a_{S,k} \geq 1$ and $b_{S,k} \geq 1$. We start with the extreme case of uniform probability distributions that will be used later to ease some proofs.

3.3.1 With a uniform probability distribution function

The probability distribution function ϖ is a collection of *uniform probability distributions* if for all coalitions $S \neq N$, all events in Ω_S occur with the same probability. Since all probabilities over Ω_S sum to 1, we have,

Uniform probability distributions ϖ : for all coalitions $S \neq N$ and for all $k \in \Omega_S$, $\varpi(S, k) = \frac{1}{|\Omega_S|}$. In this case, we simply say that ϖ is uniform.

Note that the collection ϖ is uniform if for all coalitions $S \neq N$, $\varpi(S, .) := (\varpi(S, k))_{k \in \Omega_S}$ is a uniform probability distribution on Ω_S. Proposition 2 holds on $\mathcal{CC}(N, \Omega, \varpi)$ for any probability distribution function ϖ. When ϖ is uniform, we now show that the first five properties in Proposition 2 completely characterize Ψ.

Lemma 1 *Consider an arbitrary probability distribution ϖ on Ω.*

Then for any two values F and F' that satisfy (E), (NP^*) and $(SSym)$ on $\mathcal{CC}(N, \Omega, \varpi)$,

$$F\left(\alpha\, (\widetilde{\gamma}_S)_\varpi\right) = F'\left(\alpha\, (\widetilde{\gamma}_S)_\varpi\right)$$

for all $S \in \mathcal{C}_N$ and for all $\alpha \in \mathbb{R}$.

Proof Members of S are symmetric players in $\alpha\gamma_S$. They are also stohastically symmetric in $\alpha\, (\widetilde{\gamma}_S)_\varpi$. Thus by $(SSym)$, players in S all have the same shares with respect to both F and F'. Furthermore, members of $N \backslash S$ are null players in γ_S. Therefore by (NP*), players in $N \backslash S$ have each a zero share in $\alpha\, (\widetilde{\gamma}_S)_\varpi$ with respect to both F and F'. The result then follows by efficiency.

Lemma 2 *Assume that ϖ is uniform on Ω. Then for any two values F and F' that satisfy (E), (A), (NP^*), $(SSym)$ and (ILR) on $\mathcal{CC}(N, \Omega, \varpi)$,*

$$F\left(\alpha \Upsilon^{k,S}\right) = F'\left(\alpha \Upsilon^{k,S}\right)$$

for all $S \in \mathcal{C}_N$, for all $k \in \Omega_S$ and for all $\alpha \in \mathbb{R}$.

Proof First suppose that $S = N$. Then any two players are stochastically symmetric in the game $\alpha\Upsilon^{k,S}$. Thus by $(SSym)$ and (E), $F\left(\alpha\Upsilon^{k,S}\right) = F'\left(\alpha\Upsilon^{k,S}\right)$. Now suppose that $S \neq N$. Then by Propositions 5 and 6, it follows that,

$$\begin{aligned}
&F\left(\sum_{k\in\Omega_S}\alpha\Upsilon^{k,S}\right)\\
&= F\left(\alpha\left(\widetilde{\gamma}_S\right)_{\varpi} + \left(|\Omega_S|-1\right)\sum_{T\in\mathcal{C}_N:S\subsetneq T}(-1)^{|T|-|S|+1}\alpha\left(\widetilde{\gamma}_T\right)_{\varpi}\right)\\
&= F\left(\alpha\left(\widetilde{\gamma}_S\right)_{\varpi}\right) + \left(|\Omega_S|-1\right)\sum_{T\in\mathcal{C}_N:S\subsetneq T}(-1)^{|T|-|S|+1}F\left(\alpha\left(\widetilde{\gamma}_T\right)_{\varpi}\right) \text{ by } (A)\\
&= F'\left(\alpha\left(\widetilde{\gamma}_S\right)_{\varpi}\right) + \left(|\Omega_S|-1\right)\sum_{T\in\mathcal{C}_N:S\subsetneq T}(-1)^{|T|-|S|+1}F'\left(\alpha\left(\widetilde{\gamma}_T\right)_{\varpi}\right) \text{ by Lemma 1}\\
&= F'\left(\sum_{l\in\Omega_S}\alpha\Upsilon^{l,S}\right) \text{ by } (A)
\end{aligned}$$

Moreover, for all $l \in \Omega_S\setminus\{k\}$, $\pi\Upsilon^{k,S} = \Upsilon^{l,S}$ where π is the transposition of k and l in Ω_S. Therefore by (ILR), $F\left(\alpha\Upsilon^{l,S}\right) = F\left(\alpha\Upsilon^{k,S}\right)$. Thus, by (A), $|\Omega_S|F\left(\alpha\Upsilon^{k,S}\right) = |\Omega_S|F'\left(\alpha\Upsilon^{k,S}\right)$. Thus $F\left(\alpha\Upsilon^{k,S}\right) = F'\left(\alpha\Upsilon^{k,S}\right)$. □

Combining the precedent results leads to the following.

Theorem 1 *Assume that ϖ is uniform on Ω. Then, the equal-surplus Shapley value Ψ is the unique value on $\mathcal{CC}(N, \Omega, \varpi)$ that simultaneously satisfies (E), (A), (NP^*), $(SSym)$ and (ILR).*

Proof Assume that ϖ is uniform on Ω.

Necessity. See Proposition 2.

Sufficiency. Suppose that F is a value that satisfies (E), (A), (NP^*), $(SSym)$ and (ILR) on $\mathcal{CC}(N, \Omega, \varpi)$. To see that $F = \Psi$, consider a game $v \in \mathcal{CC}(N, \Omega, \varpi)$. By Proposition 4, the family $\left(\Upsilon^{k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is a basis of the space vector $\mathcal{CC}(N, \Omega, \varpi)$. Therefore there exists a family of reals $(\alpha_{k,S})_{S\in\mathcal{C}_N, k\in\Omega_S}$ such that

$$v = \sum_{S\in\mathcal{C}_N}\sum_{k\in\Omega_S}\alpha_{k,S}\Upsilon^{k,S}.$$

By Lemma 2, $F\left(\alpha_{k,S}\Upsilon^{k,S}\right) = \Psi\left(\alpha_{k,S}\Upsilon^{k,S}\right)$ for all $k \in \Omega_S$, since F and Ψ both satisfy (E), (A), (NP^*), $(SSym)$ and (ILR) on $\mathcal{CC}(N, \Omega, \varpi)$. Therefore, $F(v) = \Psi(v)$ by additivity. □

In Theorem 1, condition (ILR) may be omitted for some specific Ω. To illustrate this, we assume that all sample spaces with proper coalitions are of the same cardinality. This is for example the case when all coalitional payoffs in the game are obtained by performing the same random experience. The following counterpart of Lemma 2 holds for this specific configuration.

Lemma 3 *Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \backslash \{N\}$. Then for any two values F and F' that satisfy (E), (A), (NP^*) and $(SSym)$ on $\mathcal{CC}(N, \Omega, \varpi)$,*

$$F\left(\alpha \Upsilon^{k,S}\right) = F'\left(\alpha \Upsilon^{k,S}\right)$$

for all $S \in \mathcal{C}_N$, for all $k \in \Omega_S$ and for all $\alpha \in \mathbb{R}$.

