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LABORATORY OF THE PHYSICS OF THE EARTH'S ENVIRONMENT

Effects of viscosity, surface tension and wind on hydrodynamic waves for shallow water

Submitted and defended publicly in partial fulfillment of the requirements for the Doctorate/Ph.D. degree in Physics

Specialty: Physics of the Earth's Environment

Option: Sciences of Earth's Atmosphere

By

Mouassom Fernand Léonel

Registration Number: 12W0902
Master in Physics

Under the Supervision of:

MVOGO Alain
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Year 2022

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ATTESTATION DE CORRECTION DE LA THÈSE DE DOCTORAT/Ph.D

Nous, Professeur **VONDOU Debertini Appolinaire,** Professeur **ENYEGUE A NYAM Françoise**, et Professeur **BEN–BOLIE Germain Hubert**, respectivement Examinateurs et Président du jury de la thèse de Doctorat/Ph.D de Monsieur **MOUASSOM Fernand Léonel**, Matricule **12W0902**, préparée sous la direction du Professeur **MBANE BIOUELE César** et Professeur **MVOGO Alain**, intitulée : « ***Effects of viscosity, surface tension and wind on hydrodynamic waves for shallow water*** », soutenue le **Mercredi, 07 Décembre 2022**, en vue de l'obtention du grade de Docteur/Ph.D en Physique, spécialité **Physique de l'Environnement Terrestre**, option **Sciences de l'Atmosphère**, attestons que toutes les corrections demandées par le jury de soutenance ont été effectuées.

En foi de quoi, la présente attestation lui est délivrée pour servir et valoir ce que de droit.

28 FEV 2023

Examinateurs

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Dedication

To my mother

Madame NGOUAMBE Pauline

Acknowledgments

I would like to thank **GOD** for providing this opportunity to learn more about the complexity and beauty of his creation. I thank all of those who have helped me and enabled me to be in a position where I can submit this thesis.

My sincerely thanks to my supervisor **Prof. Mbane Biouele César** who, from the time I enrolled in the master's program to the completion of this thesis, has meticulously followed me. This work would not have been possible without the many long hours he spent discussing nonlinear questions in shallow water with me. While leaving me a great freedom to work, he encouraged me to persevere in the scientific research way. I take this opportunity to tell him that his support was decisive.

My sincerely thanks to my director **Dr. Mvogo Alain** for his time and patience, his humour and passion for all things integrable. The opportunity to undertake research with him in this area has given me the honor of enriching my experience in both research and writing methodologies. Here are a so many other virtues that I have received from this great hero of nonlinear sciences. I hope that he will find in this work the height of the efforts he has made.

I thank **Prof. Ndjaka Jean Marie Bienvenu** the head of Physics departement, as well as the authorities of our university institution for their objectivity in the organization of the research.

I would also like to thank all the teachers of the University of Yaounde I, especially the teachers of the Department of Physics, for their valuable teaching throughout my academic career.

I would like to thank the members of the pre-defense jury and the members of the defense jury namely, **Prof. Ben-Bolie Germain Hubert**, **Prof. Zekeng Serge**, **Prof. Vondou Derbetini Appolinaire**, **Prof. Enyegue A Nyam Francoise**, **Prof. Hona Jacques**, **Prof. Essimbi Zobo Bernard**, **Prof. Djuidje Kenmoe Aloyem** of the University of Yaounde 1, and **Prof. Kenfack Sadem Christian** of the University of Dschang, who, in spite of their multiple occupations, agreed to examine this work in order to give him the benefit of their experiences and thus to bring ample clarifications to the work carried out. I was extremely honored to discuss my results with these eminent experts.

I would like to thank **Prof. Tchoundja Edgard Landry** of the Department of Mathematics of the University of Yaoundé 1, for having helped and advised me throughout my studies since my first year at the university. For his prodigious advice and encouragement. His help in the mathematical aspect of the work has been of great help in the realization of this thesis.

I thank **Prof. Vondou Derbetini Appolinaire** who gave me the opportunity to work in the Laboratory for Environmental Modelling and Atmospheric Physics in which reigns an atmosphere of research and a scientific spirit. I thank him for his numerous advices and his constructive remarks during the sessions of presentation of the progress of the works organized within the submentioned laboratory.

I especially thank my aunt **Manikak Georgette** (my second mother to whom I owe a lot) and her husband **Leumassom Pagal** for their patience and their financial support during these last years. They never stopped encouraging me in difficult moments, and congratulating me in moments of success, I thank them very much.

My profound gratitude goes to my whole family and freinds. They have always supported and encouraged me, I take the opportunity to tank them. In particular, I acknowledge the infallible support and encouragements from my brother **Deni Yannick Raphael**, my sisters **Ngouetou Marthe Blandine**, **Nzakou Djianou Kévine** and **Manikak Nzakou Francline**.

I thank in a general way all the members of the LAMEPA laboratory. I do not forget to thank particularly the teachers and the academic seniors who constitute it, for their professionalism. Their undeniable contributions to the good functioning of the laboratory, and especially for their availability and their kindness.

I would like to thank my comrades of promotion for their multiform contributions, especially **Tchoungui Hervé** for his unconditional friendship and our fruitful exchanges, which have contributed in an indisputable way to the realization of this work.

May all those who have contributed in any manner their efforts to this work, see in these words the expression of may deep gratefulness.

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List of symbols

x : is the horizontal spatial coordinate along the (Ox) axis,

y : is the horizontal spatial coordinate along the (Oy) axis,

z : is the vertical spatial coordinate along the (Oz) axis,

t : is the time coordinate,

g : is the acceleration of the gravity,

c_0 : is the velocity of propagation of linear waves in the limit large wavlengths,

h : is the depth of the fluid,

η : is the surface elevation of the fluid,

σ^2 : is the variance of the random variable η,

H_s : is the significant height of a sea state, which is a height commonly used in physical oceanography and coastal engineering,

P_a : is the atmospheric pressure,

P : is the pressure of the fluid,

μ : is the dynamics viscosity of the fluid,

ν : is the kinematics viscosity of the fluid,

σ : is the surface tension parameter,

ρ_a : is the air density,

ρ : is the fluid density,

s : is the shelter coefficient,

v_{ph} : is the wave phase velocity,

v_{gr} : is the wave group velocity,

U_∞ : is the wind velocity,

U_1 : is the characteristic velocity,

$U = U(z)$: is the air flow velocity as a function of altitude,

u_* : is the roughness velocity,

κ : is the constant of Von Kàmàn, $\kappa \approx 0.4$,

α_{ch} : is the Charnock parameter,

z_0 : is roughness velocity,

k : is the wavenumber,

∇ : is nabla operator,

α : is the amplitude parameter, which measures the ratio of wave amplitude to undisterbed fluid depth,

β : is the wavelenght parameter which measures the square of the ratio of fluid depth to wavelenght,

τ : is the Bohm number (which are considered as the surface tension parameter in this work),

δ : is the viscosity parameter,

χ : is the wind parameter.

List of acronyms

DNA : deoxyribonucleic acid,

fKdV : fifth-order Korteweg-De Vries,

gNNV : generalized Nizhnik-Novikov-Veselov,

KdV : Korteweg-De Vries,

KP : Kadomtsev-Petviashvili,

MI : modulational instability,

mKdV : modified Korteweg-De Vries,

mNV : modified Novikov-Veselov,

nKdV : ninth-order Korteweg-De Vries,

NLS : nonlinear Schrödinger,

NNV : Nizhnik-Novikov-Veselov,

NV : Novikov-Veselov,

sKdV : seventth-order Korteweg-De Vries,

SSE : Sasa-Satsuma equation,

UK : United Kingdom,

UNESCO : United Nations Educational, Scientific and Cultural Organization,

ZK : Zakharov-Kuznetsov,

Abstract

This thesis is devoted to the theoretical study of the effects of viscosity, surface tension and wind on hydrodynamic waves for shallow water. Various theories have been formulated for the study of weakly damped free-surface flows. These theories have essentially focused on forces relatively perpendicular to the fluid volume such as gravity forces, while neglecting forces relatively parallel to the fluid volume such as pressure forces due to wind and shear forces due to viscosity. In this work, some corrections due to viscosity are applied to the kinematics boundary condition at the surface and the dynamics condition modeled by Bernoulli's equation. By using a linear approximation applied to the Navier-Stokes equation, we obtain a system of equations for the potential flow that includes the dissipative effect due to visosity for right- and left-moving waves. The pertubation theory applied to the Boussinesq system leads to some new higher-order generalized Korteweg De Vries (KdV) equations with nonlinear, dissipative and wind forcing terms. The wind effects are integrated into the model equations through the expression of atmospheric pressure proposed by the Miles model in which only the terms participating in the energy transfer (terms in phase quadrature with the surface elevation) are considered. In the absence of wind effects, these model equations describe the propagation of solitons in the viscous medium. We find the soliton solutions of each corresponding equation and then investigate on the effects of surface tension, viscosity and wind on the waves dynamics. The results show that such effects can strongly impact the group and phase velocities and the soliton dynamics. We can conclude that, these new equations obtained in this thesis, can be considered as improved versions of the KdV equation and can better describe the shallow water soliton dynamics. In addition, these equations can lead to several applications in various fields of nonlinear science.

Key words: Shallow water; KdV equation; Soliton; Viscosity; Surface tension; Wind effect.

Résumé

Cette thèse est consacrée à l'étude théorique des effets de la viscosité, de la tension de surface et du vent sur les ondes hydrodynamiques en eaux peu profondes. Diverses théories ont été formulées pour l'étude de l'écoulement des fluides faiblement amortis dans un canal à surface libre. Ces théories se sont essentiellement concentrées sur les forces relativement perpendiculaires à un volume de fluide telles que les forces de gravité, tout en négligeant les forces relativement parallèles à un volume de fluide telles que les forces de pression dues au vent et les forces de cisaillement dues à la viscosité. Dans ce travail, des corrections dues à la viscosité sont appliquées à la condition limite cinématique à la surface et aussi à la condition dynamique modélisée par l'équation de Bernoulli. En utilisant une approximation linéaire appliquée à l'équation de Navier-Stokes, nous obtenons un système d'équations pour un écoulement potentiel incluant l'effet dissipatif dû à la viscosité pour les vagues se déplaçant à droite et à gauche. La théorie de perturbation appliquée au system de Boussinesq conduit à de nouvelles équations de type Korteweg De Vries (KdV) généralisées d'ordre supérieur avec des termes non linéaires, dissipatifs et de forçage du vent. Les effets du vent sont intégrés dans nos équations à travers l'expression de la pression atmosphérique proposée par le modèle de Miles dans lequel seuls les termes participant au transfert d'énergie (termes en quadrature de phase avec l'élévation de la surface) sont considérés. En l'abscence des effets du vent, ces modèles décrivent la propagation des solitons dans un milieu visqueux. Nous étudions les solutions de type solitons de chaque équation et investiguons les effets de la tension de surface, de la viscosité et du vent sur la dynamique des vagues. Les résultats montrent que de tels effets peuvent avoir un impact important sur les vitesses de phase et de groupe et sur la dynamique des solitons. Nous pouvons conclure que, ces nouvelles équations obtenues dans cette thèse, peuvent être considérées comme des versions améliorées de l'équation de KdV et peuvent permettre de mieux décrire la dynamique des solitons en eau peu profonde. De plus ces équations peuvent conduire à plusieurs applications dans le domaines des sciences non linéaires.

Mots clés: Eau peu profonde; Equation de KdV; Soliton; Viscosité; Tension superficielle; Effet du vent.

General Introduction

Shallow water is defined as water whose the depth is such that propagating surface waves are significantly affected by the bottom topography. In general, the depth of the water is less than half the wavelength of the waves propagating in these medium. This type of configuration is generally found in coastal or littoral areas. Today, according to UNESCO, about half of the world's population lives in coastal areas and this amount is expected to increase to 75% or 6.3 billion people in 2025 [1]. Human interactions with shallow waters are increasing due to maritime trade, mass tourism and energy industries: oil, gas and renewable energy. In addition, climate change and the intensification of human activities, services, and population in the world's coastal areas are leading to a gradual rise in sea level and a change in the physico-chemical properties of waters in shallow areas. Infrastructure and human lives are regularly endangered by hydrodynamic waves associated with extreme weather events such as cyclones or storms. These extreme events mainly result in the formation of exceptional waves with a strong rise in water level on the coast and inundation of the land. The human and material damage caused by these types of waves can be considerable as the tsunami of December 26, 2004 in the Indian Ocean, which is undoubtedly the most deadly in history with more than 285,000 deaths. In shallow water, the wave motion is usually reflected in a visible deformation on the water surface. These deformations give rise to exceptional waves of different natures including tides, tsunamis, swells and many others. Thus, the understanding and modeling of the propagation of hydrodynamic waves for shallow water, related to these extreme events is a subject of great interest for the scientific community.

Among these exceptional hydrodynamic waves, the most interesting is the solitary wave because it has several intriguing properties. Indeed, a solitary wave is a wave localized in space with a very particular shape (regular and symmetrical), able to propagate with a constant velocity over a long distance without changing its shape or amplitude [2]. Moreover, contrary to all expectations, these waves emerge from a collision with each other keeping exactly their initial shape and velocity [3]. The first observation of a wave with characteristics similar to those of solitary waves was made in 1834 by John Scott Russell. He reproduced these waves experimentally and presented the results to the scientific community, which has been doubtful. Indeed, it was hard to believe that a wave could exist with such properties, at that time there were two types of models to explain the propagation of waves on the water surface. The first model is linear dispersif and predicts that a wave with an initial profile localized in space will tend to sag and generate ripples in its wake. The second model is nonlinear nondispersif and predicts that an initial profile localized in space will tend to straighten out to create a shock wave. However, these two type of models do not reproduce the results obtained by John Scott Russell experiments. Several years later, Korteweg and de Vrie following the work of Boussinesq and Rayleigh on solitary waves, have established the first descriptive equation called Korteweg-de Vries (KdV) equation [4]. This equation has both nonlinear and dispersive terms. It should be note that, it is the subtle balance between the nonlinear effect and the dispersif effect that is at the origin of the solitary waves corresponding to John Scott Russell's observation.

In the same line, several equations have been derived including, the Boussinesq equation [5], the Kadomtsev-Petviashvili equation [6], the nonlinear Schrödinger equation [7], just name a few. All these integrable equations provide remarkable solitary wave solutions which, have found many applications in several fields of nonlinear sciences. For example, in optical fibers, solitons can be used to transfer large amounts of information over long distances while minimizing errors in the signal [8]. In biophysics, the storage of light and information in DNA is done by means of waves, so a soliton will allow the long term storage of information patterns [9]. In the context of shallow water, the KdV equation which describes the motion of small but finite amplitude waves that propagate in the positive x-direction [10], has been the subject of intensive work. Indeed, the KdV equation is obtained by applying the pertubation theory to the Boussinesq system. Since the beginning of the 1990's, many formulations have been developed to jointly improve the nonlinear and dispersive properties. In that sense, several authors have improved the KdV equation by introducing high-order terms, leading generally to near partially integrable or integrable high-order equations with quasi-soliton solutions with new effects [11]. Fokou et al. [12], with the help of the Boussinesq perturbation theory, have derived a higher-order KdV equation. This equation contains many nonlinear and nonlocal terms that describe well the long, small-amplitude, unidirectional wave motion in shallow water with surface tension. The modification of the KdV equation by taking into account the viscosity has been also investigated by Dias et al.[13], Depassier and Letelier [14] and Sajjadi and Smith [15], just to cite a few. As result, these authors concluded that taking into account the effects of viscosity in the modeling of the KdV equation can better describes the wave dynamics. In the same line, several works have been devoted to study the impact of the wind on the dynamics of nonlinear waves. The Jeffreys model [16] assumes that, waves move in the opposite direction to the wind and thus constitute a barrier to the wind flow. In this model, the windward side of the wave receives the wind frontally, while the leeward side is sheltered. This creates a pressure difference and thus a force exerted on the wave [17]. The model developed by Phillips [18], assumes that the fluid flow is potential and the air flow is turbulent. The turbulence of the air creates random pressure variations which create waves. The Miles model [19] is based on an analysis of the stability of parallel flows in both air and water. It consider air to be incompressible and non-viscous, with a logarithmic velocity profile [17].

