UNIVERSITE DE YAOUNDE 1

Centre de Recherche et de Formation Doctorale en Sciences, Technologie et Géosciences

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Unité de Recherche et de Formation Doctorale en Mathématiques, Informatique, Bioinformatique et Applications

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Postgraduate School of Science, Technology & Geosciences

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A COALGEBRAIC STUDY OF BL-ALGEBRAS

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Dedication

To the God of my Father

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Contents

Abstract

This thesis is intended to develop the theory of coalgebras over an endofunctor by investigating coalgebraic structure of BL-algebras via two functors. The first assigns every BL-algebra to its MV-center, and the second assigns every local BL-algebra to its quotient by its unique maximal filter. Functorial coalgebras have been mainly studied on the category of sets, topological spaces and also in arbitrary categories. Our aim is to prove that the categories of logical algebras are also good candidates as base categories of coalgebras.

We study some categorical properties of BL-algebras and show that BL-algebras have good properties enough to obtain a rich structure of coalgebra on them ((co)completeness, adequate factorization structure, coreflective subcategory) .

We introduce the $\mathcal{MV}$-functor, and investigate its coalgebras. We characterize homomophisms , subcoalgebras, bisimulations and prove that the category of coalgebras of the $\mathcal{MV}$-functor is complete and cocomplete. Moreover, we add a topological structure based on filters of the underlined BL-algebras and obtain topological $\mathbb{MV}$-coalgebras. We construct an inverse system in the category of $\mathbb{MV}$-coalgebras and show that the category of topological $\mathbb{MV}$-coalgebras is complete, cocomplete and strong-monotopological over the category of $\mathbb{MV}$-coalgebras.

We also introduce $\prod$-coalgebras over local BL-algebras (which are BL-algebras with a unique maximal filter) and local BL-frames. We show that the corresponding categories are isomorphic, establishing a link between coalgebras over BL-algebras and modal logic.

Keywords: Coalgebra, BL-algebra, filter, Modal logic, Topological space. .

Résumé

Cette thèse vise à développer la théorie des coalgèbres sur un endofoncteur en explorant la structure coalgébrique des BL-algèbres à travers deux foncteurs. Le premier qui à toute BL-algèbre associe son MV-centre, et le second qui à toute BL-algèbre locale associe son quotient par son unique filtre maximal. Les coalgèbres fonctorielles ont toujours été étudiées principalement sur la catégorie des ensembles , celle des espaces topologiques ou encore sur des catégories arbitraires. Notre but est de prouver que les catégories d'algèbres logiques sont aussi des bons candidats comme catégories de base de coalgèbres.

Nous étudions des propriétés catégoriques des BL-algèbres et nous démontrons qu'elles admettent de bonnes propriétés ((co)complétude, système de factorization adéquat, sous catégories coreflexives) pour obtenir une riche structure de coalgèbre sur elles.

Nous présentons le $\mathcal{MV}$-foncteur et étudions les coalgèbres correspondantes. Nous caractérisons les homomorphismes, les sous-coalgebres, les bisimulations pour ces coalgebres et démontrons que la catégorie des $\mathbb{MV}$-coalgèbres est complète et cocomplète. D'autre part, nous munissons ces coalgèbres d'une structure topologique et obtenons les $\mathbb{MV}$-coalgèbres topologiques. Nous construisons un système inverse dans la catégorie des $\mathbb{MV}$-coalgèbres topologiques et démontrons que la catégorie correspondante est complète, cocomplète et monotopologique forte sur celle des $\mathbb{MV}$-coalgèbres.

Nous introduisons les $\prod$-coalgèbres sur les BL-algèbres locales (ce sont des BL-algèbres qui ne possèdent qu'un seul filtre maximal) et les environnements BL-locaux. Nous démontrons que les catégories correspondantes sont isomorphes, établissant ainsi le lien entre les coalgèbres sur les BL-algèbres et la logique modale.

Mots clés: Coalgèbre, BL-algèbre, filtre, logique modale, espace topologique.

INTRODUCTION

Theoretical computer science is a subset of general computer science and mathematics that focuses on mathematical aspects of computer science such as the theory of computation, lambda calculus, and type theory. It covers a wide variety of topics including algorithms, data structures, computational complexity, parallel and distributed computation, probabilistic computation, quantum computation, automata theory, information theory, cryptography, program semantics and verification, machine learning, computational biology, computational economics, computational geometry, and computational number theory and algebra. Work in this field is often distinguished by its emphasis on mathematical technique and rigor.

In the last decades, coalgebra has arisen as a prominent candidate for a mathematical framework to specify and reason about computer systems. It has found its usefulness as mathematical and categorical presentations of state based systems such as automata. There are several introductory articles to the view of "coalgebras as state based systems" such as Aczel 1989 [1], Barr 1996 [7], Rutten 2000 [49]. We refer to them for a historical account of the field. Coalgebras arise naturally, as Kripke models for modal logic, as objects for object oriented programming languages in computer science, etc. Till now coalgebras have been mainly studied on the category $\mathcal{SET}$ of sets and mappings (see e.g. Adamek 2010 [3], Gumm 2001 [24], Gumm 2002 [21], Gumm 2005 [22]), on topological spaces (mesurable spaces [50], Haussdorf spaces [34], Stone spaces [40]) and also arbitrary categories (see, Ad*á*mek 2005 [4], Hughes (2001) [32], Kianpi 2020 [36]).

In his thesis [41], Alexander Kurz pointed out the importance of the study of coalgebras over various specific categories. One reason is to provide examples for the general study of coalgebras, another is the construction of a large variety of predicate liftings for coalgebraic logics and many more type of automata. In [14] Cabrera and al defined a new kind of coalgebras named ND-coalgebras which allows to formalize non-determinism and show that several concepts, widely used in computer science, are indeed ND-coalgebras. In [16] Davey and Galati gave a coalgebraic view of the restricted Priestley duality between Heyting algebras and Heyting spaces.

One of the aims of this thesis is to argue that, besides $\mathcal{SET}$ and $\mathcal{TOP}$ the category of topological spaces, the categories of logical algebras in general, and that of BL-algebras in particular are interesting base categories for coalgebras. For this purpose, we investigate the category of coalgebras over this algebraic structure, both for a deterministic and a non deterministic cases.

BL-algebras were invented by Petr Há́jek (1998) [27] in order to prove the completeness theorem of basic fuzzy logic, BL-logic in short. The language of propositional Há́jek's basic logic contains the binary connectives $\circ$ and $\Rightarrow$ and the constant $\bar{0}$. Axioms of basic fuzzy logic are:

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow \omega))$

(A2) $(\varphi \circ \psi) \Rightarrow \varphi$

(A3) $(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$

(A4) $(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$

(A5a) $(\varphi \Rightarrow (\psi \Rightarrow \omega)) \Rightarrow ((\varphi \circ \psi) \Rightarrow \omega)$

(A5b) $((\varphi \circ \psi) \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \omega))$

(A6) $((\varphi \Rightarrow \psi) \Rightarrow \omega) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \omega) \Rightarrow \omega))$

(A7) $\bar{0} \Rightarrow \omega$.

Soon after Cignoli et al. (2000) [15] proved that Há́jek's logic really is the logic of continuous t-norms as conjectured by Há́jek in [27]. At the same time, started a systematic study of BL-algebras (see Turunen (2001) [50], Di Nola (2003) [18], Haveshki 2008 [29], Turunen (2011) [52]), mainly via deductive systems, also called filters, which correspond to subsets closed under modus ponens. The main objective of this thesis, is to investigate BL-algebras with coalgebraic techniques.

For an efficient study of coalgebras over a category, we must have enough information on the latter ((co)completeness, special morphisms, factorization systems, subcategories) in order to be able to deduce some properties of the corresponding categories of coalgebras. In chapter 2, we study the category of BL-algebras and BL-morphisms as a concrete category over $\mathcal{SET}$, and prove that it has very nice properties such as essential algebraicity. However, we also show that this category fails to be cartesian closed and topological. Moreover, we investigate the relation between the category of BL-algebras and two of its most studied subcategories, namely the category of MV-algebras and the category of Gödel-algebras.

In [24] , [25] and [23] H. P. Gumm et al. have introduced, in the setting of $\mathcal{SET}$, some $\mathcal{SET}$-functors preserving weak pullbacks as they provide the basis of an rich structure theory of coalgebras. In chapter 3, we show that this kind of results also hold on coalgebras over BL-algebras by presenting a limit-preserving non-trivial endofunctor on the category of BL-algebras and BL-morphisms and characterize the corresponding coalgebras.

The connection of coalgebra with modal logic is one of the main reasons why coalgebras are studied. Indeed, since coalgebras can be seen as a very general model of state based systems, and modal logics as logic for dynamical systems, there is a tight relation between modal logic and coalgebras. It is well known that the category of Kripke frames and p-morphisms is isomorphic to the category of coalgebras of the covariant powerset functor (see, e.g Bezhanishvili 2010 [9] and [10]). In [40] Kupke et al. prove that the category of descriptive general frames is isomorphic to the category of coalgebras of the Vietoris functor. In chapter 4, we show a similar result for coalgebras over local BL-algebras.

The outline of this thesis is as follows:

Chapter 1: In this chapter, we introduce basic results about BL-algebras and the categorical definition of coalgebra. We also present some basic properties of modal logic and topological algebras which we will used in our work.

Chapter 2: This chapter is devoted to the study of the category of BL-algebras as a concrete category. We show how some limits and colimits are constructed and prove that the category of BL-algebras is essentially algebraic, but is neither topological nor cartesian closed. Moreover, we establish a hierarchy for some types of momomorphisms on the one hand and for some types of epimorphisms on the other hand. Finally, we study the relations between the category of BL-algebras and two of its subcategories: we show that the category of Gödel-algebras is an isomorphism-closed subcategory of the category of BL-algebras, and the category of MV-algebras is a coreflective subcategory of the category of BL-algebras.

Chapter 3: In this chapter, we introduce MV-coalgebras, which are coalgebras of the functor which assigns to every BL-algebra its MV-center. We prove the (co)completeness of the category of MV-coalgebras and characterize homomorphisms, MV-subcoalgebras and bisimulations. Moreover, we investigate topological MV-coalgebras and construct an inverse system in the category of MV-coalgebras.

Chapter 4: This chapter is devoted to the link between BL-algebras and modal logic. We introduce a new type of modal frames (models), namely local BL-frames and show that the category of local BL-frames is isomorphic to the category of $\prod$-coalgebras over local BL-algebras.

Chapter 1

PRELIMINARIES

Our purpose in this chapter is to summarize, the relevant materials on category theory, coalgebras and BL-algebras that are needed in the rest of the thesis. We also introduce some basic facts on modal logic and topology.

1.1 Categorical notions

In this section, we present some basic category theoretic results notions and facts. Most of them can be found in Adámek 1990 [2] or Maclane (1998) [43].

In addition to mathematical objects modern mathematics investigates more and more the maps defined between them. One familiar example is given by sets. Besides the sets, which form the mathematical objects in set theory, the set maps are very important. Much information about a set is available if only the maps into this set from all other sets are known. For example, the set containing only one element can be characterized by the fact that, from every other set, there is exactly one map into this set. Let us first summarize in a definition those properties of mathematical objects and maps.

1.1.1 Categories

Definition 1.1.1. A *category* is a quadruple $\mathcal{C} = (\mathrm{Ob(C)}, Hom, id, \circ)$ consisting of

1. a class $Ob(\mathcal{C})$, whose members are called $\mathcal{C}$-objects,
2. for each pair (A, B) of $\mathcal{C}$-*objects*, a set $Hom(A, B)$, whose members are called $\mathcal{C}$-*morphisms* from A to B (the statement "$f \in Hom(A, B)$" is expressed more graphically by using arrows; e.g., by statements such as "$f : A \longrightarrow B$ is a morphism " or "$A \xrightarrow{f} B$ is a morphism"),
3. for each $\mathcal{C}$-object A, a morphism $A \xrightarrow{id_A} A$, called the A-*identity* on A,
4. a composition law associating with each $\mathcal{C}$-morphism $A \xrightarrow{f} B$ and each A-morphism $B \xrightarrow{g} C$ a $\mathcal{C}$-morphism $A \xrightarrow{g \circ f} C$, called the *composite* of f and g, subject to the following conditions:

(a) composition is associative; i.e., for morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $C \xrightarrow{h} D$, the equation $h \circ (g \circ f) = (h \circ g) \circ f$ holds,

(b) $\mathcal{A}$-identities act as identities with respect to composition; i.e., for $\mathcal{C}$-morphisms $A \xrightarrow{f} B$, we have $id_B \circ f = f$ and $f \circ id_A = f$,

(c) the sets $Hom(A, B)$ are pairwise disjoint.

Example 1.1.2. The following are categories:

1. The category $\mathcal{SET}$ whose object class is the class of all sets; $Hom(A, B)$ is the set of all functions from A to B.

2. The following constructs; i.e., categories of structured sets and structure-preserving functions between them :

   (a) $\mathcal{VEC}$ with objects all real vector spaces and morphisms all linear transformations between them.

   (b) $\mathcal{GRP}$ with objects all groups and morphisms all homomorphisms between them

   (c) $\mathcal{TOP}$ with objects all topological spaces and morphisms all continuous functions between them.

   (d) $\mathcal{REL}$ with objects all pairs (X, ρ), where X is a set and ρ is a (binary) relation on X. Morphisms $f : (X, \rho) \longrightarrow (Y, \mu)$ are relation-preserving maps; i.e., maps $f : X \longrightarrow Y$ such that $x\rho y$ implies $f(x)\mu f(y)$.

1.1.2 Functors and natural transformations

If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a *functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a function that assigns to each $\mathcal{C}$-object A a $\mathcal{D}$-object $F(A)$, and to each $\mathcal{C}$-morphism $A \xrightarrow{f} B$ a $\mathcal{D}$-morphism $F(A) \xrightarrow{F(f)} F(B)$, in such a way that

(i) F preserves composition, i.e., $F(f \circ g)) = F(f) \circ F(g)$ whenever $f \circ g$ is defined, and

(ii) F preserves identity morphisms, i.e., $F(id_A) = id_{F(A)}$ for each $\mathcal{C}$-object A.

Definition 1.1.3. (i) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *faithful* provided that all the hom-set restrictions

$$F : hom_{\mathcal{C}}(A, A') \longrightarrow hom_{\mathcal{D}}(F(A), F(A'))$$

are injective.

(ii) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called an *isomorphism* provided that there is a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $G \circ F = id_{\mathcal{C}}$ and $F \circ G = id_{\mathcal{D}}$.

(iii) The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic* provided that there is an isomorphism $F : \mathcal{C} \longrightarrow \mathcal{D}$.

Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors, a transformation $\eta : F \longrightarrow G$ is a class of morphisms $(\eta_X : F(X) \longrightarrow G(X))_{X \in \mathcal{C}}$ in $\mathcal{D}$, called *components of the transformation.* Saying mono, epi or iso about a transformation, we mean a component-wise such. Every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ gives rise to a square in $\mathcal{D}$ called the *transformation square* for f:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array} .$$

η is called natural in case it is commutative, i.e. $G(f) \circ \eta_X = \eta_Y \circ F(f)$. When the natural transformation η is iso, then F and G are said *naturally isomorphic* and it is denoted by $F \cong G$.

Definition 1.1.4. A functor $L : \mathcal{C} \longrightarrow \mathcal{D}$ is said to be *left adjoint* to a functor $R : \mathcal{D} \longrightarrow \mathcal{C}$ if there exists a natural isomorphism $Hom_{\mathcal{D}}(L(-), -) \cong Hom_{\mathcal{C}}(-, R(-))$, i.e., for any $\mathcal{C}$-object A and $\mathcal{D}$-object B, there is a natural bijection

$$\phi_{A,B} : Hom_{\mathcal{D}}(L(A), B) \longrightarrow Hom_{\mathcal{C}}(A, R(B)).$$

In this case, L is a *left adjoint functor.* A *right adjoint functor* is the dual notion, so a functor is left adjoint if it has a right adjoint and vice versa. We use the notation $L \dashv R$ to express "L is left adjoint to R".

Example 1.1.5. 1. For any category $\mathcal{C}$, there is the identity functor $Id_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ defined by $Id_{\mathcal{C}}(A \xrightarrow{f} B) = A \xrightarrow{f} B$.

2. For any of the constructs $\mathcal{C}$ mentioned above (Example 1.1.2) there is the *forgetful* functor (or *underlying* functor) $U : \mathcal{C} \longrightarrow \mathcal{SET}$, where in each case $U(A)$ is the underlying set of A, and $U(f) = f$ is the underlying function of the morphism f. This forgetful functor is an example of faithful and adjoint functor.

3. The covariant *power-set* functor $P : \mathcal{SET} \longrightarrow \mathcal{SET}$ is defined by

$$P(A \xrightarrow{f} B) = P(A) \xrightarrow{P(f)} P(B)$$

where $P(A)$ is the power-set of A;i.e. the set of all subsets of A; and for each $X \subseteq A$, $P(f)(X)$ is the image $f(X)$ of X under f.

1.1.3 Some special morphisms

Most of the main definitions and results in chapter [2] and chapter [3] of this thesis are based on special morphisms. In this subsection, we define them and give some properties thereof. They can be found in Adámek 1990 [2] for instance.

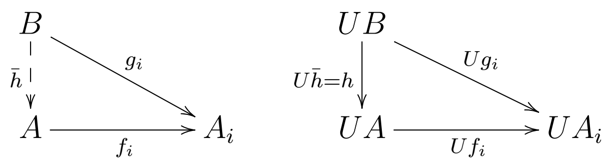
Definition 1.1.6. Let $\mathcal{C}$ be a category.

1. A *source* in $\mathcal{C}$ is a pair $(A, (f_i)_{i \in I})$ consisting of a $\mathcal{C}$-object A and a family of $\mathcal{C}$-morphisms $f_i : A \longrightarrow A_i$ with domain A, indexed by some class I. A is called the *domain of the source* and the family $(A_i)_{i\in I}$ is called the *codomain of the source* .

2. A *source* $(A, (f_i)_{i\in I})$ is called a *mono-source* provided that for all pair $r, s : B \longrightarrow A$ of $\mathcal{C}$-morphisms,

$$f_i \circ r = f_i \circ s \text{ for each } i \in I \text{ implies } r = s.$$

3. Let $U : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. A source $\mathsf{S} = (A \xrightarrow{f_i} A_i)_I$ in $\mathcal{C}$ is called U- *initial* provided that for each source $\mathsf{T} = (B \xrightarrow{g_i} A_i)_I$ in $\mathcal{C}$ with the same codomain as S and each $\mathcal{D}$-morphism $UB \xrightarrow{h} UA$ with $U\mathsf{T} = U\mathsf{S} \circ h$, there exists a unique $\mathcal{C}$-morphism $B \xrightarrow{\bar{h}} A$ with $\mathsf{T} = \mathsf{S} \circ \bar{h}$ and $h = U\bar{h}$.



Definition 1.1.7. Let $\mathcal{C}$ be a category.

1. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is called a *regular epimorphism* if it is a coequalizer of some parallel pair of C-morphisms.

2. A monomorphism $m : C \longrightarrow D$ in $\mathcal{C}$ is called:

   (i) *strong* provided that for all epimorphism $e : A \longrightarrow B$ and all morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow D$ such that $g \circ e = m \circ f$, there exists a unique morphism $d : B \longrightarrow D$ such that $d \circ e = f$ and $m \circ d = g$.

   (ii) extremal provided that whenever $f = m \circ e$, where e is an epimorphism, then e must be an isomorphism.

3. The concepts of *strong (extremal) epimorphism* and *regular monomorphism* in $\mathcal{C}$ are define dually.

4. The kernel equivalence of a morphism is the pullback of that morphism with itself.

5. Let Ω be a class of morphisms in $\mathcal{C}$. A functor $\mathbb{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is said to

   (i) *preserve* morphisms of the class Ω, if $F(f)$ is in Ω whenever f is.

   (ii) *create* morphisms of the class Ω if for any $\mathcal{D}$-morphism in g in Ω, there exists a unique $\mathcal{C}$-morphism f in Ω such that $F(f) = g$.

1.1.4 Factorization structures

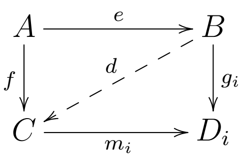
In this section we present factorization structures for sources. These notions will be mainly used in chapter 2.

Definition 1.1.8. Let E be a class of morphisms and let $\mathbf{M}$ be a conglomerate of sources in a category $\mathcal{C}$. $(E, \mathbf{M})$ is called a *factorization structure* on $\mathcal{C}$, and $\mathcal{C}$ is called an $(E, \mathbf{M})$-*category* provided that:

1. each of E and M is closed under compositions with isomorphisms

(ii) $\mathcal{C}$ has $(E, \mathbf{M})$-factorizations (of sources); i.e., each source S in $\mathcal{C}$ has a factorization $S = M \circ e$, with $e \in E$ and $M \in \mathbf{M}$.

(iii) $\mathcal{C}$ has the unique $(E, \mathbf{M})$-diagonalization property; i.e., whenever $A \xrightarrow{e} B$ and $A \xrightarrow{f} C$ are $\mathcal{C}$-morphisms with $e \in E$ and $(S = B \xrightarrow{g_i} D_i)_I$ and $(M = C \xrightarrow{m_i} D_i)_I$ are sources in $\mathcal{C}$ with $M \in \mathbf{M}$ such that $M \circ f = S \circ e$, then there exists a unique diagonal, i.e., a morphism $B \xrightarrow{d} C$ such that for each $i \in I$ the diagram



commutes.

Example 1.1.9. *1. Every category is an (Iso, Source)-category. This factorization structure is called trivial. Also, every category has the (unique) (RegEpi, Mono-Source)-diagonalization property.*

2. $\mathcal{SET}$ is an (Epi, Mono-Source)-category. For Set this is the only nontrivial factorization structure.

3. $\mathcal{VEC}$ and $\mathcal{GRP}$ are (RegEpi, Mono-Source)-categories.

Remark 1.1.10. *[2]* If $(E, \mathbf{M})$ is a factorization structure on $\mathcal{C}$ and $\mathbf{M}$ is the class of those $\mathcal{C}$-morphisms that (considered as 1-sources) belong to $\mathbf{M}$, then $(E, \mathbf{M})$ is a factorization structure for morphisms on $\mathcal{C}$.

It is well known (see e.g. [2], Corollary 7.63) that in any category, regular monomorphisms are extremal and by [[2], exercise 14C.f] we have:

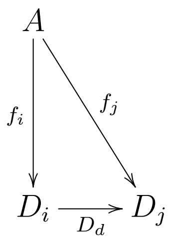
Lemma 1.1.11. *If $\mathcal{C}$ is $(Epi, \boldsymbol{M})$-structured for some class $\boldsymbol{M}$ of monomorphisms, then strong monomorphisms in $\mathcal{C}$ are precisely extremal monomorphisms.*

1.1.5 Limits and colimits

A *diagram* in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \longrightarrow \mathcal{C}$ with codomain $\mathcal{C}$. The domain, $\mathcal{I}$, is called the *scheme* of the diagram. A diagram with a small (or finite) scheme is said to be small (or finite).

Definition 1.1.12. Let $D : \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram.

(i) A $\mathcal{C}$-source $(A \xrightarrow{f_i} D_i)_{i\in Ob(\mathcal{I})}$ is said to be *natural* for D provided that for each $\mathcal{I}$-morphism $i \xrightarrow{d} j$, the triangle



commutes.

(ii) A *limit* of D is a natural source $(L \xrightarrow{l_i} D_i)_{i\in Ob(\mathcal{I})}$ for D with the (universal) property that each natural source $(A \xrightarrow{f_i} D_i)_{i\in Ob(\mathcal{I})}$ for D uniquely factors through it; i.e., for every such source there exists a unique morphism $f : A \longrightarrow L$ with $f_i = l_i \circ f$ for each $i \in Ob(I)$.

Definition 1.1.13. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to *preserve a limit* $L = (L \xrightarrow{l_i} D_i)_{i\in Ob(\mathcal{I})}$ of a diagram $D : \mathcal{I} \longrightarrow \mathcal{C}$ provided that $FL = (FL \xrightarrow{Fl_i} FD_i)_{i\in Ob(\mathcal{I})}$ is a limit of the diagram $F \circ D : \mathcal{I} \longrightarrow \mathcal{D}$.

Definition 1.1.14. A category $\mathcal{C}$ is said to be *complete* if for each small diagram in $\mathcal{C}$ there exists a limit.

Colimits and cocompleteness are defined dually.

Example 1.1.15. A (initial) terminal object is a particular of (co)limit.

An object A is called a *terminal* (resp. *initial*) object provided that for each object B there is exactly one morphism from B to A (resp. A to B).

1. Every singleton set is a terminal object for Set.

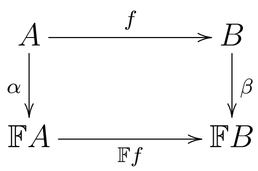
2. Frequently for constructs, there is only one structure on the singleton set, and in these cases the corresponding object is a terminal object. This is the case, for example, in $\mathcal{VEC}$, $\mathcal{GRP}$ and $\mathcal{TOP}$.
3. The empty set $\emptyset$ is the unique initial object for Set $\mathcal{SET}$.
4. Every one-element group is an initial object for $\mathcal{GRP}$; likewise for $\mathcal{VEC}$.

1.2 Coalgebras

In this section, we introduce the main notion of the thesis and give some basic insights thereof.

1.2.1 Definition and examples

Definition 1.2.1. Let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be a functor from the category $\mathcal{C}$ to itself. A *coalgebra* of type $\mathbb{F}$ is a pair $\mathbf{A} = (A, \alpha)$, consisting of a $\mathcal{C}$-object A and a $\mathcal{C}$-morphism $\alpha : A \longrightarrow \mathbb{F}A$. A is called the *carrier* and α is called the *structure morphism* of $\mathbf{A}$. If $\mathbf{A} = (A, \alpha)$ and $\mathbf{B} = (B, \beta)$ are $\mathbb{F}$ -coalgebras, then a map $f : A \longrightarrow B$ is called a *homomorphism*, if $\beta \circ f = \mathbb{F}(f) \circ \alpha$, i.e. such that the following diagram commutes:



Coalgebras of type $\mathbb{F}$ and homomorphisms between them form a category which will be denoted by $Coalg(\mathbb{F})$. Below are some examples of coalgebras.

Example 1.2.2. When $\mathcal{C} = \mathcal{SET}$,

1. The identity functor is the functor $\mathbf{Id} : \mathcal{SET} \longrightarrow \mathcal{SET}$ defined by $\mathbf{Id}(X) = X$ and $\mathbf{Id}(f) = f$ for every set X and every map $f : X \longrightarrow Y$. $\mathbf{Id}$-coalgebras are just pairs (X, α) where $\alpha : X \longrightarrow X$ is a map.
2. Let $\mathbf{P}$ be the covariant *powerset functor* defined by $\mathbf{P}(X) := \{A \mid A \subseteq X\}$, and for all map $f : X \longrightarrow Y$, $\mathbf{P}(f) : \mathbf{P}(X) \longrightarrow \mathbf{P}(Y)$ where $\mathbf{P}(f)(A) = f(A)$. $\mathbf{P}$-coalgebras are called non-deterministic transition systems. $\mathbf{P}$-coalgebras are just pairs (X, α) where $\alpha : X \longrightarrow \mathbf{P}(X)$ is a unary hyperoperation.
3. We take this example from Gumm 2001 [24]. Let X be a set. A filter on $\mathbf{P}(X)$ is a collection $U \subseteq \mathbf{P}(X)$ if U is closed under finite intersections and supersets. In other words, U is a filter on $\mathbf{P}(X)$ just in case
   - If $S, T \in U$, then $S \cap T \in U$, and
   - If If $S \in U$ and $S \subseteq T$, then If $T \in U$.

We define a functor $\mathbf{F} : \mathcal{SET} \longrightarrow \mathcal{SET}$ taking each set X to the collection of filters on X. If $f : X \longrightarrow Y$ is a map in SET, then for each $A \in \mathbf{F}(X)$, $\mathbf{F}(f)(A)$ is the filter generated by $\mathbf{F}(f)(X)$. $\mathbf{F}$ is called the *filter functo*r on SET. Each topological space (X, τ_X) gives rise to an $\mathbf{F}$-coalgebra, as follows. We define the structure map $\alpha : X \longrightarrow \mathbf{F}X$ on elements $x \in X$ by

$$\alpha(x) = \{A \subseteq X \mid \exists U \in \tau_X,\ x \in U \subseteq A\}.$$

4. An important example of coalgebras are labeled transition systems consisting of a state set Q and transitions $q \stackrel{s}{\longrightarrow} \bar{q}$ for $q, \bar{q} \in Q$ and $s \in \sum$ (where $\sum$ is the set of possible actions). More precisely, for every action s a binary relation $\stackrel{s}{\longrightarrow}$ is given on Q. This can be viewed as a coalgebra for the functor $\mathbf{P}(\sum \times -)$. In fact, define $\alpha : Q \longrightarrow \mathbf{P}(\sum \times -)$ by assigning to every state q the set $\alpha(q)$ of all pairs $(s, \bar{q}) \in \sum \times Q$ with $q \stackrel{s}{\longrightarrow} \bar{q}$. Coalgebra homomorphisms $f : (Q, \alpha) \longrightarrow (Q', \alpha')$ are precisely the functions which preserve and reflect transitions.

Example 1.2.3. Let $\mathcal{T}op$ denote the category of topological spaces and continuous maps, and $\mathcal{S}tone$ the category of Stone spaces and continuous maps (recall that a topological space $X = (X, \tau)$ is a Stone space if it is compact Hausdorff and has a basis of clopen sets.):

1. Hughes 2001 [32] Consider the functor $\Gamma(A) = Z \times A$ on the category $\mathcal{T}op$, where Z is a fixed T_1 space (so points are topologically distinguishable). A Γ-coalgebra consists of a pair (A, α) where A is a topological space and $\alpha : A \longrightarrow \Gamma A$ is continuous. Let I be the unit interval $[0, 1]$. Then a Γ-coalgebra with carrier I is just a path in the space $Z \times I$.

2. Let $\mathbf{X} = (X, \tau)$ be a topological space. We let $K(\mathbf{X})$ denote the collection of all closed subsets of X. Define for any subset U of X the sets

$$\hat{U} := \{F \in K(\mathbf{X}) \mid F \subseteq U\}$$

and

$$\check{U} := \{F \in K(\mathbf{X}) \mid F \cap U \neq \emptyset\}.$$

Given a subset $Q \subseteq \mathbf{P}(X)$, define

$$V_Q = \left\{\hat{U} \mid U \in Q\right\} \cup \left\{\check{U} \mid U \in Q\right\}.$$

The Vietoris Space $\mathbf{V(X)}$ associated with $\mathbf{X}$ is given by the topology $\upsilon_{\mathbf{X}}$ on $K(\mathbf{X})$ which is generated by V_τ as subbasis. The Vietoris functor $\mathbb{V} : \mathcal{S}tone \longrightarrow \mathcal{S}tone$ assigns each Stone space to its Vietoris Space and every continuous morphism $f : \mathbf{X} \longrightarrow \mathbf{X'}$ to $\mathbb{V}(f) : K(\mathbf{X}) \longrightarrow K(\mathbf{X'})$ by

$$\mathbb{V}(f)(F) := f(F).$$

Coalgebras of the Vietoris functor are called *Vietoris-coalgebras*. It follows from the duality between $\mathcal{C}oalg(\mathbb{V})$ and the category $\mathcal{MA}$ of modal algebras (see Kurz (2000) [41]) that $\mathcal{C}oalg(\mathbb{V})$ provides an adequate semantic for finitary modal logics.

1.2.2 Limits and colimits in the category of coalgebras

In what follows, some results about the construction of limits and colimits in categories of coalgebras are introduced. They can be found in Hughes (2001) [32], Adámek (2005) [4] .

Just as in algebraic semantics the initial algebra plays a central role, in coalgebra the terminal coalgebras (i.e., terminal objects of the category of coalgebras) are of major importance. Recall that a terminal **F**-coalgebra is a coalgebra $T \stackrel{\epsilon}{\longrightarrow} \mathbf{F}T$ such that for every coalgebra $Q \stackrel{\alpha}{\longrightarrow} \mathbf{F}Q$, there exists a unique homomorphism $\lambda : Q \longrightarrow T$.

Theorem 1.2.4. *The coalgebraic forgetful functor* $\mathbb{U} : \mathcal{C}oalg(\mathbb{F}) \longrightarrow \mathcal{C}$ *creates colimits and all limits preserved by* $\mathbb{F}$.

This yields the following consequence:

Corollary 1.2.5. *Let* $\mathcal{C}$ *be an arbitrary category.*

1. *If* $\mathcal{C}$ *is cocomplete, then so is* $\mathcal{C}oalg(\mathbb{F})$.
2. *A homomorphism in* $\mathcal{C}oalg(\mathbb{F})$ *is an isomorphism (resp. epimorphism) iff its underlying morphism is an isomorphism (resp. an epimorphism).*

Monomorphisms in $\mathcal{C}oalg(\mathbb{F})$ are more difficult (dually to the diffculties with epimorphisms well known from General Algebra) to characterize in coalgebra theory. It is clear that every homomorphism carried by a monomorphism in $\mathcal{C}$ is a monomorphism in $\mathcal{C}oalg(\mathbb{F})$ (since U is faithful), but not conversely. for example, if $\mathcal{C} = \mathcal{SET}$, then homomorphisms which are injective maps (i.e., are monomorphisms in A) are precisely the regular monomorphisms of $\mathcal{C}oalg(\mathbb{F})$, see Theorem 3.4 in Gumm (1999) [26];

A *subcoalgebra* of a coalgebra (A, α) is a strong subobject in $\mathcal{C}oalg(\mathbb{F})$, i.e., one represented by a strong monomorphism with codomain (A, α). This is substantiated by the following:

Lemma 1.2.6. *[4] If* $\mathcal{C}$ *has (epi, strong mono)-factorizations and* $\mathbb{F}$ *preserves strong monomorphisms, then strong monomorphisms in* $\mathcal{C}oalg(\mathbb{F})$ *are precisely the morphisms carried by strong monomorphisms in* $\mathcal{C}$.

