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CENTRE DE RECHERCHE ET DE FORMATION DOCTORALE EN SCIENCES, TECHNOLOGIES ET GEOSCIENCES

UNITE DE RECHERCHE ET DE FORMATION DOCTORALES EN MATHEMATIQUES, INFORMATIQUES, BIOINFORMATIQUES ET APPLICATIONS

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REASEACH AND POSTGRADUATE TRAINING UNIT FOR MATHEMATICS, COMPUTER SCIENCES AND APPLICATIONS

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GLOBAL EXISTENCE OF SOLUTIONS FOR THE COUPLED MAXWELL - BOLTZMANN SYSTEM WITH A HARD POTENTIAL

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BY

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Yaoundé, le ___________________

ATTESTATION DE CORRECTION DU DOCTORAT /PhD

Les soussignés Professeurs ANDJIGA Nicolas Gabriel, AYISSI Raoul Domingo et NOUNDJEU Pierre attestons que Monsieur **NANA MBAJOUN Aubin**, Matricule **17U5440,** ayant soutenu publiquement le 17 Décembre 2021 à la Faculté des Sciences sa thèse de Doctorat/PhD en mathématiques intitulée

Global existence of solutions for the coupled Maxwell-Boltzmann system with a hard potential

A effectué toutes les corrections exigées par le jury de soutenance.

En foi de quoi lui est délivré cette attestation pour servir et valoir ce que de droit.

Membre	RAPPOTEUR	Président
NOUNDJEU Pierre, MC.	**AYISSI Raoul Domingo, Pr.**	**ANDJIGA Nicolas Gabriel, Pr.**

Global Existence Of Solutions For The Coupled Maxwell-Boltzmann System With A Hard Potential

This thesis is submitted in fulfillment of the academic requirement for the degree of

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in Mathematics.

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By:

NANA MBAJOUN Aubin

Master in Mathematics

Registration number: 17U5440

Supervisors:

Pr. ETOUA Rémy Magloire **and** **Pr. AYISSI Raoul**

Professors

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Dedicace

Dedicated to my parents.

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Abstract

A global existence theorem and uniqueness of solution of the coupled spatially homogeneous relativistic Maxwell-Boltzmann system is proved in a Bianchi type I spacetime back-ground , in a hard potential case. The proof relies in the use of a particular form of Povzner inequality.

Keys words: Bianchi type I spacetime, Pozner inequality, relativistic Boltzmann equation, Maxwell equations, energy estimates, hard potential, global existence.

Résumé

Dans un espace-temps de Bianchi de type I, un théorème d'existence globale de solutions du système couplé homogème de Maxwell-Boltzmann est établi. Ce résultat provient de l'utilisation d'une forme particulière d'inégalité de Povzner.
Mots clés: espace temps de Bianchi type I, inégalité de Povzner, équation relativiste de Boltzmann, équations de Maxwell, estimation d'énergie, noyau dur, existence globale.

Introduction

The main purpose of this work is to prove a global existence and uniqueness theorem for a classical solution in the Bianchi type I space time of a magnetized relativistic Boltzmann equation with a hard potential. This result is based on a particular form of Povzner inequality.

The relativistic Boltzmann equation we consider here is one of the basic equations of relativistic kinetic theory. This equation rules the dynamics of the considered charged particles which are subject to mutual collisions, by determining their distribution function f which is a non-negative real-valued function of both the position and the momentum of the particles. The Boltzmann equation generalizes the Vlasov equation which governs the collisionless case, by introducing the collision operator. At the contrary of the Vlasov equation which is widely studied in the literature, there are few works on the Boltzmann equation. The Maxwell equations are the basic equations of electromagnetism and determine the electromagnetic field F created by the fast-moving charged particles. We consider the case where the electromagnetic field F is generated, through the Maxwell equations by the Maxwell current defined by the distribution function f of the colliding particles, a charge density e and a future pointing unit vector u tangent at any point to the temporal axis. The system is coupled in the sens that, f, which is subject to the relativistic Boltzmann equation generates the Maxwell current in the Maxwell equations, whereas the electromagnetic field F, which is subject to the Maxwell equations is in the Lie derivative of f with respect to the vectors field tangent to the trajectories of particles, which are deviated in the presence of F and are no longer the geodesics of the space-time.

Povzner introduced in 1962, the Povzner lemmas for the treatment of the moments of solution for homogeneous Boltzmann equation. These techniques have been extensively used in the last 5 years to greatly develop the homogeneous Boltzmann equation theory, confer [37].

The scattering kernel is a quantity that determines the nature of collisions between particles, and in the non-relativistic case several different types of scattering kernels have been found to be attractive. For instance, the inverse power law gives the best- known types of scattering kernel, and they are further classified into hard and soft cases. In the relativistic setting it is not so clear which types of scattering kernel should be of interest, but a classification of (special) relativistic hard and soft potentials has been proposed in [9] and [39] by applying arguments similar to those used in the non-relativistic case.

Some authors proved local existence theorems for the relativistic Boltzmann equation, considering this equation alone as K. Bichteler. in [4] ; D. Bancel. in [2] or coupling it to other fields equations and looking for local in time existence as D. Bancel. and Y. Choquet-Bruhat. In [3] , R.T Glassey and W. Strauss obtained a global result in[14], in the case of data near to that of equilibrium solution with non-zero density. P.B. Mucha studied the relativistic Boltzmann equation coupled to Einstein's equation in [27] and [26], confusing unfortunately with the non relativistic formulation. More recently, N. Noutchegueme and E. Takou in [32] and N. Noutchegueme and D. Dongo in [30], studied the relativistic Boltzmann equation coupled to Einstein's equations in a Robertson-Walker space-time and in a Bianchi type I space-time, respectively, But only the uncharged case. N. Noutchegueme, D. Dongo and E. Takou proved the global existence of solutions to the single Boltzmann equation in [31] , in the uncharged case.

The motivations to make this study are too many. The first one was essentially a mathematical challenge. Removing the hypothesis that the initial datum of the Boltzmann equation is invariant under a subgroup of $\mathcal{O}_3$ in [29] will transform the problem into a difficult one. Another problem is to solve the new system in a very different framework. A supplementary motivation was to extend the method used by Ho Lee in [21] in a single equation, to a coupled system of differential equations namely the transformed Boltzmann equation, the Maxwell equations and newly equations involving the momentum that becomes a variable. The study of Ho Lee was done in Robertson-Walker space time, here we work in a Bianchi type I space time and this makes the proofs more difficult to elaborate. In short, we have had many motivations to make this study. Now this brings out the question of the method to use.

The method used here is a combination of the one used in [29] and [21], with the change of variables into the covariant coordinates as in [21] and [41] , the Povzner inequality type, and the construction of a sequence of solutions of a modified Maxwell-Boltzmann system.

This study is organized in four chapters:

- In chapter 1 we present the non-relativistic and the relativistic Boltzmann equations in the first two sections. We stress on the hard and soft potential interactions in the third section. The fourth section is devoted to the history of Povzner inequality and the last section of the chapter talks about the Cauchy Lipschitz and the Banach fixed point theorems together with the Gronwall inequality

- In chapter 2 we settle the framework which is a Bianchi type I space-time, we present the phases space, the distribution function and the collision operator as well as the change of variables. Sections 2.2 and 2.3 are devoted to the presentation of the Maxwell and the relativistic Boltzmann equations. In the fourth section we give the assumptions of the work. Section five is devoted to the coupled Maxwell-Boltzmann system and we end the chapter by section 6 in which we define the functional spaces.

- In chapter 3 we study the Maxwell-Boltzmann system for the $\mu - N$ regularity. This chapter is organized in two sections. In section 3.1 we give a local existence theorem and in section 3.2 we extend this result to a global existence and uniqueness theorem for the $\mu - N$ regularity.

- In chapter 4 we address the problem of global existence of solutions to the relativistic Maxwell-Boltzmann system for hard potential case. We present the method, we give preliminary results and we study the modified Maxwell-Boltzmann system. The strategy is to construct a sequence of solutions of the modified Maxwell-Boltzmann system in other to prove that this sequence of solutions converges to a solution of the initial system for the hard potential case.

CHAPTER 1

Preliminaries

This introductory chapter is devoted to the presentation in detail of the Boltzmann equations. The first section talks about the non relativistic Boltzmann equation, the second section gives important informations related to the relativistic Boltzmann equation. In the third section we stress on the nature of interactions between particles, so we consequently present the relativistic hard and soft interactions. We dedicated the fourth section to the Povzner inequality history and we end the chapter by recalling the Cauchy-Lipschitz theorem, the Banach fixed point theorem and the Gronwall inequality. The following contain comes from the book [6] but here the presentation has been improved in order to make reading easy and pleasant.

1.1 The non relativistic Boltzmann equation

Consider a monoatomic gas with N molecules enclosed in a recipient of volume V. One molecule of this gas can be specified at a given time by its position $x = (x_1, x_2, x_3)$ and velocity $c = (c_1, c_2, c_3)$. Hence, a molecule can be specified as a point in a six-dimensional space spanned by its coordinates and velocity components, the so-called $\mu-$phase space. In the $\mu-$phase space, a system of N molecules is described by N points with coordinates (x_α, c_α) for each $\alpha = 1, 2, \cdots, N$.

The state of a gas in the $\mu-$phase space is characterized by a distribution function

$f(x,c,t)$ such that

$$f(x,c,t)\,dxdc \equiv f(x,c,t)\,dx_1dx_2dx_3dc_1dc_2dc_3 \tag{1.1}$$

gives, at time t, the number of molecules in the volume element with position vectors within the range x and $x+dx$ and with velocity vectors within the range c and $c+dc$.

By denoting the volume element in the $\mu-$phase space at time t as

$$d\mu(t) = dxdc, \tag{1.2}$$

the number of molecules in this volume element is given by

$$N(t) = f(x,c,t)d\mu(t). \tag{1.3}$$

Furthermore, let $d\mu(t+\Delta t)$ denote the volume element in the $\mu-$phase space at time $t+\Delta t$ where

$$N(t+\Delta t) = f(x+\Delta x, c+\Delta c, t+\Delta t) \tag{1.4}$$

represents the number of molecules in this volume element.

If during the time interval Δt the molecules do not collide, the quantities $N(t)$ and $N(t+\Delta t)$ should be equal to each other. However, by considering time intervals that are large than the mean free time–i.e., for $\Delta t \gg \tau$ –collisions between the gas molecules occur and the difference between the two numbers of molecules does not vanish. Let this difference be denoted by

$$\begin{aligned}\Delta &= N(t+\Delta t) - N(t)\\ &= f(x+\mathit{\Delta} x, c+\Delta c, t+\Delta t)\,d\mu(t+\Delta t) - f(x,c,t)\,d\mu(t).\end{aligned} \tag{1.5}$$

The changes in the position and velocity vectors of the molecules during the time interval Δt are given by

$$\Delta x = c\Delta t, \Delta c = F\Delta t, \tag{1.6}$$

where $F(x,c,t)$ denotes a specific external force which acts on the molecules. It is a force per unit of mass, i.e., has the dimension of an acceleration.

The relationship between the two volume elements $d\mu(t+\Delta t)$ and $d\mu(t)$ reads

$$d\mu(t+\Delta t) = |J|\,d\mu(t) \tag{1.7}$$

where J is the Jacobian of the transformation, i.e.,

$$J = \frac{\partial \left(x_1 \left(t + \Delta t\right), x_2 \left(t + \Delta t\right), \cdots, c_3 \left(t + \Delta t\right)\right)}{\partial \left(x_1 \left(t\right), x_2 \left(t\right), \cdots c_3 \left(t\right)\right)}. \tag{1.8}$$

Up to linear terms in Δt, the Jacobian is approximated by

$$J = 1 + \frac{\partial F_i}{\partial c_i}\Delta t + \mathcal{O}\left[(\Delta t)^2\right]. \tag{1.9}$$

Furthermore, the expansion of $f\left(x + \mathit{\Delta} x, c + \Delta c, t + \Delta t\right)$ in Taylor series around (x, c, t) and by considering also linear terms up to Δt, becomes

$$\begin{aligned} &f\left(x + \mathit{\Delta} x, c + \Delta c, t + \Delta t\right) \\ &\qquad \approx f(x,c,t) + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x_i}\Delta x_i + \frac{\partial f}{\partial c_i}\Delta c_i + \mathcal{O}\left[(\Delta t)^2\right]. \end{aligned} \tag{1.10}$$

The combination of (1.5) through (1.10) yields

$$\begin{aligned} \frac{\Delta N}{\Delta t} &= \left[\frac{\partial f}{\partial t} + c_i\frac{\partial f}{\partial x_i} + F_i\frac{\partial f}{\partial c_i} + f\frac{\partial F_i}{\partial c_i}\right] d\mu\left(t\right) \\ &= \left[\frac{\partial f}{\partial t} + c_i\frac{\partial f}{\partial x_i} + \frac{\partial f F_i}{\partial c_i}\right] d\mu\left(t\right). \end{aligned} \tag{1.11}$$

The determination of $\frac{\Delta N}{\Delta t}$ when the collisions are taken into account are based on the following hypotheses :

1st Hypothesis : For a rarefied gas, the probability of occurrence of collisions in which more than two molecules participate is much smaller than one which corresponds to binary encounters;

2nd Hypothesis : The effect of external forces on the molecules during the mean collision time τ_c is negligible in comparison with the interaction molecular forces;

3rd Hypothesis :The asymptotic pre-collisional velocities of two velocities . This hypothesis is known as the molecular chaos assumption and

4th Hypothesis : The distribution function $f(x, c, t)$ does not change very much over a time interval which is larger than the mean collision time but smaller than the mean free time. The same assumption applies to the variation of f over a distance of the order of the range of the intermolecular forces.

Consider now two gas molecules whose asymptotic pre-collisional velocities are denoted by c and c_1. In figure (1.1) , the molecule which has velocity c is at the point

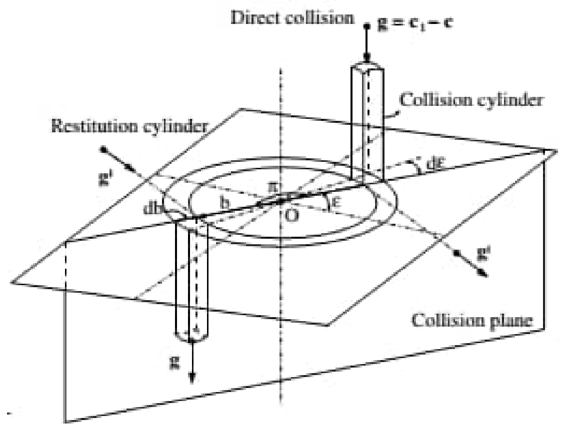

Figure 1.1:

O, while the other molecule is approaching the plane according to a right angle and with relative velocity $g = c_1 - c$. The relative motion is also characterized by the impact parameter b and by the azimuthal angle ε.

From Figure (1.1), one can infer that- during the time interval $\Delta t-$all molecules with velocities within the range c_1 and $c_1 + dc_1$,and that are inside the cylinder of volume $g\Delta t\, bdb\, d\varepsilon$,will collide with the molecules located in a volume element dx around the point O and whose velocities are within the range c and $c + dc$. The number of molecules with velocities within the range $c_1$and $c_1 + dc_1$ inside the collision cylinder is given by $f\left(x, c_1, t\right) dc_1 g\Delta t\, bdb\, d\varepsilon$. These molecules will collide with all molecules with velocities within the range c and $c+dc$ and which are in the volume element dx around the point O, i.e., $f\left(x, c, t\right) dxdc$. Hence, the number of collisions, during the time interval Δt, which occur in the volume element dx, reads

$$f\left(x, c_1, t\right) dc_1 g\Delta t\, bdb\, d\varepsilon f\left(x, c, t\right) dxdc. \tag{1.12}$$

By dividing (1.12) by Δt and integrating the resulting formula over all components of the velocity $-\infty < c_i^1 < +\infty\, (i = 1, 2, 3)$, over the azimuthal angle $0 \leq \varepsilon \leq 2\pi$ and over all values of the impact parameter $0 \leq b < \infty$, it follows the total number of collisions per time interval Δt in the μ-phase space that annihilates points with velocity c in the volume element $d\mu\left(t\right)$, namely,

$$\left(\frac{\Delta N}{\Delta t}\right)^{-} = d\mu\left(t\right) \int \left(x, c_1, t\right) f\left(x, c, t\right) gb\, db\, d\varepsilon\, dc_1. \tag{1.13}$$

In (1.13), all the five integrals described above were represented by only one symbol of integration.

However, there exist collisions which create points with velocity c in the volume

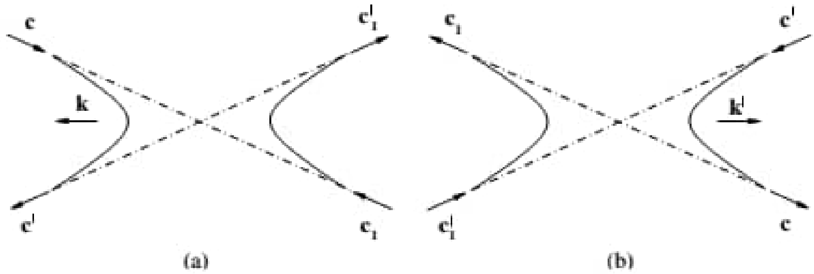

Figure 1.2:

element $d\mu(t)$. Indeed, they result from collisions of molecules with the following characteristics:

$i)$ asymptotic post-collisional velocities c' and c'_1,

$ii)$ asymptotic post-collisional velocities c and c_1,

$iii)$ apsidal vector $k' = -k$,

$iv)$ impact parameter $b' = b$ and

$v)$ azimuthal angle $\varepsilon' = \pi + \varepsilon$.

Such collisions are known as restitution collisions, whereas the former are called direct collisions. The geometry of these two collisions are represented in figures (1.1) and (1.2).

By taking into account the previous analysis, one can infer that the number of collisions, during the time interval Δt, which occur in the volume element dx for the restitution collisions is given by

$$f\left(x, c'_1, t\right) dc'_1 g' \Delta t \, b \, db \, d\varepsilon f\left(x, c', t\right) dx dc'. \tag{1.14}$$

The above expression can be rewritten in a modified form as follows. First, recall that the modulus of the Jacobian for the equations that relate the post-and pre-collisional asymptotic velocities are equal to each other, i.e., $g' = g$. Hence, (1.14) becomes

$$f\left(x, c'_1, t\right) f\left(x, c', t\right) dx \Delta t \, g \, b \, db d\varepsilon dc dc_1. \tag{1.15}$$

Now, it follows from the above expression that the total number of collisions per time interval Δt, which creates points in the μ-phase space with velocity c in the volume element $d\mu(t)$,reads

$$\left(\frac{\Delta N}{\Delta t}\right)^+ = d\mu(t) \int f\left(x, c'_1, t\right) f\left(x, c', t\right) g \, b \, db d\varepsilon dc dc_1. \tag{1.16}$$

The four hypotheses above and the arguments which lead to (1.13) and (1.16) are designated frequently in the literature by the German word Stobzahlansatz, which means supposition about the number of collisions.

By taking into account the above results, (1.11) with

$$\frac{\Delta N}{\Delta t} = \left(\frac{\Delta N}{\Delta t}\right)^{+} - \left(\frac{\Delta N}{\Delta t}\right)^{-} \tag{1.17}$$

is written in a final form as

$$\frac{\partial f}{\partial t} + \underbrace{c_i \frac{\partial f}{\partial x_i} + \frac{\partial F_i f}{\partial c_i}}_{\text{streaming}} = \underbrace{\int (f_1' f' - f_1 f)\, gb\, db\, d\varepsilon\, dc_1,}_{\text{collision}} \tag{1.18}$$

which is the Boltzmann equation, a non-linear integro-differential equation for the distribution function f. It describes the evolution of the distribution function in the μ-phase space, and one can infer that the temporal change of f has two terms, one of them is a collision term due to the motion of the molecules, whereas the other is a collision term related to the encounters of the molecules. Above, the following abbreviations were introduced :

$$f' \equiv f\left(x, c', t\right), \; f_1' \equiv f\left(x, c_1', t\right) \; f \equiv f\left(x, c, t\right) \; f_1 \equiv f\left(x, c_1, t\right). \tag{1.19}$$

If the specific external force does not depend on the velocities of the molecules–as in the case of the gravitational acceleration—or does depend on the velocity through a cross product–like the cases of coriolis acceleration in non-inertial frames or Lorentz' force in ionized gases–the velocity divergence $\frac{\partial F_i}{\partial c_i}$ vanishes and the Boltzmann equation (1.18) reduces to

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} + F_i \frac{\partial f}{\partial c_i} = \int (f_1' f' - f_1 f)\, gb\, db\, d\varepsilon\, dc_1. \tag{1.20}$$

1.2 The relativistic Boltzmann equation

In this section we shall consider a single non-degenerate relativistic gas, i.e., a gas where quantum effects are not taken into account.

A gas particle of rest mass m is characterized by the space-time coordinates $(x^\alpha) = (ct, x)$ and by the momentum four- vector $(p^\alpha) = (p^0, p)$. Due to the constraint that the length of the momentum four-vector is mc, p^0 is given in terms of p by

$$p^0 = \sqrt{|p| + m^2c^2}.$$

The one-particle distribution function, defined in terms of the space-time and momentum coordinates $f(x^\alpha, p^\alpha) = f(x, p, t)$, is such that

$$f(x, p, t)\, d^3x\, d^3p = f(x, p, t)\, dx^1 dx^2 dx^3 dp^1 dp^2 dp^3 \tag{1.21}$$

gives at time t the number of particles in the volume element d^3x about x and with momenta in a range d^3p about p.

The number of particles in the volume element is a scalar invariant since all observers will count the same particles. Let us examine the invariance of the volume element $d^3x\, d^3p$. We have

$$\frac{d^3p'}{p'_0} = \frac{d^3p}{p_0}. \tag{1.22}$$

Let us choose the primed frame of reference as a rest frame, i.e., where $p' = 0$. In this case d^3x' is the proper volume and we have that

$$d^3x = \sqrt{1 - v^2/c^2}\, d^3x' = \frac{1}{\gamma} d^3x', \tag{1.23}$$

where $\gamma = \frac{1}{\sqrt{1-|v|^2/c^2}}$.

On the other hand, from the transformation of the components of the momentum four-vector we have that

$$p'_0 = \frac{1}{\gamma} p_0, \tag{1.24}$$

since $p^0 = p_0$. Now we build the product

$$d^3x\, d^3p = \frac{1}{\gamma} d^3x' \frac{p^0}{p'^0} d^3p' = d^3x'\, d^3p' \tag{1.25}$$

and conclude that $d^3x\, d^3p$ is a scalar invariant. Since the number of the particles in the volume element $d^3x\, d^3p$ is also a scalar invariant we conclude that the one particle distribution function $f(x, p, t)$ is a scalar invariant.

Let us denote the volume element at time t by

$$d\mu(t) = d^3x\, d^3p. \tag{1.26}$$

The number of particles in this volume element at time t is

$$N(t) = f(x, p, t)\, d\mu(t). \tag{1.27}$$

Further, the number of particles in the volume element $d\mu\left(t+\Delta t\right)$ at time $t+\Delta t$ is

$$N\left(t+\Delta t\right)=f\left(x+\Delta x,p+\Delta p,t+\Delta t\right)d\mu\left(t+\Delta t\right). \tag{1.28}$$

The collisions between the particles imply that $N(t)$ is not equal to $N\left(t+\Delta t\right)$ and the change in the number of particles is given by

$$\begin{aligned}\Delta N&=N\left(t+\Delta t\right)-N\left(t\right)\\&=f\left(x+\Delta x,p+\Delta p,t+\Delta t\right)d\mu\left(t+\Delta t\right)-f\left(x,p,t\right)d\mu\left(t\right),\end{aligned} \tag{1.29}$$

where the increments in the position and in the momentum read

$$\Delta x=v\Delta t,\qquad \Delta p=F\Delta t, \tag{1.30}$$

$F\left(x,p,t\right)$ denotes the external force that acts on the particles and $v=cp/p^0$ is the velocity of the particle with momentum p.

The relationship between $d\mu\left(t+\Delta t\right)$ and $d\mu\left(t\right)$ is given by:

$$d\mu\left(t+\Delta t\right)=\left|J\right|d\mu\left(t\right), \tag{1.31}$$

with Jdenoting the Jacobian of the transformation

$$J=\frac{\partial\left(x^1\left(t+\Delta t\right),x^2\left(t+\Delta t\right),\cdots p^3\left(t+\Delta t\right)\right)}{\partial\left(x^1\left(t\right),x^2\left(t\right),\cdots,p^3\left(t\right)\right)}. \tag{1.32}$$

If we consider up to linear terms in Δt we get from (1.32) that the Jacobian reduces to

$$J=1+\frac{\partial F^i}{\partial p^i}\Delta t+\mathcal{O}\left[\left(\Delta t\right)^2\right]. \tag{1.33}$$

Now by expanding $f\left(x+\Delta x,p+\Delta p,t+\Delta t\right)$ in Taylor series about the point $\left(x,p,t\right)$ and by considering only linear terms in Δt it follows that

$$\begin{aligned}&f\left(x+\Delta x,p+\Delta p,t+\Delta t\right)\\&\approx f(x,p,t)+\frac{\partial f}{\partial x^i}\Delta x^i+\frac{\partial f}{\partial t}\Delta t+\frac{\partial f}{\partial p^i}\Delta p^i+\mathcal{O}\left[\left(\Delta t\right)^2\right].\end{aligned} \tag{1.34}$$

We combine equations (1.29) through (1.34) and get the total change in the number of particles per unit of time interval:

$$\begin{aligned}\frac{\Delta N}{\Delta t} &= \left[\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + F^i \frac{\partial f}{\partial p^i} + f \frac{\partial F^i}{\partial p^i}\right] d\mu\,(t) \\ &= \left[\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \frac{\partial f F^i}{\partial p^i}\right] d\mu\,(t)\,. \end{aligned} \tag{1.35}$$

ΔN is a scalar invariant as well as the proper time $\Delta\tau = \Delta t/\gamma$, hence

$$\gamma \frac{\Delta N}{\Delta t} = \frac{\Delta N}{\Delta \tau} = \gamma \left[\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i} + \frac{\partial f F^i}{\partial p^i}\right] d\mu\,(t) \tag{1.36}$$

is also a scalar invariant. We have shown in (1.25) that $d\mu = d^3x\, d^3p$ is a scalar invariant, and as a consequence the expression multiplying it in (1.36) must have the same property as we shall show. We first consider the term

$$\gamma \left[\frac{\partial f}{\partial t} + v^i \frac{\partial f}{\partial x^i}\right] = \gamma \left[\frac{\partial f}{\partial t} + \frac{c p^i}{p^0} \frac{\partial f}{\partial x^i}\right] = \frac{c\gamma}{p^0} p^\alpha \frac{\partial f}{\partial x^\alpha} = \frac{1}{m} p^\alpha \frac{\partial f}{\partial x^\alpha}. \tag{1.37}$$

Due to the fact that f is a scalar invariant, $\partial f/\partial x^\alpha$ is a four-vector and the scalar product $p^\alpha \partial f/\partial x^\alpha$ is a scalar invariant. We need only to prove that ,$\gamma \partial f F^i/\partial p^i$ has the same property. For the proof we consider the Minkowski force $K^\alpha = \left(\frac{F.v}{c(1-v^2/c^2)^{\frac{1}{2}}}, \frac{F}{(1-v^2/c^2)^{\frac{1}{2}}}\right)$, that satisfies

$$K^\alpha p_\alpha = K^0 p_0 - K.p = 0, \tag{1.38}$$

and the relationship

$$F = \frac{K}{\gamma} = \frac{mcK}{p^0}, \tag{1.39}$$

where m is the rest mass. If we consider p^0 as an independent variable and make use of the chain rule:

$$\frac{\partial}{\partial p^i} \longrightarrow \frac{\partial p^0}{\partial p} \frac{\partial}{\partial p^0} + \frac{\partial}{\partial p} = \frac{p}{p^0} \frac{\partial}{\partial p^0} + \frac{\partial}{\partial p}, \tag{1.40}$$

we can write the following expression

$$\gamma \frac{\partial f F^i}{\partial p^i} = \gamma mc \left[\frac{p}{p^0} \frac{\partial}{\partial p^0} \left(\frac{fK}{p^0}\right) + \frac{\partial}{\partial p}. \left(\frac{fK}{p^0}\right)\right]. \tag{1.41}$$

Since p^0 and p are treated as independent variables, the above equation reduces to

$$\begin{aligned}\gamma\frac{\partial fF^i}{\partial p^i} &= \gamma mc\left[\frac{1}{p^0}\frac{\partial}{\partial p^0}\left(\frac{fK}{p^0}\right)+\frac{1}{p^0}\frac{\partial}{\partial p}.\left(fK\right)\right]\\ &= \frac{\gamma mc}{p^0}\left[\frac{\partial fK^0}{\partial p^0}+\frac{\partial}{\partial p}.\left(fK\right)\right]\\ &= \frac{\gamma mc}{p^0}\frac{\partial fK^\alpha}{\partial p^\alpha}=\frac{\partial fK^\alpha}{\partial p^\alpha},\end{aligned} \tag{1.42}$$

which is a scalar invariant.

Now according to (1.37) and (1.42) equation (1.35) reads

$$\frac{\Delta N}{\Delta t}=\frac{c}{p^0}\left[p^\alpha\frac{\partial f}{\partial x^\alpha}+m\frac{\partial fK^\alpha}{\partial p^\alpha}\right]d\mu\left(t\right). \tag{1.43}$$

To determine $\Delta N/\Delta t$ we decompose it in two terms

$$\frac{\Delta N}{\Delta t}=\frac{\left(\Delta N\right)^+}{\Delta t}-\frac{\left(\Delta N\right)^-}{\Delta t}, \tag{1.44}$$

where $\left(\Delta N\right)^-/\Delta t$ corresponds to the particles that leave the volume element $d^3x\,d^3p$, whereas $\left(\Delta N\right)^+/\Delta t$ corresponds to those particles that enter in the same volume element. Further we assume the following:

a) Only collisions between pairs of particles are taken into account, i.e., only binary collisions are considered (this is reasonable if the gas is dilute, i.e., if the volume occupied by the molecules is much smaller than the volume of the gas);

b) if p and p_*denote the momenta of two particles before collision they are not correlated. This will be applied to the momenta p of the particle that we are following, andp_* of its collision partner, as well as to two momenta p' and p'_* possessed by two particles before a collision that will transform them into particles with momenta p and p_* after collision. This hypothesis is the so-called molecular chaos assumption;

c) the one-particle distribution function $f(x,p,t)$ does not vary very much over a time interval which is larger than the duration of a collision but smaller than the time between collisions. The same applies to the change of $f$over a distance of the order of the interaction range.

The German word StobJzahlansatz, which means supposition of number of collisions, is frequently employed in the literature to indicate this set of assumptions

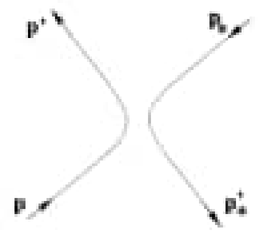

Figure 1.3:
Representation of a binary collision

that will be presently used to determine $\Delta N/\Delta t$.

We consider a collision between two beams of particles with velocities $v = cp/p^0$ and $v_* = cp_*/p_*^0$. The particle number densities of these two beams in their own frames are denoted by dn and dn_*. The d in front of n and n_* indicates that these number densities are infinitesimal because they refer to volume elements d^3p and d^3p_* of momentum space ($dn = fd^3p$ and $dn_* = f_*d^3p_*$). We first consider a reference frame where the particles without label are at rest, i.e., $v = 0$. The total number of these particles about x is $dn\, d^3x$. The total number of particles that will collide with the former and are in a volume element dV_* will be $dn_*dV_* = dn_*dV/\sqrt{1 - v_{rel}^2/c^2}$, where v_{rel} is the relative speed and
$dV/\sqrt{1 - v_{rel}^2/c^2}$ is a proper volume.

The particles with density dn_* in the volume dV are differently scattered by their partners in the collision through different angles. Each collision will occur in a plane with some scattering angle Θ, another angle is needed to single out the plane (which must contain the relative velocity) and two infinitesimal neighborhoods of the two angles together single out a solid angle element $d\Omega$. The volume element dV can be written in terms of the so-called collision cylinder of base $\sigma d\Omega$ and height $v_{rel}\Delta t$, Δt is identified with the differential of the proper time, because of the choice of the reference frame. The factor σ has clearly the dimensions of an area and is called the differential cross-section of the scattering process corresponding to the relative speed v_{rel} and the scattering angle Θ. In another reference system where $v \neq 0$, $d^3x\Delta t$,σ, $d\Omega$ and v_{rel} are scalar invariants.

The total number of collisions will be given then by the product of the particle numbers corresponding to the velocities v and v_*:

$$dn\, d^3x \frac{dn_*}{\sqrt{1 - v_{rel}^2/c^2}} dV = dn\, d^3x \frac{dn_*}{\sqrt{1 - v_{rel}^2/c^2}} \left(\sigma d\Omega v_{rel} \Delta t\right), \tag{1.45}$$

where we have rewritten the volume element dV in terms of the collision cylinder as discussed above.

Let us consider the product of the particle number densities in a system where

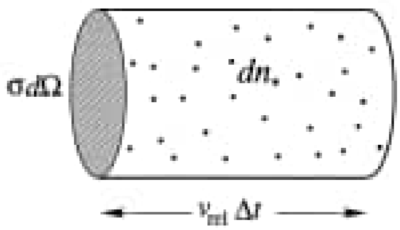

Figure 1.4:
Representation of the collision cylinder

$v \neq 0$:

$$\begin{aligned}
\frac{dn\, dn_*}{\sqrt{1-v_{rel}^2/c^2}} &= \frac{dn}{\sqrt{1-v^2/c^2}} \frac{dn_*}{\sqrt{1-v_*^2/c^2}} \frac{\sqrt{1-v^2/c^2}\sqrt{1-v_*^2/c^2}}{\sqrt{1-v_{rel}^2/c^2}} \\
&= f(x,p,t)d^3p f_*(x,p,t) \frac{\sqrt{1-v^2/c^2}\sqrt{1-v_*^2/c^2}}{\sqrt{1-v_{rel}^2/c^2}} \\
&= f(x,p,t)d^3p f_*(x,p,t)d^3p_* \frac{p_\alpha p_*^\alpha}{p^0 p_*^0}. \qquad (1.46)
\end{aligned}$$

The particle number densities above were written in terms of the one-particle distribution functions

$$\frac{dn}{\sqrt{1-v^2/c^2}} \equiv f(x,p,t)d^3p, \qquad \frac{dn_*}{\sqrt{1-v_*^2/c^2}} \equiv f_*(x,p,t)d^3p_*. \qquad (1.47)$$

Hence instead of (1.45) we have that the total number of collisions reads

$$\begin{aligned}
& f(x,p,t)d^3p f_*(x,p,t)d^3p_* v_{rel} \frac{p_\alpha p_*^\alpha}{p^0 p_*^0} \sigma d\Omega d^3x \Delta t \\
&= f(x,p,t)d^3p f_*(x,p_*,t)d^3p_* g_\phi \sigma d\Omega d^3x \Delta t \qquad (1.48)
\end{aligned}$$

where we have introduced Mϕller's relative speed g_ϕ.