Proof Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \backslash \{N\}$. Consider two values F and F' that satisfy (E), (A), (NP^*) and (SS). As above, $F\left(\alpha \Upsilon^{k,S}\right) = F'\left(\alpha \Upsilon^{k,S}\right)$ for $S = N$ by $(SSym)$ and (E). Now, let S be a coalition other than N; and $k, l \in \Omega_S$. Choose an arbitrary event k_T for each possible coalition T such that $|T| = |S|$ and define the c.c. game v by

$$v = \sum_{T \neq S : |T| = |S|} \alpha \Upsilon^{k_T, T}.$$

Taking into the account that the probability distribution is uniform, it follows that for both $\alpha \Upsilon^{k,S} + v$ and $\alpha \Upsilon^{l,S} + v$, any pair of players are stochastically symmetric. Therefore by efficiency, $F(\alpha \Upsilon^{k,S} + v) = F'(\alpha \Upsilon^{l,S} + v)$. Thus by additivity, one gets $F(\alpha \Upsilon^{k,S}) = F'(\alpha \Upsilon^{l,S})$. □

Theorem 2 *Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all $S, T \in \mathcal{C}_N \backslash \{N\}$.*

The equal-surplus Shapley value Ψ is the unique value on $\mathcal{CC}(N, \Omega, \varpi)$ that satisfies axioms (E), (A), (NP^*) and $(SSym)$.

Proof Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \backslash \{N\}$.

Necessity. See Proposition 2.

Sufficiency. Very similar to the proof of Theorem 1 using Lemma 3 instead of Lemma 2. □

The four axioms in Theorem 2 are independent as stated below. For all independence issues in this paper, we only present, in the main text, an allocation rule that meets all the axioms listed except one. More details are relegated to appendix sections.

Proposition 7 *Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \backslash \{N\}$. Then axioms (E), (A), (NP^*) and $(SSym)$ are independent on $\mathcal{CC}(N, \Omega, \varpi)$.*

Proof For any three axioms out of (E), (A), (NP^*) and $(SSym)$, we provide a value that meets all these three axioms, but which does not satisfy the fourth. Details are provided in Appendix 5.1.

1. $F_i^1(v,k) = \frac{1}{n}v(N,k)$ for all $v \in \mathcal{CC}(N,\Omega,\varpi)$, for all $k \in \Omega_N$ and for all $i \in N$. Then F^1 is (E), (A) and $(SSym)$; but not (NP^*).
2. $F^2(v) = 2\Psi(v)$ for all $v \in \mathcal{CC}(N,\Omega,\varpi)$. Then F^2 is (NP^*), (A) and $(SSym)$; but not (E).
3. Given $v \in \mathcal{CC}(N,\Omega,\varpi)$, denote by $N^*(v)$ the set of all null players in the expectation game E_v. Define the value F^3 for all $v \in \mathcal{CC}(N,\Omega,\varpi)$ and for all $i \in N$ by
$$F_i^3(v,k) = 0 if i \in N^*(v); \text{ and } F_i^3(v,k) = \frac{v(N,k)}{|N \backslash N^*(v)|} \text{otherwise.}$$
Then F^3 is (E), (NP^*) and $(SSym)$; but not (A).
4. Given two distinct players i and j in N, denote by a the n-tuple defined by $a_i = 1$, $a_j = -1$ and $a_t = 0$ for all $t \in N\backslash\{i,j\}$. Define F^4 for all $v \in \mathcal{CC}(N,\Omega,\varpi)$ and for all $k \in \Omega_N$ by
$$F^4(v,k) = \Psi(v,k) + \left[\sum_{l\in\Omega_{\{i,j\}}} \varpi(\{i,j\},l)v(\{i,j\},l) - \sum_{l\in\Omega_{\{i\}}} \varpi(\{i\},l)v(\{i\},l) \right.$$
$$\left. - \sum_{l\in\Omega_{\{j\}}} \varpi(\{j\},l)v(\{j\},l)\right] a.$$
The value F^4 is (E), (NP^*) and (A); but not $(SSym)$.

In summary, the four axioms are independent. □

In Theorem 1, (ILR) is a necessary condition for some sample spaces. An illustration is provided in the following example.

Example 5 Let $N = \{1,2\}$ be a set of two players. Consider the coalitional sample spaces $\Omega_{\{1\}} = \{a,b\}$, $\Omega_{\{2\}} = \{c\}$ and $\Omega_{\{1,2\}} = \{x,y\}$. A probability distribution function ϖ that is uniform on Ω is such that $\varpi(\{1\},a) = \varpi(\{1\},b) = \frac{1}{2}$ and $\varpi(\{2\},c) = 1$. Define the value F on $\mathcal{CC}(N,\Omega,\varpi)$ as follows:

$$F(v) = \left(\begin{array}{|c|c|} \hline F_1(v,x) & F_2(v,x) \\ \hline F_1(v,y) & F_2(v,y) \\ \hline \end{array}\right)$$
$$= \left(\begin{array}{|c|c|} \hline v_{1,b} - \frac{1}{2}v_{1,a} - \frac{1}{2}v_{2,c} + \frac{1}{2}v_{12,x} & \frac{1}{2}v_{1,a} - v_{1,b} + \frac{1}{2}v_{2,c} + \frac{1}{2}v_{12,x} \\ \hline \frac{1}{2}v_{1,a} - \frac{1}{2}v_{2,c} + \frac{1}{2}v_{12,y} & \frac{1}{2}v_{2,c} - \frac{1}{2}v_{1,a} + \frac{1}{2}v_{12,y} \\ \hline \end{array}\right)$$

where for simplicity $v_{1,k} = v(\{1\}, k)$ for $k \in \{a, b\}$, $v_{2,c} = v(\{2\}, c)$ and $v_{12,k} = v(\{1, 2\}, k)$ for $k \in \{x, y\}$. Note that F satisfies (E), (A), (NP*) and $(SSym)$; but not (ILR). This shows that for Theorem 1 to hold with this specific sample space, one can no longer rule out condition (ILR).

3.3.2 Over the full class of rational probability distributions

A uniform probability distribution is completely described by the collection of its sample spaces. This is no longer the case for non uniform probability distribution functions. Also note that conditions (E), (A) and (NP*) are entirely normative considerations on individual shares with respect to coalitional payoffs in a game. Thus, apart from $(SSym)$, axioms in Theorem 1 do not fully capture the full strength of non uniform probability distributions. Condition (ILR) is some type of neutral treatment of elementary events in a sample space. Thus none of the five properties in Theorem 1 tells about how changes on the collection of probability distributions impact on individual shares. Further properties are needed to characterize the equal-surplus Shapley value on $\mathcal{CC}^r(N)$.

Remark 5 Consider a pair $\{a, b\}$ of integers and the collection Ω^0 of sample spaces such that $\Omega^0_S = \{a, b\}$ for all $S \in \mathcal{C}_N$. Let $p \geq 3$ be a prime number. Define the probability distribution function ϖ_p for all coalitions S, by $\varpi_p(S, a) = 2/p$ and $\varpi_p(S, b) = 1 - 2/p$. Now, define the value F^5 for all $v \in \mathcal{CC}(N, \Omega, \varpi) \subseteq \mathcal{CC}^r(N)$ by

$$F^5(v) = \begin{cases} \Psi(\widehat{v}) & \text{if} \Omega = \Omega^0 \text{and } \varpi = \varpi_p \\ \Psi(v) & \text{otherwise} \end{cases}$$

where the game $\widehat{v}$ is obtained from v by substituting to ϖ_p the uniform probability distribution function on Ω^0. It can be checked that both F^5 and Ψ meet (E), (A), (NP^*), $(SSym)$ and (ILR). Since $F^5 \neq \Psi$, one needs further requirements to characterize Ψ on $\mathcal{CC}^r(N)$.