During the 20th century, starting from one or the other of the overcited models, several authors have studied the impact of wind on the dynamics of extreme waves. For example, by using a pressure distribution over the steep crests given by the Jeffreys model, Kharif et al. [20] have investigated experimentally and numerically the influence of wind on extreme wave events in deep water. Touboul et al. [21] have used the Jeffreys model and investigated experimentally without wind and in presence of wind the rogue wave formation due to the dispersive focusing mechanism. The authors conclude that, the duration of the rogue wave event increases with the wind velocity. In the same line, amplification of nonlinear surface waves by wind have been investigated by Leblanc using the Miles' model [22]. Despite all these interesting works, the question of the interaction between wind and waves remains an open subject. Indeed, all these works listed above have been carried out in order to study the effects of wind on the dynamics of extreme waves such as rogue waves. Most of these works deal only with the generalized nonlinear Schrödinger equations. Motivated by the obtained results, we go beyond by extending the study to the generalized KdV equation in the context of shallow water. We simultaneously combine the dissipation due to viscosity, surface tension and wind effects and show that, in the context of wave motion in shallow water, an expansion of the Boussinesq system can be decomposed into a set of coupled equation. We show that, for any order of expansion, one of these equations depends only on the surface elevation for the right-moving, while the other depends simultaneous on the surface elevation for the right- and left-moving wave. These equation have been shown to better describes the propagation of solitary waves in shallow water. The wave equation corresponding to the pure right-moving has the form of a generalized KdV equation that includes higher diffusion and instability effects. We solve this equation using Hirota's bilinear method and the results show that, such effects have considerable impacts on shallow water wave dynamics. The combination of such effects has never been studied in the literature to the best of our knowledge.

The rest of this dissertation is presented in three chapters organized as follows.

In chapter I, we present a literature review on hydrodynamic waves. We present some categories of waves that can be encountered in shallow water. We recall their history, main characteristics, conditions of existence and mechanisms of generation. We also present some physico-chemical and atmospheric parameters that can have an impact on the wave dynamics.

In Chapter II, we present the basic equations related to the physical and mathematical modeling of the equations describing the dynamics of waves propagating in shallow water. Thereafter, we present the methodologies applied to derive the new generalized higher-order KdV equations, taking into account the effects of viscosity, surface tension and wind.

In Chapter III, we present the main results of this thesis. We clearly present the effects of viscosity, surface tension, and wind on phase and group velocities and on the solitons dynamics in general. We show that, the physical parameters as mentioned above have a considerable impact on solitons dynamics. We also demonstrate in this charter that, taking into account such parameters leads to some improved versions of the KdV equation and then can better describe the waves dynamics in shallow water.

The present thesis ends with a general conclusion. We summarize our results and give some future directions that could be investigared.

Chapter I

Literature review on hydrodynamic waves

Introduction

The wave motion is usually reflected in a visible deformation on the water surface. These deformations give rise to waves of different natures including tides, tsunamis, solitons and rogue waves. This chapter presents the literature review on these hydrodynamic waves. In fact, we present some categories of waves that can be encountered in shallow water. Theirs history, main characteristics, conditions of existence and mechanisms of generation are recall. We also presents some physico-chemical and atmospheric parameters that can have an impact on the wave dynamics.

1.1 Hydrodynamic waves

Since man has been sailing, he has always been impressed by the presence and even the nature of the waves encounters on the sea and ocean surface. Over the years, numerous testimonies of sailors referring to walls of water rising without reason in the middle of the sea, and hitting the ships with extraordinary violence has been collected by the scientific community. The increase of these testimonies gave more and more necessity to the study of the dynamics of surface water waves. This is how hydrodynamics field became more and more interesting to researchers. Hydrodynamics is the branch of science or physics that deals with the study of fluid motion in relation to the forces that cause it. The study of hydrodynamic waves at the water surface is very interesting because, it corresponds to many phenomena directly accessible to observation. It include many different classes of waves depending on the boundary conditions, ranging from nondispersive and linear waves to dispersive and nonlinear waves. There are several types of hydrodynamic waves, the best known are the tsunamis. Indeed, since the giant tsunami that occurred in the Indian Ocean on December 26, 2004, this phenomenon is the object of particular interest. But the socalled rogue waves, solitary waves as well as tidal waves are also hydrodynamic waves, certainly less publicized, but just as deadly.

1.1.1 Solitary waves

Solitary wave is a wave localized in space with a very particular shape (regular and symmetrical), able to propagate with a constant velocity over a long distance without changing its shape or amplitude [2]. Moreover, contrary to all expectations, these waves emerge from a collision with each other keeping exactly their initial shape and velocity [3].

1.1.1.1 History of Solitary waves

Since the 19th century, the scientific community has been greatly interested in the study of natural phenomena not only from mathematics and physics point of view, but also from the perspective of engineering and scientific sciences. One of these important and spectacular phenomena provided by nature first called the “great Wave of Translation” or solitary wave, was uncovered by the mathematician and naval engineering John Scott Russell. Indeed in the 1830s, the engineer John Scott Russell observed a strange wave on the Union Canal in Scotland that didn’t disperse in the normal fashion but instead held its form as it traveled down the canal. He tried to reproduce this type of wave experimentally and as illustrated in Figure 1.1 (takes in [2]) this partially successful attempt on the Union Canal presents a John Scott Russell wave called solitary waves.

Figure 1.1: Soliton in water original 1834 phenomenon reproduced on the Scott Russell Aqueduct over the Union Canal near Heriot-Watt University (UK), 12 July 1995.

His laboratory results show a wave localized in space with a very particular, regular and symmetrical shape that moves at a constant velocity over a very long distance without changing its shape. These results have been presented to the scientific community which, at first, was doubtful to say the least. It was hard to believe that a wave could exist with such properties, because at that time there were two types of models to explain the propagation of waves on the water surface. The first model is linear dispersif in the sense that, a component of the flow frequency moves with a velocity that depends solely on the frequency of this component. Indeed, this type of model predicts that a wave with an initial profile localized in space will tend to sag and generate ripples in its wake as shown in Figure 1.2 (takes in [2]).

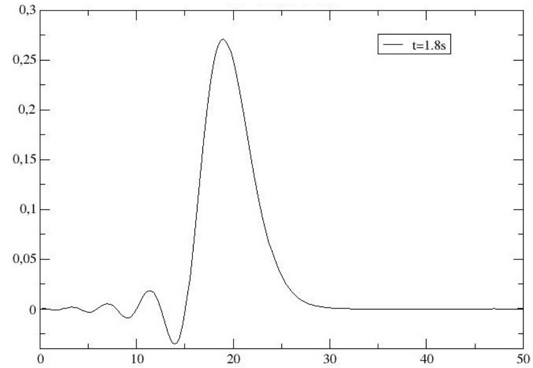


Figure 1.2: Evolution of a wave whose dynamics is governed by a linear and dispersive model.

The second model is nonlinear nondispersif in the sense that, a spatial component of the flow moves with a velocity which depends only on the amplitude. Here, an initial profile localized in space will tend to straighten out to create a shock wave as shown in Figure 1.3 (takes in [2]).

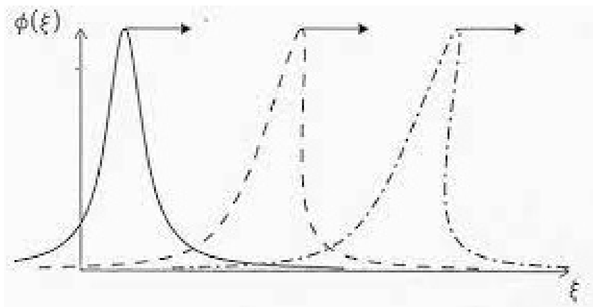


Figure 1.3: Evolution of a wave whose dynamics is governed by a nonlinear and nondispersive model (Burgers equation).

It is therefore several years later that Korteweg and De Vries [4], following the work of Rayleigh [23] and Boussinesq [5] (who showed that if dissipation is neglected, the increase in local wave velocity associated with finite amplitude is balanced by the decrease associated with dispersion, leading to a wave of permanent form) on the solitary waves, will establish the first descriptive equation called KdV equation, having both a nonlinear and dispersive terms. It is worth nothing that, it is the subtle equilibrium between the nonlinear effect and the dispersif effect that is at the origin of the waves corresponding to John Scott Russell's observation (solitary waves). This balance is generally very stable, which explains the various applications of the solitons theory, even if the real physical situations are only approximately described by the equations having soliton as solutions and in particular by the KdV equation [24].

For almost 70 years after the work of Korteweg and de Vries, the theory of solitary wave has been considered a relatively unimportant research subject in the field of nonlinear wave theory. Indeed, from a mathematical point of view, it was generally thought that the collision of two solitary waves would leads to a strong nonlinear interaction and eventually destroy them. The fact that this is not true has left many surprises for future workers in this field. Indeed, the works of Zabusky and Kruskal [3] and Gardner et al. [25] proved that it is possible to create explicit solutions of the KdV equation which behave as a superposition of two solitary waves with variable amplitudes and thus with variable velocities generating solitary wave collisions. They have shown therefore contrary to what was thought from the mathematical point of view that, during the collision of two solitary waves, these two waves immerse from the collision keeping exactly the same shape as shown in Figure 1.4 (takes in Soomere [26]). Thus, we only retain from the collision between two solitary waves a certain change of phase which is explained by the fact that their position in space is not rectilinear.

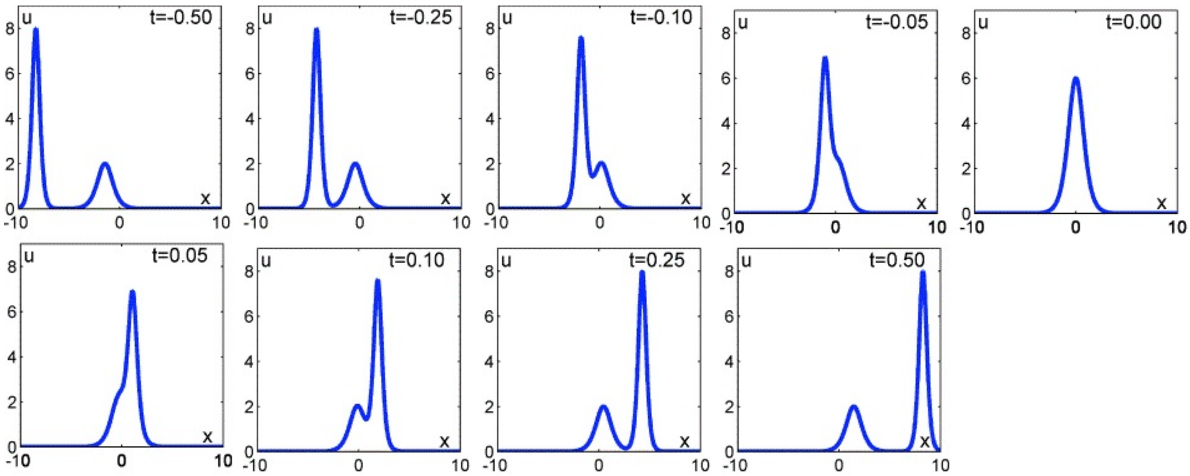


Figure 1.4: Illustration of the interaction between two solitons of different amplitudes

It is fair to say that the study of nonlinear waves, during the first half of the 20th century, was not considered as an important research field by many physicists or mathematicians. Because during this period, the attention of the researchers was held by fields such as quantum mechanics and nuclear physics. Practical applications of water waves were enhanced by activities during second World War [10]. The resurgence of activity followed with the invention of the electronic computer and the use of linear Fourier analysis to spectrally analyze the wave trains first measured. However, the study of the solitary wave has remained an important and unfinished field of research, with interesting applications in many areas of the nonlinear sciences.

1.1.1.2 Modelling of solitary waves in shallow water: The KdV equations

Euler's equation for an incompressible and non viscous fluid, bottom and surface boundary conditions, and the assumption of irrotational flow lead to the classical KdV equation given as

$$\frac{1}{c_0}\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{3\eta}{2h}\frac{\partial \eta}{\partial x} + \frac{h^2}{6}\frac{\partial^3 \eta}{\partial x^3} = 0, \tag{1.1}$$

where $c_0 = \sqrt{gh}$ is the velocity of propagation of linear waves in the limit large wavelengths, h the depth and η the height of the liquid surface above its level of equilibrium. We notice that the soliton resulting from the model of the KdV equation given by equation (1.1) is supersonic because its propagation velocity is greater than the velocity c_0 [2]. Thus, by performing the transformation $X = x - c_0 t$ and $T = t$, which will allow us to move to a moving frame of reference at velocity c_0, it is possible to eliminate the second term of the equation thus, we have

$$\frac{1}{c_0}\frac{\partial \eta}{\partial T} + \frac{3\eta}{2h}\frac{\partial \eta}{\partial X} + \frac{h^2}{6}\frac{\partial^3 \eta}{\partial X^3} = 0. \tag{1.2}$$

Since the study deals with small amplitude and long length waves, it is preferable to scale the variables to avoid any ambiguity corresponding to a different physical situation. Thus, by scaling the variables so that $u = \eta/h$, $x = X/X_0$ and $t = T/T_0$ (where X_0 is the length and T_0 the time), we can obtain the KdV equation in its standard form as follows

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1.3}$$

The KdV equation (1.3) describes the motion of small but-finite amplitude shallow water waves that propagate in the positive x-direction [10]. In recent years, various and powerful direct methods such as Darboux transformation Bäcklund transformation, Inverse Scattering Transformation, Hirota bilinear method, Symmetry method, just to cite a few, have been developed to investigate the analytical solutions of KdV equation. In addition, some other constructive approaches, which use the solutions of simple nonlinear differential equation namely ansätz to express the corresponding solutions of complicated nonlinear wave equations have been also constructed to investigate the analytical solutions of KdV equation. As example, there are Similarity transformation, Riccati equation expansion method, Tanh method, Extended tanh function method, Modified extended tanh function method, Generalized hyperbolic function method, Tanh-coth method, Jacobi elliptic function expansion method, just to cite a few. All these methods inform that, the KdV equation has among others, the following spatially localized solutions

$$u(x,t) = A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{2}}\,(x-2At)\right] \qquad (A>0), \tag{1.4}$$

where A is a positive coefficient and the function $\mathrm{sech}(x) = 1/\cosh(x)$. A graphical representation of these solutions as shown in Figure 1.5 (takes in [10]) confirms that, these solutions are in quantitative agreement with the observations of Jonh Scott Russell.

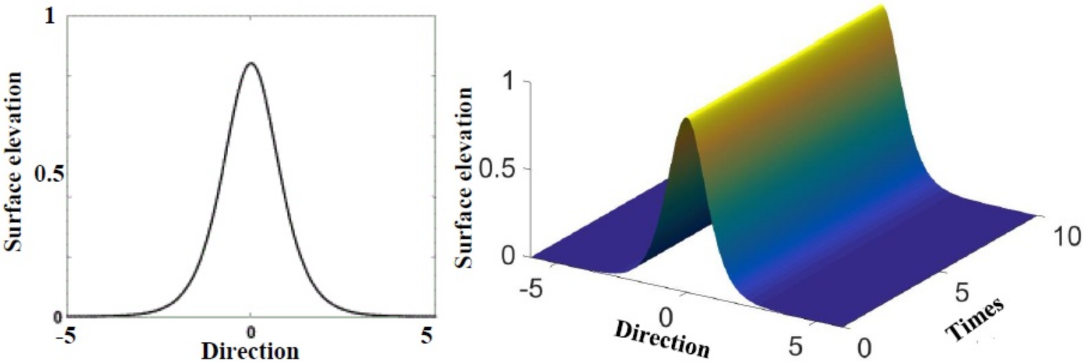


Figure 1.5: Graphical illustration of the soliton solution for the KdV equation

One of the particularities of this solution is the decrease of the width of the soliton with the amplitude as shown in figure 1.6 (takes in [10]). If we go back to variable dimensions and in a fixed reference frame, the corresponding solution is given by

$$\eta(x,t) = \eta_0\,\mathrm{sech}^2\left[\frac{1}{2h}\sqrt{\frac{3\eta_0}{h}}\left(x - c_0\left[1+\frac{\eta_0}{2h}\right]t\right)\right]. \tag{1.5}$$

It is clearly shown that, the velocity of the soliton is greater than the velocity c_0, then confirm the assumption stating that the KdV soliton is supersonic [10].

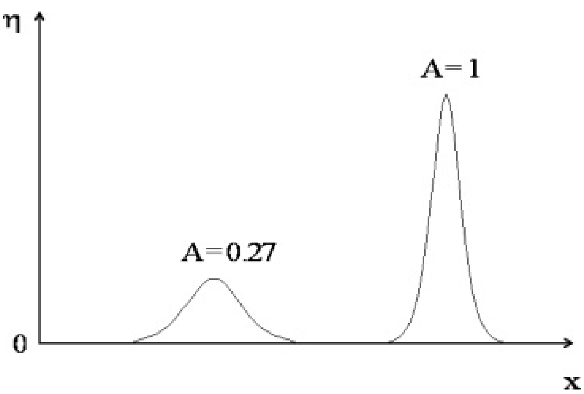


Figure 1.6: Comparison between two solutions of the KdV equation with different amplitude.

1.1.1.3 Some apllications of solitary waves

The KdV equation established by Diederik Johannes Korteweg and Gustav de Vries appears in many fields of physics, whenever waves can propagate in a weakly nonlinear and weakly dispersive medium. Let us now consider some of the most typical solitary waves applications.