Now the total number of particles that leave the volume element d^3xd^3p is obtained from (1.48) by integrating it over all momenta p_* and over all solid angle $d\Omega$, yielding

$$(\Delta N)^- = \int_\Omega \int_{p_*} f(x,p,t) f_*(x,p_*,t) g_\phi \sigma d\Omega d^3p_* d^3x d^3p \Delta t. \qquad (1.49)$$

This is frequently called the loss term because it describes the loss of particles in the volume element d^3xd^3p in phase space, due to collisions.

We remark that sometimes this equation is written with a factor 1/2 in front of the above integral. This is clearly related to the definition of the cross-section. The fact is that we are dealing with identical particles. Whereas in non-quantum mechanics identical particles can be regarded as distinguishable (because we can follow their motion in a continuous way), this is not the case in quantum mechanics. Thus if we start with two states for the colliding particles we find for each pair of final states a number which is the double of what we should expect from an analogy with a non-quantum calculation. This is due to the fact that we compute the two scattering processes leading the particles from the states (p', p'_*) to the states (p, p_*) and from the states (p_*, p) to the states $(p', p'_*)\, t$, respectively, as the same process, due to the indistinguishability of the particles involved. The definition of cross-section in quantum mechanics thus leads to a result which is twice as much the result expected from an analogy with classical mechanics. Thus if we use the quantum cross-section, we must divide by two the result for the loss term obtained above. We shall mainly deal with non-quantum effects and write the collision terms without the factor 1/2. We remark that, when considering a mixture, one deals with collisions of distinguishable particles and then our convention agrees with the opposite one; this means that authors using the latter have a factor 1/2 in front of collision terms associated with particles of the same species, whereas this factor is absent in the collision terms referring to different species.

The same reasoning and comments apply to the computation of the total number of particles that leave the volume element $d^3x'd^3p'$ and enter the volume element d^3xd^3p. We consider a collision between two beams of particles with velocities $v'_* = cp'_*/p'^0_*$ and by taking into account (1.49) we write the total number of particles that leave the volume element $d^3x'd^3p'$ as

$$(\Delta N)^+ = \int_{\Omega'} \int_{p'_*} f(x, p', t) f_*(x, p'_*, t) g'_\phi \sigma' d\Omega' d^3p'_* d^3x' d^3p' \Delta t', \tag{1.50}$$

which is called the gain term since it describes the gain of particles in the volume element d^3xd^3p. For relativistic particles we have that $g_\phi \neq g'_\phi$. However, Liouville's theorem asserts that if we follow the evolution of a volume element in phase space its volume does not change in the course of time. Here we have that

$$g_\phi \Delta t \sigma d\Omega d^3p_* d^3x d^3p = g'_\phi \Delta t' \sigma' d\Omega' d^3p'_* d^3x' d^3p'. \tag{1.51}$$

Since $d^3x\Delta t = d^3x'\Delta t'$ is an invariant it follows from (1.51) that

$$\int_\Omega g_\phi \sigma d\Omega d^3p_* d^3p = \int_\Omega g'_\phi \sigma' d\Omega' d^3p'_* d^3p'. \tag{1.52}$$

Now we get from (1.43) together with (1.44), (1.49), (1.50) and (1.52):

$$\frac{c}{p^0}\left[p^\alpha\frac{\partial f}{\partial x^\alpha}+m\frac{\partial fK^\alpha}{\partial p^\alpha}\right]d^3xd^3p=\frac{(\Delta N)^+-(\Delta N)^-}{\Delta t}$$
$$=\int\left(f'_*f'-f_*f\right)g_\phi\sigma d\Omega d^3p_*d^3xd^3p \tag{1.53}$$

where we have introduced the abbreviations

$$f'_*\equiv f\left(x,p'_*,t\right),\quad f'\equiv f\left(x,p',t\right)\quad f_*\equiv f\left(x,p_*,t\right)\quad f\equiv f\left(x,p,t\right). \tag{1.54}$$

If we denote by F the invariant flux

$$F=\frac{p^0p^0_*}{c}g_\phi=\frac{p^0p^0_*}{c}\sqrt{\left(v-v_*\right)^2-\frac{1}{c^2}\left(v\times v_*\right)^2}=\sqrt{\left(p^\alpha_*p_\alpha\right)^2-m^4c^4}, \tag{1.55}$$

equation (1.53) reduces to

$$p^\alpha\frac{\partial f}{\partial x^\alpha}+m\frac{\partial fK^\alpha}{\partial p^\alpha}=\int\left(f'_*f'-f_*f\right)F\sigma d\Omega\frac{d^3p_*}{p^0_*}, \tag{1.56}$$

which is the final form of the relativistic Boltzmann equation for a single nondegenerate relativistic gas. In (1.56) we have denoted by only one symbol the integrals over Ω and p_*.

Another expression for the Boltzmann equation (1.56) is obtained by the combination of (1.35), (1.43) and (1.53), yielding

$$\frac{\partial f}{\partial t}+v^i\frac{\partial f}{\partial x^i}+\frac{\partial fF^i}{\partial p^i}=\int\left(f'_*f'-f_*f\right)g_\phi\sigma d\Omega d^3p_*. \tag{1.57}$$

The above equation has the same expression as that of the classical Boltzmann equation.

1.3 The relativistic Hard and Soft Interactions

From physicists point of view, it is well known that the precise structure of the collision kernel has hardly any influence on the behavior of the solutions of the Boltzmann equation. Fortunately for mathematicians, this belief has proven to be wrong in several respects, [43]. In [15] is given a classification of the cross in the non relativistic case, this classification is extended to the relativistic case in [9] and [39].

From the non-relativistic kinetic theory, it is well known that refine analysis of the Boltzmann equation seems heavily sensitive to assumptions on the cross section

σ [10, 18]. Thus following the Grad's procedure [15], we shall classify the cross section σ into the so call hard and soft interactions. This distinction between both of them is due to a different collision frequency behavior: for hard interactions the collision frequency (1.66) satisfies the relation

$$\nu\,(p) \geqq \nu_0 > 0, \tag{1.58}$$

while for soft interactions we have

$$\nu\,(p) \leqq v_0 \text{ and } \nu\,(p) \to 0 \text{ for } |p| \to \infty. \tag{1.59}$$

Hard interactions are especially interesting not only because of some physical motivations, but also from mathematical point of view.

In the non-relativistic physics Grad [15] has defined as hard interactions those for which the cross section obeys:

$$\sigma\,(g,\theta) > B\frac{g^{\varepsilon}}{1+g}, \tag{1.60}$$

and as the soft interaction these with

$$\sigma\,(g,\theta) < Bg^{\varepsilon-1}, \tag{1.61}$$

where $0 < \varepsilon < 1$. Grad has shown the dependence of the collision frequency ν. For the hard interaction, ν is bounded from below:

$$\nu\,(p) > \nu_0\,,$$

while for soft interaction ν is bounded from above

$$\nu\,(p) < \nu'\,(p) \leqq \nu_0\,,$$

where ν_0 is a positive constant and $\nu\,(p) \to 0$ as $|p| \to \infty$.

The following issues coming from [15], examine the behaviour of the collision frequency $\nu\,(p)$ in the relativistic case and establish the meaning of relativistic hard and soft interactions.

Theorem 1.1. *Let us assume that* $\exists\,\gamma > -2$, $0 \leqq \beta < \gamma + 2$,

$$\sigma\,(g,\theta) > \frac{g^{\beta+1}}{c_0+g}\sin^{\gamma}\theta. \tag{1.62}$$

Then the collision frequency obeys:

$$\nu\left(p\right)>\nu_0\left[\frac{p^0}{M}\right]^{\beta/2}\geqq\nu_0, \tag{1.63}$$

where c_0 and ν_0 are constants.

Theorem 1.2. *Let us assume existence of $\alpha/\ 0<\alpha<4$, and $\gamma>-2$ so that :*

$$\sigma\left(g,\theta\right)<Bg^{-\alpha}\sin^{\gamma}\theta.$$

Then

$$\nu\left(p\right)<\left[\frac{p^0}{M}\right]^{-\varepsilon/2}\leqq\nu_0, \tag{1.64}$$

where

$$\varepsilon=\begin{cases}\alpha & \textit{for } 0<\alpha<3,\\ \alpha-2 & \textit{for } 3<\alpha<4,\\ \delta+1 & \textit{where } 0<\delta<1,\ \textit{for } \alpha=3.\end{cases}$$

Since Grad's distinction between hard and soft interactions is based on the different properties of the collision frequency for corresponding types of interactions, in the relativistic theory the meaning of hard and soft interactions should be redefined using relations (1.62) and (1.63).

The preceding issues have been reformulated by Strain [39] as follows:

- for soft potential we assume that there exist $\gamma>-2$ and $0<b<\min\left\{4,\gamma+4\right\}$ satisfying

$$\left(\frac{g}{\sqrt{s}}\right)g^{-b}\sigma_0\left(\omega\right)\lesssim\sigma\left(g,\omega\right)\lesssim g^{-b}\sigma_0\left(\omega\right),$$

- while for the hard potentials we assume that there exist $\gamma>-2$, $0\leq a\leq\gamma+2$ and $0<b<\min\left\{4,\gamma+4\right\}$ satisfying

$$\left(\frac{g}{\sqrt{s}}\right)g^{-b}\sigma_0\left(\omega\right)\lesssim\sigma\left(g,\omega\right)\lesssim\left(g^a+g^{-b}\right)\sigma_0\left(\omega\right),$$

$$\sigma_0\left(\omega\right)\lesssim\sin^{\gamma}\theta.$$

Here for any two quantities A, B the relation $A\lesssim B$ means that there exists a constant C such that $A\leq CB$.

We consider a one-component classical relativistic gas of particles with rest mass $m\neq0$ in the flat space-time and in the absence of all external forces. We assume that the system is close to a global equilibrium and that in order to determine its state it suffices to know the one-particle distribution function.

It is convenient to introduce dimensionless variables $x^\mu = y^\mu/ct$ and $p^\mu = q^\mu c/kT$, where y^μ and q^μ represent the usual, dimensional four-vectors of position and momentum respectively. the dimensionless mass is $M = mc^2/kT$. It is convenient to interpret T as the temperature. In this frame we decompose x^μ and p^μ as : $x^\mu = (t, r)$ and $p^\alpha = (p_0, p)$ .The signature is $(-, +, +, +)$.

We define here

- $s^{1/2} = |p_1 + p|$ is the total energy;
- $2g =|p_1 - p|$ is the value of the relative momentum;
- $\cos\theta = 1 - 2(p_\mu - p_{1\mu})(p^\mu - p'^\mu)(4M^2 - s)^{-1}$ defines the angle of scattering; $d\Omega = \sin\theta \, d\theta \, d\varphi$;
- $\sigma(g, \theta)$ is the differential scattering cross section.

All the above variables refer to the center of mass frame.

$$K_i\left[f(r,p,t)\right] = \int d^3p_1 k_i\left(p, p_1\right) f\left(r, p_1, t\right), \tag{1.65}$$

$$v\left(p\right) = \int d^3p_1 k_1\left(p, p_1\right) exp\left[\left(\tau - \tau_1\right)/2\right], \tag{1.66}$$

where $\tau = U^\mu p_\mu, \quad \tau_1 = U^\mu p_{1\mu}$.

The integral kernels k_i have the following form :

$$k_1\left(p, p_1\right) = \frac{1}{2M^2 K_2\left(M\right)} \frac{g s^{1/2}}{p_0 p_{10}} \exp\left[-\left(\tau + \tau_1\right)/2\right] \int_0^\pi d\theta sin\theta\sigma\left(g, \theta\right), \tag{1.67}$$

$$k_2\left(p, p_1\right) = \frac{1}{8M^2 K_2\left(M\right)} \frac{s^{3/2}}{g p_0 p_{10}} \int_0^\infty dx \exp\left[-\left(1 + x^2\right)^{1/2}\left(\tau + \tau_1\right)/2\right]$$
$$\times \sigma\left[\frac{g}{sin\left(\varphi/2\right)}, \varphi\right] x \frac{1 + \left(1 + x^2\right)^{1/2}}{\left(1 + x^2\right)^{1/2}} I_0\left[\frac{|p \wedge p_1|}{2g} x\right] \tag{1.68}$$

where I_0 is the Bessel function of purely imaginary argument of index zero and φ is connected with x by the relation

$$\sin\left(\varphi/2\right) = 2^{1/2} g\left[g^2 - M^2 + \left(g^2 + M^2\right)\left(1 + x^2\right)^{1/2}\right]^{-1/2}. \tag{1.69}$$

$p \wedge p_1$ is a vector product of p and p_1 calculated in the rest frame of our gas [in this frame $U^\alpha = (1, 0, 0, 0)$], so the explicit expression for $|p \wedge p_1|$has a form :

$$|p \wedge p_1| = \left[4g^2\left(\tau\tau_1 - g^2 - M^2\right) - M^2\left(\tau - \tau_1\right)^2\right]^{1/2}. \tag{1.70}$$

1.4 Povzner inequality

Acording to [37] Povzner introduced in 1962, the Povzner lemmas for the treatment of the moments of solution for homogeneous Boltzmann equation. These techniques have been extensively used in the last 5 years to greatly develop the homogeneous Boltzmann equation theory.

The Povzner equation was first introduced for purely mathematical reasons and usually ignored by the physicists. However, when considering the Grad limit of a system of N interacting 'soft spheres', Cercignani [6] obtained a hierarchy of equations factorized by a Povzner-like equation. Lichowicz and Pulvirenti [19] considered a system of N spheres colliding at a stochastic distance . They proved that when N tends to infinity, the one-particle distribution function converges to a local Maxwellian with density, velocity and temperature satisfying the Euler equations. At an intermediate step the Povzner equation appears.

A number of results are known concerning the cases of the non-linear stationary Boltzmann equation close to equilibrium, and solutions of the corresponding linearized equation. There, more general techniques - such as contraction mapping based ones - can be utilized. So e.g. in an $\mathbb{R}^n$ setting, the solvability of boundary value problems for the Boltzmann equation in situations close to equilibrium is studied in [16], [17]. Stationary problems in small domains for the non-linear Boltzmann equation are also studied in [34]. The unique solvability of internal stationary problems for the Boltzmann equation at large Knudsen numbers is established in [22]. Existence and uniqueness of stationary solutions for the linearized Boltzmann equation in a bounded domain are proven in [23] and for the linear Boltzmann equation uniqueness and existence are also proven in [42].

Moreover, existence results far from equilibrium have been obtained for the stationary nonlinear Povzner equation in a bounded region in ([1]). The Povzner collision operator is a modified Boltzmann operator with a 'smearing' process for the pair collisions, whereas in the derivation of the Boltzmann collision operator, each separate collision between two molecules occurs at one point in space.

Recently Lee and Rendall in [21] use povzner inequality proved by Povzner in [36]. This inequality has been crucially used to prove existence theorems for the non-relativistic spatially homogeneous Boltzmann equation by Elmroth [12] and Mischler and Wennberg [25]. The sharpest form of the Povzner inequality is given by Mischler and Wennberg, but Lemma 3.5 in [21]corresponds to a relativistic extension of Elmroth's result.

1.5 The Cauchy- Lipschitz theorem, the Banach fixed point theorem and Gronwall inequality

Definition 1.1. Let E be a Banach space and Ω be an open subset of $\mathbb{R} \times E$.

One say that $f : \Omega \longrightarrow E$ is locally $k-$Lipschitz if for all point $(t_0, x_0) \in \Omega$, there exists a neighborhood U of (t_0, x_0) in Ω and $k > 0$ such that one has :$\forall (t_1, x_1) \in U, \forall (t_2, x_2) \in U$,

$$||f(t_1, x_1) - f(t_2, x_2)|| \leq k||x_1 - x_2|| \tag{1.71}$$

Theorem 1.3. *(Cauchy-Lipschitz : Local version)*

If $f : \Omega \longrightarrow E$ is continuous and locally $k-$Lipschitz, and if $(t_0, x_0) \in \Omega$, then there exists a real number $\alpha > 0$ such that the differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1.72}$$

has a unique solution $\varphi :]t_0 - \alpha, t_0 + \alpha[\longrightarrow E$, which is $C^1$and satisfy the initial data $\varphi(t_0) = x_0$.

Proof. See [5] □

Theorem 1.4. *(Cauchy-Lipschitz : Global version) One suppose that the hypothesis of theorem 1.3 are satisfied and that all solution of Cauchy problem*

$$\begin{cases} \frac{dx}{dt} & = f(t, x) \\ x(t_0) & = x_0 \end{cases} \tag{1.73}$$

is inside a fixed ball of E.

Then (1.73) has a unique global solution.

Proof. See [38] □

Theorem 1.5. *(Banach fixed point theorem) Let X be a complete metric space in which the distance between two points x and y is denoted $d(x, y)$. And let $f : X \longrightarrow X$*

be a contraction (i.e there exists $c \in]0,1[$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq cd(x,y)$), *then* f *has a unique fixed point i.e there exists a unique* $z \in X$ *such that*

$$f(z) = z. \tag{1.74}$$

CHAPTER 2

The relativistic Maxwell-Boltzmann system in a Bianchi type I Space-time

In this chapter we settle the framework which is a Bianchi type I space-time, we give the formulations of the Boltzmann and the Maxwell equations. We also give different parametrizations of the post collisional momentum. We then present the coupled Boltzmann-Maxwell system of equations. Finally we define the functional spaces.

2.1 Bianchi type I space-time and phase space

2.1.1 Definitions

Let us consider (M, g) a lorentzian spacetime with metric tensor of signature $(-, +, +, +)$.

Definition 2.1. The manifold (M, g) is spatially homogeneous if it possesses a group of isometries G_4 which operates transitively on the spatial hypersurfaces.

If the group G_4 has a subgroup G_3 operating simply transitively, the manifold (M, g) is called a Bianchi space-time.

There exists nine types of Bianchi space-times - Bianchi type I to IX, classified according to the structure constants of the Lie group G_3. If G_4 does not have any subgroup G_3 operating simply transitively, one deduces the existence of the space-times of Kantowski-Sachs. Bianchi space-times are spacially homogeneous.

Definition 2.2. A Bianchi type I space-time (M, g) is a space time where the isometries group G_4 is Abelian.

In this work, Greek indexes run from 0 to 3 and Latin indexes run from 1 to 3. We adopt the Einstein convention:

$$a_\alpha b^\alpha = \sum_\alpha a_\alpha b^\alpha. \tag{2.1}$$

In a time oriented Bianchi type I space-time, we consider the collisional evolution of a kind of fast moving massive and charged particles and denote by $x^\alpha = (x^0, x^i) = (t, x^i)$,the usual coordinates in $\mathbb{R}^{3+1}$,where $t = x^0$ represents the time and (x^i) the space, g stands for the metric tensor of Lorentzian signature $(-,+,+,+)$ which writes in local coordinates:

$$g = -(dt)^2 + a^2(t)\left(dx^1\right)^2 + b^2(t)\left(\left(dx^2\right)^2 + \left(dx^3\right)^2\right), \tag{2.2}$$

where a and b are two differentiable increasing functions on $\mathbb{R}^+$.

Note that the metric tensor g given by (2.2) generalizes the metric of the Robertson-Walker space time which is very important since it modelizes our universe in cosmological expansion. Note also that in (2.2) the components $g_{\alpha\beta}$ of g and $g^{\alpha\beta}$ of g^{-1} are given by :

$$\begin{cases} g_{00} = -1; \\ g_{11} = a^2; \\ g_{22} = g_{33} = b^2; \\ g^{00} = -1; \end{cases} \quad \begin{cases} g^{11} = \frac{1}{a^2}; \\ g^{22} = g^{33} = \frac{1}{b^2}; \\ g_{\alpha\beta} = g^{\alpha\beta} = 0 \qquad \text{if otherwise.} \end{cases} \tag{2.3}$$

The metric g being taken in the form (2.2) , the matrix $(g_{\alpha\beta})$ is diagonal and is written :

$$(g_{\alpha\beta}) = \text{diag}(-1, a^2, b^2, b^2). \tag{2.4}$$

The computation of the determinant of this matrix gives us

$$|g| = |\text{det} g| = a^2 b^4 > 0. \tag{2.5}$$

Hence, $(g_{\alpha\beta})$ is invertible and its inverse matrix $(g^{\alpha\beta})$ is written:

$$(g^{\alpha\beta}) = \mathrm{diag}(-1, \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{b^2}). \tag{2.6}$$

The Christoffel symbols of the Levi-Civita connection ∇ associated to a metric tensor g are defined by:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}(\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}). \tag{2.7}$$

In the present case, using (2.3) and (2.7) we obtain :

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2} g^{0\mu} \left[\partial_1 g_{\mu 1} + \partial_1 g_{1\mu} - \partial_\mu g_{11}\right] \\
&= -\frac{1}{2} \left(-\partial_0 g_{11}\right) \\
&= \frac{1}{2} \frac{\mathrm{d}a^2 \left(t\right)}{\mathrm{d}t} \\
&= a\dot{a} \left(t\right)
\end{aligned}$$

$$\begin{aligned}
\Gamma^0_{22} = \Gamma^0_{33} =& \frac{1}{2} g^{0\mu} \left[\partial_2 g_{\mu 2} + \partial_2 g_{2\mu} - \partial_\mu g_{22}\right] \\
=& -\frac{1}{2} \left(-\partial_0 g_{22}\right) \\
=& \frac{1}{2} \frac{db^2 \left(t\right)}{dt} \\
=& \dot{b}b \left(t\right)
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{01} =& \frac{1}{2} g^{1\mu} \left[\partial_0 g_{\mu 1} + \partial_1 g_{0\mu} - \partial_\mu g_{01}\right] \\
=& \frac{1}{2} \times \frac{1}{a^2 \left(t\right)} \left(\frac{da^2 \left(t\right)}{dt}\right) \\
=& \frac{\dot{a}}{a}
\end{aligned}$$

$$\begin{aligned}\Gamma^2_{02} = \Gamma^3_{03} =& \frac{1}{2} g^{2\mu} \left[\partial_0 g_{\mu 2} + \partial_2 g_{0\mu} - \partial_\mu g_{02}\right] \\ =& \frac{1}{2} \times \frac{1}{b^2(t)} \left(\frac{db^2(t)}{dt}\right) \\ =& \frac{\dot{b}}{b}.\end{aligned}$$

In summary, we obtain that the Christoffel symbol are :

$$\begin{cases} \Gamma^1_{10} = \frac{\dot{a}}{a}; \\ \Gamma^2_{20} = \Gamma^3_{30} = \frac{\dot{b}}{b}; \\ \Gamma^0_{11} = a\dot{a}; \\ \Gamma^0_{22} = \Gamma^0_{33} = b\dot{b}; \\ \Gamma^\lambda_{\alpha\beta} = 0 \qquad \text{otherwise.} \end{cases} \tag{2.8}$$

In (2.8) the dot stands for the derivative with respect to t.

2.1.2 The distribution function

The particles are statistically described by their distribution function, denoted f ,which is a non-negative unknown real-valued function of both the position x^α and the 4-momentum of the particles p^α. So let $T(\mathbb{R}^4)$ be the tangent bundle. Due to the form of the metric, we can write $T(\mathbb{R}^4) \cong \mathbb{R}^4 \times \mathbb{R}^4$. Then

$$f : T\left(\mathbb{R}^4\right) \cong \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^+, \ (x^\alpha, p^\alpha) \longmapsto f(x^\alpha, p^\alpha). \tag{2.9}$$

We define an inner product on $\mathbb{R}^3$ by setting for $\overline{p} = (p^1, p^2, p^3)$ and $\overline{q} = (q^1, q^2, q^3)$:

$$\overline{p}.\overline{q} = a^2 p^1 q^1 + b^2 \left(p^2 q^2 + p^3 q^3\right). \tag{2.10}$$

We consider that the massive particles have a same rest mass $m > 0$ normalized to the unity, i.e $m = 1$. The particles are then required to move on the future sheet of the mass-shell or the mass hyperboloid, whose equation is

$$g(p, p) = -1 \tag{2.11}$$

or equivalently, using the expression (2.2) of g :

$$p^0 = \sqrt{1 + a^2 \left(p^1\right)^2 + b^2 \left(\left(p^2\right)^2 + \left(p^3\right)^2\right)} \tag{2.12}$$

where $p^0 > 0$ symbolizes the fact that, naturally, the particles eject towards the future. The relation (2.12) shows that f is in fact defined on the sub-bundle of $T\left(\mathbb{R}^4\right)$ whose local coordinates are x^α and p^i.

In the present work, we consider the homogeneous Boltzmann equation for which f depends only on the time $x^0 = t$ and $\overline{p} = (p^i)$.

In the presence of electromagnetic field F, the trajectories

$$s \longmapsto \left(x^\alpha(s), p^\alpha(s)\right) \tag{2.13}$$

of the charged particles are solutions of the differential system :

$$\begin{cases} \frac{dx^\alpha}{ds} = p^\alpha \\ \frac{dp^\alpha}{ds} = P^\alpha \end{cases} \tag{2.14}$$

where

$$P^\alpha = P^\alpha(F) = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + e p^\beta F^\alpha_\beta \tag{2.15}$$

with

$$e(t) > 0 \tag{2.16}$$

denotes the charge density of the particles.

Notice that the differential system (2.14) shows that the vectors field $X(F)$ defined locally by :

$$X(F) = \left(p^\alpha, P^\alpha(F)\right) \tag{2.17}$$

where P^α is given by (2.15), is tangent to the trajectories.

2.1.3 Change of variables

For simplicity, we now consider the covariant variables. The covariant variables (p_0, p_i) are obtained by lowering indexes as:

$$p_0 = g_{0\alpha} p^\alpha = -p^0$$

$$p_i = g_{i\alpha} p^\alpha = g_{ii} p^i$$

and this implies that

$$p_1 = a^2 p^1$$

and

$$p_i = b^2 p^i, \;\; i = 2, 3.$$

For the sake of simplicity, we will note in the sequel

$$v^0 = p^0, \quad v^i = g_{ii} p^i$$

so that

$$\overline{v} = \left(v^1, v^2, v^3\right).$$

We let the distribution function f depending on the time t and on the covariant variables $\overline{v} = (v^1, v^2, v^3)$ as in [20] and [33], instead of $\overline{p}$.

Really:

$$\begin{cases} v^1 = a^2 p^1 \\ v^2 = b^2 p^2 \\ v^3 = b^2 p^3, \end{cases} \tag{2.18}$$

and we see that

$$d\overline{v} = a^2 b^4 d\overline{p.} \tag{2.19}$$

In fact

$$\frac{d\overline{v}}{d\overline{p}} = a^2 b^4 \tag{2.20}$$

and setting

$$v^0 = \sqrt{1 + a^{-2}\left(v^1\right)^2 + b^{-2}\left(\left(v^2\right)^2 + \left(v^3\right)^2\right)} \tag{2.21}$$

we get :

$$v^0 = p^0 \tag{2.22}$$

as indicated above.

2.2 The Maxwell system in F

The Maxwell system in F can be written as :

$$\nabla_\alpha F^{\alpha\beta} = J^\beta \tag{2.23}$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0. \tag{2.24}$$

(2.23) and (2.24) are respectively the first and second group of the Maxwell equations , $F = (F^{0i}, F_{ij})$ is the electromagnetic field where F^{0i} and F_{ij} stand for the electric and magnetic parts respectively , and ∇_α stands for the covariant derivative in g. In (2.23), J^β represents the Maxwell current whose local expression is given by :

$$J^\beta = \int_{\mathbb{R}^3} \frac{p^\beta f(t, \overline{p})\, ab^2 dp^1 dp^2 dp^3}{p^0} - eu^\beta \tag{2.25}$$

in which

$$ab^2 = (\det g)^{\frac{1}{2}} \tag{2.26}$$

$$e(t) = e \geq 0 \tag{2.27}$$

is the charge density which also appears in (2.15),

$$u = \left(u^\beta\right) \tag{2.28}$$

is a unit futur pointing timelike vector tangent to the time axis at any point which means that

$$u^0 = 1,\ u^i = 0,\ i = 1, 2, 3. \tag{2.29}$$

The particles are then suppose to be spatially at rest.

(2.24) is an identity expressing the fact F is closed, or equivalently

$$dF = 0. \tag{2.30}$$

Now the identity $\nabla_\alpha \nabla_\beta F^{\alpha\beta} = 0$ (see[28]) implies that we must have the conservation law of current

$$\nabla_\alpha J^\beta = 0. \tag{2.31}$$

2.3 The spatially homogeneous Boltzmann equation in f

For charged particles, and as obtained in section 1.2, the relativistic Boltzmann equation in Bianchi type I space-time can be written as

$$\frac{p^{\alpha}}{p^{0}}\frac{\partial f}{\partial x^{\alpha}}+\frac{P^{i}}{p^{0}}\frac{\partial f}{\partial p^{i}}=Q\left(f,f\right) \tag{2.32}$$

where

$$Q\left(f,f\right)=\int_{\mathbb{R}^{3}}\int_{S^{2}}v_{\phi}\sigma\left(k,\theta\right)\left(f'f'_{*}-ff_{*}\right)ab^{2}dwd\overline{q} \tag{2.33}$$

with

$$v_{\phi}=\frac{k\sqrt{\delta}}{p^{0}q^{0}},\ f'=f\left(t,\overline{p'}\right),\ f'_{*}=f\left(t,\overline{q'}\right),\ f=f\left(t,\overline{p}\right),\ f_{*}=f\left(t,\overline{q}\right). \tag{2.34}$$

Here Q is the collision operator, v_{ϕ} the Mϕller velocity, σ the scattering kernel, θ the scattering angle, δ and k are given by

$$\delta=\delta\left(p^{\alpha},q^{\alpha}\right)=-\left(p_{\alpha}+q_{\alpha}\right)\left(p^{\alpha}+q^{\alpha}\right),$$

$$k=k\left(p^{\alpha},q^{\alpha}\right)=\sqrt{\left(p_{\alpha}-q_{\alpha}\right)\left(p^{\alpha}-q^{\alpha}\right)},$$

and are called total energy and relative momentum respectively. In the instantaneous, binary and elastic scheme due to Lichnerowicz and Chernikov, we consider that at a given position $x^{\alpha}=(t,\overline{x})$, only two particles collide each other, without destroying each one, the collision affecting only the two momenta, but the energy momentum being conserved. In this scheme, p and q stand for the two momenta before the shock, and p' and q' for two momenta after the shock. The collisions operator Q is defined using functions f and h on $\mathbb{R}^{3}$ by :

$$Q\left(f,h\right)=Q_{+}\left(f,h\right)-Q_{-}\left(f,h\right), \tag{2.35}$$

where

$$Q_{+}\left(f,h\right)=\int\int_{\mathbb{R}^{3}\times S^{2}}ab^{2}f\left(\overline{p'}\right)h\left(\overline{q'}\right)v_{\phi}\sigma\left(k,\theta\right)d\overline{q}dw, \tag{2.36}$$

$$Q_{-}\left(f,h\right)=\int\int_{\mathbb{R}^{3}\times S^{2}}ab^{2}f\left(\overline{p}\right)h\left(\overline{q}\right)v_{\phi}\sigma\left(k,\theta\right)d\overline{q}dw. \tag{2.37}$$

The energy momentum conservation is written as

$$p^{0}+q^{0}=p'^{0}+q'^{0} \tag{2.38}$$

$$\overline{p}+\overline{q}=\overline{p'}+\overline{q'} \tag{2.39}$$

As suggested in [21] and [33], we can parametrize the post-collisional momenta as

follows.
For p^α and q^α been given, we first consider

$$n^\alpha = p^\alpha + q^\alpha, \quad t^\alpha = \left(n_i w^i, -n_0 w\right), w \in S^2. \tag{2.40}$$

Then, for $w \in S^2$, the post-collisional momenta are represented by :

$$p'^\alpha = \frac{p^\alpha + q^\alpha}{2} + \frac{k}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}}, \qquad q'^\alpha = \frac{p^\alpha + q^\alpha}{2} - \frac{k}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}} \tag{2.41}$$

It can be easily checked that they satisfy the mass shell condition and energy momentum conservation.

Lemma 2.1. *The Jacobian of the change of variable* $(\overline{p}, \overline{q}) \longmapsto \left(\overline{p'}, \overline{q'}\right)$ *defined by (2.41) is computed to be*

$$\frac{\partial\left(\overline{p'}, \overline{q'}\right)}{\partial\left(\overline{p}, \overline{q}\right)} = -\frac{p'^0 q'^0}{p^0 q^0} \tag{2.42}$$

Proof. See the Appendix □

Now it appears clearly, using (2.12) and (2.41) that the functions in the integrals (2.36) and (2.37) depend only on $\overline{p}$ and $\overline{q}$ and that these integrals with respect to $\overline{q}$ and w give functions $Q_+(f, h)$ and $Q_-(f, h)$ of the single variable $\overline{p}$.