Recall that condition (ILD) allows to reshape the sample space as well as the probability distribution function.

Theorem 3 *A value F on $\mathcal{CC}^r(N)$ satisfies (E), (A), (NP^*), $(SSym)$ and (ILD) if and only if $F = \Psi$.*

Proof *Sufficiency*. See Proposition 2.

Necessity. Suppose that a value F on $\mathcal{CC}^r(N)$ satisfies(E), (A), (NP^*), $(SSym)$ and (ILD). Then by Theorem 2, it follows that $F(v) = \Psi(v)$ whenever v is a game with a uniform probability distribution function on a sample space Ω such that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \backslash \{N\}$. Now, consider an arbitrary game $v \in \mathcal{CC}(N, \Omega, \varpi) \subseteq \mathcal{CC}^r(N)$. Given a coalition $S \neq N$, pose $\varpi(S, k) = \frac{a_{S,k}}{b_{S,k}}$ with $a_{S,k} \geq 1$ and $b_{S,k} \geq 1$. Denote by q, the least common multiple of the collection $\{b_{S,k}\colon S \in \mathcal{C}_N \backslash \{N\}$ and $k \in \Omega_S\}$. Then given a coalition $S \neq N$, $q = q_{S,k} b_{S,k}$ for some integer $q_{S,k} \geq 1$. By duplicating $q_{S,k} a_{S,k} - 1$ times each event $k \in \Omega_S$ for all coalitions $S \neq N$ and by applying (ILD) as indicated in Remark 2, one constructs a new game $v^d \in \mathcal{CC}\left(N, \Omega^d, \varpi^d\right)$ such that ϖ^d is uniform on Ω^d, $F(v) = F(v^d)$ and for all coalitions $S, T \neq N$, $|\Omega_S^d| = |\Omega_T^d| = q$. Since ϖ^d is uniform, then by Theorem 2, $F(v^d) = \Psi(v^d)$. Hence $F(v) = \Psi(v^d) = \Psi(v)$. □

Proposition 8 *The axioms (E), (A), (NP^*), $(SSym)$ and (ILD) are independent on $\mathcal{CC}^r(N)$.*

Proof Each of the four values presented in Proposition 7 satisfies (ILD) but fails to satisfy exactly one axiom among (E), (A), (NP^*) and $(SSym)$. Therefore, we only have to prove that (ILD) can not be deduced from the other four axioms in consideration on $\mathcal{CC}^r(N)$. To see this, we note that the value introduced in Remark 5 is (E), (A), (NP^*) and $(SSym)$, but not (ILD). See Appendix 5.2 for further details. □

The arguments in the proof of Theorem 3 are strongly related to the fact that the probability of each coalitional event is a rational number. The main step of the proof consists in moving from any game with rational probability distributions to a game with a uniform probability distribution with sample spaces of equal sizes.

3.4 The general case

In this section, the whole set $\mathcal{CC}(N)$ of all c.c. games on N is considered (there is no restriction on the probability distribution function). New axioms are introduced to capture how a value behaves when some specific changes on the payoffs or on the probability distribution occur. But before, we extend the scope of Theorem 3 from $\mathcal{CC}^r(N)$ to $\mathcal{CC}(N)$. To this end, the following lemma is introduced.

Lemma 4 *If a value F satisfies (E), (A), (NP^*), $(SSym)$ and (ILD) on $\mathcal{CC}(N)$, then*

$$F\left(\alpha\Upsilon^{k,S}\right) = \Psi\left(\alpha\Upsilon^{k,S}\right)$$

for all collections $\Omega = (\Omega)_{S\in\mathcal{C}_N}$ of sample spaces, for all probability distribution functions ϖ on Ω, for all $S \in \mathcal{C}_N$, for all $k \in \Omega_S$ and for all $\alpha \in \mathbb{R}$.

Proof Consider a value F that satisfies (E), (A), (NP^*), $(SSym)$ and (ILD) on $\mathcal{CC}(N)$; a collection Ω of sample spaces, a probability distribution function ϖ on Ω, a coalition $S \in \mathcal{C}_N$, an event $k \in \Omega_S$ and a real number $\alpha \in \mathbb{R}$. First note that by Theorem 3, the result holds when ϖ is rational-valued. Now, suppose that the probability distribution function ϖ is not rational-valued and cannot be reduced to a rational-valued function by merging only a few duplicate events. We prove that, even in this case, $F\left(\alpha\Upsilon^{k,S}\right) = \Psi\left(\alpha\Upsilon^{k,S}\right)$. By the definition of $\Upsilon^{k,S}$, for all coalitions $T \neq S$, all events in Ω_T lead to the same coalitional payment of 1 when $S \subsetneq T$, and 0 when $S \nsubseteq T$. Thus by (ILD), $F\left(\alpha\Upsilon^{k,S}\right) = F(u_0)$ where the game u_0 is obtained from $\Upsilon^{k,S}$ by merging all duplicate events in each sample space Ω_T, with

$T \notin \{S, N\}$, into a single event e_T. By so doing, we move from (Ω, ϖ) to (Ω_1, ϖ_1) such that for $T \in \{S, N\}$, $(\Omega_1)_T = \Omega_T$, $\varpi_1(T, k) = \varpi(T, k)$ for all $k \in \Omega_T$; and for all $T \in \mathcal{C}_N \backslash \{S, N\}$, $(\Omega_1)_T = \{e_T\}$ and $\varpi_1(T, e_T) = 1$. Since ϖ cannot be reduced to a rational-valued probability distribution, Ω_S necessarily contains at least two events. First suppose that $\Omega_S = \{k, l\}$ for some events k and l. Let $\pi(S, k) = p_1$ and $\varpi(S, l) = p_2$ with $p_1 + p_2 = 1$. To continue, we consider three new games. To this end, we consider two distinct events $k', l' \notin \{k, l\}$ together with an arbitrary rational number q such that $p_1 - \frac{1}{2} < q < \min\{p_1, \frac{1}{2}\}$. Such a rational number q necessarily exists since $p_1 - \frac{1}{2} < p_1$ and $p_1 - \frac{1}{2} < \frac{1}{2}$. The new games are:

- $u_1 \in \mathcal{CC}(N, \Omega_1, \varpi_1)$ such that $u_1(T, t) = u_0(T, t)$ and $\varpi_1(T, t) = \varpi(T, t)$ for all $T \in\in \mathcal{C}_N \backslash \{S\}$ together with

$$u_1(S, k) = 0, u_1(S, l) = 1 \text{and } \varpi_1(S, k) = \varpi_1(S, l) = \frac{1}{2}.$$

- $u_2 \in \mathcal{CC}(N, \Omega_2, \varpi_2)$ obtained from u_0 by only duplicating k into k and k', and l into l and l' in such a way that $(\Omega_2)_S = \{k, k', l, l'\}$ and

$$\begin{aligned}\varpi_2(S, k) = q, \varpi_2(S, k') = p_1 - q; \varpi_2(S, l') = \frac{1}{2} - q; \varpi_2(S, l)\\ = q + \frac{1}{2} - p_1 = p_2 - \left(\frac{1}{2} - q\right).\end{aligned}$$

- $u_3 \in \mathcal{CC}(N, \Omega_2, \varpi_3)$ obtained from u_1 by only duplicating k into k and l', and l into l and k' in such a way that

$$\begin{aligned}\varpi_3(S, k) = q, \varpi_3(S, l') = \frac{1}{2} - q; \varpi_3(S, k') = p_1 - q;\\ \varpi_3(S, l) = q + \frac{1}{2} - p_1 = \frac{1}{2} - (p_1 - q).\end{aligned}$$