1.2.3 Bisimulation

Motivating Example: Consider labelled transition systems as coalgebras of $\mathbf{P}(\sum \times -)$. The concept of (strong) bisimulation goes back to R. Milner [44]: it is an equivalence between states "based intuitively on the idea that we wish to distinguish between two states if the distinction can be detected by an external agent interacting with each of them". Formally, a state a_1 in a labeled transition system A_1 is bisimilar to a state a_2 in A_2 provided that there exists a relation R between the state sets A_1 and A_2 such that:

(i) a_1 is related to a_2, i.e., $a_1 R a_2$

(ii) for every related pair b_1Rb_2 and every transition $b_1 \overset{s}{\longrightarrow} b'_1$ in A_1 there exists a transition $b_2 \overset{s}{\longrightarrow} b'_2$ in A_2 with $b'_1Rb'_2$ and

(iii) for every related pair b_1Rb_2 and every transition $b_2 \overset{s}{\longrightarrow} b'_2$ in A_2 there exists a transition $b_1 \overset{s}{\longrightarrow} b'_1$ in A_1 with $b_1Rb'_1$. The conditions (ii) and (iii) can be elegantly summarized by saying that there is a dynamics on the relation R, i.e., a function $R \longrightarrow \mathbf{P}(R)$ for which both projections $r_i : R \longrightarrow A_i$, $i(i = 1, 2)$ become coalgebra homomorphisms.

Definition 1.2.7. A *bisimulation* between $\mathbb{F}$-coalgebras (A, α) and (B, β) is a relation $r_i : R \longrightarrow A_i$, $(i = 1, 2)$ such that there exists a dynamics on R making both r_1 and r_2 homomorphisms of $\mathbb{F}$-coalgebras.

Proposition 1.2.8. *[4] Let $\mathcal{C}$ be a well-powered, complete category with coproducts, and let $\mathbb{F}$ be an endofunctor preserving strong monomorphisms. Then for every pair of coalgebras there exists a largest bisimulation between them.*

Corollary 1.2.9. *[4] If H moreover preserves weak pullbacks, then on every coalgebra there exists a largest bisimulation which is an equivalence relation.*

1.3 BL-algebras

BL -algebras were invented by P. Hajek [27] in order to provide an algebraic proof of the completeness theorem of basic logic (*BL* , for short), arising from the continuous triangular norms, familiar in the fuzzy logic framework [15]. We present in this section some basic facts about BL-algebras, which can be found in Turunen (1999) [51].

1.3.1 Definitions and examples

An algebraic structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *BL-algebra* if it satisfies the following conditions:

(BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(BL2) $(L, *, 1)$ is a commutative monoid;

(BL3) $*$ is a left adjoint of $\rightarrow$, that is $x * z \leq y$ if and only if $z \leq x \rightarrow y$;

(BL4) $x \wedge y = x * (x \rightarrow y)$;

(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A BL-algebra L is called a *BL-chain* if it is totally ordered and a *Gödel algebra* if $x^2 = x * x = x$ for every $x \in L$. L is called an MV-*algebra* if $\bar{\bar{x}} = x$ for all $x \in L$, where $\bar{x} = x \rightarrow 0$. The subset $MV(L) = \{\bar{x}/x \in L\}$ is called the MV-*center* of L. It is the greatest MV-algebra contained in L.

The following holds in any BL-algebra L:

Lemma 1.3.1. *For all $x, y, z \in L$*

(1) $x \leq y$ *if and only if* $x \rightarrow y = 1$*;*

(2) $x * y \leq x \wedge y$*;*

(3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$*;*

(4) If $x \leq y$ *then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$*;*

(5) $x \leq y \rightarrow (x * y)$*;* $x * (x \rightarrow y) \leq y$*;*

(6) $x * \bar{x} = 0$*;*

(7) $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$*;*

(8) $1 \rightarrow x = x$*;* $x \rightarrow 1 = 1$*;* $x \rightarrow x = 1$*;* $x \leq y \rightarrow x$*;* $x \leq \bar{\bar{x}}$*;* $\bar{\bar{\bar{x}}} = \bar{x}$*;*

(9) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$*;*

(10) $x \rightarrow \bar{y} = y \rightarrow \bar{x}$ *.*

Let L_1 and L_2 be two BL-algebras, a map $f : L_1 \longrightarrow L_2$ is called a *homomorphism of BL-algebras (BL-morphism)*, if $f(0) = 0$ and

$$f(x \propto y) = f(x) \propto f(y) \text{ for all } \propto \in \{*, \rightarrow\}.$$

We obviously have $f(1) = 1$ for any BL-morphism f and it is shown that for any BL-morphism f, $f(x \propto y) = f(x) \propto f(y)$ with $\propto \in \{\vee, \wedge\}$ and if $x \leq y$ then $f(x) \leq f(y)$.

The class of BL-algebras, equipped with BL-morphisms forms a category which will be denoted by $\mathcal{BL}$.

Example 1.3.2. BL-algebras with less than 3 elements:

- The one-element BL-algebra $\{0 = 1\}$ is called the *degenerate BL-algebra* [[51], Remark 8], we will denote it by $\mathbf{G}_1$. The two-element non degenerate BL-algebra $\{0, 1\}$ is called the *trivial BL-algebra*, we will denote it by $\mathbf{G}_2$. These two algebras are examples of BL-algebras which are both Gödel-algebras and MV-algebras.
- The chain $\{0, x, 1\}$, with the operations $*$ and $\rightarrow$ defined by the following tables

$*$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

$\rightarrow$	0	x	1
0	1	1	1
x	0	1	1
1	0	x	1

  is the unique Gödel-algebra with three elements and we will denote it by $\mathbf{G}_3$.
- The chain $\{0, x, 1\}$, with the operations $*$ and $\rightarrow$ defined by the following tables:

$*$	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

$\rightarrow$	0	x	1
0	1	1	1
x	x	1	1
1	0	x	1

is the unique MV-algebra with three elements and we will denote it by $\mathbf{M}_3$.

Let $\mathcal{BL}(L, L')$ denote the set of BL-morphisms from L to L'. The following observations will be useful in the sequel:

Lemma 1.3.3. *For $\boldsymbol{G}_2$, $\boldsymbol{G}_3$ and $\boldsymbol{M}_3$ defined as above we have:*

(i) $\mathcal{BL}(\boldsymbol{G}_3, \boldsymbol{M}_3)$ is a singleton and $\mathcal{BL}(\boldsymbol{M}_3, \boldsymbol{G}_3) = \emptyset$

(ii) $MV(\boldsymbol{G}_3) = \boldsymbol{G}_2$

Proof. (i) Let $\mathbf{G}_3 \xrightarrow{f} \mathbf{M}_3$ be a map such that $f(0) = 0$, $f(1) = 1$. If $f(x) = 0$, then

$$f(x \to 0) = f(0) = 0 \neq 1 = f(x) \to f(0)$$

and for $f(x) = x$,

$$f(x * x) = f(x) = x \neq 0 = f(x) * f(x).$$

Hence in the both cases f is not a BL-morphism. For $f(x) = 1$, it easily checked that f preserves $*$ and $\to$ and so it is the unique BL-morphism from $\mathbf{G}_3$ to $\mathbf{M}_3$. With similar computations, one proves that there is no BL-morphism from $\mathbf{M}_3$ to $\mathbf{G}_3$.

(ii) Straightforward.

□

Example 1.3.4. The set $\{0, a, b, c, d, 1\}$, with the operations $*$ and $\to$ defined by the following tables:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

is an example of finite BL-algebra which is not a chain.

Example 1.3.5. A *t-norm* is a binary operation $*$ on $[0, 1]$ (i.e. $* : [0, 1]^2 \longrightarrow [0, 1]$) satisfying the following conditions: for all $x, x', y \in [0, 1]$

(i) $*$ is commutative and associative

(ii) $*$ is non-decreasing in both arguments, i.e. $x \leq x'$ implies $x * y \leq x' * y$ and $x \leq x'$ implies $y * x \leq y * x'$

(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0, 1]$.

$*$ is a *continuous t-norm* if it is a t-norm and is a continuous mapping of $* : [0,1]^2 \longrightarrow [0,1]$ (in the usual sense).

t-algebras $([0,1], \wedge, \vee, *_t, \rightarrow_t, 0, 1)$, where $([0,1], \wedge, \vee, 0, 1)$ is the usual lattice on the real unit interval $[0,1]$ *and* $*_t$ is a continuous t-norm, whereas $\rightarrow_t$ is the corresponding residuum are BL-algebras. The most known t-algebras are the following:

$$\textit{Gödel algebra}\text{: } x *_t y = min\{x, y\}, \text{ and } x \rightarrow_t y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}.$$

$$\textit{Product algebra}\text{: } x *_t y = xy, \text{ and } x \rightarrow_t y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}.$$

$$\textit{Lukasiewicz algebra}\text{: } x *_t y = max\{0, x + y - 1\}, \text{ and } x \rightarrow_t y = min\{1, 1 - x + y\}$$

These three examples are fundamental as any BL-algebra on the real unit interval $[0,1]$ can be constructed by using them. Any BL-algebra is, up to isomorphism, a subdirect product of linear BL-algebras (for details, see [27]).

Example 1.3.6. For any set X, define for $A \subseteq X$ and $B \subseteq X$, $A * B = A \cap B$ and $A \rightarrow B = A^C \cup B$. Then the structure $(P(X), \cap, \cup, *, \rightarrow, \emptyset, X)$ where $P(X)$ is the powerset of X is a BL-algebra called the *power BL-algebra* of X.

Example 1.3.7. It is well known that $\mathbb{Z}$, the set of integer with its usual operations is a Noetherian multiplication ring i.e. a Noetherian ring in which for every ideals I, J such that $I \subseteq J$, there exists an ideal K of $\mathbb{Z}$ such that $I = J \cdot K$. Hence by Remark 2.5 in [31], the lattice of ideals of $\mathbb{Z}$, $(Id(\mathbb{Z}), \wedge, \vee, *, \rightarrow, \{0\}, \mathbb{Z})$ is a BL-algebra, where

$$I \wedge J = I \cap J, I \vee J = I + J, I * J = I \cdot J, I \rightarrow J = \{n \in \mathbb{Z} \mid nI \subseteq J\},$$

for all $I, J \in Id(\mathbb{Z})$. Such rings are called BL-rings and have been widely studied in the litterature (see, e.g. [8], [31], [46]).

Example 1.3.8. Let $(I, \leq)$ be a totally ordered set. Consider a family $((L_i, \wedge_i, \vee_i, *_i, \rightarrow_i, 0, 1))_{i \in I}$ of BL-chains such that for all $i, j \in I$ with $i \neq j$, $L_i \cap L_j = \{1\}$. Then the *ordinal sum* of the family $(L_i)_{i \in I}$, denoted by $\oplus_{i \in I} L_i$ is a BL-algebra $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ defined by the following:

- The base set is $L = \bigcup_{i \in I} L_i$
- The ordering is: $x \leq y$ iff $\begin{cases} x, y \in L_i \text{ and } x \leq_i y \\ x \in L_i \setminus \{1\}, y \in L_j \text{ and } i < j \end{cases}$ for all $x, y \in L$.
- $x \rightarrow y = \begin{cases} x \rightarrow_i y & \text{if } x, y \in L_i \\ y & \text{if } x \in L_i \text{ and } y \in L_j \text{ with } i > j \\ 1 & \text{if } x \in L_i \setminus \{1\} \text{ and } y \in L_j \text{ with } i < j \end{cases}$ for all $x, y \in L$.
- $x * y = \begin{cases} x *_i y & \text{if } x, y \in L_i \\ y & \text{if } x \in L_i \text{ and } y \in L_j \setminus \{1\} \text{ with } i > j \\ x & \text{if } x \in L_i \setminus \{1\} \text{ and } y \in L_j \text{ with } i < j \end{cases}$ for all $x, y \in L$.

1.3.2 Filters in BL-algebras

Filters have been widely studied in BL-algebras namely to characterize fragments of Basic fuzzy logic (see Haveshki (2008) [29] and Turunen (2011) [52]) or to construct topological BL-algebras (see Haveshki (2007) [28] and Zahiri (2016) [56]). In logic they have a natural interpretation as sets of provable formulas. In this section, we introduce some basic facts about filter theory of BL-algebras.

A *filter* of L is a non empty subset F of L such that for all $x, y \in L$,

(F1) $x, y \in F$ implies $x * y \in F$;

(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A subset D of a BL-algebra L is called a *deductive system* (ds for short) if

(DS1) $1 \in D$;

(DS2) $x \in D$ and $x \to y \in D$ imply $y \in D$.

Remark 1.3.9. In logic, deductive systems appear as sets of formulas stable under the modus ponens. For a non-empty subset F of L, F is a deductive system if and only if it is a filter.

The *kernel* of a BL-morphism $f : L_1 \longrightarrow L_2$ is the set

$$Ker(f) := \{x \in L_1 : f(x) = 1\}.$$

Clearly, f is injective iff $Ker(f) = \{1\}$. $Ker(f)$ is always a deductive system.

Definition 1.3.10. Let L be a BL-algebra.

(1) A deductive system F of L is *proper* if $0 \notin F$.

(2) A proper deductive system M of L is said *maximal* if it is not contained in any other proper deductive system.

For any deductive system D of a BL-algebra $L = (L, \wedge, \vee, *, \to, 0, 1)$, we can define a binary relation θ_D on L as follows: for all $x, y \in L$,

$$x\theta_D y \iff (x \to y) \wedge (y \to x) \in D.$$

It is well known that θ_D is a congruence on L (see, e.g. [[27], Theo 2.7]) and since the class of BL-algebras is a variety, the quotient structure L/θ_D is also a BL-algebra for which for all $x, y \in L$, $[x \propto y]_D := [x]_D \propto [y]_D$ where $\propto \in \{\wedge, \vee, *, \to\}$, and $[x]_D := [x]_{\theta_D}$.

Example 1.3.11. (i) Let $(L, \wedge, \vee, *, \to, 0, 1)$ be a BL-algebra. $\{1\}$ and L are filters of L called trivial filters.

(ii) For $\mathbf{G}_3$ and $\mathbf{M}_3$, $\{x, 1\}$ is a non trivial filter.

(iii) $]0; 1]$ is a non trivial filter of the Lukasiewicz algebra.

1.4 Modal logic

Modal logics play an important role in many areas of computer science. In recent years, the connection of modal logic and coalgebra received a lot of attention, see e.g. [53]. In particular, it has been recognised that modal logic is to coalgebras what equational logic is to algebras.

1.4.1 Algebraic counterpart of a logic

Logic is the science that studies correct reasoning. For a logic $\mathcal{L}$, we write $\Gamma \vdash_{\mathcal{L}} \varphi$ if there is a proof of φ from Γ, where φ is a formula and Γ is a set of formulas. A set of formulas T is called a *theory* in $\mathcal{L}$ if it is closed under $\vdash_{\mathcal{L}}$, that is, if for every formula φ such that $T \vdash_{\mathcal{L}} \varphi$, we have $\varphi \in T$.

Given a theory T of $\mathcal{L}$, we can define the following binary relation $\theta(T)$ between formulas:

$$(\varphi, \psi) \in \theta(T) \text{ iff } T \vdash_{\mathcal{L}} \varphi \leftrightarrow \Psi.$$

Then $\theta(T)$ is a congruence relation on the formula algebra $Fm_{\mathcal{L}}$. Each logic has an algebraic counterpart called its *Lindenbaum-algebra* and defined as the quotient algebra $Fm_{\mathcal{L}}/\theta(T)$.

Logic studies the notion of logical consequence. There are many kinds of logical consequences, i.e. many different logics:

(1) Propositional classical logic whose Lindenbaum-algebras are Boolean algebras;

(2) Non classical logics:

- Modal logics whose Lindenbaum-algebras are modal algebras;
- Intuitionistic logic whose Lindenbaum-algebras are Heyting algebras;
- fuzzy logics whose Lindenbaum-algebras are semi linear residuated lattices;
- Basic fuzzy logic (A fragment of fuzzy logic) whose Lindenbaum-algebras are BL-algebras;
- ...

1.4.2 Kripke frames and models

Modal logic is the logic of modalities, which indicate the mode in which a statement is said to be true. These are not easily handled by the truth tables of classical logic, so logicians have developped an augmented form called modal logic. There is a very strong link between coalgebras and modal logic. In 1999, L. Moss [45] proposed Coalgebraic modal logic, in order to provide a uniform framework to various semantics of modal logics using the theory of coalgebra. This connection has been up to now intensively investigated (see, e.g. Y. Venema (2007) [53] and Bezhanishvili (2020) [10]). In this section, we introduce some basic features about modal logic wich will be useful in Chapter 4.

The language of modal logic have the following set of symbols:

- A countably infinite set of letters, called propositional variables;
- the unary operators $\Box$, $\Diamond$, $\neg$;
- the binary operators $\Rightarrow$, $\vee$, $\wedge$;
- brackets (and) .

$\Box$ and $\Diamond$ are *modal operators.*

In modal logic, "necessary true" is read as true in all possible world. Under this interpretation the truth of a statement is relative to the world in question. This means that $\Box P$ is defined to be true whenever P is true in all conceivable world.

$\Diamond P$ is similar, although in this case the modality is that of possibility. If P is true in at least one accessible world, $\Diamond P$ is true (true somewhere means not impossible).

Definition 1.4.1. Let W be a non empty set of what we will call "possible worlds" . Let R be a binary relation on W, which we call an *accessibilty relation.* Together, (W, R) form a *(Kripke) frame.*

Let (W, R) and (W', R') be two Kripke frames. A *p-morphism* is a function $f : W \longrightarrow W'$ satisfying

$$f([x]_R) = [f(x)]_{R'} \text{ for all } x \in W$$

.

Kripke frames and p-morphisms form a category denoted by $\mathcal{KF}r$.

Definition 1.4.2. Let (W, R) be a frame and let $\models$ be a binary relation between W and the set of wff. Let $\Gamma \in W$, we will assume that $\models$ obeys the following rules.

- For all propositional variable p, either $\Gamma \models p$ or $\Gamma \models \neg p$;
- If F is a wff, then $(\Gamma \models \neg F)$ iff $\neg(\Gamma \models F)$;
- If F and G are wff, then $(\Gamma \models F \vee G)$ iff ($\Gamma \models F$ or $\Gamma \models G$);
- If F and G are wff, then $(\Gamma \models F \wedge G)$ iff ($\Gamma \models F$ and $\Gamma \models G$);
- If F and G are wff, then $(\Gamma \models F \Rightarrow G)$ iff ($\neg(\Gamma \models F)$ or $\Gamma \models G$);
- If F is a wff, then $(\Gamma \models \Box F)$ iff (for any $\Delta \in W$, $\Gamma R \Delta$ implies $\Delta \models F$);
- If F is a wff, then $(\Gamma \models \Diamond F)$ iff (there exists $\Delta \in W$, such that $\Gamma R \Delta$ and $\Delta \models F$);

$(W, R, \models)$ is called a *model. definition*

The logicians distinguish some particular collections of frames based on the properties of R.

- The system **K** places no retsrictions on the frame;
- In the system **D**, R is serial ie For every world $\Gamma \in W$, there exists at least one $\Delta \in W$ such that Δ is accessible to Γ;
- In the system **T** R is reflexive;

- In the system **B** , R is reflexive and symmetric;
- In the system $\mathbf{K}_4$, R is transitive;
- In the system $\mathbf{S}_4$, R is reflexive and transitive;
- In the system $\mathbf{S}_5$, R is reflexive, symetric and transitive.

1.4.3 Modal algebras

Modal logics have an algebraic semantics based on a Boolean algebra, but with additional operators that model the modal operators.

Definition 1.4.3. A *Boolean algebra* is an algebraic structure $(A, \wedge, \vee, \neg, 0, 1)$ of type $(2, 2, 1, 0, 0)$, such that for all elements a, b and c of A, the following axioms hold:

- $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associativity)
- $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ (commutativity)
- $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ (absortion)
- $a \vee 0 = a$ and $a \wedge 1 = a$ (identity)
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (distributivity)
- $a \vee \neg a = 1$ and $a \wedge \neg a = 0$ (complements)

Definition 1.4.4. In algebra and logic, a *modal algebra* is a structure $(A, \wedge, \vee, \neg, 0, 1, \Box)$ such that:

- $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra.
- $\Box$ is a unary operation on A satisfying $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \Box y$, for all $x, y \in A$.

Modal algebras provide models of propositional modal logics in the same way as Boolean algebras are models of classical logic. In particular, the variety of all modal algebras is the equivalent algebraic semantics of the modal logic **K** in the sense of abstract algebraic logic, and the lattice of its subvarieties is dually isomorphic to the lattice of normal modal logics.

Example 1.4.5. (i) Let $(\{0, 1\}, \wedge, \vee, \neg 0, 1)$ be the trivial Boolean algebra, defined by the following tables:

$\wedge$	0	1
0	0	0
1	0	1

	0	1
0	0	1
1	1	1

a	0	1
$\neg a$	1	0

Then $(\{0, 1\}, \wedge, \vee, \neg, 0, 1, \Box)$ is a modal algebra, where the operator $\Box$ acts as identity.

(ii) The power set $P(S)$ of a set S, together with the operations of union, intersection and complement, can be viewed as the prototypical example of a Boolean algebra. Therefore, Let $\mathbb{N}$ be the set of all integers. Define for any integer n the set

$$\underline{n} = \{m \in \mathbb{N} \mid m \leq n\}.$$

Then $(P(\mathbb{N}), \cap, \cup, \neg, \emptyset, \mathbb{N}, \Box)$ is a modal algebra, where for all subset X of $\mathbb{N}$,

$$\Box X = \{n \in \mathbb{N} \mid \underline{n} \subseteq X\}.$$

Chapter 2

ON CONCRETE CATEGORIES OF BL-ALGEBRAS

Many familiar categories such as $\mathcal{VEC}$ and $\mathcal{TOP}$ are constructs (i.e., categories of structured sets and structure-preserving functions between them). If we regard such constructs as purely abstract categories, some valuable information (concerning underlying sets of objects and underlying functions of morphisms) is lost. Fortunately, category theory enables us to retain this information by providing a means for a formal definition of construct (a construct being a pair $(\mathcal{C}, U)$ consisting of a category $\mathcal{C}$ and a faithful functor $U : \mathcal{C} \longrightarrow \mathcal{SET}$). A careful analysis reveals that, for instance, many of the interesting properties of the constructs of vector spaces and topological spaces are not properties of the corresponding abstract categories $\mathcal{VEC}$ and $\mathcal{TOP}$ but rather of the corresponding constructs $(\mathcal{VEC}, U)$ and $(\mathcal{TOP}, V)$, where U and V denote the obvious underlying functors. In fact, they are often properties of just the underlying functors U and V . For example, the facts that the construct of vector spaces is "algebraic" and the construct of topological spaces is "topological" are very conveniently expressed by specific properties of the underlying functors, rather than by properties of the abstract categories $\mathcal{VEC}$ and $\mathcal{TOP}$. This leads to the concept of concrete categories over a category $\mathcal{X}$ as pairs $(\mathcal{C}, U)$ consisting of a category $\mathcal{C}$ and a faithful functor $U : \mathcal{C} \longrightarrow \mathcal{X}$. The concept of concrete categories over arbitrary base categories provides a suitable language to carry out such investigations. In this chapter, we investigate $\mathcal{BL}$ as a concrete category over $\mathcal{SET}$. We show how some limits and colimits are constructed and also that $\mathcal{BL}$ is essentially algebraic, but not topological nor cartesian closed. Moreover we present a hierarchy on some special morphisms and the categorical relation between $\mathcal{BL}$ and the categories $\mathcal{GOD}$ and $\mathcal{MV}$ of Gödel-algebras and MV-algebras respectively.

Definition 2.0.6. Given a category $\mathcal{X}$, a *concrete category* over $\mathcal{X}$ is a pair $(\mathcal{C}, U)$, where $U : \mathcal{C} \longrightarrow \mathcal{X}$ is a faithful functor. Sometimes U is called the *forgetful (or underlying) functor* of the concrete category and $\mathcal{X}$ is called the *base category* for $(\mathcal{C}, U)$. When $\mathcal{X} = \mathcal{SET}$, $(\mathcal{C}, U)$ is called a *construct.*

We consider the concrete category $(\mathcal{BL}, U)$ over $\mathcal{SET}$, where U is the standard forgetful functor.

Let $(\mathcal{C}, U)$ be a concrete category over $\mathcal{X}$. The *fibre* of an $\mathcal{X}$-object X is the preordered class consisting of all $\mathcal{C}$-objects A with $U(A) = X$ ordered by:

$$A \preceq B \text{ if and only if } id_X : UA \longrightarrow UB \text{ is (can be lifted to) a } \mathcal{C}\text{-morphism.}$$

Example 2.0.7. In the concrete category $(\mathcal{BL}, U)$, $\mathbf{M}_3$ and $\mathbf{G}_3$ are in the fibre of the set $\{0, x, 1\}$. Since by Lemma 1.3.3(i), $id : \{0, x, 1\} \longrightarrow \{0, x, 1\}$ is not a $\mathcal{BL}$-morphism between $\mathbf{M}_3$ and $\mathbf{G}_3$, they are not comparable.

Definition 2.0.8. A concrete category $(\mathcal{C}, U)$ over $\mathcal{X}$ is said to be:

(i) *fibre-discrete* provided that its fibres are ordered by equality.

(ii) *(uniquely) transportable* provided that for every $\mathcal{C}$-object A and every $\mathcal{X}-$isomorphism $UA \xrightarrow{f} X$, there exists a (unique) $\mathcal{C}$-object B with $UB = X$ such that $A \xrightarrow{f} B$, is a $\mathcal{C}$-isomorphism. In that case U is said to be (uniquely) transportable.

(iii) *strongly complete* if it is complete and has intersections;

(iv) is called *wellpowered* if no $\mathcal{C}$-object has a proper class of pairwise non-isomorphic subobjects.

Definition 2.0.9. Let $\mathcal{C}$ be a category.

Let $A \xrightarrow{f} B$ and $C \underset{q}{\overset{p}{\rightrightarrows}} A$ be $\mathcal{C}$-morphisms. (p, q) is called a *congruence relation* on f if (C, p, q) is a pullback of (f, f).

Definition 2.0.10. Let $(\mathcal{C}, U)$ be a concrete category over $\mathcal{X}$.

(i) A $\mathcal{C}$-morphism $A \xrightarrow{f} B$ is called *initial* provided that for any $\mathcal{C}$-object C an $\mathcal{X}$-morphism $UC \xrightarrow{g} A$ is an $\mathcal{C}$-morphism whenever $UC \xrightarrow{Uf\circ g} UA$ is a $\mathcal{C}$-morphism.

(ii) Let X be an object in $\mathcal{X}$. A *U-structured arrow* with domain X is a pair (f, A) consisting of an $\mathcal{C}$-object A and a $\mathcal{X}$-morphism $f : X \longrightarrow UA$.

(iii) A U-structured arrow (f, A) with domain X is called:

  a. *generating* provided that for any pair of $\mathcal{C}$-morphisms $A \underset{s}{\overset{r}{\rightrightarrows}} B$, the equality $Ur \circ f = Us \circ f$ implies that $r = s$,

  b. *extremally generating* provided that it is generating and whenever $B \xrightarrow{m} A$ is a $\mathcal{C}$-monomorphism and (g, B) is a U-structured arrow with $f = U(m) \circ g$, then m is an $\mathcal{C}$-isomorphism.

  c. *U-universal* for A provided that for each U-structured arrow (g, C) with domain B there exists a unique $\mathcal{C}$-morphism $A \xrightarrow{\bar{f}} C$ with $g = U(\bar{f}) \circ f$.

Definition 2.0.11. Let E be a class of morphisms and let $\mathbf{M}$ be a conglomerate of sources in a category $\mathcal{C}$:

(1) $\mathcal{C}$ has $(E, \mathbf{M})$-*factorizations* provided that each source S in $\mathcal{C}$ has a factorization $M \circ e$ with $e \in E$ and $M \in \mathbf{M}$.

(2) A functor $U : \mathcal{C} \longrightarrow \mathcal{X}$ has $(E, \mathbf{M})$-*factorizations* provided that for each U-structured source $(X \xrightarrow{f_i} UA_i)_I$ there exists $X \xrightarrow{e} UA \in E$ and $(A \xrightarrow{m_i} A_i)_I \in \mathbf{M}$ such that $f_i = Um_i \circ e$ for each $i \in I$.

Definition 2.0.12. A functor $U : \mathcal{C} \longrightarrow \mathcal{X}$ is called:

(i) *topological* provided that every U-structured source $(X \xrightarrow{f_i} UA_i)_I$ has a unique U-initial lift $(A \xrightarrow{f_i} A_i)_I$ (i.e., there exists a unique U-initial source $(A \xrightarrow{\tilde{f}_i} A_i)_I$ such that for each $i \in I$, $U(\tilde{f}_i) = f_i$).

(ii) *essentially algebraic* provided that it creates isomorphisms and is (Generating, Mono-Source)-factorizable.

Definition 2.0.13. Let $(\mathcal{C}, U)$ be a concrete category.

(i) $\mathcal{C}$ is *topological (essentially algebraic)* provided that U is topological (essentially algebraic).

(ii) $\mathcal{C}$ is called *cartesian closed* if it has finite products and for each $\mathcal{C}$-object A the functor $A \times -$ is left-adjoint.

2.1 Categorical properties of $\mathcal{BL}$

2.1.1 Limits and colimits in $\mathcal{BL}$

As a particular case of finitary algebraic category, $\mathcal{BL}$ is complete and cocomplete. In this section, we show how some (co)limits are constructed in $\mathcal{BL}$.

Proposition 2.1.1. *In $\mathcal{BL}$, the initial object is $\boldsymbol{G}_2$ and the final object is $\boldsymbol{G}_1$;*

Proof. Let L be a BL-algebra. The unique BL-morphism from $\mathbf{G}_2$ to L is the one which assigns 0 to 0 and 1 to 1. The unique morphism from L to $\mathbf{G}_1$ is the constant morphism. □

Proposition 2.1.2. *In $\mathcal{BL}$, the equalizer of a pair $L_1 \overset{f}{\underset{g}{\rightrightarrows}} L_2$ of $\mathcal{BL}-$morphisms is the embedding $E \xrightarrow{e} L_1$, where $E = \{x \in L_1 / f(x) = g(x)\}$;*

Proof. Let $L_1 \overset{f}{\underset{g}{\rightrightarrows}} L_2$ be a pair of BL-morphisms. Consider the set

$$E = \{x \in L_1 \mid f(x) = g(x)\} .$$

Then E is obviously a BL-subalgebra of L_1 and $f \circ e = g \circ e$, where $e : E \longrightarrow L_1$ is the standard embedding. For another BL-morphism $L' \xrightarrow{e'} L_1$, such that $f \circ e = g \circ e$, let $\varphi : L' \longrightarrow L$ defined by $\varphi(x) = e'(x)$. Clearly, φ is a BL- morphism and

$$e \circ \varphi(x) = \varphi(x) = e'(x) \text{ for all } x \in L'.$$

Now, let ψ be another morphism such that $e \circ \psi = e'$. Since e is a mono, $\psi = \varphi$. Therefore, $E \xrightarrow{e} L_1$ is an equalizer of f and g. □

Proposition 2.1.3. *In $\mathcal{BL}$, the product of a family $(L_i)_I$ of BL-algebras is the source $(P, P \xrightarrow{p_i} L_i)_I$ where*

$$P = \{I \xrightarrow{f} U_{i \in I} L_i / f(i) \in L_i \text{ for all } x \in I\}$$

and $P \xrightarrow{p_i} L_i$ is defined by $p_i(f) = f(i)$ for all $i \in I$.

Proof. P is clearly a BL-algebra and the mappings $P \xrightarrow{p_i} L_i$ such that $p_i(f) = f(i)$ are BL-morphisms. Let $(Q \xrightarrow{q_i} L_i)_I$ be a source. Then the map $\varphi : Q \longrightarrow P$ such that for all $x \in Q$, $\varphi(x)(i) = q_i(x)$, $i \in I$ is a BL-morphism such that $q_i = p_i \circ \varphi$. For the uniqueness, let $\psi : Q \longrightarrow P$ be another BL-morphism such that $q_i = p_i \circ \psi$. Let $x \in Q$, for all $i \in I$,

$$\varphi(x)(i) = p_i \circ \varphi(x) = p_i \circ \psi(x) = \psi(x)(i).$$

Hence $\varphi(x) = \psi(x)$ for all $x \in Q$ and then $\varphi = \psi$. This proves that the source $(P \xrightarrow{p_i} L_i)_I$ is a product of the family $(L_i)_I$. □

Proposition 2.1.4. *In $\mathcal{BL}$, the pullback of the morphisms $L_1 \xrightarrow{f} L \xleftarrow{g} L_2$ is the triple $(Pb(f, g), \pi_1, \pi_2)$ where $Pb(f, g) = \{(x, y) \in L_1 \times L_2 / f(x) = g(y)\}$ and π_i is the projection on L_i.*

Proof. Let $x, y \in Pb(f, g)$. We have $f \circ \pi_1(x, y) = g \circ \pi_2(x, y)$. Let $\pi_1' : X \longrightarrow L_1$ and $\pi_2' : X \longrightarrow L_2$ be two BL-morphisms such that $f \circ \pi_1' = g \circ \pi_2'$. Consider $\varphi : X \longrightarrow Pb(f, g)$ such that $\varphi(x) = (\pi_1'(x), \pi_2'(x))$. Then for any $x \in X$, $f(\pi_1'(x)) = g(\pi_2'(x))$ which implies that φ is well defined. φ is a BL-morphism since so are π_1' and π_2'.

Moreover, for all $x \in X$, $\pi_1 \circ \varphi(x) = \pi_1'(x)$ and $\pi_2 \circ \varphi(x) = \pi_2'(x)$. Let φ' be another BL-morphism such that $\pi_1 \circ \varphi' = \pi_1'$ and $\pi_2 \circ \varphi' = \pi_2'$, we have $\pi_1 \circ \varphi'(x) = \pi_1 \circ \varphi(x)$ and $\pi_2 \circ \varphi'(x) = \pi_2 \circ \varphi(x)$, for all $x \in X$. Thus, $\varphi = \varphi'$. So φ is unique and we can conclude that $(Pb(f, g), \pi_1, \pi_2)$ is a pullback of f and g.