In practice we will consider functions f on $\mathbb{R}^4$ that induce for $t \in \mathbb{R}$, functions $f(t)$ on $\mathbb{R}^3$ defined by

$$f(t)(\overline{p}) = f(t, \overline{p}).$$

2.4 The Maxwell-Boltzmann system in (F, f) in the hard potential case

Setting $\beta = 0$ in the Maxwell system (2.23), we easily deduce that

$$\nabla_\alpha F^{\alpha 0} = 0 \tag{2.43}$$

since

$$F^{00} = F_{00} = 0; \quad F = F(t); \quad \Gamma^0_{\alpha\lambda} = \Gamma^0_{\lambda\alpha}, \quad F^{\alpha\lambda} = -F^{\lambda\alpha}, \tag{2.44}$$

and from (2.8)

$$\Gamma^\alpha_{\alpha i} = 0. \tag{2.45}$$

Equation (2.23) then imposes that

$$J^0 = 0. \tag{2.46}$$

By the relation (2.46), the expression (2.25) of J^β in which we set $\beta = 0$ then allows to compute e and gives, since $u^0 = 1$:

$$e(t) = \int_{\mathbb{R}^3} f(t, \overline{p})\, ab^2 d\overline{p} \tag{2.47}$$

where

$$d\overline{p} = dp^1 dp^2 dp^3. \tag{2.48}$$

This relation shows that e is determined when f is known.
For the equation (2.24) the usual formula of the covariant derivative of g gives :

$$\nabla_0 F_{ij} + \nabla_i F_{j0} + \nabla_j F_{0i} = 0 \tag{2.49}$$

and

$$\nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0 \tag{2.50}$$

using

$$\Gamma^k_{ij} = \Gamma^k_{ji} \quad \text{and} \quad F_{ij} = -F_{ji}, \tag{2.51}$$

(2.50) and (2.51) write:

$$\partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = 0 \tag{2.52}$$

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0. \tag{2.53}$$

The second set of equations is identically satisfied since $F = F(t)$, and the first set reduces to :

$$\partial_0 F_{ij} = 0. \tag{2.54}$$

Then F_{ij} is constant and :

$$F_{ij} = F_{ij}(0) := \varphi_{ij}. \tag{2.55}$$

It remains to determine the electric part

$$F^{0i} := E^i. \tag{2.56}$$

Writing (2.25) for $\beta = i$, we obtain

$$J^i = \int_{\mathbb{R}^3} \frac{p^i f(t, \overline{p})\, ab^2 d\overline{p}}{p^0}. \tag{2.57}$$

Writing (2.23) for $\alpha = 0$ and using (2.8) allows to get :

$$\nabla_0 F^{0i} = \partial_0 F^{0i} + \Gamma^i_{0\lambda} F^{0\lambda} \tag{2.58}$$
$$\nabla_0 F^{0i} = J^i \tag{2.59}$$

thus

$$\nabla_0 F^{0i} = \partial_0 F^{0i} + \Gamma^i_{0j} F^{0j} = J^i. \tag{2.60}$$

Then, (2.57) and (2.60) imply

$$\dot{E}^i + \Gamma^i_{0j} E^j = \int_{\mathbb{R}^3} \frac{p^i f\left(t, \overline{p}\right) ab^2 d\overline{p}}{p^0}. \tag{2.61}$$

Since $f = f\left(t, \overline{p}\right)$, then the homogeneous Boltzmann equation (2.32) can be written:

$$\frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} = Q\left(f, f\right) \tag{2.62}$$

and is equivalent to

$$\frac{\partial f}{\partial t} + \frac{P^1}{p^0}\frac{\partial f}{\partial p^1} + \frac{P^2}{p^0}\frac{\partial f}{\partial p^2} + \frac{P^3}{p^0}\frac{\partial f}{\partial p^3} = Q\left(f, f\right). \tag{2.63}$$

Solve the non linear first order partial differential equation (2.63) is equivalent to solve the first order characteristic differential system :

$$\frac{dt}{1} = \frac{dp^1}{\frac{P^1}{p^0}} = \frac{dp^2}{\frac{P^2}{p^0}} = \frac{dp^3}{\frac{P^3}{p^0}} = \frac{df}{Q\left(f, f\right)} = ds \tag{2.64}$$

which allows to take t as parameter.
(2.64) give the following system :

$$\begin{cases} \frac{dp^1}{dt} & = \frac{P^1}{p^0} \\ \frac{dp^2}{dt} & = \frac{P^2}{p^0} \\ \frac{dp^3}{dt} & = \frac{P^3}{p^0} \\ \frac{df}{dt} & = Q\left(f, f\right) \end{cases} \tag{2.65}$$

Using the expression $\widetilde{f}\left(t, \overline{v}\right) = f\left(t, \overline{p}\right)$, it follows directly that the left hand side of (2.63) is equal to $\partial_t \widetilde{f}\left(t, \overline{v}\right)$. (2.63) is written now :

$$\frac{\partial \widetilde{f}}{\partial t}\left(t, \overline{v}\right) = Q\left(\widetilde{f}, \widetilde{f}\right). \tag{2.66}$$

For simplicity of notation, it will cause no confusion if we use the same letter f to designate $\widetilde{f}$ in the remainder of the work.

On the other hand, we obtain from (2.15)

$$P^i = -\Gamma^i_{\lambda\mu} p^\lambda p^\mu + e\left[p^0 F^i_0 + p^k F^i_k\right], \ i = 1, 2, 3$$

which gives using expression (2.8) of $\Gamma^\lambda_{\alpha\beta}$,

$$F^i_0 = -F^{0i} \tag{2.67}$$

and

$$\begin{aligned} F^i_j &= g^{i\lambda} F_{j\lambda} && (2.68)\\ &= g^{ii} F_{ji} && (2.69)\\ &= -g^{ii} F_{ij} && (2.70) \end{aligned}$$

$i, j = 1, 2, 3$:

$$\frac{P^i}{p^0} = -2\Gamma^i_{0j} p^j - e\left[F^{0i} + g^{ii}\frac{p^k F_{ik}}{p^0}\right], \ i, = 1, 2, 3. \tag{2.71}$$

Using relation (2.47), (2.61), (2.64) and (2.71) the spatially homogeneous Maxwell-Boltzmann system takes the following form :

$$\dot{E}^i = -\Gamma^i_{0j} E^j + \int_{\mathbb{R}^3} \frac{q^i f(t, \overline{q})\, ab^2 d\overline{q}}{q^0}, \tag{2.72}$$

$$\dot{p}^i = -2\Gamma^i_{0j} p^j - \left[E^i + g^{ii}\frac{p^k \varphi_{ik}}{p^0}\right] \int_{\mathbb{R}^3} f(t, \overline{q})\, ab^2 d\overline{q}, \tag{2.73}$$

$$\frac{df}{dt} = Q(f, f), \tag{2.74}$$

$$F_{ij} = F_{ij}(0) = \varphi_{ij} \quad i, j = 1, 2, 3, \ f(0, \overline{p}) = f_0(\overline{p}). \tag{2.75}$$

Note that f and $\overline{p}$ are independent variables for the integro-differential system (2.72)-(2.75).

In this context, the collision operator Q defined by (2.33) will depend on $\overline{p}$ only through the collision kernel and we show it clearly by writing now $Q(f, f, \overline{p})$ instead of $Q(f, f)(\overline{p})$.

In what follows, it will be useful to use covariant variables. We can express the collision operator in terms of covariant variables as

$$Q(f,f)(t,v) = a^{-1}b^{-2}\int_{S^2} dw \int_{\mathbb{R}^3} d\overline{u}\frac{k\sqrt{\delta}}{v^0u^0}\sigma(k,\theta) \times \left[f\left(t,\overline{v'}\right)f\left(t,\overline{u'}\right) - f(t,\overline{v})f(t,\overline{u})\right]. \quad (2.76)$$

Thus, the Boltzmann equation (2.62) can be written in the following equivalent form :

$$\frac{df(t,v)}{dt} = Q(f,f)(t,v). \quad (2.77)$$

The above equivalent form will be used later for the hard potential case.

2.5 Assumptions on the work

In this work, as in [21], we assume that the scattering kernel for the spatially homogeneous Maxwell-Boltzmann system in (F, f) in the hard potential case has the form

$$\sigma(k,\theta) = k^{\beta} sin^{\gamma}\theta, \ \ -2 < \gamma \leq 1, \ \ 0 \leq \beta < \gamma + 2 \quad (2.78)$$

Since $\frac{k}{\delta}$ is a bounded quantity, because $\delta = 4 + k^2$, a scattering kernel of this form falls into the hard potential case.

We also assume the following assumptions on the potential of gravitations $a(t)$ and $b(t)$:

$$\begin{cases} a \leq b \\ a(0) = a_0 \geq \frac{3}{2} \end{cases} \quad (2.79)$$

and that there exists a positive constant C such that :

$$\left|\frac{\dot{a}}{a}\right| \leq C, \left|\frac{\dot{b}}{b}\right| \leq C \quad (2.80)$$

where the dot stands for derivative with respect to t.

The last assumption is physically justified and represent an expanding universe.

As a consequence, we have:

$$a(t) \leq a_0 e^{Ct}; \quad b(t) \leq b_0 e^{Ct}; \quad \frac{1}{a}(t) \leq \frac{1}{a_0}e^{Ct}; \quad \frac{1}{b}(t) \leq \frac{1}{b_0}e^{Ct} \quad (2.81)$$

where

$$a_0 = a(0); \quad b = b(0). \quad (2.82)$$

Notice that the two first inequalities in (2.81) are obtained by integrating (2.80) over $[0,t]$,$t>0$ and that one deduces from

$$\dot{\widehat{\left(\frac{1}{a}\right)}} = -\frac{\dot{a}}{a^2} = (-\frac{\dot{a}}{a}) \times \frac{1}{a} \tag{2.83}$$

and using (2.8) that

$$\left|\dot{\widehat{\left(\frac{1}{a}\right)}}\right| \leq C(\frac{1}{a}), \text{ or } \left|\dot{\widehat{\left(\frac{1}{a}\right)}}/\left(\frac{1}{a}\right)\right| \leq C, \tag{2.84}$$

which yields the two last inequalities in (2.81) by integrating over $[0,t]$, $t>0$.

For the moment, we need to introduce some useful functional spaces.

2.6 Functional spaces

The framework we will refer to for the distribution function f is $L^1_r(\mathbb{R}^3)$, the subspace of the Lebesgue space $L^1(\mathbb{R}^3)$ whose norm is denoted $||.||_{1,r}$, $r\geq 0$ and defined by :

$$L^1_r\left(\mathbb{R}^3\right) = \left\{ f \in L^1\left(\mathbb{R}^3\right) : ||f||_{1,r} = \int_{\mathbb{R}^3} |f\left(\overline{p}\right)| \left(p^0\right)^r d\overline{p} < +\infty \right\}. \tag{2.85}$$

For $r=1$, we will simply denote $||.||_{1,r}$ by $||.||$ we also define

$$|f\left(t\right)|_{1,r} = \int_{\mathbb{R}^3} |f\left(t,\overline{v}\right)| \langle \overline{v} \rangle^r d\overline{v}, \quad \langle \overline{v} \rangle = \sqrt{1+|\overline{v}|^2}. \tag{2.86}$$

Consequently, we have the following useful relations

$$||f\left(t\right)||_{1,r} \leq |f\left(t\right)|_{1,r} \leq b^r\left(t\right) ||f\left(t\right)||_{1,r}, \tag{2.87}$$

where we use the inequalities

$$v^0 \leq \langle \overline{v} \rangle \leq b\left(t\right) v^0. \tag{2.88}$$

Now ,we set for $r\in\mathbb{R}, r>0$:

$$X_r = \left\{ f \in L^1_1\left(\mathbb{R}^3\right), f \geq 0 \quad \text{a.e}, ||f|| \leq r \right\} \tag{2.89}$$

For any real interval I, we set :

$$C\left(\left[I, L^1_1\left(\mathbb{R}^3\right)\right]\right) = \left\{f : I \longrightarrow L^1_1\left(\mathbb{R}^3\right), f \text{ continuous and bounded}\right\}, \tag{2.90}$$

$$C\left([I, X_r]\right) = \left\{f \in C\left(\left[I, L^1_1\left(\mathbb{R}^3\right)\right]\right), f(t) \in X_r, \forall t \in I\right\} \tag{2.91}$$

with its usual norm

$$|||f||| = \sup\left\{||f(t)||, t \in I\right\}. \tag{2.92}$$

The framework we will refer to for $\overline{p}$ and $\overline{E}$ is $\mathbb{R}^3$, whose norm is denoted $||.||$ or $||.||_{\mathbb{R}^3}$.
$C\left([I, \mathbb{R}^3]\right)$ is a Banach space for the norm

$$|||m||| = \sup\left\{||m(t)||, t \in I\right\}. \tag{2.93}$$

We consider on $\mathbb{R}^3 \times \mathbb{R}^3 \times L^1_1\left(\mathbb{R}^3\right)$ the norm :

$$||\left(\overline{p}, \overline{E}, f\right)|| = ||\overline{p}|| + ||\overline{E}|| + ||f||. \tag{2.94}$$

We consider on $C\left([I, \mathbb{R}^3]\right) \times C\left([I, \mathbb{R}^3]\right) \times C\left([I, L^1_1\left(\mathbb{R}^3\right)]\right)$ the norm

$$|||\left(\overline{p}, \overline{E}, f\right)||| = |||\overline{p}||| + |||\overline{E}||| + |||f|||. \tag{2.95}$$

Lemma 2.2. *We have :*

1) $\left(L^1_r\left(\mathbb{R}^3\right); ||.||_{1,r}\right)$ *is a Banach space.*

2) Endowed with the distance induced by the norm $||.||$, X_r *is a complete and connected metric space.*

3) $C\left([I, L^1_1\left(\mathbb{R}^3\right)]; |||.|||\right)$ *is a Banach space.*

4) Endowed with the distance induced by the norm $|||.|||$, $C\left([I, X_r]\right)$ *is a complete metric space.*

Proof. See the Appendix □

CHAPTER 3

The homogeneous Maxwell-Bolzmann system for $\mu - N$ regularity

In this chapter we are concerned with the homogeneous Maxwell-Boltzmann system for $\mu - N$ regularity. This chapter is very important in the method adopted to attain the main issue of the work. It is a reformulation and a readjustment of the result coming from [29]. The change of hypotheses imposes to change the framework in which we look for solutions. The results obtained here are very important for the next chapter. The concept of $\mu - N$ regularity introduced by Choquet and Bancel many years ago in [3] consists on putting some boundary and Lipschitzian assumptions on the scattering kernel. For more details on this concept, see [3]. The chapter is organized as follows:

- Firstly, we begin by the proof of the local existence theorem of solution to the Maxwell-Boltzmann-Momentum system for $\mu - N$ regularity.
- In the second part, we deduce a global existence theorem using a special technique.

3.1 Local existence of solutions

3.1.1 Properties of the collision operator

For technical purposes in the method adopted in this work, in this section we change the scattering kernel $k\sqrt{\delta}\sigma$ into a bounded kernel $S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right)$, considering that it is a non negative continuous real valued function of all its arguments, on which we additionally require the Lipschitz continuity assumption as in [29]:

$$\begin{cases} 0 \leq S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right) \leq C_1 \\ |S\left(\overline{p_1},\overline{q},\overline{p'},\overline{q'}\right) - S\left(\overline{p_2},\overline{q},\overline{p'},\overline{q'}\right)| \leq C_1 ||\overline{p_1} - \overline{p_2}||, \end{cases} \tag{3.1}$$

where C_1 is a positive constant.
We recall that the relativistic-Boltzmann equation (2.32) reads:

$$\frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0}\overline{Q}\left(f,f\right), \tag{3.2}$$

where, using functions $f, h : T\left(\mathbb{R}^4\right) \longrightarrow \mathbb{R}$

$$\overline{Q}\left(f,h\right) = \overline{Q_+}\left(f,h\right) - \overline{Q_-}\left(f,h\right), \tag{3.3}$$

$$\overline{Q_+}\left(f,h\right) = \int_{\mathbb{R}^3}\int_{S^2} \frac{ab^2}{q^0} f\left(\overline{p'}\right) h\left(\overline{q'}\right) S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right) dw d\overline{q}, \tag{3.4}$$

$$\overline{Q_-}\left(f,h\right) = \int_{\mathbb{R}^3}\int_{S^2} \frac{ab^2}{q^0} f\left(\overline{p}\right) h\left(\overline{q}\right) S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right) dw d\overline{q}, \tag{3.5}$$

in which $f\left(\overline{p}\right) = f\left(t,\overline{p}\right)$.

The Maxwell-Boltzmann-Momentum system (2.72)-(2.75) transforms into the following integro-differential system :

$$\dot{E}^i = -\Gamma^i_{0j}E^j + \int_{\mathbb{R}^3} \frac{q^i f\left(t,\overline{q}\right) ab^2 d\overline{q}}{q^0}, \quad i,j = 1,2,3 \tag{3.6}$$

$$\dot{p}^i = -2\Gamma^i_{0j}p^j - \left[E^i + g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right]\int_{\mathbb{R}^3} f\left(t,\overline{q}\right) ab^2 d\overline{q}, \tag{3.7}$$

$$\frac{df}{dt} = \frac{1}{p^0}\overline{Q}\left(f,f,\overline{p}\right), \tag{3.8}$$

$$F_{ij} = F_{ij}\left(0\right) = \varphi_{ij} \quad i,j = 1,2,3,\ f\left(0,\overline{p}\right) = f_0\left(\overline{p}\right). \tag{3.9}$$

In order to state the local existence theorem, we first prove the following important propositions:

Proposition 3.1. *Let $f, g \in L_1^1(\mathbb{R}^3)$ be given. Then*

$$\frac{1}{p^0}\overline{Q_+}(f,g),\ \frac{1}{p^0}\overline{Q_-}(f,g),\ \frac{1}{p^0}\overline{Q}(f,g)$$

belong to $L_1^1(\mathbb{R}^3)$, moreover,

$$\begin{cases} \left|\left|\frac{1}{p^0}\overline{Q_+}(f,g)\right|\right| \leq C(t)\,||f||\;||g|| \\ \left|\left|\frac{1}{p^0}\overline{Q_-}(f,g)\right|\right| \leq C(t)\,||f||\;||g|| \end{cases} \tag{3.10}$$

$$\left|\left|\frac{1}{p^0}\overline{Q_+}(f,f) - \frac{1}{p^0}\overline{Q_+}(g,g)\right|\right| \leq C(t)\,||f-g||\,(||f|| + ||g||) \tag{3.11}$$

$$\left|\left|\frac{1}{p^0}\overline{Q_-}(f,f) - \frac{1}{p^0}\overline{Q_-}(g,g)\right|\right| \leq C(t)\,||f-g||\,(||f|| + ||g||) \tag{3.12}$$

$$\left|\left|\frac{1}{p^0}\overline{Q}(f,f) - \frac{1}{p^0}\overline{Q}(g,g)\right|\right| \leq C(t)\,||f-g||\,(||f|| + ||g||) \tag{3.13}$$

where

$$C(t) = 8\pi C_1 a b^2(t)$$

and C_1 is the constant provided by assumption (3.1).

Proof. Let $f, g \in L_1^1(\mathbb{R}^3)$.

$i)$ We have

$$\overline{Q_-}(f,g) = \int_{\mathbb{R}^3}\int_{S^2} \frac{ab^2}{q^0} f(\overline{p})\,g(\overline{q})\,S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right) dw d\overline{q}.$$

Taking the norm, we have :

$$\left|\left|\frac{1}{p^0}\overline{Q_-}(f,g)\right|\right| = \int_{\mathbb{R}^3}\sqrt{1+|\overline{p}|^2}\left|\frac{1}{p^0}\overline{Q_-}(f,g)\right| d\overline{p}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\sqrt{1+|\overline{p}|^2}ab^2 d\overline{q}d\overline{p}}{p^0 q^0}\int_{S^2}|f(\overline{p})\,g(\overline{q})|\,S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right) dw$$

$$\leq 4\pi C_1 a b^2(t)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\sqrt{1+|\overline{p}|^2}\sqrt{1+|\overline{q}|^2}}{p^0 q^0}\,|f(\overline{p})\,g(\overline{q})|\,d\overline{q}d\overline{p}$$

$$\leq 4\pi C_1 a b^2(t)\int_{\mathbb{R}^3}\sqrt{1+|\overline{p}|^2}\,|f(\overline{p})|\,d\overline{p}\int_{\mathbb{R}^3}\sqrt{1+|\overline{q}|^2}\,|f(\overline{q})|\,d\overline{q}$$

$$\leq 4\pi C_1 a b^2(t)\,\|f\|\;\|g\|$$

$$\leq C(t)\|f\|\ \|g\|$$

where

$$C(t) = 4\pi C_1 ab^2(t).$$

ii) We also have :

$$\frac{1}{p^0}\overline{Q_+}(f,g) = \int_{\mathbb{R}^3}\int_{S^2}\frac{ab^2}{p^0q^0}f\left(\overline{p'}\right)g\left(\overline{q'}\right)S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right)dwd\overline{q}.$$

Taking the norm we get :

$$\left\|\frac{1}{p^0}\overline{Q_+}(f,g)\right\| = \int_{\mathbb{R}^3}\sqrt{1+|\overline{p}|^2}\left|\frac{1}{p^0}\overline{Q_+}(f,g)\right|d\overline{p}$$

then

$$\left\|\frac{1}{p^0}\overline{Q_+}(f,g)\right\| =$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\sqrt{1+|\overline{p}|^2}ab^2d\overline{q}d\overline{p}}{p^0q^0}\int_{S^2}\left|f\left(\overline{p'}\right)g\left(\overline{q'}\right)\right|S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right)dw$$

$$\leq C_1ab^2(t)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\sqrt{1+|\overline{p}|^2}d\overline{q}d\overline{p}}{p^0q^0}\int_{S^2}\left|f\left(\overline{p'}\right)g\left(\overline{q'}\right)\right|dw.$$

The Jacobian of the change of variables $(\overline{p},\overline{q}) \longmapsto \left(\overline{p'},\overline{q'}\right)$ defined by (2.41) is

$$d\overline{p}d\overline{q} = \frac{p^0q^0}{p'^0q'^0}d\overline{p'}d\overline{q'}.$$

Then we get:

$$\left\|\frac{1}{p^0}\overline{Q_+}(f,g)\right\| \leq C_1ab^2(t)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\sqrt{1+|\overline{p}|^2}d\overline{q'}d\overline{p'}}{p'^0q'^0}$$
$$\times\left|f\left(\overline{p'}\right)g\left(\overline{q'}\right)\right|\int_{S^2}dw.$$

We set

$$A = C_1ab^2(t)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\sqrt{1+|\overline{p}|^2}d\overline{q'}d\overline{p'}}{p'^0q'^0}\left|f\left(\overline{p'}\right)g\left(\overline{q'}\right)\right|\int_{S^2}dw$$

therefore

$$A \leq 4\pi C_1 ab^2(t) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\sqrt{1+|\overline{p}|^2}\sqrt{1+|\overline{p'}|^2}\sqrt{1+|\overline{q'}|^2}}{p'^0 q'^0} \times \left|f\left(\overline{p'}\right) g\left(\overline{q'}\right)\right| d\overline{q'}d\overline{p'}.$$

But we have

$$\begin{aligned}\sqrt{1+|\overline{p}|^2} &= \sqrt{1+\frac{1}{a^2}a^2\left(p^1\right)^2+\frac{1}{b^2}b^2\left(\left(p^2\right)^2+\left(p^3\right)^2\right)}\\ &\leq \sqrt{1+\frac{1}{a^2}+\frac{1}{b^2}}\sqrt{1+a^2\left(p^1\right)^2+b^2\left(\left(p^2\right)^2+\left(p^3\right)^2\right)}\\ &= p^0\sqrt{1+\frac{1}{a^2}+\frac{1}{b^2}}.\end{aligned}$$

Since $a \leq b$ we have

$$\frac{1}{a^2}+\frac{1}{b^2} \leq \frac{2}{a^2}.$$

So:

$$\left|\left|\frac{1}{p^0}\overline{Q_+}(f,g)\right|\right| \leq 4\pi C_1 ab^2(t) \times \sqrt{1+\frac{2}{a^2}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{p^0\sqrt{1+|\overline{p'}|^2}\sqrt{1+|\overline{q'}|^2}}{p'^0 q'^0}\left|f\left(\overline{p'}\right) g\left(\overline{q'}\right)\right| d\overline{q'}d\overline{p'}.$$

Otherwise, we have :

$$\begin{aligned}\frac{p^0}{p'^0 q'^0} &\leq \frac{p^0+q^0}{p'^0 q'^0}\\ &= \frac{p'^0+q'^0}{p'^0 q'^0}\\ &= \frac{1}{q'^0}+\frac{1}{p'^0}\\ &\leq 1+1=2;\end{aligned}$$

Accordingly:

$$\left|\left|\frac{1}{p^0}\overline{Q_+}(f,g)\right|\right| \leq B$$

where

$$B = 8\pi C_1 ab^2(t)\sqrt{1+\frac{2}{a^2}}\int_{\mathbb{R}^3}\sqrt{1+|\overline{p'}|^2}\left|f\left(\overline{p'}\right)\right|d\overline{p'}$$
$$\times\int_{\mathbb{R}^3}\sqrt{1+|\overline{q'}|^2}\left|f\left(\overline{q'}\right)\right|d\overline{q'}.$$

Then

$$B \leq 8\pi C_1 ab^2(t)\sqrt{1+\frac{2}{a^2}}||f||\,||g||,$$

Consequently

$$\left|\left|\frac{1}{p^0}\overline{Q_+}(f,g)\right|\right| \leq C(t)\,||f||\,||g||,$$

where

$$C(t) = 8\pi C_1 ab^2(t)\sqrt{1+\frac{2}{a^2}}.$$

iii) To show inequality (3.11) we use the bilinearity of the collision operator and we write

$$\frac{1}{p^0}\overline{Q_+}(f,f) - \frac{1}{p^0}\overline{Q_+}(g,g) = \frac{1}{p^0}\overline{Q_+}(f,f-g) + \frac{1}{p^0}\overline{Q_+}(f-g,g).$$

We obtain using (3.10)

$$\begin{aligned}
\left|\left|\frac{1}{p^0}\overline{Q_+}(f,f) - \frac{1}{p^0}\overline{Q_+}(g,g)\right|\right| &= \left|\left|\frac{1}{p^0}\overline{Q_+}(f,f-g) + \frac{1}{p^0}\overline{Q_+}(f-g,g)\right|\right| \\
&\leq \left|\left|\frac{1}{p^0}\overline{Q_+}(f,f-g)\right|\right| + \left|\left|\frac{1}{p^0}\overline{Q_+}(f-g,g)\right|\right| \\
&\leq C(t)\,||f||\,||f-g|| + C(t)\,||g||\,||f-g|| \\
&\leq C(t)\,||f-g||\,(||f||+||g||).
\end{aligned}$$

iv) By the same way for (3.12) we write

$$\left|\left|\frac{1}{p^0}\overline{Q_-}(f,f) - \frac{1}{p^0}\overline{Q_-}(g,g)\right|\right| = \left|\left|\frac{1}{p^0}\overline{Q_-}(f,f-g) + \frac{1}{p^0}\overline{Q_-}(f-g,g)\right|\right|$$

and we set

$$D = \left|\left|\frac{1}{p^0}\overline{Q_-}(f,f-g) + \frac{1}{p^0}\overline{Q_-}(f-g,g)\right|\right|.$$

Then

$$
\begin{aligned}
D &\le \left|\left|\frac{1}{p^0}\overline{Q_-}\left(f,f-g\right)\right|\right| + \left|\left|\frac{1}{p^0}\overline{Q_-}\left(f-g,g\right)\right|\right| \\
&\le C\left(t\right)||f||\;||f-g|| + C\left(t\right)||g||\;||f-g|| \\
&\le C\left(t\right)||f-g||\left(||f||+||g||\right).
\end{aligned}
$$

v) For the last inequality, since :

$$\overline{Q} = \overline{Q_+} - \overline{Q_-},$$

we have:

$$\left|\left|\frac{1}{p^0}\overline{Q}\left(f,f\right) - \frac{1}{p^0}\overline{Q}\left(g,g\right)\right|\right| = E$$

where

$$E = \left|\left|\frac{1}{p^0}\overline{Q_+}\left(f,f\right) - \frac{1}{p^0}\overline{Q_-}\left(f,f\right) - \frac{1}{p^0}\overline{Q_+}\left(g,g\right) + \frac{1}{p^0}\overline{Q_-}\left(g,g\right)\right|\right|.$$

Then

$$
\begin{aligned}
E &= \left|\left|\frac{1}{p^0}\overline{Q_+}\left(f,f\right) - \frac{1}{p^0}\overline{Q_+}\left(g,g\right) - \frac{1}{p^0}\overline{Q_-}\left(f,f\right) + \frac{1}{p^0}\overline{Q_-}\left(g,g\right)\right|\right| \\
&\le \left|\left|\frac{1}{p^0}\overline{Q_+}\left(f,f\right) - \frac{1}{p^0}\overline{Q_+}\left(g,g\right)\right|\right| + \left|\left|\frac{1}{p^0}\overline{Q_-}\left(f,f\right) - \frac{1}{p^0}\overline{Q_-}\left(g,g\right)\right|\right| \\
&\le C\left(t\right)||f-g||\left(||f||+||g||\right) + C\left(t\right)||f-g||\left(||f||+||g||\right) \\
&\le C\left(t\right)||f-g||\left(||f||+||g||\right).
\end{aligned}
$$

□

Proposition 3.2. *Let* $\overline{p} = \left(p^i\right), \overline{p_j} = \left(p^i_j\right) \in \mathbb{R}^3, j = 1,2, f \in L^1_1\left(\mathbb{R}^3\right), k \in \{1,2,3\}.$ *Then*

$$p^0 \ge a\left|p^1\right|; \quad p^0 \ge b\left|p^2\right|; \quad p \ge b\left|p^3\right|, \tag{3.14}$$

$$\left|\frac{p^k_1}{p^0_1} - \frac{p^k_2}{p^0_2}\right| \le 5\left(1 + \frac{a}{b} + \frac{b}{a}\right)||\overline{p_1} - \overline{p_2}||, \tag{3.15}$$

$$\left|\frac{1}{p^0_1} - \frac{1}{p^0_2}\right| \le \left(2a+4b\right)\frac{||\overline{p_1} - \overline{p_2}||}{p^0_j}, j = 1,2 \tag{3.16}$$

$$||\frac{1}{p^0_j}\overline{Q}\left(f,f,\overline{p_1}\right) - \frac{1}{p^0_j}\overline{Q}\left(f,f,\overline{p_2}\right)|| \le 4\pi C_1 ab^2||f||^2||\overline{p_1} - \overline{p_2}|| \tag{3.17}$$

where $C_1 > 0$ *is the constant appearing in (3.1).*

Proof. (3.14) is a direct consequence of (2.12).

Let $k \in \{1,2,3\}$. We have

$$\begin{aligned}
\left|\frac{p_1^k}{p_1^0} - \frac{p_2^k}{p_2^0}\right| &= \left|\frac{p_1^k p_2^0 - p_1^0 p_2^k}{p_1^0 p_2^0}\right| \\
&= \left|\frac{p_1^k p_2^0 - p_1^k p_1^0 + p_1^k p_1^0 - p_1^0 p_2^k}{p_1^0 p_2^0}\right| \\
&= \left|\frac{p_1^k \left(p_2^0 - p_1^0\right) + p_1^0 \left(p_1^k - p_2^k\right)}{p_1^0 p_2^0}\right|.
\end{aligned}$$

Using the fact that $p_2^0 \geq 1$ we deduce that

$$\left|\frac{p_1^k}{p_1^0} - \frac{p_2^k}{p_2^0}\right| \leq \frac{\left|p_1^k\right| \left|p_2^0 - p_1^0\right|}{p_1^0 p_2^0} + \left|p_1^k - p_2^k\right|. \tag{3.18}$$

For the first term, using expression (2.12) of p^0:

$$p_1^0 - p_2^0 = \frac{\left(p_1^0\right)^2 - \left(p_2^0\right)^2}{p_1^0 + p_2^0}$$

$$\begin{aligned}
&= \frac{a^2 \left(p_1^1 + p_2^1\right)\left(p_1^1 - p_2^1\right) + b^2 \left(p_1^2 + p_2^2\right)\left(p_1^2 - p_2^2\right)}{p_1^0 + p_2^0} \\
&\quad + \frac{b^2 \left(p_1^3 + p_2^3\right)\left(p_1^3 - p_2^3\right)}{p_1^0 + p_2^0}.
\end{aligned} \tag{3.19}$$

□

From (3.18) and (3.19) we have :

$$\begin{aligned}
&\left|\frac{p_1^k}{p_1^0} - \frac{p_2^k}{p_2^0}\right| \leq \\
&\quad \left[\frac{a^2\left(\left|p_1^k p_1^1\right| + \left|p_1^k p_2^1\right|\right) + b^2\left(\left|p_1^k p_1^2\right| + \left|p_1^k p_2^2\right|\right) + b^2\left(\left|p_1^k p_1^3\right| + \left|p_1^k p_2^3\right|\right)}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} + 1\right] \\
&\qquad \times ||\overline{p_1} - \overline{p_2}||.
\end{aligned} \tag{3.20}$$

Now we have, using conveniently the inequalities (3.14) :

for $k = 1$:

$$\frac{\left|p_1^1\right|^2 + \left|p_1^1 p_2^1\right|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{a^2};$$

$$\frac{\left|p_1^1 p_1^2\right| + \left|p_1^1 p_2^2\right|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{ab};$$

$$\frac{|p_1^1 p_1^3| + |p_1^1 p_2^3|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{ab},$$

Proof. for $k = 2$:

$$\frac{|p_1^2|^2 + |p_1^2 p_2^2|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{b^2};$$

$$\frac{|p_1^2 p_1^1| + |p_1^2 p_2^1|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{ab};$$

$$\frac{|p_1^2 p_1^3| + |p_1^2 p_2^3|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{b^2},$$

for $k = 3$:

$$\frac{|p_1^3|^2 + |p_1^3 p_2^3|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{b^2};$$

$$\frac{|p_1^3 p_1^1| + |p_1^3 p_2^1|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{ab};$$

$$\frac{|p_1^3 p_1^2| + |p_1^3 p_2^2|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{b^2}.$$

Then, the relation (3.20) gives :

$$\left|\frac{p_1^1}{p_1^0} - \frac{p_2^1}{p_2^0}\right| \leq \left(3 + 4\frac{b}{a}\right) ||\overline{p_1} - \overline{p_2}|| \tag{3.21}$$

$$\left|\frac{p_1^2}{p_1^0} - \frac{p_2^2}{p_2^0}\right| \leq \left(5 + 2\frac{a}{b}\right) ||\overline{p_1} - \overline{p_2}|| \tag{3.22}$$

$$\left|\frac{p_1^3}{p_1^0} - \frac{p_2^3}{p_2^0}\right| \leq \left(5 + 2\frac{a}{b}\right) ||\overline{p_1} - \overline{p_2}||. \tag{3.23}$$

Using (3.21), (3.22) and (3.23) we obtain (3.15).

We have, using expression (2.12) of p^0 and (3.19):

$$\begin{aligned} \left|\frac{1}{p_1^0} - \frac{1}{p_2^0}\right| &= \left|\frac{\left(p_1^0\right)^2 - \left(p_2^0\right)^2}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)}\right| \\ &\leq \frac{a^2 \left(|p_1^1| + |p_2^1|\right) + b^2 \left(|p_1^2| + |p_2^2|\right) + b^2 \left(|p_1^3| + |p_2^3|\right)}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} ||\overline{p_1} - \overline{p_2}||. \end{aligned} \tag{3.24}$$

But using conveniently inequalities (3.14) and $p_j^0 > 1, \quad j = 1, 2$:

$$\frac{|p_1^1| + |p_2^1|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{a p_j^0}; \tag{3.25}$$

$$\frac{|p_1^2| + |p_2^2|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{b p_j^0}; \tag{3.26}$$

$$\frac{|p_1^3| + |p_2^3|}{p_1^0 p_2^0 \left(p_1^0 + p_2^0\right)} \leq \frac{2}{b p_j^0}. \tag{3.27}$$

Then we have using (3.24), (3.25), (3.26) and (3.27) the relation (3.16).