In the atmosphere, based on the similarity between the behavior of a shallow fluid with a free surface and that of a shallow layer of cold air on which rests a deep layer of potentially warmer air, Abdullah [27] was led to propose a hypothesis that solitary waves could exist in the atmosphere. Because of their relatively very fast movements and their small dimensions, it was difficult to detect these disturbances [27]. The first clear indication of the existence of atmospheric solitary waves has been point out by Fawbush and Miller [28], who observed that small migratory anticyclones, which they called "bubbles", are sometimes observed in connection with tornado formation. These authors have hypothesized that those "bubbles" are of the same nature as the atmospheric solitary waves. Several studies [27, 29, 30] have been carried out in order to propose the mechanism of formation of these atmospheric solitary waves. Thus, these studies show that a stagnant inversion layer can be in contact with a stationary front. The stationary front can move through this air by means of an impulse it receives from the air behind it, and can stop after moving a short distance. A single rise can then be created in the cold air that can move as a solitary wave. Abdullah [27] go further and shows that if, during the formation of atmospheric solitary waves, the appropriate vertical accelerations are produced, it can move a considerable distance without much change in shape. If the associated vertical accelerations are negligible, the wave will eventually break up and behave as a small pressure jump. We can therefore speak of two different types of atmospheric solitary waves: the breaking wave and the permanent wave. Because a atmospheric solitary wave is an elevated mass of cold air progressing on a layer of inversion, it is expected that it lifts the less dense air which lies above that layer as it passes through it. Its effect may be, therefore, simulated to that of a mountain which forces the wind to rise on its forward slope. Condensation may result because of this lift if the lifted air contains enough humidity so that it may reach its lifting condensation level while it is being raised. If the upper air is stable, the effect of a passing solitary wave would be limited to the formation of a cloud which appears to move with the same velocity as that of the solitary wave [29]. If the wave is of the breaking type the turbulence caused during the process of breaking would result in convective type cumulus clouds, but if it is of the permanent type the clouds tend to be of the stratus form and no turbulence may be expected as long as the air remains stable [29].

In optical fiber communication using linear pulses, dispersion and losses in the fiber limit the information capacity which can be transported and the distance of transmission. Through the work of Hasegawa and Tappert [8], it is now known that solitons can be used to transfer large amounts of information over long distances while minimizing errors in the signal.

In biophysics, the storage of light and information in DNA is done by means of waves, so a soliton will allows the long term storage of information patterns [31]. The soliton model in neuroscience is a recent developed model that attempts to expalain how signals are conducted within neurons. This model proposes that the signals travel along the cell's membrane in the form of pulse solitons [32].

In solid state physics, solitons allow the interpretation of the properties of dielectric materials. Moreover, magnetic materials are interesting examples to verify experimentally and in a very precise way the theory of solitons at the atomic scale [33]. The concept of soliton in polymer physics is a very nice case of interdisciplinary approach. Their appearance was suggested in 1988 by theoretical physicists [34].

1.1.2 Rogue waves

Nowadays, the study of rogue waves attracts a great deal of interest from scientific community, especially in nonlinear sciences. Rogue waves are giant single waves that may suddenly appear in oceans [35]. These are also known as monstrous waves, deadly waves or extreme waves. Their appearance can be quite unexpected and their origin is mysterious. Contrary to the dispersive behaviour adopted by traditional waves of low amplitude, rogue waves are self-reinforcing packets of solitary waves. The rogue wave phenomenon is not just a spectacular event accessible to routine observations and satellite images but also a combination of complex physical processes that occur under the accuracy conditions [36].

1.1.2.1 History and discovery of rogue waves

Among the legends that have long circulated within the community of sailors to testify the disappearance and sinking of ships in the ocean, we find that of the rogue waves. Numerous testimonies of sailors have long alluded to walls of water rising for no reason in the middle of the sea and hitting the ships with extraordinary violence. It is in 1978 following the disappearance of the cargot "Munchen" that all these stories have had a meaning. Indeed, this cargot, at the cutting edge of naval technology, sank completely one night when the weather forecast had not recorded any storm. Among the traces of the sinking, the rescue crew found a lifeboat which had been torn off violently 20 meters above the waterline. A rogue wave was a good candidate to explain this sinking.

In 1980, Philippe Lijour, captain of the tanker "Esso Languedoc" put an end to suppositions and superstitions by producing a photograph of an extraordinary wave, special and much higher than the others, which had struck the tanker by surprise and broke over the deck. This photograph was the first proof of the existence of rogue waves. From there, the testimonies and the accounts of the events of rogue waves multiplied, providing precious information to the scientific community to better understand this phenomenon. In this way, several studies were born among which, those of Mallory [37] and Lavrenov [38] which lists a series of events that occurred in the Agulhas Current, along the South-East African coast. Kharif and Pelinovsky [39], on the other hand, relates testimonies from many parts of the world and in various conditions of wind, current and depth. Figure (1.7) (takes in Touboul [21]) presents on a planisphere some of these events, which occurred during the period 1968-1994.

Figure 1.7: Location of several collisions related to rogue waves that occurred during the period 1968-1994.

In addition, the rapid development of the tanker industry and the improvement of observation methods in the ocean environment over the last thirty years have made it possible to obtain new and increasingly reliable data [21]. Large tankers and platforms have been equipped with numerous probes, allowing to record in a quasi-permanent way the rise of the sea level. The best known example is certainly the recording of the Draupner platform, on January 15, 1995 presented in figure (1.8) (takes in Touboul [21]), reporting a wave whose crest-height was about 26 meters while the surrounding sea state had a significant height of about 12 meters.

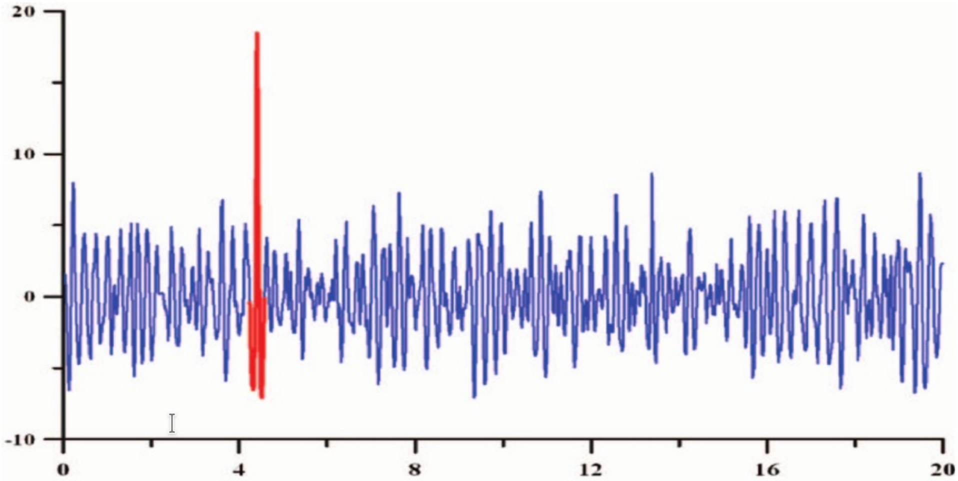


Figure 1.8: Time record of the "New Year wave", recorded on 01/01/95 by the Draupner platform, in the North Sea.

The existence of rogue waves is now universally recognized. Numerous images are available, for example Figure (1.9) (takes in Touboul [40]) shows some of them and some examples of damage caused by these waves. However, the understanding of the phenomenon, as well as its prediction, are not completely mastered.

Figure 1.9: Examples of rogue waves in accordance with maritime legends, top line and examples of damage caused by rogue waves, bottom line.

1.1.2.2 Rogue waves properties

As we have previously mentioned, the rogue waves which were still particularly unknown a few years ago have for a few decades been the subject of numerous works in order to better understand their dynamics and properties. We present here the context, and the current knowledge about rogue waves. Among the approaches to represent sea surface deformations, the simplest is to consider waves as a sum of sinusoïdes of different amplitudes and phases. In the linear approximation, a random sea state obeys a stationary Gaussian random distribution [40]. The probability density of the sea surface elevations is then

$$f(\eta) = \frac{1}{2\pi\sigma}\exp\left(- \frac{\eta^2}{2\sigma^2}\right), \tag{1.6}$$

where σ^2 represents the variance of the random variable η, which itself denotes the sea surface elevation. The variance is obtained from the frequency spectrum, $S(\omega)$,

$$\sigma^2 =< \eta^2 >= \int_0^\infty S(\omega). \tag{1.7}$$

Traditionally, the wind sea spectrum is assumed to be a narrow band spectrum. Therefore, the wave heights follow a Rayleigh distribution

$$f(H) = \frac{H}{2\sigma^2}\exp\left(- \frac{H^2}{8\sigma^2}\right). \tag{1.8}$$

The probability density $f(H)$ is shown in Figure 1.10(a) (takes in Touboul [40]). Assuming that the associated probability distribution function (i.e., the probability that a wave, for a given sea state, exceeds a certain height H^*) as shown in Figure 1.10(b) (takes in Touboul [40]), can be put into the form

$$P(H > H^*) = \int_{H^*}^\infty f(H)dH = 2\exp\left(- \frac{H^{*2}}{8\sigma^2}\right), \tag{1.9}$$

we can therefore introduce the notion of significant height of a sea state noted H_s which is a height commonly used in physical oceanography and coastal engineering. The notion of significant height H_s was introduced by Sverdrup [41], who defined it as the average of the higher one-third of wave heights in time series. Indeed, it is the average of the heights (measured between crest and trough) of the top third of waves. To calculate it from a surface elevation record, the waves are ranked in order of height, and the average of the heights of the top third gives the H_s. This historical definition comes from the estimation of wave height by visual observation, the significant height being close to that estimated by an observer. Using a Rayleigh distribution, Massel [42] showed that the significant height can be expressed as follows

$$H_s = 3\sqrt{2\pi}\ \mathrm{erfc}(\sqrt{\ln(3)} + 2\sqrt{2\ln(3)})\sigma \simeq 4\sigma, \tag{1.10}$$

where erfc(.) denotes the complementary Gaussian error function. Indeed, the height H^* of the third of the highest waves is given by $P(H > H^*) = 1/3$, which thus means that

$$H^* = 2\sqrt{2\ln(3)}\ \sigma. \tag{1.11}$$

Thus, the average height of the considered waves can be written in the following form

$$H_s = \int_{H^*}^{\infty} Hf(H)dH. \tag{1.12}$$

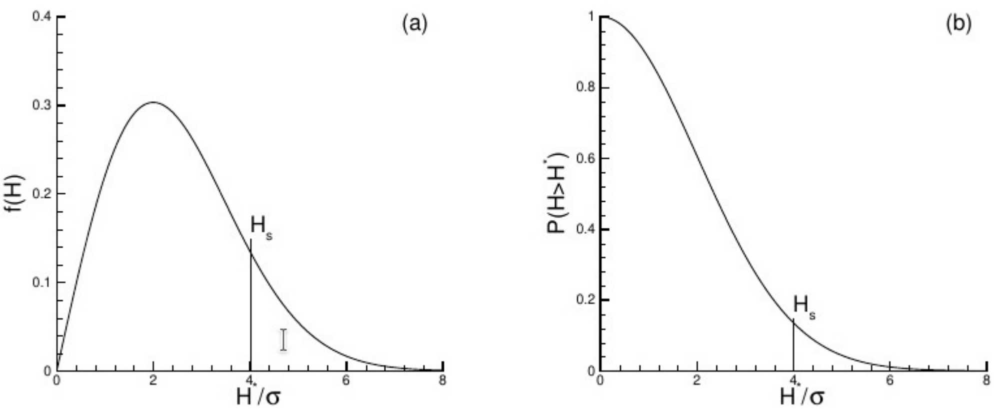


Figure 1.10: (a): Rayleigh distribution, corresponding to the probability density of wave heights of waves. (b): Probability distribution function associated with this distribution.

The significant height is about four times the standard deviation. This height corresponds approximately to the average height of a wave field estimated by the human eye. By introducing this quantity, equation (1.9) is rewritten as follows

$$P(H > H^*) = \exp\left(\frac{2H^{*2}}{H_s^2}\right), \tag{1.13}$$

In general, a wave can be considered as a rogue wave whenever its height H is greater than twice the significant height H_s of the sea [39, 40, 43].

$$H > 2H_s. \tag{1.14}$$

Statistically, this corresponds to the formation of a rogue wave every 16000 waves. There would be a rogue wave every 44 hours if we consider a characteristic wave period of 10 seconds.

1.1.2.3 Generation of rogue wave mechanisms

Rogue waves are events that occur in the middle of the oceans, at the coast, in the presence of strong currents, or not, or with or without the action of strong winds. Therefore, it seems impossible to establish a direct correlation between a particular geophysical phenomenon and these waves. For this reason, many physical mechanisms have been advanced to explain the formation of rogue waves. We focus here on presenting these different mechanisms, described by Kharif and Pelinovsky [39].

- Spatio-Temporal Focusing

In linear theory, a given wave field can be interpreted as a sum of monochromatic sinusoïdal wave groups. Therefore, the geometry of the wave field may well lead to a constructive interaction of these different components. In the two-dimensional setting, focusing is due solely to the dispersive character of the waves. Considering at a given initial moment short waves moving with a low group velocity and located in front of long waves which move with a large group velocity, a spatio-temporal focusing can then occur. Indeed, in the development phase, the long waves, which are faster, will catch up with the short waves, which are slower, leading to a constructive interaction of these waves, generating a wave of much higher amplitude. Afterward, the long waves will be in front of the short waves, and the amplitude of wave train will decrease. It is obvious, that a significant focusing of the wave energy can occur only if all the quasi-monochromatic groups merge at a fixed location. As is well known, real wind waves are not uniform in space and time, they correspond to wave groups with varying amplitude and frequency (wave number). It means that specific locations of transitional wave groups should sometimes occur, leading to the rogue wave formation. This scenario can also explain why the rogue wave phenomenon is a rare event with short "life time". Baldock et al. [44] have experimentally studied the behavior of highly nonlinear waves obtained by dispersive focusing. Johannessen and Swan [45] have reproduced these experiments numerically, obtaining more details on the deviation from the linear theory of highly cambered waves. In the context of rogue waves, more specifically, Pelinovsky et al.[46] have studied this scenario in the context of shallow water theory.

- Geometric Focusing

In the three-dimensional case, geometric focusing can also be observed. The geometric focusing is obtained from wave trains propagated in a beam of different directions. This phenomenon also exists in the natural state. Thus Whitham [47], have studied the evolution of the wave front as a function of bathymetry, and showed that the topography curved the rays of propagation of the wave, leading to the formation of caustics. In a natural environment, over variable bottoms, the interactions between wave fields become much more complex, and can lead to the formation of numerous focal points, as illustrated by Kharif and Pelinovsky [39]. This phenomenon can justify the formation of rogue waves.

- The modulational instability

There is another mechanism for the formation of rogue waves corresponding to the modulation of wave groups. Among the remarkable phenomena related to the nonlinearity of surface waves, we can cite the modulational instability (MI) highlighted by Benjamin and Feir [48]. This MI, known as the Benjamin-Feir instability, corresponds to the progressive modulation of a Stokes wave train. More clearly, we speak of MI when a wave of high intensity propagates in a dispersive and non-linear medium. It results in an amplitude modulation of the wave at a frequency determined by the characteristics of the medium, and which amplifies exponentially during the propagation. If the incident wave is weakly modulated by a signal with an adequate frequency shift, then the instability can exponentially amplify this modulation: we speak in this case of a process of induced MI. On the other hand, if the incident wave is continuous, then the modulation increases from the noise: we speak in this case of a spontaneous MI process. A wave train subjected to this instability presents a modulation-demodulation cycle, the Fermi-Pasta-Ulam recurrence. Many authors, such as Dysthe and Trulsen [49], Osborne et al.[50] and Calini and Schober [51], have suggested that at the maximum modulation of the Fermi-Pasta-Ulam recurrence a rogue wave could form.

- Wave and current interactions

Historically, the first confirmed observations of rogue waves were made in the Agulhas current, along the east coast of South Africa. Indeed, as testified by Mallory [37], this area which is very frequented by commercial shipping, has been the scene of several accidents. In 1976, Smith [52] suggested that these giant or rogue waves are formed where wave groups are blocked by the current. This suggestion was observed and confirmed in 2004 by experimental studies of Wu and Yao [53]. Using a more global linear approach, Lavrenov [38] showed that the transformation of waves by the current led to the focusing of rays, forming caustics that could justify the appearance of rogue waves.