□

Proposition 2.1.5. *In $\mathcal{BL}$, the coequalizer of a pair $L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2$ of BL-morphisms is the pair $(L_2/\theta, L_2 \xrightarrow{\pi} L_2/\theta)$ where θ is the smallest congruence on L_2 containing the set $X = \{(f(x), g(x)), x \in L_1\}$, π is the canonical surjection and L_2/θ is the quotient algebra.*

Proof. Let $L_1 \overset{f}{\underset{g}{\rightrightarrows}} L_2$ be a pair of BL-morphisms. We have for all $x \in L_1$,

$$\pi \circ f(x) = \pi \circ g(x),$$

since

$$(f(x), g(x)) \in X \subseteq \theta.$$

Moreover, let $h : L_2 \longrightarrow L_3$ be another BL-morphism such that $h \circ f = h \circ g$. For all $x \in L_1$, we have

$$h(f(x) \to g(x)) = h \circ f(x) \to h \circ g(x) = 1$$

and also

$$h(g(x) \to f(x)) = 1$$

which means that $(f(x), g(x)) \in \theta_h$. So $X \subseteq \theta_h$ and then $\theta \subseteq \theta_h$. Consider $\varphi : L_2/\theta \longrightarrow L_3$ such that $\varphi([y]_\theta) = h(y)$. Then for $y, y' \in L_2$ such that $[y]_\theta = [y']_\theta$, we have $(y, y') \in \theta \subseteq \theta_h$. Hence

$$(h(y) \to h(y')) \wedge (h(y') \to h(y)) = 1$$

which means that $h(y) = h(y')$ and then φ is well defined. It is not difficult to check that φ is a BL-morphism such that $h = \varphi \circ \pi$. The uniqueness of φ follows from the fact that π is an epimorphism. □

We denote by **BL** the variety of BL-algebras and by $F_{\mathbf{BL}}(X)$ the free BL-algebra over the set X. Since **BL** is a nontrivial equational class, free BL-algebras exists in **BL** and it is clear that these free BL-algebras are the free objects of the category $\mathcal{BL}$. It is well known that in such a category, the free functor $F_{\mathbf{BL}}$ which assigns to each set X the free BL-algebra $F_{\mathbf{BL}}(X)$ generated by X and to each function $X \xrightarrow{f} Y$ the $\mathcal{BL}$-morphism $F_{\mathbf{BL}}(f)$ defined for each element $(x_1, x_2, ..., x_n) \in F_{\mathbf{BL}}(X)$ by $F_{\mathbf{BL}}(f)(x_1, x_2...x_n) = (f(x_1), f(x_2), ..., f(x_n))$ is the left adjoint to the forgetful functor. Since left adjoint functors preserves colimits, in $\mathcal{BL}$ we have:

Proposition 2.1.6. *$(F_{\mathbf{BL}}(\sum_I X_i), F_{\mathbf{BL}}(\alpha_i))_I$ is the coproduct of the family of the free algebras $(F_{\mathbf{BL}}(X_i))_I$, where $(\sum_I X_i, \alpha_i)_I$ is the coproduct of the family $(X_i)_I$ in $\mathcal{SET}$.*

Open problem: Construct the coproduct of an arbitrary family of BL-algebras.

2.1.2 Properties of the forgetful functor

Remark 2.1.7. In a concrete category $(\mathcal{C}, U)$ over a category $\mathcal{X}$, a *lifting* of an $\mathcal{X}$-morphism $X \stackrel{f}{\longrightarrow} Y$, whenever it exists, which will be denoted by $\bar{f}$ is a $\mathcal{C}$-morphism such that $U(\bar{f}) = f$. By the faithfulness of U we have:

(i) the lifting $\bar{f}$ of an $\mathcal{X}$-morphism $UA \stackrel{f}{\longrightarrow} UB$ is unique and $\bar{f}$ coincides with f in UA if $\mathcal{X} = \mathcal{SET}$;

(ii) for any $\mathcal{X}$-morphisms $X \stackrel{g}{\longrightarrow} Y \stackrel{f}{\longrightarrow} Z$, $\overline{f \circ g} = \bar{f} \circ \bar{g}$, whenever $\bar{f}$ and $\bar{g}$ exist.

Now we establish the transportability of the concrete category of BL-algebras:

Theorem 2.1.8. *$(\mathcal{BL}, U)$ is fibre-discrete and uniquely transportable.*

Proof. Let X be a set and $L = (X, \wedge, \vee, *, \rightarrow, 0, 1)$, $L' = (X, \wedge', \vee', *', \rightarrow', 0', 1')$ be two BL-algebras in the fibre of X. Suppose that $L \preceq L'$. Then for all $x, y \in X$ we have $x \propto y \in X$ since L is a BL-algebra, where $\propto \in \{\wedge, \vee, *, \rightarrow\}$. Thus by Remark 2.1.7(i)

$$x \propto y = \overline{id_X}(x \propto y).$$

By the fact that $\overline{id_X}$ is a $\mathcal{BL}$-morphism, we obtain

$$x \propto y = \overline{id_X}(x) \propto' \overline{id_X}(y) = x \propto' y$$

and we can conclude that $L = L'$. Conversely, if $L = L'$ then it is obvious that $id_L = \overline{id_X}$. Therefore, $(\mathcal{BL}, U)$ is fibre-discrete.

For the unique transportability, let $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ be a BL-algebra, X be a set and $L \stackrel{f}{\longrightarrow} X$ be a bijective function. For $y_1, y_2 \in X$, define for $\propto' \in \{\wedge', \vee', *', \rightarrow'\}$,

$$y_1 \propto' y_2 = f(x_1 \propto x_2) \ (\propto \in \{\wedge, \vee, *, \rightarrow\}),$$

where $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $L' = (X, \wedge', \vee', *', \rightarrow', f(0), f(1))$ is the unique BL-algebra in the fibre of X such that $L \stackrel{f}{\longrightarrow} L'$ is a $\mathcal{BL}$-isomorphism.

□

The above Theorem and [[2], Proposition 5.8] lead to the following result:

Corollary 2.1.9. *The forgetful functor $U : \mathcal{BL} \longrightarrow \mathcal{SET}$ reflects identities.*

In $\mathcal{SET}$, mono-sources are exactely the point-separating sources, i.e., sources $(X, f_i)_I$ such that for any two different elements x and y of X there exists some $i \in I$ with $f_i(x) \neq f_i(y)$. Since faithful functors reflect mono-sources [[2], Proposition 10.7], we have:

Lemma 2.1.10. *In $\mathcal{BL}$, point separating sources are mono-sources.*

Lemma 2.1.11. *The category $\mathcal{BL}$ has $(Epi, Mono - Source)$-factorizations.*

Proof. Let $(L \xrightarrow{f_i} L_i)_I$ be a source in $\mathcal{BL}$. Consider the congruence θ defined by

$$(x, y) \in \theta \Longleftrightarrow f_i(x) = f_i(y) \text{ for all } i \in I$$

and let $L \xrightarrow{\pi} L/\theta$ be the canonical epimorphism. Then the map $L/\theta \xrightarrow{m_i} L_i$ defined by $m_i([x]_\theta) = f_i(x)$ is the unique BL-morphism such that $f_i = m_i \circ \pi$ (since π is an epimorphism). Let $[x]_\theta$ and $[y]_\theta$ be two distincts classes. Then $f_i(x) \neq f_i(y)$ for some $i \in I$, and so $m_i([x]_\theta) \neq m_i([y]_\theta)$. It follows that $(L/\theta \xrightarrow{m_i} L_i)_I$ is a point-separating source and by Lemma 2.1.10 a mono-source. □

The following summarizes a combination of Proposition 8.24 and Remark 13.26 from [2].

Lemma 2.1.12. *[2] The following hold:*

(i) every universal arrow is extremally generating;

(ii) a functor creates isomorphisms if and only if it reflects isomorphisms and is uniquely transportable;

Proposition 2.1.13. *The forgetful functor $U : \mathcal{BL} \longrightarrow \mathcal{SET}$ has $(Generating, Mono-Source)$-factorizations.*

Proof. Let $(X \xrightarrow{f_i} UL_i)_I$ be an U-structured source in $\mathcal{BL}$. Since $F_{\mathbf{BL}}(X)$ is a free object in $\mathcal{BL}$, there exists an universal arrow $X \xrightarrow{u} U(F_{\mathbf{BL}}(X))$. Hence for each f_i, there exists a unique BL-morphism $F_{\mathbf{BL}}(X) \xrightarrow{\tilde{f}_i} L_i$ such that $f_i = U\tilde{f}_i \circ u$. By Lemma 2.1.11, it follows that for each $i \in I$,

$$\tilde{f}_i = F_{\mathbb{BL}}(X) \xrightarrow{e} L \xrightarrow{m_i} L_i$$

where e is an epimorphism and $(L \xrightarrow{m_i} L_i)_I$ is a mono-source in $\mathcal{BL}$. Thus

$$f_i = X \xrightarrow{Ue\circ u} UL \xrightarrow{Um_i} UL_i.$$

Let's show that $Ue \circ u$ is generating. Let $L \underset{s}{\overset{r}{\rightrightarrows}} L'$ be a pair of $\mathcal{BL}$-morphisms such that

$$Ur \circ (Ue \circ u) = Us \circ (Ue \circ u).$$

Then by the fact that u is universal and so generating (Lemma 2.1.12 (i)), we have $Ur \circ Ue = Us \circ Ue$ and since U is faithful, $r \circ e = s \circ e$. Thus $r = s$ since e is an epimorphism. □

Theorem 2.1.14. *The construct $(\mathcal{BL}, U)$ is essentially algebraic.*

Proof. Let $L \xrightarrow{f} L'$ be a $\mathcal{BL}$-morphism such that $U(f)$ is bijective. Then

$$f^{-1}(0) = f^{-1}(f(0)) = 0$$

and for any binary operation $\propto$, we have:

$$f^{-1}(x \propto y) = f^{-1}(f(x') \propto f(y')) = f^{-1} \circ f(x' \propto y') = x' \propto y' = f^{-1}(x) \propto f^{-1}(y)$$

since f is a $\mathcal{BL}$-morphism and is surjective. Therefore f^{-1} is a $\mathcal{BL}$-morphism and so U reflects isomorphisms. Moreover, U is uniquely transportable (Theorem 2.1.8), it follows from Lemma 2.1.12(ii) that U creates isomorphisms. Taking into account the Proposition 2.1.13, we conclude that U is essentially algebraic. □

Essentially algebraic categories have some nice properties. For example, they inherit some properties of the base category as we can see in the following proposition [2]:

Proposition 2.1.15. *If $(\mathcal{C}, U)$ is essentially algebraic over $\mathcal{C}$, then the following hold:*

(i) If $\mathcal{X}$ is (strongly) complete, then $\mathcal{C}$ is (strongly) complete.

(ii) If $\mathcal{X}$ has coproducts, then $\mathcal{C}$ is cocomplete.

(iii) If $\mathcal{X}$ is wellpowered, then $\mathcal{C}$ is wellpowered.

Therefore, since $\mathcal{SET}$ is strongly complete and wellpowered, we have:

Proposition 2.1.16. *$\mathcal{BL}$ is strongly complete and wellpowered.*

Moreover, by [[2], Corollary 14.21], we have:

Corollary 2.1.17. *$\mathcal{BL}$ is (ExtrEpi, Mono)-structured and (Epi, ExtrMono)-structured.*

Concerning the topologicity of $\mathcal{BL}$, we obtain the following result:

Proposition 2.1.18. *Let $\mathcal{POS}$ denote the category of posets and order-preserving maps. The forgetful functors $\mathcal{BL} \xrightarrow{V} \mathcal{POS}$ and $\mathcal{BL} \xrightarrow{U} \mathcal{SET}$ are not topological.*

Proof. For the functor V, the morphism (1-source) f from the poset $\{0, z, x, y, 1\}$, with $z = x \wedge y$ to the poset $\{0, 1\}$, such that $f(x) = 0$ if $x \neq 1$ and $f(1) = 1$ cannot be lifted to a BL-morphism. If it were the case we would have

$$f(x \to y) = f(x) \to f(y) = 0 \to 0 = 1$$

which means that $x \to y = 1$ and by Lemma 1.3.1(1) $x \leq y$, which contradicts the hypothesis.

For the functor U, for any U-structured 1-source $\{0, x, 1\} \xrightarrow{g} U\mathbf{G}_2$ such that $g(0) = 1$ and $g(1) = 0$, g cannot be lifted to a $\mathcal{BL}$-morphism from $\mathbf{G}_3$ or $\mathbf{M}_3$ to $\mathbf{G}_2$.

□

Proposition 2.1.19. *$\mathcal{BL}$ is not cartesian closed.*

Proof. Let L be a BL-algebra, and $f, g : \mathbf{G}_2 \times \mathbf{G}_2 \longrightarrow L$ be two maps defined by:

$$f(a) = f(b) = 0, f(c) = f(d) = 1 \text{ and } g(a) = g(c) = 0, g(b) = g(d) = 1$$

where $a = (0,0), b = (0,1), c = (1,0)$ and $d = (1,1)$. Then f and g are $\mathcal{BL}$-morphisms. It follows that the functor $\mathbf{G}_2 \times -$ doesn't preserve initial object and hence, is not left adjoint.

□

2.2 Some classes of morphisms in $\mathcal{BL}$

2.2.1 Monomorphisms in $\mathcal{BL}$

Definition 2.2.1. Let $(\mathcal{C}, U)$ be a concrete category. A $\mathcal{C}$-morphism $A \xrightarrow{f} B$ is called *initial* provided that for any $\mathcal{C}$-object C, an $\mathcal{X}$-morphism $UC \xrightarrow{g} UA$ is (i.e. can be lifted to) a $\mathcal{C}$-morphism whenever $UC \xrightarrow{Uf \circ g} UB$ is a $\mathcal{C}$-morphism i.e. there exists $C \xrightarrow{h} B$ such that $U(h) = Uf \circ g$. A $\mathcal{C}$-morphism is called a *regular monomorphism* if it is the equalizer of some pair of $\mathcal{C}$-morphisms.

In this section we consider the concrete category $(\mathcal{BL}, U)$ over $\mathcal{SET}$, where U is the standard forgetful functor.

Proposition 2.2.2. *In $\mathcal{BL}$, we have:*

$$RegMono(\mathcal{BL}) \subseteq ExtrMono(\mathcal{BL}) = StrongMono(\mathcal{BL}) \subseteq Mono(\mathcal{BL})$$

and

$$Mono(\mathcal{BL}) = Inj(\mathcal{BL}) = Init(\mathcal{BL})$$

where (Reg, Strong, Extremal)Mono($\mathcal{BL}$), Init($\mathcal{BL}$) and Inj($\mathcal{BL}$) are the classes of (regular, Strong, Extremal)monomorphisms, initial and injective morphisms , respectively.

Proof. It is clear that every injective BL-morphism is a monomorphism and since U is right adjoint and hence preserves monomorphisms [[2], Proposition 18.6], every monomorphism is injective. Hence $Mono(\mathcal{BL}) = Inj(\mathcal{BL})$. Let's show that injective morphisms are initial in $\mathcal{BL}$. Let $f : L_1 \longrightarrow L_2$ be an injective $\mathcal{BL}$-morphism. Let L_3 be a BL-algebra and $g : UL_3 \longrightarrow UL_1$ be a function. Suppose that $Uf \circ g : UL_3 \longrightarrow UL_2$ is a $\mathcal{BL}$-morphism. Then for all $x, y \in UL_3$, and $\propto \in \{*, \rightarrow\}$, we have

$$\overline{Uf \circ g}(x \propto y) = \overline{Uf \circ g}(x) \propto \overline{Uf \circ g}(y)$$

and since $x, y, x \propto y \in UL_3$, we have by Remark 2.1.7(i)

$$Uf \circ g(x \propto y) = Uf \circ g(x) \propto Uf \circ g(y)$$

. Therefore, because $g(x), g(y), g(x \propto y) \in UL_1$ the same remark leads to:

$$\begin{aligned} f \circ g(x \propto y) &= f \circ g(x) \propto f \circ g(y) \\ &= f(g(x) \propto g(y)) \ (f \text{ is a BL-morphism}). \end{aligned}$$

Since f is injective, we obtain $g(x \propto y) = g(x) \propto g(y)$. With similar arguments, we show that $g(0) = 0$ and we conclude that g is a BL-morphism and so f is an initial morphism. Conversely, suppose that f is an initial morphism. Let $x, y \in L_1$ such that $f(x) = f(y)$. Define $g : \{0, 1\} \longrightarrow L_1$ by

$$g(0) = 0 \text{ and } g(1) = (x \to y) \wedge (y \to x).$$

It is obvious that $f \circ g(0) = 0$ and $f \circ g(1) = 1$, which means that $f \circ g$ is a BL-morphism. By hypothesis, it follows that g is a BL-morphism and then $g(1) = 1$, which leads to

$$(x \to y) \wedge (y \to x) = 1$$

and by Lemma 1.3.1(1), we obtain $x = y$. Therefore, $inj(\mathcal{BL}) = init(\mathcal{BL})$.

$ExtrMono(\mathcal{BL}) = StrongMono(\mathcal{BL}) \subseteq Mono(\mathcal{BL})$ follows from Lemma 1.1.11 and Corollary 2.1.17. $RegMono(\mathcal{BL}) \subseteq ExtrMono(\mathcal{BL})$ follows from [[2], Corollary 7.63].

□

2.2.2 Epimorphisms in $\mathcal{BL}$

Since BL-algebras form a variety, $L \times L$ is a BL-algebra for any BL-algebra L. We recall that for any BL-morphism $L \stackrel{f}{\longrightarrow} L'$, θ_f denote the congruence induced by the deductive system $Ker(f)$. It is easily checked that θ_f is a BL-subalgebra of $L \times L$ and we have the following:

Lemma 2.2.3. *For any BL-morphism $L \stackrel{f}{\longrightarrow} L'$, $\theta_f = \{(x, y) \in L \times L; f(x) = f(y)\}$.*

Proof. Let $(x, y) \in L \times L$.

$$\begin{aligned} (x \to y) \wedge (y \to x) \in Ker(f) &\Leftrightarrow f((x \to y) \wedge (y \to x)) = 1 \\ &\Leftrightarrow (f(x) \to f(y)) \wedge (f(y) \to f(x)) = 1 \\ &\Leftrightarrow f(x) \to f(y) = 1 \text{ and } f(y) \to f(x) = 1 (\text{ by Lemma 1.3.1}) \\ &\Leftrightarrow f(x) \leq f(y) \text{ and } f(y) \leq f(x) (\text{ by Lemma 1.3.1}) \\ &\Leftrightarrow f(x) = f(y). \end{aligned}$$

□

The following result will be useful in the sequel:

Lemma 2.2.4. *If* $L \xrightarrow{f} L'$ *is a regular epimorphism in* $\mathcal{BL}$*, then* f *is the coequalizer of the pair* $\theta_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} L$ *where* $\pi_1(x_1, x_2) = x_1$ *and* $\pi_2(x_1, x_2) = x_2$*.*

Proof. Let $L \xrightarrow{f} L'$ be a regular epimorphism in $\mathcal{BL}$. It follows from Lemma 2.2.3 and Proposition 2.1.4 that (θ_f, π_1, π_2) is a pullback of (f, f). Thus (π_1, π_2) is a congruence relation of f. Hence, by [[2], Proposition 11.22], f is a coequalizer of π_1 and π_2. □

Proposition 2.2.5. *In the construct* $(\mathcal{BL}, U)$*, we have:*

$$RegEpi(\mathcal{BL}) = Surj(\mathcal{BL}) \subsetneq Epi(\mathcal{BL})$$

where $(Reg)Epi(\mathcal{BL})$*, and* $Surj(\mathcal{BL})$ *are the classes of (regular)epimorphisms and surjective morphisms, respectively.*

Proof. Since faithfull functors reflect epimorphisms (see [[2], proposition 7.44]), every surjective BL-morphism is an epimorphism in $\mathcal{BL}$. The converse does not hold. Indeed, consider $L = \{0, x, y, 1\}$ and $L' = \{0, z, x, y, 1\}$, where $z = x \vee y$ and x and y are not comparable. Define $*, \odot, \rightharpoondown$ and $\rightharpoonup$ as follows:

$*$	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

$\rightharpoonup$	0	x	y	1
0	1	1	1	1
x	0	1	y	1
y	0	x	1	1
1	0	x	y	1

$\odot$	0	x	y	z	1
0	0	0	0	0	0
x	0	x	0	x	x
y	0	0	y	y	y
z	0	x	y	z	z
1	0	x	y	z	1

$\rightharpoondown$	0	x	y	z	1
0	1	1	1	1	1
x	0	1	y	1	1
y	0	x	1	1	1
z	0	x	y	1	1
1	0	x	y	z	1

Then $(L, \wedge, \vee, *, \rightharpoonup, 0, 1)$ and $(L', \wedge, \vee, \odot, \rightharpoondown, 0, 1)$ are BL-algebras. Consider the function $L \xrightarrow{m} L'$ such that $m(t) = t$ for all $t \in L$. Then m is an epimorphism but it is not surjective. Thus we have the strict inclusion $Surj(\mathcal{BL}) \subset Epi(\mathcal{BL})$.

Let $L \xrightarrow{f} L'$ be a surjective $\mathcal{BL}$-morphism. Consider the pair $\theta_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} L$. Then we have $f \circ \pi_1 = f \circ \pi_2$. Let $L \xrightarrow{g} L''$ be another BL-morphism such that $g \circ \pi_1 = g \circ \pi_2$ and consider the map $L' \xrightarrow{u} L''$ such that for all $y = f(x) \in L'$, $u(y) = g(x)$. Let $f(x_1), f(x_2) \in L'$ such that $f(x_1) = f(x_2)$. Then $(x_1, x_2) \in \theta_f$. So we have

$$g \circ \pi_1(x_1, x_2) = g \circ \pi_2(x_1, x_2)$$

i.e., $g(x_1) = g(x_2)$ which means that

$$u(f(x_1)) = u(f(x_2))$$

and thus u is well defined. u is clearly a $\mathcal{BL}$-morphism and we have $u \circ f = g$. For another $\mathcal{BL}$-morphism v such that $v \circ f = g$, we have $u \circ f = v \circ f$ and so $u = v$ since f is an epimorphism. Hence f is the coequaliser of the pair $\theta_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} L$.

To complete the proof, we have to show that regular epimorphisms are surjective. Let $L \xrightarrow{f} L'$ be a regular epimorphism in $\mathcal{BL}$. Then by Lemma 2.2.4, f is the coequalizer of the pair $\theta_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} L$. For all $(x, y) \in \theta_f$, we have

$$\pi \circ \pi_1(x, y) = [x]_{\theta_f} = [y]_{\theta_f} = \pi \circ \pi_2(x, y)$$

, where $L \xrightarrow{\pi} L/\theta_f$ is the canonical surjection. Thus there exists a unique BL-morphism $L' \xrightarrow{\varphi} L/\theta_f$ such that $\varphi \circ f = \pi$. Since π is surjective, it is a regular epimorphism. So π is the coequalizer of the pair $\theta_\pi \underset{\pi'_2}{\overset{\pi'_1}{\rightrightarrows}} L$. Let $(x, y) \in \theta_\pi$. Then $\pi(x) = \pi(y)$ which means that $[x]_{\theta_f} = [y]_{\theta_f}$ and we get that $f(x) = f(y)$. So $f \circ \pi'_1 = f \circ \pi'_2$ and thus there exists an unique $\mathcal{BL}$-morphism $L/\theta_f \xrightarrow{\phi} L'$ such that $\phi \circ \pi = f$. Hence

$$(\varphi \circ \phi) \circ \pi = \varphi \circ (\phi \circ \pi) = \varphi \circ f = \pi.$$

Since π is an epimorphism, we obtain $\varphi \circ \phi = 1_{L/\theta_f}$. Moreover

$$(\phi \circ \varphi) \circ f = \phi \circ (\varphi \circ f) = \phi \circ \pi = f.$$

Since f is a regular epimorphism, it is an epimorphism and we get $\phi \circ \varphi = 1_{L'}$. It follows that ϕ is a $\mathcal{BL}$-isomorphism and hence is surjective. Therefore $f = \phi \circ \pi$ is surjective as composition of such morphisms. □

2.2.3 Some subcategories of $\mathcal{BL}$

Let $\mathcal{GOD}$ and $\mathcal{MV}$ denote the categories of Gödel and MV-algebras respectively. We present the relations between $\mathcal{BL}$ and the categories $\mathcal{GOD}$ and $\mathcal{MV}$.

Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is said *coreflective* if the inclusion functor $i : \mathcal{D} \hookrightarrow \mathcal{C}$ has a right adjoint R. In this case, R is called a *reflector*. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is said *isomorphism-closed* if every object of $\mathcal{C}$ that is isomorphic to a $\mathcal{D}$-object is itself a $\mathcal{D}$-object.

The morphisms in $\mathcal{GOD}$ and $\mathcal{MV}$ are exactly BL-morphisms. Thus these categories are full subcategories of $\mathcal{BL}$. Moreover, we have:

Proposition 2.2.6. *$\mathcal{GOD}$ and $\mathcal{MV}$ are isomorphism-closed subcategories of $\mathcal{BL}$.*

Proof. Let L be a BL-algebra isomorphic to a Gödel-algebra G. Then let $L \xrightarrow{f} G$ be that BL-isomorphism. For all $x \in L$, there exists $y \in G$ such that $x = f(y)$. We have

$$x * x = f(y) * f(y) = f(y * y) = f(y) = x.$$

Thus L is a Gödel-algebra and so $\mathcal{GOD}$ is an isomorphism-closed subcategory of $\mathcal{BL}$. By a similar method, one can easily prove that $\mathcal{MV}$ is also an isomorphism-closed subcategory of $\mathcal{BL}$. □

Lemma 2.2.7. *Let L and L' be two BL-algebras. For all BL-morphism $MV(L) \xrightarrow{f} L'$, $Im(f) \subseteq MV(L')$*

Proof. Let $MV(L) \xrightarrow{f} L'$ be a BL-morphism and $y \in Im(f)$. Then there exists $x \in MV(L)$ such that $y = f(x)$. By the definition of the MV-center, it means there exists $z \in L$ such that $y = f(\overline{z}) = \overline{f(z)}$. So $y \in MV(L')$. □

Proposition 2.2.8. *The correspondance* $\mathbb{MV} : \mathcal{BL} \longrightarrow \mathcal{MV}$ *which assigns to every BL-algebra its MV-center extends to a functor.*

Proof. The Lemma 2.2.7 shows that $\mathbb{MV}$ is well defined. Let $L \xrightarrow{f} L' \xrightarrow{g} L''$ be two $\mathcal{BL}$-morphisms. Then for all $x \in MV(L)$, we have

$$\mathbb{MV}(g \circ f)(x) = g \circ f(x) = \mathbb{MV}(g) \circ \mathbb{MV}(f)(x)$$

and

$$\mathbb{MV}(1_L)(x) = x = 1_{MV(L)}(x).$$

□

In the sequel, this functor will be called the *$\mathcal{MV}$-functor.*

Proposition 2.2.9. *The* $\mathcal{MV}$-functor *is neither faithful nor conservative.*

Proof. Let $\mathbf{G}_4$ be the BL-algebra defined by the following tables:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Consider now the functions $\mathbf{G}_3 \overset{f}{\underset{g}{\rightrightarrows}} \mathbf{G}_4$ such that $f(0) = 0$, $f(x) = a$, $f(1) = 1$ and $g(0) = 0$, $g(x) = b$, $g(1) = 1$. Then it is easily checked that f and g are BL-morphisms. We have $\mathbb{MV}(f) = \mathbb{MV}(g)$, but $f \neq g$. Thus $\mathbb{MV}$ is not faithful. Moreover, $\mathbb{MV}(f)$ is a BL-isomorphism but it is not the case for f. So $\mathbb{MV}$ is not conservative. □

Theorem 2.2.10. *$\mathcal{MV}$ is a coreflective subcategory of $\mathcal{BL}$.*

Proof. We first prove that the $\mathcal{MV}$-functor is the right adjoint to the inclusion functor $i : \mathcal{MV} \hookrightarrow \mathcal{BL}$. Let M be an MV-algebra and L a BL-algebra. Consider the function

$$\begin{aligned} \Phi_{M,L} : \quad & \mathcal{BL}(M, L) \longrightarrow \mathcal{MV}(M, MV(L)) \\ & M \xrightarrow{f} L \longmapsto \mathbb{MV}(f) \end{aligned}$$

where $\mathcal{BL}(M, L)$ (respectively $\mathcal{MV}(M, MV(L))$) denote the set of BL-morphisms from M to L (respectively from M to $MV(L)$). Since $MV(M) = M$, by Lemma 2.2.7, $\Phi_{M,L}(f)$ is well defined. $\Phi_{M,L}$ is bijective and its inverse $\Phi^{-1}_{M,L}$ is defined by $\Phi^{-1}_{M,L}(g)(x) = g(x)$ for all $\mathcal{MV}$-morphism $g : M \longrightarrow MV(L)$. Moreover, for any $\mathcal{MV}$-morphism $M' \xrightarrow{f} M$, we have for all $M \xrightarrow{\alpha} L$ and $x \in M'$:

$$\begin{aligned} \mathcal{MV}(f, MV(L)) \circ \Phi_{M,L}(\alpha)(x) &= \Phi_{M,L}(\alpha) \circ f(x) \\ &= \mathbb{MV}(\alpha) \circ f(x) \\ &= \alpha \circ f(x) \\ &= \mathbb{MV}(\alpha \circ f)(x) \\ &= \Phi_{M',L}(\alpha \circ f)(x) \\ &= \Phi_{M',L} \circ \mathcal{BL}(f, L)(\alpha)(x) \end{aligned}$$

which proves the naturality of $\Phi_{M,L}$ in the first variable. With similar computations, one can easily checked that $\Phi_{M,L}$ is also natural in the second variable. Therefore $i \dashv \mathbf{MV}$. Moreover, since $\mathcal{MV}$ is a full subcategory of $\mathcal{BL}$ we have the result.

□

Since right-adjoint functors preserve limits, we have:

Corollary 2.2.11. *The $\mathcal{MV}$-functor preserves limits.*

2.2.4 Conclusion

In this chapter, we studied concrete categorical properties of BL-algebras. Although this category is not cartesian closed or topological, we proved that it is essentially algebraic over $\mathcal{SET}$, meaning that many properties of $\mathcal{SET}$, can be transferred to $\mathcal{BL}$, in particular

(Epi, ExtrMono)-factorization, which will be useful to define some coalgebraic notions in next chapter. We also establish the hierarchy between monomorphisms on a hand and epimorphisms in another hand in $\mathcal{BL}$. We end the chapter by the relation between $\mathcal{BL}$ and the categories of Gödel and MV- algebras respectively. We showed that $\mathcal{MV}$ is a coreflective subcategory of $\mathcal{BL}$. In the next chapter we will establish a coalgebraic lift of this relation.

Chapter 3

MV-COALGEBRAS

Coalgebras (Adamek (2005) [4], Jacobs (2016) [35], Rutten (2000) [49]) form a powerful theory of state- based transition systems where definitions and results are formulated at a high level of genericity that covers several families of systems at once, from deterministic automata and Kripke frames to different kinds of probabilistic models. Traditionally, these formulations are elaborated in a set-based context, i.e. no further structure in the system's state space than that of a set is assumed. In many cases, however, a switch of context is needed. The projects on the coalgebraic foundations of stochastic systems, where the Giry functor and measurable spaces have a central role (cf. Doberkat 2009 [18]; Panangaden (2009) [47]; Viglizzo (2005) [55]), are evident examples of this. Research on coalgebras over Stone spaces (e.g. Bezhanishvili (2010) [9] ; Kupke (2004) [40], Venema (2014) [54]) and coalgebras over pseudometric spaces (Balan (2019) [5]) forms equally important cases. In [40], [9] and [54], the aim is to provide a suitable coalgebraic semantics for finitary modal logics by taking advantage of a Vietoris functor, while in Baldan (2018) [6] and Balan (2019) [5] is to introduce a notion of distance between states.

In this chapter, we investigate $\mathbb{MV}$-coalgebras, which are pairs (L, α) where L is a BL-algebra and $L \xrightarrow{\alpha} MV(L)$ is a $\mathcal{BL}$-morphism. We show that the properties of $\mathcal{BL}$ are good enough to obtain a rich coalgebraic structure on this category. We also use deductive systems in $\mathcal{BL}$ to construct a topology and introduce topological $\mathbb{MV}$-coalgebras.

3.1 The category of $\mathbb{MV}$-coalgebras

The composite $\mathbf{i} \circ \mathbb{MV}$, where $\mathbf{i} : \mathcal{MV} \hookrightarrow \mathcal{BL}$ is the inclusion functor shall also be denoted by $\mathbb{MV}$ and it is a covariant $\mathcal{BL}$-endofunctor, which has the preservation properties of the $\mathcal{MV}$-functor. In this section, we characterize homomorphisms, $\mathbb{MV}$-subcoalgebras and Bisimulations in the category of $\mathbb{MV}$-coalgebras and prove its (co)completeness.

The following observations are easily checked:

Remark 3.1.1. *Let (L, α) be an $\mathbb{MV}$-coalgebra.*

(i) For all $x \in L$, $\alpha(x) = \alpha(\overline{\overline{x}})$;

(ii) If α is injective, then L is an MV-algebra;

(iii) For all $x \in L$, there exists $z \in L$ such that $\alpha(x) = \overline{z}$.

3.1.1 $\mathbb{MV}$-homomorphisms

We introduce an arrow notation similar to transition system as in Gumm (2001) [24]. We write

$$x \stackrel{\alpha}{\to} y \text{ iff } \alpha(x) = y.$$

We say that a map $(L, \alpha) \stackrel{f}{\longrightarrow} (L', \beta)$:

(i) *preserves transitions* if for all $x, y \in L$, $x \stackrel{\alpha}{\to} y \Longrightarrow f(x) \stackrel{\beta}{\to} f(y)$;

(ii) *reflects transitions* if for all $x \in L$ and $y \in L'$, $f(x) \stackrel{\beta}{\to} y \Longrightarrow x \stackrel{\alpha}{\to} t,$ with $f(t) = y$.

The following results provides a chraracterization of $\mathbb{MV}$-homomorphisms:

Proposition 3.1.2. *Let (L, α) and (L', β) be two $\mathbb{MV}$-coalgebras and $L \stackrel{f}{\longrightarrow} L'$ be a BL-morphism. The following are equivalent:*

(i) f is a homomorphism;

(ii) For all $x \in L$, $\beta \circ f(x) = \overline{f(z)}$ where $\overline{z} = \alpha(x)$;

(iii) f preserves and reflects transitions.

Proof. $(i) \Leftrightarrow (ii)$ Suppose f is a homomorphism. Then for all $x \in L$ such that $\alpha(x) = \overline{z}$, we have

$$\beta \circ f(x) = MV(f)(\overline{z}) = f(\overline{z}).$$

Since f is a BL-morphism, we obtain

$$\beta \circ f(x) = \overline{f(z)}.$$

Conversely, suppose that $\beta \circ f(x) = \overline{f(z)}$ where $\overline{z} = \alpha(x)$, for all $x \in L$. Then

$$\mathbb{MV}(f) \circ \alpha(x) = \mathbb{MV}(f)(\overline{z}) = \overline{f(z)}.$$

So $\beta \circ f(x) = \mathbb{MV}(f) \circ \alpha(x)$.