Note that u_2 and u_3 are defined on the same sample space Ω_2. Moreover, from ϖ_2 to ϖ_3, only the probabilities of k' and l' are permuted. Let π be the transposition of Ω_2 that interchanges k' and l'; that is $\pi = (k', l')$. Then $\pi\varpi_3 = \varpi_2$. We deduce that $\pi u_3 \in \mathcal{CC}(N, \Omega_2, \varpi_2)$. Furthermore,

$$(u_2 + \pi u_3)(S, t) = 1 \text{ for all } t \in (\Omega_2)_S.$$

In words, all events in $(\Omega_2)_S$ lead to the same coalitional payment of 1. Therefore, by (ILD), k, k', l and l' can be merged from $u_2 + \pi u_3$ into a single event e_S that occurs with probability 1. This leads us to the game $u_4 = u_2 + \pi u_3 \in \mathcal{CC}(N, (\Omega_3, \varpi_4)$ such that

$$(\Omega_2)_T = \{e_T\} \text{ and } \varpi_4(T, e_T) = 1 \text{ for all coalitions } T \neq N.$$

Thus, u_1 and u_4 belongs to $\mathcal{CC}^r(N)$. Therefore

$$\begin{aligned}
F\left(\alpha\Upsilon^{k,S}\right) &= F(u_0) \text{ by (ILD)}\\
&= F(u_2) \text{ by (ILD)}\\
&= F(u_2+\pi u_3) - F(\pi u_3) \text{ by additivity of } F\\
&= \Psi(u_2+\pi u_3) - F(\pi u_3) \text{ by Theorem 3 since } u_4 = u_2\\
&\quad +\pi u_3 \in \mathcal{CC}^r(N)\\
&= \Psi(u_2) + \Psi(\pi u_3) - F(\pi u_3) \text{ by additivity of } \Psi\\
&= \Psi(u_2) + \Psi(u_3) - F(u_3) \text{ by } (ILD) \text{ and Proposition 1}\\
&= \Psi(u_0) + \Psi(u_1) - F(u_1) \text{ by } (ILD)\\
&= \Psi(u_0) \text{ by Theorem 3 since } u_1 \in \mathcal{CC}^r(N)\\
&= \Psi\left(\alpha\Upsilon^{k,S}\right) \text{ by } (ILD)
\end{aligned}$$

Now, suppose that Ω_S contains more than two events. By definition, $\Upsilon^{k,S}(S,k) = 1$ and $\Upsilon^{k,S}(S,t) = 0$ for all $t \in \Omega_S \backslash \{k\}$. Thus by merging all events $t \in \Omega_S \backslash \{k\}$ into a single event l, one returns to the previous case by applying (ILD). □

Theorem 4 *A value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), $(SSym)$ and (ILD) if and only if $F = \Psi$.*

Proof *Necessity*. See Proposition 2.
Sufficiency. For a given collection Ω of sample spaces and a probability distribution function ϖ on Ω, $\left(\Upsilon^{k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is, by Proposition 4, a basis of $\mathcal{CC}(N,\Omega,\varpi)$. Thus, the result follows from Lemma 4 and additivity using very similar arguments to those used in the proof of Theorem 1. □

Proposition 9 *The axioms (E), (A), (NP^*), $(SSym)$ and (ILD) are independent on $\mathcal{CC}(N)$.*

Proof Since $\mathcal{CC}^r(N) \subseteq \mathcal{CC}(N)$, each of the five values invoked in the proof of Proposition 8 also permit to prove that none of the five axioms in consideration on $\mathcal{CC}(N)$ can not be deduced from the four others. □

In Theorem 4 , axioms (E), (A) and (NP^*) are related to how, independently of the probability distribution, the information on coalitional payoffs in a game impacts on the shares of players. Axioms $(SSym)$ and (ILD) describe some specific patterns or changes that may be observed either on the coalitional payoffs; or on the probability distribution function separately. Some more mixture changes that combine changes on the probability distribution and the coalitional payoffs are explored below. The first change consists in merging two events as follows: given $v \in \mathcal{CC}(N)$, a coalition $S \neq N$ and $\{k,l\} \subseteq \Omega_S$, the game $\left(\Omega^{S,k\sim l}, v^{S,k\sim l}, \varpi^{S,k\sim l}\right) \in \mathcal{CC}(N)$ is defined by

(i) $\Omega_T^{S,k\sim l} = \Omega_T \backslash \{l\}$ if $T = S$ and $\Omega_T^{S,k\sim l} = \Omega_T$ if $T \neq S$;

(ii) $\varpi^{S,k\sim l}(S,k) = \varpi(S,k) + \varpi(S,l)$ and

$$v^{S,k\sim l}(S,k) = \frac{\varpi(S,k)}{\varpi(S,k)+\varpi(S,l)} v(S,k) + \frac{\varpi(S,l)}{\varpi(S,k)+\varpi(S,l)} v(S,l);$$

(iii) $\varpi^{S,k\sim l}(T,t) = \varpi(T,t)$ and $v^{S,k\sim l}(S,t) = v(S,t)$ if $T \neq S$ or ($T = S$ and $t \in \Omega_S \backslash \{k,l\}$).

To obtain the game $v^{S,k\sim l}$, event l is deleted, the probability of event k is updated to the sum of the probabilities of events k and l while the corresponding payoff is obtained by aggregating the payoffs associated with k and l to their weighted mean. Hereafter, we say that moving from v to $v^{S,k\sim l}$ is an MC-merging.

Merge-and-Cancel Invariance (MCI). For all $v \in \mathcal{CC}(N)$, for all coalitions $S \neq N$ and for all $k, l \in \Omega_S$, $F\left(v^{S,k\sim l}\right) = F(v)$.

In contrast with (ILD), mergeability invariance captures changes that combine probabilities with payoffs of coalitions. Although, (MCI) seems to be almost a mathematical disposition, it turns out in the following result that (MCI) is equivalent to (ILD) when one assumes (E), (A), (NP^*) and $(SSym)$.

Theorem 5 *A value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), $(SSym)$ and (MCI) if and only if $F = \Psi$.*

Proof Consider a value F on $\mathcal{CC}(N)$.

Necessity. Thanks to Proposition 2, we only need to prove that Ψ satisfies (MCI). By merging two events $k, l \in \Omega_S$ into k in a c.c. game v, it can be checked that the expectation game in the new game $v^{S,k\sim l}$ coincides with that of v. Therefore, by equation (5), $F\left(v^{S,k\sim l}\right) = F(v)$.