To establish (3.17), write using $\overline{Q} = \overline{Q_+} - \overline{Q_-}$:

$$\begin{aligned}
&\left|\frac{\overline{Q}\left(f, f, \overline{p_1}\right)}{p_j^0} - \frac{\overline{Q}\left(f, f, \overline{p_2}\right)}{p_j^0}\right| \\
&\qquad = \left|\frac{\overline{Q_+}\left(f, f, \overline{p_1}\right)}{p_j^0} - \frac{\overline{Q_-}\left(f, f, \overline{p_1}\right)}{p_j^0} - \frac{\overline{Q_+}\left(f, f, \overline{p_2}\right)}{p_j^0} + \frac{\overline{Q_-}\left(f, f, \overline{p_2}\right)}{p_j^0}\right| \\
&\qquad\qquad \leq \left|\frac{\overline{Q_+}\left(f, f, \overline{p_1}\right) - \overline{Q_+}\left(f, f, \overline{p_2}\right)}{p_j^0}\right| \\
&\qquad\qquad\qquad + \left|\frac{\overline{Q_-}\left(f, f, \overline{p_1}\right) - \overline{Q_-}\left(f, f, \overline{p_2}\right)}{p_j^0}\right|.
\end{aligned} \tag{3.28}$$

Now the expression (3.4) of $\overline{Q_+}$ shows that :

$$\begin{aligned}
&\left|\frac{\overline{Q_+}\left(f, f, \overline{p_1}\right) - \overline{Q_+}\left(f, f, \overline{p_2}\right)}{p_j^0}\right| \leq \\
&\qquad \frac{1}{p_j^0} \int_{\mathbb{R}^3} \frac{ab^2}{q^0} d\overline{q} \int_{S^2} \left|f\left(\overline{p'}\right)\right| \left|f\left(\overline{q'}\right)\right| \left|S\left(\overline{p_1}, \overline{q}, \overline{p'}, \overline{q'}\right)\right| \left|S\left(\overline{p_2}, \overline{q}, \overline{p'}, \overline{q'}\right)\right| dw.
\end{aligned}$$

Then using the second assumption (3.1) on the collision kernel S and proceeding as in proof of the first inequality (3.10), we obtain :

$$\left\|\frac{\overline{Q_+}\left(f, f, \overline{p_1}\right) - \overline{Q_+}\left(f, f, \overline{p_2}\right)}{p_j^0}\right\| \leq 4\pi C_1 ab^2\left(t\right) ||f||^2 ||\overline{p_1} - \overline{p_2}||. \tag{3.29}$$

Next, using once more the second assumption (3.1) on the collision kernel S and proceeding this time as in the proof of the second inequality (3.10) we obtain :

$$\left\|\frac{\overline{Q_-}\left(f, f, \overline{p_1}\right) - \overline{Q_-}\left(f, f, \overline{p_2}\right)}{p_j^0}\right\| \leq 4\pi C_1 ab^2\left(t\right) ||f||^2 ||\overline{p_1} - \overline{p_2}||. \tag{3.30}$$

(3.17) follows then from (3.28), (3.29) and (3.30).

This completes the proof of this proposition. □

3.1.2 Local existence solutions of the system (3.6)-(3.9)

Let $I = [t_0, t_0 + T]$ a fixed interval of $\mathbb{R}^+$ with $t_0 \geq 0, T > 0$ arbitrarily given.

To solve the system (3.6)-(3.9) we begin by determine a local solution in interval I.

Let us consider the map

$$H : [t_0, t_0+T] \times \mathbb{R}^3 \times \mathbb{R}^3 \times L^1_1\left(\mathbb{R}^3\right) \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \times L^1_1\left(\mathbb{R}^3\right)$$

$$\left(t, \overline{p}, \overline{E}, f\right) \longmapsto H\left(t, \overline{p}, \overline{E}, f\right)$$

where

$$H\left(t, \overline{p}, \overline{E}, f\right) = (H_1, H_2, H_3)\left(t, \overline{p}, \overline{E}, f\right)$$

denotes the right hand side of equations (3.6)-(3.8), or equivalently :

$$H_1\left(t, \overline{p}, \overline{E}, f\right) = -\Gamma^i_{0j}E^j + \int_{\mathbb{R}^3} \frac{q^i f\left(t, \overline{q}\right) ab^2 d\overline{q}}{q^0}, \tag{3.31}$$

$$H_2\left(t, \overline{p}, \overline{E}, f\right) = -2\Gamma^i_{0j}p^j - \left[E^i + g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right]\int_{\mathbb{R}^3} f\left(t, \overline{q}\right) ab^2 d\overline{q}, \tag{3.32}$$

$$H_3\left(t, \overline{p}, \overline{E}, f\right) = \frac{1}{p^0}\overline{Q}\left(f, f, \overline{p}\right) \tag{3.33}$$

in which $i, j, k = 1, 2, 3,\ \Gamma^i_{0j}$ depends on t through $\frac{\dot{a}}{a}$ if $i = j = 1$ and $\frac{\dot{b}}{b}$ if $i = j = 2, 3$.

Additionally, we will need to estimate the difference in $f, \overline{E}$ and $\overline{p}$ in L^1_1 and $\mathbb{R}^3$ norms.

Proposition 3.3. *Let* $\bar{p_1}, \overline{p_2}, \overline{E_1}, \overline{E_2} \in \mathbb{R}^3, f_1, f_2 \in L^1_1\left(\mathbb{R}^3\right)$. *Then :*

$$\left|\left|H_1\left(t, \overline{p_1}, \overline{E_1}, f_1\right) - H_1\left(t, \overline{p_2}, \overline{E_2}, f_2\right)\right|\right|_{\mathbb{R}^3} \leq C_2\left(\left|\left|\overline{E_1} - \overline{E_2}\right|\right|_{\mathbb{R}^3} + ||f_1 - f_2||\right), \tag{3.34}$$

$$\left|\left|H_2\left(t, \overline{p_1}, \overline{E_1}, f_1\right) - H_2\left(t, \overline{p_2}, \overline{E_2}, f_2\right)\right|\right|_{\mathbb{R}^3} \leq C_3\left(\left|\left|\overline{E_1} - \overline{E_2}\right|\right|_{\mathbb{R}^3} + ||\overline{p_1} - \overline{p_2}||_{\mathbb{R}^3} + ||f_1 - f_2||\right) \tag{3.35}$$

$$\left|\left|H_3\left(t, \overline{p_1}, \overline{E_1}, f_1\right) - H_3\left(t, \overline{p_2}, \overline{E_2}, f_2\right)\right|\right|_{\mathbb{R}^3} \leq C_4\left(||\overline{p_1} - \overline{p_2}||_{\mathbb{R}^3} + ||f_1 - f_2||\right) \tag{3.36}$$

$$\left|\left|H\left(t, \overline{p_1}, \overline{E_1}, f_1\right) - H\left(t, \overline{p_2}, \overline{E_2}, f_2\right)\right|\right| \leq C_5\left(\left|\left|\overline{E_1} - \overline{E_2}\right|\right|_{\mathbb{R}^3} + ||\overline{p_1} - \overline{p_2}||_{\mathbb{R}^3} + ||f_1 - f_2||\right) \tag{3.37}$$

where

$$\begin{cases} C_2 = 3C + b^2 \\ C_3 = 5\left(6C+1\right)\left(1+a+\frac{b^2}{a}\right) \\ \times\left(1+\frac{a}{b}+\frac{b}{a}+\frac{1}{a}+\frac{1}{b}\right)\left(1+||f_2||\right)\left(1+ab^2\right)\left(1+||f_2||+\left|\left|\overline{E_1}\right|\right|\right) \\ \\ C_4 = 8\pi C_1 ab^2\left(1+a+2b\right)\left(1+||f_1||+||f_2||+||f_2||^2\right) \\ C_5 = C_2+C_3+C_4. \end{cases} \tag{3.38}$$

Proof. Write, using the expression of H_1 in (3.31)

$$H_1\left(t,\overline{p_1},\overline{E_1},f_1\right) - H_1\left(t,\overline{p_2},\overline{E_2},f_2\right) = \Gamma^i_{0j}\left(E^j_2 - E^j_1\right) +$$

$$\int_{\mathbb{R}^3}\frac{q^i}{q^0}ab^2\left[f_1\left(t,\overline{q}\right) - f_2\left(t,\overline{q}\right)\right]d\overline{q}. \tag{3.39}$$

For the first term in (3.39), we have, since $\Gamma^1_{01} = \frac{\dot{a}}{a}$, $\Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{b}}{b}$ and using (2.80) :

$$\left|\Gamma^i_{0j}\left(E^j_2 - E^j_1\right)\right| \leq 3C\left|\left|\overline{E_1}-\overline{E_2}\right|\right|.$$

For the second term in (3.39), we have, since

$$\frac{|q^1|}{q^0} \leq \frac{1}{a}, \quad \frac{|q^2|}{q^0} \leq \frac{1}{b}, \quad \frac{|q^3|}{q^0} \leq \frac{1}{b},$$

$$\left|\int_{\mathbb{R}^3}\frac{q^i}{q^0}ab^2\left[f_1\left(t,\overline{q}\right) - f_2\left(t,\overline{q}\right)\right]d\overline{q}\right| = F$$

where

$$F = \left|\int_{\mathbb{R}^3}\frac{q^i}{q^0}ab^2\frac{\sqrt{1+|\overline{q}|^2}}{\sqrt{1+|\overline{q}|^2}}f_1\left(t,\overline{q}\right) - f_2\left(t,\overline{q}\right)d\overline{q}\right|.$$

Then

$$\begin{aligned} F \leq & ab^2 \times \frac{1}{a}\int_{\mathbb{R}^3}\frac{\sqrt{1+|\overline{q}|^2}}{q^0}\left|f_1\left(t,\overline{q}\right) - f_2\left(t,\overline{q}\right)\right|d\overline{q} \\ \leq & b^2\int_{\mathbb{R}^3}\sqrt{1+|\overline{q}|^2}\left|f_1\left(t,\overline{q}\right) - f_2\left(t,\overline{q}\right)\right|d\overline{q} \\ \leq & b^2\left|\left|f_1-f_2\right|\right|, \text{ since } \ \frac{1}{q^0} \leq 1. \end{aligned}$$

We have now :

$$\left|\left|H_1\left(t,\overline{p_1},\overline{E_1},f_1\right)-H_1\left(t,\overline{p_2},\overline{E_2},f_2\right)\right|\right|\leq 3C\left|\left|\overline{E_1}-\overline{E_2}\right|\right|+b^2\left|\left|f_1-f_2\right|\right|$$

since

$$3C\left|\left|\overline{E_1}-\overline{E_2}\right|\right|+b^2\left|\left|f_1-f_2\right|\right|\leq\left(3C+b^2\right)\left(\left|\left|\overline{E_1}-\overline{E_2}\right|\right|+\left|\left|f_1-f_2\right|\right|\right),$$

then (3.34) yields.

Write, using the expression of H_2 in (3.32)

$$\begin{aligned} H_2\left(t,\overline{p_1},\overline{E_1},f_1\right)-H_2\left(t,\overline{p_2},\overline{E_2},f_2\right)&=2\Gamma^i_{0j}\left(p^j_2-p^j_1\right)\\ +ab^2\left[E^i_2\int_{\mathbb{R}^3}f_2\left(t,\overline{q}\right)d\overline{q}-E^i_1\int_{\mathbb{R}^3}f_1\left(t,\overline{q}\right)d\overline{q}\right]&\\ +ab^2g^{ii}\varphi_{ki}\left[\frac{p^k_2}{p^0_2}\int_{\mathbb{R}^3}f_2\left(t,\overline{q}\right)d\overline{q}-\frac{p^k_1}{p^0_1}\int_{\mathbb{R}^3}f_1\left(t,\overline{q}\right)d\overline{q}\right].& \end{aligned} \tag{3.40}$$

For the first term in (3.40) , we have :

$$\left|2\Gamma^i_{0j}\left(p^j_2-p^j_1\right)\right|\leq 6C\left|\left|\overline{p_1}-\overline{p_2}\right|\right|. \tag{3.41}$$

We write the second term in (3.40) in the form

$$ab^2\left[E^i_2\int_{\mathbb{R}^3}f_2\left(t,\overline{q}\right)d\overline{q}-E^i_1\int_{\mathbb{R}^3}f_1\left(t,\overline{q}\right)d\overline{q}\right]=$$

$$=ab^2\left(E^i_2-E^i_1\right)\int_{\mathbb{R}^3}f_2\left(t,\overline{q}\right)d\overline{q}+ab^2\left(E^i_1\right)\int_{\mathbb{R}^3}\left[f_2\left(t,\overline{q}\right)-f_1\left(t,\overline{q}\right)\right]d\overline{q}$$

$$\begin{aligned}=ab^2\left(E^i_2-E^i_1\right)\int_{\mathbb{R}^3}\frac{\sqrt{1+\left|\overline{q}\right|^2}}{\sqrt{1+\left|\overline{q}\right|^2}}f_2\left(t,\overline{q}\right)d\overline{q}+ab^2\left(E^i_1\right)&\\ \times\int_{\mathbb{R}^3}\frac{\sqrt{1+\left|\overline{q}\right|^2}}{\sqrt{1+\left|\overline{q}\right|^2}}\left[f_2\left(t,\overline{q}\right)-f_1\left(t,\overline{q}\right)\right]d\overline{q}.&\end{aligned}$$

Now we then have :

$$\left|ab^2\left[E^i_2\int_{\mathbb{R}^3}f_2\left(t,\overline{q}\right)d\overline{q}-E^i_1\int_{\mathbb{R}^3}f_1\left(t,\overline{q}\right)d\overline{q}\right]\right|\leq G$$

where

$$G = ab^2 \left(||\overline{f_2}|| + ||\overline{E_1}|| \right) \left(||\overline{E_1} - \overline{E_2}|| + ||f_1 - f_2|| \right) \tag{3.42}$$

Now we write the third term in (3.40) in the form

$$ab^2 g^{ii} \varphi_{ki} \left[\frac{p_2^k}{p_2^0} \int_{\mathbb{R}^3} f_2(t, \overline{q})\, d\overline{q} - \frac{p_1^k}{p_1^0} \int_{\mathbb{R}^3} f_1(t, \overline{q})\, d\overline{q} \right]$$

$$= ab^2 g^{ii} \varphi_{ki} \left[\left(\frac{p_2^k}{p_2^0} - \frac{p_1^k}{p_1^0} \right) \int_{\mathbb{R}^3} f_2(t, \overline{q})\, d\overline{q} + \frac{p_1^k}{p_1^0} \int_{\mathbb{R}^3} \left(f_2(t, \overline{q}) - f_1(t, \overline{q}) \right) d\overline{q} \right]$$

$$= ab^2 g^{ii} \varphi_{ki} \left[\left(\frac{p_2^k}{p_2^0} - \frac{p_1^k}{p_1^0} \right) \int_{\mathbb{R}^3} \frac{\sqrt{1 + |\overline{q}|^2}}{\sqrt{1 + |\overline{q}|^2}} f_2(t, \overline{q})\, d\overline{q} \right]$$

$$+ ab^2 g^{ii} \varphi_{ki} \left[\frac{p_1^k}{p_1^0} \int_{\mathbb{R}^3} \frac{\sqrt{1 + |\overline{q}|^2}}{\sqrt{1 + |\overline{q}|^2}} \left(f_2(t, \overline{q}) - f_1(t, \overline{q}) \right) d\overline{q} \right]. \tag{3.43}$$

Using (2.3) and (3.14) :

$$\begin{cases} ab^2 g^{11} = \frac{b^2}{a}, \ \ ab^2 g^{22} = ab^2 g^{33} = a \\ \frac{|p_1^1|}{p_1^0} \leq \frac{1}{a}, \ \ \frac{|p_1^2|}{p_1^0} \leq \frac{1}{b}, \ \ \frac{|p_1^3|}{p_1^0} \leq \frac{1}{b}. \end{cases} \tag{3.44}$$

Using (3.44), (3.15), we then deduce from (3.43) for the first term in brackets that, we have for $i \in \{1, 2, 3\}$:

$$\left| ab^2 g^{ii} \varphi_{ki} \left[\frac{p_2^k}{p_2^0} \int_{\mathbb{R}^3} f_2(t, \overline{q})\, d\overline{q} - \frac{p_1^k}{p_1^0} \int_{\mathbb{R}^3} f_1(t, \overline{q})\, d\overline{q} \right] \right|$$

$$\leq K_1 \left(||\overline{p_1} - \overline{p_2}|| + ||f_1 - f_2|| \right) \tag{3.45}$$

where :

$$K_1 = 5 \left(\frac{b^2}{a} + a \right) \left(2 + \frac{a}{b} + \frac{b}{a} + \frac{1}{a} + \frac{1}{b} \right) \left(1 + ||f_2|| \right) \sum_{i,k} |\varphi_{ik}| \, . \tag{3.46}$$

(3.35) and the expression of C_3 in (3.38) then follow from (3.42), (3.43), (3.44), (3.45) and (3.46).

Write, using the expression of H_3 in (3.33)

$$\begin{aligned}
H_3\left(t,\overline{p_1},\overline{E_1},f_1\right)-H_3\left(t,\overline{p_2},\overline{E_2},f_2\right) &= \frac{1}{p_1^0}\overline{Q}\left(f_1,f_1,\overline{p_1}\right)-\frac{1}{p_2^0}\overline{Q}\left(f_2,f_2,\overline{p_2}\right)\\
&= \frac{1}{p_1^0}\left[\overline{Q}\left(f_1,f_1,\overline{p_1}\right)-\overline{Q}\left(f_2,f_2,\overline{p_1}\right)\right]\\
&+ \frac{1}{p_1^0}\left[\overline{Q}\left(f_2,f_2,\overline{p_1}\right)-\overline{Q}\left(f_2,f_2,\overline{p_2}\right)\right]\\
&+ \left(\frac{1}{p_1^0}-\frac{1}{p_2^0}\right)\overline{Q}\left(f_2,f_2,\overline{p_2}\right). \qquad (3.47)
\end{aligned}$$

For the first term in (3.47) in which $\overline{p_1}$ is fixed, use (3.13), (3.1) to obtain setting in (3.13) $f=f_1, \quad g=f_2$

$$\left|\left|\frac{1}{p_1^0}\left[\overline{Q}\left(f_1,f_1,\overline{p_1}\right)-\overline{Q}\left(f_2,f_2,\overline{p_1}\right)\right]\right|\right| \leq 8\pi C_1 ab^2\left(||f_1||+||f_2||\right)||f_1-f_2||. \qquad (3.48)$$

For the second term in (3.47) in which f_2 is fixed, by using inequality (3.17) , with $j=1,\ f=f_2$ we obtain

$$\left|\left|\frac{1}{p_1^0}\left[\overline{Q}\left(f_2,f_2,\overline{p_1}\right)-\overline{Q}\left(f_2,f_2,\overline{p_2}\right)\right]\right|\right| \leq 4\pi C_1 ab^2\,||f_2||^2\,||\overline{p_1}-\overline{p_2}||. \qquad (3.49)$$

For the third term in (3.47) in which f_2 is fixed, we use inequality (3.16) with $j=2$ to obtain

$$\left|\left|\left(\frac{1}{p_1^0}-\frac{1}{p_2^0}\right)\overline{Q}\left(f_2,f_2,\overline{p_2}\right)\right|\right| \leq (2a+4b)\left|\left|\frac{\overline{Q}\left(f_2,f_2,\overline{P_2}\right)}{p_2^0}\right|\right|. \qquad (3.50)$$

Now we deduce from (3.50) using inequality (3.13) in which we set $f=f_2,\ g=0$ that :

$$\left|\left|\left(\frac{1}{p_1^0}-\frac{1}{p_2^0}\right)\overline{Q}\left(f_2,f_2,\overline{p_2}\right)\right|\right| \leq 8\pi C_1 ab^2\left(a+2b\right)||f_2||^2\,||\overline{p_1}-\overline{p_2}||. \qquad (3.51)$$

(3.36) and the expression of C_4 in (3.38), then follow from (3.49), (3.50) and (3.51).

Combining (3.34) ,(3.35) and (3.36) and using the definition (2.94) of the norm on $\mathbb{R}^3\times\mathbb{R}^3\times L_1^1(\mathbb{R}^3)$ we have (3.37).

This completes the proof of the proposition . □

The following theorem gives the local existence of solution of the Maxwell-Boltzmann system and is stated as follows:

Theorem 3.1. *Let $t_0\geq 0$, $\left(\overline{p_{t_0}},\overline{E_{t_0}},f_{t_0}\right)\in\mathbb{R}^3\times\mathbb{R}^3\times L_1^1(\mathbb{R}^3)$ be given. Then:*

1. There exists a real number $\delta > 0$ *such that the differential system* (3.6)-*(3.9) has a unique solution* $\left(\overline{p}, \overline{E}, f\right) \in C\left(\left[t_0; t_0+\delta\right]; \mathbb{R}^3\right) \times C\left(\left[t_0; t_0+\delta\right]; \mathbb{R}^3\right) \times C\left(\left[t_0; t_0+\delta\right]; L_1^1\left(\mathbb{R}^3\right)\right)$ *satisfying*

$$\left(\overline{p}, \overline{E}, f\right)(t_0) = \left(\overline{p_{t_0}}, \overline{E_{t_0}}, f_{t_0}\right).$$

Moreover f *satisfies*

$$|||f||| = \sup\left\{||f(t)||, t \in [t_0; t_0+\delta]\right\} \leq ||f_{t_0}||. \tag{3.52}$$

2. The Maxwell-Boltzmann system (2.23)-(2.24)-(3.2) has a unique local solution (F, f) *on* $[t_0; t_0+\delta]$ *such that*

$$F^{i0}(t_0) = E^i_{t_0},\ F_{ij}(t_0) = \varphi_{ij},\ f(t_0) = f_{t_0},\ |||f||| \leq ||f_{t_0}||. \tag{3.53}$$

Proof. We apply the Cauchy-Lipschitz theorem to the first order differential system (3.6)-(3.8).

1) Since the functions : $a, b, \dot{a}, \dot{b}, \frac{1}{a}, \frac{1}{b}, S$ are continuous functions of t, so is the right hand side $H = (H_1, H_2, H_3)$ of system (3.6)-(3.8).

By the continuity of the functions $z = a, b, \frac{1}{a}, \frac{1}{b}$ at $t = t_0$, there exists a real number $\delta_0 > 0$ such that :

$$t \in]t_0 - \delta_0, t_0 + \delta_0[\Longrightarrow |z(t)| \leq |z(t_0)| + 1. \tag{3.54}$$

(3.54) implies using (2.81) to control $z = a, b, \frac{1}{a}, \frac{1}{b}$ that :

$$t \in]t_0 - \delta_0, t_0 + \delta_0[\Longrightarrow |z(t)| \leq \left(a_0 + b_0 + \frac{1}{a_0} + \frac{1}{b_0}\right) e^{Ct_0} + 1. \tag{3.55}$$

Next, we set

$$B(f_{t_0}, 1) = \left\{f \in L_1^1\left(\mathbb{R}^3\right), ||f - f_{t_0}|| \leq 1\right\}.$$

Then :

$$f \in B(f_{t_0}, 1) \Longrightarrow ||f|| \leq ||f_{t_0}|| + 1. \tag{3.56}$$

Now consider the neighborhood

$$V_{t_0} =]t_0 - \delta_0, t_0 + \delta_0[\times \mathbb{R}^3 \times \mathbb{R}^3 \times B(f_{t_0}, 1)$$

of $\left(t_0, \overline{p_{t_0}}, \overline{E_{t_0}}, f_{t_0}\right)$ in the Banach space $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times L_1^1\left(\mathbb{R}^3\right)$ and take

$$\left(t, \overline{p_1}, \overline{E_1}, f_1\right), \left(t, \overline{p_2}, \overline{E_2}, f_2\right) \in V_{t_0}. \tag{3.57}$$

We deduce from the inequality (3.37), the definition (3.38) of C_2, C_3, C_4, and C_5, the relation (3.55) for $z = a, b, \frac{1}{a}, \frac{1}{b}$, the relation (3.56) which implies since $f_j \in B\left(f_{t_0}, 1\right), \;\; j = 1,2 \quad ||f_j|| \leq ||f_{t_0}|| + 1$, that there exists a constant

$$C_6 = C_6\left(a_0, b_0, t_0, f_{t_0}, \varphi_{ij}, ||\overline{E_1}||\right)$$

such that :

$$\begin{aligned} &\left|\left|H\left(t, \overline{p_1}, \overline{E_1}, f_1\right) - H\left(t, \overline{p_2}, \overline{E_2}, f_2\right)\right|\right| \\ &\qquad\qquad \leq C_6\left(\left|\left|\overline{E_1} - \overline{E_2}\right|\right|_{\mathbb{R}^3} + ||\overline{p_1} - \overline{p_2}||_{\mathbb{R}^3} + ||f_1 - f_2||\right). \end{aligned}$$

This shows that H is locally Lipschitz in $\left(\overline{p}, \overline{E}, f\right)$ with respect to the norm of the Banach space $\mathbb{R}^3 \times \mathbb{R}^3 \times L^1_1\left(\mathbb{R}^3\right)$.

The existence of a unique solution $\left(\overline{p}, \overline{E}, f\right)$ of the differential system (3.6)-(3.8) on an interval $[t_0; t_0 + \delta]$, $\delta > 0$, such that $\left(\overline{p}, \overline{E}, f\right)(t_0) = \left(\overline{p_{t_0}}, \overline{E_{t_0}}, f_{t_0}\right)$ is then guaranteed by the Cauchy- Lipschitz theorem on first order differential systems.

2) The relation (3.52) is established in [31] , which studied the single Boltzmann equation.

3) The existence of a local solution (F, f) of the Maxwell-Boltzmann system is a direct consequence of the first part of the proof, the equivalence between the Boltzmann equation (3.2) and the differential system 3.6-(3.8) and given (3.9). This completes the proof of theorem 3.1. □

We end this section by the following useful result, which is an immediate consequence of theorem 3.1 for $t_0 = 0$.

Theorem 3.2. *Let $\overline{p_0}, \overline{E_0} \in \mathbb{R}^3$, $f_0 \in L^1_1\left(\mathbb{R}^3\right)$,$\varphi_{ij} \in \mathbb{R}$ be given.*

Then there exists a real number $T > 0$ such that :

1) The differential system (3.6)-*(3.9) has a unique solution*

$$\left(\overline{p}, \overline{E}, f\right) \in C\left([0;T]; \mathbb{R}^3\right)^2 \times C\left([0;T]; L^1_1\left(\mathbb{R}^3\right)\right)$$

such that

$$\left(\overline{p}, \overline{E}, f\right)(0) = \left(\overline{p_0}, \overline{E_0}, f_0\right).$$

2) Moreover f satisfies

$$|||f||| \leq ||f_0||. \tag{3.58}$$

3) The Maxwell- Boltzmann system (2.23)-(2.24)-(3.2) has a unique solution (F, f) satisfying

$$F^{i0}(0) = E^i_0, \;\; F_{ij}(0) = \varphi_{ij}, \;\; f(0) = f_0.$$

3.2 Global existence theorem for the Boltzmann equation for $\mu - N$ regularity

3.2.1 The strategy of proof of global existence theorem

Denote by $[0,T[$ the maximal existence domain of solution of the system (3.6)-(3.9) denoted here by $\left(\widetilde{\overline{p}}, \widetilde{\overline{E}}, \widetilde{f}\right)$ and given by theorem 3.2 with the initial data

$$\left(\overline{p_0}, \overline{E_0}, f_0\right) \in C\left([0;T]\,;\mathbb{R}^3\right)^2 \times C\left([0;T]\,;L^1_1\left(\mathbb{R}^3\right)\right).$$

We want to prove that $T = +\infty$.

1. If we already have $T = +\infty$, then the problem of global existence is solved.

2. If we suppose $0 < T < +\infty$, then we prove that the solution $\left(\widetilde{\overline{p}}, \widetilde{\overline{E}}, \widetilde{f}\right)$ can be extended beyond T, which contradicts the maximality of T.

3. The strategy is as follows :
Suppose $0 < T < +\infty$ and let $t_0 \in [0,T[$. We will show that there exists a strictly positive number $\tau > 0$ independent on t_0 such that the system (3.6)-(3.9) has a unique solution $\left(\overline{p}, \overline{E}, f\right)$ on $[t_0, t_0+\tau]$, with the initial data $\left(\widetilde{\overline{p_0}}, \widetilde{\overline{E_0}}, \widetilde{f_0}\right)$ at $t = t_0$. Then taking t_0 sufficiently close to T , for example, to such that $0 < T - t_0 < \frac{\tau}{2}$ and hence $T < t_0 + \frac{\tau}{2}$, we can extend the solution $\left(\widetilde{\overline{p_0}}, \widetilde{\overline{E_0}}, \widetilde{f_0}\right)$ to $[0, t_0+\frac{\tau}{2}]$, which strictly contains $[0,T[$, and this contradicts the maximality of T. In order to simplify the notation, it will be sufficiently enough if we could look for a number τ such that $0 < \tau < 1$.

4. In what follows we fix a number $r > 0$ and we take f_0 such that

$$||f_0|| \leq r.$$

By (3.58) we have :

$$\left|\left|\widetilde{f_0}\right|\right| \leq ||f_0||\,. \tag{3.59}$$

We deduce from (3.59), using (3.52) that, any solution f of the Boltzmann equation on $[t_0, t_0+\tau]$ such that $f(t_0) = \widetilde{f}\,(t_0)$, satisfies :

$$||f\,(t)|| \leq r, t \in [t_0, t_0+\tau]. \tag{3.60}$$

Notice that (3.60) shows that a solution $\left(\overline{p}, \overline{E}, f\right)$ of system (3.6)-(3.9) on

$[t_0, t_0 + \tau]$,$\tau > 0$, such that

$$\left(\overline{p}, \overline{E}, f\right)(t_0) = \left(\widetilde{\overline{p}}(t_0), \widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right)$$

satisfies :

$$\left(\overline{p}, \overline{E}, f\right) \in C\left([t_0; t_0 + \tau]; \mathbb{R}^3\right)^2 \times C\left([t_0; t_0 + \tau]; X_r\right) \tag{3.61}$$

where X_r is defined by (2.89), $C([t_0; t_0 + \tau]; X_r)$ by (2.91), with $I = [t_0; t_0 + \tau]$.

In what follows, $[0, T[$, $T > 0$, is the maximal existence domain of the solution $\left(\widetilde{\overline{p}}, \widetilde{\overline{E}}, \widetilde{f}\right)$ of (3.6)-(3.9) such that

$$\left(\widetilde{\overline{p}}, \widetilde{\overline{E}}, \widetilde{f}\right)(0) = \left(\widetilde{\overline{p_0}}, \widetilde{\overline{E_0}}, \widetilde{f_0}\right) \in \mathbb{R}^3 \times \mathbb{R}^3 \times L_1^1\left(\mathbb{R}^3\right), ||f_0|| \leq r.$$

3.2.2 Preliminary results

We prove the following result which will be useful in what follows.

Lemma 3.1. *The maps* $t \longmapsto \widetilde{\overline{E}}(t), t \longmapsto \widetilde{\overline{p}}(t)$ *are uniformly bounded over* $[0, T[$.

Proof. 1) Let $t \in [0, T[$.
Let us show that the map $t \longmapsto \widetilde{\overline{E}}(t)$ is uniformly bounded i.e there exists $k_1 > 0$ such that $\forall t \in [0, T[$, we have

$$\left|\widetilde{\overline{E}}(t)\right| \leq k_1.$$

Consider (3.6) in which we set $\overline{E} = \widetilde{\overline{E}}$ and $f = \widetilde{f}$ i.e, the inequality

$$\dot{\widetilde{E^i}} = -\Gamma^i_{0i}\widetilde{E^i} + \int_{\mathbb{R}^3} \frac{q^i \widetilde{f}(t, \overline{q})\, ab^2 d\overline{q}}{q^0} \tag{3.62}$$

on $[0, T[$.

We have, using (2.8)

$$\Gamma^1_{01} = \frac{\dot{a}}{a}, \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{b}}{b},$$

then (2.80) implies that

$$\left|\Gamma^i_{0i}\right| \leq C. \tag{3.63}$$

Since

$$\frac{|q^1|}{q^0} \leq \frac{1}{a}, \quad \frac{|q^2|}{q^0} \leq \frac{1}{b}, \quad \frac{|q^3|}{q^0} \leq \frac{1}{b},$$

using (2.81) by which

$$z = a, b, \frac{1}{a}, \frac{1}{b},$$

we have

$$\left|\int_{\mathbb{R}^3} \frac{q^i}{q^0}\widetilde{f}(t,\overline{q})\,ab^2 d\overline{q}\right| \leq C_{6,}^i \tag{3.64}$$

where

$$C_6^i = C_6^i(a_0, b_0, r, T).$$

We then deduce from(3.62),(3.63) and (3.64) that

$$\left|\dot{\widetilde{E^i}}\right| \leq C\left|\widetilde{E^i}\right| + C_6^i. \tag{3.65}$$

Now integrating (3.65) over $[0,t]$, yields, using $\widetilde{E^i}(0) = E_0^i$

$$\left|\widetilde{E^i}\right| \leq \left(\left|E_0^i\right| + C_6^i T\right) + C\int_0^t \left|\widetilde{E^i}\right|(s)\,ds, \;\; t \in [0,T[, \;\; i = 1,2,3 \tag{3.66}$$

applying Gronwall lemma to (3.66) we obtain

$$\left|\widetilde{E^i}(t)\right| \leq \left(\left|E_0^i\right| + C_6^i T\right)e^{Ct}, \;\; t \in [0,T[, \;\; i = 1,2,3. \tag{3.67}$$

Then the map $t \longmapsto \widetilde{\overline{E}}(t)$ is uniformly bounded.