$(i) \Rightarrow (iii)$ suppose f is a homomorphism. Let $x, y \in L$ such that $x \stackrel{\alpha}{\to} y$ and $z \in L$ such that $\alpha(x) = \overline{z}$. Then $\overline{z} = y$. Since f is a BL-morphism, $\overline{f(z)} = f(y)$ and by hypothesis

$$\alpha \circ f(x) = f(y),$$

which proves that f preserves transitions. Moreover, let $x \in L$ with $\overline{z} = \alpha(x)$ and $y \in L'$. Suppose $f(x) \stackrel{\alpha'}{\to} y$. We have $x \stackrel{\alpha}{\to} \overline{z}$ and by hypothesis

$$f(\overline{z}) = \overline{f(z)} = \beta \circ f(x) = y$$

Therefore, f reflects transitions.

$(iii) \Rightarrow (ii)$ Suppose f preserves and reflects transitions. Let $x \in L$ with $\alpha(x) = \overline{z}$. Then $x \stackrel{\alpha}{\to} \overline{z}$ and by hypothesis, $f(x) \stackrel{\beta}{\to} f(\overline{z})$ which means that $\beta \circ f(x) = \overline{f(z)}$. $\square$

Example 3.1.3. Consider the BL-algebra $\mathbf{G}_4$ from the proof of Proposition 2.2.9 and consider the function $\mathbf{G}_3 \xrightarrow{f} \mathbf{G}_4$ such that

$$f(0) = 0,\ f(x) = a,\ f(1) = 1.$$

Then it is easily checked that f is a BL-morphism. Define $\mathbf{G}_i \xrightarrow{\alpha_i} \mathbf{G}_2$ by

$$\alpha_i(0) = 0 \text{ and } \alpha_i(x) = 1 \text{ for } x \neq 0,\ i \in \{3, 4\}.$$

α_3 and α_4 are BL-morphisms and it is obvious that $\mathbf{G}_2 = MV(\mathbf{G}_i)$ for $i \in \{3, 4\}$. Let $x \in \mathbf{G}_3$ with $x \neq 0, 1$. We have:

$$\begin{aligned} \alpha_4 \circ f(x) &= \alpha_4(a) \\ &= 1 \\ &= \mathbb{MV}(f)(1) \\ &= \mathbb{MV}(f) \circ \alpha_3(x) \end{aligned}$$

Hence f is a homomorphism of coalgebras between $(\mathbf{G}_3, \alpha_3)$ and $(\mathbf{G}_4, \alpha_4)$.

The category of **MV**-coalgebras and homomorphisms shall be denoted by $\mathcal{BL}_{\mathbb{MV}}$.

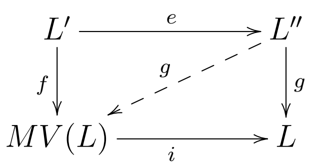
3.1.2 $\mathbb{MV}$-subcoalgebras

Since $\mathcal{BL}$ has (Epi, ExtrMono)=(Epi, StrongMono)-factorizations, and following [[4], page 171], we give the following definition of $\mathbb{MV}$-subcoalgebra.

Definition 3.1.4. Let (L', β) be an $\mathbb{MV}$-coalgebra. An $\mathbb{MV}$-*subcoalgebra* of (L', β) is an $\mathbb{MV}$-coalgebra (L, α) together with a strong mono homomorphism (i.e. a homomorphism which is a strong monomorphism in $\mathcal{BL}$) $(L, \alpha) \xrightarrow{m} (L', \beta)$.

Proposition 3.1.5. *Let (L, α) be an $\mathbb{MV}$-coalgebra. Then $(MV(L), MV(\alpha))$ is an $\mathbb{MV}$-subcoalgebra of (L, α).*

Proof. Let $i : MV(L) \hookrightarrow L$ be the inclusion morphism. Let $L' \xrightarrow{e} L''$ be an epimorphism, f and g are morphisms such that the following square commutes



Then $f = i \circ f = g \circ e$. Since $g = i \circ g$, it follows that both triangles commute. Therefore i is a strong-monomorphism. Moreover, for all $x \in MV(L)$,

$$\mathbb{MV}(i) \circ \mathbb{MV}(\alpha)(x) = i \circ \mathbb{MV}(\alpha)(x) = i \circ \alpha(x).$$

Hence i is a homomorphism from $(MV(L), \mathbb{MV}(\alpha))$ to (L, α).

□

We give now a characterization of BL-subalgebras of a BL-algebra which can be endowed with a transition structure making them $\mathbb{MV}$-subcoalgebras.

Proposition 3.1.6. *Let (L', β) be an $\mathbb{MV}$-coalgebra. A BL-subalgebra L of L' is an $\mathbb{MV}$-subcoalgebra of (L', β) iff there exists a strong monomorphism $L \xrightarrow{m} L'$ such that for all $x \in L$, there exists $z \in L$ such that $m(x) \xrightarrow{\beta} \overline{m(z)}$.*

Proof. Let L be a BL-subalgebra of L'. Suppose (L, α) is a subcoalgebra of (L', β). Then by definition, there exists a strong mono homomorphism $(L, \alpha) \xrightarrow{m} (L', \beta)$. Let $x \in L$. By Remark 3.1.1, there exists $z \in L$ such that $\alpha(x) = \overline{z}$. It follows from Proposition 3.1.2 that $\beta \circ m(x) = \overline{m(z)}$ which means that $m(x) \xrightarrow{\beta} \overline{m(z)}$.

Conversely, assume that there is a strong monomorphism $m : L \longrightarrow L'$ such that for all $x \in L$, there exists $z \in L$ with $m(x) \xrightarrow{\beta} \overline{m(z)}$. Then observe that $\overline{z}$ is unique since m is injective (Proposition 2.2.2) and define

$$\begin{aligned} \alpha : \ L &\longrightarrow MV(L) \\ x &\longmapsto \overline{z} \end{aligned}$$

Let $x, y \in L$ such that $\alpha(x) = \overline{z}$ and $\alpha(y) = \overline{z'}$. If $x = y$ then we have $m(x) \xrightarrow{\beta} \overline{m(z)}$ and $m(x) \xrightarrow{\beta} \overline{m(z')}$. It follows that $\overline{m(z)} = \overline{m(z')}$ and then $m(\overline{z}) = m(\overline{z'})$. Since m is a mono, we obtain $\overline{z} = \overline{z'}$ i.e., $\alpha(x) = \alpha(y)$. Therefore, α is well defined. Moreover,

$$\begin{aligned} \beta \circ m(0) &= 0 \text{ (since } \beta \text{ and } m \text{ are BL-morphisms)} \\ &= \overline{1} \text{(by Lemma 1.3.1(8))} \\ &= \overline{m(1)} \ (m, \text{ is a BL-morphism}). \end{aligned}$$

Thus $m(0) \xrightarrow{\beta} \overline{m(1)}$ and so $\alpha(0) = 0$. Let $x, y \in L$ such that $\alpha(x \propto y) = \overline{t}$, $\alpha(x) = \overline{z}$ and $\alpha(y) = \overline{z'}$, where $\propto \in \{*, \rightarrow\}$. We have

$$m(x \propto y) \xrightarrow{\beta} \overline{m(t)}.$$

Since $\beta \circ m$ is a BL-morphism, we obtain

$$\beta \circ m(x) \propto \beta \circ m(y) = m(\overline{t}).$$

So

$$m(\overline{z}) \propto m(\overline{z'}) = m(\overline{t})$$

and therefore

$$m(\overline{z} \propto \overline{z'}) = m(\overline{t}).$$

Using the fact that m is injective (Proposition 3.22), we obtain $\overline{z} \propto \overline{z'} = \overline{t}$ and then,

$$\alpha(x) \propto \alpha(y) = \alpha(x \propto y).$$

Hence α is a $\mathcal{BL}$-morphism. It follows that α is a transition structure on L making m a strong mono homomorphism. □

3.1.3 $\mathbb{MV}$-bisimulations and (co)limits in $\mathcal{BL}_{\mathbb{MV}}$

Definition 3.1.7. *Let R be a strong relation between two $\mathbb{MV}$-coalgebras (L_1, α_1) and (L_2, α_2), that is, there is a strong-mono $m : R \hookrightarrow L \times L'$. R is called an* $\mathbb{MV}$-bissimulation *provided that there is a structure map on R making the projections $\pi_i : R \longrightarrow L_i$ MV-homomorphisms.*

Proposition 3.1.8. *In $\mathcal{BL}_{\mathbb{MV}}$, bisimulations are precisely strong-mono relations.*

Proof. Let R be a strong-mono relation on (L_1, α_1) and (L_2, α_2). Consider

$$\begin{aligned} \delta : \qquad R &\longrightarrow \mathbb{MV}(R) \\ (x, y) &\longmapsto (\alpha_1(x), \alpha_2(x)) \end{aligned}$$

Then δ is a $\mathcal{BL}$-morphism since α_1 and α_2 are so. Moreover, for all $(x, y) \in R$,

$$\begin{aligned} \alpha \circ \pi_1(x, y) &= \alpha_1(x) \\ &= \mathbb{MV}(\pi_1)(\alpha_1(x), \alpha_2(x)) \\ &= \mathbb{MV}(\pi_1) \circ \delta(x, y). \end{aligned}$$

Thus π_1 is an $\mathbb{MV}$-homomorphism. Similarly, one can show that π_2 is an $mathbbMV$-homomorphism.

□

Proposition 3.1.9. *The largest bissimulation between two $\mathbb{MV}$-coalgebras (L, α) and (L', β) always exists. Moreover, when $(L, \alpha) = (L', \beta)$ that largest bisimulation (called the bisimilarity on (L, α)) is an equivalence relation.*

Proof. $\mathcal{BL}$ is wellpowered, complete, cocomplete and the $\mathbb{MV}$-functor preserves limits. It follows from [[4], Proposition 5.5] that the largest bisimulation between any two **MV**-coalgebras exists. The second part of the Proposition is a consequence of [[4], Corollary 5.6].

□

Lemma 3.1.10. *Let $FIX(\mathbb{MV})$ denote the class of fixed points of $\mathbb{MV}$ and **MV** the class of MV-algebras. The following hold:*

(i) $FIX(\mathbb{MV}) = \boldsymbol{MV}$.

(ii) $(\boldsymbol{G}_1, id_{\boldsymbol{G}_1})$ *is the final coalgebra for the functor* $\mathbb{MV}$.

Proof. (i) Since any MV-algebra is its own MV-center, we just have to prove that $FIX(\mathbb{MV}) \subseteq$ **MV**. Let $L \in FIX(\mathbb{MV})$ be a BL-algebra. Then there exists a $\mathcal{BL}$-isomorphism $\varphi : L \longrightarrow MV(L)$. For any $x \in L$, there exists $y \in MV(L)$ such that $x = \varphi^{-1}(y)$. This means that

$$\bar{\bar{x}} = \overline{\overline{\varphi^{-1}(y)}} = \varphi^{-1}(\bar{\bar{y}}) = \varphi^{-1}(y) = x,$$

since the converse φ^{-1} of φ is a $\mathcal{BL}$-morphism. Therefore L is an MV-algebra.
(ii) follows from the fact that $\mathbb{MV}$ preserves final objects. □

Since $\mathbb{MV}$ preserves pullbacks, the composition of $\mathbb{MV}$-bissimulations is again an $\mathbb{MV}$-bisimulation (see [[4], Example 5.4]). Therefore (or see [32], Theorem 2.5.7]), pullbacks of $\mathbb{MV}$-homomorphisms are $\mathbb{MV}$-bisimulations. In that case, it is stated in [[4], Remark 5.8] for an arbitrary weak pullback preserving endofunctor that the largest bisimulation on a given coalgebra is the kernel equivalence of the unique homomorphism from that coalgebra to the final coalgebra. The following result comes from that observation:

Proposition 3.1.11. *Let (L, α) be an $\mathbb{MV}$-coalgebra. The largest bisimulation on (L, α) is $L \times L$.*

Proof. By [[4], Remark 5.8], the largest bisimulation on (L, α) is the kernel pair of the morphism $(L, \alpha) \xrightarrow{\tilde{1}} (\mathbf{G}_1, id_{\mathbf{G}_1})$, which is clearly $L \times L$. □

It is well known (see, e.g., [[4], Proposition 4.3] or [[32], Theorem 1.2.4]) that the forgetful functor from the category of coalgebras to the base category creates colimits. It follows that $\mathcal{BL}_{\mathbb{MV}}$ has whatever colimit $\mathcal{BL}$ has. On the other hand, $\mathcal{BL}_{\mathbb{MV}}$ has all limits preserved by $\mathbb{MV}$. Since $\mathcal{BL}$ is complete, cocomplete and $\mathbb{MV}$ is a limit-preserving functor, we have:

Theorem 3.1.12. *$\mathcal{BL}_{\mathbb{MV}}$ is complete and cocomplete.*

3.1.4 Coalgebraic relation between $\mathcal{BL}$ and $\mathcal{MV}$

Every natural transformation $\eta : \mathbb{F} \longrightarrow \mathbb{G}$ between two endofunctors (or types) of a category $\mathcal{C}$ induces a functor between categories of coalgebras $\mathcal{C}_{\mathbb{F}}$ and $\mathcal{C}_{\mathbb{G}}$. This was already observed by Rutten in [49] for the category $\mathcal{SET}$ and other facts about natural transformations and coalgebras were proved by Gumm in [22] and [21]. Our aim in this subsection is to lift the coreflectivity of $\mathcal{MV}$ to an coalgebraic one.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, $\mathbb{F} : \mathcal{C} \longrightarrow \mathcal{C}$ and $\mathbb{G} : \mathcal{D} \longrightarrow \mathcal{D}$ be two functors. The following result has been proven by Kianpi (2016) [37] in the case where $\mathcal{C} = \mathcal{D}$.

Proposition 3.1.13. *Let $\mathbb{H} : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Every natural transformation $\eta : \mathbb{H} \circ \mathbb{F} \longrightarrow \mathbb{G} \circ \mathbb{H}$ induces a functor $\mathbb{H}_\eta : \mathcal{C}_{\mathbb{F}} \longrightarrow \mathcal{D}_{\mathbb{G}}$ defined as:*

- *for any $\mathbb{F}$-coalgebra (A, α), $\mathbb{H}_\eta(A, \alpha) = (\mathbb{H}(A), \eta_A \circ \mathbb{H}(\alpha))$;*
- *For any homomorphism $(A, \alpha) \xrightarrow{f} (B, \beta)$, $\mathbb{H}_\eta(f) = \mathbb{H}(f)$.*

If $\mathbb{H}$ is faithful, then so is $\mathbb{H}_\eta$.

Proof. Let $A \in ob(\mathcal{C})$. Then

$$\eta_A \circ \mathbb{H}(\alpha) : \mathbb{H}(A) \xrightarrow{\mathbb{H}(\alpha)} \mathbb{H} \circ \mathbb{F}(A) \xrightarrow{\eta_A} \mathbb{G} \circ \mathbb{H}(A)$$

is a clearly a $\mathcal{D}$-morphism and so, $(\mathbb{H}(A), \eta_A \circ \mathbb{H}(\alpha))$ is a $\mathbb{G}$-coalgebra. Let $(A, \alpha) \xrightarrow{f} (B, \beta)$ be a $\mathcal{C}_{\mathbb{F}}$-morphism. Let show that $\mathbb{H}_\eta(f)$ is a $\mathcal{D}_{\mathbb{G}}$-morphism.

$$\begin{array}{ccc}
\mathbb{H}(A) & \xrightarrow{\mathbb{H}(f)} & \mathbb{H}(B) \\
\downarrow{\scriptstyle \mathbb{H}(\alpha)} & & \downarrow{\scriptstyle \mathbb{H}(\beta)} \\
\mathbb{H}\circ\mathbb{F}(A) & \xrightarrow[\mathbb{H}\circ\mathbb{F}(f)]{} & \mathbb{H}\circ\mathbb{F}(B) \\
\downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\
\mathbb{G}\circ\mathbb{H}(A) & \xrightarrow[\mathbb{G}\circ\mathbb{H}(f)]{} & \mathbb{G}\circ\mathbb{H}(B)
\end{array}$$

Since f is a homomorphism, we have $\beta \circ f = \mathbb{F}(f) \circ \alpha$. Since $\mathbb{H}$ is a functor, we obtain

$$\mathbb{H}(\beta) \circ \mathbb{H}(f) = \mathbb{H} \circ \mathbb{F}(f) \circ \mathbb{H}(\alpha).$$

This shows that the up rectangle of the above diagram commutes. The naturality of η yields the commutativity of the down rectangle. Thus the whole diagram commutes, meaning that

$$\eta_B \circ \mathbb{H}(\beta) \circ \mathbb{H}(f) = \mathbb{G} \circ \mathbb{H}(f) \circ \eta_A \circ \mathbb{H}(A).$$

Therefore, $\mathbb{H}(f)$ is a homomorphism of $\mathbb{G}$-coalgebras.

The functoriality and the faithfulness of $\mathbb{H}_\eta$ follows straightforwardly from that of $\mathbb{H}$. □

Since the $\mathcal{MV}$-functor is idempotent, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{BL} & \xrightarrow{\mathbb{MV}} & \mathcal{MV} \\
\downarrow{\scriptstyle \mathbb{MV}} & & \downarrow{\scriptstyle Id_{\mathcal{MV}}} \\
\mathcal{BL} & \xrightarrow[\mathbb{MV}]{} & \mathcal{MV}
\end{array} .$$

Therefore, $Id_{MV} : \mathbb{MV} \circ \mathbb{MV} \longrightarrow Id_{\mathcal{MV}} \circ \mathbb{MV}$ is obviously a natural transformation, which will be denoted η in the sequel.

In Theorem 2.2.10 we showed that the $\mathcal{MV}$-functor is right adjoint to the inclusion functor $i : \mathcal{MV} \hookrightarrow \mathcal{BL}$. In what follows, we lift this result to the categories of coalgebras.

Proposition 3.1.14. $\mathcal{MV}_{\mathbb{I}d_{\mathcal{MV}}}$ *is a coreflective subcategory of* $\mathcal{BL}_{\mathbb{MV}}$

Proof. Let $i^* : \mathcal{MV}_{\mathbb{I}d_{\mathcal{MV}}} \hookrightarrow \mathcal{BL}_{\mathbb{MV}}$ be the inclusion functor. By Proposition 3.1.13, $\mathbb{MV}_\eta$ is a functor, and we have for any $\mathbb{MV}$-coalgebra (L, α), $\mathbb{MV}_\eta(L, \alpha) = (MV(L), \mathbb{MV}(\alpha))$. Let $\mathbf{M} = (M, \alpha) \in \mathcal{MV}_{\mathbb{I}d_{\mathcal{MV}}}$ and $\mathbf{L} = (L, \epsilon) \in \mathcal{BL}_{\mathbb{MV}}$. Define

$$\begin{array}{rccc}
\Phi^*_{(\mathbf{M},\mathbf{L})} : & \mathcal{BL}_{\mathbb{MV}}(\mathbf{M}, \mathbf{L}) & \longrightarrow & \mathrm{MV}_{\mathbb{I}d_{\mathcal{MV}}}(\mathbf{M}, \mathbb{MV}_\eta(\mathbf{L})) \\
& \mathbf{M} \xrightarrow{f} \mathbf{L} & \longmapsto & \mathbf{M} \overset{\mathbb{MV}_\eta(f)}{\longrightarrow} (MV(L), \mathbb{MV}(\epsilon))
\end{array}$$

$\Phi^*_{(\mathbf{M},\mathbf{L})}$ is well defined since $\mathbb{MV}_\eta$ is a functor. Moreover, for any $f \in \mathcal{BL}_{\mathbb{MV}}(\mathbf{M}, \mathbf{L})$, we have

$$\Phi^*_{(\mathbf{M},\mathbf{L})}(f) = \mathbb{MV}_\eta(f) = \mathbb{MV}(f) = \Phi_{(M,L)}(f),$$

where $\Phi_{(M,L)}$ is the map defined in the proof of Theorem [?]. Thus, the bijectivity and the naturality of $\Phi^*_{(\mathbf{M},\mathbf{L})}$ follows from that of $\Phi_{(M,L)}$. Therefore, $\mathbb{MV}_\eta$ is a right adjoint of i^*.

□

3.2 Topological $\mathbb{MV}$-coalgebras

According to the viewpoint of the School of Bourbaki, there are three mother structures in mathematics from which all other mathematical structures can be generated and they are not reducible from one to the other: algebraic structures, topological structures, and order structures. The interaction between topological structures and order structures is a stimulating topic in mathematics and computer science, e.g., in the theory of domains and the theory of locales. Topologies and algebras naturally come in contact in representation theory and topological groups . In recent decades, several researchers have proposed a number of algebraic structures associated with logical systems equipped with topologies (further details may be found in Di Nola (2003) [18], L. Leustean (2004) [42], Haveshki (2007) [28], Borzooei (2011) [12]). Recently Haveshki (2007) [28] and also Zahiri (2016) [56] applied filter theory to construct a topology on BL-algebras. In this section, we introduce and investigate topological **MV**-coalgebras. Moreover, we construct an inverse system in the category of $\mathbb{MV}$-coalgebras.

Definition 3.2.1. Let $L = (L, \wedge, \vee, *, \to, 0, 1)$ be a BL-algebra with a topology τ. Then, (L, τ) is called a *topological BL-algebra* if the operations $\wedge, \vee, *, \to$ are continuous.

3.2.1 Some facts about deductive systems in BL-algebras

Definition 3.2.2. A ds D of a BL-algebra L is said *Boolean* if for all $x \in L$, $x \vee \bar{x} \in D$.

Example 3.2.3. In Example 1.3.11 (the third case), for any $x \in [0; 1]$, $\bar{x} = 1 - x$. So $x \vee \bar{x} \geq \frac{1}{2}$. Thus $\left[\frac{1}{2}; 1\right]$ is a Boolean filter of the Lukasiewicz-algebra.

These filters have been deeply studied by many authors (see, e.g. [E. Turunen (2001) [50], Haveshki (2008) [29], Turunen (2011) [52]]) and the following has been proved:

Proposition 3.2.4. *In any BL-algebra, the following are equivalent:*

(i) D is a Boolean ds;

(ii) L/D is a Boolean algebra;

(iii) If $\bar{x} \to x \in D$, then $x \in D$.

The following result will be useful in the sequel:

Lemma 3.2.5. *Let D be a Boolean ds a BL-algebra L. For all $x, y \in L$,*

$$\bar{y} \to \bar{x} \in D \text{ implies } x \to y \in D.$$

Proof. Let $x, y \in L$ such that $\bar{y} \to \bar{x} \in D$. Since D is Boolean, $y \vee \bar{y} \in D$. Then by Lemma 1.3.1(9),

$$y \vee \bar{y} \leq ((y \to \bar{y}) \to \bar{y}) \wedge ((\bar{y} \to y) \to y).$$

So $(\bar{y} \to y) \to y \in D$ and since $\bar{\bar{y}} \leq \bar{y} \to y$, we obtain $(\bar{y} \to y) \to y \leq \bar{\bar{y}} \to y$. Thus, $\bar{\bar{y}} \to y \in D$ implying $[y]_D = \left[\bar{\bar{y}}\right]_D$. Therefore,

$$\begin{aligned} [x \to y]_D &= [x]_D \to [y]_D \\ &= [x]_D \to \left[\bar{\bar{y}}\right]_D \\ &= [\bar{y}]_D \to [\bar{x}]_D \text{ by Lemma 1.3.1(10)} \\ &= [\bar{y} \to \bar{x}]_D . \end{aligned}$$

It follows that $(\bar{y} \to \bar{x}) \to (x \to y) \in D$, which implies by hypothesis that $x \to y \in D$. □

Lemma 3.2.6. *Let D be a ds of L, $L \xrightarrow{f} L'$ be a BL-morphism. Then for all $x \in L$, $f([x]_D) \subseteq [f(x)]_{f(D)}$. The equality holds when $x \in D$.*

Proof. Let $y \in f([x]_D)$. Then there exists $z \in L$ such that

$$y = f(z) \text{ and } (z \to x) \wedge (x \to z) \in D.$$

We have

$$(y \to f(x)) \wedge (f(x) \to y) = f(z \to x) \wedge f(x \to z) = f((z \to x) \wedge (x \to z)) \in f(D).$$

Hence $y \in [f(x)]_{f(D)}$. For the converse, suppose $x \in D$. Let $y \in [f(x)]_{f(D)}$. Then

$$(y \to f(x)) \wedge (f(x) \to y) \in f(D).$$

Since $f(D)$ is a ds of L, $y \in f(D)$, that is, there exists $z \in D$ such that $y = f(z)$. Since x and z are in D, $(z \to x) \wedge (x \to z) \in D$. Thus $z \in [x]_D$ and so $y \in f([x]_D)$.

□

3.2.2 Topology on $\mathbb{MV}$-coalgebras

Definition 3.2.7. Let (L, α) be an $\mathbb{MV}$-coalgebra.

(i) Let τ be a topology on L. $((L, \alpha), \tau)$ is called a *topological MV-coalgebra* if (L, τ) is a topological BL-algebra and α is continuous, i.e., for any $x \in L$ and any subset V of L containing $\alpha(x)$, there exists an open set U containing x such that $\alpha(U) \subseteq V$.

(ii) Let D be a ds of L. D is said *α-stable* if $\alpha(D) \subseteq D$.

For any $\mathbb{MV}$-coalgebra(L, α), the class of α-stable ds of L is not empty since it contains $\{1\}$.

A poset $(I, \leq)$ is said to be *upward directed* provided that for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let I be an upward directed set and $D = \{D_i, i \in I\}$ be a family of dss of a BL-algebra L. Then D is called a *system of dss* or simply a *system* of L if $i \leq j$ implies $D_j \subseteq D_i$, for any $i, j \in I$. An *inverse system* in a category $\mathcal{C}$, is a family $(B_i, \varphi_{i,j})_{i,j \in I}$ of objects indexed by an upward directed set I, with a family of morphisms $\varphi_{i,j} : B_i \longrightarrow B_j$, for $i \leq j$, satisfying the following conditions:

(1) $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$, for any $k \leq j \leq i$

(2) $\varphi_{i,i} = id_{B_i}$, for any $i \in I$.

Definition 3.2.8. Let (L, τ) be a topological space. The topology τ is called a *linear topology* on L if there exists a base B for τ such that any element of B containing 1 is a ds of L.

Theorem 3.2.9. *Let (L, α) be an $\mathbb{MV}$-coalgebra and $D = \{D_i, i \in I\}$ be a α-stable system of L (i.e. each D_i is a α-stable ds of L, $i \in I$). Then*

(i) The set $B = \{[x]_{D_i}, x \in L, i \in I\}$ is a base for a topology on L and τ_B, the topology induced by B is linear.

(ii) $((L, \alpha), \tau_B)$ is a topological $\mathbb{MV}$-coalgebra.

Proof. (i) For all $x \in L$, and $i \in I$, $x \in [x]_{D_i}$. So, $L = \bigcup_{x \in L} \left\{[x]_{D_i}, i \in I\right\}$. Moreover let $x \in [y]_{D_i} \cap [z]_{D_j}$, for some $y, z \in L$ and $i, j \in I$. Since I is upward directed, there exists $k \in I$ such that $i \leq k$ and $j \leq k$. So $D_k \subseteq D_i$ and $D_k \subseteq D_j$. Let $t \in [x]_{D_k}$. Then

$$(t \to x) \wedge (x \to t) \in D_k,$$

implying

$$(t \to x) \wedge (x \to t) \in D_i$$

and

$$(t \to x) \wedge (x \to t) \in D_j.$$

So

$$t \in [x]_{D_i} = [y]_{D_i}$$

and

$$t \in [x]_{D_j} = [z]_{D_j}.$$

It follows that

$$t \in [y]_{D_i} \cap [z]_{D_j}$$

and then

$$[x]_{D_k} \subseteq [x]_{D_i} \cap \{[y]_{D_j}$$

. Therefore, B is a base for a topology on L.

Moreover, let $[x]_{D_i} \in B$ such that $1 \in [x]_{D_i}$. Then $[x]_{D_i} = [1]_{D_i} = D_i$. Thus $[x]_{D_i}$ is a filter of L and we can conclude that τ_B is linear.

(ii) Let first show that (L, τ_B) is a topological BL-algebra. Let $x, y \in L$ and $i \in I$. Then $[x]_{D_i} \times [y]_{D_i}$ is an open subset of $L \times L$. Let $z \in L$ such that $(x, y) \in f_*^{-1}([z]_{D_i})$, i.e. $x * y \in [z]_{D_i}$. For any $(u, v) \in [x]_{D_i} \times \{[y]_{D_i}$, since θ_{D_i} is a congruence relation, the compatibilty property leads to $(x * y; u * v) \in \theta_{D_i}$, which implies by transitivity $u * v \in [z]_{D_i}$, i.e. $(u, v) \in f_*^{-1}([z]_{D_i})$. So

$$[x]_{D_i} \times \{[y]_{D_i} \subseteq f_*^{-1}([z]_{D_i}).$$

It follows that $f_*^{-1}([z]_{D_i})$ is an open subset of L and so, f_* is a continuous. The continuity of $f_\propto$ is obtained in an analogue manner, for $\propto \in \{\wedge, \vee, \rightarrow\}$. Therefore, (L, τ_B) is a topological BL-algebra.

Now, we just have to show that α is continuous. Let $x \in L$, such that $\alpha(x) \in [z]_{D_i}$, $z \in L$. For all $y \in \alpha([x]_{D_i})$, we have by Lemma 3.2.6

$$(y \rightarrow \alpha(x)) \wedge (\alpha(x) \rightarrow y) \in \alpha(D_i) \subseteq D_i.$$

So $y \in [\alpha(x)]_{D_i} = [z]_{D_i}$. Therefore, $[x]_{D_i}$ is an open subset containing x such that $\alpha([x]_{D_i}) \subseteq [z]_{D_i}$. Thus α is continuous.

□

Theorem 3.2.10. *Let* (L, α) *be an* $\mathbb{MV}$*-coalgebra,* $D = \{D_i, i \in I\}$ *be a* α*-stable system of Boolean filters of* L*. For each* $i \in I$*, define* $\alpha_i : L/D_i \longrightarrow \alpha(L/D_i)$ *by:*

$$\alpha_i([x]_{D_i}) = [\alpha(x)]_{D_i}$$

. Consider the family of maps $(\varphi_{ij})_{i \leq j \in I}$ *defined by* $\varphi_{ij} : (L/D_j, \alpha_j) \longrightarrow (L/D_i, \alpha_i)$ *such that for all* $x \in L$*,*

$$\varphi_{ij}([x]_{D_j}) = [x]_{D_i}$$

. Then

(i) $((L/D_i, \alpha_i)_{i \in I)}; (\varphi_{ij})_{i \leq j \in I})$ *is an inverse system in* $\mathcal{BL}_{\mathbb{MV}}$*.*

(ii) $\overset{lim}{\longleftarrow} ((L/D_i, \alpha_i)_{i \in I)}; (\varphi_{ij})_{i \leq j \in I}) = ((X, \xi), \phi_j)$*, where*

$$X = \left\{ ([x]_{D_i})_{i \in I} \in \prod_{i \in I} L/D_i \mid x_i \in [x_j]_{D_j}, j < i \right\},$$

$$\begin{array}{rccc} \xi : & X & \longrightarrow & MV(X) \\ & ([x_i]_{D_i})_{i \in I} & \longmapsto & ([\alpha(x_i)]_{D_i})_{i \in I} \end{array}$$

and

$$\begin{array}{rccl}\phi_j: & X & \longrightarrow & L/D_j\\ & ([x_i]_{D_i})_{i\in I} & \longmapsto & [x_j]_{D_j}\end{array}$$

.

Proof. (i) Let $x \in L$. By Remark 3.1.1, there exists $z \in L$ such that $\alpha(x) = \bar{z}$. So for all $i \in I$,

$$\alpha_i([x]_{D_i}) = [\alpha(x)]_{D_i} = [\bar{z}]_{D_i} = \overline{[z]_{D_i}}.$$

Moreover, for $x, x' \in L$ such that $[x]_{D_i} = [x']_{D_i}$. Then $x \to x' \in D_i$ and $x' \to x \in D_i$. It follows that $\alpha(x \to x') \in D_i$ and $\alpha(x' \to x) \in D_i$. So $[\alpha(x)]_{D_i} = [\alpha(x')]_{D_i}$. Thus each α_i is well defined. Since α is a $\mathcal{BL}$-morphism, it is easily checked that each α_i is a $\mathcal{BL}$-morphism and therefore, for all $i \in I$, $(L/D_i, \alpha_i)$ is an $\mathbb{MV}$-coalgebra.

Let $i, j \in I$ such that $i \leq j$. Since $F_j \subseteq F_i$, φ_{ij} is well defined and is clearly a $\mathcal{BL}$-morphism. Moreover, let $x \in L$ and let $z \in L$ such that $\alpha(x) = \bar{z}$. Then for all $j \in I$,

$$\begin{aligned}\overline{\varphi_{ij}([z]_{D_j})} &= \varphi_{ij}(\overline{[z]_{D_j}})\\ &= \varphi_{ij}([\alpha(x)]_{D_j})\\ &= [\alpha(x)]_{D_i}\\ &= \alpha_i([x]_{D_i})\\ &= \alpha_i \circ \varphi_{ij}([x]_{D_j}).\end{aligned}$$

Hence, by Proposition 3.1.2, φ_{ij} is an $\mathbb{MV}$-homomorphism. It is clear that $\varphi_{ii} = 1_{L/D_i}$ and for $i \leq j \leq k \in I$,

$$\varphi_{ij} \circ \varphi_{jk}([x]_{D_k}) = \varphi_{jk}([x]_{D_j}) = [x]_{D_k} = \varphi_{ik}([x]_{D_i}).$$

Thus, $((L/D_i, \alpha_i)_{i\in I)}; (\varphi_{ij})_{i\leq j\in I})$ is an inverse system in $\mathcal{BL}_{\mathbb{MV}}$.

(ii) It is easily checked that X is a BL-subalgebra of the product $\prod_{i\in I} L/D_i$. Let show that ξ is a BL-morphism. Let $([x_i]_{D_i})_{i\in I}$ and $([y_i]_{D_i})_{i\in I}$ in X such that $[x_i]_{D_i} = [y_i]_{D_i}$ for all $i \in I$. We have

$$\begin{aligned}\xi(([x_i]_{D_i})_{i\in I}) &= ([\alpha(x_i)]_{D_i})_{i\in I}\\ &= (\alpha_i([x_i]_{D_i}))_{i\in I}\\ &= (\alpha_i([y_i]_{D_i}))_{i\in I}\\ &= ([\alpha(y_i)]_{D_i})_{i\in I}\\ &= \xi(([y_i]_{D_i})_{i\in I}).\end{aligned}$$

Moreover, For all $i \in I$, there exists $z_i \in L$ such that $\alpha(x_i) = \overline{z_i}$. So

$$\xi(([x_i]_{D_i})_{i\in I}) = ([\overline{z_i}]_{D_i})_{i\in I} = \overline{([z_i]_{D_i})_{i\in I}}.$$

Let $j < i$ then $x_i \in [x_j]_{D_j}$ by definition of X. Hence,

$$(x_i \to x_j) \wedge (x_j \to x_i) \in D_j.$$

Since α is a BL-morphism, it follows from the stability of the dss that

$$(\alpha(x_i) \to \alpha(x_j)) \wedge (\alpha(x_j) \to \alpha(x_i)) \in D_j.$$

Thus,

$$(\overline{z_i} \to \overline{z_j}) \wedge (\overline{z_j} \to \overline{z_i}) \in D_j.$$

This leads by Lemma 3.2.5 to

$$(z_i \to z_j) \wedge (z_j \to z_i) \in D_j.$$

Hence $z_i \in [z_j]_{D_j}$, implying that $([z_i]_{D_i})_{i\in I} \in X$. This means that $\xi(([x_i]_{D_i})_{i\in I}) \in MV(X)$ and so, ξ is well defined. Since α is a BL-morphism, ξ preserves the BL-algebras operations and is therefore a BL-morphism.