Sufficiency. Suppose that a value F on $\mathcal{CC}(N)$ satisfies (E), (A), (NP^*), $(SSym)$ and (MCI). Given a coalition $S \neq N$, all events in Ω_S can be merged into a single event, say k_S, by a successive use of the merging operation. By MCI, this operation leaves unchanged all individual shares with respect to F and Ψ from v to the new game. By iterating this procedure for all coalitions $S \neq N$, one moves from v to the game $\Upsilon \in \mathcal{CC}(N, \Omega', \varpi')$:

$$\Upsilon = \left(\widetilde{E}_v\right)_{\varpi'} + \sum_{k \in \Omega_N} \left(v(N,k) - E_v(N)\right) \Upsilon^{k,N}$$

such that for all $k \in \Omega_N$, the payoff and the probability of event k are the same in Υ as in v; for all coalitions $S \neq N$, k_S is the unique event for S in Υ while the payoff of the members of S is the expectation $E_v(S)$ of S in v. By (MCI), $F(v) = F(\Upsilon)$ and $\Psi(\Upsilon) = \Psi(v)$. Moreover, the probability distribution in Υ is uniform on Ω'. Thus by Theorem 2, $F(\Upsilon) = \Psi(\Upsilon)$. Therefore $F(v) = \Psi(v)$. □

Proposition 10 *Axioms (E), (A), (NP^*), $(SSym)$ and (MCI) are independent on $\mathcal{CC}(N)$.*

Proof Each of the first three values presented in Proposition 7 satisfies (MCI) but fails to satisfy exactly one axiom among (E), (A) and (NP^*). Now, consider two distinct players i and j. Denote by a the n-tuple defined by $a_i = 1$, $a_j = -1$ and $a_t = 0$ for all $t \in N\backslash\{i, j\}$. Let the value F^5 be defined for all $v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all $k \in \Omega_N$ by

$$F^6(v, k) = \Psi(v, k) + \left[\sum_{l \in \Omega_{\{i,j\}}} \varpi(\{i, j\}, l)\, v(\{i, j\}, l) - \sum_{l \in \Omega_{\{i\}}} \varpi(\{i\}, l)\, v(\{i\}, l) - \sum_{l \in \Omega_{\{j\}}} \varpi(\{j\}, l)\, v(\{j\}, l) \right] a.$$

The value F^6 is (E), (NP^*), (A) and (MCI); but not $(SSym)$. Moreover, we note that the value introduced in Remark 5 is (E), (A), (NP^*) and $(SSym)$, but not (MCI). See Appendix 5.3 for further details. □

By an MC-merging operation, the size of a sample space is reduced by canceling some events. An alternative operation consists in maintaining all events. To define such an operation, associate to a game $v \in \mathcal{CC}(N, \Omega, \varpi)$ the game $v^{S,k\wedge l} \in \mathcal{CC}(N, \Omega, \varpi)$ defined by $v^{S,k\wedge l}(T, t) = v(T, t)$ if $T \neq S$ or ($T = S$ and $t \in \Omega_S\backslash\{k, l\}$); together with

$$v^{S,k\wedge l}(S, k) = v^{S,k\wedge l}(S, l) = \frac{\varpi(S, k)}{\varpi(S, k) + \varpi(S, l)} v(S, k) + \frac{\varpi(S, l)}{\varpi(S, k) + \varpi(S, l)} v(S, l).$$

For the game $v^{S,k\wedge l}(S, k)$, one merges the payoffs of events k and l to the their weighted mean; but keeps all events with their respective probabilities. This contrasts with the game $v^{k\sim l}$, in which one merges the payoffs of two events and removes one from the sample space of S.

Merge-and-Keep Invariance (MKI). For all $v \in \mathcal{CC}(N)$, for all coalitions $S \neq N$ and for all $k, l \in \Omega_S$, $F\left(v^{S,k\wedge l}\right) = F(v)$.

Both (MKI) and (MCI) are based on the same intuition that merging some events in a consistent way should not affect individual shares in a game. Hereafter, we say that moving from v to $v^{S,k\wedge l}$ is an MK-merging operation.

To prove the next result, we consider the basis $\left(\Upsilon^{c,k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ of $\mathcal{CC}(N, \Omega, \varpi)$ where the collection c is such that for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$,

$$c_{k,S} = \frac{1}{\varpi(S, k)}. \tag{12}$$

The collection $c = \left(c_{k,S}\right)_{S \in \mathcal{C}_N, k \in \Omega_S}$ is well-defined since only events with positive probabilities are considered. It can also be checked that all games $\Upsilon^{c,k,S}$ for $k \in \Omega_S$

yield the same expectation of one unit to coalition S. Moreover, the following result holds:

Lemma 5 *Let Ω be an arbitrary collection of sample spaces and ϖ a probability distribution function on Ω. Suppose that $c = \left(c_{k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is defined by (12).*
Then, all values F on $\mathcal{CC}\,(N,\Omega,\varpi)$ that satisfy (MKI) are such that

$$F\left(\alpha\Upsilon^{c,k,S}\right) = F\left(\alpha\Upsilon^{c,l,S}\right) \tag{13}$$

for all coalitions S, for all events $k, l \in \Omega_S$ and for all real numbers α.

Proof Suppose that $c = \left(c_{k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ is defined by (12). Consider a value F that satisfies (MKI) on $\mathcal{CC}\,(N,\Omega,\varpi)$, a coalition S, two events $k, l \in \Omega_S$ and a real number α. Pose $u = \alpha\Upsilon^{k,S}$ and $v = \alpha\Upsilon^{l,S}$. By the definition of $\alpha\Upsilon^{k,S}$ and $\alpha\Upsilon^{l,S}$, $u(T,t) = v(T,t)$ for all coalitions T and for all $t \in \Omega_T$ such that $T \neq S$ or ($T = S$ and $t \in \Omega_S\backslash\{k,l\}$). Therefore $u^{S,k\wedge l}(T,t) = v^{S,k\wedge l}(T,t)$ for all coalitions T and for all $t \in \Omega_T$ such that $T \neq S$ or ($T = S$ and $t \in \Omega_S\backslash\{k,l\}$). Furthermore, for $t \in \{k,l\}$,

$$u^{S,k\wedge l}(S,t) = v^{S,k\wedge l}(S,t) = \frac{1}{\varpi(S,k) + \varpi(S,l)}.$$

This proves that $u^{S,k\wedge l} = v^{S,k\wedge l}$. Therefore,

$$F(\alpha\Upsilon^{k,S}) = F(u^{S,k\wedge l}) = F(v^{S,k\wedge l}) = F(\alpha\Upsilon^{l,S}).$$

since F satisfies (MKI). □

Theorem 6 *Let Ω be an arbitrary collection of sample spaces and ϖ a probability distribution function on Ω.*
A value F on $\mathcal{CC}\,(N,\Omega,\varpi)$ satisfies (E), (A), (NP^*), $(SSym)$ and (MKI) if and only if $F = \Psi$.

Proof *Sufficiency*. Due to Proposition 2, we only needs to prove that Ψ satisfies (MKI). Consider $v \in \mathcal{CC}(N)$. By the definition of an MK-merging operation, the expectation game of $v^{S,k\wedge l}$ coincides with that of v. Therefore, by equation (5), $\Psi\left(v^{S,k\wedge l}\right) = \Psi\,(v)$.

Necessity. Suppose that a value F on $\mathcal{CC}\,(N)$ satisfies (E), (A), (NP^*), $(SSym)$ and (MKI). Consider $v \in \mathcal{CC}\,(N,\Omega,\varpi)$. Since F satisfies (A) and the collection $\left(\Upsilon^{c,k,S}\right)_{S\in\mathcal{C}_N, k\in\Omega_S}$ with c defined by (12) is a basis of $\mathcal{CC}\,(N,\Omega,\varpi)$, to prove that $F(v) = \Psi(v)$, we only have to prove that $F\left(\alpha\Upsilon^{c,k,S}\right) = \Psi\left(\alpha\Upsilon^{c,k,S}\right)$ for all $S \in \mathcal{C}_N$ and for all $k \in \Omega_S$.