2) Consider (3.7) in which we set $\overline{p} = \widetilde{\overline{p}}, \overline{E} = \widetilde{\overline{E}}$ and $f = \widetilde{f}$ i.e the equality :

$$\dot{\widetilde{p^i}} = -2\Gamma_{0i}^i \widetilde{p^i} - \left[\widetilde{E^i} + g^{ii}\frac{\widetilde{p^k}\varphi_{ik}}{p^0}\right]\int_{\mathbb{R}^3} \widetilde{f}(t,\overline{q})\,ab^2 d\overline{q}. \tag{3.68}$$

We already have

$$\left|\Gamma_{0i}^i\right| \leq C. \tag{3.69}$$

Since

$$ab^2 g^{11} = \frac{b^2}{a}, \;\; ab^2 g^{22} = ab^2 g^{33} = a,$$

$$\frac{\left|\widetilde{p^1}\right|}{p^0} \leq \frac{1}{a}, \;\; \frac{\left|\widetilde{p^2}\right|}{p^0} \leq \frac{1}{b}, \;\; \frac{\left|\widetilde{p^3}\right|}{p^0} \leq \frac{1}{b},$$

and using (2.80) by which

$$|z(t)| \leq \left(a_0 + b_0 + \frac{1}{a_0} + \frac{1}{b_0}\right)e^{CT}, \forall t \in [0,T[,$$

for

$$z = a, b, \frac{1}{a}, \frac{1}{b},$$

we have :

$$\left|\left[\widetilde{E^i} + g^{ii}\frac{\widetilde{p^k}\varphi_{ik}}{p^0}\right]\int_{\mathbb{R}^3}\widetilde{f}\left(t,\overline{q}\right)ab^2 d\overline{q}\right| \leq C_7^i \tag{3.70}$$

where

$$C_7^i = C_7^i\left(a_0, b_0, r, T, \sum_{i,k}\left|\varphi_{ik}\right|, \left|E_0^i\right|\right).$$

We deduce from (3.68) , (3.69) and (3.70) that :

$$\left|\dot{\widetilde{p^i}}\right| \leq 2C\left|\widetilde{p^i}\right| + C_7^i, \ \ i = 1,2,3. \tag{3.71}$$

Now integrating (3.71) over $[0,t]$, $t \in [0,T[$, yields, using $\widetilde{p^i}\left(0\right) = p_0^i$:

$$\left|\widetilde{p^i}\left(t\right)\right| \leq \left(\left|p_0^i\right| + C_7^i T\right) + 2C\int_0^t \left|\widetilde{p^i}\right|\left(s\right)ds, \ i = 1,2,3. \tag{3.72}$$

Applying Gronwall lemma 1.1 to (3.71) we obtain

$$\left|\widetilde{p^i}\left(t\right)\right| \leq K\left(\left|p_0^i\right| + C_7^i T\right)e^{2Ct}, \ K \in \mathbb{R}, \ t \in [0,T[, \ i = 1,2,3. \tag{3.73}$$

This completes the proof of Lemma 3.1. □

3.2.3 Global existence of solutions for $\mu - N$ regularity

We first consider, for $t_0 \in [0,T[$ and $\tau > 0$,

$$\left(\overline{\overline{p}}, \overline{\overline{E}}, \overline{f}\right) \in C\left(\left[t_0; t_0+\tau\right]; \mathbb{R}^3\right)^2 \times C\left(\left[t_0; t_0+\tau\right]; X_r\right).$$

We built from system (3.6)-(3.8) by setting in right hand side $H = (H_1, H_2, H_3)$ which is given by (3.31)-(3.32) :

$\bar{p} = \overline{\overline{p}}, f = \overline{f}$ in H_1, $\overline{E} = \overline{\overline{E}}, f = \overline{f}$ in H_2 and $\overline{E} = \overline{\overline{E}}, \bar{p} = \overline{\overline{p}}$ in H_3; the following differential system :

$$\frac{dE^i}{dt} = \overline{H_1}\left(t, \overline{\overline{p}}, \overline{E}, \overline{f}\right) \tag{3.74}$$

$$\frac{dp^i}{dt} = \overline{H_2}\left(t, \overline{p}, \overline{\overline{E}}, \overline{f}\right) \tag{3.75}$$

$$\frac{df}{dt} = \overline{H_3}\left(t, \overline{\overline{p}}, \overline{\overline{E}}, f\right) \tag{3.76}$$

where

$$\overline{H_1}\left(t,\overline{\overline{p}},\overline{E},\overline{f}\right) = -\Gamma^i_{0j}E^j + \int_{\mathbb{R}^3} \frac{q^i\overline{f}\left(t,\overline{q}\right)ab^2 d\overline{q}}{q^0} \tag{3.77}$$

$$\overline{H_2}\left(t,\overline{P},\overline{\overline{E}},\overline{f}\right) = -2\Gamma^i_{0j}p^j - \left[\overline{E^i} + g^{ii}\frac{p^k\varphi_{ik}}{p^0}\right]\int_{\mathbb{R}^3} \overline{f}\left(t,\overline{q}\right)ab^2 d\overline{q} \tag{3.78}$$

$$\overline{H_3}\left(t,\overline{\overline{P}},\overline{\overline{E}},f\right) = \frac{1}{p^0\left(\overline{\overline{p}}\right)}\overline{Q}\left(f,f,\overline{\overline{p}}\right) \quad i=1,2,3. \tag{3.79}$$

We prove :

Proposition 3.4. *Let* $t_0 \in [0,T[, \tau \in]0,1[$, *and* $\left(\overline{\overline{p}},\overline{\overline{E}},\overline{f}\right) \in C\left([t_0;t_0+\tau];\mathbb{R}^3\right)^2 \times C\left([t_0;t_0+\tau];X_r\right)$ *be given.*

Then the differential system (3.74)-(3.76) has a unique solution $\left(\overline{p},\overline{E},f\right) \in C\left([t_0;t_0+\tau];\mathbb{R}^3\right)^2 \times C\left([t_0;t_0+\tau];X_r\right)$ *such that :*

$$\left(\overline{p},\overline{E},f\right)(t_0) = \left(\widetilde{\overline{p}}(t_0),\widetilde{\overline{E}}(t_0),\widetilde{f}(t_0)\right).$$

Proof. - Firstly we consider equation (3.74) in $\overline{E}$, with $\overline{H_1}$ defined by (3.77) in which $\overline{\overline{p}}$ and $\overline{f}$ are fixed.
Since a , b , $\dot{a}$, $\dot{b}$, $\frac{1}{a}$, $\frac{1}{b}$, $\overline{f}$ are continuous functions of t so is $\overline{H_1}$.
Next, we deduce from (3.34) in which we set

$$f_1 = f_2 = \overline{f}$$

that :

$$\left|\left|\overline{H_1}\left(t,\overline{\overline{p}},\overline{E_1},\overline{f}\right) - \overline{H_1}\left(t,\overline{\overline{p}},\overline{E_2},\overline{f}\right)\right|\right|_{\mathbb{R}^3} \leq C_2\left(\left|\left|\overline{E_1}-\overline{E_2}\right|\right|_{\mathbb{R}^3}\right) \tag{3.80}$$

where

$$C_2 = 3C + b^2. \tag{3.81}$$

Now we can use (2.81) to bound

$$z = a, b, \frac{1}{a}, \frac{1}{b}$$

and we obtain, for $t \in [t_0;t_0+\tau]$ then $t \leq t_0+\tau \leq T+1$

$$|z(t)| \leq \left(a_0 + b_0 + \frac{1}{a_0} + \frac{1}{b_0}\right)e^{C(T+1)}, t \in [t_0;t_0+\tau], z = a, b, \frac{1}{a}, \frac{1}{b}. \tag{3.82}$$

We then deduce from (3.81) that :

$$C_2 \leq C_2'$$

where

$$C_2' = C_2'(a_0, b_0, T). \tag{3.83}$$

By (3.80) and (3.83), $\overline{H_1}$ is (globally) Lipschitz with respect to the $\mathbb{R}^3-$ norm and the local existence of a solution $\overline{E}$ of (3.74) such that $\overline{E}(t_0) = \widetilde{\overline{E}}(t_0)$ is guaranteed by the Cauchy-Lipschitz theorem on first order differential systems.

Now, since $\overline{E}$ satisfies (3.74) in which $\overline{H_1}$ is given by (3.77), following the same way as in the proof of Lemma 3.1, substituting $\overline{E}$ to $\widetilde{\overline{E}}$, $\overline{p}$ to $\widetilde{\overline{p}}$,$\overline{f}$ to $\widetilde{f}$, using (3.82) and integrating (3.74) this time over $[t_0; t_0 + t]\,, t \in [0; \tau]\,,$ we obtain :

$$\left|E^i(t_0+t)\right| \leq \left(\left|\widetilde{E^i}(t_0)\right| + C_8^i T\right) + C\int_{t_0}^{t_0+t}\left|E^i\right|(s)\,ds,$$
$$t \in [0,\tau[,\ \ i = 1,2,3$$

where

$$C_8^i = C_8^i(a_0, b_0, r, T).$$

But by Lemma 3.1 and more precisely (3.67) , we have since $t_0 \in [0, \tau[$:

$$\left|\widetilde{E^i}(t)\right| \leq \left(\left|E_0^i\right| + C_6^i T\right)e^{Ct}.$$

But, by the Gronwall Lemma

$$\left|E^i(t_0+t)\right| \leq \left[C_8^i T + \left(\left|E_0^i\right| + C_6^i T\right)e^{Ct}\right]e^{C(T+1)},$$

$$t \in [0,\tau[, \qquad i = 1,2,3$$

which shows that, every solution $\overline{E}$ of (3.74) satisfying $\overline{E}(t_0) = \widetilde{\overline{E}}(t_0)$ and defined in $[t_0; t_0 + \tau]$ is uniformly bounded. By the Cauchy-Lipschitz theorem, the solution $\overline{E}$ is defined all over $[t_0; t_0 + \tau[$ and $\overline{E} \in C([t_0; t_0 + \tau[; \mathbb{R}^3)$.

- Secondly, we consider equation (3.75) in $\overline{p}$,with $\overline{H_2}$ defined by (3.78) in which $\overline{\overline{E}}, \overline{f}$ are fixed.

Since a, b, $\dot{a}$, $\dot{b}$, $\frac{1}{a}$, $\frac{1}{b}$, $\overline{f}, \overline{\overline{E}}$ are continuous functions of t, so is $\overline{H_2}$. Next , we deduce from (3.35) in which we set

$$\overline{E_1} = \overline{E_2} = \overline{\overline{E}}, \qquad f = f_2 = f$$

that :

$$\left|\left|H_2\left(t, \overline{p_1}, \overline{\overline{E}}, f\right) - H_2\left(t, \overline{p_2}, \overline{\overline{E}}, f\right)\right|\right|_{\mathbb{R}^3} \leq C_3\left|\left|\overline{p_1} - \overline{p_2}\right|\right|_{\mathbb{R}^3} \tag{3.84}$$

where using (3.38) :

$$C_3 = (6C+1)\left(1+a+\frac{b^2}{a}\right)\left(1+\frac{a}{b}+\frac{b}{a}+\frac{1}{a}+\frac{1}{b}\right)$$
$$\times(1+\|f\|)\left(1+ab^2\right)\left(1+\|f\|+\left\|\overline{\overline{E}}\right\|\right). \quad (3.85)$$

Now we can use (2.81) to bound

$$z = a, b, \frac{1}{a}, \frac{1}{b}$$

and we obtain, for $t \in [t_0; t_0+\tau]$, then $t \leq t_0+\tau \leq T+1$:

$$|z(t)| \leq \left(a_0+b_0+\frac{1}{a_0}+\frac{1}{b_0}\right)e^{C(T+1)},$$
$$t \in [t_0; t_0+\tau], z = a, b, \frac{1}{a}, \frac{1}{b}. \quad (3.86)$$

We then deduce from (3.85) using

$$\|\overline{f}\| \leq |||\overline{f}||| \leq r$$

since

$$\overline{f} \in ([t_0; t_0+\tau]; X_r),$$

that

$$C_3 \leq C_3'$$

where

$$C_3' = C_3'\left(a_0, b_0, r, \left|E^i\right|, \varphi_{ij}\right). \quad (3.87)$$

By (3.84) and (3.87), $\overline{H_2}$ is (globally) Lipschitz with respect to the $\mathbb{R}^3-$ norm and the local existence of a solution $\overline{p}$ of (3.75) such that $\overline{p}(t_0) = \widetilde{\overline{p}}(t_0)$ is guaranteed by the Cauchy-Lipschitz theorem on first order differential systems. □

Now, since $\overline{p}$ satisfies (3.75) in which $\overline{H_2}$ is given by (3.78), following the same way as in the proof of Lemma 3.1, substituting $\overline{p}$ to $\widetilde{\overline{p}}$, $\overline{\overline{E}}$ to $\widetilde{\overline{E}}$, $\overline{f}$ to $\widetilde{f}$, using (3.86) and integrating this time over $[t_0; t_0+t], t \in [0, \tau[$, we are let to:

$$\left|\widetilde{p^i}(t_0+t)\right| \leq \left(\left|\widetilde{p^i}(t_0)\right| + C_9^i T\right) + 2C\int_0^t \left|\widetilde{p^i}\right|(s)\,ds,$$
$$t \in [0, \tau[, \qquad i = 1, 2, 3$$

where

$$C_9^i = C_9^i \left(a_0, b_0, r, T, \left|E^i\right|, \sum_{i,k} |\varphi_{ik}| \right).$$

But by Lemma 3.1 and more precisely (3.73), we have, since $t_0 \in [0, T[$

$$\left|\widetilde{p^i}\,(t)\right| \leq \left(\left|p_0^i\right| + C_7^i T\right) e^{2Ct},$$

Then by Lemma 1.1

$$\left|p^i\,(t_0 + t)\right| \leq \left[C_9^i T + \left(\left|p_0^i\right| + C_7^i T\right) e^{2Ct}\right] e^{C(T+1)}$$

$$t \in [0, \tau[, \qquad i = 1, 2, 3,$$

which shows that , every solution $\overline{p}$ of (3.75) satisfying $\overline{p}(t_0) = \widetilde{\overline{p}}\,(t_0)$ and defined in $[t_0; t_0 + \tau]$ is uniformly bounded. By the Cauchy-Lipschitz theorem of first order differential systems, the solution $\overline{p}$ of (3.75) is defined all over $[t_0; t_0{+}\tau[$ and $\overline{p} \in C\left([t_0; t_0 + \tau[; \mathbb{R}^3\right).$

Finally, under same assumption it is proved in [31] that the single equation (3.76) in f, has a unique solution $f \in C\left([t_0; t_0 + \tau[; X_r\right)$ such that $f\,(t_0) = \widetilde{f}\,(t_0)$. This completes the proof of proposition 3.4

In what follows we set

$$Y_\tau = C\left([t_0; t_0 + \tau]\,; \mathbb{R}^3\right)^2 \times C\left([t_0; t_0 + \tau]\,; X_r\right), \; r \in \mathbb{R},\; r > 0. \tag{3.88}$$

Y_τ is a complete metric subspace of the Banach space

$$C\left([t_0; t_0 + \tau]\,; \mathbb{R}^3\right)^2 \times C\left([t_0; t_0 + \tau]\,; L_1^1\left(\mathbb{R}^3\right)\right)$$

Proposition 3.4 allows us to define the map

$$g : Y_\tau \longrightarrow Y_\tau, \quad \left(\overline{\overline{p}}, \overline{\overline{E}}, \overline{f}\right) \longmapsto \left(\overline{p}, \overline{E}, f\right). \tag{3.89}$$

We now prove

Proposition 3.5. *Let $t_0 \in [0, T[$.*

There exists a number $\tau \in]0, 1[$, independent of t_0, such that the system (3.6)-(3.8) has a unique solution $\left(\overline{p}, \overline{E}, f\right) \in Y_\tau$ satisfying

$$\left(\overline{p}, \overline{E}, f\right)(t_0) = \left(\widetilde{\overline{p}}\,(t_0)\,, \widetilde{\overline{E}}\,(t_0)\,, \widetilde{f}\,(t_0)\right).$$

Proof. We will prove that there exists a number $\tau \in]0, 1[$, independent of t_0, such that the map g, defined by (3.89) is a contraction of the complete metric space Y_τ defined by (3.88), which will then have a unique fixed point $\left(\overline{p}, \overline{E}, f\right)$ solution of system (3.58)-(3.8).

With the initial data $\left(\widetilde{\overline{p}}(t_0), \widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right)$ at $t = t_0$, the differential system (3.74)-(3.76) with $\overline{H_1}, \overline{H_2}, \overline{H_3}$ is equivalent to the integral system :

$$E^i(t_0+t) = \widetilde{E^i}(t_0) + \int_{t_0}^{t_0+t} \left\{ -\Gamma^i_{0j} E^j + \int_{\mathbb{R}^3} \frac{q^i \overline{f}(t, \overline{q})\, ab^2 d\overline{q}}{q^0} \right\}(\delta)\, d\delta \qquad (3.90)$$

$$p^i(t_0+t) = \widetilde{p^i}(t_0) + \int_{t_0}^{t_0+t} \left\{ -2\Gamma^i_{0j} p^j - \left[\overline{E^i} + g^{ii} \frac{p^k \varphi_{ik}}{p^0} \right] \int_{\mathbb{R}^3} \overline{f}(t, \overline{q})\, ab^2 d\overline{q} \right\}(\delta)\, d\delta \quad (3.91)$$

$$f(t_0+t) = f(t_0) + \int_{t_0}^{t_0+t} \frac{1}{p^0\left(\overline{\overline{p}}\right)} \overline{Q}\left(f, f, \overline{\overline{p}}\right) \qquad (3.92)$$

$$t \in]0, \tau[, \ \ i = 1, 2, 3.$$

To $\left(\overline{\overline{p_l}}, \overline{\overline{E_l}}, \overline{f_l}\right) \in Y_\tau, l = 1, 2$ corresponds the solution $\left(\overline{p_l}, \overline{E_l}, f_l\right) \in Y_\tau, l = 1, 2$, whose existence is proved in proposition 3.4 . We now write the integral system (3.90)-(3.92) for $l = 1$ and $l = 2$, and taking the differences , using notations (3.74)-(3.76), we get:

$$\left(E^i_1 - E^i_2\right)(t_0+t) = \int_{t_0}^{t_0+t} \overline{f_2}(t) \left[\overline{H_1}\left(\delta, \overline{\overline{p_1}}, \overline{\overline{E_1}}, \overline{f_1}\right) - \overline{H_1}\left(\delta, \overline{\overline{p_2}}, \overline{\overline{E_2}}, \overline{f_2}\right) \right](\delta)\, d\delta \quad (3.93)$$

$$\left(p^i_1 - p^i_2\right)(t_0+t) = \int_{t_0}^{t_0+t} \left[\overline{H_2}\left(\delta, \overline{\overline{p_1}}, \overline{\overline{E_1}}, \overline{f_1}\right) - \overline{H_2}\left(\delta, \overline{\overline{p_2}}, \overline{\overline{E_2}}, \overline{f_2}\right) \right](\delta)\, d\delta \quad (3.94)$$

$$(f_1 - f_2)(t_0+t) = \int_{t_0}^{t_0+t} \left[\overline{H_3}\left(\delta, \overline{\overline{p_1}}, \overline{\overline{E_1}}, \overline{f_1}\right) - \overline{H_3}\left(\delta, \overline{\overline{p_2}}, \overline{\overline{E_2}}, \overline{f_2}\right) \right](\delta)\, d\delta. \quad (3.95)$$

- Since $\left(\overline{\overline{p_2}}, \overline{\overline{E_2}}, \overline{f_2}\right) \in Y_\tau$ we deduce from (3.34) in which we set

$$\overline{p_1} = \overline{\overline{p_1}}, \overline{p_2} = \overline{\overline{p_2}}, f_1 = \overline{f_1}, f_2 = \overline{f_2} :$$

$$\begin{aligned} & \left|\left|\overline{H_1}\left(\delta, \overline{\overline{p_1}}, \overline{E_1}, \overline{f_1}\right) - \overline{H_1}\left(\delta, \overline{\overline{p_2}}, \overline{E_2}, \overline{f_2}\right)\right|\right| \\ & \qquad \leq C_2'\left(\left|\left|\overline{\overline{p_1}} - \overline{\overline{p_2}}\right|\right| + \left|\left|\overline{E_1} - \overline{E_2}\right|\right| + \left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right) \end{aligned} \quad (3.96)$$

where $C_2' = C_2'\left(a_0, b_0, T\right)$ is still given by (3.83).

- Since $\left(\overline{\overline{p_2}}, \overline{\overline{E_2}}, \overline{f_2}\right) \in Y_\tau$, we have in (3.58) in which we set $\overline{f} = f_2$:

$$\left(\overline{p}, \overline{E}, f\right)(t_0) = \left(\widetilde{\overline{p}}(t_0), \widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right) :$$

$$\left|\left|\overline{f_2}(t)\right|\right| \leq \left|\left|\left|\overline{f_2}(t)\right|\right|\right| \leq r, t \in [t_0; t_0 + \tau].$$

So we can deduce from (3.35) in which we set

$$\overline{E_1} = \overline{\overline{E_1}}, \overline{E_2} = \overline{\overline{E_2}}, f_1 = \overline{f_1}, f_2 = \overline{f_2} :$$

$$\begin{aligned} & \left|\left|\overline{H_2}\left(\delta, \overline{p_1}, \overline{\overline{E_1}}, \overline{f_1}\right) - \overline{H_2}\left(\delta, \overline{p_2}, \overline{\overline{E_2}}, \overline{f_2}\right)\right|\right| \\ & \qquad \leq C_3'\left(\left|\left|\overline{p_1} - \overline{p_2}\right|\right| + \left|\left|\overline{\overline{E_1}} - \overline{\overline{E_2}}\right|\right| + \left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right) \end{aligned} \quad (3.97)$$

where

$$C_3' = C_3'\left(a_0, b_0, r, T, \left|E^i\right|, \varphi_{ij}\right)$$

is still given by (3.87).

- Since $\left(\overline{p_l}, \overline{E_l}, f_l\right) \in Y_\tau$, we deduce from (3.36) in which we set

$$\overline{p_1} = \overline{\overline{p_1}}, \overline{p_2} = \overline{\overline{p_2}}, \overline{E_1} = \overline{\overline{E_1}}, \overline{E_2} = \overline{\overline{E_2}}$$

and using in C_4 given in (3.38)

$$\left|\left|\overline{f_l}(t)\right|\right| \leq \left|\left|\left|\overline{f_l}(t)\right|\right|\right| \leq r,$$

since $\left(\overline{p_l}, \overline{E_l}, f_l\right) \in Y_\tau \ , l = 1, 2$:

$$\left|\left|\overline{H_3}\left(\delta, \overline{p_1}, \overline{\overline{E_1}}, \overline{f_1}\right) - \overline{H_3}\left(\delta, \overline{p_2}, \overline{\overline{E_2}}, \overline{f_2}\right)\right|\right| \\ \leq C_4'\left(\left|\left|\overline{\overline{p_1}} - \overline{\overline{p_2}}\right|\right| + \left|\left|\overline{\overline{E_1}} - \overline{\overline{E_2}}\right|\right| + \left|\left|f_1 - f_2\right|\right|\right) \quad (3.98)$$

where

$$C_4' = C_4'\left(a_0, b_0, r, T\right).$$

Already notice that the constants C_2', C_3' and C_4' are independent of t_0.

Now using the inequalities (3.96),(3.97) and (3.98), we deduce from (3.93),(3.94) and (3.95),

using the norm $|||.|||$ and since $t \in [0, \tau]$:

$$\left|\left|\left|\overline{E_1} - \overline{E_2}\right|\right|\right| \leq C_2'\tau\left(\left|\left|\left|\overline{E_1} - \overline{E_2}\right|\right|\right| + \left|\left|\left|\overline{\overline{p_1}} - \overline{\overline{p_2}}\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right|\right) \quad (3.99)$$

$$\left|\left|\left|\overline{p_1} - \overline{p_2}\right|\right|\right| \leq C_3'\tau\left(\left|\left|\left|\overline{\overline{E_1}} - \overline{\overline{E_2}}\right|\right|\right| + \left|\left|\left|\overline{p_1} - \overline{p_2}\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right|\right) \quad (3.100)$$

$$\left|\left|\left|f_1 - f_2\right|\right|\right| \leq C_4'\tau\left(\left|\left|\left|\overline{\overline{E_1}} - \overline{\overline{E_2}}\right|\right|\right| + \left|\left|\left|\overline{\overline{p_1}} - \overline{\overline{p_2}}\right|\right|\right| + \left|\left|\left|f_1 - f_2\right|\right|\right|\right). \quad (3.101)$$

Now add (3.99),(3.100) and (3.101) to obtain

$$\left|\left|\left|\overline{p_1} - \overline{p_2}\right|\right|\right| + \left|\left|\left|\overline{E_1} - \overline{E_2}\right|\right|\right| + \left|\left|\left|f_1 - f_2\right|\right|\right|$$

$$\leq \left(C_2' + C_3' + C_4'\right)\tau\left(\left|\left|\left|\overline{p_1} - \overline{p_2}\right|\right|\right| + \left|\left|\left|\overline{E_1} - \overline{E_2}\right|\right|\right| + \left|\left|\left|f_1 - f_2\right|\right|\right|\right)$$

$$+ \left(C_2' + C_3' + C_4'\right)\tau\left(\left|\left|\left|\overline{\overline{p_1}} - \overline{\overline{p_2}}\right|\right|\right| + \left|\left|\left|\overline{\overline{E_1}} - \overline{\overline{E_2}}\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right|\right). \quad (3.102)$$

Then if we take τ such that :

$$0 < \tau < \inf\left\{1, \frac{1}{4\left(C_2' + C_3' + C_4'\right)}\right\} \quad (3.103)$$

(3.103) implies in particular

$$0 < \left(C_2' + C_3' + C_4'\right)\tau < \frac{1}{4}$$

from which we deduce, by sending the first term of right hand side of (3.102) to the left hand side

$$\frac{3}{4}\left(\left|\left|\left|\overline{p_1} - \overline{p_2}\right|\right|\right| + \left|\left|\left|\overline{E_1} - \overline{E_2}\right|\right|\right| + \left|\left|\left|f_1 - f_2\right|\right|\right|\right)$$

$$\leq \frac{1}{4}\left(\left|\left|\left|\overline{\overline{p_1}} - \overline{\overline{p_2}}\right|\right|\right| + \left|\left|\left|\overline{\overline{E_1}} - \overline{\overline{E_2}}\right|\right|\right| + \left|\left|\left|\overline{f_1} - \overline{f_2}\right|\right|\right|\right) \quad (3.104)$$

and (3.104) gives :

$$|||\overline{p_1} - \overline{p_2}||| + \left|\left|\left|\overline{E_1} - \overline{E_2}\right|\right|\right| + |||f_1 - f_2|||$$

$$\leq \frac{1}{3}\left(|||\overline{\overline{p_1}} - \overline{\overline{p_2}}||| + \left|\left|\left|\overline{\overline{E_1}} - \overline{\overline{E_2}}\right|\right|\right| + |||\overline{f_1} - \overline{f_2}|||\right). \tag{3.105}$$

(3.105) shows that

$$g : \left(\overline{\overline{p}}, \overline{\overline{E}}, \overline{f}\right) \longmapsto \left(\overline{p}, \overline{E}, f\right)$$

is a contracting map in the complete metric space Y_τ which then has a unique fixed point $\left(\overline{p}, \overline{E}, f\right)$, solution of the integral system (3.90)-(3.92) and hence, of the differential system (3.6)-(3.8) such that

$$\left(\overline{p}, \overline{E}, f\right)(t_0) = \left(\widetilde{\overline{p}}(t_0), \widetilde{\overline{E}}(t_0), \widetilde{f}(t_0)\right).$$

This completes the proof of proposition 3.5. □

Based on the method detailed in section 3.3.1, we have proved the following result :

Theorem 3.3. *Let* $\overline{p_0} \in \mathbb{R}^3, \overline{E_0} \in \mathbb{R}^3, f_0 \in L^1_1(\mathbb{R}^3), \varphi_{ij} \in \mathbb{R}, i, j = 1, 2, 3$ *be given, such that* $||f_0|| \leq r$, *where* $r > 0$ *is a given real number. Then :*

1) The differential system (3.6)-*(3.8) has a unique global solution* $\left(\overline{p}, \overline{E}, f\right)$ *defined all over* $[0, +\infty[$ *and such that*

$$\left(\overline{p}, \overline{E}, f\right)(0) = \left(\overline{p_0}, \overline{E_0}, f_0\right)$$

and

$$|||f||| \leq ||f_0||, \; f(t) \geq 0, \; t \in [0, +\infty[.$$

2) The Maxwell- Boltzmann system (2.23)-(2.24)-(3.2) has a unique global solution (F, f) *defined all over the interval* $[0, +\infty[$ *and satisfying :*

$$F^{i0}(0) = E^i_0, \;\; F_{ij}(0) = \varphi_{ij}, \;\; f(0) = f_0, \;\; |||f||| \leq ||f_0||.$$

CHAPTER 4

The modified relativistic Maxwell-Boltzman system for a hard potential case

In this Chapter, we give the main existence result of this thesis using the results of chapter 3 and a Povzner inequality type. The use of $\mu - N$ regularity does not allow a very good physical description of the collision operator. In fact, this operator depends on several terms including the collision kernel, the relative momentum and the energy in the center of momentum. Furthermore, one of the main terms in the collision kernel is the scattering kernel which measures interactions between particles. In the newtonian Boltzmann equation, scattering kernels are usually classified into soft and hard potentials. This classification was originally adapted in the relativistic case by Dudynski and Ekiel-Jerzewska [9] and recently reformulated by Strain in [40]. This reformulation increases the importance and the interest of the relativistic Boltzmann equation. We consider in this chapter a case of collision kernel which falls into hard potential. Such collision kernels modelize strong shocks. That is why the strategy adopted in the sequel will be to construct a sequence of solutions of a modified Maxwell-Boltzmann system in order to prove that this sequence of solutions converges to a particular solution of the Maxwell-Boltzmann system with hard potential. Here, almost all the theorems, propositions, lemmas are original and have been exposed using clear processes. We have also given a reformulation of the Povzner inequality.

4.1 The method

We will consider the equivalent Maxwell-Boltzmann-Momentum system (3.6)-(3.9) in which the collision operator is now given by:

$$Q(f,f) = \int_{\mathbb{R}^3}\int_{S^2} v_\phi \sigma(k,\theta)\left(f' f'_\star - f f_\star\right) ab^2 \, d\omega \, d\overline{q},$$

where for simplicity of the notation we let

$$v_\phi = \frac{k\sqrt{\delta}}{p^0 q^0}, f' = f'\left(t,\overline{p'}\right), f'_\star = f'_\star\left(t,\overline{q'}\right), f = f\left(t,\overline{p}\right), f'_\star = f'_\star\left(t,\overline{q}\right).$$

Similarly, using the covariant variables as indicated in the change of variables in chapter 2, we get:

$$Q(f,f)(t,v) = a^{-1}b^{-2}\int_{\mathbb{R}^3} d\overline{u}\int_{S^2} d\omega \, v_\phi \sigma(k,\theta)\left(f' f'_\star - f f_\star\right).$$

The argument is to construct a modified system by truncating a certain part of the collision kernel in the equation (3.8). Because in the truncated system the scattering kernel is easily controlled, then global existence of solution for this system is insured by theorem 3.3. Therefore we obtain a sequence of solutions of the truncated systems, and showing that this sequence is a Cauchy sequence, we obtain a solution of the initial system (3.6)-(3.9).

4.2 Preliminaries results

4.2.1 Estimates on the energy and relative momentum

We start by the following useful lemmas.

Lemma 4.1. *The following inequalities hold*

$$k \leq \sqrt{\delta}, \tag{4.1}$$

$$k \leq 2\sqrt{u^0 v^0}, \tag{4.2}$$

$$\sqrt{\delta} \leq 2\sqrt{u^0 v^0}, \tag{4.3}$$

$$k \leq a^{-1} |v - u| . \tag{4.4}$$

Proof. We recall that:

$$\delta = \delta (p^\alpha, q^\alpha) = - (p_\alpha + q_\alpha) (p^\alpha + q^\alpha)$$

and

$$k = k (p^\alpha, q^\alpha) = \sqrt{(p_\alpha - q_\alpha) (p^\alpha - q^\alpha)}.$$

Let us show that $k \leq \sqrt{\delta}$. We have

$$\begin{aligned}
\delta &= - (p_\alpha + q_\alpha) (p^\alpha + q^\alpha) \\
&= p^\alpha p_\alpha - 2p^\alpha q_\alpha + q^\alpha q_\alpha - 2p^\alpha p_\alpha - 2q^\alpha q_\alpha \\
&= p^\alpha p_\alpha - 2p^\alpha q_\alpha + q^\alpha q_\alpha - 2 (p^\alpha q_\alpha + q^\alpha q_\alpha) \\
&= k^2 + 4.
\end{aligned}$$

Then $\delta = k^2 + 4$ implies $k \leq \sqrt{\delta}$.

Let us show that $\sqrt{\delta} \leq 2\sqrt{u^0 v^0}$ and $k \leq 2\sqrt{u^0 v^0}$.

We have

$$\delta = -p^\alpha p_\alpha - q^\alpha q_\alpha - 2p^\alpha q_\alpha$$

$$= 2 - 2p^\alpha g_{\alpha\lambda} q^\lambda$$

$$= 2p^0 q^0 + 2 \left(1 - a^2 p^1 q^1 - b^2 p^2 q^2 - b^2 p^3 q^3\right)$$

$$\leq 2p^0 q^0 + 2 \left(1 + a^2 \left|p^1\right| \left|q^1\right| + b^2 \left|p^2\right| \left|q^2\right| + b^2 \left|p^3\right| \left|q^3\right|\right)$$

$$\leq 2p^0 q^0 + 2 \left(1, a \left|p^1\right|, b \left|p^2\right|, b \left|p^3\right|\right) \cdot \left(1, a \left|q^1\right|, b \left|q^2\right|, b \left|q^3\right|\right)$$

$$\leq 2p^0 q^0$$

$$+2\sqrt{1 + a^2 \left(p^1\right)^2 + b^2 \left(p^2\right) + b^2 \left(p^3\right)} \cdot \sqrt{1 + a^2 \left(q^1\right)^2 + b^2 \left(q^2\right) + b^2 \left(q^3\right)}$$

$$= 4p^0q^0.$$

Then $\sqrt{\delta} \leq 2\sqrt{p^0q^0} = 2\sqrt{u^0v^0}$.

We have $k \leq \sqrt{\delta}$ and $\sqrt{\delta} \leq 2\sqrt{u^0v^0}$, then $k \leq 2\sqrt{u^0v^0}$.

Let us show that $k \leq a^{-1}|v-u|$.