It is easily checked that ϕ_j is a homomorphism of 𝕄𝕍-coalgebras, for all $j \in I$. Let $i, j \in I$, such that $j < i$. For any $([x_i]_{D_i})_{i\in I} \in X$, we have

$$\begin{aligned} \varphi_{ij} \circ \phi_i([x_i]_{D_i})_{i\in I}) &= \varphi_{ij}([\alpha(x_i)]_{D_i}) \\ &= [x_i]_{D_j} \\ &= [x_j]_{D_j} \text{ by definition of } X \\ &= \phi_j(([x_i]_{D_i})_{i\in I}) \end{aligned}$$

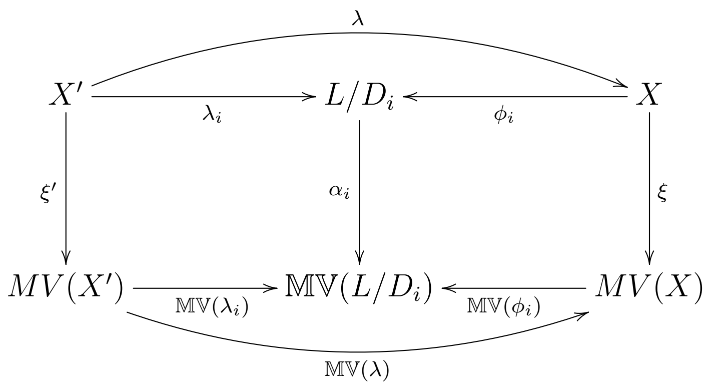
Thus, $\varphi_{ij} \circ \phi_i = \phi_j$.

It remains to show the universal property. Let (X', ξ') be a 𝕄𝕍-coalgebra, and $(X' \xrightarrow{\lambda_i} L/D_i)_{i\in I}$ a family of homomorphisms of 𝕄𝕍-coalgebras such that $\varphi_{ij} \circ \lambda_i = \lambda_j$, for all $i, j \in I$ such that $j < i$. Define

$$\begin{aligned} \lambda : \; X' &\longrightarrow X \\ x &\longmapsto (\lambda_i(x))_{i\in I} \end{aligned}$$

For any $x \in X'$, $\lambda_i(x) = [y_i]_{D_i}$ with $y_i \in L$ for all $i \in I$. For any $j \in I$ such that $j < i$, we have $\varphi_{ij} \circ \lambda_i(x) = \lambda_j(x)$. So $\varphi_{ij}([y_i]_{D_i}) = [y_j]_{D_j}$, which implies that $[y_i]_{D_j} = [y_j]_{D_j}$ and then $y_i \in [y_j]_{D_j}$, i.e. $(\lambda_i(x))_{i\in I} \in X$. This proves that λ is well defined. Since λ_i

is a BL-morphism, then so is λ. We obviously have $\phi_i \circ \lambda = \lambda_i$ for all $i \in I$.



Let $x \in X'$. We have:

$$\begin{aligned}
\mathbb{MV}(\lambda) \circ \xi'(x) &= \lambda \circ \xi'(x) \\
&= (\lambda_i(\xi'(x)))_{i\in I} \\
&= (\alpha_i \circ \lambda_i(x))_{i\in I} \text{ by commutativity of the first square} \\
&= (\alpha_i \circ \phi_i \circ \lambda(x))_{i\in I} \\
&= (\mathbb{MV}(\phi_i) \circ \xi \circ \lambda(x))_{i\in I} \text{ by commutativity of the second square} \\
&= (\phi_i \circ \xi \circ \lambda(x))_{i\in I} \\
&= \xi \circ \lambda(x).
\end{aligned}$$

Thus, λ is an $\mathbb{MV}$-homomorphism.
Let $\lambda' : (X', \xi') \longrightarrow (X, \xi)$ be another $\mathbb{MV}$-homomorphism such that $\phi_i \circ \lambda' = \lambda_i$, for all $i \in I$. Then $\phi_i \circ \lambda'(x) = \phi_i \circ \lambda(x)$ for all $x \in X'$. Suppose that $\lambda(x) = ([y_k]_{D_k})_{k\in I}$ and $\lambda'(x) = ([y'_k]_{D_k})_{k\in I}$. Then

$$\phi_i(([y_k]_{D_k})_{k\in I}) = \phi_i(([y'_k]_{D_k})_{k\in I}).$$

So $[y_i]_{D_i} = [y'_i]_{D_i}$ for all $i \in I$. Therefore, $\lambda = \lambda'$.

□

Let $((L, \alpha), \tau)$ denote a topological MV -coalgebra where τ is the topology induced by a system $\{D_i, i \in I\}$ of α-stable dss. Let C be the set of all Cauchy sequences on (L, τ) and C_1 the set of all sequences which converges to 1. It has been shown in Zahiri (2016) [[56], Theorem 4.7 and proposition 4.8] that C is a BL-algebra and C_1 is a filter of C. Thus the quotient C/C_1 is a BL-algebra called the *completion* of L with respect to the topology induced by the system of dss. In what follows we equip the completion with a structure morphism and show that the coalgebra obtained is isomorphic to the inverse limit of the system defined above.

With the notations of Theorem 3.2.10, it has been shown in Zahiri (2016) [[56], Theorem 4.11] that the following result holds:

Lemma 3.2.11. *Let $f : X \longrightarrow C/C_1$ defined by: $f(([(x_i]_{D_i})_i) = [(x_i)_i]_{C_1}$. Then f is an isomorphism of BL-algebras.*

Theorem 3.2.12. *Let (L, α) be an MV-coalgebra. With the notation of Theorem 3.2.10, (X, ξ) is isomorphic to $(C/C_1, \gamma)$, where $\gamma([(x_i)_i]_{C_1}) = [(\alpha(x_i))_i]_{C_1}$.*

Proof. Since α is continuous, $(\alpha(x_i))_i$ is a Cauchy sequence. Now let $(x_i)_i$ and $(y_i)_i$ be two Cauchy sequences such that $[(x_i)_i]_{C_1} = [(y_i)_i]_{C_1}$. Then $((x_i)_i \to (y_i)_i) \wedge ((x_i)_i \to (y_i)_i) \in C_1$. By continuity of α and the fact that α is a BL-morphism, we have $(\alpha(x_i))_i \to (\alpha(y_i))_i) \wedge (\alpha(x_i))_i \to (\alpha(y_i))_i) \in C_1$. Thus γ is well defined. With similar arguments, one proves that γ is a BL-morphism and thus, $(C/C_1, \gamma)$ is an MV-coalgebra.

Consider $f : X \longrightarrow C/C_1$ defined by: $f(([(x_i]_{D_i})_i) = [(x_i)_i]_{C_1}$. It follows from Lemma 3.2.11 that f is an isomorphism of BL-algebras. Moreover, $MV(f) \circ \xi(([x_i]_{D_i})_{i\in I}) = f(([\alpha(x_i)]_{D_i})_{i\in I}) = [(\alpha(x_i))_i]_{C_1} = \gamma \circ f(([x_i]_{D_i})_{i\in I})$. So f is a homomorphism of coalgebras. Therefore, (X, ξ) and $(C/C_1, \gamma)$ are isomorphic.

□

Definition 3.2.13. A concrete category $(\mathcal{C}, U)$ over an $(E, \mathbf{M})$-category $\mathcal{X}$ is said to be $\mathbf{M}$-*topological* provided that every structured source in $\mathbf{M}$ has a unique initial lift. If for example, $\mathbf{M}$ = Mono- Sources, the term *monotopological* is used.

The following properties of $\mathbf{M}$-topological categories can be found in [Herrlich (1974) [30], Corollary 5.2, and Corollary 6.4,]:

Lemma 3.2.14. *Let $\mathcal{C}$ be a $\boldsymbol{M}$-topological category over $\mathcal{X}$. If $\mathcal{X}$ is (co)complete, then so is $\mathcal{C}$.*

The following result can be found in Kianpi (2020) [36] and will be useful in the sequel.

Lemma 3.2.15. *If $\mathcal{C}$ is an $(Epi, StrongMono)$-category and F preserves strong monosources, then $\mathcal{C}_{\mathbb{F}}$ is an $(Epi, StrongMono)$-category with $(Epi, StrongMono)$- factorizations created by the forgetful functor $U_{\mathbb{F}}$.*

It is well known (see, e.g. Brummer (1984) [13]) that for any topological functor $\mathbb{V} : \mathcal{C} \longrightarrow \mathcal{SET}$, and any forgetful functor $\mathbb{U} : \mathcal{X} \longrightarrow \mathcal{SET}$, Where $\mathcal{X}$ is a category of universal algebra, the category $\mathcal{CX}$ with objects (A, X), $(A \in ob(\mathcal{C})$, $X \in ob(\mathcal{X}))$ such that $\mathbb{V}A = \mathbb{U}X$ is topological over $\mathcal{X}$. This leads to the following result:

Lemma 3.2.16. *Let $\mathcal{T}op\mathcal{BL}$ be the category of topological BL-algebras with continuous BL-morphisms. Then $\mathcal{T}op\mathcal{BL}$ is topological over $\mathcal{BL}$.*

The question rising up from the above observation is: does the result holds if $\mathcal{X}$ is a category of coalgebras over universal algebras? The following theorem is a partial answer, in the setting of $\mathbb{MV}$-coalgebras.

Theorem 3.2.17. *The category $\mathcal{T}op\mathcal{BL}_{\mathbb{MV}}$ of topological $\mathbb{MV}$-coalgebras and continuous $\mathbb{MV}$-homomorphisms is strong mono-topological over $\mathcal{BL}_{\mathbb{MV}}$.*

Proof. Since $\mathcal{BL}$ is an $(Epi, StrongMono)$-category and $\mathbb{MV}$ is limit-preserving, it follows from Lemma 3.2.15 that $\mathcal{BL}_{\mathbb{MV}}$ is an $(Epi, StrongMono)$-category.

Let $\mathbb{T} : \mathcal{T}op\mathcal{BL}_{\mathbb{MV}} \longrightarrow \mathcal{BL}_{\mathbb{MV}}$ and $\mathbb{U} : \mathcal{T}op\mathcal{BL} \longrightarrow \mathcal{BL}$ be the standard forgetful functors. We have to show that $\mathbb{T}$ is strong monotopological.

Let $\mathsf{S} = ((L,\alpha) \xrightarrow{f_i} \mathbb{T}((L_i,\alpha_i),\tau_i))_I$ be a $\mathbb{T}$-structured strong monosource in $\mathcal{BL}_{\mathbb{MV}}$. Then $(L \xrightarrow{f_i} U(L_i,\tau_i))_I$ is an $\mathbb{U}$-structured source in $\mathcal{BL}$. So, by Lemma 3.2.16 it admits a unique U-initial lift $(L,\alpha) \xrightarrow{\bar{f}_i} (L_i,\tau_i))_I$. Let show that $\mathsf{S}' = ((L,\alpha),\tau) \xrightarrow{\bar{f}_i} ((L_i,\alpha_i),\tau_i))_I$ is a $\mathbb{T}$-initial lift of S. It is clear that $\mathsf{S}' = T\mathsf{S}$.

Let $x \in MV(L)$, and V a subset of $MV(L)$ such that $\alpha(x) \in V$. Since the following diagram commutes, for all $i \in I$,

$$\begin{array}{ccc} L & \xrightarrow{f_i} & L_i \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha_i} \\ MV(L) & \xrightarrow[\mathbb{MV}(f_i)]{} & MV(L_i) \end{array}$$

and by the fact that α_i and f_i are continuous, we obtain that $\mathbb{MV}(f_i)\circ\alpha$ is a continuous function and $\mathbb{MV}(f_i)\circ\alpha(x) \in f_i(V)$ for all $i \in I$. Hence, there exists an open subset U of L containing x such that

$$\mathbb{MV}(f_i)\circ\alpha(U) \subseteq \mathbb{MV}(f_i)(V) \text{ for all } i \in I,$$

i.e. $f_i \circ \alpha(U) \subseteq f_i(V)$ for all $i \in I$. Since S is a strong monosource, we obtain $\alpha(U) \subseteq V$. Thus, α is continuous and therefore, $((L,\alpha),\tau)$ is a topological $\mathbb{MV}$-coalgebra.

Let show now that S' is $\mathbb{T}$-initial. Let $\mathsf{S}'' = ((L',\alpha'),\tau') \xrightarrow{g_i} ((L_i,\alpha_i),\tau_i))_I$. Let $(L',\alpha') \xrightarrow{h} (L,\alpha)$ be a $\mathbb{MV}$-homomorphism such that $T\mathsf{S}'' = T\mathsf{S}' \circ h$.Then h is a BL-morphism from $\mathbb{U}(L',\tau')$ to $\mathbb{U}(L,\tau)$ such that

$$\mathbb{U}((L',\tau') \xrightarrow{g_i} (L_i,\tau_i))_I = \mathbb{U}((L,\tau) \xrightarrow{\bar{f}_i} (L_i,\tau_i))_I \circ h.$$

Since $((L,\tau) \xrightarrow{\bar{f}_i} (L_i,\tau_i))_I$ is initial, there exists $\bar{h}$ $(L',\tau') \xrightarrow{\bar{h}} (L,\tau)$ such that

$$((L',\tau') \xrightarrow{g_i} (L_i,\tau_i))_I = ((L,\tau) \xrightarrow{\bar{f}_i} (L_i,\tau_i))_I \circ \bar{h}$$

and $h = U\bar{h}$. $\bar{h}$ is a $\mathbb{MV}$-homomorphism since so is h; moreover, $\bar{h}$ is continuous by construction. Thus, $\bar{h}$ is a $\mathcal{T}op\mathcal{BL}_{\mathbb{MV}}$-morphism. It is easily checked that $\mathsf{S}' = \mathsf{S}\circ\bar{h}$ and $T\bar{h} = h$. Therefore, S' is a T-initial source. The unicity of S' follows from the faithfulness of $\mathbb{T}$.

□

Since $\mathcal{BL}_{\mathbb{MV}}$ is complete and cocomplete, the above Theorem, combined with Lemma 3.2.14 leads to the following result:

Corollary 3.2.18. *$\mathcal{T}op\mathcal{BL}_{\mathbb{MV}}$ is complete and cocomplete.*

3.2.3 Conclusion

In this chapter, we proved that the category of MV-coalgebras is complete and cocomplete, meaning that a final $\mathbb{MV}$-coalgebra exists. We also characterize homomorphisms, subcoalgebras and bisimulations in the category $\mathcal{BL}_{\mathbb{MV}}$. We established a coalgebraic lift of the coreflectivity of $\mathcal{MV}$ using an adequat natural transformation. Moreover, we constructed an inverse system in the category of $\mathbb{MV}$-coalgebras and show that the category of topological $\mathbb{MV}$-coalgebras is strong-monotopological over the category of $\mathbb{MV}$-coalgebras.

Chapter 4

FRAME REPRESENTATION OF Π-COALGEBRAS

One of the main interests of the study of coalgebras is the development of coalgebraic logical foundations over base categories, as a way of reasoning in a quantitive way about transition systems. There is a strong link between coalgebras and modal logic (see [39] or [48]). We investigate this link in the framework of BL-algebras. We introduce local BL-frames based on local BL-algebras, and show that the category of local BL-frames is isomorphic to the category of $\prod$-coalgebras, where $\prod$ is the endofunctor on the category of local BL-algebras and BL-morphisms which assigns to each local BL-algebra its quotient by its unique maximal filter.

In this chapter, the category of coalgebras of a functor $\mathbb{F}$ will be denoted by $Coalg(\mathbb{F})$.

4.1 Some facts about local BL-algebras

In this section, we present some properties of local BL-algebras which are BL-algebras with a unique maximal filter. We define a non trivial endofunctor of the category of local BL-algebras and investigate the corresponding coalgebras.

4.1.1 Basic properties

Theorem 4.1.1 (Turunen (2001) [50], Theorem 5)**.** *Let L be a BL-algebra. Define*

$$D(L) = \{x \in L / x^n \neq 0 \textit{ for all integer } n\}.$$

The following are equivalent:

(i) $D(L)$ is a deductive system of L;

(ii) L is local;

(iii) $D(L)$ is the unique maximal deductive system of L.

Example 4.1.2. (i) $D(\mathbf{G}_3) = \{x, 1\}$, $D(\mathbf{M}_3) = D(\mathbf{G}_2) = \{1\}$ are deductive systems. So by the above theorem, $\mathbf{G}_3$, $\mathbf{M}_3$ and $\mathbf{G}_2$ are local BL-algebras;

(ii) Consider $A = ([0;1], \wedge, \vee, *, \rightarrow, 0, 1)$ the BL-algebra such that for all $x, y \in L$, $x * y = x \cdot y$ and $x \rightarrow y = 1$ if $x \leq y$ and $x \rightarrow y = \frac{y}{x}$ else. then $D(A) =]0;1]$ is a deductive system of A. Thus A is a local BL-algebra.

(iii) $\mathbf{G}_1$ is not local.

Proposition 4.1.3 (Di Nola (2003) [18], Proposition 1.10)**.** *Let $f : L \longrightarrow L'$ be a BL-morphism. If M' is a maximal filter of L', then $f^{-1}(M')$ is a maximal filter of L.*

Lemma 4.1.4 (Di Nola (2003) [18], Lemma 1.9)**.** *Let L be a nontrivial BL-algebra and M a proper deductive system of L. The following are equivalent:*

(i) M is maximal;

(ii) for any $x \in L$, $x \notin M \Rightarrow (x^n)^- \in M$ for some integer n.

Lemma 4.1.5. *Let f be a BL-morphism between two local BL-algebras L and L' whose maximal filters are M and M' respectively. If f is surjective, then $f(M) = M'$.*

Proof. Let f be a surjective morphism between two local BL-algebras L and L'. By Proposition 4.1.3, $f^{-1}(M') = M$. So, $f(f^{-1}(M')) = f(M)$. Since f is surjective we obtain $M' = f(M)$. □

Lemma 4.1.6. *Let L be a BL-algebra and F be a filter of L. Then θ_F is the unique congruence on L induced by F.*

Proof. Let θ be a congruence on L induced by F. We have to show that $\theta_F = \theta$. Let $(x, y) \in \theta_F$. Then $x \rightarrow y \in [1]_\theta$ and $y \rightarrow x \in [1]_\theta$. So by compatibility,

$$(x * (x \rightarrow y), x * 1) \in \theta \text{ and } (y * (y \rightarrow x), y * 1) \in \theta.$$

Hence by BL4 we obtain $(x \wedge y, x) \in \theta$ and $(y \wedge x, y) \in \theta$. Since θ is symetric and $\wedge$ commutative, it follows that $(x, x \wedge y) \in \theta$ and $(x \wedge y, y) \in \theta$. By transitivity, we have $(x, y) \in \theta$. Conversely, Let $(x, y) \in \theta$. Then $(x \rightarrow y, y \rightarrow y) \in \theta$ and $(y \rightarrow x, y \rightarrow y) \in \theta$. So $(x \rightarrow y, 1) \in \theta$ and $(y \rightarrow x, 1) \in \theta$. It follows that $x \rightarrow y \in F$ and $y \rightarrow x \in F$ and therefore, $(x, y) \in \theta_F$. □

In the sequel we will denote L/θ_F by L/F and $[x]_{\theta_F}$ by $[x]_F$.

Let M be the maximal deductive system of a local BL-algebra L.Then by [Di Nola (2003) [18], Proposition 1.13], since M is the unique maximal deductive system which contains M, L/M is a local BL-algebra. Therefore we have:

Lemma 4.1.7. *Let M be the maximal deductive system of a local BL-algebra L. Then L/M is a local BL-algebra and $D(L/M) = \{M\}$.*

Proof. We have $M^n = [1]_M^n = [1]_M \neq [0]_M$, which means that $M \in D(L/M)$. Let $[x]_M \in D(L/M)$. Then $[x^n]_M = [x]_M^n \neq [0]_M$, for all integer n. It follows that $x^n \rightarrow 0 \notin M$, for all integer n. Thus by Lemma 4.1.4, $x \in M$; That is, $[x]_M = M$. □

Local BL-algebras and BL-morphisms form a category which will be denoted by $l\mathcal{BL}$.

Proposition 4.1.8. *$l\mathcal{BL}$ is an isomorphism-closed subcategory of $\mathcal{BL}$.*

Proof. Let $f : L \longrightarrow G$ be an isomorphism between a BL-algebra L and a local BL-algebra G, whose inverse is g. Then by Proposition 4.1.3, $f^{-1}(M')$ is a maximal filter of L, where M' is the unique maximal filter of G. Moreover, let H be another maximal filter of L. Then $g^{-1}(H) = M'$ and so $H = g(M') = f^{-1}(M')$. Thus L is a local BL-algebra. □

Remark 4.1.9. Let L and L' be two local BL-algebras, M and M' their respectives maximal filters. Then $M \times L'$ and $L \times M'$ are maximal filters of $L \times L'$. Thus, $L \times L'$ is not a local BL-algebra. It follows that $l\mathcal{BL}$ has no (co)products and therefore $l\mathcal{BL}$ is not complete, nor cocomplete.

4.1.2 $\prod$-coalgebras

Proposition 4.1.10. *Consider the correspondance $\prod$ such that $\prod(L) = L/M$ for any local BL-algebra L whose unique maximal filter is M and $\prod(f) : L/M \longrightarrow L/M'$ such that*

$$\prod(f)([x]_M) = [f(x)]_{M'}.$$

Then $\prod$ is a covariant endofunctor on $l\mathcal{BL}$.

Proof. By Lemma 4.1.7 and the fact that θ_M is a congruence, $\prod$ is well defined. Moreover, let $L \xrightarrow{f} L'$ and $L' \xrightarrow{g} L''$ be two BL-morphisms. Let $x \in L$. We have

$$\prod(g) \circ \prod(f)([x]_M) = \prod(g)([f(x)]_{M'}) = [g \circ f(x)]_{M'} = \prod(g \circ f)([x]_M)$$

and also

$$\prod(id_L)([x]_M) = [x]_M = id_{\prod(L)}([x]_M).$$

□

Let $Coalg(\prod)$ be the category of $\prod$-coalgebras and $\prod$-homomorphisms. Let (L, α) be a $\prod$-coalgebra. For any x, y in a BL-algebra L, we denote $x \overset{\alpha}{\to} y$ by $\alpha(x) = [y]_M$. Then one can observe that $\prod$-coalgebras mimic non deterministic transition systems, contrary to MV-coalgebras.

Let (L, α) and (L', α') be two $\prod$-coalgebras. A BL-morphism $f : L \longrightarrow L'$ *weakly reflects* transition systems if for all $x \in L$ and $y \in L'$, $f(x) \overset{\alpha'}{\to} y$ implies $x \overset{\alpha}{\to} t$, with $f(t) \in [y]_M$, $t \in L$.

Proposition 4.1.11. *Let (L, α) and (L', α') be two $\prod$-coalgebras, and $f : L \longrightarrow L'$ a BL-morphism. The following are equivalent:*

(i) f is a $\prod$-homomorphism;

(ii) for all $x \in L$, $\alpha'(f(x)) = [f(z)]_{M'})$, whenever $\alpha(x) = [z]_M$;

(iii) f preserves and weakly reflects transitions.

Proof. $(i) \Leftrightarrow (ii)$ Straightforward.

$(ii) \Rightarrow (iii)$ Suppose for all $x \in L$, $\alpha'(f(x)) = [f(z)]_{M'})$, whenever $\alpha(x) = [z]_M$. Let $x, y \in L$ such that $x \stackrel{\alpha}{\to} y$. then $\alpha(x) = [y]_M$. So by hypothesis, $\alpha'(f(x)) = [f(y)]_{M'})$ implying $f(x) \stackrel{\alpha'}{\to} f(y)$. So f preserves transitions. Moreover, Let $x \in L$ and $y \in L'$ such that $f(x) \stackrel{\alpha'}{\to} y$. Then $\alpha'(f(x) = [y]_M$. Let $z \in L$ such that $\alpha(x) = [z]_M$. Then $x \stackrel{\alpha}{\to} z$ and by hypothesis, $[f(z)]_{M'} = \alpha'(f(x)) = [y]_M$, so $f(z) \in [y]_M$. Thus, f weakly preserves transitions.

$(iii) \Rightarrow (ii)$ Let $x \in L$, such that $\alpha(x) = [z]_M$. Then $x \stackrel{\alpha}{\to} z$, which implies by hypothesis that $f(x) \stackrel{\alpha'}{\to} f(z)$, i.e. $\alpha'(f(x)) = [f(z)]_{M'})$.

□

Proposition 4.1.12. *Let (L', α') be a $\prod$-coalgebra. A local BL-subalgebra L of L' is a $\prod$-subcoalgebra of (L', α') iff there exists a strong mono $L \stackrel{m}{\longrightarrow} L'$, such that for all $x \in L$, there exists $z \in L$ such that for all $x \in L$, there exists $z \in L$ such that $m(x) \stackrel{\alpha'}{\to} m(z)$.*

Proof. Suppose that (L, α) is a $\prod$-subcoalgebra of (L', α'), and m the corresponding strong mono. Let $x \in L$. Since m is a $\prod$-homomorphism, it follows from Proposition 4.1.11 that $\alpha' \circ m(x) = [m(z)]_{M'}$, where $\alpha(x) = [z]_M$. So $m(x) \stackrel{\alpha'}{\to} m(z)$, $z \in L$.

Conversely, assume that there is a strong mono $m : L \longrightarrow L'$ such that for all $x \in L$, there exists $z \in L$ such that $m(x) \stackrel{\alpha'}{\to} m(z)$. Define $\alpha : L \longrightarrow \prod(L)$ by $\alpha(x) = [z]_M$, where $m(x) \stackrel{\alpha'}{\to} m(z)$. Let $x, x' \in L$ such that $\alpha(x) = [z]_M$ and $\alpha(x') = [z']_M$. If $x = x'$, then $\alpha' \circ m(x) = \alpha' \circ m(x')$. So by Proposition 4.1.11 (ii), we obtain $[m(z)]_{M'} = [m(z')]_{M'}$. Hence

$$(m(z) \to m(z')) \wedge (m(z') \to m(z)) \in M'.$$

Thus

$$(z \to z') \wedge (z' \to z) \in m^{-1}(M').$$

It follows from Proposition 4.1.3 that $(z \to z') \wedge (z' \to z) \in M$. So $[z]_M = [z']_M$. Thus α is well defined. Moreover, since α' and m are BL-morphisms, we have

$$\alpha' \circ m(0) = \alpha'(0) = [1]_M = [m(1)]_M .$$

Hence $m(0) \stackrel{\alpha'}{\to} m(1)$ implying $\alpha'(0) = [1]_M$. On another hand, let $x, y \in L$ such that $\alpha(x \propto y) = [t]_M$, $\alpha(x) = [u]_M$ and $\alpha(y) = [v]_M$ where $\propto \in \{*, \to\}$. Then we have $m(x \propto y) \stackrel{\alpha'}{\to} m(t)$ i.e. $\alpha' \circ m(x \propto y) = [m(t)]_{M'}$. Since $\alpha' \circ m$ is a BL-morphism, we have

$$\alpha' \circ m(x) \propto \alpha' \circ m(y) = [m(t)]_{M'} ,$$

i.e.

$$[m(u)]_{M'} \propto [m(v)]_{M'} = [m(t)]_{M'} .$$

Thus $m([u]_M \propto [v]_M) = m([t]_M)$. Since m is a mono, $[u]_M \propto [v]_M = [t]_M$ and so $\alpha(x \propto y) = \alpha(x) \propto \alpha(y)$. Therefore, α is a BL-morphism. It follows that (L, α) is a $\prod$-subcoalgebra of (L', α'). $\square$

It follows from Remark 4.1.9 that $l\mathcal{BL}$ has no products and therefore bisimulations cannot be defined on $\prod$-coalgebras in the sense of Definition 3.1.7. We will use in this case a more general definition due to Aczel and Mendler (1989) and adapted to our setting:

Definition 4.1.13. *A* $\prod$-bisimulation *between two* $\prod$*-coalgebras* (L, α) *and* (L', α') *is a mono-source* $(R, R \xrightarrow{p} L, R \xrightarrow{q})L'$ *in* $l\mathcal{BL}$ *such that there exists a BL-morphism* $\rho : R \longrightarrow \prod R$ *making* p *and* q *homomorphisms of coalgebras.*

In what follows we give a characterization of $\prod$-bisimulations by the means of the arrows notation:

Proposition 4.1.14. *Let* (L, α) *and* (L', α') *be two* $\prod$*-coalgebras. Let* R *be a local BL-algebra with maximal ds* N*. A mono-source* $(R, R \xrightarrow{p} L, R \xrightarrow{q} L')$ *in* $l\mathcal{BL}$ *is a bisimulation iff there exists a BL-morphism* $\rho : R \longrightarrow \prod R$ *such that for all* $x \in R$*,* $p(x) \overset{\alpha}{\to} p(z)$ *and* $q(x) \overset{\beta}{\to} q(z)$*, where* $\rho(x) = [z]_N$*,* $z \in R$*.*

Proof. Suppose that $(R, R \xrightarrow{p} L, R \xrightarrow{q})L'$ is a $\prod$-bisimulation between two $\prod$-coalgebras (L, α) and (L', α'). Let $x \in R$. Since $\rho(x) \in \prod R$ there exists $z \in R$ such that $\rho(x) = [z]_N$. Since p and q are homomorphisms of coalgebras, we have $\alpha \circ p(x) = \prod p \circ \rho(x)$ and $\beta \circ q(x) = \prod q \circ \rho(x)$ i.e. $\alpha \circ p(x) = \prod p([z]_N)$ and $\beta \circ q(x) = \prod q([z]_N)$. Hence $\alpha \circ p(x) = [p(z)]_M$ and $\alpha \circ q(x) = [q(z)]_{M'}$ which means that $p(x) \overset{\alpha}{\to} p(z)$ and $q(x) \overset{\beta}{\to} q(z)$. The converse is straightforward. $\square$

Since limits and colimits in the categories of coalgebras are carried by limits and colimits in the base categories, we obtain the following result:

Proposition 4.1.15. $Coalg(\prod)$ *is not complete, nor cocomplete.*

4.2 Local BL-frames as $\prod$-coalgebras

4.2.1 Local BL-frames and models

Definition 4.2.1. (1) A *local BL-frame* is a structure (L, θ_M) where L is a local BL-algebra and M is the maximal filter of L;

(2) A *local BL-model* is a structure (L, θ_M, ν) where (L, θ_M) is a local BL-frame and $\nu : Prop \longrightarrow \prod(L)$ is a compatible valuation, that is for all $x, y \in L$, we have

(i) $\nu^{-1}(\{[x]_M * [y]_M\}) = \nu^{-1}(\{[x]_M\}) \cap \nu^{-1}(\{[y]_M\})$;

(ii) $\nu^{-1}(\{[x]_M \to [y]_M\}) = \nu^{-1}(\{[x]_M\})^C \cup \nu^{-1}(\{[y]_M\})$;

(iii) $\nu^{-1}(\{[0]_M\}) = \emptyset$.

local BL-frames (models) and BL-morphisms form a category which will be denoted by $\mathcal{F}r(lBL)$ ($\mathcal{M}od(lBL)$).

Remark 4.2.2. It is well known that the normal modal logic S_5 is characterized by the class of reflexive, symmetric, and transitive Kripke frames, that is, the frames for S_5 are exactly that Kripke frames in which the accessibility relation is an equivalence relation. Therefore S_5 is sound and complete in the class of local BL-frames.

The validity of modal formulas at a world x in a local BL-model (L, θ_M, ν) is defined recursively as:

$\mathcal{M}, x \models p$ iff $x \in \nu(p)$

$\mathcal{M}, x \models \neg\varphi$ iff not $\mathcal{M}, x \models \varphi$

$\mathcal{M}, x \models \varphi \wedge \psi$ iff $\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \varphi \vee \psi$ iff $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \varphi \to \psi$ iff not $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \Box\varphi$ iff for every $y \in [x]_M$, $\mathcal{M}, y \models \varphi$

$\mathcal{M}, x \models \Diamond\varphi$ iff there exists $y \in [x]_M$, $\mathcal{M}, y \models \varphi$

The *truth set* of a formula φ in a model $\mathcal{M}$ is the set $[[\varphi]]^{\mathcal{M}} = \{x \in L/\mathcal{M}, x \models \varphi\}$. For any subset K of L, we define the operators $\triangleleft$ and $\tilde{\Box}$ by:

$$\triangleleft K = L \setminus K \text{ and } \tilde{\Box} K = \{x \in L/\, [x]_M \subseteq K\}.$$

By checking the semantics clause above, we have the following result:

Lemma 4.2.3. *For any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$,*

(i) $[[p]]^{\mathcal{M}} = \nu(p)$;

(ii) $[[\neg\varphi]]^{\mathcal{M}} = \triangleleft\, [[\varphi]]^{\mathcal{M}}$;

(iii) $[[\varphi \wedge \psi]]^{\mathcal{M}} = [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$;

(iv) $[[\Box\varphi]]^{\mathcal{M}} = \tilde{\Box}\, [[\varphi]]^{\mathcal{M}}$.

The following result shows how to construct modal algebras with any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$:

Theorem 4.2.4. *For any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$, define the set*

$$\tau(\mathcal{M}) = \{[[\varphi]]^{\mathcal{M}}, \varphi \in Prop\}.$$

Then the structure $(\tau(\mathcal{M}), \cap, \cup, \triangleleft, \emptyset, L, \tilde{\Box})$ is a modal algebra.