For this purpose, consider a coalition S and $k \in \Omega_S$. On the one hand, we have

$$F\left(\sum_{l\in\Omega_S}\varpi(S,l)\alpha\Upsilon^{c,l,S}\right) = F\left(\alpha\sum_{l\in\Omega_S}\varpi(S,l)\left(c_{l,S}g^{l,S} + \Upsilon^{\star,S}\right)\right) \text{ since } \Upsilon^{c,l,S}$$

$$\begin{aligned}
&= c_{l,S} g^{l,S} + \Upsilon^{\star,S} \\
&= F\left(\alpha \sum_{l\in\Omega_S}\left(g^{l,S} + \varpi(S,l)\Upsilon^{\star,S}\right)\right) \text{ since } c_{l,S} \\
&= \frac{1}{\varpi(S,l)} \text{ from (12)} \\
&= F\left(\alpha \sum_{l\in\Omega_S}\left(\Upsilon^{l,S} - \Upsilon^{\star,S} + \varpi(S,l)\Upsilon^{\star,S}\right)\right) \text{ since } g^{l,S} \\
&= \Upsilon^{l,S} - \Upsilon^{\star,S} \text{ from (6)} \\
&= F\left(\left(\alpha \sum_{l\in\Omega_S}\Upsilon^{l,S}\right) - \alpha|\Omega_S|\Upsilon^{\star,S}\right. \\
&\quad \left. +\alpha\Upsilon^{\star,S}\right) \text{ since } \sum_{l\in\Omega_S}\varpi(S,l) = 1 \\
&= F\left(\alpha\left(\widetilde{\gamma}_S\right)_{\varpi}\right) \text{ by Proposition 5}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
F\left(\sum_{l\in\Omega_S}\varpi(S,l)\alpha\Upsilon^{c,l,S}\right) &= \sum_{l\in\Omega_S} F\left(\varpi(S,l)\alpha\Upsilon^{c,l,S}\right) \text{ by additivity} \\
&= \sum_{l\in\Omega_S} F\left(\varpi(S,l)\alpha\Upsilon^{c,k,S}\right) \text{ by Lemma 5} \\
&= F\left(\sum_{l\in\Omega_S}\varpi(S,l)\alpha\Upsilon^{c,k,S}\right) \text{ by additivity} \\
&= F\left(\alpha\Upsilon^{k,S}\right) \text{ since } \sum_{l\in\Omega_S}\varpi(S,l) = 1
\end{aligned}$$

This proves that $F\left(\alpha\Upsilon^{k,S}\right) = F\left(\alpha\left(\widetilde{\gamma}_S\right)_{\varpi}\right)$. Since Ψ also satisfies (E), (A), (NP^*), $(SSym)$ and (MKI), we deduce that $\Psi\left(\alpha\Upsilon^{k,S}\right) = \Psi\left(\alpha\left(\widetilde{\gamma}_S\right)_{\varpi}\right)$. In the c.c. game $\alpha\left(\widetilde{\gamma}_S\right)_{\varpi}$, all players in S are stochastically symmetric while all players out of S are null players in γ_S. The result follows by applying (E), (NP^*) and $(SSym)$. □

Proposition 11 *Axioms (E), (A), (NP^*), $(SSym)$ and (MKI) are independent on $\mathcal{CC}(N)$.*

Proof Each of the five values invoked in the proof of Proposition 10 also permit to prove that none of the five axioms in consideration here on $\mathcal{CC}(N)$ can not be deduced from the four others. See Appendix 5.4 for further details. □

Remark 6 It appears from Theorem 5 and Theorem 6 that (MCI) and (MKI) are equivalent axioms for values that satisfy (E), (A), (NP^*) and (SS). Furthermore, the intuition behind both (MCI) or (MKI) seems quite to be the preservation of the expectations of proper coalitions in a c.c. game. Each of the two axioms is equivalent to (ILD) for values that satisfy (E), (A), (NP^*) and (SS). Hence (ILD), which simply requires that merging duplicate events should not affect individual shares in a c.c. game, tells us a more perceivable story than (MCI) or (MKI).

Remark 7 For an overview of the results presented, it is worth noticing that in Theorem 3-6, the collection Ω of sample spaces varies as well as the probability distribution function ϖ. In contrast, Theorems 1, 2 and 6 hold for a given Ω and ϖ. Furthermore, Theorems 1 and 2 rely only on uniform probability distribution functions while the scope of Theorem 6 is not restricted. Similarly, Theorem 3 holds on the subclass of c.c. games with rational-valued probability distributions while Theorem 5 and 6 are obtained on the whole class of c.c. games.

4 Conclusion

In this paper, a value for c.c. games is presented together with some of its key features. Firstly, it can be interpreted as a two-stage contract in Charnes and Granot sense. Given a c.c. game, players are first promised their prior Shapley shares from the expectation game associated with the initial game. When an event for the grand coalition is observed, the surplus is equally re-allocated among players to obtain the final shares. Secondly, a very simple and compact formula is provided and shows how the payoff vector of a game is obtained from the Shapley value of the expectation game. Thirdly, a procedure that tells the story behind the determination of individual shares is built up and follows the same spirit as the Shapley procedure. Fourthly and finally, characterizations that exhibit the normative requirements behind the equal-surplus Shapley value are presented. Some of these characterization results are obtained thanks to a new presentation of c.c. games with sample spaces embedded.

Still, in our framework, it is for some interest to pursue investigations on a proportional-surplus Shapley value that will maintain the same prior shares we use, but will proportionally split the surplus according to a collection of predetermined individual weights. Nowak and Radzik (1995) or Béal et al. (2016) are some appropriate references on this issue. In c.c. games, coalitional worths are independent random variables. The TUU-game model by Habis and Herings (2011) encompasses this limitation since it makes it possible to have correlated coalitional worths. It would also be interesting to construct single-valued solutions for TUU-games and study their prominent properties. Our current framework develops some useful tools for such inquiries.

Acknowledgements We are particularly indebted to the Editor and two anonymous reviewers for valuable and helpful comments. We would also like to thank Philippe Solal, Bertrand Tchantcho, Louis A. Fono, Aurelien Y. Mekuko, and Roland Pongou for their encouraging comments on a preliminary version of this paper.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

5 Appendices

In the main text, we have presented some allocation rules and assert that each of them meets some properties. We show here that each such rule effectively satisfies the announced properties.

5.1 Proof of Proposition 7

Proposition 7 *Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all $S, T \in \mathcal{C}_N \backslash \{N\}$.*

Axioms (E), (A), (NP^*) and $(SSym)$ are independent on $\mathcal{CC}(N, \Omega, \varpi)$.

Proof Assume that ϖ is uniform on Ω and that $|\Omega_S| = |\Omega_T|$ for all coalitions $S, T \in \mathcal{C}_N \backslash \{N\}$.