We have

$$\begin{aligned} k^2 &= (p_\alpha - q_\alpha)(p^\alpha - q^\alpha) \\ &= -\left(p^0 - q^0\right)^2 + a^2\left(p^1 - q^1\right)^2 + b^2\left(p^2 - q^2\right)^2 + b^2\left(p^3 - q^3\right)^2 \\ &= -\left(v^0 - u^0\right)^2 + a^{-2}\left(v^1 - u^1\right)^2 + b^{-2}\left(v^2 - u^2\right)^2 + b^{-2}\left(v^3 - u^3\right)^2 \\ &\leq a^{-2}|v-u|^2. \end{aligned}$$

Then $k \leq a^{-1}|v-u|$ and the proof is completed. □

Lemma 4.2. *The total energy k and the relative momentum δ are invariant quantities under the collision process.*

Proof. One has

$$\begin{aligned} \delta\left(p'^\alpha, q'^\alpha\right) &= -\left(p'_\alpha + q'_\alpha\right)\left(p'^\alpha + q'^\alpha\right) \\ &= -\left(g_{\alpha\beta}p'^\beta + g_{\alpha\beta}q'^\beta\right)\left(p'^\alpha + q'^\alpha\right) \\ &= -g_{\alpha\beta}\left(p'^\beta + q'^\beta\right)\left(p'^\alpha + q'^\alpha\right). \end{aligned}$$

Since

$$\begin{aligned} p'^\beta + q'^\beta &= \frac{p^\beta + q^\beta}{2} + \frac{k}{2}\frac{t^\beta}{\sqrt{t_\alpha t^\alpha}} + \frac{p^\beta + q^\beta}{2} - \frac{k}{2}\frac{t^\beta}{\sqrt{t_\alpha t^\alpha}} \\ &= p^\beta + q^\beta \end{aligned}$$

and

$$\begin{aligned} p'^\alpha + q'^\alpha &= \frac{p^\alpha + q^\alpha}{2} + \frac{k}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}} + \frac{p^\alpha + q^\alpha}{2} - \frac{k}{2}\frac{t^\alpha}{\sqrt{t_\beta t^\beta}} \\ &= p^\alpha + q^\alpha, \end{aligned}$$

we have

$$\begin{aligned}\delta\left(p'^{\alpha}, q'^{\alpha}\right) &= -g_{\alpha\beta}\left(p^{\beta}+q^{\beta}\right)\left(p^{\alpha}+q^{\alpha}\right)\\ &= -\left(p_{\alpha}+p_{\alpha}\right)\left(p^{\alpha}+q^{\alpha}\right)\\ &= \delta\left(p^{\alpha}, q^{\alpha}\right).\end{aligned}$$

By definition, we have

$$\begin{aligned}k\left(p'^{\alpha}, q'^{\alpha}\right) &= \sqrt{\left(p'_{\alpha}-q'_{\alpha}\right)\left(p'^{\alpha}-q'^{\alpha}\right)}\\ &= \sqrt{g_{\alpha\beta}\left(p'^{\beta}-q'^{\beta}\right)\left(p'^{\alpha}-q'^{\alpha}\right)}\end{aligned}$$

Since

$$\begin{aligned}p'^{\beta}-q'^{\beta} &= \frac{p^{\beta}+q^{\beta}}{2}+\frac{k}{2}\frac{t^{\beta}}{\sqrt{t_{\alpha}t^{\alpha}}}-\frac{p^{\beta}+q^{\beta}}{2}+\frac{k}{2}\frac{t^{\beta}}{\sqrt{t_{\alpha}t^{\alpha}}}\\ &= k\frac{t^{\beta}}{\sqrt{t_{\alpha}t^{\alpha}}}\end{aligned}$$

and

$$\begin{aligned}p'^{\alpha}-q'^{\alpha} &= \frac{p^{\alpha}+q^{\alpha}}{2}+\frac{k}{2}\frac{t^{\alpha}}{\sqrt{t_{\beta}t^{\beta}}}-\frac{p^{\alpha}+q^{\alpha}}{2}+\frac{k}{2}\frac{t^{\alpha}}{\sqrt{t_{\beta}t^{\beta}}}\\ &= k\frac{t^{\alpha}}{\sqrt{t_{\beta}t^{\beta}}}\end{aligned}$$

we deduce that

$$\begin{aligned}k\left(p'^{\alpha}, q'^{\alpha}\right) &= \sqrt{g_{\alpha\beta}\left(k\frac{t^{\beta}}{\sqrt{t_{\alpha}t^{\alpha}}}\right)\left(k\frac{t^{\alpha}}{\sqrt{t_{\beta}t^{\beta}}}\right)}\\ &= \sqrt{g_{\alpha\beta}\frac{k^{2}t^{\alpha}t^{\beta}}{\sqrt{g_{\alpha\beta}t^{\beta}t^{\alpha}}\sqrt{g_{\alpha\beta}t^{\alpha}t^{\beta}}}}\\ &= \sqrt{k^{2}}\\ &= k\left(p^{\alpha}, q^{\alpha}\right)\end{aligned}$$

□

4.2.2 Estimates on the collision operator

Lemma 4.3. *Using the Bianchi type I spacetime, the following properties hold:*

For any measurable function h depending only of k, g and w, we have:

$$\int\int\int \frac{h(k,\delta,w)}{p^0q^0}\left(f'f'_\star - ff_\star\right)\left(p^0\right)^r \, dw\, d\overline{q}\, d\overline{p}$$
$$= \frac{1}{2}\int\int\int \frac{h(k,\delta,w)}{p^0q^0} ff_\star \left(\left(p'^0\right)^r + \left(q'^0\right)^r - \left(p^0\right)^r - \left(q^0\right)^r\right) dw\, d\overline{q}\, d\overline{p}.$$

Proof. We use lemma 2.1 to make the change of variables between pre- and post-collisional momenta as follow

$$\frac{1}{p^0q^0}dp\, dq = \frac{1}{p'^0q'^0}dp'\, dq',$$

and note that k and δ are invariant quantities under the collision process and symmetric for p and q. Hence, the gain term can be written as

$$\int\int\int \frac{h(k,\delta,w)}{p^0q^0} f'f'_\star \left(p^0\right)^r \, dw\, d\overline{q}\, d\overline{p}$$

$$= \int\int\int \frac{h(k,\delta,w)}{p'^0q'^0} f'f'_\star \left(p^0\right)^r \, dw\, d\overline{q'}\, d\overline{p'}$$

$$= \int\int\int \frac{h(k,\delta,w)}{p^0q^0} ff_\star \left(p'^0\right)^r \, dw\, d\overline{q}\, d\overline{p}.$$

Interchanging p and q , this can also be rewritten as

$$\int\int\int \frac{h(k,\delta,w)}{p^0q^0} f'f'_\star \left(p^0\right)^r \, d\omega\, d\overline{q}\, d\overline{p} = L$$

where

$$L = \int\int\int \frac{h(k,\delta,w)}{p^0q^0} ff_\star \left(q'^0\right)^r \, dw\, d\overline{q}\, d\overline{p}.$$

Hence, we obtain the following representation for the gain term:

$$\int\int\int \frac{h(k,\delta,w)}{p^0q^0} f'f'_\star \left(p^0\right)^r \, dw\, d\overline{q}\, d\overline{p} = M$$

where

$$M = \int\int\int \frac{h(k,\delta,w)}{p^0q^0} ff_\star \left(\left(p'^0\right)^r + \left(q'^0\right)^r\right) dw\, d\overline{q}\, d\overline{p}.$$

After applying the same argument to the loss term, we obtain the desired result. □

Lemma 4.4. *Consider the collisional process in the Bianchi type I spacetime. Let*

(p^α, q^α) and (p'^α, q'^α) be pre- and post- collisional momenta respectively.

Consider the following quantity for $r > 1$:

$$G = \left(p'^0\right)^r + \left(q'^0\right)^r - \left(p^0\right)^r - \left(q^0\right)^r.$$

Then G satisfies

$$G \leq C_r \left(\left(p^0\right)^{r-1} q^0 + p^0 \left(q^0\right)^{r-1}\right). \tag{4.5}$$

If ω is restricted to the subset

$$\left\{\omega \in S^2 : |n \cdot \omega| \leq \frac{a^2(t)}{\sqrt{2} b^2(t)} |n|\right\},$$

then

$$G \leq C_r \left(\left(p^0\right)^{r-\frac{1}{2}} \left(q^0\right)^{\frac{1}{2}} + \left(p^0\right)^{\frac{1}{2}} \left(q^0\right)^{r-\frac{1}{2}}\right) - c_r \left(\left(p^0\right)^r + \left(q^0\right)^r\right), \tag{4.6}$$

where C_r and c_r are both different non-negative constants depending on r.

Proof. By the energy momentum conservation we have

$$p^0 + q^0 = p'^0 + q'^0$$

for each p^0 and q^0. Let p^α and q^α be given. By the inequality

$$\alpha^r + \beta^r \leq (\alpha + \beta)^r \leq \alpha^r + \beta^r + C_r \left(\alpha^{r-1}\beta + \alpha\beta^{r-1}\right), \qquad \alpha, \beta \geq 0,\ r > 1, \tag{4.7}$$

we deduce that

$$\left(p'^0\right)^r + \left(q'^0\right)^r \leq \left(p^0\right)^r + \left(q^0\right)^r + C_r \left(\left(p^0\right)^{r-1} \left(q^0\right) + \left(p^0\right) \left(q^0\right)^{r-1}\right).$$

Then

$$G \leq C_r \left(\left(p^0\right)^{r-1} q^0 + p^0 \left(q^0\right)^{r-1}\right). \tag{4.8}$$

To prove the second result, we make the assumption that
$|n \cdot \omega| \leq \frac{a^2}{\sqrt{2}b^2} |n|$ and suppose that $p'^0 \geq q'^0$.
Then p'^0 is estimated as

$$p'^0 \leq \frac{p^0 + q^0}{2} + \frac{\frac{k}{2} \left|a^2(t)\, n^1\omega^1 + b^2\left(n^2\omega^2 + n^3\omega^3\right)\right|}{\sqrt{B(t)}}$$

where

$$B(t) = \left(n^0\right)\left(a^2(t)\left(\omega^1\right)^2 + b^2(t)\left(\left(\omega^2\right)^2 + \left(\omega^3\right)^2\right)\right) \\ \times \left(a^2(t)\, n^1 w^1 + b^2(t)\left(n^2\omega^2 + n^3\omega^3\right)\right)$$

And we notice that

$$\frac{\left|a^2(t)\, n^1\omega^1 + b^2\left(n^2\omega^2 + n^3\omega^3\right)\right|}{\sqrt{B(t)}} \leq 1$$

if and only if

$$2\left(a^2 n^1\omega^1 + b^2\left(n^2\omega^2 + n^3\omega^3\right)\right)^2 \leq \left(n^0\right)^2\left(a^2\left(\omega^1\right)^2 + b^2\left(\left(\omega^2\right)^2 + \left(\omega^3\right)^2\right)\right).$$

Now using the fact that $a \leq b$ and

$$\delta = \left(n^0\right)^2 - \left(a^2\left(n^1\right)^2 + b^2\left(\left(n^2\right)^2 + \left(n^3\right)^2\right)\right) \geq 0,$$

we easily deduce that:

$$|n\cdot\omega| \leq \frac{a^2}{\sqrt{2b^2}}|n| \Rightarrow 2b^2\left(n\cdot\omega\right)^2 \leq a^4|n|^4.$$

This implies that

$$2b^2\left(n\cdot\omega\right)^2 \leq a^4|n|^4 \leq \left(a^2\left(n^1\right)^2 + b^2\left(\left(n^2\right)^2 + \left(n^3\right)^2\right)\right)a^2,$$

and then

$$2\left(a^2(t)\, n^1 w^1 + b^2(t)\left(n^2\omega^2 + n^3\omega^3\right)\right) \leq N$$

where

$$N = a^2\left(a^2\left(n^1\right)^2 + b^2\left(\left(n^2\right)^2 + \left(n^3\right)^2\right)\right).$$

The inequality $|n\cdot\omega| \leq \frac{a^2}{\sqrt{2b^2}}|n|$ together with lemma 4.1 imply :

$$p'^0 \leq \frac{p^0+q^0}{2} + \frac{k}{2} \leq \frac{\left(\sqrt{p^0}+\sqrt{q^0}\right)^2}{2}.$$

Then G is estimated as

$$
\begin{aligned}
G &\leq 2\left(p'^0\right)^r - \left(p^0\right)^r - \left(q^0\right)^r \\
&\leq \frac{1}{2^{r-1}}\left(\sqrt{p^0}+\sqrt{q^0}\right)^{2r} - \left(p^0\right)^r - \left(q^0\right)^r \\
&\leq \frac{\left(p^0\right)^r}{2^{r-1}} + \frac{\left(q^0\right)^r}{2^{r-1}} + C_r\left(\left(p^0\right)^{r-\frac{1}{2}}\left(q^0\right)^{\frac{1}{2}} + \left(p^0\right)^{\frac{1}{2}}\left(q^0\right)^{r+\frac{1}{2}}\right) - \left(p^0\right)^r - \left(q^0\right)^r \\
&\leq C_r\left(\left(p^0\right)^{r-\frac{1}{2}}\left(q^0\right)^{\frac{1}{2}} + \left(p^0\right)^{\frac{1}{2}}\left(q^0\right)^{r+\frac{1}{2}}\right) - c_r\left(\left(p^0\right)^r - \left(q^0\right)^r\right),
\end{aligned}
$$

where (4.7) is used and C_r and c_r are two positive constants depending on r. This completes the proof. □

4.3 The modified Maxwell-Boltzmann-Momentum system

Let m be any positive integer. Now we modify the Maxwell-Boltzmann-Momentum system (3.6)-(3.9) by setting

$$
\begin{cases}
\dot{E}^i_m = -\Gamma^i_{0j}E^i_m + \int_{\mathbb{R}^3} q^i \frac{f_m(t,\overline{q})ab^2}{q^0}\, d\overline{q} \\
\dot{p}^i_m = -\Gamma^i_{0j}p^j_m - \left[E^i_m + g^{ii}\frac{p^k_m\varphi_{ki}}{p^0_m}\right]\int_{\mathbb{R}^3} f_m\left(t,\overline{q}\right)ab^2\, d\overline{q} \\
\dot{f}_m = Q_m\left(f_m, f_m\right) \\
F_{ij} = F_{ij}\left(0\right) = \varphi_{ij}\,, \quad \overline{E}_m\left(0\right) = \overline{E}_0, \quad \overline{p}_m\left(0\right) = \overline{p}_0, \quad f_m\left(0\right) = f_0
\end{cases}
\tag{4.9}
$$

where

$$
Q_m\left(f_m, f_m\right) = ab^2 \iint_{\mathbb{R}^3\times S^2} v_{\phi,m}\left(k_m\right)^\beta \sigma_{0,m}\left(f'_m f'_{m\star} - f_m f_{m\star}\right) d\omega d\overline{q},
$$

$$
v_{\phi,m} := \frac{\min\left\{k\sqrt{\delta}, m\right\}}{p^0q^0}, \;\; k_m = \min\left\{k, m\right\}, \;\; \sigma_{0,m} := \min\left\{\sigma_0\left(\omega\right), m\right\}.
$$

Since the scattering kernel for the modified collision operator $Q_m\left(f_m, f_m\right)$ satisfies the $\mu - N$ regularity properties (3.1) ,we conclude by theorem 3.3 that the truncated equation (4.9) has a unique global solution

$$
\left(\overline{p}_m, \overline{E}_m, f_m\right) \in C\left([0,+\infty[, \mathbb{R}^3\right)^2 \times C\left([0,+\infty[, L^1_1\left(\mathbb{R}^3\right)\right)
$$

such that $\left(\overline{p}_m, \overline{E}_m, f_m\right)(0) = \left(\overline{p}_0, \overline{E}_0, f_0\right)$.

The following lemma establishes that the sequence f_m is a Cauchy sequence.

Lemma 4.5. *For any $r \geq 0$ and $T > 0$, there exists a constant C_r, which does not depend on m such that if $||f_0||_{1,r}$ is bounded, then*

$$\sup_m \sup_{t\in[0,T]} |f_m(t)|_{1,r} + ||f_m(t)||_{1,r} \leq C_r. \tag{4.10}$$

Proof. We first estimate $||f_m(t)||_{1,r}$ and then obtain the result by the relation (2.87).

By theorem 3.3 we have

$$\sup_{t\in[0,T]} ||f_m(t)||_{1,r} \leq C$$

where $C = ||f_0||_{1,1}$ does not depend on m for $0 \leq r \leq 1$ because for $r \leq s \leq 1$,

$$||f_m(t)||_{1,r} \leq ||f_m(t)||_{1,s}.$$

Now we assume that $r > 1$.

In Bianchi type I space time, v^0 depends on time and decreases as the time evolves for each $\overline{v}$. To be precise,

$$v^0 = \left(1 + a^{-2}(t)\left(v^1\right)^2 + b^{-2}(t)\left(\left(v^2\right)^2 + \left(v^3\right)^2\right)\right).$$

So,

$\partial_t v^0 = -\left(\frac{\dot{a}(t)}{a^3(t)}\left(v^1\right)^2 + \frac{\dot{b}(t)}{b^3(t)}\left(\left(v^2\right)^2 + \left(v^3\right)^2\right)\right)\cdot\frac{1}{v^0} \leq 0$

because we assumed $b(t) \geq a(t) \geq \frac{3}{2}$, $\dot{a}(t) \geq 0$, $\dot{b}(t) \geq 0$. By direct calculation using equation (2.77), we have

$$\frac{d}{dt}|f_m(t)|_{1,r} =$$

$$a^{-1}b^{-2}\int\int\int v_{\phi,m}(k_m)^\beta \sigma_{0,m}(\omega)\left(f'_m f'_{m\star} - f_m f_{m\star}\right)\left(v^0\right)^r \, d\omega\, d\overline{v}\, d\overline{u}$$

$$+\int f_m(t,\overline{v})\frac{\partial v^0}{\partial t}\, d\omega.$$

By lemma 4.3, we have

$$\frac{d}{dt}|f_m(t)|_{1,r} \leq \frac{a^{-1}b^{-2}}{2}\int\int\int v_{\phi,m}(k_m)^\beta f_m f_{m\star}$$
$$\left[\left(v'^0\right)^r + \left(u'^0\right)^r - \left(v^0\right)^r - \left(u^0\right)^r\right] d\omega\, d\overline{v}\, d\overline{u}.$$

Using the fact that $a^{-1}b^{-2}$ is bounded , we apply lemma 4.4 to obtain

$$\frac{d}{dt}\left|f_m\left(t\right)\right|_{1,r} \leq I_1 + I_2 - I_3$$

where

$$I_1 = C_r \int\int\int_{|n\cdot\omega|\geq \frac{a^2(t)}{b^2(t)\sqrt{2}}|n|} v_{\phi,m}\left(k_m\right)^{\beta}\sigma_{0,m}\left(\omega\right) f_m f_{m\star}\left(v^0\right)^{r-1}u^0 d\omega d\overline{v}d\overline{u}$$

$$+C_r \int\int\int_{|n\cdot\omega|\geq \frac{a^2(t)}{b^2(t)\sqrt{2}}|n|} v_{\phi,m}\left(k_m\right)^{\beta}\sigma_{0,m}\left(\omega\right) f_m f_{m\star} v^0\left(u^0\right)^{r-1} d\omega d\overline{v}d\overline{u}$$

$$I_2 = C_r \iiint_{|n\cdot\omega|\leq \frac{a^2(t)}{b^2(t)\sqrt{2}}|n|} v_{\phi,m}\left(k_m\right)^{\beta}\sigma_{0,m}\left(\omega\right) f_m f_{m\star}\left(v^0\right)^{r-\frac{1}{2}}\left(u^0\right)^{\frac{1}{2}} d\omega d\overline{v}d\overline{u}$$

$$+C_r \iiint_{|n\cdot\omega|\leq \frac{a^2(t)}{b^2(t)\sqrt{2}}|n|} v_{\phi,m}\left(k_m\right)^{\beta}\sigma_{0,m}\left(\omega\right) f_m f_{m\star}\left(v^0\right)^{\frac{1}{2}}\left(u^0\right)^{r-\frac{1}{2}} d\omega d\overline{v}d\overline{u}$$

$$I_3 = c_r \iiint_{|n\cdot\omega|\leq \frac{a^2(t)}{b^2(t)\sqrt{2}}|n|} v_{\phi,m}\left(k_m\right)^{\beta}\sigma_{0,m}\left(\omega\right) f_m f_{m\star} \times\left[\left(v^0\right)^{r}+\left(u^0\right)^{r}\right] d\omega d\overline{v}d\overline{u}.$$

The second term I_2 is easily estimated using lemma 4.1 as

$$\begin{aligned} I_2 &\leq C_r \iint f_m f_{m\star}\left(v^0\right)^{r-\frac{1}{2}+\frac{\beta}{2}}\left(u^0\right)^{\frac{1}{2}+\frac{\beta}{2}} d\omega d\overline{v}d\overline{u} \\ &\leq C_r\left|f_m\left(t\right)\right|_{1,r-\frac{1}{2}+\frac{\beta}{2}}\left|f_m\left(t\right)\right|_{1,\frac{1}{2}+\frac{\beta}{2}}. \end{aligned}$$

Consider now $\sigma_{0,m}\left(\omega\right)$ defined by

$$\sigma_{0,m}\left(\omega\right) = \min\left\{\sin^{\gamma}\theta,\, m\right\} \quad \text{for } -2<\gamma<-1.$$

Note that $\sigma_{0,m}\left(\omega\right)$ is integrable on S^2 for $\gamma > -2$, and there exists a constant C_γ satisfying $\int_{S^2}\sigma_{0,m}\left(\omega\right) d\omega \leq C_\gamma$, where the constant C_γ does not depend on m. On the other hand, since γ is negative, we have $\sigma_{0,m}\left(\omega\right)\geq 1$ for any m.
Since

$$\left\{\omega\in S^2,\, |n\cdot\omega| \leq \frac{a^2\left(t\right)1}{b^2\left(t\right)\sqrt{2}}|n|\right\} \subset \left\{\omega\in S^2,\, |n\cdot\omega| \leq \frac{1}{\sqrt{2}}|n|\right\},$$

then

$$\mu\left\{|n\cdot\omega|\leq\frac{a^{2}(t)\,1}{b^{2}(t)\sqrt{2}}|n|\right\}\leq\mu\left\{|n\cdot\omega|\leq\frac{1}{\sqrt{2}}|n|\right\}.$$

Now the integration domain of I_3 is a set with Lebesgue measure

$$\mu\left\{|n\cdot\omega|\leq\frac{1}{\sqrt{2}}|n|\right\}=2\sqrt{2}\pi,$$

which does not depend on m. Hence, I_1 and I_3can be estimated as

$$I_1\leq D_r\iiint v_{\phi,m}\left(k_m\right)^{\beta}f_m f_{m\star}\left(v^0\right)^{r-1}u^0\,d\bar{v}d\bar{u},$$

$$I_3\geq d_r\iiint v_{\phi,m}\left(k_m\right)^{\beta}f_m f_{m\star}\left(u^0\right)^{r-1}\,d\bar{v}d\bar{u},$$

for some constants D_r and d_r. We now fix the constants D_r and d_r to split the domain by $\left\{D_r\left(v^0\right)^{r-1}\leq d_r\left(u^0\right)^{r-1}\right\}$ and $\left\{D_r\left(v^0\right)^{r-1}\geq d_r\left(u^0\right)^{r-1}\right\}$, and then obtain

$$I_1\leq I_{11}+I_{12}$$

where

$$I_{11}=D_r\iiint v_{\phi,m}\left(k_m\right)^{\beta}f_m f_{m\star}\left(v^0\right)^{r-1}u^0\,d\bar{v}\,d\bar{u},$$

$$I_{12}=D_r\iiint v_{\phi,m}\left(k_m\right)^{\beta}f_m f_{m\star}\left(v^0\right)^{r-1}u^0\,d\bar{v}\,d\bar{u}.$$

We now obtain

$$I_{11}\leq I_3.$$

In the case of I_{12}, we may simply use

$$\left(k_m\right)^{\beta}\leq C\left(v^0u^0\right)^{\frac{\beta}{2}}\leq C_r\left(v^0\right)^{\beta}.$$

Then I_{12} is simply estimated as

$$I_{12}\leq C_r\left|f_m(t)\right|_{1,r-1+\beta}.$$

Combining the above estimates, we obtain

$$\begin{aligned}\frac{d}{dt}\left|f_m(t)\right|_{1,r}&\leq C_r\left(\left|f_m(t)\right|_{1,r-\frac{1}{2}+\frac{\beta}{2}}\left|f_m(t)\right|_{1,\frac{1}{2}+\frac{\beta}{2}}+\left|f_m(t)\right|_{1,r-1+\beta}\right)\\&\leq C_r\left|f_m(t)\right|_{1,r}\end{aligned}$$

where we use the fact that $0\leq\beta\leq\gamma+2$ and $-2<\gamma\leq-1$.

Integrating over $[0,t]$, $t \in [0,T[$, we obtain

$$|f_m(t)|_{1,r} \leq |f_m(0)|_{1,r} + C_r \int_0^t |f_m(s)|_{1,r}\, ds.$$

By Gronwall lemma, we obtain

$$|f_m(t)|_{1,r} \leq |f_m(0)|_{1,r}\, e^{C_r T}.$$

Then

$$\sup_m \sup_{t\in[0,T]} |f_m(t)|_{1,r} \leq C_r.$$

Using equation (2.87) we obtain the desired result. □

Lemma 4.6. *Consider the sequence $\{f_m\}$ on any finite interval $[0,T]$. For any small number $\varepsilon > 0$, there exists a positive integer M such that if $l, m \geq M$, then*

$$\sup_{t\in[0,T]} |f_l(t) - f_m(t)|_{1,1} \leq \varepsilon. \tag{4.11}$$

Proof. Firstly we estimate $||f_l(t) - f_m(t)||_{1,1}$. We have using the relation (2.77) :

$$\begin{aligned}
&\frac{d}{dt}||f_l(t) - f_m(t)|| \\
&= \int \frac{\partial}{\partial t}\left(|f_l(t,\bar v) - f_m(t,\bar v)|\right) v^0 + |f_l(t,\bar v) - f_m(t,\bar v)| \frac{\partial v^0}{\partial t}\, d\bar v \\
&= \int \operatorname{sgn}(f_l - f_m)\left[Q_l(f_l,f_l) - Q_m(f_m,f_m)\right] v^0\, d\bar v \\
&- \int \left(\frac{\dot a(t)}{a^3(t)}\left(v^1\right)^2 + \frac{\dot b(t)}{b^3(t)}\left(\left(v^2\right)^2 + \left(v^3\right)^2\right)\right) |f_l(t,\bar v) - f_m(t,\bar v)| \frac{1}{v^0}\, d\bar v \\
&\leq \int \operatorname{sgn}(f_l - f_m)\left[Q_l(f_l,f_l) - Q_m(f_m,f_m)\right] v^0\, d\bar v \\
&\leq I + J
\end{aligned}$$

where

$$I = \int \operatorname{sgn}(f_l - f_m)\left[Q_l(f_l,f_l) - Q_l(f_m,f_m)\right] v^0\, d\bar v,$$

$$J = \int \operatorname{sgn}(f_l - f_m)\left[Q_l(f_m,f_m) - Q_m(f_m,f_m)\right] v^0\, d\bar v,$$

and I and J will be estimated separately. The first term I is split again as

$$\begin{aligned}I = \frac{1}{2}\iiint \operatorname{sgn}\,(f_l - f_k) v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega) \times \\ (f'_l - f'_m)\,(f'_{l\star} + f'_{m\star}) + (f'_l + f'_m)\,(f'_{l\star} - f'_{m\star}) \\ - (f_l - f_m)\,(f_{l\star} + f_{m\star}) - (f_l + f_m)\,(f_{l\star} - f_{m\star})]\,v^0\,d\omega\,d\overline{v}\,d\overline{u} \\ = I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Each I_i is estimated as follow :

$$\begin{aligned}I_1 &\leq \frac{1}{2}\iiint v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,|f'_l - f'_m|\,(f'_{l\star} + f'_{m\star})\,v^0\,d\omega\,d\overline{v}\,d\overline{u} \\ &= \frac{1}{2}\iiint v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,|f_l - f_m|\,(f_{l\star} + f_{m\star})\,v'^0\,d\omega\,d\overline{v}\,d\overline{u},\end{aligned}$$

$$\begin{aligned}I_2 &\leq \frac{1}{2}\iiint v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,(f'_l + f'_m)\,|f'_{l\star} - f'_{m\star}|\,v^0\,d\omega\,d\overline{v}\,d\overline{u} \\ &= \frac{1}{2}\iiint v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,(f_l + f_m)\,|f_{l\star} - f_{m\star}|\,v'^0\,d\omega\,d\overline{v}\,d\overline{u}\end{aligned}$$

$$\begin{aligned}I_3 &= -\frac{1}{2}\iiint \operatorname{sgn}\,(f_l - f_k)\,v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,(f_l - f_m)\,(f_{l\star} + f_{m\star})\,v^0 d\omega d\overline{v} d\overline{u} \\ &= -\frac{1}{2}\iiint v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,|f_l - f_m|\,(f_{l\star} + f_{m\star})\,v^0\,d\omega\,d\overline{v}\,d\overline{u}\end{aligned}$$

and finally

$$\begin{aligned}I_4 &\leq \frac{1}{2}\int\int\int v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,(f_l + f_m)\,|f_{l\star} - f_{m\star}|\,v^0\,d\omega\,d\overline{v}\,d\overline{u} \\ &\leq \frac{1}{2}\int\int\int v_{\phi;l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,(f_{l\star} + f_{m\star})\,|f_l - f_m|\,u^0\,d\omega\,dp\,dq.\end{aligned}$$

Therefore, I is estimated as

$$\begin{aligned}I &\leq \frac{1}{2}\iiint v_{\phi,l}\,(k_l)^\beta\,\sigma_{0,l}\,(\omega)\,|f_l - f_m|\,(f_l + f)\left(v^0 - u^0 + v'^0 + u'^0\right)\,d\omega\,d\overline{v}\,d\overline{u} \\ &\leq C\iint (k_l)^\beta\,|f_l - f_m|\,(f_l + f)\,u^0\,d\overline{v}\,d\overline{u}\end{aligned}$$

where we use $v^0 + u^0 = v'^0 + u'^0$.

Using $k_l \leq 2\sqrt{v^0 u^0}$, we obtain for I :

$$
\begin{aligned}
I &\leq C \iint (k_l)^\beta |f_l - f_m| (f_l + f) \left(v^0\right)^{\frac{\beta}{2}} \left(u^0\right)^{1+\frac{\beta}{2}} \, d\bar{v}\, d\bar{u} \\
&\leq C \sup_n ||f_n(t)||_{1,1+\frac{\beta}{2}} ||f_l(t) - f_n(t)||_{1,\frac{\beta}{2}} . \qquad (4.12)
\end{aligned}
$$

To estimate the second term J, note that

$$
\begin{aligned}
|v_{\phi,l} - v_{\phi,m}| &= \frac{1}{v^0 u^0} \left| \min\left\{k\sqrt{\delta}, l\right\} - \min\left\{k\sqrt{\delta}, m\right\}\right| \\
&\leq 1_{\left\{k\sqrt{\delta}\geq l\right\}} \min\left\{\frac{k\sqrt{\delta}}{v^0 u^0}, m\right\} = 1_{\left\{k\sqrt{\delta}\geq l\right\}} v_{\phi,m}
\end{aligned}
$$

and similarly

$$
\left|(k_l)^\beta - (k_m)^\beta\right| \leq 1_{\{k>l\}} (k_m)^\beta ,
$$

$$
|\sigma_{0,l}(\omega) - \sigma_{0,m}(\omega)| \leq 1_{\{\sin\gamma,\gamma\geq l\}} \sigma_{0,m}(\omega) .
$$

Hence J can be estimated as

$$
\begin{aligned}
J &\leq \iiint \left| v_{\phi,l} (k_l)^\beta \sigma_{0,l}(\omega) - v_{\phi,m} (k_m)^\beta \sigma_{0,m}(\omega) \right| |f'_m f'_{m\star} - f_m f_{m\star}| \, v^0 d\omega d\bar{v} d\bar{u} \\
&\leq \iiint 1_{\left\{k\sqrt{\delta}\geq l\right\}} v_{\phi,m} (k_l)^\beta \sigma_{0,l}(\omega) \left(f'_m f'_{m\star} + f_m f_{m\star}\right) v^0 \, d\omega\, d\bar{v}\, d\bar{u} \\
&+ \iiint 1_{\{k\geq l\}} v_{\phi,m} (k_l)^\beta \sigma_{0,l}(\omega) \left(f'_m f'_{m\star} + f_m f_{m\star}\right) v^0 \, d\omega\, d\bar{v}\, d\bar{u} \\
&+ \iiint 1_{\{\sin^\gamma\theta,\gamma\geq l\}} 1_{\left\{k\sqrt{\delta}\geq l\right\}} v_{\phi,m} (k_m)^\beta \sigma_{0,m}(\omega) \left(f'_m f'_{m\star} + f_m f_{m\star}\right) v^0 d\omega d\bar{v} d\bar{u} \\
& J_1 + J_2 + J_3.
\end{aligned}
$$

Note that each J_i can be separated into two terms: a gain containing $f'_m f'_{m\star}$ and a loss term containing $f_m f_{m\star}$. The gain and loss terms are estimated in the same way after making the change of variables $(v, u) \longmapsto (v', u')$, hence we only present the estimates for the loss terms. To estimate J_1, we take a small number $\varepsilon > 0$ and use

$k\sqrt{\delta} \leq 4v^0 u^0$ from lemma 4.1:

$$\begin{aligned}
J_1 &\leq C \iint 1_{\{4v^0u^0 \geq l\}} (k_l)^\beta f_m f_{m\star} v^0 \, d\bar{u} \, d\bar{v} \\
&\leq C \iint 1_{\{4v^0u^0 \geq l\}} f_m \left(v^0\right)^{1+\frac{\beta}{2}} f_{m\star} \left(u^0\right)^{\frac{\beta}{2}} \, d\bar{u} \, d\bar{v} \\
&\leq \frac{C}{l^\varepsilon} \iint 1_{\{4v^0u^0 \geq l\}} f_m \left(v^0\right)^{1+\frac{\beta}{2}+\varepsilon} f_{m\star} \left(u^0\right)^{\frac{\beta}{2}+\varepsilon} \, d\bar{u} \, d\bar{v} \\
&\leq \frac{C}{l^\varepsilon} \left|\left|f_m(t)\right|\right|_{1,1+\frac{\beta}{2}+\epsilon} \left|\left|f_m(t)\right|\right|_{1,\frac{\beta}{2}+\varepsilon} . \qquad (4.13)
\end{aligned}$$

To estimate J_2, we use $k \leq |v-u|$ to obtain

$$\begin{aligned}
J_2 &\leq C \iint 1_{|v-u| \geq l} f_m \left(v^0\right)^{1+\frac{\beta}{2}} f_{m\star} \left(u^0\right)^{\frac{\beta}{2}} \, d\bar{u} \, d\bar{v} \\
&\leq C \iint 1_{\left\{|v| \geq \frac{l}{2}\right\} \cup \left\{|u| \geq \frac{l}{2}\right\}} f_m \left(v^0\right)^{1+\frac{\beta}{2}} f_{m\star} \left(u^0\right)^{\frac{\beta}{2}} \, d\bar{u} \, d\bar{v} \\
&\leq C \left|\left|f_m(t)\right|\right|_{1,1+\frac{\beta}{2}+\varepsilon} \int 1_{\left\{|u| \geq \frac{l}{2}\right\}} f_{m\star} \left(u^0\right)^{\frac{\beta}{2}} \, d\bar{u} \\
&\leq \frac{C}{l} \left|\left|f_m(t)\right|\right|^2_{1,1+\frac{\beta}{2}} . \qquad (4.14)
\end{aligned}$$

For J_3 term, we use $\sin\theta \approx \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Hence, the condition $\sin^\gamma \theta \leq l$ is equivalent to $\theta \leq Cl^{\frac{\gamma+2}{\gamma}}$, since γ is negative. We first estimate J_3 as

$$\begin{aligned}
J_3 &\leq C \iiint 1_{\left\{\gamma \geq Cl^{\frac{1}{\gamma}}\right\}} \sigma_{0,m}(\omega) f_m \left(v^0\right)^{1+\frac{\beta}{2}} \, d\omega \, d\bar{u} \, d\bar{v} \\
&\leq C \left|\left|f_m(t)\right|\right|_{1,1+\frac{\beta}{2}} \left|\left|f_m(t)\right|\right|_{1,\frac{\beta}{2}} \int 1_{\left\{\gamma \geq Cl^{\frac{1}{\gamma}}\right\}} \sigma_{0,m}(\omega) \, d\omega.
\end{aligned}$$

The integration on S^2 above is estimated as

$$\int 1_{\left\{\theta \geq Cl^{\frac{1}{\gamma}}\right\}} \sigma_{0,m}(\omega) \, d\omega \leq 2\pi \int_0^{Cl^{\frac{1}{\gamma}}} \sin^{\gamma+1}\theta \, d\theta \leq Cl^{\frac{\gamma+2}{\gamma}},$$

where the constant depends on γ. Note that $-1 \leq (\gamma+2)/\gamma$, and the third term J_3 is estimated as

$$J_3 \leq Cl^{\frac{\gamma+2}{\gamma}} \left|\left|f_m(t)\right|\right|_{1,1+\frac{\beta}{2}} \left|\left|f_m(t)\right|\right|_{1,\frac{\beta}{2}} . \qquad (4.15)$$

We combine (4.12), (4.13), (4.14) and (4.15) and apply lemma 4.6 on any finite time interval $[0,T]$ to obtain:

$$\begin{aligned}\frac{d}{dt}\left\|f_l(t)-f_m(t)\right\|_{1,1} &\leq C\left(l^{-\varepsilon}+l^{-1}+l^{\frac{\gamma+2}{\gamma}}\right)+C\left\|f_l(t)-f_m(t)\right\|_{1,\frac{\beta}{2}}\\ &\leq C\left(l^{-\varepsilon}+l^{\frac{\gamma+2}{\gamma}}\right)+C\left\|f_l(t)-f_m(t)\right\|_{1,1}.\end{aligned}$$

Integrating over $[0,T]$, we obtain

$$\begin{aligned}\left\|f_l(t)-f_m(t)\right\|_{1,1} \leq\ & C\left(l^{-\varepsilon}+l^{\frac{\gamma+2}{\gamma}}\right)T+\left\|f_l(t)-f_m(t)\right\|_{1,1}\\ &+C\int_0^t\left\|f_l(s)-f_m(s)\right\|_{1,1}ds.\end{aligned}$$

Since

$$f_l(0)=f_m(0),$$

and $\frac{\gamma+2}{\gamma}$ is negative, applying Gronwall lemma we obtain

$$\left\|f_l(t)-f_m(t)\right\|_{1,1}\leq\left[C\left(l^{-\varepsilon}+l^{\frac{\gamma+2}{\gamma}}\right)T\right]e^{CT}.$$

Then

$$\sup_{t\in[0,T]}\left\|f_l(t)-f_m(t)\right\|_{1,1}\leq C.$$

Using relation (2.87) we obtain the desired result. □

Lemma 4.7. *Consider the sequence* $\{\bar{E}_m\}$ *and* $\{\bar{p}_m\}$ *on any finite interval* $[0;T]$. *For any small number, there exists a positive integer* M *such that if* $k,m\geq M$*, then*

$$\sup_{t\in[0,T]}\left\|\bar{E}_k(t)-\bar{E}_m(t)\right\|\leq\varepsilon, \tag{4.16}$$

$$\sup_{t\in[0,T]}\left\|\bar{p}_k(t)-\bar{p}_m(t)\right\|\leq\varepsilon. \tag{4.17}$$

Proof. We consider the relations (3.6), (3.31), (3.34) to deduce that:

$$\left\|\dot{\bar{E}}_k(t)\text{ -}\dot{\bar{E}}_m(t)\right\|\leq C_2\left(\left\|\bar{E}_k(t)-\bar{E}_m(t)\right\|+\left\|\bar{f}_k(t)-\bar{f}_m(t)\right\|\right).$$

using the expression of C_2 given by (3.38), relations (2.87) and (2.81) , we easily deduce that there exists a positive absolute constant C_6 such that :

$$C_2\leq C_6,$$

where $C_6=C_6(a_0,b_0,T,C_1)$.