- Soliton collision

The Kadomtsev-Petviashvili (KP) equation, which is the generalization of the KdV equation to the (2+1)-dimensional case, allows to represent the collision of two solitons propagating in different directions. This equation is written

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}+c_0\left(1+\frac{3\eta}{2h}\right)\frac{\partial \eta}{\partial t}+c_0\frac{h^2}{6}\frac{\partial^3\eta}{\partial x^3}\right)=-\frac{c_0}{2}\frac{\partial^2\eta}{\partial y^2}, \tag{1.15}$$

where h is the depth and $c_0 = \sqrt{gh}$ is the velocity of the wave in which g is the acceleration due to the gravity. This equation, integrable in the same way as the nonlinear Schrödinger and Korteweg-De Vries equations, has been solved by Ohkuma and Wadati [54] and in even more detail by Pelinovsky [55]. As results, they obtained a (2+1)-dimensional solution with the evolution of two solitons given by

$$\begin{cases} \eta(x,y,t) = h^3\dfrac{\partial^2 log[1+\exp(\zeta_1)+\exp(\zeta_2)+d\exp(\zeta_1+\zeta_1)]}{\partial x^2}, \\ \zeta_i = k_i x - p_i y - V_i t, \quad i=1,2, \\ V_i = c_g(k_i^2+p_i^2) \quad i=1,2, \\ d=\dfrac{(k_1+k_2)^2-(p_1-p_2)^2)}{(k_1-k_2)^2-(p_1-p_2)^2)}. \end{cases} \tag{1.16}$$

In the particular case where $k_1 = k_2$ and $p_1 = -p_2$, an interaction of two solitons of equal amplitudes and velocities can be observed. This problem is similar to the problem of a soliton reflecting on a wall located at y = 0, i.e. parallel to the longitudinal component of the wave vector. Thus, the amplitude at the point of contact is given by

$$\frac{a}{a_0}=\frac{4}{1+\sqrt{1-\frac{3a_0}{4h\tan^2(\theta)}}}, \tag{1.17}$$

where a_0 is the amplitude of the incident wave, and where θ denotes the angle between the wave vector and the x axis. We observe that for small values of the angles, of the order of the nonlinearity parameter a_0/h, the amplification factor becomes significant. This result was proved by Porubov et al. [56], who proposed another solution of the KP equation, and observed a similar behavior in a more general framework. Peterson et al. [57] and Soomere and Engelbrecht [58] suggested that N-soliton solutions of the KP equation explained the formation of (2+1)-dimensional shallow water rogue waves very well. Figure 1.11 (takes in Peterson et al. [57]) represents different examples of these waves for different values of the angle of incidence. It is important to note that these waves of extreme amplitudes have an infinite lifetime, and propagate at constant velocity. However, these solitons only exist at finite depths, and such waves can only form in shallow areas. This approach is only applicable to coastal areas.

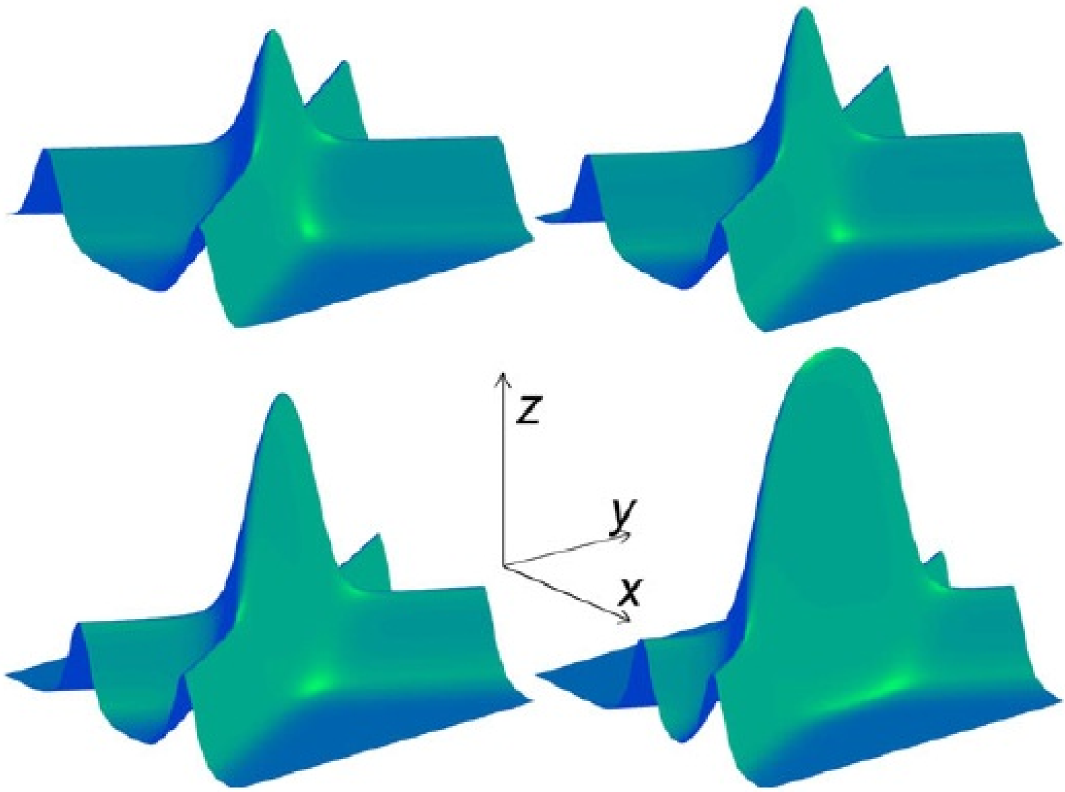


Figure 1.11: Waves of strong amplitudes (rogue waves) related to the interaction of solitons in shallow water

1.1.3 Tsunamis

The tsunami can be defined as a giant ocean wave that due to the sudden movement of a large mass of water, spreads and causes a tidal wave, (the term tsunami meaning "harbor wave" in Japanese). This phenomenon can be broken down into three phases, (i) generation, (ii) propagation and (iii) amplification and flooding (runup in English) [43]. A tsunami is not usually created by the wind as the swell, but by other natural mechanisms of great magnitude such as earthquakes, submarine landslides or sub-aerial, volcanic eruptions or the fall of asteroids. There are, however, meteorological tsunamis, long waves caused by an atmospheric disturbance that pushes the water and forces it to accumulate on one side of the basin, like water in a bathtub. But in general, tsunamis are usually generated by strong submarine earthquakes (magnitude >7) and very shallow (depth < 50 km). The intensity of the tsunami will depend on both the magnitude of the earthquake and the geometry of the sea floor.

In 2004, following a magnitude 9.3 earthquake off Indonesia's coast, the Indian Ocean was the starting point of the deadliest tsunami in history (more than 285,000 deaths), spread in the oceans from East to West flooded the African coasts. Since then, all the seas of the world have seen their levels change. After this spectacular and catastrophic event, countries bordering oceans and seas such as Cameroun, South Africa, Benin and Togo, just to cite a few, are led to assess such a wave's risks. In general, tsunami waves are often modeled as soliton waves [43].

1.1.4 Tidal waves and Storm tides

A tidal wave is a regularly reoccurring shallow water wave caused by effects of the gravitational interactions between the Sun, Moon, and Earth on the ocean. The term "tidal wave" is often used to refer to tsunamis; however, this reference is incorrect as tsunamis have nothing to do with tides. Indeed, the tidal wave is the variation of the sea level due to the gravitational action of the Moon and the Sun. The gravitational attraction between the Moon and the Earth acts in the opposite direction of the centrifugal force, due to the rotation of the Earth. On the surface of the planet, these two forces do not compensate each other and their difference is at the origin of the tides: when in a point, the centrifugal force is less intense than the gravitational force, this point will tend to move towards the Moon; conversely, when in a point, the centrifugal force is greater than the force exerted by the Moon, this point will tend to move away. This is why there is a tide on Earth twice a day.

A storm surge or storm tide is a sudden inundation caused by wind and low pressure; it is often associated with hurricanes, cyclones or typhoons. Storm surge is the most deadly aspect of a hurricane, responsible for 90% of the deaths that result from it. The strong winds that accompany a hurricane blow across the surface of the sea around the cyclonic core creating a very strong current by friction, normally compensated at depth by a counter-current, beyond 50 to 60 meters depth. When the cyclone reaches the continental shelf or very close to land, this counter-current no longer exists, only the surface current remains strongly established. There is therefore a natural mechanical push of the surface water and its accumulation towards the shores is all the more important as the continental shelf is marked. These winds push the water rapidly, causing the formation of a huge wave. At the same time, the depression caused by the cyclone causes the water level to rise in the lower pressure areas and the wells in the higher pressure areas. This is the intumescence or inverted barometer effect. This accentuates the giant wave generated by the wind. The "surcharge" is maximum in the part where all the effects are combined. This "peak" can last a few tens of minutes, two hours maximum. In areas where there is a large continental shelf, i.e. where the sea remains shallow for kilometers offshore, intense cyclones can cause a storm surge of 5, 6 or even 7 metres. And the victims are then counted in thousands. This was the case in China in 1881 and in Bangladesh in 1970, when typhoons killed more than 300,000 people each time, surprised by the sudden rise in water levels. The largest reported storm surge occurred in 1899 with Cyclone Mahina, which hit Bathurst Bay, Australia, with a tide of 13 meters. The largest in the United States was over 10 meters with Hurricane Katrina in 2005 in St. Louis Bay.

1.2 Nonlinear evolution equations describing the dynamics of hydrodynamic waves

There are several differential and nonlinear equations that model the nonlinear propagation dynamics of waves in physical systems, which vary according to the properties of the medium i.e. dispersion and nonlinearity. In this section, we present some equations that admit soliton and or rogue waves as solutions. The Korteweg and de Vries equation, the nonlinear Schrödinger equation, the Sasa-Satsuma equation, the Camassa-Holm equation and the sine-Gordon equation, just to cite a few.

1.2.1 The Korteweg-De Vries (KdV) equation

Following the experimental observations of Scott Russell in 1834, many works such as the theoretical work of Lord Rayleigh and Joseph Boussinesq in 1871 have seen the light of day with the aim of physically modeling these observations. Finally in 1895, two Dutch scientists Diedrik Korteweg and Gustav de Vries derived analytically a nonlinear partial differential equation that describes the propagation of waves on the surface of a shallow water channel. Subsequently, the equation was named after its discoverers by the Korteweg-de Vries (KdV) equation [4]. This classical nonlinear dispersive equation was formulated by Korteweg and de Vries in the simplest form:

$$u_t + cu_x + \beta uu_x + \gamma u_{xxx}, \tag{1.18}$$

where c is the wave velocity, γ is the dispersion coefficient and the parameter β represents the nonlinear coefficient which can take any real number. The subscripts of the form "nx" denote derivatives of the order n with respect to x and t, where x an t are the space and time variables. Especially, in the limit case where $\beta = \pm 1$ and $\gamma = \pm 6$, we obtain the commonly used form of this equation. The term u_t characterizes the time evolution of the wave propagating in one direction, the nonlinear term uu_x describes the resdess of the wave and the linear term u_{xxx}

represents the spread or dispersion of the wave. This equation is a simple nonlinear equation involving two effects: the nonlinearity represented by uu_x, and the linear dispersion represented by u_{xxx}. The nonlinearity tends to localize the wave while the dispersion extends the wave. The subtle balance between the weak nonlinearity and the dispersion defines the soliton formulation. The stability of solitons is a result of the perfect balance between the effect of nonlinearity and the effect of dispersion.

The KdV equation has played an important role in soliton theory, however, the term "Soliton" was only introduced in 1965 by Zabusky and Kruskal [3] who demonstrated that the KdV equation reveals linear properties allowing a solution in the form of a solitary wave propagating without profile change. The KdV equation models a variety of nonlinear phenomena, including ion acoustic waves in plasmas, atmospheric waves and shallow water waves, just to cite a few. This completely integrable has a solitary wave solution illustrated by the figure 1.12 (takes in [3]) below and given by the following equation [59].

$$u(x,t) = \frac{3c}{\beta}\,\mathrm{sech}^2\left[\frac{\sqrt{c/\gamma}}{2}(x - x_0 - ct)\right], \tag{1.19}$$

where x_0 is an arbitrary integration constant.

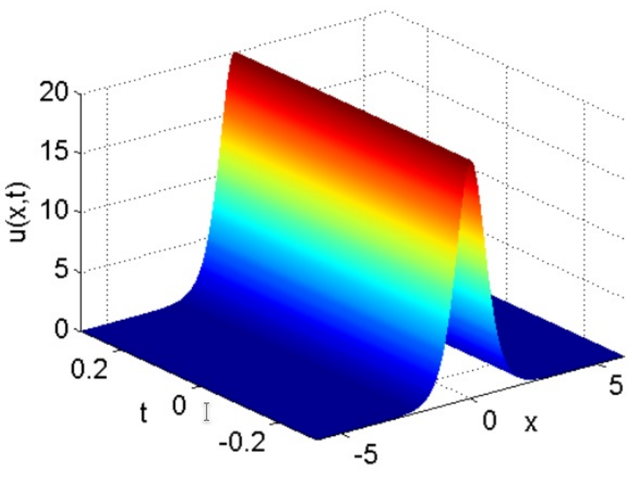


Figure 1.12: Evolution of the Soliton solution of equation (1.19) with : $\gamma = 0.5$, $\beta = 0.3$ and $c = 2$.

1.2.2 The nonlinear Schrödinger (NLS) equation

The NLS equation governs the spatial and temporal evolution of the envelope of a weakly nonlinear and weakly dispersive wave train. It was established in infinite depth by Zakharov [7], then in finite depth by Hasimoto and Ono [60]. Davey and Stewartson [61] extended this work in a three-dimensional model in 1973. They obtain a system of two coupled equations, which reduces to the single NLS equation when one dimension is removed. The commonly used form of this equation is the following

$$i\left(\frac{\partial\Psi}{\partial t}+v_{gr}\frac{\partial\Psi}{\partial x}\right)+\vartheta\frac{\partial^2\Psi}{\partial x^2}+\Lambda|\Psi|^2\Psi=0, \tag{1.20}$$

where $v_{gr}=\omega_0/2k_0$, $\vartheta-\omega_0/8k_0^2$ and $\Lambda=\omega k_0^2/2$. The associated linear, deep-water dispersion relation is given by $\omega_0^2=gk_0$. The function $\Psi\equiv\Psi(x,t)$ is the complex envelope function of a narrow-banded wave train whose amplitude is given by

$$\eta(x,t)=\frac{1}{2}\left[\Psi(x,t)e^{i(k_0x-\omega_0t)}+c.c.\right]. \tag{1.21}$$

By placing in a frame of reference that moves with the group velocity, and making the following variable changes: $T=-\frac{\omega_0}{8k_0^2}\,t$, $X=x-v_{gr}\,t$ and $Q=\sqrt{2}\,k_0^2\Psi$, the NLS equation can be put into the following form:

$$iQ_T+Q_{XX}+2|Q|^2Q=0. \tag{1.22}$$

The equation (1.22) thus obtained is an integrable equation, this type of equation can be solved by using the inverse scattering method [62]. Kuznetsov [62] and Ma [63] found a family of solutions of the type pulsed soliton, periodic in time and tending to a plane wave when $X\to\pm\infty$ as shown in figure 1.13

$$Q_M(X,T)=\frac{\cos(\Omega T-2i\varphi)-\cosh(\varphi)\cosh(pX)}{\cos(\Omega T)-\cosh(\varphi)\cosh(pX)}\exp(2iT), \tag{1.23}$$

where $\Omega=2\sinh(2\varphi)$ and $p=2\sinh(\varphi)$ and $\varphi\in\mathbb{R}$.

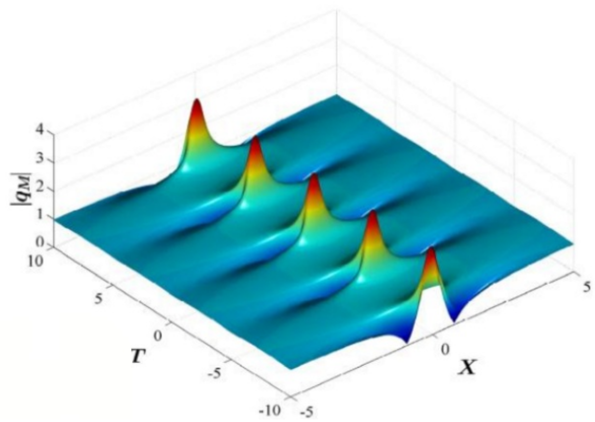


Figure 1.13: Evolution of the Ma-Kuznetsov breather $Q_M(X,T)$ given by equation (1.23).

Akhmediev and Korneev [64] have found another family of solutions, periodic in space, and which tends to a plane wave when $T \to \pm\infty$ as shown in figure 1.14

$$Q_A(X,T) = \frac{\cosh(\Omega T - 2i\varphi) - \cos(\varphi)\cos(pX)}{\cosh(\Omega T) - \cos(\varphi)\cos(pX)} \exp(2iT), \tag{1.24}$$

where $\Omega = 2\sin(2\varphi)$ and $p = 2\sin(\varphi)$ and $\varphi \in \mathbb{R}$.