1. $F_i^1(v, k) = \frac{1}{n} v(N, k)$ for all $v \in \mathcal{CC}(N, \Omega, \varpi)$, for all coalitions S, for all $k \in \Omega_N$ and for all $i \in N$. Then F^1 is (E), (A) and $(SSym)$; but not (NP^*). Proving this is straightforward and is omitted.
2. $F^2(v) = 2\Psi(v)$ for all $v \in \mathcal{CC}(N, \Omega, \varpi)$. Then F^2 is (NP^*), (A) and $(SSym)$; but not (E). Proving this is straightforward and is omitted.
3. Given $v \in \mathcal{CC}(N, \Omega, \varpi)$, denote by $N^*(v)$ the set of all null players in the expectation game E_v and for all $S \in \mathcal{C}_N$, let $V(S) = \{v(S, l) : l \in \Omega_S\}$ be the set of all possible worths of coalition S. Define the value F^3 for all $v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all $i \in N$ by

$$F_i^3(v, k) = 0 if i \in N^*(v); \text{ and } F_i^3(v, k) = \frac{v(N, k)}{|N \backslash N^*(v)|} \text{otherwise.}$$

   (a) It is clear that F^3 satisfies (E) and (NP^*).
   (b) Suppose that i and j are two stochastically symmetric players in a c.c. game $v \in \mathcal{CC}(N, \Omega, \varpi)$. To see that i and j are symmetric players in the expectation game, consider $S \subseteq N \backslash \{i, j\}$.

$$\begin{aligned} E_v(S \cup \{i\}) &= \sum_{l \in \Omega_{S \cup \{i\}}} \varpi((S \cup \{i\}, l) v((S \cup \{i\}, l) \\ &= \sum_{x \in V(S \cup \{i\})} x \sum_{l \in \Omega_{S \cup \{i\}} : v(S \cup \{i\}, l) = x} \varpi(S \cup \{i\}, l) \\ &= \sum_{x \in V(S \cup \{j\})} x \sum_{l \in \Omega_{S \cup \{j\}} : v(S \cup \{j\}, l) = x} \varpi(S \cup \{j\}, l) \\ &\quad \text{since } i \text{ and} j \text{ are stochastically symmetric in } v \end{aligned}$$

$$\begin{aligned} &= \sum_{l \in \Omega_{S\cup\{j\}}} \varpi((S \cup \{j\}, l) v((S \cup \{j\}, l) \\ &= E_v(S \cup \{j\}) \end{aligned}$$

Thus, i and j are symmetric players in E_v. By the definition of F^3, $F_i^3(v,k) = F_j^3(v,k)$ for all $k \in \Omega_N$. Therefore F^3 satisfies $(SSym)$.

(c) Consider $\{i, j\} \in \mathcal{C}_N$. Pose $u = \left(\widetilde{\gamma}_{\{i\}}\right)_\varpi$ and $v = \left(\widetilde{\gamma}_{\{i,j\}}\right)_\varpi$. Let $k \in \Omega_S$. We have $N^*(u) = N\backslash\{i\}$ and $N^*(u+v) = N\backslash\{i, j\}$. By the definition of F^3,

$$F_i^3(u,k) + F_i^3(v,k) = 1 + \frac{1}{2} = \frac{3}{2} \text{ and } F_i^3(u+v,k) = 1.$$

Therefore $F^3(v+u) \neq F^3(u) + F^3(v)$ since $F_i^3(u+v,k) \neq F_i^3(u,k) + F_i^3(v,k)$. This proves that F^3 does not satisfy (A).

4. Given two distinct players i and j in N, denote by a the n-tuple defined by $a_i = 1$, $a_j = -1$ and $a_h = 0$ for all $h \in N\backslash\{i, j\}$. Let the value F^4 be defined for all $v \in \mathcal{CC}(N, \Omega, \varpi)$ and for all $k \in \Omega_N$ by

$$\begin{aligned} F^4(v,k) = \Psi(v,k) + \Bigg[&\sum_{l \in \Omega_{\{i,j\}}} \varpi(\{i,j\},l) v(\{i,j\},l) - \sum_{l \in \Omega_{\{i\}}} \varpi(\{i\},l) v(\{i\},l) \\ &- \sum_{l \in \Omega_{\{j\}}} \varpi(\{j\},l) v(\{j\},l) \Bigg] a. \end{aligned}$$

(a) Since the terms of $\Psi(v,k)$ sum to $v(N,k)$ and the terms of a sum to zero, it follows that the terms of $F^4(v,k)$ sum to $v(N,k)$. Therefore F^4 satisfies (E).

(b) Suppose that u is a TU-game on N and $h \in N$ is a null player in u. If $h \in N\backslash\{i, j\}$, then by the definition of a, $a_h = 0$ and $F_h^4(\widetilde{u}_\varpi, k) = \Psi_h(\widetilde{u}_\varpi, k) = 0$ since Ψ is (NP^*). Now, without lost generality, suppose that $h = i$. Since Ψ is (NP^*), $\Psi_i(\widetilde{u}_\varpi, k) = 0$. Moreover, $\widetilde{u}_\varpi(S,l) = u(S)$ for all coalitions S and for all $l \in \Omega_S$. Thus, by the definition of F^4, we have:

$$\begin{aligned} F_i^4(\widetilde{u}_\varpi, k) &= \Bigg[\sum_{l \in \Omega_{\{i,j\}}} \varpi(\{i,j\},l) u(\{i,j\}) - \sum_{l \in \Omega_{\{i\}}} \varpi(\{i\},l) u(\{i\}) \\ &\quad - \sum_{l \in \Omega_{\{j\}}} \varpi(\{j\},l) u(\{j\}) \Bigg] a_i \\ &= \Bigg[\underbrace{u(\{i,j\}) - u(\{j\})}_{0} - \underbrace{u(\{i\})}_{0} \Bigg] a_i \text{ since} \\ &\quad \varpi \text{ is a probability distribution function} = 0. \end{aligned}$$

We conclude that F^4 satisfies (NP^*).

(c) F^4 verifies (A) since Ψ verifies (A) and the coefficient of vector a in the definition of F^4 is linear.

(d) Pose $v = \Upsilon^{k,\{i,j\}}$ for some $k \in \Omega_{\{i,j\}}$. Players i and j are stochastically symmetric in v. Moreover,

$$F_i^4(v,k) = \Psi_i(v,k) + 1 \text{ and } F_j^4(v,k) = \Psi_j(v,k) - 1.$$

Since $\Psi_i(v,k) = \Psi_j(v,k)$ by symmetry of Ψ, we deduce that $F_i^4((v,k) \neq F_j^4(v,k)$. Therefore F^4 does not satisfy $(SSym)$.

In summary, the four axioms are independent. □

5.2 Proof of Proposition 8

Proposition 8 *The axioms* (E), (A), (NP^*), $(SSym)$ *and* (ILD) *are independent on* $\mathcal{CC}^r(N)$.

Proof Each of the four values presented in Proposition 7 fails to satisfy exactly one axiom among (E), (A), (NP^*) and $(SSym)$. Each of those four values obviously satisfies (ILD). Therefore, we only have to prove that (ILD) can not be deduced from the other four axioms in consideration on $\mathcal{CC}^r(N)$. To prove this, consider a pair $\{a,b\}$ of integers and the collection of sample spaces Ω^0 such that $\Omega_S^0 = \{a,b\}$ for all $S \in \mathcal{C}_N$. Denote by ϖ^0 the uniform probability distribution function on Ω^0 and let ϖ_p be the probability distribution function defined for all coalitions S by $\varpi_p(S,a) = \frac{2}{p}$ and $\varpi(S,b) = 1 - \frac{2}{p}$ where p is a prime number such that $p \geq 3$. Now, define the value F^5 on $\mathcal{CC}^r(N)$ by

$$F^5(v) = \begin{cases} \Psi(\widehat{v}) & \text{if } v \in \mathcal{CC}(N,\Omega^0,\varpi_p) \\ \Psi(v) & \text{otherwise} \end{cases}$$

where the game $\widehat{v}$ is obtained from v by substituting to ϖ_p the uniform probability distribution function ϖ^0 on Ω^0.