Then

$$\left|\left|\dot{\bar{E}}_k(t)\text{ -}\dot{\bar{E}}_m(t)\ \right|\right| \leq C_6 \left(\left|\left|\bar{E}_k(t) - \bar{E}_m(t)\right|\right| + \left|\left|\bar{f}_k(t) - \bar{f}_m(t)\right|\right|\right).$$

Integrating over $[0,T]$,we easily obtain:

$$\left|\left|\bar{E}_k(t) - \bar{E}_m(t)\right|\right| \leq$$

$$C_6 \left(T \sup_{t\in[0,T]} \left|\left|f_k(t) - f_m(t)\right|\right| + \int_0^t \left|\left|\bar{E}_k(s) - \bar{E}_m(s)\right|\right| ds \right),$$

$t \in [0,T]$.

By Gronwall inequality, we obtain:

$$\left|\left|\bar{E}_k(t) - \bar{E}_m(t)\right|\right| \leq TC_6 \sup_{t\in[0,T]} \left|\left|f_k(t) - f_m(t)\right|\right| e^{C_6 T}.$$

Then (2.87) and lemma 4.5 allow to obtain the inequality (4.16.).

Using the same scheme and invoking this time relations (3.7), (3.32),(3.35), we obtain

$||\dot{\bar{p}}_k(t)\text{ -}\dot{\bar{p}}_m(t)\ ||$

$\leq C_3 \left(\left|\left|\bar{p}_k(t) - \bar{p}_m(t)\right|\right| + \left|\left|\bar{E}_k(t) - \bar{E}_m(t)\right|\right| + \left|\left|\bar{f}_k(t) - \bar{f}_m(t)\right|\right|\right).$

Using the expression of C_3 given by (3.38), relations (2.80) and , (2.81) invoking lemma 3.1 and theorem 3.3 to bound $\left|\left|\bar{E}_m(t)\right|\right|$ and $\left|\left|\bar{f}_m(t)\right|\right|$, we easily deduce that there exists a positive absolute constant C_7 such that

$$C_3 \leq C_7$$

where $C_7 = C_7\left(a_0, b_0, ||f_0||, \left|\left|\bar{E}_0\right|\right|, T, C_1, C\right)$. Then

$||\dot{\bar{p}}_k(t)\text{ -}\dot{\bar{p}}_m(t)\ ||$

$\leq C_7 \left(\left|\left|\bar{p}_k(t) - \bar{p}_m(t)\right|\right| + \left|\left|\bar{E}_k(t) - \bar{E}_m(t)\right|\right| + \left|\left|\bar{f}_k(t) - \bar{f}_m(t)\right|\right|\right)$

Integrating over $[0,T]$ we easily obtain

$$\begin{aligned}\left|\left|\bar{p}_k(t) - \bar{p}_m(t)\right|\right| \leq C_7 \Bigg[&\sup_{t\in[0,T]} \left(\left|\left|\bar{E}_k(t) - \bar{E}_m(t)\right|\right| + \left|\left|\bar{f}_k(t) - \bar{f}_m(t)\right|\right|\right) \\ &+ \int_0^t \left|\left|\bar{p}_k(s) - \bar{p}_m(s)\right|\right| ds \Bigg], \qquad t \in [0,T].\end{aligned}$$

By Gronwall inequality, we obtain

$$\left|\left|\bar{p}_k(t) - \bar{p}_m(t)\right|\right| \leq TC_7 \sup_{t\in[0,T]} \left(\left|\left|\bar{E}_k(t) - \bar{E}_m(t)\right|\right| + \left|\left|\bar{f}_k(t) - \bar{f}_m(t)\right|\right|\right) e^{C_7 T}.$$

Then the inequalities (2.87) , lemma 4.6 and the inequality (4.16) give the relation (4.17) .

So the proof is completed. □

Theorem 4.1. *Let* $\bar{p}_0, \bar{E}_0 \in \mathbb{R}^3$*,* $\varphi_{ij} \in \mathbb{R}$*,*$f \in L^1_r(\mathbb{R}^3)$ *be given, in which* $r > 1 + \frac{\beta}{2}$ *and* $f_0 \geq 0$*, and suppose that the scattering kernel has the form* (2.78)*:*

- Then the equivalent Maxwell-Boltzmann- Momentum system (3.6)-(3.9) *has a unique global solution* $\left(\bar{F}, \bar{p}, f\right)$ *such that* $f \in C\left([0, +\infty[, L^1_1(\mathbb{R}^3)\right)$ *with* $f(t) \geq 0$ *and satisfying*

$$F^{i0} = F^{i0}(0) = E^i_0, \quad F_{ij} = F_{ij}(0) = \varphi_{ij}, \quad f(0, \cdot) = f_0;$$

- (F, f) *is the unique global solution of the Maxwell-Boltzmann system* (2.23)-(2.24)*,* (2.32)*.*

Proof. Lemmas (4.6) and (4.7) show that the sequence $\left\{\left(\bar{p}_m, \bar{E}_m, f_m\right)\right\}$ is a Cauchy sequence in the Banach space $\left(\mathbb{R}^3\right)^2 \times L^1_1(\mathbb{R}^3)$. Hence, there exists $\left(\bar{p}, \bar{E}, f\right)$ a solution of the system (2.72)-(2.74) with initial condition $\left(\bar{p}_0, \bar{E}_0, f_0\right)$. The initial condition $f_0 \in L^1_r(\mathbb{R}^3)$ with $r > 1 + \frac{\beta}{2}$ comes from Theorem (3.3). The non-negativity of f is guaranteed by the same theorem. The uniqueness is obtained by following the proofs of lemmas (4.6) and (4.7). This completes the proof. □

Conclusion

This work was devoted to extend the work of [29] who considered the homogeneous relativistic Maxwell-Boltzmann system for $\mu - N$ regularity with an additional hypothesis of invariance under a subgroup of $\mathcal{O}_3$. In the present work, we have discarded this hypothesis of invariance. After presenting the Boltzmann equations, the hard and soft intercations, the background spacetime, the unknowns and the equations, we have considered in chapter 3 the Maxwell-Boltmann system for $\mu - N$ regularity, readjusting results of [29]and [20]. The same system were also considered in case of hard potential kernel in the last chapter. The method followed was the one used in [20], relying in the use of a particular form of Povzner inequality, but in a more difficult situation, since the Boltzmann equation was coupled with the Maxwell equations and because the momentum raised as an unknown. Some energy estimates in particular functionnal spaces allowed us to obtain a global existence theorem and uniqueness of mild solutions.

In our future investigations, we will consider a generalized inhomogeneous and magnetized relativistic Boltzmann equation for both the soft and hard potential cases in a curved space time.

APPENDIX A

Proof of Lemma 2.1

Proof. To prove the lemma, we use a parametrisation different from (2.41)

$$p'^{\alpha} = p^{\alpha} + 2\frac{t_\beta q^\beta}{t_\gamma t^\gamma} t^\alpha, \quad q'^{\alpha} = q^{\alpha} - 2\frac{t_\beta q^\beta}{t_\gamma t^\gamma} t^\alpha,$$

where t^α is the same as in (2.41). For convenience we write

$$p'^{k} = p^{k} + Aw^{k}, \quad q'^{k} = q^{k} - Aw^{k}, \quad A = -2\frac{t_\beta q^\beta}{t_\gamma t^\gamma} n_0$$

and we obtain

$$\begin{aligned}
\frac{w\left(p', q'\right)}{w\left(p, q\right)} &= det \begin{pmatrix} d^i_j + \left(w_{p^j} A\right) w^i & \left(w_{q^j} A\right) w^i \\ -\left(w_{p^j} A\right) w^i & d^i_j - \left(w_{q^j} A\right) w^i \end{pmatrix} \\
&= det\left(d^i_j + \left(w_{p^j} A - w_{q^j} A\right) w^i\right) \\
&= 1 + \left(w_{p^j} A - w_{q^j} A\right) w^i. \qquad (1)
\end{aligned}$$

Differentiating the conserved energy

$$p'^0 + q'^0 = p^0 + q^0$$

with respect to p^j, and multiplying by w^j yields:

$$\left(\frac{-p'_k}{p'_0}\frac{wp'^k}{wp^j}-\frac{-q'_k}{q'_0}\frac{wq'^k}{wp^j}\right)w^j=-\frac{p_j}{p_0}w^j.$$

Then

$$\left(\frac{p_j}{p_0}-\frac{p'_j}{p'_0}\right)w^j=\left(\frac{p'_k}{p'_0}-\frac{q'_k}{q'_0}\right)w^k\left(w_{p^j}A\right)w^j.$$

Similarly we obtain

$$\left(\frac{q_j}{q_0}-\frac{q'_j}{q'_0}\right)w^j=\left(\frac{p'_k}{p'_0}-\frac{q'_k}{q'_0}\right)w^k\left(w_{q^j}A\right)w^j.$$

Hence (1) is given by

$$\frac{w\left(p',q'\right)}{w\left(p,q\right)}=\left(\frac{p'_k}{p'_0}w^k-\frac{q'_k}{q'_0}w^k\right)^{-1}\left(\frac{p_j}{p_0}w^j-\frac{q_j}{q_0}w^j\right).\quad (2)$$

Recall that

$$n^\alpha=p^\alpha+q^\alpha,\quad t^\alpha=\left(n_jw^j,-n_0w\right),\text{for } w\in S^2,$$

and then the above quantities are rewritten as follows :

$$\begin{aligned}\frac{p_j}{p_0}w^j-\frac{q_j}{q^0}w^j&=\frac{1}{p_0q_0}\left(q_0n_j-n_0q_j\right)w^j\\&=\frac{1}{p_0q_0}q_\alpha t^\alpha.\qquad (3)\end{aligned}$$

Similarly we obtain

$$\begin{aligned}\frac{p'_k}{p'_0}w^k-\frac{q'_k}{q'_0}w^k&=\frac{1}{p'_0q'_0}\left(q'_0p'_k+q'_0q'_k-q'_0q'_k-p'_0q'_k\right)w^k\\&=\frac{1}{p'_0q'_0}\left(q'_0n_k-n_0q'_k\right)w^k\\&=\frac{1}{p'_0q'_0}q'_\alpha t^\alpha\\&=\frac{1}{p'_0q'_0}q_\alpha t^\alpha,\qquad (4)\end{aligned}$$

where we use the energy-momentum conservation and the following equality :

$$\begin{aligned}t_\alpha q'^\alpha&=t_\alpha q^\alpha-2\frac{t_\beta q^\beta}{t_\gamma t^\gamma}t_\alpha t^\alpha\\&=-t_\alpha q^\alpha.\end{aligned}$$

We plug (4) and (3) into (2) to obtain the desired result and this completes the proof. □

APPENDIX B

Proof of lemma 2.2

Proof. 1) $L_r^1(\mathbb{R}^3)$ is already a normed vector space .
Now it suffices to show that $L_r^1(\mathbb{R}^3)$ is complete i.e in $L_r^1(\mathbb{R}^3)$ any Cauchy sequence converges.
Let then $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L_r^1(\mathbb{R}^3)$.
Let us show that $(f_n)_{n\geq 1}$ converges in $L_r^1(\mathbb{R}^3)$ towards a function f belonging to $L_r^1(\mathbb{R}^3)$.
Since $(f_n)_{n\geq 1}$ is a Cauchy sequence in $L_r^1(\mathbb{R}^3)$, one has :

$$\left[||f_n - f_m||_{1,r} \longrightarrow 0, \;\; m, n \longrightarrow +\infty\right].$$

Thus $\left(\left(\sqrt{1+|\overline{p}|^2}\right)^r f_n\right)_{n\geq 1}$ is a Cauchy sequence in $L^1(\mathbb{R}^3)$ which is a complete space.
Then there exists $g \in L^1(\mathbb{R}^3)$ such that

$$\left(\sqrt{1+|\overline{p}|^2}\right)^r f_n \longrightarrow g \quad (a)$$

in $L^1(\mathbb{R}^3)$.
Otherwise, we have

$$||f_n - f_m||_{L^1(\mathbb{R}^3)} \leq ||f_n - f_m||_{1,r} \longrightarrow 0,$$

then $(f_n)_{n\geq 1}$ is also a Cauchy sequence in $L^1(\mathbb{R}^3)$ which is a complete space. Consequently, there exists $f \in L^1(\mathbb{R}^3)$ such that

$$f_n \longrightarrow f$$

in $L^1(\mathbb{R}^3)$.

Now

$$\left(\left(\sqrt{1+|\overline{p}|^2}\right)^r f_n \longrightarrow g\right)$$

in $L^1(\mathbb{R}^3)$ implies

$$f_n \longrightarrow \frac{1}{\left(\sqrt{1+|\overline{p}|^2}\right)^r} g$$

in $L^1(\mathbb{R}^3)$.

Since

$$\begin{aligned} \left\| f_n - \frac{1}{\left(\sqrt{1+|\overline{p}|^2}\right)^r} g \right\|_{L^1(\mathbb{R}^3)} &= \left\| \frac{\left(\sqrt{1+|\overline{p}|^2}\right)^r f_n - g}{\left(\sqrt{1+|\overline{p}|^2}\right)^r} \right\|_{L^1(\mathbb{R}^3)} \\ &\leq \left\| \left(\sqrt{1+|\overline{p}|^2}\right)^r f_n - g \right\|_{L^1(\mathbb{R}^3)} \end{aligned}$$

and

$$\left\| \left(\sqrt{1+|\overline{p}|^2}\right)^r f_n - g \right\|_{L^1(\mathbb{R}^3)} \longrightarrow 0$$

then, we have

$$\begin{cases} f_n \longrightarrow f \quad \text{in} \quad L^1(\mathbb{R}^3) \\ f_n \longrightarrow \frac{1}{\left(\sqrt{1+|\overline{p}|^2}\right)^r} g \quad \text{in} \quad L^1(\mathbb{R}^3). \end{cases}$$

Accordingly

$$f = \frac{1}{\left(\sqrt{1+|\overline{p}|^2}\right)^r} g,$$

and we have that:

$$g = \left(\sqrt{1+|\overline{p}|^2}\right)^r f \in L^1\left(\mathbb{R}^3\right)$$

In conclusion, we have :

$$f \in L^1\left(\mathbb{R}^3\right) \text{ and } \left(\sqrt{1+|\overline{p}|^2}\right)^r f \in L^1\left(\mathbb{R}^3\right).$$

Now if we replace in (a) g by $\left(\sqrt{1+|\overline{p}|^2}\right)^r f$, we have

$$\left(\sqrt{1+|\overline{p}|^2}\right)^r f_n \longrightarrow \left(\sqrt{1+|\overline{p}|^2}\right)^r f$$

in $L^1\left(\mathbb{R}^3\right)$,
which simply means

$$f_n \longrightarrow f \text{ in } L^1_r\left(\mathbb{R}^3\right).$$

Then $L^1_r\left(\mathbb{R}^3\right)$ is complete and consequently is a Banach space.

2) $L^1_1\left(\mathbb{R}^3\right)$ is obviously a complete normed space.
It suffices to show that the subset X_r is closed in $L^1_1\left(\mathbb{R}^3\right)$.
Let $(f_n)_{n\geq 1}$ be a sequence in X_r which converges to f in $L^1_1\left(\mathbb{R}^3\right)$. Let us show that $f \in X_r$ i.e

$$f \geq 0 \text{ a.e and } ||f|| \leq r.$$

Since

$$f_n \longrightarrow f \text{ in } L^1_1\left(\mathbb{R}^3\right)$$

we deduce that

$$||f_n|| \longrightarrow ||f||.$$

Thus

$$||f|| \leq r.$$

It remains to prove that $f \geq 0$ a.e.
Since

$$f_n \longrightarrow f \text{ in } L^1_1\left(\mathbb{R}^3\right),$$

then

$$\left(\sqrt{1+|\overline{p}|^2}\right)^r f_n \longrightarrow \left(\sqrt{1+|\overline{p}|^2}\right)^r f$$

in $L^1_1\left(\mathbb{R}^3\right)$. So $f_n \longrightarrow f$ in $L^1\left(\mathbb{R}^3\right)$.

We can extract a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\geq 1}$ such that

$$f_{n_k}(x) \longrightarrow f(x) \text{ a.e }.$$

But we have $f_{n_k} \geq 0$ a.e, so we can conclude that $f \geq 0$ a.e .
Now let us show that X_r is connected.
Let $f, g \in X_r$. Then for all $t \in [0,1]$, we set

$$h = (1-t) f + tg.$$

We then have $h \in X_r$ because :

i) $h \geq 0$ a.e

ii) $||h|| \leq (1-t) ||f|| + t ||g|| \leq (1-t) r + tr = r.$

Thus X_r is a convex subset of $L_1^1 (\mathbb{R}^3)$.
Since the space $L_1^1 (\mathbb{R}^3)$ is a topological vector space, the map $t \longmapsto (1-t) f + tg$ is continuous from $[0,1]$ to $L_1^1 (\mathbb{R}^3)$. But X_r being convex, any segment joingning two points $f, g \in X_r$ is a path strictly included in X_r.
Thus X_r is piecewise connected and so X_r is a connected metric subspace of $L_1^1 (\mathbb{R}^3)$.

3) $C ([I, L_1^1 (\mathbb{R}^3)] ; |||.|||)$ is a Banach space since $L_1^1 (\mathbb{R}^3)$ is a Banach space.

4) By the same way, $C ([I, X_r])$ is a complete metric space because X_r is a complete metric space.

Consequently $C ([I, X_r])$ is a complete metric subspace of $C ([I, L_1^1 (\mathbb{R}^3)] ; |||.|||)$. $\square$

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The Magnetized Relativistic Boltzmann Equation with a Hard Potential

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Abstract

A global existence theorem and uniqueness of solution of the coupled spatially homogeneous relativistic Maxwell-Boltzmann system is proved, in a Bianchi type I spacetime back-ground, in a hard potential case. The proof relies in the use of a particular form of Povzner inequality.

Mathematics Subject Classification: 83Cxx

Keywords: Bianchi type I spacetime, Povzner inequality, relativistic Boltzmann equation, Maxwell equations, energy estimates, hard potential, global existence

1. INTRODUCTION

One of the most important models which rules the dynamic of dilute charged particles is expressed by the coupled relativistic Maxwell-Boltzmann system, in which particles interact with themselves through collisions and with their self consistence electromagnetic field. The study encludes the case of fast moving particles of gas being submitted to binary collisions. We restrict the study to homogeneous case, which means that the unknown in the equation depends only on time and velocity variables. In [5] , Noutchegueme and R. Ayissi have studied the Maxwell-Boltzmann system, in a Bianchi type I spacetime, with a bounded scattering kernel. One of the purpose of this article is to extend this result in a more physically relevant situation.

The nature of collisions between particles is determined by the scattering kernel. In the relativistic setting, a classification of the scattering kernel into hard and soft potential has been proposed in [2, 9]. As in [7], we consider a scattering kernel of hard potential type, which is more physically relevant.

The relativistic Boltzmann equation rules the dynamic of the considered charged particles which are subject to collide with themselves, by determining their distribution function, denoted f, a non-negative real valued function of both the time and the momentum of particles.

The Maxwell equations are the equations of electromagnetism and determine the electromagnetic field F created by the fast moving and charged particles. We consider the case where this field is generated by a Maxwell current defined by the distribution function, a charge density e and a future pointing unit vector u, tangent at any point to the temporal axis.

The main objective of the present paper is to extend the results of [5] in two points. Firstly, we consider a hard potential case, instead of bounded kernel. Secondly, we remove the assumption that the initial datum of the Boltzmann equation is invariant under a subgroup of $\mathcal{O}_3$. But for the sake of method we first consider a bounded kernel in section 3, in order to extend to the hard potentiel case after obtaining an existence theorem. The main tool here is a particular form of Povzner inequality.

The paper organises as follows :

- In section 2, we introduce the equations .
- In section 3, we study the bounded case.
- In section 4, we study the hard potential case.

2. THE MAXWELL-BOLTZMANN SYSTEM IN A BIANCHI TYPE I SPACETIME

2.1. **Notations.** In a time oriented Bianchi type I spacetime, we consider the collisional evolution of fast moving massive and charged particles and denote by $x^\alpha = (x^0; x^i)$ the usual coordinates in $\mathbb{R}^4$, where $t = x^0$ represents the time and (x^i) the space, g is the metric tensor of Lorentzian signature $(-,+,+,+)$ which writes:

$$g = -dt^2 + a^2(t)(dx^1)^2 + b^2(t)\left((dx^2)^2 + (dx^3)^2\right), \tag{2.1}$$

where a and b are two differentiable increasing functions on $\mathbb{R}^+$ such that:

$$a \leq b,\ a(0) = a_0 \geq \frac{3}{2}. \tag{2.2}$$

The expression of the Christoffel symbols of the Levi-Civita connection ∇ of g is:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} \left[\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right]. \tag{2.3}$$

As in [5] , we require that there exists a constant $C > 0$ such that:

$$\left|\frac{1}{a}\frac{da}{dt}\right| \leq C, \ \left|\frac{1}{b}\frac{db}{dt}\right| \leq C. \tag{2.4}$$

The particles are statistically described by their distribution function, denoted $f = f(x^\alpha, p^\alpha)$ in which (x^α) is the position $(p^\alpha) = (p^0, \overline{p})$is the the $4-$momentum of the particle. So:

$$f : T\left(\mathbb{R}^4\right) \cong \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^+, \ (x^\alpha, p^\alpha) \mapsto f(x^\alpha, p^\alpha). \tag{2.5}$$

On $\mathbb{R}^3$ a scalar product is defined by

$$\overline{p}.\overline{q} = a^2 p^1 q^1 + b^2 \left(p^2 q^2 + p^3 q^3\right). \tag{2.6}$$

The particles whose mass $m = 1$ is normalized to the unity move on the futur sheet of the mass-shell

$$p^0 = \sqrt{1 + a^2 (p^1)^2 + b^2 \left((p^2)^2 + (p^3)^2\right)}. \tag{2.7}$$

The trajectories $s \longmapsto (x^\alpha(s), p^\alpha(s))$ of particles solve the differential system:

$$\frac{dx^\alpha}{ds} = p^\alpha, \ \frac{dp^\alpha}{ds} = P^\alpha := -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + e p^\beta F^\alpha_\beta \tag{2.8}$$

where $e = e(t)$ denotes the charge density of particles.

With the covariant variables, the distribution function f will be seen sometimes as a function of t and $\overline{v} = (v^1, v^2, v^3)$ as in [4] , instead of $\overline{p}$, where:

$$\begin{cases} v^1 = a^2 p^1, \ v^2 = b^2 p^2, \ v^3 = b^2 p^3, d\overline{v} = a^2 b^4 d\overline{p} \\ v^0 = \sqrt{1 + a^{-2} (p^1)^2 + b^{-2} \left((p^2)^2 + (p^3)^2\right)}. \end{cases} \tag{2.9}$$

2.2. **The Maxwell system in** F**.** The Maxwell system in F can be written as:

$$\nabla_\alpha F^{\alpha\beta} = J^\beta, \ \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \tag{2.10}$$

where ∇_α stands for the covariant derivative in g, J^β represents the Maxwell current whose local expression is given by:

$$J^\beta = \int_{\mathbb{R}^3} \frac{p^\beta f(t, \overline{p}) (det g)^{\frac{1}{2}} d\overline{p}}{p^0} - e u^\beta, \ u^0 = 1, \ u^i = 0 \tag{2.11}$$

in which $u = \left(u^{\beta}\right)$ is a unit futur pointing timelike vector tangent to the time axis at any point.

The particles are then spatially at rest. Now the identity

$$\nabla_{\alpha}\nabla_{\beta}F^{\alpha\beta} = 0$$

imposes, given (2.10) that

$$\nabla_{\alpha}J^{\beta} = 0. \qquad (2.12)$$

2.3. **The Boltzmann equation in f.** The Boltzmann equation in a Bianchi type I spacetime writes:

$$\frac{p^{\alpha}}{p^{0}}\frac{\partial f}{\partial x^{\alpha}} + \frac{P^{i}}{P^{0}}\frac{\partial f}{\partial p^{i}} = Q(f,f) = \int_{\mathbb{R}^3}\int_{S^2} v_{\phi}\sigma(k,\theta)\left(f' f'_{*} - f f_{*}\right) ab^2 d\omega d\overline{q} \qquad (2.13)$$

where

$$v_{\phi} = \frac{k\sqrt{\delta}}{p^0 q^0},\ f' = f\left(t,\overline{p'}\right),\ f'_{*} = f\left(t,\overline{q'}\right),\ f = f\left(t,\overline{p}\right),\ f_{*} = f\left(t,\overline{q}\right).$$

Here Q is the collision operator, v_{ϕ} the Mï¿œller velocity, σ the scattering kernel, θ the scattering angle, δ and k are given by

$$\delta = -\left(p_{\alpha} + q_{\alpha}\right)\left(p^{\alpha} + q^{\alpha}\right),\ k = \sqrt{\left(p_{\alpha} - q_{\alpha}\right)\left(p^{\alpha} - q^{\alpha}\right)},$$

and are called total energy and relative momentum respectively. In the instantaneous, binary and elastic scheme , if p, q and p', q' stand for the two momenta before and after shock , the collision operator Q is defined by:

$$Q\left(f,h\right) = Q_{+}\left(f,h\right) - Q_{-}\left(f,h\right),\ \ f,h : \mathbb{R}^3 \to \mathbb{R} \qquad (2.14)$$

$$Q_{+}\left(f,h\right) = \int\int_{\mathbb{R}^3\times S^2} ab^2 f\left(\overline{p'}\right) h\left(\overline{q'}\right) v_{\phi}\sigma(k,\theta) d\overline{q} d\omega, \qquad (2.15)$$

$$Q_{-}\left(f,h\right) = \int\int_{\mathbb{R}^3\times S^2} ab^2 f\left(\overline{p}\right) h\left(\overline{q}\right) v_{\phi}\sigma(k,\theta) d\overline{q} d\omega. \qquad (2.16)$$

The energy momentum conservation is written as

$$p^0 + q^0 = p'^0 + q'^0, \qquad \overline{p} + \overline{q} = \overline{p'} + \overline{q'}. \qquad (2.17)$$

As suggested in [4] and [8], we parametrize the post-collisional momenta as follows: p^{α} and q^{α} being given, we first consider

$$n^{\alpha} = p^{\alpha} + q^{\alpha},\ t^{\alpha} = \left(n_i\omega^i,\ -n_o\omega\right),\ \ \omega \in S^2 \qquad (2.18)$$

then, the post-collisional momenta are represented by:

$$p'^{\alpha} = \frac{p^{\alpha} + q^{\alpha}}{2} + \frac{k}{2}\frac{t^{\alpha}}{\sqrt{t_{\beta}t^{\beta}}}, \quad q'^{\alpha} = \frac{p^{\alpha} + q^{\alpha}}{2} - \frac{k}{2}\frac{t^{\alpha}}{\sqrt{t_{\beta}t^{\beta}}}. \tag{2.19}$$

They satisfy the mass shell condition and energy momentum conservation.

As shown in [1] , the Jacobian of the change of variable $\left(\bar{p}, \bar{q}\right) \longmapsto \left(\overline{p'}, \overline{q'}\right)$ is

$$\frac{\partial\left(\overline{p'}, \overline{q'}\right)}{\partial\left(\overline{p}, \overline{q}\right)} = \frac{p'^{0} q'^{0}}{p^{0} q^{0}}. \tag{2.20}$$

2.4. **Assumptions on the scattering kernel.** In this work, we assume that the scattering kernel has the form

$$\sigma\left(k, \theta\right) = k^{\beta}\sin^{\gamma}\theta, \; -2 < \gamma \leq -1, \; 0 \leq \beta < \gamma + 2. \tag{2.21}$$

Since $\frac{k}{\delta}$ is a bounded, a kernel of this form falls into the hard potential case.

2.5. **The Maxwell-Boltzmann system in** (F, f)**.** Setting $\beta = 0$ in the first equation (2.10) , we easily deduce that

$$J^{0} = 0. \tag{2.22}$$

By (2.22) , the expression (2.11) of J^{β} with $\beta = 0, \; u^{0} = 1$ gives:

$$e\left(t\right) = \int_{\mathbb{R}^{3}} f\left(t, \overline{p}\right) ab^{2}\, d\overline{p}, \tag{2.23}$$

and shows that f determines e.

The second set of the Maxwell equations is identically satisfied and the first set reduces to $\partial F_{ij} = 0$, so:

$$F_{ij} = F_{ij}(0) := \varphi_{ij}. \tag{2.24}$$

(2.24) means that the magnetic part F_{ij} does not evolve during time.

It remains to determine the electric part $F^{0i} := E^{i}$.

Writing (2.11) for $\beta = i$, using (2.10) for $\alpha = 0$ and (2.3) gives:

$$J^{i} = \int_{\mathbb{R}^{3}} \frac{p^{i} f\left(t, \overline{p}\right) ab^{2}}{p^{0}} d\overline{p}, \tag{2.25}$$

$$\dot{E}^{i} + \Gamma^{i}_{0j}E^{j} = \int_{\mathbb{R}^{3}} \frac{p^{i} f\left(t, \overline{p}\right) ab^{2}}{p^{o}} d\overline{p}. \tag{2.26}$$

Since $f = f\left(t, \overline{p}\right)$, the Boltzmann equation (2.13) can be written:

$$\frac{\partial f}{\partial t} + \frac{P^{i}}{P^{0}}\frac{\partial f}{\partial p^{i}} = Q\left(f, f\right). \tag{2.27}$$

Still using the letter f instead of $f^{\#}$ usually used in the standard notation , solving the non linear PDE (2.27) is equivalent to solve the characteristic system:

$$\frac{dt}{1}=\frac{dp^1}{\frac{P^1}{p^0}}=\frac{dp^2}{\frac{P^2}{p^0}}=\frac{dp^3}{\frac{P^3}{p^0}}=\frac{df}{Q\left(f,f\right)}=ds, \tag{2.28}$$

which allows to take t as parameter. Now we obtain from (2.8) and (2.3) :

$$\frac{P^i}{p^0}=-2\Gamma^i_{0j}p^j-e\left[F^{0i}+g^{ii}\frac{p^kF_{ik}}{p^0}\right],\ i=1,2,3. \tag{2.29}$$

Using relations (2.23) , (2.25) , (2.28) and (2.29) , the Maxwell-Boltzmann system transforms into a Maxwell-Boltzmann -Momentum system of the form:

$$\begin{cases} \dot{E}^i=-\Gamma^i_{0j}E^j+\int_{\mathbb{R}^3}\frac{q^if(t,\overline{q})ab^2}{q^0}d\overline{q}\ \ (a)\\ \dot{p}^i=-2\Gamma^i_{0j}p^j-\left[E^i+g^{ii}\frac{p^k}{p^0}\varphi_{ki}\right]\int_{\mathbb{R}^3}f\left(t,\overline{q}\right)ab^2d\overline{q}\ \ (b)\\ \frac{df}{dt}=Q\left(f,f\right)\ \ (c)\\ F_{ij}=F_{ij}(0)=\varphi_{ij}.\ \ (d)\end{cases} \tag{2.30}$$

Now f and $\overline{p}$ are independant variables for the integro-differential system (2.30) .

The collision operator expresses in terms of covariant variables using (2.9) as

$$Q\left(f,f\right)\left(t,v\right)=a^{-1}b^{-2}\int_{S^2}d\omega\int d\overline{u}v_{\phi}\sigma(k,\theta)\left[f\left(t,\overline{v'}\right)f\left(t,\overline{u'}\right)-f\left(t,\overline{v}\right)f\left(t,\overline{u}\right)\right]$$

and the Boltlzmann equation (2.27) becomes:

$$\frac{\partial f\left(t,v\right)}{\partial t}=Q\left(f,f\right)\left(t,v\right). \tag{2.31}$$

Now we introduce some useful functional spaces.

2.6. **Functional spaces.** The framework for the distribution function f is $L^1_r\left(\mathbb{R}^3\right)$, the subspace of $L^1\left(\mathbb{R}^3\right)$ whose norm, denoted $\|\ .\ \|_{1,r}$, $r\geq 0$ is defined by:

$$L^1_r\left(\mathbb{R}^3\right)=\left\{f\in L^1\left(\mathbb{R}^3\right):\|\ f\ \|_{1,r}=\int_{\mathbb{R}^3}\left|f\left(\overline{p}\right)\right|\left(p^0\right)^r d\overline{p}<+\infty\right\}.$$

We will denote $\|\ .\ \|_{1,1}$ by $\|\ .\ \|$ and we define:

$$\left|f\left(t\right)\right|_{1,r}=\int_{\mathbb{R}^3}\left|f\left(t,\overline{v}\right)\right|<\overline{v}>^r d\overline{v},\quad <\overline{v}>=\sqrt{1+\left|\overline{v}\right|^2}.$$

Consequently, we have:

$$\| f(t) \|_{1,r} \leq |f|_{1,r} \leq b^r(t) \| f(t) \|_{1,r}. \tag{2.32}$$

Now, we set for $r \in \mathbb{R},\ r > 0$:

$$X_r = \left\{ f \in L^1_1\left(\mathbb{R}^3\right), f \geq 0 \ \text{ a.e } \ \| f \| \leq r \right\}. \tag{2.33}$$

X_r is a complete and connected metric space for the induced norm.