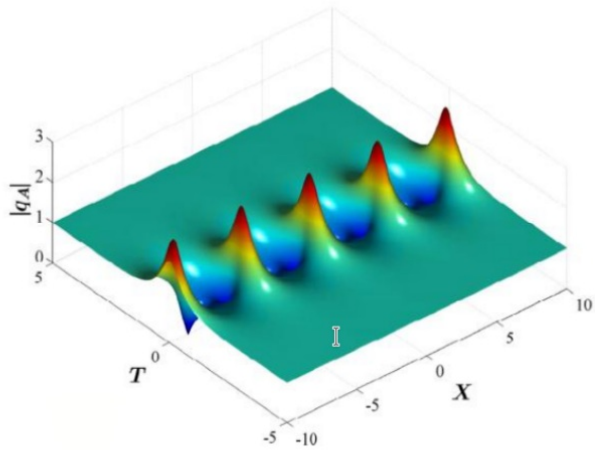


Figure 1.14: Evolution of the Akhmediev breather $Q_A(X,T)$ given by equation (1.24).

The pulsed soliton of Peregrine [65] corresponds to the limiting case of Ma and Akhmediev when $\varphi \to 0$, i.e., when the time and space periods tend to infinity as shown in figure 1.15

$$Q_P(X,T) = \lim_{\varphi\to 0} Q_M(X,T) = \lim_{\varphi\to 0} Q_A(X,T) = \left(1 - \frac{4(1+4iT)}{1+4X^2+16T^2}\right)e^{2iT}. \tag{1.25}$$

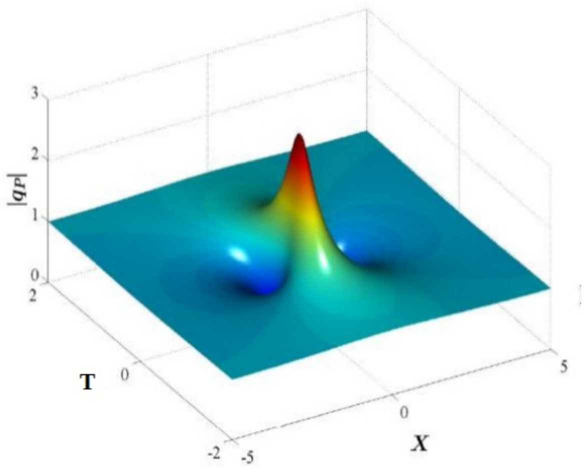


Figure 1.15: Evolution of the Peregrine breather $Q_P(X,T)$ given by equation (1.25).

1.2.3 The Sasa-Satsuma equation (SSE)

The SSE was established by Sasa and Satsuma in 1991 [66]. As one of the multiple extended form of the nonlinear Schrödinger equation, this equation contains additional terms explaining the third-order dispersion, the self-steepening and the self-frequency shift as often found in many fields of physics. It contains the most essential contributions often found in important physical applications, such as dynamics of deep and shallow water waves, pulse propagation in optical fibers. According to the original work of Sasa and Satsuma [66], the equation can be written in the form

$$iu_t + \frac{1}{2}\, u_{xx} + |u|^2 u + i\epsilon \left[u_{xxx} + 3(|u|^2)_x u + 6|u|^2 u_x\right] = 0, \tag{1.26}$$

where $u \equiv u(x,t)$ is a complex function, x and t are two independent variables. The indications in indices represent partial derivatives. The parameter ϵ is a real positive multiplying the higher terms. The term $6|u|^2 u_x$ represents the self-steepening, the term related to the self-frequency shift, the term $3(|u|^2)_x u$ and the 3-rd order dispersion term is given by the term u_{xxx}. When $\epsilon = 0$, equation equation (1.26) is reduced to the classical NLS equation. Many works have been devoted to the construction of analytical solutions of the SSE, and the results revealed that this equation has very rich dynamical behavior and can describe the propagation of nonlinear waves in many fields of physics espacially in shallow water wave.

Among the above equations, one of the most important is the KdV equation because, the study of their solitary wave solutions provide a significant help to explain the physical mechanism of some complex phenomena occurring in many areas of physics. Especially in the area of shallow water waves, the KdV equation, which describes the motion of small but finite amplitude waves that propagate in the positive x-direction [10], has been the subject of intensive works [10, 67, 68, 69], and experiments [12].

1.3 Some parameters that influence the surface waves dynamics

Many of the laws known up to now do not always succeed in finding or accurately representing the natural experiences or phenomena that occur in everyday life. It is therefore by taking into account the properties of the interfaces, in particular the effects of surface tension, viscosity and wind, that we come as close as possible to reality.

1.3.1 Surface tension

The assertion that a liquid always takes the shape of the container in which it is contained is not entirely correcte. In fact, a small amount of liquid can take the form of a drop: a drop of oil in water or a soap bubble forms a perfect sphere, smooth and very little deformable. These examples show that the surface of a liquid is like a taut membrane, characterized by a surface tension that opposes its deformations. The surface tension is a physico-chemical phenomenon linked to the molecular interactions of a fluid. It results from the increase of energy at the interface between two fluids. The system tends towards an equilibrium which corresponds to the configuration of lower energy, it thus modifies its geometry to decrease the area of this interface. The force $\vec{R}$ shown in figure 1.16 that maintains the system in this configuration is proportional to the surface tension.

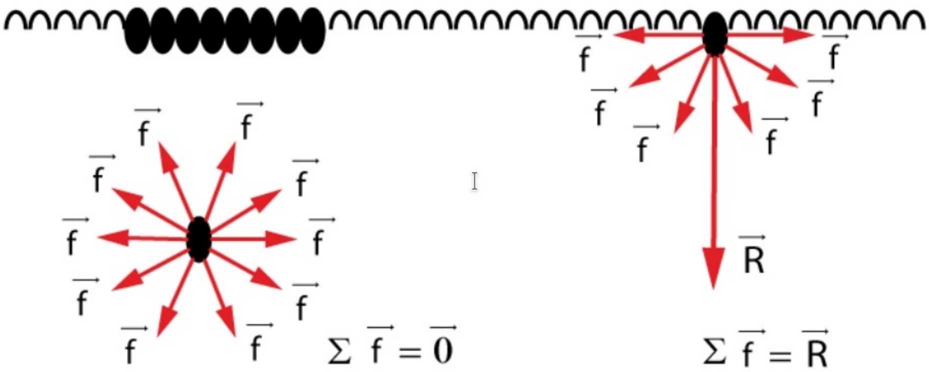


Figure 1.16: Forces exerted on a molecule within and at the surface of a liquid.

This force always tends to reduce the area of the interface, it is for example the basis of the quasi-spherical shape of fine liquid droplets suspended in a gas. The surface tension generally noted σ is expressed in units of force per unit length (N/m) and its value is relative to the interface between two given fluids. Thus, the increasing of the pressure Δp when crossing a separation surface between two fluids whose radii of curvature are R and R' is

$$\Delta P = P_{int} - P_{ext} = \sigma \left(\frac{1}{R} + \frac{1}{R'} \right). \tag{1.27}$$

The surface tension have generally been ignored in the description of water waves around large floating bodies, as its effect is considered significant only for rather short waves such as ripples. However, gravity wave theory can produce very short waves that cannot be ignored and pose substantial difficulties for their modeling. Since these singular and strongly oscillatory properties are clearly non-physical, surface tension is expected to play an important role in the modeling of surface waves. Thus, Wehausen and Laitone [70] have been the first to introduce an attempt to study the effects of surface tension in the surface wave dynamics. Crapper [71] have studied the effect of surface tension on the dynamics of capillary gravity water waves. Indeed the author considers only the case of infinite depth of water and shows that only for very small speeds are the gravity waves changed appreciably by surface tension. In the same line, Xiao et al. [72] show that, by taking into account the effect of surface tension, the wave form strongly changes. Afterwards, several studies have been done in order to investigated the effetcs of surface tension on the gravity wave dynamics. As results, these studies show that taking into account the effects of surface tension in the description of water wave models has an important impact on the surface wave dynamics.

1.3.2 Viscosity

In the same way as the density, the notion of viscosity of a fluid is the object of common observation: everyone has indeed noticed that two fluids of different nature do not necessarily flow with the same speed and that for some fluids (oils in particular), the temperature has a great influence. Viscosity is linked to the existence of inter molecular forces that result in the adherence of the fluid to a wall and in the resistance to the relative movement (sliding) of two neighboring particles of fluid. This resistance, corresponding to a loss of kinetic energy of the fluid, can be dissipated as heat within the fluid. The viscosity of a moving fluid is the property that expresses its resistance to a tangential force. Let's imagine a layer of fluid placed between two plate, parallel and horizontal plates as shown in figure 1.17. One is fixed and the other is in uniform motion with a velocity V_{max}. In order for the velocity to occur, a force F must be exerted on the upper plate.

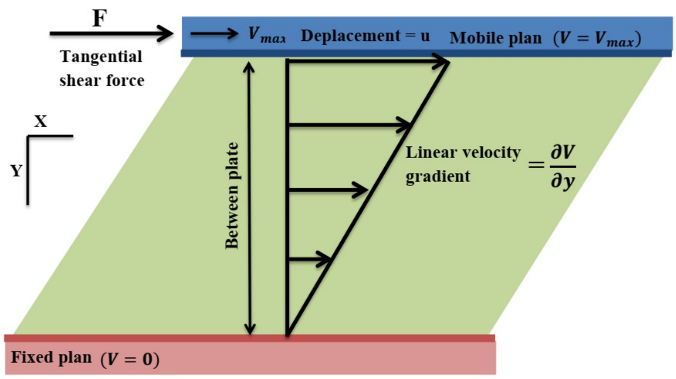


Figure 1.17: Illustration of a viscous friction.

This force is the resultant of the viscous friction forces, which is :

$$F = \mu A \frac{dV_{max}}{dy}, \tag{1.28}$$

where the coefficients $A[m^2]$ and $\mu[N.s/m^2]$ are the area of the plate and the viscosity, respectively.

Consider this time two coaxial cylinders separated by a gap containing a fluid as shown in figure 1.18. If we make the external cylinder rotate at a constant velocity (ω) while keeping the internal cylinder fixed, the fluid in contact with the external cylinder will adhere and will thus be animated by a linear velocity V. The fluid in contact with the fixed cylinder will have a zero velocity. It occurs in the fluid shear forces or viscosity force due to the interaction between the molecules of the fluid.

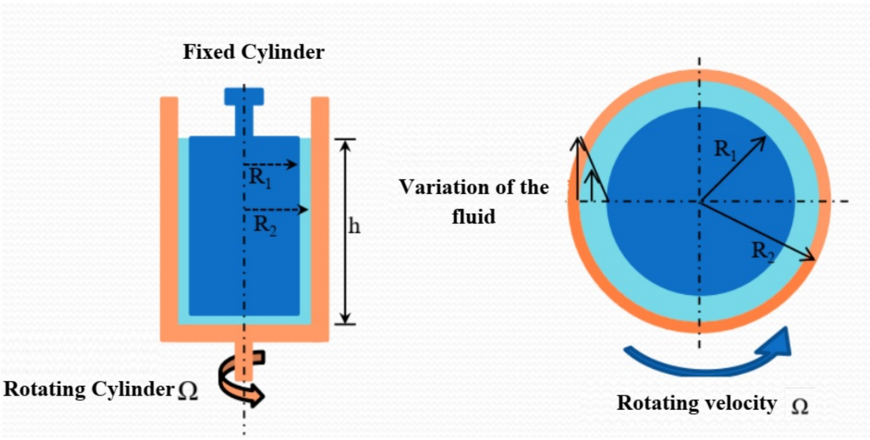


Figure 1.18: Cell of Couette.

This interaction gives rise to a frictional force that is measured by the torque M. Experiments have shown that the torque M varies proportionally to the velocity.

In general, the introduction of a dissipative term due to viscosity is done strategically [73]. Chester [74] have been the first to introduce an attempt to study the effects of dissipation and dispersion on nonlinear wave dyanmics. However, the first true formulation of nonlinear wave equations with viscosity effect has been presented by Ott and Sudan [75]. In that direction, Lundgren [76] and Dias et al. [77] have established a set of equations in the context of both linearity and nonlinearity governing flow taking into account dissipation due to viscosity, just to cite few. All these works prove that, taking into account the viscosity in the modeling of nonlinear equations especially in the context of shallow water can highlight phenomena that had not been relatively explored and therefore unknown and produce at the same time to the scientific community interesting and innovative results.

1.3.3 Wind

Everyone realizes by observing the sea on a stormy day that the energy of the wind can be transmitted to the waves. In fact, the effect of wind blowing on surface wave packets is twofold: it produces a variation in the pressure exerted on the surface which translates into a flow of energy from the wind to the waves and it generates a rotational current in the water. It remains to specify how this transfer occurs, and to measure this energy transfer as a function of wind intensity and surface roughness. In recent years, several works have been devoted to study the impact of the wind on the dynamics of nonlinear waves. Precise studies on this question began in 1924 with Jeffreys [16] then continued in the 1950s with Phillips [18], and Miles [19].

The Jeffreys model assumes that, waves move in the opposite direction to the wind and thus constitute a barrier to the wind flow. In this model, the windward side of the wave receives the wind frontally, while the leeward side is sheltered. This creates a pressure difference and thus a force exerted on the wave [17]. The air layers are generally laminar, but are deformed by the presence of the waves and for a certain steepness, one can notice the existence of detachments of the boundary layer on the side downwind of the waves. This detachment gives rise to an overpressure whereas the attachment of the same layer on the windward side of the following wave produces, on the contrary, an underpressure. The pressure difference thus created on the surface of the waves produces a force which transmits the energy of the wind to the water flow. Physical assumptions and dimensional analysis led to an expression for atmospheric pressure given by

$$P_a = \rho_a s (U_\infty - v_{ph})^2 \frac{\partial \eta}{\partial x}, \tag{1.29}$$

where ρ_a is the air density, s the shelter coefficient, U_∞ the wind speed, c the wave phase velocity and η the surface elevation. At the beginning, this model did not seem to be in good agreement with the experiments. Indeed, the differences in pressure created by the wind on a rigid wave profile led to calculations of energy transfer whose order of magnitude did not match the experiments. Moreover, this model does not explain the formation of waves when the wind blows on a sea initially at rest. However, this mechanism is the first to explain the necessary phase shift between atmospheric pressure and ocean surface elevation for energy transfer to occur between wind and waves. Touboul [40] have modified Jeffreys' theory by introducing a critical threshold of wave slope beyond which the relation given by equation (1.29) can be applied.

The model developed by Phillips is based on the assumption that the air flow is turbulent and the fluid flow is potential. The turbulence creates random pressure variations which in turn create waves. Thus, the pressure variations have the same spatial frequency as the emerging waves, hence a resonance phenomenon appears and an increase in amplitude. Experience has confirmed that this model is quite good at quantitatively justifying wave formation, but once the waves are formed, it no longer accurately accounts for the energy transferred between the wind and the waves.

The Miles model is based on an analysis of the stability of parallel flows in air and water. It considers air to be incompressible and non-viscous, with a logarithmic velocity profile. One mechanism for wave formation and growth is the resonant interaction between waves and induced pressure fluctuations. The wave amplification then follows an exponential law. Miles reduced the calculation of the transfer of energy of wind to waves to the solution of a Rayleigh equation, Thomas [17] has detailed the calculation leading to the model of Miles in his thesis. The wave amplification rate is calculated as a function of the wind velocity profile. He assumes that the airflow is a sheared current defined by the wind velocity as a function of altitude $U = U(z)$. The most common model to represent the reality while remaining quite simple and manipulable is the logarithmic $U(z) = U_1 \ln(z/z_0)$. U_1 is the characteristic velocity defined by $U_1 = u_*/\kappa$, where u_* is the roughness velocity, κ the Von kármán velocity (which is about 0.4) and z_0 is the roughness length guiven by $z_0 = \alpha_{ch} u_*^2/g$, where α_{ch} is the Charnock parameter and g the acceleration due to the gravity. The Miles model proposes the following form for atmospheric pressure

$$P_a = (\xi + i\beta)\rho_a U_1^2 k\eta, \tag{1.30}$$

where ξ and β are two coefficients depending on both the wavenumber k. The coefficient ξ that is in phase with the surface elevation $\eta(x,t)$, does not participate in the energy transfer between wind an wave and will therefore be neglected. The coefficient β is guiven by

$$\beta = -\frac{\pi}{k}\frac{U_c''}{\mid U_c' \mid}\frac{\overline{\omega_c^2}}{U_1^2\overline{\eta_x^2}}, \tag{1.31}$$

where the overline mean an average over a spatial period and the subscripts c mean that the corresponding quantities are evaluated at the altitude $z = z_c$ where we have the equality $U = c$, celerity of the waves. This altitude, called critical height, intervenes in the Rayleigh equation because it causes a singularity, hence the importance of its role in these formulas: it is at this altitude that the energy transfer occurs. it is worth noting that for the transfer of energy from the wind to the waves to be possible, it is necessary that $U_c'' < 0$ (second derivative evaluated at $z = z_c$).