Proof. Using Lemma 4.2.3, it is easily checked that $(\tau(\mathcal{M}), \cap, \cup, \triangleleft, \emptyset, L)$ is a Boolean algebra and that $\tilde{\Box}L = L$. We only show that $\tilde{\Box}$ preserves intersections. Let $\varphi, \psi \in Prop$. We have

$$\tilde{\Box}([[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}) = \{x \in L/ \, [x]_M \subseteq [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}\} \subseteq \tilde{\Box}\,[[\varphi]]^{\mathcal{M}} \cap \tilde{\Box}\,[[\psi]]^{\mathcal{M}}.$$

Conversely, let $x \in \tilde{\Box}\,[[\varphi]]^{\mathcal{M}} \cap \tilde{\Box}\,[[\psi]]^{\mathcal{M}}$. Then $[x]_M \subseteq [[\varphi]]^{\mathcal{M}}$ and $[x]_M \subseteq [[\psi]]^{\mathcal{M}}$. Thus for all $y \in [x]_M$, we have $\mathcal{M}, y \models \varphi$ and $\mathcal{M}, y \models \psi$. So $\mathcal{M}, y \models \varphi \wedge \psi$. By Lemma 4.2.3 we obtain $y \in [[\varphi \wedge \psi]]^{\mathcal{M}} = [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$. It follows that $[x]_M \subseteq [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$ and so $x \in \tilde{\Box}([[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}})$.

□

For each BL-algebra L, let $\underline{L}$ denote the carrier.

4.2.2 Categorical relations between local BL-frames and coalgebras over local BL-algebras

In what follows, we give a link between local BL-frames and well known Kripke frames:

Proposition 4.2.5. *Let $\mathcal{F}r(lBL)^*$ be the category of local BL-frames with surjective morphisms. Then the correspondance $U : \mathcal{F}r(lBL)^* \longrightarrow \mathcal{KF}r$ which sends every (L, θ_M) to $(\underline{L}, \theta_M))$ and acts on morphisms as identity is a faithful functor.*

Proof. For any local BL-frame (L, θ_M), $U((L, \theta_M)) = (\underline{L}, \theta_M)$ is clearly a Kripke frame. Let $f : (L, \theta_M) \longrightarrow (L', \theta_{M'})$ be a surjective morphism. In order to show that U is well defined, we have to show that f is a p-morphism. Let $x \in L$ and let $y \in f([x]_M)$. Then $y = f(z)$ with $z \in [x]_M$. So

$$(z \to x) \wedge (x \to z) \in M.$$

Thus

$$f((z \to x) \wedge (x \to z)) \in f(M).$$

It follows from Lemma 4.1.5 that

$$(y \to f(x)) \wedge (f(x) \to y) \in M'.$$

So $y \in [f(x)]_{M'}$ and we have $f([x]_M) \subseteq [f(x)]_{M'}$. Moreover, let $y \in [f(x)]_{M'}$. Since f is surjective, there exists $z \in L$ such that $y = f(z)$ and we have

$$(f(z) \to f(x)) \wedge (f(x) \to f(z)) \in M',$$

that is

$$(f((z \to x) \wedge (x \to z)) \in M'$$

so that

$$(z \to x) \wedge (x \to z) \in f^{-1}(M') = M.$$

Thus $z \in [x]_M$. Therefore $y \in f([x]_M)$. Hence $f([x]_M) = [f(x)]_M$. So U is well defined. The functoriality and the faithfulness of U are straightforward.

□

We present now the result which allows to see local BL-frames as $\prod$-coalgebras:

Theorem 4.2.6. *$\mathcal{F}r(lBL)$ is isomorphic to $Coalg(\prod)$.*

Proof. Consider the correspondance $\mathbb{F}$ which assigns to each local BL-frame (L, θ_M) the pair $(L, L \xrightarrow{\alpha_L} L/M)$ such that $\alpha(x) = [x]_M$ for all $x \in L$ and to each BL-morphism $f : L \longrightarrow L'$, $\mathbb{F}(f) = f$. Let (L, θ_M) be a local BL-frame. Since θ_M is a congruence, α is a BL-morphism and so $(L, L \overset{\alpha_L}{\to} L/M)$ is a $\prod$-coalgebra. Moreover, let $(L, \theta_M) \xrightarrow{f} (L', \theta_{M'})$ be a BL-morphism. For all $x \in L$,

$$\alpha' \circ f(x) = [f(x)]_{M'} = \prod(f)([x]_M) = \prod(f) \circ \alpha(x).$$

So f is a $\prod$-homomorphism between $(L, L \xrightarrow{\alpha_L} L/M)$ and $(L', L' \xrightarrow{\alpha_L} L'/M')$. Hence, $\mathbb{F}$ is well defined. By spelling out the definitions, one shows that $\mathbb{F}$ preserves composition and identity. Thus $\mathbb{F} : \mathcal{F}r(lBL) \longrightarrow Coalg(\prod)$ is a covariant functor.

Moreover The correspondance $\mathbb{G}$ which assigns to each $\prod$-coalgebra $(L, L \xrightarrow{\alpha_L} L/M)$ the local BL-frame (L, θ_M) and which acts as identity on homomorphisms is functorial. Finally, Lemma 3.7 allows to prove that the two functors above statisfy the identities $\mathbb{F} \circ \mathbb{G} = id_{Coalg(\prod)}$ and $\mathbb{G} \circ \mathbb{F} = id_{\mathcal{F}r(lBL)}$. So $\mathcal{F}r(lBL)$ and $Coalg(\prod)$ are isomorphic. □

It follows from Lemma 4.1.6 that the categories $\mathcal{F}r(lBL)$ and $l\mathcal{BL}$ are isomorphic. Therefore, the above theorem yields to the following consequence:

Corollary 4.2.7. *$l\mathcal{BL}$ and $Coalg(\prod)$ are isomorphic.*

4.2.3 conclusion

After the study of a deterministic transition system in the previous chapter, we introduced $\prod$-coalgebras over local BL-algebras, which are non deterministic ones. In spite the fact that the category of local BL-algebras is not complete, nor cocomplete, it offers coalgebraic semantics for the normal modal logic $\mathbf{S}_5$.

Concluding remarks and further research

In this thesis, we have showed that the category of BL-algebras is relevant for the study of coalgebras. We first study BL-algebras in a categorical perspective, showed that it is essentially algebraic over the category of sets and mapping. We also show that the category of MV-algebras is a coreflective subcategory of $\mathcal{BL}$.

We introduced in the sequel the $\mathcal{MV}$-functor, which is a limit-preserving functor and a coreflector on the category of BL-algebras, and prove that the category of the corresponding $\mathbb{MV}$-coalgebras is complete and cocomplete, so the final $\mathbb{MV}$-coalgebra exists. Final coalgebras are of special interest because they have number of pleasant properties as for instance, they allow inductive proofs which is a very good feature for proof theorists. We also established the relation topology and show that the category of topological $\mathbb{MV}$-coalgebras is strong-monotopological over the category of $\mathbb{MV}$-coalgebras, generalising the result of Brummer on categories of universal algebras. Moreover, we lift the coreflectivity of the category of MV-algebras in $\mathcal{BL}$ shown in the previous chapter to a coalgebraic one using the identity functor on the category of MV-algebras.

We ended this work by introducing the fucntor $\prod$ on local BL-algebras and show that $\prod$-coalgebras are suitable coalgebraic semantics for the normal modal logic $\mathbf{S}_5$.

We plan in a future work, to investigate how coinduction could be define on coalgebras over BL-algebras in order to obtain some applications in automata theory.

In another hand, it will be interessant to find a cartesian closed category of logical algebras which allow a deeper investigation of the notion of the notion of bisimulation, in order to present some new coalgebraic semantics of certain fuzzy modal logics.

Another interesting research line is the study of a dual notion of filter (of a BL-algebras) in coalgebras over BL-algebras, for an arbitrary endofunctor and observe their impact on the corresponding coalgebraic logics.

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FUZZY SYSTEMS AND THEIR MATHEMATICS

On MV-coalgebras over the category of BL-algebras

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Abstract

We establish some concrete properties of the category of BL-algebras and use them to introduce **MV**-coalgebras, the coalgebras of the functor which assigns each BL-algebra to its MV-centre. Homomorphisms of **MV**-coalgebras, sub-**MV**-coalgebras and bisimulations are characterized, and we show that the final **MV**-coalgebra exists. Moreover, we applied this notion in topology and constructed an inverse system in the category of **MV**-coalgebras.

Keywords BL-algebra · MV-centre · Essentially algebraic functor · MV-coalgebra · Topological MV-coalgebra

1 Introduction

Coalgebras were introduced by Aczel and Mendler (1989) to model various types of transition system. They offer a rich field of mathematics since they arise as Kripke models for modal logic, as automata and objects for object oriented programming languages in computer sciences, etc. This theory is usually investigated in a set-based context. However, research on structured sets is becoming an important research line. For example, Doberkat (2009) and Kupke et al. (2004) studied coalgebras on measurable spaces, Bezhanishvili et al. (2010), Hofmann et al. (2019) studied coalgebras over Stone spaces, and Haveshki and Eslami (2008) investigated coalgebras over arbitrary topological spaces. The aim of these generalizations is mainly to provide coalgebraic semantics to some modal logics.

This paper introduces coalgebras over BL-algebras. BL-algebras are the Lindenbaum–Tarski algebras of the basic logic (BL for short), the logic arising from the continuous triangular norms familiar in the framework of fuzzy set theory (Cignoli et al. 2000). These algebras have very important algebraic properties (see, for example, Gumm 1999; Hughes 2001; Panangaden 2009).

In particular, we study coalgebras whose underlying functor is the **MV**-functor which is the correspondence which assigns to every BL-algebra its MV-centre. We characterize homomorphisms, subcoalgebras and bisimulations on **MV**-coalgebras and show that the final **MV**-coalgebra exists. We also apply this notion to topology and introduce topological **MV**-coalgebras. The paper is organized as follows: in Sect. 2, we recall some definitions and facts about BL-algebras and coalgebras that we use in the sequel. Section 3 is devoted to the study of the category of BL-algebras as a concrete category over the category of sets and mappings: we show that the forgetful functor is transportable, essentially algebraic, but not topological. This essential algebraicity is used in the next section to show that this category has a good factorization system enough to define bisimulation on sub-**MV**-coalgebras in an easy manner. In Sect. 4, we introduce **MV**-coalgebras and characterize homomorphisms, sub-**MV**-coalgebras and bisimulations in the category of **MV**-coalgebras and show that the category of **MV**-coalgebras is complete and cocomplete. The last section is devoted to topological **MV**-coalgebras: we use a system of filters compatible with the coalgebraic structure to construct a topology on **MV**-coalgebras and an inverse system in the category of **MV**-coalgebras.

This paper is a final form and no version of it will be submitted to publication elsewhere.

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2 Preliminaries

We recall some definitions and basic results; some of them can be found in Adámek et al. (1990), Gumm (1999) and Hughes (2001).

An algebraic structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type (2, 2, 2, 2, 0, 0) is called a *BL-algebra* if it satisfies the following conditions:

(BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(BL2) $(L, *, 1)$ is a commutative monoid;
(BL3) $*$ is a left adjoint of $\rightarrow$, that is $x * z \leq y$ if and only if $z \leq x \rightarrow y$;
(BL4) $x \wedge y = x * (x \rightarrow y)$;
(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A BL-algebra L is called a *Gödel algebra* if $x^2 = x * x = x$ for every $x \in L$. L is called an *MV-algebra* if $\bar{\bar{x}} = x$ for all $x \in L$, where $\bar{x} = x \rightarrow 0$. The subset $MV(L) = \{\bar{x} \mid x \in L\}$ is called the *MV-centre* of L. It is the greatest MV-algebra contained in L.

The following properties holds in any BL-algebra L:

Lemma 2.1 *For all* $x, y, z \in L$

(1) $x \leq y$ *if and only if* $x \rightarrow y = 1$;
(2) $x * y \leq x \wedge y$;
(3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(4) *If* $x \leq y$ *then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$;
(5) $x \leq y \rightarrow (x * y)$; $x * (x \rightarrow y) \leq y$;
(6) $x * \bar{x} = 0$;
(7) $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
(8) $1 \rightarrow x = x$; $x \rightarrow 1 = 1$; $x \rightarrow x = 1$; $x \leq y \rightarrow x$; $x \leq \bar{\bar{x}}$; $\bar{\bar{\bar{x}}} = \bar{x}$.

A *filter* of L is a non-empty subset F of L such that for all $x, y \in L$,

(F1) $x, y \in F$ implies $x * y \in F$;
(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A subset D of a BL-algebra L is called a *deductive system* (ds for short) if

(DS1) $1 \in D$;
(DS2) $x \in D$ and $x \rightarrow y \in D$ imply $y \in D$.

Deductive systems have been widely studied in BL-algebras namely to characterize fragments of basic fuzzy logic (see Hájek 1998; Panangaden 2009); it is obvious that for a non-empty subset F of L, F is a deductive system if and only if it is a filter.

Let L_1 and L_2 be two BL-algebras, a map $f : L_1 \longrightarrow L_2$ is called a *homomorphism of BL-algebras (BL-morphism)*, if $f(0) = 0$ and $f(x \propto y) = f(x) \propto f(y)$ for all $\propto \in \{*, \rightarrow\}$. We obviously have $f(1) = 1$ for any BL-morphism f and it is shown in Turunen (1999) that for any BL-morphism f, $f(x \propto y) = f(x) \propto f(y)$ with $\propto \in \{\vee, \wedge\}$ and if $x \leq y$ then $f(x) \leq f(y)$.

The *kernel* of a BL-morphism $f : L_1 \longrightarrow L_2$ is the set Ker(f):= $\{x \in L_1 \mid f(x) = 1\}$. Clearly, f is injective iff Ker(f)= $\{1\}$. Ker(f) is always a deductive system.

For any deductive system D of a BL-algebra $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$, we can define a binary relation θ_D on L as follows: for all $x, y \in L$,

$$x\theta_D y \iff x \rightarrow y, y \rightarrow x \in D.$$

It is well known that θ_D is a congruence relation on L (see e.g. Hájek1998, Theorem 2.7) and since the class of BL-algebras is a variety, the quotient structure L/θ_D is also a BL-algebra for which for all $x, y \in L$, $[x \propto y]_D := [x]_D \propto [y]_D$ where $\propto \in \{\wedge, \vee, *, \rightarrow\}$, and $[x]_D := [x]_{\theta_D}$.

The one-element BL-algebra $\{0 = 1\}$ is called the *degenerate BL-algebra* (Turunen 1999, Remark 8), we will denote it by $\mathbf{G}_1$. The two-element non-degenerate BL-algebra $\{0, 1\}$ is called the *trivial BL-algebra*, we will denote it by $\mathbf{G}_2$. These two algebras are examples of BL-algebras which are both Gödel-algebras and MV-algebras. It is easily checked that the chain $\{0, x, 1\}$, with the operations $*$ and $\rightarrow$ defined by the following tables

$*$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

$\rightarrow$	0	x	1
0	1	1	1
x	0	1	1
1	0	x	1

is the unique Gödel-algebra with three elements and we will denote it by $\mathbf{G}_3$. The chain $\{0, x, 1\}$, with the operations $*$ and $\rightarrow$ defined by the following tables:

$*$	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

$\rightarrow$	0	x	1
0	1	1	1
x	x	1	1
1	0	x	1

is the unique MV-algebra with three elements and we will denote it by $\mathbf{M}_3$.

Let $\mathcal{BL}(L, L')$ denote the set of BL-morphisms from L to L'. The following observations will be useful in the sequel:

Lemma 2.2 *For* $\mathbf{G}_2, \mathbf{G}_3$ *and* $\mathbf{M}_3$ *defined as above we have:*

(1) $\mathcal{BL}(\mathbf{G}_3, \mathbf{M}_3)$ *is a singleton and* $\mathcal{BL}(\mathbf{M}_3, \mathbf{G}_3) = \emptyset$
(2) $MV(\mathbf{G}_3) = \mathbf{G}_2$

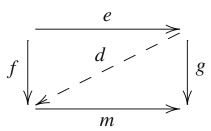
Proof (i) Let $\mathbf{G}_3 \stackrel{f}{\longrightarrow} \mathbf{M}_3$ be a map such that $f(0) = 0$, $f(1) = 1$. If $f(x) = 0$, then $f(x \rightarrow 0) = f(0) = 0 \neq 1 = f(x) \rightarrow f(0)$ and for $f(x) = x$, $f(x * x) = f(x) = x \neq 0 = f(x) * f(x)$. Hence, in the both cases f is not a BL-morphism. For $f(x) = 1$, it easily checked that f preserves $*$ and $\rightarrow$ and so it is the unique BL-morphism from $\mathbf{G}_3$ to $\mathbf{M}_3$. With similar computations, one proves that there is no BL-morphism from $\mathbf{M}_3$ to $\mathbf{G}_3$.

(ii) Straightforward. □

The class of BL-algebras equipped with the class of BL-morphisms form a category which will be denoted by $\mathcal{BL}$.

Let $F : \mathcal{C} \longrightarrow \mathcal{C}$ be a functor from the category $\mathcal{C}$ to itself. A *coalgebra* of type F is a pair $A = (A, \alpha)$, consisting of a set A and a map $\alpha : A \longrightarrow FA$. A is called the *carrier* set and α is called the *structure map* of A. If $A = (A, \alpha)$ and $B = (B, \beta)$ are F -coalgebras, then a map $f : A \longrightarrow B$ is called a *homomorphism*, if $\beta \circ f = F(f) \circ \alpha$.

A monomorphism m is called *extremal* if it satisfies the following extremal condition: If $m = f \circ e$, where e is an epimorphism, then e must be an isomorphism. *Extremal epimorphisms* are defined dually. A $\mathcal{C}$-morphism is called a *regular mono* if it is the equalizer of some pair of $\mathcal{C}$-morphisms. A $\mathcal{C}$-morphism is called a *regular monomorphism* if it is the equalizer of some pair of $\mathcal{C}$-morphisms. A monomorphism m is called *strong* in a category $\mathcal{C}$ if for every epimorphism e and every commutative square



there exists a diagonal d such that $g = m \circ d$ and $f = d \circ e$. It is well known (see e.g. Adámek et al. 1990, Corollary 7.63) that in any category, regular monomorphisms are extremal and by Adámek et al. (1990, exercise 14C.f) we have:

Lemma 2.3 *If $\mathcal{C}$ is (Epi, M)-structured (see Adámek et.al 1990, Definition14.1)) for some class M of monomorphisms, then strong monomorphisms in $\mathcal{C}$ are precisely extremal monomorphisms.*

The *kernel equivalence* of a morphism is the pullback of that morphism with itself. As a particular case of finitary algebraic category, $\mathcal{BL}$ is complete and cocomplete. By using standard computations, (see, for example, Ghita 2009; Ghorbani et al. 2009), one can easily check the following:

Proposition 2.4 *In the category $\mathcal{BL}$,*

(1) *The initial object is $\mathbf{G}_2$ and the final object is $\mathbf{G}_1$;*

(2) *The equalizer of a pair $L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2$ of $\mathcal{BL}$−morphisms is the embedding $E \stackrel{e}{\longrightarrow} L_1$, where $E = \{x \in L_1 \mid f(x) = g(x)\}$;*

(3) *the product of a family $(L_i)_I$ of BL-algebras is the source $(P, P \stackrel{p_i}{\longrightarrow} L_i)_I$ where*

$$P = \{I \stackrel{f}{\longrightarrow} U_{i \in I} L_i \mid f(i) \in L_i \text{ for all } x \in I\}$$

and $P \stackrel{p_i}{\longrightarrow} L_i$ is defined by $p_i(f) = f(i)$ for all $i \in I$;

(4) *The pullback of the morphisms $L_1 \stackrel{f}{\longrightarrow} L \stackrel{g}{\longleftarrow} L_2$ is the triple $(Pb(f, g), \pi_1, \pi_2)$ where $Pb(f, g) = \{(x, y) \in L_1 \times L_2 \mid f(x) = g(y)\}$ and π_i is the projection on L_i.*

(5) *The coequalizer of a pair $L_1 \underset{g}{\overset{f}{\rightrightarrows}} L_2$ of $\mathcal{BL}$-morphisms is the pair $(L_2/\theta, L_2 \stackrel{\pi}{\longrightarrow} L_2/\theta)$ where θ is the smallest congruence on L_2 containing the set $X = \{(f(x), g(x)) \mid x \in L_1\}$, π is the canonical surjection and L_2/θ is the quotient algebra.*

3 Some properties of $\mathcal{BL}$

3.1 Concrete categories of BL-algebras

Definition 3.1 Given a category $\mathcal{X}$, a *concrete category* over $\mathcal{X}$ is a pair $(\mathcal{C}, U)$, where $U : \mathcal{C} \longrightarrow \mathcal{X}$ is a faithful functor. Sometimes U is called the *forgetful (or underlying) functor* of the concrete category and $\mathcal{X}$ is called the *base category* for $(\mathcal{C}, U)$. When $\mathcal{X} = \mathcal{SET}$, $(\mathcal{C}, U)$ is called a *construct*.

We consider the concrete category $(\mathcal{BL}, U)$ over $\mathcal{SET}$, where U is the standard forgetful functor.

Let $(\mathcal{C}, U)$ be a concrete category over $\mathcal{X}$. The *fibre* of an $\mathcal{X}$-object X is the preordered class consisting of all $\mathcal{C}$-objects A with $U(A) = X$ ordered by:

$A \preceq B$ if and only if $id_X : UA$
$\longrightarrow UB$ is (can be lifted to) a $\mathcal{C}$-morphism.

Example 3.2 $\mathbf{M}_3$ and $\mathbf{G}_3$ are in the fibre of the set $\{0, x, 1\}$. Since by Lemma 2.2(1), $id : \{0, x, 1\} \longrightarrow \{0, x, 1\}$ is not a $\mathcal{BL}$-morphism between $\mathbf{M}_3$ and $\mathbf{G}_3$, they are not comparable.

Definition 3.3 A concrete category $(\mathcal{C}, U)$ over $\mathcal{X}$ is said to be:

(1) *Fibre-discrete* provided that its fibres are ordered by equality.

(2) *(Uniquely) transportable* provided that for every $\mathcal{C}$-object A and every $\mathcal{X}$−isomorphism $UA \xrightarrow{f} X$, there exists a (unique) $\mathcal{C}$-object B with $UB = X$ such that $A \xrightarrow{f} B$, is a $\mathcal{C}$-isomorphism. In that case U is said to be (uniquely) transportable.
(3) *Strongly complete* if it is complete and has intersections;
(4) Is called *well powered* if no $\mathcal{C}$-object has a proper class of pairwise non-isomorphic subobjects.

Definition 3.4 Let $\mathcal{C}$ be a category.

Let $A \xrightarrow{f} B$ and $C \underset{q}{\overset{p}{\rightrightarrows}} A$ be $\mathcal{C}$-morphisms. (p, q) is called a *congruence relation* on f if (C, p, q) is a pullback of (f, f).

Definition 3.5 Let E be a class of morphisms and let $\mathbf{M}$ be a conglomerate of sources in a category $\mathcal{C}$:

(1) $\mathcal{C}$ has $(E, \mathbf{M})$*-factorizations* provided that each source S in $\mathcal{C}$ has a factorization $M \circ e$ with $e \in E$ and $M \in \mathbf{M}$.
(2) A functor $U : \mathcal{C} \longrightarrow \mathcal{X}$ has $(E, \mathbf{M})$*-factorizations* provided that for each U-structured source $(X \xrightarrow{f_i} UA_i)_I$ there exists $X \xrightarrow{e} UA \in E$ and $(A \xrightarrow{m_i} A_i)_I \in \mathbf{M}$ such that $f_i = Um_i \circ e$ for each $i \in I$.

Definition 3.6 A functor $U : \mathcal{C} \longrightarrow \mathcal{X}$ is called:

(1) *Topological* provided that every U-structured source $(X \xrightarrow{f_i} UA_i)_I$ has a unique U-initial lift $(A \xrightarrow{f_i} A_i)_I$ (i.e. there exists a unique U-initial source $(A \xrightarrow{\bar{f_i}} A_i)_I$ such that for each $i \in I$, $U(\bar{f_i}) = f_i$).
(2) *Essentially algebraic* provided that it creates isomorphisms and is (Generating, Mono-Source)-factorizable.

Definition 3.7 Let $(\mathcal{C}, U)$ be a concrete category.

(1) $\mathcal{C}$ is *topological (essentially algebraic)* provided that U is topological (essentially algebraic).
(2) $\mathcal{C}$ is called *Cartesian closed* if it has finite products and for each $\mathcal{C}$-object A the functor $A \times -$ is left-adjoint.

Remark 3.8 In a concrete category $(\mathcal{C}, U)$ over a category $\mathcal{X}$, a *lifting* of an $\mathcal{X}$-morphism $X \xrightarrow{f} Y$, whenever it exists, which will be denoted by $\bar{f}$ is a $\mathcal{C}$-morphism such that $U(\bar{f}) = f$. By the faithfulness of U, we have:

(1) The lifting $\bar{f}$ of an $\mathcal{X}$-morphism $UA \xrightarrow{f} UB$ is unique and $\bar{f}$ coincides with f in UA if $\mathcal{X} = \mathcal{SET}$;
(2) For any $\mathcal{X}$-morphisms $X \xrightarrow{g} Y \xrightarrow{f} Z$, $\overline{f \circ g} = \bar{f} \circ \bar{g}$, whenever $\bar{f}$ and $\bar{g}$ exist.

Now, we establish the transportability of the concrete category of BL-algebras:

Theorem 3.9 *$(\mathcal{BL}, U)$ is fibre-discrete and uniquely transportable.*

Proof Let X be a set and $L = (X, \wedge, \vee, *, \rightarrow, 0, 1)$, $L' = (X, \wedge', \vee', *', \rightarrow', 0', 1')$ be two BL-algebras in the fibre of X. Suppose that $L \preceq L'$. Then for all $x, y \in X$ we have $x \propto y \in X$ since L is a BL-algebra, where $\propto \in \{\wedge, \vee, *, \rightarrow\}$. Thus, by Remark 3.8(1) $x \propto y = \overline{id_X}(x \propto y)$. By the fact that $\overline{id_X}$ is a $\mathcal{BL}$-morphism, we obtain $x \propto y = \overline{id_X}(x) \propto' \overline{id_X}(y) = x \propto' y$ and we can conclude that $L = L'$. Conversely, if $L = L'$, then it is obvious that $id_L = \overline{id_X}$. Therefore, $(\mathcal{BL}, U)$ is fibre-discrete.

For the unique transportability, let $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ be a BL-algebra, X be a set and $L \xrightarrow{f} X$ be a bijective function. For $y_1, y_2 \in X$, define for $\propto' \in \{\wedge', \vee', *', \rightarrow'\}$, $y_1 \propto' y_2 = f(x_1 \propto x_2)$ ($\propto \in \{\wedge, \vee, *, \rightarrow\}$), where $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $L' = (X, \wedge', \vee', *', \rightarrow', f(0), f(1))$ is the unique BL-algebra in the fibre of X such that $L \xrightarrow{f} L'$ is a $\mathcal{BL}$-isomorphism. □

The above theorem and Adámek et al. (1990, Proposition 5.8) lead to the following result:

Corollary 3.10 *The forgetful functor $U : \mathcal{BL} \longrightarrow \mathcal{SET}$ reflects identities.*

In $\mathcal{SET}$, mono-sources are exactly the point-separating sources, i.e. sources $(X, f_i)_I$ such that for any two different elements x and y of X there exists some $i \in I$ with $f_i(x) \neq f_i(y)$. Since faithful functors reflect mono-sources (Adámek et al. 1990, Proposition 10.7), we have:

Lemma 3.11 *In $\mathcal{BL}$, point separating sources are mono-sources.*

Lemma 3.12 *The category $\mathcal{BL}$ has $(Epi, Mono - Source)$-factorizations.*

Proof Let $(L \xrightarrow{f_i} L_i)_I$ be a source in $\mathcal{BL}$. Consider the congruence θ defined by $(x, y) \in \theta \iff f_i(x) = f_i(y)$ for all $i \in I$ and let $L \xrightarrow{\pi} L/\theta$ be the canonical epimorphism. Then, the map $L/\theta \xrightarrow{m_i} L_i$ defined by $m_i([x]_\theta) = f_i(x)$ is the unique BL-morphism such that $f_i = m_i \circ \pi$ (since π is an epimorphism). Let $[x]_\theta$ and $[y]_\theta$ be two distinct classes. Then, $f_i(x) \neq f_i(y)$ for some $i \in I$, and so $m_i([x]_\theta) \neq m_i([y]_\theta)$. It follows that $(L/\theta \xrightarrow{m_i} L_i)_I$ is a point-separating source and by Lemma 3.11 a mono-source. □

The following summarizes a combination of Proposition 8.24 and Remark 13.26 from Adámek et al. (1990).

Lemma 3.13 *The following hold:*

(1) *every universal arrow is extremally generating;*
(2) *a functor creates isomorphisms if and only if it reflects isomorphisms and is uniquely transportable;*

Proposition 3.14 *The forgetful functor* $U : \mathcal{BL} \longrightarrow \mathcal{SET}$ *has* $(Generating, Mono - Source)$*-factorizations.*

Proof Let $(X \xrightarrow{f_i} UL_i)_I$ be an U-structured source in $\mathcal{BL}$. Since $F_{\mathbb{BL}}(X)$ is a free object in $\mathcal{BL}$, there exists an universal arrow $X \xrightarrow{u} U(F_{\mathbb{BL}}(X))$. Hence for each f_i, there exists a unique BL-morphism $F_{\mathbb{BL}}(X) \xrightarrow{\tilde{f}_i} L_i$ such that $f_i = U\tilde{f}_i \circ u$. By Lemma 3.12, it follows that for each $i \in I$, $\tilde{f}_i = F_{\mathbb{BL}}(X) \xrightarrow{e} L \xrightarrow{m_i} L_i$ where e is an epimorphism and $(L \xrightarrow{m_i} L_i)_I$ is a mono-source in $\mathcal{BL}$. Thus, $f_i = X \xrightarrow{Ue \circ u} UL \xrightarrow{Um_i} UL_i$. Let us show that $Ue \circ u$ is generating. Let $L \underset{s}{\overset{r}{\rightrightarrows}} L'$ be a pair of $\mathcal{BL}$-morphisms such that $Ur \circ (Ue \circ u) = Us \circ (Ue \circ u)$. Then, by the fact that u is universal and so generating (Lemma 3.13 (1)), we have $Ur \circ Ue = Us \circ Ue$ and since U is faithful, $r \circ e = s \circ e$. Thus, $r = s$ since e is an epimorphism. □

Theorem 3.15 *The construct* $(\mathcal{BL}, U)$ *is essentially algebraic.*

Proof Let $L \xrightarrow{f} L'$ be a $\mathcal{BL}$-morphism such that $U(f)$ is bijective. then $f^{-1}(0) = f^{-1}(f(0)) = 0$ and for any binary operation $\propto$, we have:

$$f^{-1}(x \propto y) = f^{-1}(f(x') \propto f(y')) = f^{-1} \circ f(x' \propto y') \\ = x' \propto y' = f^{-1}(x) \propto f^{-1}(y)$$

since f is a $\mathcal{BL}$-morphism and is surjective. Therefore, f^{-1} is a $\mathcal{BL}$-morphism and so U reflects isomorphisms. Moreover, U is uniquely transportable (Theorem 3.9), it follows from Lemma 3.13(2) that U creates isomorphisms. Taking into account the Proposition 3.14, we conclude that U is essentially algebraic. □

Essentially algebraic categories have some nice properties. For example, they inherit some properties of the base category. Indeed, since $\mathcal{SET}$ is strongly complete and well powered, we have by Adámek et al. (1990, Proposition 23.12):

Proposition 3.16 *$\mathcal{BL}$ is strongly complete and well powered.*

Moreover, by Adámek et al. (1990, Corollary 14.21), we have:

Corollary 3.17 *$\mathcal{BL}$ is (ExtrEpi, Mono)-structured and (Epi, ExtrMono)-structured.*

Proposition 3.18 *Let $\mathcal{POS}$ denote the category of posets and order-preserving maps. The forgetful functors $\mathcal{BL} \xrightarrow{V} \mathcal{POS}$ and $\mathcal{BL} \xrightarrow{U} \mathcal{SET}$ are not topological.*

Proof For the functor V, the morphism (1-source) f from the poset $\{0, z, x, y, 1\}$, with $z = x \wedge y$ to the poset $\{0, 1\}$, such that $f(x) = 0$ if $x \neq 1$ and $f(1) = 1$ cannot be lifted to a BL-morphism. If it were the case, we would have $f(x \to y) = f(x) \to f(y) = 0 \to 0 = 1$ which means that $x \to y = 1$ and by Lemma 2.1 $x \leq y$, which contradicts the hypothesis.

For the functor U, for any U-structured 1-source $\{0, x, 1\} \xrightarrow{g} U\mathbf{G}_2$ such that $g(0) = 1$ and $g(1) = 0$, g cannot be lifted to a $\mathcal{BL}$-morphism from $\mathbf{G}_3$ or $\mathbf{M}_3$ to $\mathbf{G}_2$. □

Proposition 3.19 *$\mathcal{BL}$ is not Cartesian closed.*

Proof Let L be a BL-algebra, and $f, g : \mathbf{G}_2 \times \mathbf{G}_2 \longrightarrow L$ be two maps defined by:

$$f(a) = f(b) = 0, f(c) = f(d) = 1 \text{ and}$$
$$g(a) = g(c) = 0, g(b) = g(d) = 1$$

where $a = (0, 0), b = (0, 1), c = (1, 0)$ and $d = (1, 1)$. Then, f and g are $\mathcal{BL}$-morphisms. It follows that the functor $\mathbf{G}_2 \times -$ does not preserve initial object and hence, is not left adjoint. □

3.2 Some classes of morphisms in $\mathcal{BL}$

Next, we investigate the relationship between some classes of monomorphisms (epimorphisms) in $\mathcal{BL}$. Since BL-algebras form a variety, $L \times L$ is a BL-algebra for any BL-algebra L. We recall that for any BL-morphism $L \xrightarrow{f} L'$, θ_f denote the congruence induced by the deductive system $Ker(f)$. It is easily checked that θ_f is a BL-subalgebra of $L \times L$, and we have the following:

Lemma 3.20 *For any BL-morphism* $L \xrightarrow{f} L'$, $\theta_f = \{(x, y) \in L \times L;\ f(x) = f(y)\}$.

Proof Straightforward □

Definition 3.21 Let $(\mathcal{C}, U)$ be a concrete category. A $\mathcal{C}$-morphism $A \xrightarrow{f} B$ is called *initial* provided that for any $\mathcal{C}$-object C, an $\mathcal{X}$-morphism $UC \xrightarrow{g} UA$ is (i.e. can be lifted to) a $\mathcal{C}$-morphism whenever $UC \xrightarrow{Uf \circ g} UB$ is a $\mathcal{C}$-morphism, i.e. there exists $C \xrightarrow{h} B$ such that $U(h) = Uf \circ g$. A $\mathcal{C}$-morphism is called a *regular monomorphism* if it is the equalizer of some pair of $\mathcal{C}$-morphisms.