For any real interval I, we set:

$$\mathcal{C}\left([I; L^1_1\left(\mathbb{R}^3\right)]\right) = \left\{ f : I \longrightarrow L^1_1\left(\mathbb{R}^3\right),\ f \text{ continuous and bounded} \right\},$$

$$\mathcal{C}\left([I; X_r]\right) = \left\{ f \in \mathcal{C}\left(\left[I; L^1_1\left(\mathbb{R}^3\right)\right]\right),\ f(t) \in X_r,\ \forall t \in I \right\}. \tag{2.34}$$

$$|||f||| = \sup \{ \| f(t) \|,\ t \in I \},\ \ f \in \mathcal{C}\left(\left[I; L^1_1\left(\mathbb{R}^3\right)\right]\right).$$

$\mathcal{C}\left([I; X_r]\right)$ is a complete metric space for the induced norm.

The frame work for $\overline{p}$ and $\overline{E}$ is $\mathbb{R}^3$, with the norm $\|.\|$ or $\|.\|_{\mathbb{R}^3}$.

$$\mathcal{C}\left(\left[I; \mathbb{R}^3\right]\right) = \left\{ m : I \longrightarrow \mathbb{R}^3,\ m \text{ continuous and bounded} \right\}$$

is a Banach space for the norm $|||m|||, = \sup \{ \|m(t)\|,\ t \in \mathbb{R} \}$.

We define on $\mathbb{R}^3 \times \mathbb{R}^3 \times L^1_1\left(\mathbb{R}^3\right)$ and on $\mathcal{C}\left([I; \mathbb{R}^3]\right) \times \mathcal{C}\left([I; \mathbb{R}^3]\right) \times \mathcal{C}\left([I; L^1_1\left(\mathbb{R}^3\right)]\right)$:

$$\left\| \left(\overline{p}, \overline{E}, f\right) \right\| = \|\overline{p}\| + \left\|\overline{E}\right\| + \|f\|, \tag{2.35}$$

$$\left|\left|\left| \left(\overline{p}, \overline{E},, f\right) \right|\right|\right| = |||\overline{p}||| + \left|\left|\left|\overline{E}, \right|\right|\right| + |||f|||. \tag{2.36}$$

3. THE MAXWELL-BOLTZMANN SYSTEM WITH A BOUNDED KERNEL

For technical purpose, in this section we change the scattering kernel $k\sqrt{\delta}\sigma$ into a bounded kernel $S\left(\overline{p}, \overline{q}, \overline{p'}, \overline{q'}\right)$, a non-negative continuous real valued function of its arguments, and on which we additionally require that:

$$\begin{cases} 0 \leq S\left(\overline{p}, \overline{q}, \overline{p'}, \overline{q'}\right) \leq C_1 \\ \left| S\left(\overline{p_1}, \overline{q}, \overline{p'}, \overline{q'}\right) - S\left(\overline{p_2}, \overline{q}, \overline{p'}, \overline{q'}\right) \right| \leq C_1 \left\|\overline{p_1} - \overline{p_2}\right\|, \end{cases} \tag{3.1}$$

where C_1 is positive constant. The Boltzmann equation (2.13) changes as:

$$\frac{\partial f}{\partial t} + \frac{P^i}{p^0}\frac{\partial f}{\partial p^i} = \frac{1}{p^0}\overline{Q}\left(f, f\right), \tag{3.2}$$

$$\overline{Q}\left(f, h\right) = \overline{Q}_+\left(f, h\right) - \overline{Q}_-\left(f, h\right), \tag{3.3}$$

$$\overline{Q}_+\left(f, h\right) = \int_{\mathbb{R}^3} \int_{S^2} \frac{ab^2}{q^0} f\left(\overline{p'}\right) h\left(\overline{q'}\right) S\left(\overline{p}, \overline{q}, \overline{p'}, \overline{q'}\right) d\omega d\overline{q}, \tag{3.4}$$

$$\overline{Q}_{-}(f,h)=\int_{\mathbb{R}^3}\int_{S^2}\frac{ab^2}{q^0}f(\overline{p})\,h(\overline{q})\,S\left(\overline{p},\overline{q},\overline{p'},\overline{q'}\right)d\omega d\overline{q}. \tag{3.5}$$

The Maxwell-Boltzmann-Momentum system (2.30) transforms in:

$$\begin{cases} \dot{E^i}=-\Gamma^i_{0j}E^j+\int_{\mathbb{R}^3}\frac{q^i f(t,\overline{q})ab^2}{q^0}d\overline{q}=H_1\left(t,\overline{p},\overline{E},f\right)(a)\\ \dot{p^i}=-2\Gamma^i_{0j}p^j-\left[E^i+g^{ii}\frac{p^k}{p^0}\varphi_{ki}\right]\int_{\mathbb{R}^3}f(t,\overline{q})\,ab^2d\overline{q}=H_2\left(t,\overline{p},\overline{E},f\right)(b)\\ \frac{df}{dt}=\frac{1}{p^0}\overline{Q}(f,f,\overline{p})=H_3\left(t,\overline{p},\overline{E},f\right)(c)\\ F_{ij}=F_{ij}(0)=\varphi_{ij}\ \ (d). \end{cases} \tag{3.6}$$

3.1. **Local existence of solutions.** In what follows and in the next, we briefly review the results of [5].

First, we estimate the differences in $f,\overline{E},\ \overline{p}$ in L^1_1 and $\mathbb{R}^3$ norms:

Proposition 1. *Let* $\overline{p_1},\overline{p_2},\overline{E_1},\overline{E_2}\in\mathbb{R}^3,\ f_1,f_2\in L^1_1(\mathbb{R}^3)$. *Then:*

$$\begin{cases} \left\|H_1\left(t,\overline{p_1},\overline{E_1},f_1\right)-H_1\left(t,\overline{p_2},\overline{E_2},f_2\right)\right\|_{\mathbb{R}^3}\le C_2\left(\left\|\overline{E_1}-\overline{E_2}\right\|_{\mathbb{R}^3}+\|f_1-f_2\|\right)\ (a)\\ \left\|H_2\left(t,\overline{p_1},\overline{E_1},f_1\right)-H_2\left(t,\overline{p_2},\overline{E_2},f_2\right)\right\|\\ \qquad\le C_3\left(\|\overline{p_1}-\overline{p_2}\|+\left\|\overline{E_1}-\overline{E_2}\right\|_{\mathbb{R}^3}+\|f_1-f_2\|\right)\ (b)\\ \left\|H_3\left(t,\overline{p_1},\overline{E_1},f_1\right)-H_3\left(t,\overline{p_2},\overline{E_2},f_2\right)\right\|_{\mathbb{R}^3}\le C_4\left(\|\overline{p_1}-\overline{p_2}\|+\|f_1-f_2\|\right)\ (c) \end{cases} \tag{3.7}$$

where

$$\begin{cases} C(t)=8\pi C_1ab^2(t),\ C_2=3C+b^2\\ C_3=5(6C+1)\left(1+a+\frac{b^2}{a}\right)\left(1+\frac{a}{b}+\frac{b}{a}+\frac{1}{a}+\frac{1}{b}\right)\times\\ \qquad(1+\|f_2\|)(1+ab^2)\left(1+\|f_2\|+\left\|\overline{E_1}\right\|\right)\\ C_4=C(1+a+2b)\left(1+\|f_1\|+\|f_2\|+\|f_2\|^2\right)\\ C_5=C_2+C_3+C_4. \end{cases} \tag{3.8}$$

In order to state the local existence theorem, we first recall this useful theorem:
Theorem 2. *Let* $t_0\ge 0$, $\left(\overline{p_{t_0}},\overline{E_{t_0}},f_{t_0}\right)\in\mathbb{R}^3\times\mathbb{R}^3\times L^1_1(\mathbb{R}^3)$ *be given. Then:*

1. There exists a real number $\tau>0$ such that the differential system (3.6) has a unique solution $\left(\overline{p},\overline{E},f\right)\in\mathcal{C}\left([t_0,t_0+\tau];\mathbb{R}^3\right)^2\times\mathcal{C}\left([t_0,t_0+\tau];L^1_1(\mathbb{R}^3)\right)$ satisfying $\left(\overline{p},\overline{E},f\right)(t_0)=\left(\overline{p_{t_0}},\overline{E_{t_0}},f_{t_0}\right)$. Moreover f satisfies:

$$|||f|||=\sup\left\{\|f(t)\|,t\in[t_0,t_0+\tau]\right\}\le\|f_{t_0}\|. \tag{3.9}$$

2. The Maxwell-Boltzmann system $(2.10), (3.2)$ has a unique local solution (F, f) on $[t_0, t_0 + \tau]$ such that

$$F^{0i}(t_0) = E^i_{t_0},\ F_{ij} = F_{ij}(t_0) = \varphi_{ij},\ f(t_0) = f_{t_0},\ |||f||| \leq \|f_{t_0}\|. \tag{3.10}$$

We end by stating the following local existence theorem coming from **theorem 2**.

Theorem 3. *Let* $\overline{p_0}, \overline{E_0} \in \mathbb{R}^3,\ f_0 \in L^1_1(\mathbb{R}^3),\ \varphi_{ij} \in \mathbb{R}$ *be given.*

Then there exists a real number $T > 0$ such that:

1. The differential system (3.6) has a unique solution $\left(\overline{p}, \overline{E}, f\right) \in \mathcal{C}([0,T]; \mathbb{R}^3)^2 \times \mathcal{C}([0,T]; L^1_1(\mathbb{R}^3))$ such that $\left(\overline{p}, \overline{E}, f\right)(0) = \left(\overline{p_0}, \overline{E_0}, f_0\right)$. Moreover:

$$|||f||| \leq \|f_0\|. \tag{3.11}$$

2. The Maxwell-Boltzmann system $(2.10), (3.2)$ has a unique solution (F, f) satisfying $F^{0i}(0) = E^i_0,\ F_{ij} = F_{ij}(0) = \varphi_{ij},\ f(0) = f_0$.

3.2. Global existence theorem.

3.2.1. ***The method***. To prove global existence, the authors in [5] used the following method:

Let $[0, T[$ be the maximal existence domain of solution of the system (3.6) denoted here by $\left(\widetilde{\overline{p}}, \widetilde{\overline{E}}, \widetilde{f}\right)$ and given by **theorem 3** with the initial data $\left(\overline{p_0}, \overline{E_0}, f_0\right) \in \mathcal{C}([0,T]; \mathbb{R}^3)^2 \times \mathcal{C}([0,T]; L^1_1(\mathbb{R}^3))$.

We want to prove that $T = +\infty$.

(a) If we already have $T = +\infty$, the problem of existence is solved.

(b) If we suppose $0 < T < +\infty$, then the solution $\left(\widetilde{\overline{p}}, \widetilde{\overline{E}}, \widetilde{f}\right)$ can be extended beyond T, which contradicts the maximality of T.

Supposing $0 < T < +\infty$ and $t_0 \in [0, T[$, it is shown in [5] that there exists a number $\tau > 0$ independant of t_0 such that the system (3.6) has a unique solution $\left(\overline{p}, \overline{E}, f\right)$ on $[t_0, t_0 + \tau]$, with the initial data $\left(\widetilde{\overline{p_0}}, \widetilde{\overline{E_0}}, \widetilde{f_0}\right)$ at $t = t_0$. Then taking t_0 sufficiently close to T, for example, t_0 such that $0 < T - t_0 < \frac{\tau}{2}$ and hence $T < t_0 + \frac{\tau}{2}$, we can extend the solution $\left(\widetilde{\overline{p}}, \widetilde{\overline{E}}, \widetilde{f}\right)$ to $\left[0, t_0 + \frac{\tau}{2}\right]$ which strictly contains $[0, T[$, and this contradicts the maximality of T.

3.2.2. ***Preliminary results***. The following preliminary results were used in [5] :

Lemma 4. *The maps* $t \mapsto \widetilde{\overline{E(t)}},\ t \mapsto \widetilde{\overline{p(t)}}$ *are uniformly bounded over* $[0, T]$.

Proposition 5. *Let* $t_0 \in [0, T[,\ \left(\widetilde{\overline{p_{t_0}}}, \widetilde{\overline{E_{t_0}}}, \widetilde{f_{t_0}}\right) \in \mathcal{C}([t_0, t_0+\tau]; \mathbb{R}^3)^2 \times \mathcal{C}([t_0, t_0+\tau]; L^1_1(\mathbb{R}^3))$ *be given. Then there exists a number* $\tau \in]0, 1[$*, independent of* t_0*, such that the* *system* (3.6) *has a unique solution* $\left(\overline{E}, \overline{p}, f\right) \in \mathcal{C}\left([t_0, t_0+\tau[\,; \mathbb{R}^3\right)^2 \times \mathcal{C}\left([t_0, t_0+\tau[\,; X_r\right)$ *such that* $\left(\overline{E}, \overline{p}, f\right)(t_0) = \left(\widetilde{\overline{p_{t_0}}}, \widetilde{\overline{E_{t_0}}}, \widetilde{f_{t_0}}\right)$.

3.2.3. ***The global existence theorem:*** Based on the method detailed above and using preliminary results, the following global existence theorem was proved in [5] :

Theorem 6. *Let* $\overline{p_0}, \overline{E_0} \in \mathbb{R}^3$, $f_0 \in L^1_1(\mathbb{R}^3)$, $\varphi_{ij} \in \mathbb{R}$ *be given, such that* $\|f_0\| \leq r$ *where* $r > 0$ *is a given real number. Then:*

1) The differential system (3.6) has a unique global solution $\left(\overline{E}, \overline{p}, f\right)$ defined all over the interval $[0, +\infty[$ and such that $= \left(\overline{E}, \overline{p}, f\right)(0) = \left(\overline{E_0}, \overline{p_0}, f_0\right)$ and

$$|||f||| \leq \|f_0\|\,,\ f(t) \geq 0,\ t \in [0, +\infty[\,.$$

2) The Maxwell-Boltzmann system $(2.10)\,,(3.2)$ has a unique global solution (F, f) on $[0, +\infty[$ satisfying :

$$F^{0i}(0) = E^i_0,\ F_{ij} = F_{ij}(0) = \varphi_{ij},\ f(0) = f_0,\ |||f||| \leq \|f_0\|\,.$$

4. THE MAXWELL-BOLTZMANN SYSTEM IN HARD POTENTIAL CASE

Here, we extend the result of **theorem 6** to some hard potential case. We still consider the Maxwell-Boltzmann-Momentum system (2.30) with the collision operator now given by:

$$\begin{aligned} Q(f,f) &= \textstyle\int_{\mathbb{R}^3}\int_{S^2} v_\phi \sigma(k,\theta)\left(f' f'_* - f f_*\right) a b^2 d\omega d\overline{q} \\ &= a^{-1} b^{-2} \textstyle\int_{S^2} d\omega \int d\overline{u} v_\phi \sigma(k,\theta)\left(f' f'_* - f f_*\right). \end{aligned}$$

4.1. **The method.** We construct a modified system by truncating a certain part of the collision kernel in the equation $(2.30 - c)$. As in the truncated system the scattering kernel is bounded, global existence of solution is insured by **theorem 6**. Then we obtain a sequence of solutions of the truncated system, and showing that this is a Cauchy sequence we obtain a solution of the initial system (2.30).

4.2. **Preliminary results.** We start by the following useful lemmas.

Lemma 7. *The following inequalities hold*

$$k \leq \sqrt{\delta},\ k \leq 2\sqrt{u^0 v^0},\ \sqrt{\delta} \leq 2\sqrt{u^0 v^0},\ k \leq a^{-1}|v-u|. \tag{4.1}$$

Proof. The results are obtained by simple calculations. □

Lemma 8. *For the collision operator, the following property holds :*

For any measurable function h depending only of k, δ and ω, we have:

$$\iiint \frac{h(k,\delta,\omega)}{p^0q^0}\left(f'f'_\star - ff_\star\right)\left(p^0\right)^r d\omega d\overline{q}d\overline{p}$$
$$= \frac{1}{2}\iiint \frac{h(k,\delta,\omega)}{p^0q^0} ff_\star \left(\left(p'^0\right)^r + \left(q'^0\right)^r - \left(p^0\right)^r - \left(q^0\right)^r\right) d\omega d\overline{q}d\overline{p}.$$

Proof. See [4] □

Lemma 9. *Consider the collisional process in the Bianchi type I spacetime. Let* (p^α, q^α) *and* (p'^α, q'^α) *be pre and post collisional momenta respectively.*

For $r > 1$, consider:

$$G = \left(p'^0\right)^r + \left(q'^0\right)^r - \left(p^0\right)^r - \left(q^0\right)^r.$$

Then G satisfies

$$G \leq \mathcal{C}_r \left((p^0)^{r-1}q^0 + p^0(q^0)^{r-1}\right). \tag{4.2}$$

If ω is restricted to the subset $\left\{\omega \in S^2 : |n.\omega| \leq \frac{a^2(t)}{\sqrt{2}b^2(t)}|n|\right\}$,then:

$$G \leq \mathcal{C}_r \left((p^0)^{r-\frac{1}{2}}(q^0)^{\frac{1}{2}} + (p^0)^{\frac{1}{2}}(q^0)^{r-\frac{1}{2}}\right) - c_r\left((p^0)^r + (q^0)^r\right) \tag{4.3}$$

where $\mathcal{C}_r$ and c_r are two different non-negative constants depending on r.
Proof. By the energy momentum conservation, we have

$$p^0 + q^0 = p'^0 + q'^0, \ \textit{for each } p^0 \textit{ and } q^0.$$

Let p^α and q^α be given. By the inequality

$$\begin{cases} \alpha^r + \beta^r \leq (\alpha+\beta)^r \leq \alpha^r + \beta^r + C_r\left(\alpha^{r-1}\beta + \alpha\beta^{r-1}\right) \\ \alpha, \beta \geq 0;\ r > 1, \end{cases} \tag{4.4}$$

we deduce that

$$(p'^0)^r + (q'^0)^r \leq (p^0)^r + (q^0)^r + \mathcal{C}_r\left((p^0)^{r-1}(q^0) + (p^0)(q^0)^{r-1}\right).$$

Then

$$G \leq \mathcal{C}_r \left((p^0)^{r-1}q^0 + p^0(q^0)^{r-1}\right). \tag{4.5}$$

For the second result, suppose that $|n.\omega| \leq \frac{a^2}{\sqrt{2}b^2}|n|$ and $p'^0 \geq q'^0$.
Then p'^0 is estimated as

$$p'^0 \leq \frac{p^0+q^0}{2}$$

$$+\frac{k}{2}\frac{|a^2(t)n^1\omega^1 + b^2\left(n^2\omega^2 + n^3\omega^3\right)|}{\sqrt{(n^0)^2\left(a^2(t)(\omega^1)^2 + b^2(t)\left((\omega^2)^2 + (\omega^3)^2\right)\right) - \left(a^2(t)n^1\omega^1 + b^2(t)\left(n^2\omega^2 + n^3\omega^3\right)\right)^2}}.$$

And we notice that

$$\frac{|a^2(t)n^1\omega^1 + b^2\left(n^2\omega^2 + n^3\omega^2\right)|}{\sqrt{(n^0)^2\left(a^2(t)(\omega^1)^2 + b^2(t)\left((\omega^2)^2 + (\omega^3)^2\right)\right) - \left(a^2(t)n^1\omega^1 + b^2(t)\left(n^2\omega^2 + n^3\omega^2\right)\right)^2}} \leq 1$$

if and only if
$2\left(a^2n^1\omega^1 + b^2\left(n^2\omega^2 + n^3\omega^3\right)\right)^2 \leq (n^0)^2\left(a^2(\omega^1)^2 + b^2\left((\omega^2)^2 + (\omega^3)^2\right)\right).$
Now using the facts that

$$a \leq b,\ \delta = \left(n^0\right)^2 - \left(a^2(n^1)^2 + b^2\left((n^2)^2 + (n^3)^2\right)\right) \geq 0,$$

we easily deduce that:

$$|n.\omega| \leq \frac{a^2}{\sqrt{2}b^2}|n| \Rightarrow 2b^4\left(n.\omega\right)^2 \leq a^4|n|^2$$

$$\Rightarrow 2\left(a^2(n^1\omega^1)^2 + b^2\left((n^2\omega^2)^2 + (n^3\omega^3)^2\right)\right)$$

$$\leq a^2\left(a^2(n^1)^2 + b^2\left((n^2)^2 + (n^3)^2\right)\right)$$

Then $|n.\omega| \leq \frac{a^2}{\sqrt{2}b^2}|n|$ implies, using lemma 10 that:

$$p'^0 \leq \frac{p^0 + q^0}{2} + \frac{k}{2} \leq \frac{\left(\sqrt{p^0} + \sqrt{q^0}\right)^2}{2}$$

So G is estimated as:

$$G \leq 2\left(p'^0\right)^r - \left(p^0\right)^r - \left(q^0\right)^r \leq \frac{1}{2^{r-1}}\left(\sqrt{p^0} + \sqrt{q^0}\right)^{2r} - \left(p^0\right)^r - \left(q^0\right)^r$$

$$\leq \frac{\left(p^0\right)^r}{2^{r-1}} + \frac{\left(q^0\right)^r}{2^{r-1}} + \mathcal{C}_r\left(\left(p^0\right)^{r-\frac{1}{2}}\left(q^0\right)^{\frac{1}{2}} + \left(p^0\right)^{\frac{1}{2}}\left(q^0\right)^{r+\frac{1}{2}}\right) - \left(p^0\right)^r - \left(q^0\right)^r$$

$$\leq \mathcal{C}_r\left(\left(p^0\right)^{r-\frac{1}{2}}\left(q^0\right)^{\frac{1}{2}} + \left(p^0\right)^{\frac{1}{2}}\left(q^0\right)^{r+\frac{1}{2}}\right) - c_r\left(\left(p^0\right)^r + \left(q^0\right)^r\right)$$

where (4.4) is used and $\mathcal{C}_r,\ c_r > 0$ are constants depending on r. □

Let m be any positive integer: Now we modify the Maxwell-Boltzmann-Momentum system (2.30) by setting :

$$\begin{cases} \dot{E}^i_m = -\Gamma^i_{0j}E^j_m + \int_{\mathbb{R}^3}\frac{q^i f_m(t,\overline{q})ab^2}{q^0}d\overline{q} \\ \dot{p}^i_m = -\Gamma^i_{0j}p^j_m - \left[E^i_m + g^{ii}\frac{p^k_m}{p^0_m}\varphi_{ki}\right]\int_{\mathbb{R}^3} f_m\left(t,\overline{q}\right)ab^2d\overline{q} \\ \frac{df_m}{dt} = ab^2\int_{\mathbb{R}^3}\int_{S^2} v_{\phi,m}\left(k_m\right)^\beta\sigma_{0,m}\left(\omega\right)\left(f'_m f'_{m\star} - f_m f_{m\star}\right)d\omega d\overline{q} = Q_m\left(f_m, f_m\right) \\ F_{ij} = F_{ij}(0) = \varphi_{ij}\ \ ,\ \overline{E_m}(0) = \overline{E_0},\ \ \overline{p_m}(0) = \overline{p_0},\ f_m(0) = f_0 \end{cases} \tag{4.6}$$

where $v_{\phi,m} := \frac{\min\left\{k\sqrt{\delta}, m\right\}}{p^0 q^0}$, $k_m := \min\left\{k, m\right\},\ \sigma_{0,m} := \min\left\{\sigma_0(\omega), m\right\}$.

Since the scattering kernel of $Q_m(f_m, f_m)$ is bounded case, we conclude by **theorem 6** that the truncated equation (4.6) has a unique global solution

$$\left(\overline{E_m}, \overline{p_m}, f_m\right) \in \mathcal{C}\left([0, +\infty[\,; \mathbb{R}^3\right)^2 \times \mathcal{C}\left([0, +\infty[\,, L^1_1(\mathbb{R}^3)\right)$$

such that $\left(\overline{E_m}, \overline{p_m}, f_m\right)(0) = \left(\overline{E_0}, \overline{p_0}, f_0\right)$.

The following lemma establishes that the sequence f_m is a Cauchy sequence.

Lemma 10. *For any $r \geq 0$ and $T > 0$, there exists a constant $\mathcal{C}_r$ which does not depend on m such that if $\| f_0 \|_{1,r}$ is bounded, then:*

$$\sup_m \sup_{t\in[0,T]} |f_m(t)|_{1,r} + \| f_m(t) \|_{1,r} \leq \mathcal{C}_r. \tag{4.7}$$

Proof. We first estimate $\|f_m(t)\|_{1,r}$ and then obtain the result using (2.32).

By **theorem 6** we have

$$\sup_{t\in[0,T]} \|f_m(t)\|_{1,r} \leq C$$

where $C = \|f_0(t)\|_{1,1}$ does not depend on m for $0 \leq r \leq 1$

because for $r \leq s \leq 1$, $\|f_m(t)\|_{1,r} \leq \|f_m(t)\|_{1,s}$. Now we assume that $r > 1$.

Since v^0 decreases with time for each $\overline{v}$, using (2.2)and (2.9), we have:

$$\partial_t v^0 = -\left(\frac{\dot{a}(t)}{a^3(t)}(v^1)^2 + \frac{\dot{b}(t)}{b^3(t)}\left((v^2)^2 + (v^3)^2\right)\right)\frac{1}{v^0} \leq 0.$$

By direct calculation using equation (2.31), we have:

$$\frac{d}{dt}|f_m(t)|_{1,r} =$$

$$a^{-1}b^{-2}\int\int\int v_{\phi,m}(k_m)^\beta \sigma_{0,m}(\omega)\left(f'_m f'_{m\star} - f_m f_{m\star}\right)\left(v^0\right)^r d\omega d\overline{v} d\overline{u} + \int f_m(t,\overline{v})\frac{\partial v^0}{\partial t} d\omega$$

and the second integral is negative. Hence, we may only consider:

$$\frac{d}{dt}|f_m(t)|_{1,r} \leq a^{-1}b^{-2}\int\int\int v_{\phi,m}(k_m)^\beta \sigma_{0,m}(\omega)\left(f'_m f'_{m\star} - f_m f_{m\star}\right)\left(v^0\right)^r d\omega d\overline{v} d\overline{u}.$$

By **lemma 8 ,** we have:

$$\frac{d}{dt}|f_m(t)|_{1,r} \leq$$

$$\frac{a^{-1}b^{-2}}{2}\int\int\int v_{\phi,m}(k_m)^\beta \sigma_{0,m}(\omega) f_m f_{m\star}\left[(v'^0)^r + (u'^0)^r - (v^0)^r - (u^0)^r\right] d\omega d\overline{v} d\overline{u}.$$

Using the fact that $a^{-1}b^{-2}$ is bounded, we apply **lemma 9** and some calculations of **lemma 3.6** in [4] to obtain

$$\sup_m \sup_{t\in[0,T]} |f_m(t)|_{1,r} \leq \mathcal{C}_r.$$

By (2.32), we obtain the desired result, and the proof is completed. □

Lemma 11. *Consider the sequence $\{f_m\}$ on any interval $[0,T]$. For each small number $\varepsilon > 0$, there exists a positive integer M such that if $k, m \geq M$:*

$$\sup_{t\in[0,T]} |f_k(t) - f_m(t)|_{1,1} \leq \varepsilon. \tag{4.8}$$

Proof. The proof is similar to the proof of the previous lemma.

We first estimate $\|f_k(t) - f_m(t)\|_{1,r}$ then by (2.32), the result follows.

Using the relation (2.31), we have :

$$\frac{d}{dt}\|f_k(t) - f_m(t)\|_{1,r} =$$

$$= \int Sgn(f_k - f_m)(Q_k(f_k, f_k) - Q_m(f_m, f_m))\, v^0 d\overline{v}$$

$$- \int \left(\frac{\dot{a}(t)}{a^3(t)}(v^1)^2 + \frac{\dot{b}(t)}{b^3(t)}\left((v^2)^2 + (v^3)^2\right)\right) |f_k(t,\overline{v}) - f_m(t,\overline{v})| \frac{1}{v^0} d\overline{v}$$

$$\leq \int Sgn(f_k - f_m)(Q_k(f_k, f_k) - Q_m(f_m, f_m))\, v^0 d\overline{v}.$$

It remains to follow the proof of **lemma 3.7** in [4] and obtain a positive integer N such that if $k, m \geq N$ then:

$$\sup_{t\in[0,T]} \|f_k(t) - f_m(t)\|_{1,1} \leq \varepsilon.$$

Thus , the desired result is obtained by (2.32) and the proof is completed. □

Lemma 12. *Consider the sequences $\left\{\overline{E_m}\right\}$ and $\{\overline{p_m}\}$ on any finite interval $[0,T]$. For any small number $\varepsilon > 0$, there exists a positive integer M such that if $k, m \geq M$, then*

$$\sup_{t\in[0,T]} \left\|\overline{E_k}(t) - \overline{E_m}(t)\right\| \leq \varepsilon, \tag{4.9}$$

$$\sup_{t\in[0,T]} \|\overline{p_k}(t) - \overline{p_m}(t)\| \leq \varepsilon. \tag{4.10}$$

Proof. We consider the relations $(3.6-a),(3.7-a)$ to deduce that:

$$\left\|\overline{E_k}^{\cdot}(t)-\overline{E_m}^{\cdot}(t)\right\| \leq C_2\left(\left\|\overline{E_k}(t)-\overline{E_m}(t)\right\|+\left\|f_k(t)-f_m(t)\right\|\right).$$

Using the expression of C_2 given by (3.8), relations (2.4) , we easily deduce that there exists a positive absolute constant C_6 such that:

$$C_2 \leq C_6 = C_6 = C_6(a_0, b_0, T, C_1).$$

Then

$$\left\|\overline{E_k}^{\cdot}(t)-\overline{E_m}^{\cdot}(t)\right\| \leq C_6\left(\left\|\overline{E_k}(t)-\overline{E_m}(t)\right\|+\left\|f_k(t)-f_m(t)\right\|\right).$$

Integrating over $[0,t]$,we obtain:

$$\left\|\overline{E_k}(t)-\overline{E_m}(t)\right\| \leq C_6\left(T\sup_{t\in[0,T]}\left\|f_k(t)-f_m(t)\right\|+\int_0^t\left\|\overline{E_k}(s)-\overline{E_m}(s)\right\|ds\right),$$

$t\in[0,T]$. By Cronwall inequality, we obtain:

$$\left\|\overline{E_k}(t)-\overline{E_m}(t)\right\| \leq TC_6\sup_{t\in[0,T]}\left\|f_k(t)-f_m(t)\right\|e^{C_6T}$$

Then (2.32) and **lemma 11** allow to conclude.
Using the same scheme, invoking this time $(3.6-b),(3.7-b)$ we obtain:

$$\left\|\overline{p_k}^{\cdot}(t)-\overline{p_m}^{\cdot}(t)\right\| \leq C_3\left(\left\|\overline{p_k}(t)-\overline{p_m}(t)\right\|+\left\|f_k(t)-f_m(t)\right\|+\left\|\overline{E_k}(t)-\overline{E_m}(t)\right\|\right).$$

Using the expression of C_3 given by (3.8), relations (2.4) , invoking **lemma 4** and **theorem 6** to bound $\left\|\overline{E_m}(t)\right\|$ and $\left\|f_m(t)\right\|$, we easily deduce that there exists a positive absolute constant C_7 such that:

$$C_3 \leq C_7 = C_7\left(a_0, b_0, \|f_0\|, \left\|\overline{E_0}\right\|, T, C_1, C\right).$$

Then

$$\left\|\overline{p_k}^{\cdot}(t)-\overline{p_m}^{\cdot}(t)\right\| \leq C_7\left(\left\|\overline{p_k}(t)-\overline{p_m}(t)\right\|+\left\|f_k(t)-f_m(t)\right\|+\left\|\overline{E_k}(t)-\overline{E_m}(t)\right\|\right)$$

Integrating over $[0,t]$ and using the Gronwall lemma, we obtain:

$$\left\|\overline{p_k}(t)-\overline{p_m}(t)\right\| \leq TC_7\sup_{t\in[0,T]}\left(\left\|\overline{E_k}(t)-\overline{E_m}(t)\right\|+\left\|f_k(t)-f_m(t)\right\|\right)e^{C_7T}$$

Then (2.32), **lemma 11** and the inequality (4.9) give the relation (4.10).
So, the proof is completed. □

4.3. **The global existence theorem.** Now we can state the main result of this work.

Theorem 13. *Let* $\overline{p_0}, \overline{E_0} \in \mathbb{R}^3$, $\varphi_{ij} \in \mathbb{R}$, $f_0 \in L^1_r(\mathbb{R}^3)$ *be given, with* $r > 1 + \frac{\beta}{2}$ *and* $f_0 \geq 0$. *Suppose that the scattering kernel has the form* (2.21).

- Then the equivalent Maxwell-Boltzmann-Momentum system (3.6) *has a unique global solution* $(F, \overline{p}, f)$ *such that* $f \in \mathcal{C}([0, +\infty[, L^1_1(\mathbb{R}^3))$ *with* $f(t) \geq 0$ *and satisfying* $F^{i0} := F^{i0}(0) = E^i_0, F_{ij} = F_{ij}(0) = \varphi_{ij}, f(0,.) = f_0$.

- (F, f) is the unique global solution of the Maxwell-Boltzmann system (2.10), (2.13).

Proof. **Lemmas 11** and **12** show that the sequence $\left\{\left(\overline{E_m}, \overline{p_m}, f_m\right)\right\}$ is a Cauchy sequence in the Banach space $(\mathbb{R}^3)^2 \times L^1_1(\mathbb{R}^3)$. Hence there exists $\left(\overline{E}, \overline{p}, f\right)$ a solution of the system (2.30) with initial condition $\left(\overline{E_0}, \overline{p_0}, f_0\right)$. The initial condition $f_0 \in L^1_r(\mathbb{R}^3)$ with $r > 1 + \frac{\beta}{2}$ comes from **lemma 3.7** in [4] and the non-negativity of f is guaranteed by the same lemma. The uniqueness is obtained by the proof of **lemmas 11** and **12**. This complete the proof. ☐

5. CONCLUSION

This work was devoted to extend the result of [5] who considered the homogeneous relativistic Maxwell- Boltzmann system for a bounded scattering kernel with an additional hypothesis of invariance under a subgroup of $\mathcal{O}_3$. In the present work, we discarded this hypothesis. After presenting the bacground spacetime, the unknowns and the equations, we considered the Maxwell-Boltzmann system for a bounded kernel and we briefly recalled the results of [5]. The same system has been considered in case of hard potential kernel. The method followed was the one used in [4], relying in the use of a particular form of Povzner inequality, but in a more difficult situation, because the Boltzmann equation was coupled with the Maxwell equations. Some energy estimates allowed us to obtain global existence theorem and uniqueness of mild solutions.

In our future investigations, we will consider an inhomogeneous magnetized Boltzmann equation for both bounded and hard potential cases.

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