The coefficient β provides a term in phase with $i\eta$ and therefore in quadrature with the surface elevation $\eta(x,t)$, which induces its participation in the energy transfer. The possible values of parameter β have been tabulated by Conte and Miles [78] by solving numerically the Rayleigh equation in the context of a logarithmic profile. These values have been used in deep water field by Kharif et al. [73] and it has been observed that for low frequency waves, a strong wind is needed to maintain the modulational instability and that, the wind needs less force to maintain this instability for higher frequency waves. From there, it is clearly seen that the wind effects are very essential and important in the water wave dynamics. For a wave in one spatial dimension, we have $\eta(x,t) = Ae^{i(kx-ct)}$ (where c is the phase velocity and A is the amplitude) which implies that $\eta_x = ik\eta$. Thus, the atmospheric pressure guiven by equation (1.30) can be written in one spatial dimensions in the following forms

$$P_a = \lambda\eta_x \quad \text{where} \quad \lambda = \frac{\mu\rho_a}{2\kappa^2}u_*^2. \tag{1.32}$$

The Jeffreys model and the Miles model are currently preferred by researchers, the first being better suited to waves of larger amplitude and the second to waves of small amplitude. During the 20th century, starting from one or the other of the overcited models, several authors have studied the impact of wind on the dynamics of extreme waves. For example, by using a pressure distribution over the steep crests given by the Jeffreys model, Kharif et al. [73], have investigated experimentally and numerically the influence of wind on extreme wave events in deep water. Touboul et al.[40] have used the Jeffreys model and investigated experimentally without wind and in presence of wind the freak wave formation due to the dispersive focusing mechanism. The authors conclude that, the duration of the freak wave event increases with the wind velocity. In the same line, amplification of nonlinear surface waves by wind have been investigated by Leblanc [22] using the Miles' model. Despite all these interesting works, the question of the interaction between wind and waves remains an open subject. Indeed, all these works listed above have been carried out in order to study the effects of wind on the dynamics of extreme waves such as rogue waves. Most of these works deal only with the generalized nonlinear Schrödinger equations. Motivated by the results obtained, extending the study to the context of shallow water and the KdV equation in particular becomes a necessity and a very interesting research axis. In addition, the simultaneous combination of dissipation due to viscosity, surface tension, and wind effects to show that the resulting model equation can lead to a generalized KdV equation that includes higher diffusion and instability effects may be an interesting study. Indeed, the combination of these effects has never been studied in the literature to our knowledge. It will be important to show that these effects have a considerable impact on the wave dynamics in shallow water.

1.4 Evolution carried out in the modeling of the KdV equation

Following the discovery of the soliton and the formulation of the KdV equation, many models aiming at improving jointly the nonlinear and dispersive properties have been developed. A decisive advance was initiated by Agnon et al. [79] after which the authors concluded that, by using Pade approximations, it is possible to double the order of the dispersion parameter. In the same line, Madsen et al. [80, 81] show that by replacing the infinite series operators by finite series approximations (of the Boussinesq type), the characteristics of linear and non linear waves become very accurate up to very high wavenumbers. These models provide the same accuracy of nonlinear properties as linear properties [80]. In general, the procedure is based on an exact formulation of the boundary condition at the free surface and at the bottom, combined with an approximate solution of the Laplace equation given in terms of a truncated series expansion. In the literature, two different formulations have been developed either in terms of velocity [81, 82] or in terms of velocity potential [83, 84]. The velocity potential formulation is the most used because the computational effort is reduced by a simpler coupling of the model with other potential flow solvers such that it is possible to use the boundary element method [84]. Based on these advances, many models have been developed by various research we present in the following some of the results obtained by these researchers and present in the literature

1.4.1 The third order KdV equation

The third order KdV equation can appear in different orders and forms, however, the general form can be written in the following form

$$\eta_t + f(\eta)\eta_x + \eta_{xxx} = 0. \tag{1.33}$$

Let us underline that the nonlinear terms $f(\eta)\eta_x$ and the dispersive terms η_{xxx} can have coefficients. The nonlinear term $f(\eta)$ can take the following forms

$$f(\eta) = \begin{cases} \alpha\eta; \\ \alpha\eta^2; \\ \alpha\eta^n; \\ \alpha\eta_x; \\ 2\alpha\eta - 3\beta\eta^2; \\ \alpha\eta^n - \beta\eta^{2n}. \end{cases} \tag{1.34}$$

In the limit case where $f(\eta) = \pm 6\eta$, we find the standard KdV equation in the form $\eta_t \pm 6\eta\eta_x + \eta_{xxx} = 0$. It is also important to note that, considering the $(+)$ sign we are in a focusing case and in the case of the (-) sign, we are in a defocusing case.

In the limit case where $f(\eta) = 6\eta^2$, we obtain the modified KdV (mKdV) equation in the form

$$\eta_t + 6\eta^2\eta_x + \eta_{xxx} = 0. \tag{1.35}$$

In the same way as the KdV equation, this equation is fully integrable and can have N-solitons as solution.

In the limit case where $f(\eta) = \alpha\eta^n$ with $n \geq 3$, equation (1.33) can turn to the generalized KdV (gKdV) equation as follows

$$\eta_t + \alpha\eta^n\eta_x + \eta_{xxx} = 0, \quad n \geq 3. \tag{1.36}$$

Undercontrary to the KdV and the mKdV equation, the gKdV equation in not intergrable for $n \geq 3$, and as a consequence, does not yield a multiple-soliton solution.

For $f(\eta) = \alpha\eta_x$, we arrive at the potential equation for KdV, given by

$$\eta_t + \alpha(\eta_x)^2 + \eta_{xxx} = 0. \tag{1.37}$$

For $f(\eta) = 2\alpha\eta - 3\beta\eta^2$, we obtain the equation generally called the Gardner equation, or the combined KdV-mKdV equation, which can be written in the following form

$$\eta_t + (2\alpha\eta - 3\beta\eta^2)\eta_x + \eta_{xxx} = 0. \tag{1.38}$$

The Gardner equation, is widely used in various branches of physics, such as plasma physics, fluid physics and quantum field theory, just to cite a few.

In the limit case where $f(\eta) = \alpha\eta^n - \beta\eta^{2n}$, we get another generalized KdV equation with two power of nonlinearities of the form

$$\eta_t + (\alpha\eta^n - \beta\eta^{2n})\eta_x + \eta_{xxx} = 0. \tag{1.39}$$

This last equation describes the nonlinear propagation of long acoustic waves. For $n = 1$, we find the equation known as the Gardner equation.

1.4.2 The fifth order KdV equation

The fifth-order KdV equation (fKdV) appears in multiple forms, but, the well known in its standard form is the following

$$\eta_t + \alpha\eta^2\eta_x + \beta\eta_x\eta_{xx} + \gamma\eta\eta_{xxx} + \eta_{xxxxx} = 0, \tag{1.40}$$

where α, β and γ are real and abitrary parameters ($\neq 0$). By changing the values of these parameters, several forms of the fKdV equation can be constructed. These equations include, for specific values of α, β and γ, the Lax, the Sawada-Kotera and the Kaup-Kupershmidt equations.

1.4.3 The seventh order KdV equation

The seventh-order KdV equation (sKdV) takes the following form

$$\eta_t + 6\eta\eta_x + \eta_{xxx} - \eta_{xxxxx} + \alpha\eta_{xxxxxxx} = 0, \tag{1.41}$$

where α is a non-zero constant. The sKdV equation was introduced by Pomeau et al.[85], to discuss the structural stability of the KdV equation under singular perturbation. The sKdV equation has the third-order dispersion term (η_{3x}) and two other dispersion terms, the fifth-order (η_{5x}) and seventh-order (η_{7x}) dispersion.

1.4.4 The ninth order KdV equation

The ninth-order KdV equation (nKdV) can be written in the following form

$$\eta_t + 6\eta\eta_x + \eta_{3x} - \eta_{5x} + \alpha\eta_{7x} + \beta\eta_{9x} = 0, \tag{1.42}$$

where α and β are arbitrary, non-zero constants. The nKdV equation has in addition third-order (η_{3x}), fifth-order (η_{5x}) and seventh-order (η_{7x}) dispersion terms, the ninth-order dispersion term (η_{9x}).

1.4.5 The KdV equation with variable coefficients

Starting from the fact that the soliton solutions have a variable propagation speed which depends on the coefficients of the equation. Many studies have been conducted to propose solutions to the KdV equation with variable coefficients. In an inhomogeneous and weakly nonlinear medium, the wave propagation is governed by the variable coefficients KdV equation can be written as follows

$$\eta_t + \alpha(t)\eta\eta_x - \beta(t)\eta^2\eta_x + \mu(t)\eta_{xx} + \delta(t)\eta_{xxx} = 0, \tag{1.43}$$

where $\alpha(t)$, $\beta(t)$, $\mu(t)$ and $\delta(t)$ are arbitrary functions of t. When $\beta(t) = 0$, $\alpha(t)$, $\mu(t)$ and $\delta(t)$ are constants, (1.43) turns to KdV-Burgers equation, which mainly describes the flow of air bladder in liquid and the flow of liquid within ballistic trajectory canal. When $\mu(t) = 0$,

$\alpha(t)$, $\beta(t)$ and $\delta(t)$ are constants, (1.43) turns to KdV-MKdV equation, which is widely used in solid physics, atom physics and quanta field theory etc. Some of the research can be referred to Khater et al. [86] or Hirota et al. [87]. When $\beta(t) = \mu(t) = 0$, $\alpha(t)$ and $\delta(t)$ are constants, it is the case of Fan [88] and Wang [89]. The applicability of the variable coefficient KdV equation (1.43) arises in many areas of physics, for example, the description of gravity-capillary and interracial-capillary wave propagation, internal waves and Rossby waves.

1.4.6 The KdV equation modified by diffusion and instability effects

The convex fluids whose first instability from the static state is oscillatory have received much attention recently. Such behavior is found, for example, in the convection of binary fluids' and in the electrohydrodynamic convection in nematic liquid crystals. A common feature of these two systems is that the transition occurs with a finite critical wavenumber and frequency. General arguments, as well as an asymptotic expansion in particular cases, show that the nonlinear behavior near the transition is governed by coupled Landau-Newell type equations. The traveling waves are governed by the Ginzburg-Landau equation. Many works have been devoted to the study of the behavior of the solutions of these equations and to their comparison with experimental results. Aspe and Depassier [67] have studied the evolution of a system which presents an oscillatory instability from the static state with a wave number and a vanity frequency. They have shown that this instability corresponds to the appearance of long surface waves. Moreover they showed that the nonlinear evolution of the system near the transition can be governed by the perturbed Korteweg-de-Vries equation written in the following form

$$\eta_t + \lambda_1 \eta \eta_x + \lambda_2 \eta_{xxx} + \epsilon \left[\frac{\sigma R_2}{15} \eta_{xx} + \lambda_3 \eta_{xxxx} + \lambda_4 (\eta \eta_x)_x \right] = 0, \tag{1.44}$$

where σ is the Prandtl number, R_2 is the coefficient of the instability and ϵ is a small parameter such that the excess of the Rayleigh number above its critical value is given by $\epsilon^2 R_2$. The coefficients λ_i, with $i = 1, ..., 4$, are functions of the deparameters of the problem. The authors study the evolution of long, shallow waves in a convective fluid by solving equation (1.44) when the critical Rayleigh number slightly exceeds its critical value. As a result the authors show that, the excess of the Rayleigh number above its critical value as well as nonlinear terms have a destabilizing effect which is balanced by diffusion.

Subsequently, Depassier and Letelier [14] extended this work to the higher order by introducing the effects of viscosity and obtain the following equation

$$\begin{aligned}\eta_t+6\eta\eta_x + \Big(\Lambda_1 + \epsilon^2 R_2 \Lambda_1^{'}\Big)\eta_{xxx} + \epsilon\Big((6/5)R_2\eta_{xx} + \Lambda_2\eta_{xxxx} + \Lambda_3(\eta\eta_x)_x\Big) \\ + \epsilon^2\Big(3\eta^2\eta_x + \Lambda_4\eta_{xxxxx} + \Lambda_5\eta_x\eta_{xx} + \Lambda_6\eta\eta_{xxx}\Big) = 0,\end{aligned} \tag{1.45}$$

where the coefficients Λ_i with $i = 1, ..., 6$ depend on parameter values, and the parameter R_2 measures the excess of the Reynolds number over the critical value.

1.4.7 The KdV equation modified by the surface tension and extended to the higher orders

In recent years, several authors have proposed improved versions of the generalized KdV equation by taking into account surface tension effects. They have shown that taking these effects into account can provide interesting nonlinear evolutionary models that describe the motion of long, small-amplitude, unidirectional waves in shallow water. For example, Burde [90] with the help of the Boussinesq perturbation expansion, have derived a fifth-order KdV equation taking into account the effects of surface tension. This equation can be written in the following form.

$$\eta_t + \eta_x + \epsilon(\alpha_1\eta\eta_x + \alpha_2\eta_{3x}) + \epsilon^2(\beta_1\eta^2\eta_x + \beta_2\eta_x\eta 2x + \beta_3\eta\eta_{3x} + \beta_4\eta_{5x}) = 0, \tag{1.46}$$

where the coefficients α_2, β_2, β_3 and β_4 depend on the Bond number τ.

In the same line, Fokou et al. [12] go beyond the fifth-order KdV equation derived by Burde (equation (1.45)), for the rightward-moving wave assumed to have smaller amplitude $0(\epsilon^2)$.They show that the wave amplitude dynamics for unidirectional long wave propagation over shallow water assumed to have smaller amplitude $0(\epsilon^4)$, is governed by a KdV equation with nonlinear and nonlocal terms given by

$$\begin{aligned}\eta_t + \eta_x + \epsilon(\alpha_1\eta\eta_x + \alpha_2\eta_{3x}) + \epsilon^2(\beta_1\eta^2\eta_x + \beta_2\eta_x\eta 2x + \beta_3\eta\eta_{3x} + \beta_4\eta_{5x}) + \epsilon^3(\gamma_1\eta^3\eta_x \\ + \gamma_2\eta^2\eta_{3x} + \gamma_3\eta_x^3 + \gamma_4\eta\eta_x\eta_{2x} + \gamma_5\eta_{2x}\eta 3x + \gamma_6\eta_x\eta 4x + \gamma_7\eta\eta_{5x} + \gamma_5\eta 7x) + \epsilon^4(\delta_1\eta^4\eta_x \\ + \delta_2\eta\eta_x^3 + \delta_3\eta\eta_{2x}^2 + \delta_4\eta^2\eta_x\eta_{2x} + \delta_5\eta_x^2\eta_{3x} + \delta_6\eta\eta_{2x}\eta_{3x} + \delta_7\eta^3\eta_{3x} + \delta_8\eta_{3x}\eta_{4x} + \delta_9\eta\eta_x\eta_{4x} \\ + \delta_{10}\eta_{2x}\eta_{5x} + \delta_{11}\eta^2\eta_{5x} + \delta_{12}\eta_x\eta_{6x} + \delta_{13}\eta\eta_{7x} + \delta_{14}\eta_{9x} + \delta_{15}\int\eta_{2x}^3dx + \delta_{16}\eta_x\int\eta_x^3dx \\ + \delta_{17}\int\eta_x^3dx + \delta_{18}\int\eta_x^2dx) = 0,\end{aligned} \tag{1.47}$$

where the coefficients δ_i, (with $i = 1, ..., 18$) also depend on the Bond number τ.

These last two equations were solved in the presence of the surface tension parameter. As results, it was observed that the presence of the surface tension and of the perturbation parameter ϵ involving the passage to higher orders strongly influenced the dynamics of the soliton obtained as solutions.

1.4.8 Multidimensional forms of the KdV equation

The different results presented above in this section have been formulated in (1+1)-dimension, more precisely in one spatial and one temporal dimension. The study of these equations has led to interesting mathematical and physical results since the end of the 1960s. But the extension of these results to multidimensional soliton equations came after 1970. The KdV equation describes nonlinear plane waves propagating in the x direction. During this time, starting from the fact that sometimes waves do not always propagate unidirectionally, an obvious question was asked: how is the propagation of waves when they move on a nonunidimensional surface? To answer this question, extensive research work has been done in the development of models with more than one dimension, in particular those in the (2+1) dimension, that is to say two spatial dimensions and one temporal one [91]. Among the results obtained, the best known two-dimensional generalizations of the KdV equation are: the Kadomtsov-Petviashivilli (KP) equation [6], the Zakharov-Kuznetsov (ZK) equation [92] and the Novikov-Veselov (NV) equation [91, 93].