1. The value F^5 satisfies (E) since Ψ verifies (E).

(a) To prove that F^5 satisfies (NP^*), suppose that u is a TU-game on N and $i \in N$ is a null player in u. We have to prove that for all $k \in \Omega_N$, $F_i^5(v,k) = 0$ where $v = \widetilde{u}_\varpi$ for an arbitrary probability distribution function ϖ on a collection Ω of sample spaces. First suppose that $v \in \mathcal{CC}(N,\Omega^0,\varpi_p)$; that is $\Omega = \Omega^0$ and $\varpi = \varpi_p$. Then $\widehat{v} = \widetilde{u}_{\varpi^0}$ and $F_i^5(v,k) = \Psi_i\left(\widetilde{u}_{\varpi^0},k\right) = 0$ since Ψ satisfies (NP^*). Now, suppose that $v \notin \mathcal{CC}(N,\Omega^0,\varpi_p)$. Then $F_i^5(v,k) = \Psi_i(\widetilde{u}_\varpi,k) = 0$ since Ψ satisfies (NP^*). Thus, we conclude that F^5 satisfies (NP^*).

(b) By noting that $\widehat{u+v} = \widehat{u} + \widehat{v}$ for all $u, v \in \mathcal{CC}(N,\Omega^0,\varpi_p)$, it follows that F^5 verifies (A) since Ψ verifies (A).

(c) Suppose that i and j are two stochastically symmetric players in a c.c. game $v \in \mathcal{CC}(N, \Omega, \varpi)$. Let $k \in \Omega_N$. First suppose that $v \notin \mathcal{CC}(N, \Omega^0, \varpi_p)$. By the definition of F^5, $F^5(v) = \Psi(v)$. Since Ψ verifies $(SSym)$, it follows that $F_i^5(v, k) = F_j^5(v, k)$ for all $k \in \Omega_N$. Now, suppose that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$. By the definition of Ω^0, i and j are such that $v(S \cup \{i\}, l) = v(S \cup \{j\}, l)$ for all $S \subseteq N \backslash \{i, j\}$ and for all $l \in \Omega_{S\cup\{i\})} = \{a, b\} = \Omega_{S\cup\{j\})}$. From v to $\widehat{v}$, only the probability distribution function changes. Therefore, $\widehat{v}(S \cup \{i\}, l) = \widehat{v}(S \cup \{j\}, l)$ for all $S \subseteq N\backslash\{i, j\}$ and for all $l = \{a, b\}$. This proves that i and j are stochastically symmetric in $\widehat{v}$ and that $F^5(v) = \Psi(\widehat{v})$. Since Ψ verifies $(SSym)$, it follows that $F_i^5(v, k) = F_j^5(v, k)$ for all $k \in \Omega_N$. This prove that F^5 satisfies $(SSym)$.

(d) Consider $i \in N$. Pose $S = N\backslash\{i\}$, $v = \Upsilon^{a,S} \in \mathcal{CC}(N, \Omega^0, \varpi_p)$ and $u = v^{S,b,b'}$ where u is obtained from v by only duplicating, in Ω_S, b into b and b'. Note that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$, $\widehat{v} \in \mathcal{CC}(N, \Omega^0, \varpi^0)$ and $u \notin \mathcal{CC}(N, \Omega^0, \varpi_p)$. Also note that,

$$E_u = E_v = \frac{2}{p}\gamma_S + \left(1 - \frac{2}{p}\right)\gamma_N \text{ and } E_{\widehat{v}} = \frac{1}{p}\gamma_S + \left(1 - \frac{1}{p}\right)\gamma_N. \tag{14}$$

Therefore

$$F_i^5(v, a) = \Psi_i(\widehat{v}, a) = Shap_i\left(\frac{1}{p}\gamma_S + \left(1 - \frac{1}{p}\right)\gamma_N\right) = \frac{1}{n} - \frac{1}{np}$$

and

$$F_i^5(u, a) = \Psi_i(u, a) = Shap_i\left(\frac{2}{p}\gamma_S + \left(1 - \frac{2}{p}\right)\gamma_N\right) = \frac{1}{n} - \frac{2}{np}$$

It follows that $F_i^5(u, a) \neq F_i^5(v, a)$. Since u is obtained from v by a duplication of b in Ω_S, we conclude that F^5 does not satisfy (ILD). The proof is thus completed.

□

5.3 Proof of Proposition 10

Proposition 10 *Axioms (E), (A), (NP^*), $(SSym)$ and (MCI) are independent on $\mathcal{CC}(N)$.*

Proof Each of the four values presented in Proposition 7 fails to satisfy exactly one axiom among (E), (A), (NP^*) and $(SSym)$. Each of those four values obviously satisfies (MCI). Now, we have proved that the value F^5 in the proof of Proposition 8 satisfies (E), (A), (NP^*) and $(SSym)$. To prove that F^5 fails to meet (MCI), consider $i \in N$. Pose $S = N\backslash\{i\}$, $v = \Upsilon^{a,S} \in \mathcal{CC}(N, \Omega^0, \varpi_p)$ and $u = v^{S,a\sim b}$ where u is obtained from v by merging, in Ω_S, a and b into a. Note that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$,

$\widehat{v} \in \mathcal{CC}(N, \Omega^0, \varpi^0)$ and $u \notin \mathcal{CC}(N, \Omega^0, \varpi_p)$. Since the expectation game does not change by applying an MC-merging operation, (14) still holds. Therefore,

$$F_i^5(v, a) = \Psi_i(\widehat{v}, a) = Shap_i\left(\frac{1}{p}\gamma_S + \left(1 - \frac{1}{p}\right)\gamma_N\right) = \frac{1}{n} - \frac{1}{np}$$

and

$$F_i^5(u, a) = \Psi_i(u, a) = Shap_i\left(\frac{2}{p}\gamma_S + \left(1 - \frac{2}{p}\right)\gamma_N\right) = \frac{1}{n} - \frac{2}{np}$$

It follows that $F_i^5(u, a) \neq F_i^5(v, a)$. Since u is obtained from v by an MC-merging operation, we conclude that F^5 does not satisfy (MCI). The proof is thus completed. □

5.4 Proof of Proposition 11

Proposition 11 *Axioms (E), (A), (NP^*), $(SSym)$ and (MKI) are independent on $\mathcal{CC}(N)$.*

Proof Each of the four values presented in Proposition 7 fails to satisfy exactly one axiom among (E), (A), (NP^*) and $(SSym)$. Each of those four values obviously satisfies (MKI). Now, we have proved that the value F^5 in the proof of Proposition 8 satisfies (E), (A), (NP^*) and $(SSym)$. To prove that F^5 fails to meet (MKI), consider $i \in N$. Pose $S = N\backslash\{i\}$, $v = \Upsilon^{a,S} \in \mathcal{CC}(N, \Omega^0, \varpi_p)$ and $u = v^{S,a\wedge b}$ where u is obtained from v by an MK-merging operation, in Ω_S. Note that $v \in \mathcal{CC}(N, \Omega^0, \varpi_p)$, $\widehat{v} \in \mathcal{CC}(N, \Omega^0, \varpi^0)$ and $u \in \mathcal{CC}(N, \Omega^0, \varpi_p)$. Since the expectation game does not change by applying an MK-merging operation, (14) still holds. Therefore,

$$F_i^5(v, a) = \Psi_i(\widehat{v}, a) = Shap_i\left(\frac{1}{p}\gamma_S + \left(1 - \frac{1}{p}\right)\gamma_N\right) = \frac{1}{n} - \frac{1}{np}$$

and

$$F_i^5(u, a) = \Psi_i((\widehat{u}, a) = Shap_i\left(\frac{2}{p}\gamma_S + \left(1 - \frac{2}{p}\right)\gamma_N\right) = \frac{1}{n} - \frac{2}{np}$$

It follows that $F_i^5(u, a) \neq F_i^5(v, a)$. Since u is obtained from v by an MK-merging operation, we conclude that F^5 does not satisfy (MKI). The proof is thus completed. □

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