In this section, we consider the concrete category $(\mathcal{BL}, U)$ over $\mathcal{SET}$, where U is the standard forgetful functor.

Proposition 3.22 *In $\mathcal{BL}$, we have:*

$$\text{RegMono}(\mathcal{BL}) \subseteq \text{ExtrMono}(\mathcal{BL}) = StrongMono(\mathcal{BL})$$

and

$$\text{StrongMono}(\mathcal{BL}) \subseteq \text{Mono}(\mathcal{BL}) = Inj(\mathcal{BL}) = \text{Init}(\mathcal{BL}),$$

where (Reg, Strong, Extremal)Mono($\mathcal{BL}$), *Init($\mathcal{BL}$) and Inj($\mathcal{BL}$) are the classes of (regular, strong, extremal) monomorphisms, initial and injective morphisms, respectively.*

Proof It is clear that every injective BL-morphism is a monomorphism and since U is right adjoint and hence preserves monomorphisms (Adámek et al. 1990, Proposition 18.6), every monomorphism is injective. Hence $Mono(\mathcal{BL}) = Inj(\mathcal{BL})$. Let us show that injective morphisms are initial in $\mathcal{BL}$. Let $f : L_1 \longrightarrow L_2$ be an injective $\mathcal{BL}$-morphism. Let L_3 be a BL-algebra and $g : UL_3 \longrightarrow UL_1$ be a function. Suppose that $Uf \circ g : UL_3 \longrightarrow UL_2$ is a $\mathcal{BL}$-morphism. Then, for all $x, y \in UL_3$, and $\propto \in \{*, \rightarrow\}$, we have

$\overline{Uf \circ g}(x \propto y) = \overline{Uf \circ g}(x) \propto \overline{Uf \circ g}(y)$ and since $x, y, x \propto y \in UL_3$, we have by Remark 3.8(1) $Uf \circ g(x \propto y) = Uf \circ g(x) \propto Uf \circ g(y)$. Therefore, because $g(x), g(y), g(x \propto y) \in UL_1$ the same remark leads to:

$$\begin{aligned} f \circ g(x \propto y) &= f \circ g(x) \propto f \circ g(y) \\ &= f(g(x) \propto g(y)) \ (f \text{ is a BL-morphism}). \end{aligned}$$

Since f is injective, we obtain $g(x \propto y) = g(x) \propto g(y)$. With similar arguments, we show that $g(0) = 0$ and we conclude that g is a BL-morphism and so f is an initial morphism. Conversely, suppose that f is an initial morphism. Let $x, y \in L_1$ such that $f(x) = f(y)$. Define $g : \{0, 1\} \longrightarrow L_1$ by $g(0) = 0$ and $g(1) = (x \rightarrow y) \wedge (y \rightarrow x)$. It is obvious that $f \circ g(0) = 0$ and $f \circ g(1) = 1$, which means that $f \circ g$ is a BL-morphism. By hypothesis, it follows that g is a BL-morphism and then $g(1) = 1$, which leads to $(x \rightarrow y) \wedge (y \rightarrow x) = 1$ and by Lemma 2.1 we obtain $x = y$. Therefore, $inj(\mathcal{BL}) = init(\mathcal{BL})$.

$ExtrMono(\mathcal{BL}) = StrongMono(\mathcal{BL}) \subseteq Mono(\mathcal{BL})$ follows from Lemma 2.3 and Corollary 3.17. $RegMono(\mathcal{BL}) \subseteq ExtrMono(\mathcal{BL})$ follows from Adámek et al. (1990, Corollary 7.63). □

The following result will be useful in the sequel:

Lemma 3.23 *If $L \xrightarrow{f} L'$ is a regular epimorphism in $\mathcal{BL}$, then f is the coequalizer of the pair $\theta_f \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} L$ where $\pi_1(x_1, x_2) = x_1$ and $\pi_2(x_1, x_2) = x_2$.*

Proof Let $L \xrightarrow{f} L'$ be a regular epimorphism in $\mathcal{BL}$. It follows from Lemma 3.20 and Proposition 2.4 (iv) that (θ_f, π_1, π_2) is a pullback of (f, f). Thus, (π_1, π_2) is a congruence relation of f. Hence, by Adámek et al. (1990, Proposition 11.22), f is coequalizer of π_1 and π_2. □

Proposition 3.24 *In the construct $(\mathcal{BL}, U)$, we have:*

$$\text{RegEpi}(\mathcal{BL}) = \text{Surj}(\mathcal{BL}) \subsetneq \text{Epi}(\mathcal{BL})$$

where (Reg)Epi($\mathcal{BL}$), *and* Surj($\mathcal{BL}$) *are the classes of (regular)epimorphisms and surjective morphisms, respectively.*

Proof Since faithful functors reflect epimorphisms (see Adámek et al. 1990, proposition 7.44), every surjective BL-morphism is an epimorphism in $\mathcal{BL}$. The converse does not hold. Indeed, consider $L = \{0, x, y, 1\}$ and $L' = \{0, z, x, y, 1\}$, where $z = x \vee y$ and x and y are not comparable. Define $*$, $\odot$, $\rightharpoondown$ and $\rightharpoonup$ as follows:

$*$	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

$\rightharpoonup$	0	x	y	1
0	1	1	1	1
x	0	1	y	1
y	0	x	1	1
1	0	x	y	1

$\odot$	0	x	y	z	1
0	0	0	0	0	0
x	0	x	0	x	x
y	0	0	y	y	y
z	0	x	y	z	z
1	0	x	y	z	1

$\rightharpoondown$	0	x	y	z	1
0	1	1	1	1	1
x	0	1	y	1	1
y	0	x	1	1	1
z	0	x	y	1	1
1	0	x	y	z	1

Then, $(L, \wedge, \vee, *, \rightharpoonup, 0, 1)$ and $(L', \wedge, \vee, \odot, \rightharpoondown, 0, 1)$ are BL-algebras. Consider the function $L \xrightarrow{m} L'$ such that $m(t) = t$ for all $t \in L$. Then, m is an epimorphism but it is not surjective. Thus, we have the strict inclusion $Surj(\mathcal{BL}) \subset Epi(\mathcal{BL})$.

Let $L \xrightarrow{f} L'$ be a surjective $\mathcal{BL}$-morphism. Consider the pair $\theta_f \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} L$. Then, we have $f \circ \pi_1 = f \circ \pi_2$. Let $L \xrightarrow{g} L''$ be another BL-morphism such that $g \circ \pi_1 = g \circ \pi_2$ and consider the map $L' \xrightarrow{u} L''$ such that for all $y = f(x) \in L'$, $u(y) = g(x)$. Let $f(x_1), f(x_2) \in L'$ such that $f(x_1) = f(x_2)$. Then $(x_1, x_2) \in \theta_f$. So we have $g \circ \pi_1(x_1, x_2) = g \circ \pi_2(x_1, x_2)$, i.e. $g(x_1) = g(x_2)$ which means that $u(f(x_1)) = u(f(x_2))$ and thus u is well defined. u is clearly a $\mathcal{BL}$-morphism and we have $u \circ f = g$. For another $\mathcal{BL}$-morphism v such that $v \circ f = g$, we have $u \circ f = v \circ f$ and so $u = v$ since f is an epimorphism. Hence f is the coequalizer of the pair $\theta_f \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} L$.

To complete the proof, we have to show that regular epimorphisms are surjective. Let $L \xrightarrow{f} L'$ be a regular epimorphism in $\mathcal{BL}$. Then, by Lemma 3.23 f is the coequalizer of the pair $\theta_f \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} L$. For all $(x, y) \in \theta_f$, we have $\pi \circ \pi_1(x, y) = [x]_{\theta_f} = [y]_{\theta_f} = \pi \circ \pi_2(x, y)$, where $L \xrightarrow{\pi} L/\theta_f$ is the canonical surjection. Thus, there exists a unique BL-morphism $L' \xrightarrow{\varphi} L/\theta_f$ such that $\varphi \circ f = \pi$. Since π is surjective, it is a regular epimorphism. So π is the coequalizer of the pair $\theta_\pi \overset{\pi'_1}{\underset{\pi'_2}{\rightrightarrows}} L$. Let $(x, y) \in \theta_\pi$. Then, $\pi(x) = \pi(y)$ which means that $[x]_{\theta_f} = [y]_{\theta_f}$ and we get that $f(x) = f(y)$. So $f \circ \pi'_1 = f \circ \pi'_2$ and thus there exists an unique $\mathcal{BL}$-morphism $L/\theta_f \xrightarrow{\phi} L'$ such that $\phi \circ \pi = f$. Hence $(\varphi \circ \phi) \circ \pi = \varphi \circ (\phi \circ \pi) = \varphi \circ f = \pi$. Since π is an epimorphism, we obtain $\varphi \circ \phi = 1_{L/\theta_f}$. Moreover $(\phi \circ \varphi) \circ f = \phi \circ (\varphi \circ f) = \phi \circ \pi = f$. Since f is a regular epimorphism, it is an epimorphism and we get $\phi \circ \varphi = 1_{L'}$. It follows that ϕ is a $\mathcal{BL}$-isomorphism and hence is surjective. Therefore, $f = \phi \circ \pi$ is surjective as composition of such morphisms. □

Now, we present the relations between $\mathcal{BL}$ and the categories of Gödel-algebras and MV-algebras.

3.3 Some subcategories of $\mathcal{BL}$

Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is said *coreflective* if the inclusion functor $i : \mathcal{D} \hookrightarrow \mathcal{C}$ has a right adjoint R. In this case, R is called a *reflector*. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is said *isomorphism-closed* if every object of $\mathcal{C}$ that is isomorphic to a $\mathcal{D}$-object is itself a $\mathcal{D}$-object.

Let $\mathcal{GOD}$ and $\mathcal{MV}$ denote the categories of Gödel and MV-algebras, respectively. The morphisms in these categories are exactly BL-morphisms. Thus, $\mathcal{GOD}$ and $\mathcal{MV}$ are full subcategories of $\mathcal{BL}$. Moreover, we have:

Proposition 3.25 *$\mathcal{GOD}$ and $\mathcal{MV}$ are isomorphism-closed subcategories of $\mathcal{BL}$.*

Proof Let L be a BL-algebra isomorphic to a Gödel-algebra G. Then, let $L \xrightarrow{f} G$ be that BL-isomorphism. For all $x \in L$, there exists $y \in G$ such that $x = f(y)$. We have $x * x = f(y) * f(y) = f(y * y) = f(y) = x$. Thus, L is a Gödel-algebra and so $\mathcal{GOD}$ is an isomorphism-closed subcategory of $\mathcal{BL}$. By a similar method, one can easily prove that $\mathcal{MV}$ is also an isomorphism-closed subcategory of $\mathcal{BL}$.

Lemma 3.26 *Let L and L' be two BL-algebras. For all BL-morphism $MV(L) \xrightarrow{f} L'$, $Im(f) \subseteq MV(L')$*

Proof Let $MV(L) \xrightarrow{f} L'$ be a BL-morphism and $y \in Im(f)$. Then, there exists $x \in MV(L)$ such that $y = f(x)$. By the definition of the MV-centre, it means there exists $z \in L$ such that $y = f(\overline{\overline{z}}) = \overline{\overline{f(z)}}$. So $y \in MV(L')$.

Proposition 3.27 *The correspondence $\boldsymbol{MV} : \mathcal{BL} \longrightarrow \mathcal{MV}$ which assigns to every BL-algebra its MV-centre extends to a functor.*

Proof Lemma 3.26 shows that **MV** is well defined. Let $L \xrightarrow{f} L' \xrightarrow{g} L''$ be two $\mathcal{BL}$-morphisms. Then, for all $x \in MV(L)$, we have $\mathbf{MV}(g \circ f)(x) = g \circ f(x) = \mathbf{MV}(g) \circ \mathbf{MV}(f)(x)$ and $\mathbf{MV}(1_L)(x) = x = 1_{MV(L)}(x)$.

In the sequel, this functor will be called the *$\mathcal{MV}$-functor*.

Proposition 3.28 *The* $\mathcal{MV}$-functor *is neither faithful nor conservative.*

Proof Let $\mathbf{G}_4$ be the BL-algebra defined by the following tables:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Consider now the functions $\mathbf{G}_3 \overset{f}{\underset{g}{\rightrightarrows}} \mathbf{G}_4$ such that $f(0) = 0$, $f(x) = a$, $f(1) = 1$ and $g(0) = 0$, $g(x) = b$, $g(1) = 1$. Then it is easily checked that f and g are BL-morphisms. We have $\mathbf{MV}(f) = \mathbf{MV}(g)$, but $f \neq g$. Thus, **MV** is not faithful. Moreover, $\mathbf{MV}(f)$ is a BL-isomorphism but it is not the case for f. So **MV** is not conservative. □

Theorem 3.29 *$\mathcal{MV}$ is a coreflective subcategory of $\mathcal{BL}$.*

Proof We first prove that the $\mathcal{MV}$-functor is the right adjoint to the inclusion functor $i : \mathcal{MV} \hookrightarrow \mathcal{BL}$. Let M be an MV-algebra and L a BL-algebra. Consider the function

$$\begin{aligned} \Phi_{M,L} :\ & \mathcal{BL}(M, L) \longrightarrow \mathcal{MV}(M, MV(L)) \\ & M \xrightarrow{f} L \longmapsto \mathbf{MV}(f) \end{aligned}$$

where $\mathcal{BL}(M, L)$ (respectively, $\mathcal{MV}(M, MV(L))$) denote the set of BL-morphisms from M to L (respectively, from M to $MV(L)$). Since $MV(M) = M$, by Lemma 3.26 $\Phi_{M,L}(f)$ is well defined. $\Phi_{M,L}$ is bijective and its inverse $\Phi^{-1}_{M,L}$ is defined by $\Phi^{-1}_{M,L}(g)(x) = g(x)$ for all $\mathcal{MV}$-morphism $g : M \longrightarrow MV(L)$. Moreover, for any $\mathcal{MV}$-morphism $M' \xrightarrow{f} M$, we have for all $M \xrightarrow{\alpha} L$ and $x \in M'$:

$$\begin{aligned} \mathcal{MV}(f, MV(L)) \circ \Phi_{M,L}(\alpha)(x) &= \Phi_{M,L}(\alpha) \circ f(x) \\ &= \mathbf{MV}(\alpha) \circ f(x) \\ &= \alpha \circ f(x) \\ &= \mathbf{MV}(\alpha \circ f)(x) \end{aligned}$$

$$= \Phi_{M',L}(\alpha \circ f)(x)$$
$$= \Phi_{M',L} \circ \mathcal{BL}(f, L)(\alpha)(x)$$

which proves the naturality of $\Phi_{M,L}$ in the first variable. With similar computations, one can easily checked that $\Phi_{M,L}$ is also natural in the second variable. Therefore, $i \dashv \mathbf{MV}$. Moreover, since $\mathcal{MV}$ is a full subcategory of $\mathcal{BL}$, we have the result. □

Since right-adjoint functors preserve limits, we have:

Corollary 3.30 *The $\mathcal{MV}$-functor preserves limits.*

4 MV-coalgebras

The composite $\mathbf{i} \circ \mathbf{MV}$, where $\mathbf{i} : \mathcal{MV} \hookrightarrow \mathcal{BL}$ is the inclusion functor shall also be denoted by **MV** and it is a covariant $\mathcal{BL}$-endofunctor, which has the preservation properties of the $\mathcal{MV}$-functor. An **MV**-coalgebra is a pair (L, α) where L is a BL-algebra and $L \xrightarrow{\alpha} MV(L)$ is a $\mathcal{BL}$-morphism.

In this section, we characterize homomorphisms, **MV**-subcoalgebras and bisimulations in the category of **MV**-coalgebras and prove its (co)completeness.

The following observations are easily checked:

Remark 4.1 Let (L, α) be an **MV**-coalgebra.

(1) For all $x \in L$, $\alpha(x) = \alpha(\overline{\overline{x}})$
(2) If α is injective, then L is an MV-algebra.
(3) For all $x \in L$, there exists $z \in L$ such that $\alpha(x) = \overline{z}$

4.1 MV-homomorphisms

We introduce an arrow notation similar to transition system as in Gumm (2001). We write

$x \overset{\alpha}{\to} y$ iff $\alpha(x) = y$.

We say that a map $(L, \alpha) \xrightarrow{f} (L', \beta)$:

(1) *Preserves transitions* if for all $x, y \in L$ $x \overset{\alpha}{\to} y \Longrightarrow f(x) \overset{\beta}{\to} f(y)$;
(2) *Reflects transitions* if for all $x \in L$ and $y \in L'$ $f(x) \overset{\beta}{\to} y \Longrightarrow x \overset{\alpha}{\to} t$, with $f(t) = y$.

The following results provides a characterization of **MV**-homomorphisms:

Proposition 4.2 *Let (L, α) and (L', β) be two **MV**-coalgebras and $L \xrightarrow{f} L'$ be a BL-morphism. The following are equivalent:*

(1) *f is a homomorphism;*
(2) *For all $x \in L$, $\beta \circ f(x) = \overline{f(z)}$ where $\overline{z} = \alpha(x)$;*
(3) *f preserves and reflects transitions.*

Proof $(i) \Leftrightarrow (ii)$ Suppose f is a homomorphism. Then for all $x \in L$ such that $\alpha(x) = \overline{z}$, we have $\beta \circ f(x) = MV(f)(\overline{z}) = \overline{f(\overline{z})}$. Since f is a BL-morphism, we obtain $\beta \circ f(x) = \overline{f(z)}$. Conversely, suppose that $\beta \circ f(x) = \overline{f(z)}$ where $\overline{z} = \alpha(x)$, for all $x \in L$. Then, $MV(f) \circ \alpha(x) = MV(f)(\overline{z}) = \overline{f(z)}$. So $\beta \circ f(x) = MV(f) \circ \alpha(x)$. $(i) \Rightarrow (iii)$ suppose f is a homomorphism. Let $x, y \in L$ such that $x \overset{\alpha}{\to} y$ and $z \in L$ such that $\alpha(x) = \overline{z}$. Then, $\overline{z} = y$. Since f is a BL-morphism, $\overline{f(z)} = f(y)$ and by hypothesis $\alpha \circ f(x) = f(y)$ which proves that f preserves transitions. Moreover, let $x \in L$ with $\overline{z} = \alpha(x)$ and $y \in L'$. Suppose $f(x) \overset{\alpha'}{\to} y$. We have $x \overset{\alpha}{\to} \overline{z}$ and by hypothesis

$$f(\overline{z}) = \overline{f(z)} = \beta \circ f(x) = y$$

Therefore, f reflects transitions.

$(iii) \Rightarrow (ii)$ Suppose f preserves and reflects transitions. Let $x \in L$ with $\alpha(x) = \overline{z}$. Then $x \overset{\alpha}{\to} \overline{z}$ and by hypothesis, $f(x) \overset{\beta}{\to} f(\overline{z})$ which means that $\beta \circ f(x) = \overline{f(z)}$. □

Example 4.3 Consider the BL-algebra $\mathbf{G}_4$ from the proof of Proposition 3.28 and consider the function $\mathbf{G}_3 \xrightarrow{f} \mathbf{G}_4$ such that $f(0) = 0$, $f(x) = a$, $f(1) = 1$. Then, it is easily checked that f is a BL-morphism. Define $\mathbf{G}_i \xrightarrow{\alpha_i} \mathbf{G}_2$ by $\alpha_i(0) = 0$ and $\alpha_i(x) = 1$ for $x \neq 0$, $i \in \{3, 4\}$. α_3 and α_4 are BL-morphisms and it is obvious that $\mathbf{G}_2 = MV(\mathbf{G}_i)$ for $i \in \{3, 4\}$. Let $x \in \mathbf{G}_3$ with $x \neq 0, 1$. We have:

$$\begin{aligned} \alpha_4 \circ f(x) &= \alpha_4(a) \\ &= 1 \\ &= MV(f)(1) \\ &= MV(f) \circ \alpha_3(x) \end{aligned}$$

Hence, f is a homomorphism of coalgebras between $(\mathbf{G}_3, \alpha_3)$ and $(\mathbf{G}_4, \alpha_4)$.

The category of **MV**-coalgebras and homomorphisms shall be denoted by $\mathcal{BL}_{\mathbf{MV}}$.

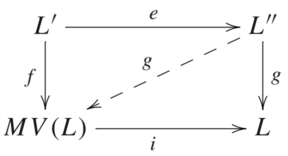
4.2 MV-subcoalgebra

Since $\mathcal{BL}$ has (Epi, ExtrMono)=(Epi, StrongMono)- factorizations, and following (Adámek 2005, p 171), we give the following definition of **MV**-subcoalgebra.

Definition 4.4 Let (L', β) be an **MV**-coalgebra. An **MV**-*subcoalgebra* of (L', β) is an **MV**-coalgebra (L, α) together with a strong mono homomorphism (i.e. a homomorphism which is a strong monomorphism in $\mathcal{BL}$) $(L, \alpha) \xrightarrow{m} (L', \beta)$.

Proposition 4.5 *Let (L, α) be an MV-coalgebra. Then, $(MV(L), MV(\alpha))$ is an MV-subcoalgebra of (L, α).*

Proof Let $i : MV(L) \hookrightarrow L$ be the inclusion morphism. Let $L' \xrightarrow{e} L''$ be an epimorphism, f and g be morphisms such that the following square commutes



Then, $f = i \circ f = g \circ e$. Since $g = i \circ g$, it follows that both triangles commute. Therefore, i is a strong-monomorphism. Moreover, for all $x \in MV(L)$,

$$MV(i) \circ MV(\alpha)(x) = i \circ MV(\alpha)(x) = i \circ \alpha(x).$$

Hence i is a homomorphism from $(MV(L), MV(\alpha))$ to (L, α). □

We give now a characterization of BL-subalgebras of a BL-algebra which can be endowed with a transition structure making them subcoalgebras.

Proposition 4.6 *Let (L', β) be an **MV**-coalgebra. A BL-subalgebra L of L' is an **MV**-subcoalgebra of (L', β) iff there exists a strong monomorphism $L \xrightarrow{m} L'$ such that for all $x \in L$, there exists $z \in L$ such that $m(x) \xrightarrow{\beta} \overline{m(z)}$.*

Proof Let L be a BL-subalgebra of L'. Suppose (L, α) is a subcoalgebra of (L', β). Then, by definition, there exists a strong mono-homomorphism $(L, \alpha) \xrightarrow{m} (L', \beta)$. Let $x \in L$. By Remark 4.1, there exists $z \in L$ such that $\alpha(x) = \overline{z}$. It follows from Proposition 4.2 that $\beta \circ m(x) = \overline{m(z)}$ which means that $m(x) \xrightarrow{\beta} \overline{m(z)}$.

Conversely, assume that there is a strong monomorphism $m : L \longrightarrow L'$ such that for all $x \in L$, there exists $z \in L$ with $m(x) \xrightarrow{\beta} \overline{m(z)}$. Then observe that $\overline{z}$ is unique since m is injective (Proposition 3.22) and define

$$\begin{aligned} \alpha : L &\longrightarrow MV(L) \\ x &\longmapsto \overline{z} \end{aligned}$$

Let $x, y \in L$ such that $\alpha(x) = \overline{z}$ and $\alpha(y) = \overline{z'}$. If $x = y$ then we have $m(x) \xrightarrow{\beta} \overline{m(z)}$ and $m(x) \xrightarrow{\beta} \overline{m(z')}$. It follows that $\overline{m(z)} = \overline{m(z')}$ and then $m(\overline{z}) = m(\overline{z'})$. Since m is a mono, we obtain $\overline{z} = \overline{z'}$ i.e. $\alpha(x) = \alpha(y)$. Therefore, α is well defined. Moreover,

$$\begin{aligned} \beta \circ m(0) &= 0 \text{ (since } \beta \text{ and } m \text{ are BL-morphisms)} \\ &= \overline{1} (\text{ by Lemma 2.1(8)}) \\ &= \overline{m(1)} \ (m, \text{ is a BL-morphism}). \end{aligned}$$

Thus, $m(0) \xrightarrow{\beta} \overline{m(1)}$ and so $\alpha(0) = 0$. Let $x, y \in L$ such that $\alpha(x \propto y) = \overline{t}$, $\alpha(x) = \overline{z}$ and $\alpha(y) = \overline{z'}$, where $\propto \in \{*, \to\}$. We have $m(x \propto y) \xrightarrow{\beta} \overline{m(t)}$. Since $\beta \circ m$ is a BL-morphism, we obtain

$\beta \circ m(x) \propto \beta \circ m(y) = m(\overline{t})$. So $m(\overline{z}) \propto m(\overline{z'}) = m(\overline{t})$ and therefore $m(\overline{z} \propto \overline{z'}) = m(\overline{t})$. Using the fact that m is injective (Proposition 3.22), we obtain $\overline{z} \propto \overline{z'} = \overline{t}$ and then, $\alpha(x) \propto \alpha(y) = \alpha(x \propto y)$. Hence, α is a $\mathcal{BL}$-morphism. It follows that α is a transition structure on L making m a strong mono homomorphism. □

4.3 MV-bisimulations and (co)limits in $\mathcal{BL}_{\mathbf{MV}}$

Definition 4.7 Let R be a strong relation between two **MV**-coalgebras (L_1, α_1) and (L_2, α_2), that is, there is a strong-mono $m : R \hookrightarrow L \times L'$. R is called an **MV**-*bisimulation* provided that there is a structure map on R making the projections $\pi_i : R \longrightarrow L_i$ **MV**-homomorphisms.

Proposition 4.8 *In $\mathcal{BL}_{MV}$, bisimulations are precisely strong-mono relations.*

Proof Let R be a strong-mono relation on (L_1, α_1) and (L_2, α_2). Consider

$$\begin{aligned} \delta : R &\longrightarrow \mathbf{MV}(R) \\ (x, y) &\longmapsto (\alpha_1(x), \alpha_2(x)) \end{aligned}$$

Then, δ is a $\mathcal{BL}$-morphism since α_1 and α_2 are so. Moreover, for all $(x, y) \in R$,

$$\begin{aligned} \alpha \circ \pi_1(x, y) &= \alpha_1(x) \\ &= \mathbf{MV}(\pi_1)(\alpha_1(x), \alpha_2(x)) \\ &= \mathbf{MV}(\pi_1) \circ \delta(x, y). \end{aligned}$$

Thus π_1 is an **MV**-homomorphism. Similarly, one can show that π_2 is an **MV**-homomorphism. □

Proposition 4.9 *The largest bisimulation between two **MV**-coalgebras (L, α) and (L', β) always exists. Moreover, when $(L, \alpha) = (L', \beta)$ that largest bisimulation (called the bisimilarity on (L, α)) is an equivalence relation.*

Proof $\mathcal{BL}$ is well powered, complete, cocomplete and the **MV**-functor preserves limits. It follows from Adámek (2005, Proposition 5.5) that the largest bisimulation between any two **MV**-coalgebras exists. The second part of the proposition is a consequence of Adámek (2005, Corollary 5.6). □

Lemma 4.10 *Let $FIX(\boldsymbol{MV})$ denote the class of fixed points of **MV** and $\mathbb{MV}$ the class of MV-algebras. The following hold:*

(1) $FIX(\boldsymbol{MV}) = \mathbb{MV}$.
(2) $(\mathbf{G}_1, id_{\mathbf{G}_1})$ *is the final coalgebra for the functor* $\boldsymbol{MV}$.

Proof (i) Since any MV-algebra is its own MV-centre, we just have to prove that $FIX(\mathbf{MV}) \subseteq \mathbb{MV}$. Let $L \in FIX(\mathbf{MV})$ be a BL-algebra. Then, there exists a $\mathcal{BL}$-isomorphism $\varphi : L \longrightarrow MV(L)$. For any $x \in L$, there exists $y \in MV(L)$ such that $x = \varphi^{-1}(y)$. This means that

$$\bar{\bar{x}} = \overline{\overline{\varphi^{-1}(y)}} = \varphi^{-1}(\bar{\bar{y}}) = \varphi^{-1}(y) = x,$$

since the converse φ^{-1} of φ is a $\mathcal{BL}$-morphism. Therefore, L is an MV-algebra.
(ii) follows from the fact that **MV** preserves final objects. □

Since **MV** preserves pullbacks, the composition of **MV**-bisimulations is again an **MV**-bisimulation (see Adámek (2005, Example 5.4)). Therefore (or see Hofmann et al. (2019, Theorem 2.5.7)), pullbacks of **MV**-homomorphisms are **MV**-bisimulations. In that case, it is stated in Adámek (2005, Remark 5.8) for an arbitrary weak pullback preserving endofunctor that the largest bisimulation on a given coalgebra is the kernel equivalence of the unique homomorphism from that coalgebra to the final coalgebra. The following result comes from that observation:

Proposition 4.11 *Let* (L, α) *be an* **MV**-*coalgebra. The largest bisimulation on* (L, α) *is* $L \times L$.

Proof By Adámek (2005, Remark 5.8), the largest bisimulation on (L, α) is the kernel pair of the morphism $(L, \alpha) \xrightarrow{\tilde{1}} (\mathbf{G}_1, id_{\mathbf{G}_1})$, which is clearly $L \times L$. □

It is well known (see e.g. Adámek (2005, Proposition 4.3) or Hofmann et al. (2019, Theorem 1.2.4)) that the forgetful functor from the category of coalgebras to the base category creates colimits. It follows that $\mathcal{BL}_{\mathbf{MV}}$ has whatever colimit $\mathcal{BL}$ has. On the other hand, $\mathcal{BL}_{\mathbf{MV}}$ has all limits preserved by **MV**. Since $\mathcal{BL}$ is complete, cocomplete and **MV** is a limit-preserving functor, we have:

Theorem 4.12 $\mathcal{BL}_{MV}$ *is complete and cocomplete.*

5 Topological MV-coalgebras

Topological BL-algebras have been studied by many authors, see e.g. Zahiri and Borzooei (2016). In this section, we introduce and investigate topological **MV**-coalgebras. Moreover, we construct an inverse system in the category of **MV**-coalgebras.

Lemma 5.1 *Let* D *be a ds of* L, $L \xrightarrow{f} L'$ *be a BL-morphism. Then, for all* $x \in L$, $f([x]_D) \subseteq [f(x)]_{f(D)}$. *The equality holds when* $x \in D$.

Proof Let $y \in f([x]_D)$. Then, there exists $z \in L$ such that $y = f(z)$ and $(z \to x) \wedge (x \to z) \in D$. We have

$$\begin{aligned}(y \to f(x)) \wedge (f(x) \to y) &= f(z \to x) \wedge f(x \to z)\\ &= f((z \to x) \wedge (x \to z)) \in f(D).\end{aligned}$$

Hence, $y \in [f(x)]_{f(D)}$. For the converse, suppose $x \in D$. Let $y \in [f(x)]_{f(D)}$. Then, $(y \to f(x)) \wedge (f(x) \to y) \in f(D)$. Since $f(D)$ is a ds of L, $y \in f(D)$, that is, there exists $z \in D$ such that $y = f(z)$. Since x and z are in D, $(z \to x) \wedge (x \to z) \in D$. Thus, $z \in [x]_D$ and so $y \in f([x]_D)$. □

Definition 5.2 Let (L, α) be an **MV**-coalgebra.

(1) Let τ be a topology on L. $((L, \alpha), \tau)$ is called a *topological* MV-*coalgebra* if (L, τ) is a topological BL-algebra and α is continuous, i.e. for any $x \in L$ and any subset V of L containing $\alpha(x)$, there exists an open set U containing x such that $\alpha(U) \subseteq V$.
(2) Let D be a ds of L. D is said α-*stable* if $\alpha(D) \subseteq D$.

For any **MV**-coalgebra(L, α), the class of α-stable ds of L is not empty since it contains $\{1\}$.

A poset $(I, \leq)$ is said to be *upward directed* provided that for any $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let I be an upward-directed set and $D = \{D_i, i \in I\}$ be a family of dss of a BL-algebra L. Then, D is called a *system of dss* or simply a *system* of L if $i \leq j$ implies $D_j \subseteq D_i$, for any $i, j \in I$. An *inverse system* in a category $\mathcal{C}$, is a family $(B_i, \varphi_{i,j})_{i,j \in I}$ of objects indexed by an upward-directed set I, with a family of morphisms $\varphi_{i,j} : B_i \longrightarrow B_j$, for $i \leq j$, satisfying the following conditions:

(1) $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$, for any $k \leq j \leq i$
(2) $\varphi_{i,i} = id_{B_i}$, for any $i \in I$.

Definition 5.3 Let (L, τ) be a topological space. The topology τ is called a *linear topology* on L if there exists a base B for τ such that any element of B containing 1 is a ds of L.

Theorem 5.4 *Let* (L, α) *be an* **MV**-*coalgebra and* $D = \{D_i, i \in I\}$ *be a* α-*stable system of* L *(i.e. each* D_i *is a* α-*stable ds of* L, $i \in I$*). Then*

(1) *The set* $B = \{[x]_{D_i}, x \in L, i \in I\}$ *is a base for a topology on* L *and* τ_B, *the topology induced by* B *is linear.*

(2) $((L, \alpha), \tau_B)$ *is a topological* ***MV****-coalgebra.*

Proof (1) Similar to the proof of Zahiri and Borzooei (2016, Lemma 3.3)

(2) It follows from Zahiri and Borzooei (2016, Theorem 3.4), that (L, τ_B) is a topological BL-algebra. We just have to show that α is continuous. Let $x \in L$, such that $\alpha(x) \in [z]_{D_i}$, $z \in L$. For all $y \in \alpha([x]_{D_i})$, we have by Lemma 5.1 $(y \to \alpha(x)) \wedge (\alpha(x) \to y) \in \alpha(D_i) \subseteq D_i$. So $y \in [\alpha(x)]_{D_i} = [z]_{D_i}$. Therefore, $[x]_{D_i}$ is an open subset containing x such that $\alpha([x]_{D_i}) \subseteq [z]_{D_i}$. Thus, α is continuous.