It should be noted that, the KP equation is integrable and describes the evolution of quasi-onedimensional waves in shallow water when the effects of surface tension and viscosity are negligible. As for the ZK equation, it is not integrable via inverse diffuse method and governs the behavior of weakly nonlinear ion acoustic waves in a plasma comprising cold ions and hot, isothermal electrons in the presence of a uniform magnetic field [94]. The NV equation is fully integrable, and it is, mathematically speaking, the most natural generalization in (2+1) dimension of the KdV equation. In what follows, we will present these three equations.

1.4.8.1 The Kadomtsev-Petviashvili equation

In 1970, two Russian physicists Kadomtsev and Petviashvili proposed a two-dimensional nonlinear and dispersive wave equation to study the stability of the solitary wave under the influence of weak perturbations, transverse to the direction of propagation. The Kadomtsev-Petviashvili (KP) equation poses as the main model for certain physical systems with weak nonlinearity, weak dispersion and quasi-two dimensionality. It admits several exact solutions called KP solitons, which are regular, non-decaying and located along distinct lines in the two-dimensional "xy" plane [95]. The KP equation is relevant for most applications in which the KdV equation arises. This equation takes the following form [6]

$$(\eta_t + 6\eta\eta_x + \eta_{xxx})_x + 3\sigma^2\eta_{yy} = 0, \tag{1.48}$$

where $\eta \equiv \eta(x, y, t)$ and $\sigma^2 = \pm 1$.

If $\sigma^2 = -1$, the equation is called KP-I, and in the case where $\sigma^2 = +1$ the equation is called KP-II. For example, to model nonlinear dispersive waves on the surface of fluids, the KP-I equation is used when the surface tension is large, while, the KP-II equation is used when this tension is small. On the other hand, the KP-I equation is known to have a focusing effect and the KP-II equation a defocusing effect [96]. Depending on the physical context, an asymptotic derivation can result in either the KP-I or KP-II equation. In all cases, these equations describe the propagation dynamics of weakly nonlinear and weakly dispersive waves whose wavelength is large compared to its amplitude.

The principle led by Kadomtsev and Petviashvili consisted in searching for a weak transverse perturbation of the one-dimensional wave equation

$$\eta_t + \eta_x = 0. \tag{1.49}$$

This perturbation amounts to adding a non-local term, leading to

$$\eta_t + \eta_x + \frac{1}{2}\,\partial_x^{-1}\eta_{yy} = 0, \tag{1.50}$$

where the operator ∂_x^{-1} is defined by the following Fourier transform:

$$\partial_x^{-1} f(\xi) = \frac{i}{\xi_1}, \text{ with } \xi = (\xi_1, \xi_2). \tag{1.51}$$

When this same procedure is applied to the KdV equation, written in the context of shallow water waves, we obtain the following KP equation

$$\eta_t + \eta_x + \eta\eta_x + (1 - \frac{1}{3}\,T)\eta_{xxx} + \frac{1}{2}\,\partial_x^{-1}\eta_{yy} = 0, \tag{1.52}$$

Where $T \geq 0$, is the Bond number measuring surface tensioneffects. By a change of values, equation (1.52) reduces to equation KP-I if $T > \frac{1}{3}$, and to equation KP-II if $T < \frac{1}{3}$.

It is important to note that the same procedure could be applied to any one-dimensional equation weakly dispersive and weakly nonlinear.

1.4.8.2 The Zakharov-Kuznetsov equation

The KdV equation has played an important role in the development of soliton theory where nonlinearity and dispersion dominate, while dissipation effects are small enough to be neglected. However, the KdV equation is considered as a spatially one-dimensional model. One of the known two-dimensional generalizations of the KdV equation is the Zakharov-Kuznetsov (ZK) equation. The latter takes the following normalized form normalized form [92, 97].

$$\eta_t + a\eta\eta_x + \nabla^2\eta_x = 0, \tag{1.53}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the isotropic Laplacian. This means that the two-dimensional ZK equation dimensions is given by

$$\eta_t + a\eta\eta_x + (\eta_{xx} + \eta_{yy})_x = 0. \tag{1.54}$$

And three-dimensional by

$$\eta_t + a\eta\eta_x + (\eta_{xx} + \eta_{yy} + \eta_{zz})_x = 0. \tag{1.55}$$

The ZK equation governs the behavior of weakly nonlinear acoustic ion waves in a plasma comprising cold ions is hot isothermal electrons in the presence of a uniform magnetic field [98]. The ZK equation was first derived to describe weakly nonlinear acoustic ion waves in a strongly magnetized plasma in two dimensions [92]. However, unlike the KP equation, the ZK equation is not integrable by inverse diffuse method. A Painlevé analysis has been done for the ZK equation [99] and it has been shown to have Painlevé properties, but there is no further evidence that this equation is integrable.

In the context of plasma physics, another derivation of the ZK equation has been by Infeld and Frycz [100] as follows

$$\eta_t + \eta^{1/2}\eta_x + a\eta_{xxx} = 0, \tag{1.56}$$

and

$$\eta_t + (1 + b\eta^{1/2})\eta_x + a\eta_{xxx} = 0. \tag{1.57}$$

These equations describe acoustic ion waves in a cold ion plasma but the electrons do not behave isothermally when the wave passes.

1.4.8.3 The Novikov-Veselov equation

The Novikov-Veselov (NV) equation, like the KdV equation, has different forms. We will briefly mention the most common forms of the NV equation

- **The generalized NV equation**

It is given by the following form [101, 102]

$$\eta_t + a\eta_{xxx} + b\eta_{yyy} + c(\eta\partial_y^{-1}\eta_x)_x + d(\eta\partial_x^{-1}\eta_y)_y = 0, \tag{1.58}$$

where a, b, c and d are constant parameters. These parameters can be variable (according to function of time), [101].

- **The modified NV equation**

The modified NV equation (mNV) is a natural generalization in (2+1) dimensions of the modified KdV equation (mKdV). The mNV equation is given by Yu et al. [103] in the following form

$$\begin{aligned} &\eta_t = (\eta_{zzz} + 3\eta_z u + \frac{3}{2}\ \eta u_z) + (\eta_{\bar{z}\bar{z}\bar{z}} + 3\eta_{\bar{z}}\bar{u} + \frac{3}{2}\ \eta\bar{u}_{\bar{z}}), \\ &u_{\bar{z}} = (\eta^2)_z. \end{aligned} \tag{1.59}$$

Bogdanov [104] have shown that the equation is related to the NV equation in a manner similar to the way in which the mKdV equation is related to the KdV equation.

- **The Nizhnik-Novikov-Veselov equation**

The Nizhnik-Novikov-Veselov (NNV) equation of (2+4)-dimensions is given by

$$\begin{aligned} &\eta_t + \eta_{xxx} + \eta_{yyy} - 3(\eta u)_x - 3(\eta v)_y = 0, \\ &u_y = \eta_x, \\ &v_x = \eta_y. \end{aligned} \tag{1.60}$$

The system equation (1.60) which represents the NNV equation is very similar to the NV equation. The NNV equation is a model for incompressible fluids.

- **The generalized Nizhnik-Novikov-Veselov equation**

The generalized NNV equation (gNNV) is a symmetric generalization of the KdV in (2+1)-dimensions and is given by Kumar et al. [105]

$$
\begin{aligned}
&\eta_t + a\eta_{xxx} + b\eta_{yyy} + c\eta_x + d\eta_y - 3a(u_x\eta + u\eta_x) - 3b(v_y\eta + \eta v_y) = 0,\\
&u_y = \eta_x,\\
&v_x = \eta_y.
\end{aligned}
\tag{1.61}
$$

where a, b, c, and d are parameters. This equation is also known to be fully integrable.

- **The Non-dispersive Novikov-Veselov equation**

The non-dispersive Novikov-Veselov equation is given by Croke et al. [106]

$$
\begin{aligned}
\eta_t &= (v\eta)_z + (\eta\bar{v})_{\bar{z}},\\
v_{\bar{z}} &= -3\eta_z.
\end{aligned}
\tag{1.62}
$$

The dNV equation was derived in Konopelchenko and Moro [107] as the geometric optical limit of Maxwell's equations in an anisotropic medium. The model governs the propagation of high frequency monochromatic electromagnetic waves. In particular, they consider nonlinear media with a frequency dependent dielectric function and magnetic permeability of the Cole-Cole type [108].

Conclusion

In this chapter,we have presented a literature review on the hydrodynamic waves. Indeed the history, main characteristics, conditions of existence and mechanisms of generation of waves encountered in shallow water have been presented. In the development of the theory of solitons, we have highlighted the importance of the KdV equation and shown that, in order to consolidate the theoretical models with the experimental or natural observations, many authors have proposed improved models of the KdV equation to the higher order taking into account some new effects. It has been clearly shown that, considering such new effects can strongly influence the dynamics of solitons. However, to the best of our knowledge, there is no KdV equation that takes into account the combined effects of viscosity due to shear forces in a fluide, surface tension due to molecular interactions of a fluide and wind due to atmospherical pressure. This will be taken into account in our this work.

Chapter II

Methodologies

Introduction

This chapter describes the methodologies applied to derive the new generalized higher-order KdV equations, taking into account the effects of surface tension, viscosity and wind. It is also present the basic equations related to the physical and mathematical modeling of the equations describing the dynamics of waves propagating in shallow water.

2.1 Basic equations using in hydrodynamic waves modeling

Water surface waves are very interesting for the physicist because they correspond to many phenomena directly accessible to observation (from waves at sea, ship wake waves to tidal waves) and include very diverse classes of equations. In order to better understand the behavior of these surface waves, several equations have been derived and taking into account in order to modeled these phenomena.

2.1.1 Forces applying on a volume of fluid

Knowing that one of the objectives of fluid dynamics is to determine the position of material particles or to study the motion under the action of the forces which solicit them, it is important to define upstream the type of forces which can act on a volume of fluid. Indeed, we distinguish two categories, the volume forces or distance forces and the surface forces or contact forces.

- **Volume forces**

Any fluid domain located in a force field (gravity, magnetic, electric and other) undergoes actions at a distance proportional to the volume of the particle, these are volume forces. In fact, if we consider an elementary volume of fluid dV on which an elementary force $\overrightarrow{F}$ is exerted, we designate by volume force $\overrightarrow{f}$ (or force density per unit volume) the limit, if it exists, of the quantity $\frac{d\overrightarrow{F}}{dV}$ we note

$$\overrightarrow{f} = \lim_{dV \to 0} \frac{d\overrightarrow{F}}{dV}. \tag{2.1}$$

The density of forces exerted by gravity (gravitational force) on a continuous medium is one of the most classical examples.

$$\overrightarrow{f} = dm\overrightarrow{g} = \rho g d\overrightarrow{V}. \tag{2.2}$$

- **Surface forces**

It manifests itself through the frictional forces exerted between the fluid particles in relative motion. Combined with the pressure forces (normal to the surfaces), these frictional forces form stresses with a normal component and a tangential component (parallel to the surface). Reduced to an elementary surface dS of normal $\overrightarrow{n}$, the force per unit area that exerts at a point M is the stress noted $\overrightarrow{T_n}$. We thus have :

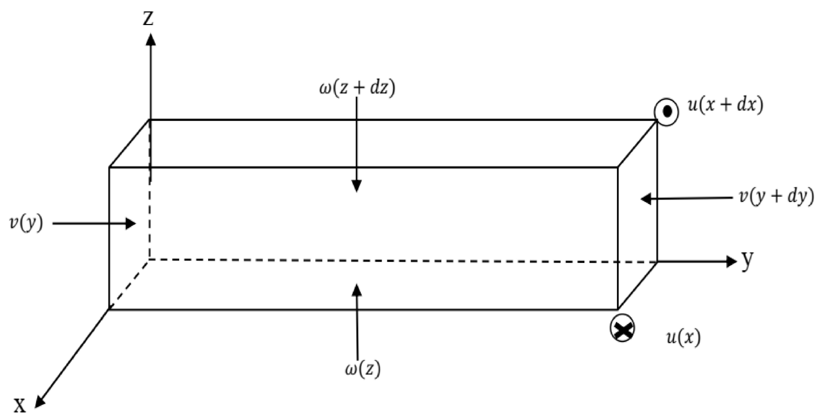
$$d\overrightarrow{F} = \overrightarrow{T_n}dS, \tag{2.3}$$

where the stress $\overrightarrow{T_n}$ has a different orientation than the normal $\overrightarrow{n}$ and the dimensions of this stress are those of a pressure.

Using one or the other or a sum of these forces, many basic equations have been formulated. In the rest of this chapter, we present some of them.

2.1.2 General continuity equation

Let us consider a volume of fluid or a parcel of fluid of mass m as follows



We consider that

$$dm = dm_x + dm_y + dm_z, \tag{2.4}$$

where $dm_x = -\dfrac{\partial(\rho u)}{\partial x}\,dxdydzdt$, $dm_y = -\dfrac{\partial(\rho v)}{\partial y}\,dxdydzdt$ and $dm_z = -\dfrac{\partial(\rho \omega)}{\partial z}\,dxdydzdt$ and ρ is the density. Remplacing dm_x, dm_y and dm_z by thier expressions in equation (2.4), we obtain

$$dm = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho \omega)}{\partial z}\right] dxdydzdt. \tag{2.5}$$

On the other hand,

$$dm = \frac{\partial \rho}{\partial t}\,dxdydzdt. \tag{2.6}$$

Equating equations (2.4) and (2.6) and assuming that the fluid is incompressible, ($\rho = cste$ imply that $\partial\rho/\partial t = 0$) we obtain the continuity equation as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0. \tag{2.7}$$

2.1.3 The Laplace equation

In a general way, to establish Laplace's equation, consists in translating the conservation of mass into an equation. The mass contained in a volume V bounded by the closed surface S is $\iiint_V \rho\, d\tau$ where ρ is the density of the fluid. The mass which leaves (algebraically) this volume during a unit of time is $\iint_S \rho\, \overrightarrow{u}.\overrightarrow{n}\, d\sigma$, where $\overrightarrow{n}$ is the normal vector to the surface S, unit and directed towards the outside. We deduce the conservation of mass equation

$$\frac{d}{dt}\iiint_V \rho\, d\tau = -\iint_S \rho\, \overrightarrow{u}.\overrightarrow{n}\, d\sigma. \tag{2.8}$$

We use the flux-divergence theorem in the right member and the derivation under the sign sum in the left one

$$\iiint_V \frac{\partial \rho}{\partial t}\, d\tau = -\iiint_V \overrightarrow{\nabla}(\rho\, \overrightarrow{u})\, d\tau. \tag{2.9}$$

We put everything in a single integral and this integral being null whatever the volume V used, it remains

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla}(\rho\, \overrightarrow{u}) = 0. \tag{2.10}$$

We develop the divergence

$$\frac{\partial \rho}{\partial t} + \overrightarrow{u}.\overrightarrow{\nabla}(\rho) + \rho\overrightarrow{\nabla}.\overrightarrow{u} = 0. \tag{2.11}$$

We see appearing a material derivative $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \overrightarrow{u}.\overrightarrow{\nabla}$ and obtains

$$\frac{D\rho}{Dt} + \rho\overrightarrow{\nabla}.\overrightarrow{u} = 0. \tag{2.12}$$

Assuming that the fluid is incompressible, $\frac{D\rho}{Dt} = 0$, it is clearly shows that $\vec{\nabla}.\vec{u} = 0$. The irrotationality of the flow allows us to assert the existence of a velocity potential, usually noted ϕ, verifying by definition

$$\vec{u} = \vec{\nabla}\phi. \tag{2.13}$$

By transferring equation (2.13) into equation (2.12) we obtain the Laplace equation, verified by the velocity potential, in the whole area occupied by the fluid as follows

$$\nabla^2\phi = 0. \tag{2.14}$$

2.1.4 The Euler equation

The Euler equation is a mathematical equation that models the motion of an incompressible, non-viscous fluid that is necessarily subject to external forces. Thus, in order to establish the equation describing the motion of a fluid, we must obtain an initial reference system. We will apply the second principle of fluid dynamics to a fluid parcel. Before doing so, it is important to note that the expression of the acceleration of the fluid parcel as a function of a velocity field according to the Euleurian approach will be considered as a function of time and space. Following this approach, we have the following equation.

$$\vec{a} = \frac{d\vec{u}}{dt} = \frac{\partial\vec{u}}{\partial t} + (\vec{u}.\vec{\nabla})\vec{u}, \tag{2.15}$$

where the term $\vec{\nabla}$ is the gradient, $\vec{u}$ is the velocity field, $\frac{d\vec{u}}{dt}$ is the total derivative and represents the acceleration of the parcel. $\frac{\partial\vec{u}}{\partial t}$ is the partial derivative of the velocity field, it represents the rate of change of velocity at fixed points in space.

Equation (2.15) clearly shows that the acceleration of the particle depends on two factors. The first factor is the local acceleration ($\frac{\partial\vec{u}}{\partial t}$). It highlights the acceleration due to the fact that, at a given point in space, the velocity of a parcel passing through it can either increase or decrease over time. The second is the advective term of acceleration ($(\vec{u}.\vec{\nabla})\vec{u}$) which reflects the acceleration due to the fact that, a parcel of fluid can move from a region of low velocity to a region of high velocity and vice versa.