□

Theorem 5.5 *Let* (L, α) *be an* ***MV****-coalgebra,* $D = \{D_i, i \in I\}$ *be a* α*-stable system of filter of* L*. For each* $i \in I$*, define* $\alpha_i : L/D_i \longrightarrow \alpha(L/D_i)$ *by:* $\alpha_i([x]_{D_i}) = [\alpha(x)]_{D_i}$*. Consider the family of maps* $(\varphi_{ij})_{i \leq j \in I}$ *defined by* $\varphi_{ij} : (L/D_j, \alpha_j) \longrightarrow (L/D_i, \alpha_i)$ *such that for all* $x \in L$*,* $\varphi_{ij}([x]_{D_j}) = [x]_{D_i}$*. Then,* $((L/D_i, \alpha_i)_{i \in I}); (\varphi_{ij})_{i \leq j \in I})$ *is an inverse system in* $\mathcal{BL}_{\mathbf{MV}}$*.*

Proof Let $x \in L$. By Remark 4.1, there exists $z \in L$ such that $\alpha(x) = \bar{z}$. So for all $i \in I$, $\alpha_i([x]_{D_i}) = [\alpha(x)]_{D_i} = [\bar{z}]_{D_i} = \overline{[z]_{D_i}}$. Moreover, for $x, x' \in L$ such that $[x]_{D_i} = [x']_{D_i}$. Then, $x \to x' \in D_i$ and $x' \to x \in D_i$. It follows that $\alpha(x \to x') \in D_i$ and $\alpha(x' \to x) \in D_i$. So $[\alpha(x)]_{D_i} = [\alpha(x')]_{D_i}$. Thus, each α_i is well defined. Since α is a $\mathcal{BL}$-morphism, it is easily checked that each α_i is a $\mathcal{BL}$-morphism and therefore, for all $i \in I$, $(L/D_i, \alpha_i)$ is an **MV**-coalgebra.

Let $i, j \in I$ such that $i \leq j$. Since $F_j \subseteq F_i$, φ_{ij} is well defined and is clearly a $\mathcal{BL}$-morphism. Moreover, let $x \in L$ and let $z \in L$ such that $\alpha(x) = \bar{z}$. Then, for all $j \in I$,

$$\begin{aligned}\overline{\varphi_{ij}([z]_{D_j})} &= \varphi_{ij}(\overline{[z]_{D_j}}) \\ &= \varphi_{ij}([\alpha(x)]_{D_j}) \\ &= [\alpha(x)]_{D_i} \\ &= \alpha_i([x]_{D_i}) \\ &= \alpha_i \circ \varphi_{ij}([x]_{D_j}).\end{aligned}$$

Hence, by Proposition 4.2, φ_{ij} is an **MV**-homomorphism. It is clear that $\varphi_{ii} = 1_{L/D_i}$ and for $i \leq j \leq k \in I$,

$$\varphi_{ij} \circ \varphi_{jk}([x]_{D_k}) = \varphi_{jk}([x]_{D_j}) = [x]_{D_k} = \varphi_{ik}([x]_{D_i}).$$

Thus, $((L/D_i, \alpha_i)_{i \in I}); (\varphi_{ij})_{i \leq j \in I})$ is an inverse system in $\mathcal{BL}_{\mathbf{MV}}$. □

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Modal representation of coalgebras over local BL-algebras

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Abstract

We consider the category $Coalg(\prod)$ of $\prod$-coalgebras where $\prod$ is the endofunctor on the category of local BL-algebras and BL-morphisms which assigns to each local BL-algebra its quotient by its unique maximal filter and we characterize homomorphisms and subcoalgebras in $Coalg(\prod)$. Moreover, we introduce local BL-frames based on local BL-algebras, and show that the category of local BL-frames is isomorphic to $Coalg(\prod)$.

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1 Introduction

Coalgebras were introduced by Aczel and Mendler [1] to model various type of transition systems. Up to now, coalgebras were studied over the category of sets and mappings (see for example [5, 11]), arbitrary categories (see for example [2, 7, 8]) or categories of topological spaces (see for example [9]), but not specially on algebraic structures. It has been shown that Kripke frames can be seen as coalgebras of the covariant powerset functor [16] and descriptive frames as coalgebras of the Vietoris functor, the topological analogue of the powerset functor, on Stone spaces [9] . These results provide a strong link between coalgebras and modal logic. The aim of this paper is to further investigate this connection for coalgebras over BL-algebras.

Coalgebras over the category $\mathcal{BL}$ of BL-algebras and BL-morphisms were introduced in [10] by the authors. They show that coalgebras of the MV-functor, which assigns each BL-algebra to its MV-center have very nice properties. In this short paper, we establish the link between modal logic and coalgebras over BL-algebras via a new type of logical frame, namely local BL-frame.

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The outline of the paper is as follows: In Section 2, we recollect some definitions and results which will be used throughout the paper. In Section 3, we state some facts about the category of local BL-algebras and introduce coalgebras of the functor $\prod$, which assigns each local BL-algebra to its quotient by its maximal filter. We characterize homomorphisms and subcoalgebras of $\prod$-coalgebras and show that the corresponding category is not complete. In the last part of the paper, we present local BL-frames and models and show that the categories of local BL-frames and $\prod$-coalgebras are isomorphic.

2 Preliminaries

BL-algebras were invented by P. Hájek [6] in order to provide an algebraic proof of the completeness theorem of basic logic (BL, for short) arising from the continuous triangular norms, familiar in the fuzzy logic framework. The language of propositional Hájek basic logic contains the binary connectives $\circ$ and $\Rightarrow$ and the constant $\bar{0}$. Axioms of BL are:

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow \omega))$

(A2) $(\varphi \circ \psi) \Rightarrow \varphi$

(A3) $(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$

(A4) $(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$

(A5a) $(\varphi \Rightarrow (\psi \Rightarrow \omega)) \Rightarrow ((\varphi \circ \psi) \Rightarrow \omega)$

(A5b) $((\varphi \circ \psi) \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \omega))$

(A6) $((\varphi \Rightarrow \psi) \Rightarrow \omega) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \omega) \Rightarrow \omega))$

(A7) $\bar{0} \Rightarrow \omega$.

We recall some definitions and basic results that can be found in [3, 6, 12, 16].

An algebraic structure $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a *bounded commutative residuated lattice* if it satisfies the following conditions:

(BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(BL2) $(L, *, 1)$ is a commutative monoid;

(BL3) $*$ is a left adjoint of $\rightarrow$, that is $x * z \leq y$ if and only if $z \leq x \rightarrow y$.

A BL-*algebra* is a bounded commutative residuated lattice which satisfies the following:

(BL4) $x \wedge y = x * (x \rightarrow y)$ (divisibilty);

(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

A BL-algebra L is called a *Gödel algebra* if $x^2 = x * x = x$ for every $x \in L$. In addition, L is called an *MV-algebra* if $\bar{\bar{x}} = x$ for all $x \in L$, where $\bar{x} = x \rightarrow 0$.

The following holds in any BL-algebra L:

Lemma 2.1. [13] *For all* $x, y, z \in L$

(1) $x \leq y$ *if and only if* $x \to y = 1$*;*

(2) $x * y \leq x \wedge y$*;*

(3) $x \to (y \to z) = y \to (x \to z)$*;*

(4) *If* $x \leq y$*, then* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$*;*

(5) $x \leq y \to (x * y)$*;* $x * (x \to y) \leq y$*;*

(6) $x * \bar{x} = 0$*;*

(7) $(x * y) \to z = x \to (y \to z)$*;*

(8) $1 \to x = x$*;* $x \to 1 = 1$*;* $x \to x = 1$*;* $x \leq y \to x$*;* $x \leq \overline{\overline{x}}$*;* $\overline{\overline{\overline{x}}} = \overline{x}$*.*

A *filter* of L is a non-empty subset F of L such that for all $x, y \in L$,

(F1) $x, y \in F$ implies $x * y \in F$;

(F2) $x \in F$ and $x \leq y$ imply $y \in F$.

A subset D of a BL-algebra L is called a *deductive system* if

(DS1) $1 \in D$;

(DS2) $x \in D$ and $x \to y \in D$ imply $y \in D$.

Deductive systems have been widely studied in BL-algebras namely to characterize fragments of Basic fuzzy logic (see [15]); it is obvious that for a non-empty subset F of L, F is a deductive system if and only if it is a filter.

Let L_1 and L_2 be two BL-algebras, a map $f : L_1 \longrightarrow L_2$ is called a *homomorphism of BL-algebras (BL-morphism)*, if $f(0) = 0$ and $f(x \propto y) = f(x) \propto f(y)$ for all $\propto \in \{*, \to\}$. We obviously have $f(1) = 1$ for any BL-homomorphism f and it is shown in [13] that for any BL-morphism f, $f(x \propto y) = f(x) \propto f(y)$ with $\propto \in \{\vee, \wedge\}$ and if $x \leq y$, then $f(x) \leq f(y)$.

For any deductive system F of a BL-algebra $L = (L, \wedge, \vee, *, \to, 0, 1)$, we can define a relation θ_F on L as follows: for all $x, y \in L$,

$$(x\theta_F y) \Longleftrightarrow ((x \to y) \wedge (y \to x) \in F).$$

It is well known that θ_F is a congruence on L (see, e.g. [6]) and since the class of BL-algebras is a variety, the quotient structure L/θ_F is also a BL-algebra for which for all $x, y \in L$, $[x \propto y] := [x] \propto [y]$ where $\propto \in \{\wedge, \vee, *, \to\}$, and $[x] := [x]_{\theta_F}$. A congruence θ on L is called *induced* by F if $[1]_\theta = F$. In addition, θ_F is clearly induced by F.

The class of BL-algebras equipped with BL-morphisms form a category. We will denote it by $\mathcal{BL}$. The one-element BL-algebra $\{0 = 1\}$ is called the *degenerate* BL-algebra (see [12], Remark 8), we will denote it by $\mathbf{G}_1$. The two-element non-degenerate BL-algebra $\{0, 1\}$ is called the *trivial BL-algebra*, we will denote it by $\mathbf{G}_2$. These two algebras are examples of BL-algebras which are both Gödel-algebras and MV-algebras.

Proposition 2.2. [10] *There are only two non-degenerate BL-algebras with three elements:*

(i) The chain $\{0, x, 1\}$*, with the operations* $*$ *and* $\to$ *defined by the following tables:*

$*$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

$\to$	0	x	1
0	1	1	1
x	0	1	1
1	0	x	1

It is the unique Gödel-algebra with three elements and we will denote it by $\boldsymbol{G}_3$*.*

(ii) The chain $\{0, x, 1\}$*, with the operations* $*$ *and* $\to$ *defined by the following tables:*

$*$	0	x	1
0	0	0	0
x	0	0	x
1	0	x	1

$\to$	0	x	1
0	1	1	1
x	x	1	1
1	0	x	1

It is the unique MV*-algebra with three elements and we will denote it by* $\boldsymbol{M}_3$*.*

Remark 2.3. For any set X, define for $A \subseteq X$ and $B \subseteq X$, $A * B = A \cap B$ and $A \to B = A^C \cup B$. Then the structure $(P(X), \cap, \cup, *, \to, \emptyset, X)$ where $P(X)$ is the powerset of X is a BL-algebra called the *powerBL-algebra* of X.

A *Kripke frame* is a pair (X, R) where X is a set and R is a binary relation on X. For $x \in X$, let $[x]_R = \{y \in X \mid xRy\}$ be the R-image of x. A *p-morphism* between two Kripke frames (X, R) and (Y, R') is a function $f : X \longrightarrow Y$ satisfying $f([x]_R) = [f(x)]_{R'}$ for each $x \in X$. Kripke frames and p-morphisms form a category denoted by $\mathcal{KF}r$.

A *Kripke model* is a tuple (W, R, ν), where (W, R) is a Kripke frame and $\nu : Prop \longrightarrow P(L)$ sends proposition letters to the set of states where they are true. A *modal algebra* is a structure $(L, \wedge, \vee, \neg, 0, 1, \Box)$ such that $(L, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and $\Box$ preserves 1 and $\wedge$.

Definition 2.4. *Let* $\mathcal{C}$ *be a category.*

(1) *A full subcategory* $\mathcal{D}$ *of* $\mathcal{C}$ *is called* isomorphism-closed *provided that every* $\mathcal{C}$*-object that is isomorphic to some* $\mathcal{D}$*-objects is itself a* $\mathcal{D}$*-object.*

(2) *A* coalgebra *for an endofunctor* $F : \mathcal{C} \longrightarrow \mathcal{C}$ *is a pair* (A, α) *where* A *is an object of* $\mathcal{C}$ *and* $\alpha : A \longrightarrow F(A)$ *is a* $\mathcal{C}$*-morphism.*

(3) *A* homomorphism *between two coalgebras* (A, α) *and* (B, β) *for* F *is a* $\mathcal{C}$*-morphism* $f : A \longrightarrow B$ *such that* $\beta \circ f = F(f) \circ \alpha$*.*

(4) *Coalgebras for* F *and their homomorphisms form a category denoted by* $\mathcal{C}oalg(F)$*.*

3 $\prod$-coalgebras

In this section, we present some properties of local BL-algebras which are BL-algebras with a unique maximal filter. We define a non-trivial endofunctor of the category of local BL-algebras and investigate the corresponding coalgebras.

Definition 3.1. Let L be a BL-algebra.

(1) A deductive system F of L is *proper* if $0 \notin F$.

(2) A deductive system M of L is called *maximal* if it is proper and not contained in any other proper deductive system.

(3) L is *local* if it has a unique maximal deductive system.

Theorem 3.2. [13] *Let L be a BL-algebra. Define*

$$D(L) = \{x \in L \mid x^n \neq 0 \text{ for all integers } n\}.$$

The following are equivalent:

(i) $D(L)$ is a deductive system of L;

(ii) L is local;

(iii) $D(L)$ is the unique maximal deductive system of L.

Example 3.3. (i) $D(\mathbf{G}_3) = \{x, 1\}$, $D(\mathbf{M}_3) = D(\mathbf{G}_2) = \{1\}$ are deductive systems. So by the above theorem, $\mathbf{G}_3$, $\mathbf{M}_3$ and $\mathbf{G}_2$ are local BL-algebras;

(ii) Consider $A = ([0;1], \wedge, \vee, *, \rightarrow, 0, 1)$ the BL-algebra such that for all $x, y \in L$, $x * y = x \cdot y$ and $x \rightarrow y = 1$ if $x \leq y$ and $x \rightarrow y = \frac{y}{x}$ else. Then $D(A) =]0;1]$ is a deductive system of A. Thus A is a local BL-algebra.

(iii) [[15], Proposition 11] Any BL-algebra such that $MV(L) = \{0, 1\}$ is local.

(iv) [[15], Example 1] The chain $\{0, x, y, 1\}$, with the operations $*$ and $\rightarrow$ defined by the following tables

$*$	0	x	y	1
0	0	0	0	0
x	0	0	x	x
y	0	x	y	y
1	0	x	y	1

$\rightarrow$	0	x	y	1
0	1	1	1	1
x	x	1	1	1
y	0	x	1	1
1	0	x	y	1

is a local BL-algebra such that $MV(L) = \{0, x, 1\}$.

(v) $\mathbf{G}_1$ is not local.

Proposition 3.4 ([4], Proposition 1.10)**.** *Let $f : L \longrightarrow L'$ be a BL-morphism. If M' is a maximal deductive system of L', then $f^{-1}(M')$ is a maximal deductive system of L.*

Lemma 3.5 ([4], Lemma 1.9)**.** *Let L be a nontrivial BL-algebra and M a proper deductive system of L. The following are equivalent:*

(i) M is maximal;

(ii) for any $x \in L$, $x \notin M \Leftrightarrow \overline{(x^n)} \in M$ for some integer n.

Lemma 3.6. *Let f be a BL-morphism between two local BL-algebras L and L' whose maximal deductive systems are M and M', respectively. If f is surjective, then $f(M) = M'$.*

Lemma 3.7. *Let L be a BL-algebra and F be a deductive system of L. Then θ_F is the unique congruence on L induced by F.*

Proof. Let θ be a congruence on L induced by F. We have to show that $\theta_F = \theta$. Let $(x, y) \in \theta_F$. Then $x \to y \in [1]_\theta$ and $y \to x \in [1]_\theta$. So by compatibility,

$$(x * (x \to y), x * 1) \in \theta \text{ and } (y * (y \to x), y * 1) \in \theta.$$

Hence by BL4 we obtain $(x \wedge y, x) \in \theta$ and $(y \wedge x, y) \in \theta$. Since θ is symmetric and $\wedge$ is commutative, it follows that $(x, x \wedge y) \in \theta$ and $(x \wedge y, y) \in \theta$. By transitivity, we have $(x, y) \in \theta$. Conversely, let $(x, y) \in \theta$. Then $(x \to y, y \to y) \in \theta$ and $(y \to x, y \to y) \in \theta$. So $(x \to y, 1) \in \theta$ and $(y \to x, 1) \in \theta$. It follows that $x \to y \in F$ and $y \to x \in F$ and therefore, $(x, y) \in \theta_F$. □

In the sequel we will denote L/θ_F by L/F and $[x]_{\theta_F}$ by $[x]_F$.

Let M be the maximal deductive system of a local BL-algebra L. Then by ([4], Proposition 1.13), since M is the unique maximal deductive system which contains M, L/M is a local BL-algebra. Therefore, we have:

Lemma 3.8. *Let M be the maximal deductive system of a local BL-algebra L. Then L/M is a local BL-algebra and $D(L/M) = \{M\}$.*

Proof. We have $M^n = [1]_M^n = [1]_M \neq [0]_M$, which means that $M \in D(L/M)$. Let $[x]_M \in D(L/M)$. Then $[x^n]_M = [x]_M^n \neq [0]_M$, for all integer n. It follows that $x^n \to 0 \notin M$, for all integer n. Thus by Lemma 3.5, $x \in M$; That is, $[x]_M = M$. □

Local BL-algebras and BL-morphisms form a category which will be denoted by $l\mathcal{BL}$.

Proposition 3.9. *$l\mathcal{BL}$ is an isomorphism-closed subcategory of $\mathcal{BL}$.*

Proof. Let $f : L \longrightarrow G$ be an isomorphism between a BL-algebra L and a local BL-algebra G, whose inverse is g. Then by Proposition 3.4, $f^{-1}(M')$ is a maximal filter of L, where M' is the unique maximal filter of G. Moreover, let H be another maximal filter of L. Then $g^{-1}(H) = M'$ and so $H = g(M') = f^{-1}(M')$. Thus L is a local BL-algebra. □

Remark 3.10. Let L and L' be two local BL-algebras, M and M' their respectives maximal filters. Then $M \times L'$ and $L \times M'$ are maximal filters of $L \times L'$. Thus, $L \times L'$ is not a local BL-algebra. It follows that $l\mathcal{BL}$ has no (co)products and therefore $l\mathcal{BL}$ is not complete, nor cocomplete.

Proposition 3.11. *Consider the correspondence $\prod : l\mathcal{BL} \longrightarrow l\mathcal{BL}$ such that $\prod(L) = L/M$ for any local BL-algebra L whose unique maximal filter is M and $\prod(f) : L/M \longrightarrow L/M'$ such that*

$$\prod(f)([x]_M) = [f(x)]_{M'}.$$

Then $\prod$ is a covariant endofunctor on $l\mathcal{BL}$.

Proof. By Lemma 3.7 and the fact that θ_M is a congruence, $\prod(L)$ is well defined. Moreover, let $L \xrightarrow{f} L'$ and $L' \xrightarrow{g} L''$ be two BL-morphisms. Let $x \in L$. We have

$$\prod(g) \circ \prod(f)([x]_M) = \prod(g)([f(x)]_{M'}) = [g \circ f(x)]_{M'} = \prod(g \circ f)([x]_M)$$

and also

$$\prod(id_L)([x]_M) = [x]_M = id_{\prod(L)}([x]_M).$$

□

Let $Coalg(\prod)$ be the category of $\prod$-coalgebras and $\prod$-homomorphisms. Let (L, α) be a $\prod$-coalgebra. For any x, y in a BL-algebra L, we denote $x \overset{\alpha}{\to} y$ by $\alpha(x) = [y]_M$. Then one can observe that $\prod$-coalgebras mimic non-deterministic transition systems.

Let (L, α) and (L', α') be two $\prod$-coalgebras. A BL-morphism $f : L \longrightarrow L'$ *weakly reflects* transition systems if for all $x \in L$ and $y \in L'$, $f(x) \overset{\alpha'}{\to} y$ implies $x \overset{\alpha}{\to} t$, with $f(t) \in [y]_M$, $t \in L$.

Proposition 3.12. *Let (L, α) and (L', α') be two $\prod$-coalgebras, and $f : L \longrightarrow L'$ a BL-morphism. The following are equivalent:*

(i) f is a $\prod$-homomorphism;

(ii) for all $x \in L$, $\alpha'(f(x)) = [f(z)]_{M'}$, whenever $\alpha(x) = [z]_M$;

(iii) f preserves and weakly reflects transitions.

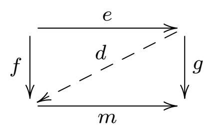
Proof. $(i) \Leftrightarrow (ii)$ Straightforward.

$(ii) \Rightarrow (iii)$ Suppose for all $x \in L$, $\alpha'(f(x)) = [f(z)]_{M'}$, whenever $\alpha(x) = [z]_M$. Let $x, y \in L$ such that $x \overset{\alpha}{\to} y$. Then $\alpha(x) = [y]_M$. So by hypothesis, $\alpha'(f(x)) = [f(y)]_{M'}$ implying $f(x) \overset{\alpha'}{\to} f(y)$. So f preserves transitions. Moreover, let $x \in L$ and $y \in L'$ such that $f(x) \overset{\alpha'}{\to} y$. Then $\alpha'(f(x)) = [y]_{M'}$. Let $z \in L$ such that $\alpha(x) = [z]_M$. Then $x \overset{\alpha}{\to} z$ and by hypothesis, $[f(z)]_{M'} = \alpha'(f(x)) = [y]_{M'}$, i.e., so $f(z) \in [y]_{M'}$. Thus, f weakly preserves transitions.

$(iii) \Rightarrow (ii)$ Let $x \in L$, such that $\alpha(x) = [z]_M$. Then $x \overset{\alpha}{\to} z$, which implies by hypothesis that $f(x) \overset{\alpha'}{\to} f(z)$, i.e. $\alpha'(f(x)) = [f(z)]_{M'}$.

□

Definition 3.13. *[3]* A monomorphism m is called *strong* in a category $\mathcal{C}$ if for every epimorphism e and every commutative square



there exists a diagonal d such that $g = m \circ d$ and $f = d \circ e$.

Proposition 3.14. *Let (L', α') be a $\prod$-coalgebra. A local BL-subalgebra L of L' is a $\prod$-subcoalgebra of (L', α') iff there exists a strong mono $L \overset{m}{\longrightarrow} L'$, verifying the following property: for all $x \in L$, there exists $z \in L$ such that $m(x) \overset{\alpha'}{\to} m(z)$.*

Proof. Suppose that (L, α) is a $\prod$-subcoalgebra of (L', α'), and m the corresponding strong mono. Let $x \in L$. Since m is a $\prod$-homomorphism, it follows from Proposition 3.12 that $\alpha' \circ m(x) = [m(z)]_{M'}$, where $\alpha(x) = [z]_M$. So $m(x) \overset{\alpha'}{\to} m(z)$, $z \in L$.

Conversely, assume that there is a strong mono $m : L \longrightarrow L'$ such that for all $x \in L$, there exists $z \in L$ such that $m(x) \overset{\alpha'}{\to} m(z)$. Define $\alpha : L \longrightarrow \prod(L)$ by $\alpha(x) = [z]_M$, where $m(x) \overset{\alpha'}{\to} m(z)$. Let $x, x' \in L$ such that $\alpha(x) = [z]_M$ and $\alpha(x') = [z']_M$. If $x = x'$, then $\alpha' \circ m(x) = \alpha' \circ m(x')$. So by Proposition 3.12 (ii), we obtain $[m(z)]_{M'} = [m(z')]_{M'}$. Hence

$$(m(z) \to m(z')) \wedge (m(z') \to m(z)) \in M'.$$

Thus

$$(z \to z') \wedge (z' \to z) \in m^{-1}(M').$$

It follows from Proposition 3.4 that $(z \to z') \wedge (z' \to z) \in M$. So $[z]_M = [z']_M$. Thus α is well defined. Moreover, since α' and m are BL-morphisms, we have

$$\alpha' \circ m(0) = \alpha'(0) = [1]_M = [m(1)]_M .$$

Hence $m(0) \overset{\alpha'}{\to} m(1)$, implying $\alpha(0) = [1]_M$. On another hand, let $x, y \in L$ such that $\alpha(x \propto y) = [t]_M$, $\alpha(x) = [u]_M$ and $\alpha(y) = [v]_M$ where $\propto \in \{*, \to\}$. Then we have $m(x \propto y) \overset{\alpha'}{\to} m(t)$, i.e. $\alpha' \circ m(x \propto y) = [m(t)]_{M'}$. Since $\alpha' \circ m$ is a BL-morphism, we have

$$\alpha' \circ m(x) \propto \alpha' \circ m(y) = [m(t)]_{M'} ,$$

i.e.

$$[m(u)]_{M'} \propto [m(v)]_{M'} = [m(t)]_{M'} .$$

Thus $m([u]_M \propto [v]_M) = m([t]_M)$. Since m is a mono, $[u]_M \propto [v]_M = [t]_M$ and so $\alpha(x \propto y) = \alpha(x) \propto \alpha(y)$. Therefore, α is a BL-morphism. It follows that (L, α) is a $\prod$-subcoalgebra of (L', α'). □

It follows from Remark 3.10 that $l\mathcal{BL}$ has no products and then bisimulations cannot be defined on $\prod$-coalgebras. Moreover, since limits and colimits in the categories of coalgebras are carried by limits and colimits in the base categories, we obtain the following result:

Proposition 3.15. *Coalg*($\prod$) *is not complete, nor cocomplete.*

4 Local BL-frames as $\prod$-coalgebras

Throughout this section, we fix a set *Prop* of proposition letters.

Definition 4.1. (1) A *local* BL-*frame* is a structure (L, θ_M) where L is a local BL-algebra and M is the maximal filter of L;

(2) A *local* BL-*model* is a structure (L, θ_M, ν) where (L, θ_M) is a local BL-frame and $\nu : Prop \longrightarrow \prod(L)$ is a compatible valuation, that is for all $x, y \in L$, we have

(i) $\nu^{-1}(\{[x]_M * [y]_M\}) = \nu^{-1}(\{[x]_M\}) \cap \nu^{-1}(\{[y]_M\})$;
(ii) $\nu^{-1}(\{[x]_M \to [y]_M\}) = \nu^{-1}(\{[x]_M\})^C \cup \nu^{-1}(\{[y]_M\})$;
(iii) $\nu^{-1}(\{[0]_M\}) = \emptyset$.

Local BL-frames (models) and BL-morphisms form a category which will be denoted by $\mathcal{F}r(lBL)$ ($\mathcal{M}od(lBL)$).

Remark 4.2. It is well known that the normal modal logic S_5 is characterized by the class of reflexive, symmetric, and transitive Kripke frames, that is, the frames for S_5 are exactly that Kripke frames in which the accessibility relation is an equivalence relation. Therefore S_5 is sound and complete in the class of local BL-frames.

The validity of modal formulas at a world x in a local BL-model (L, θ_M, ν) is defined recursively as:

$\mathcal{M}, x \models p$ iff $x \in \nu(p)$

$\mathcal{M}, x \models \neg\varphi$ iff not $\mathcal{M}, x \models \varphi$

$\mathcal{M}, x \models \varphi \wedge \psi$ iff $\mathcal{M}, x \models \varphi$ and $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \varphi \vee \psi$ iff $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \varphi \rightarrow \psi$ iff not $\mathcal{M}, x \models \varphi$ or $\mathcal{M}, x \models \psi$

$\mathcal{M}, x \models \Box\varphi$ iff for every $y \in [x]_M$, $\mathcal{M}, y \models \varphi$

$\mathcal{M}, x \models \Diamond\varphi$ iff there exists $y \in [x]_M$, $\mathcal{M}, y \models \varphi$

The *truth set* of a formula φ in a model $\mathcal{M}$ is the set $[[\varphi]]^{\mathcal{M}} = \{x \in L / \mathcal{M}, x \models \varphi\}$. For any subset K of L, we define the operators $\triangleleft$ and $\widetilde{\Box}$ by:

$$\triangleleft K = L \setminus K \text{ and } \widetilde{\Box} K = \{x \in L / [x]_M \subseteq K\}.$$

By checking the semantics clause above, we have the following result:

Lemma 4.3. *For any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$,*

(i) $[[p]]^{\mathcal{M}} = \nu(p)$;

(ii) $[[\neg\varphi]]^{\mathcal{M}} = \triangleleft [[\varphi]]^{\mathcal{M}}$;

(iii) $[[\varphi \wedge \psi]]^{\mathcal{M}} = [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$;

(iv) $[[\Box\varphi]]^{\mathcal{M}} = \widetilde{\Box} [[\varphi]]^{\mathcal{M}}$.

The following result shows how to construct modal algebras with any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$:

Theorem 4.4. *For any $l\mathcal{BL}$-model $\mathcal{M} = (L, \theta_M, \nu)$, define the set*

$$\tau(\mathcal{M}) = \{[[\varphi]]^{\mathcal{M}}, \varphi \in Prop\}.$$

Then the structure $(\tau(\mathcal{M}), \cap, \cup, \triangleleft, \emptyset, L, \widetilde{\Box})$ is a modal algebra.

Proof. Using Lemma 4.3, it is easily checked that $(\tau(\mathcal{M}), \cap, \cup, \triangleleft, \emptyset, L)$ is a Boolean algebra and that $\widetilde{\Box} L = L$. We only show that $\widetilde{\Box}$ preserves intersections. Let $\varphi, \psi \in Prop$. We have

$$\widetilde{\Box}([[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}) = \{x \in L \mid [x]_M \subseteq [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}\} \subseteq \widetilde{\Box} [[\varphi]]^{\mathcal{M}} \cap \widetilde{\Box} [[\psi]]^{\mathcal{M}}.$$

Conversely, let $x \in \widetilde{\Box} [[\varphi]]^{\mathcal{M}} \cap \widetilde{\Box} [[\psi]]^{\mathcal{M}}$. Then $[x]_M \subseteq [[\varphi]]^{\mathcal{M}}$ and $[x]_M \subseteq [[\psi]]^{\mathcal{M}}$. Thus for all $y \in [x]_M$, we have $\mathcal{M}, y \models \varphi$ and $\mathcal{M}, y \models \psi$. So $\mathcal{M}, y \models \varphi \wedge \psi$. By Lemma 4.3 we obtain $y \in [[\varphi \wedge \psi]]^{\mathcal{M}} = [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$. It follows that $[x]_M \subseteq [[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}}$ and so $x \in \widetilde{\Box}([[\varphi]]^{\mathcal{M}} \cap [[\psi]]^{\mathcal{M}})$. □

For each BL-algebra L, let $\underline{L}$ denote the carrier.

In what follows, we give a link between local BL-frames and well known Kripke frames:

Proposition 4.5. *Let $\mathcal{F}r(lBL)^*$ be the category of local BL-frames with surjective morphisms. Then the correspondance $U : \mathcal{F}r(lBL)^* \longrightarrow \mathcal{KF}r$ which sends every (L, θ_M) to $(\underline{L}, \theta_M))$ and acts on morphisms as identity is a faithful functor.*

Proof. For any local BL-frame (L, θ_M), $U((L, \theta_M)) = (\underline{L}, \theta_M)$ is clearly a Kripke frame. Let $f : (L, \theta_M) \longrightarrow (L', \theta_{M'})$ be a surjective morphism. In order to show that U is well defined, we have to show that f is a p-morphism. Let $x \in L$ and $y \in f([x]_M)$. Then $y = f(z)$ with $z \in [x]_M$. So

$$(z \to x) \wedge (x \to z) \in M.$$

Thus

$$f((z \to x) \wedge (x \to z)) \in f(M).$$

It follows from Lemma 3.6 that

$$(y \to f(x)) \wedge (f(x) \to y) \in M'.$$

So $y \in [f(x)]_{M'}$ and we have $f([x]_M) \subseteq [f(x)]_{M'}$. Moreover, let $y \in [f(x)]_{M'}$. Since f is surjective, there exists $z \in L$ such that $y = f(z)$ and we have

$$(f(z) \to f(x)) \wedge (f(x) \to f(z)) \in M',$$

that is

$$f((z \to x) \wedge (x \to z)) \in M'$$

so that

$$(z \to x) \wedge (x \to z) \in f^{-1}(M') = M.$$

Thus $z \in [x]_M$. Therefore $y \in f([x]_M)$. Hence $f([x]_M) = [f(x)]_M$. So U is well defined. The functoriality and the faithfulness of U are straightforward.

□

We present now the result which allows to see local BL-frames as $\prod$-coalgebras:

Theorem 4.6. *$\mathcal{F}r(lBL)$ is isomorphic to $Coalg(\prod)$.*

Proof. Consider the correspondance $\mathbb{F}$ which assigns to each local BL-frame (L, θ_M) the pair $(L, L \xrightarrow{\alpha_L} L/M)$ such that $\alpha(x) = [x]_M$ for all $x \in L$ and to each BL-morphism $f : L \longrightarrow L'$, $\mathbb{F}(f) = f$. Let (L, θ_M) be a local BL-frame. Since θ_M is a congruence, α is a BL-morphism and so $(L, L \overset{\alpha_L}{\to} L/M)$ is a $\prod$-coalgebra. Moreover, let $(L, \theta_M) \xrightarrow{f} (L', \theta_{M'})$ be a BL-morphism. For all $x \in L$,

$$\alpha' \circ f(x) = [f(x)]_{M'} = \prod(f)([x]_M) = \prod(f) \circ \alpha(x).$$

So f is a $\prod$-homomorphism between $(L, L \xrightarrow{\alpha_L} L/M)$ and $(L', L' \xrightarrow{\alpha_L} L'/M')$. Hence, $\mathbb{F}$ is well defined. By spelling out the definitions, one shows that $\mathbb{F}$ preserves composition and identity. Thus $\mathbb{F} : \mathcal{F}r(lBL) \longrightarrow Coalg(\prod)$ is a covariant functor.

Moreover The correspondance $\mathbb{G}$ which assigns to each $\prod$-coalgebra $(L, L \xrightarrow{\alpha_L} L/M)$ the local BL-frame (L, θ_M) and which acts as identity on homomorphisms is functorial. Finally, Lemma 3.7 allows to prove that the two functors above statisfy the identities $\mathbb{F} \circ \mathbb{G} = id_{Coalg(\prod)}$ and $\mathbb{G} \circ \mathbb{F} = id_{\mathcal{F}r(lBL)}$. So $\mathcal{F}r(lBL)$ and $Coalg(\prod)$ are isomorphic. □

5 Conclusion

One of the main interests of the study of coalgebras is the development of coalgebraic logical foundations over base categories, as a way of reasoning in a quantitive way about transition systems. There is a strong link between coalgebras and modal logic. In this paper, we investigate this relation in the framework of BL-algebras. After the characterization of $\prod$-homomorphisms and $\prod$-subcoalgebras, where $\prod$ is the endofunctor on the category of local BL-algebras and BL-morphisms which assigns to each local BL-algebra its quotient by its unique maximal filter, we introduced local BL-frames based on local BL-algebras, and shown that the category of local BL-frames is isomorphic to the category of $\prod$-coalgebras.

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