By applying the second principle of dynamics, we obtain an equation that provides an expression for the acceleration of a parcel of fluid as follows

$$\rho\,\frac{d\vec{u}}{dt}\,\delta x\delta y\delta z = \rho\vec{F}\,\delta x\delta y\delta z, \tag{2.16}$$

where ρ is the density of the fluid, $\delta x \delta y \delta z$ is the volume of the parcel such that $dm = \delta x \delta y \delta z$ with dm the mass, $\overrightarrow{F}$ the force per unit mass applied to the fluid. In accordance with the fact that, the force of gravity is not altered by the motion, the remaining force especially the surface pressure will be the subject of particular attention here because any change in the value of the pressure with respect to its hydrostatic distribution will give rise to another form of motion.

In fact, it is difficult to think that as soon as a fluid starts moving, the stress forces acting on it are deeply affected or disappear. It seems reasonable to assume that the structure of the stresses acting on the parcel of fluid remains the same as that crossed in the hydrostatic case, even for a fluid in motion. In other words, we must assume that the stresses between the parcels are always normal to their separating surfaces and are independent of their orientation. The symbol used to designate the scalar defining the magnetude of these forces will remain the same as well as the name pressure or more exactly dynamic pressure, to indicate that now the pressure varies not only in space, but also in time. In this sense, the expression of the force acting on the parcel of fluid is written as a function of the hydrostatic pressure as follows

$$\overrightarrow{F} = -\frac{1}{\rho}\overrightarrow{\nabla} P + \overrightarrow{g}. \tag{2.17}$$

By introducing equation (2.17) into equation ((2.16) and using the expression of the equation obtained in equation (2.15) we have

$$\frac{\partial \overrightarrow{u}}{\partial t} + (\overrightarrow{u}.\overrightarrow{\nabla})\overrightarrow{u} = -\frac{1}{\rho}\overrightarrow{\nabla} P + \overrightarrow{g}. \tag{2.18}$$

This equation is known as the Euler equation, after the name of the author that derived them for the first time. written in component form, it becomes

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x},\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y},\\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g.
\end{aligned} \tag{2.19}$$

It turns out that this equation can describe the structure of the motion only in some circunstance. In many cases the solution is a good approximation of the real flow only in certain regions of space but not in others. In others cases the real motion in completly different. The simplest assumption we can make is that pressure forces are not the only forces present in a fluid. Therefore, there must be other forces that certain types of motions could bring to light.

2.1.5 The Navier Stockes equation

The time variation of the momentum of an elementary fluid of volume V bounded by a surface S as illustrated in Figure (2.1) is given by the sum of three terms including:

- the net flow of momentum through the surface S

- the sum of the volume forces acting on V

- the sum of the surface forces acting on S.

The volume of fluid crossing the boundary S per unit area per unit time is $-\overrightarrow{u}.\overrightarrow{n}$ and the momentum carried per unit volume is $\rho\overrightarrow{u}$. The momentum balance in the volume V is motion in the volume V is therefore written :

$$\int_V \frac{\partial\rho\overrightarrow{u}}{\partial t}\, dV = -\int_S \rho\overrightarrow{u}\,\overrightarrow{u}.\overrightarrow{n}\ dS + \int_V \rho\overrightarrow{f}dV + \int_S \overrightarrow{T_n}.\overrightarrow{n}dS, \tag{2.20}$$

where $\overrightarrow{f}$ is the volume force per mass unit (such as $\overrightarrow{g}$), $\overrightarrow{T_n}$ is the stress tensor and $\overrightarrow{n}$ is the unit vector normal to S.

Applying the divergence theorem to equation (2.20), we obtain :

$$\int_V \frac{\partial\rho\overrightarrow{u}}{\partial t}\, dV + \int_V \overrightarrow{\nabla}\rho\overrightarrow{u}\,\overrightarrow{u}dV = \int_V \rho\overrightarrow{f}dV + \int_V \overrightarrow{\nabla}.\overrightarrow{T_n}\ dV, \tag{2.21}$$

such as

$$\begin{aligned} \overrightarrow{\nabla}\rho\overrightarrow{u}\,\overrightarrow{u} &= \frac{\partial}{\partial x_j}(\rho u_i u_j) = u_i\frac{\partial\rho u_j}{\partial x_j} + \rho u_j\frac{\partial u_i}{\partial x_j}, \\ \overrightarrow{\nabla}.\overrightarrow{T_n} &= \frac{\partial T_{ij}}{\partial x_j}. \end{aligned} \tag{2.22}$$

Since this equality holds regardless of the volume V, it can be written in local rather than integral:

$$\rho\frac{\partial\overrightarrow{u}}{\partial t} + \overrightarrow{u}\frac{\partial\rho}{\partial t} + (\overrightarrow{u}\overrightarrow{\nabla}).\rho\overrightarrow{u} + \rho(\overrightarrow{u}\overrightarrow{\nabla})\overrightarrow{u} = \rho\overrightarrow{f} + \overrightarrow{\nabla}.\overrightarrow{T_n}. \tag{2.23}$$

This equation can be rewritten as follows

$$\rho\frac{\partial\overrightarrow{u}}{\partial t} + \overrightarrow{u}\left(\frac{\partial\rho}{\partial t} + \overrightarrow{\nabla}(\rho\overrightarrow{u})\right) + \rho(\overrightarrow{u}.\overrightarrow{\nabla})\overrightarrow{u} = \rho\overrightarrow{f} + \overrightarrow{\nabla}.\overrightarrow{T_n}. \tag{2.24}$$

The conservation of mass of the fluid imposes: $\frac{\partial\rho}{\partial t} + \overrightarrow{\nabla}(\rho\overrightarrow{u}) = 0$ and the equation for the momentum motion therefore reduces to :

$$\rho\frac{\partial\overrightarrow{u}}{\partial t} + \rho(\overrightarrow{u}.\overrightarrow{\nabla})\overrightarrow{u} = \rho\overrightarrow{f} + \overrightarrow{\nabla}.\overrightarrow{T_n}. \tag{2.25}$$

The term in brackets in the left-hand member is the acceleration of a fluid particle. It involves, on the one hand, the unsteady character of the flow by the temporal variation of the Eulerian velocity field and, on the other hand, the convective acceleration due to the spatial variation of the velocity field.Taking into account the expression of the stress tensor for an incompressible Newtonian fluid, $\overrightarrow{\nabla}.\overrightarrow{T_n}$ is written: $-\overrightarrow{\nabla}P + \eta\Delta\overrightarrow{u}$ and the equation for the evolution of the momentum is the Navier-Stokes equation which is written, in vector notation for the Eulerian velocity field $\overrightarrow{u}$:

$$\frac{\partial\overrightarrow{u}}{\partial t} + (\overrightarrow{u}.\overrightarrow{\nabla})\overrightarrow{u} = -\frac{1}{\rho}\overrightarrow{\nabla}P + \nu\Delta\overrightarrow{u} + \overrightarrow{f}, \tag{2.26}$$

where η is the dynamics viscosity, $\nu = \eta/\rho$ is the kinematics viscosity and $\overrightarrow{f}$ the external force exerted on an element of volume of unit mass (gravity, electric, magnetic field...).

In index notation for the component i of velocity we can write:

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_i \partial x_j}, \tag{2.27}$$

where the summation over the repeated indices is implicit, in this case :

$$u_j\frac{\partial u_i}{\partial x_j} = \sum_{j=1}^{3} u_j\frac{\partial u_i}{\partial x_j}. \tag{2.28}$$

In cartesian coordinates (x, y, z), the three components of the Navier-Stokes equation are written in the following form:

$$\begin{cases} \frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + f_x, \\ \frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + f_y, \\ \frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + f_z. \end{cases} \tag{2.29}$$

It is important to note that, in a direction where there is no velocity component, the pressure gradient in that direction is zero, in the absence of an external force.

2.1.6 Boundary conditions

In general, to solve the partial differential equations, it is necessary to define boundary conditions in the fluid domain. The considered domain includes the free surface, two lateral conditions and a bottom condition (figure 2.1). The boundary conditions must be set on each of these surfaces. Several conditions can be used depending on the physical problem considered.

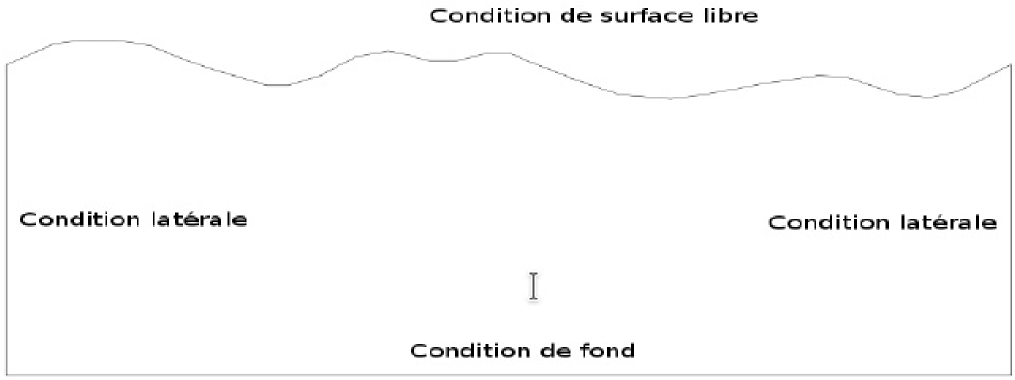


Figure 2.1: Geometry of the problem

2.1.7 The dynamics condition

The dynamic condition consists in writing that the pressure is the same on both sides of the free surface of the fluid.Thus, the free surface condition or dynamics condition is of great importance for wave problems. It is obtained from Euler's equation (2.18), from equation (2.13) in the following way

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\overrightarrow{\nabla}\phi)^2+g\eta=-\frac{P}{\rho} \quad \text{for} \quad z=\eta(x,y,t) \quad \text{(at the surface)}, \tag{2.30}$$

where p is the atmospheric pressure, $\eta(x,y,t)$ is the free surface elevation, and x, y and z are the horizontal and vertical spatial coordinates respectively. This condition is known as the dynamic condition. It is important to recall that in the presence of surface tension, this dynamic condition is rewritten as

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\overrightarrow{\nabla}\phi)^2+g\eta-\frac{\sigma}{R}=-\frac{P}{\rho} \quad \text{for} \quad z=\eta(x,y,t), \tag{2.31}$$

with σ the surface tension coefficient and R the radius of curvature of the free surface. Moreover, by expressing the radius of curvature is given as a function of the surface elevation, we can obtain the dynamics condition in present of the surface tension as follows

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\overrightarrow{\nabla}\phi)^2+g\eta+\sigma\frac{[(1+\eta_y^2)\eta_{xx}+(1+\eta_x^2)\eta_{yy}-2\eta_x\eta_{xy}\eta_y]}{(1+\eta_x^2+\eta_y^2)^{\frac{3}{2}}}=-\frac{P}{\rho} \quad \text{for} \quad z=\eta(x,y,t), \tag{2.32}$$

The position of the free surface is not known, so a second free surface condition is required. The kinematic condition that each particle of the free surface remains at the free surface. This condition corresponds to an impermeability condition.

2.1.8 The kinematics condition

The kinematic condition expresses that a fluid particle located at a given time on the surface cannot cross the surface; its velocity is therefore tangential. The surface is represented for more convenience by $S(x, y, z, t) = 0$, it is enough for example to set

$$S(x, y, z, t) = \eta(x, y, t) - z. \tag{2.33}$$

The normal velocity of a fluid particle is given by the relation:

$$V_n = -\frac{1}{\left|\vec{\nabla} S\right|}\frac{\partial S}{\partial t}. \tag{2.34}$$

This is the numerical velocity, measured on the normal directed outwards, i.e. generally upwards. The velocity of water is u and at a point on the surface its normal component is

$$U_n = \frac{\vec{u}.\vec{\nabla} S}{\left|\vec{\nabla} S\right|}. \tag{2.35}$$

We write the kinematics condition in the form $U_n = V_n$

$$\frac{1}{\left|\vec{\nabla} S\right|}\frac{\partial S}{\partial t} + \frac{\vec{u}\vec{\nabla} S}{\left|\vec{\nabla} S\right|} = 0 \Leftrightarrow \frac{\partial S}{\partial t} + \vec{u}.\vec{\nabla} S = 0, \tag{2.36}$$

and this equation is written simply $\frac{DS}{Dt} = 0$ where we note in this usual way the particle derivative.

Recall that we write the equation of the surface in the form (2.30). The kinematics condition or the boundary condition on the surface is then written as

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} - w = 0, \tag{2.37}$$

and if we want to translate it for the velocity potential (see (2.13)), we obtain the equation of the kinematic condition as follows

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \phi}{\partial z} = 0. \tag{2.38}$$

2.1.9 The condition at the bottom

Depending on the problem to be solved, we can consider two cases for the background condition. In infinite depth the bottom condition is

$$\frac{\partial \phi}{\partial z} \to 0 \quad \text{when} \quad z \to 0. \tag{2.39}$$

In the case of finite depth, it is written

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{when} \quad z = -h(x, y), \tag{2.40}$$

where $z = -h(x, y)$ is the bottom equation.

2.1.10 Lateral conditions

The choice of the lateral condition also depends on the physical problem considered. We consider here a closed geometry where the lateral surfaces are treated as solid and impermeable walls. The impermeability condition is written as

$$\frac{\partial \phi}{\partial z} = V, \tag{2.41}$$

where V is the wall velocity. This wall velocity can be taken equal to 0 in the case of immobile walls, or can obey a temporal law to simulate a beater.

2.2 Derivation of the classical KdV equation

Before presenting the methodology we used to derive the different new versions of the generalized KdV equation improved with the viscosity, surface tension and wind effects, it is important to recall the different steps leading to the establishment of the classical KdV equation. In fact, it has been shown that, the classical KdV equation can be derived using Euler's equation for an incompressible and nonviscous fluid, the bottom and surface boundary conditions and the assumption of irrotational flows. These different phases are organized as follows

2.2.1 Formulation of the problem

The KdV equation describes the propagation of long waves in shallow water. It is the simplest equation that incorporates both nonlinearity and dispersion. The equation is derived from Euler's equations by assuming that the amplitude is small compared to the depth assumed small compared to the wavelength. The mathematical theory of hydrodynamic waves goes back to Stokes [109], who first wrote the equations of motion of a perfect and incompressible fluid, subjected to a constant gravitational force, where the fluid has been bounded below by a rigid bottom and above by a free surface. above by a free surface. If the motion is non-rotational, then the velocity of the fluid can be written in terms of a velocity potential [110].

Considering the movement in one dimension (x direction) of the waves of an incompressible and non-viscous fluid (water), in a shallow channel of height h and sufficient width with a uniform cross section, leading to the formation of a propagating under gravity. It is assumed that the effect of surface tension is negligible. Let L be the length of the wave and A the maximum value of its amplitude above the horizontal surface (Figure 2.2), which is represented by $\eta(x,t)$ [111]. Assuming that $A << h$ (shallow water) and $h << L$ (long waves), we can introduce two small natural parameters into the problem ϵ and δ, which is defined by

$$\epsilon = \frac{A}{h} << 1 \quad \text{and} \quad \delta = \frac{h}{L} << 1. \tag{2.42}$$

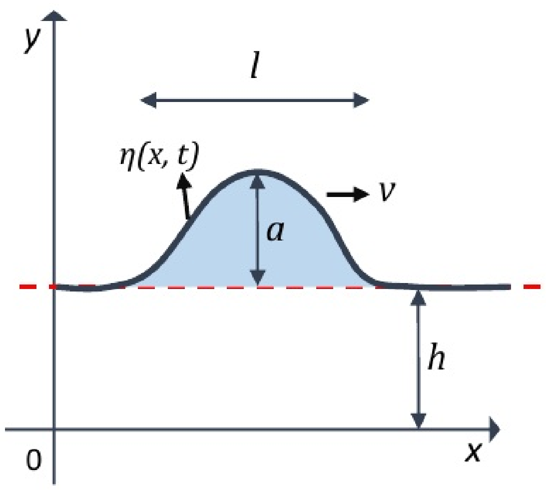


Figure 2.2: Motion of a one-dimensional wave in a shallow water channel

We choose the coordinates such that the movement of the fluid is two-dimensional. The properties of the system are independent of the y coordinate and the component of the velocity along y is zero. Then we assume that the velocity field inside the fluid is given by $\overrightarrow{v}(u,0,w)$, where $u \equiv u(x,z,t)$,$w \equiv w(x,z,t)$ and that the surface equation is given by $z(x,t) = h+\eta(x,t)$, where the function $\eta(x,t)$ is the surface elevation, thus we can write

$$\overrightarrow{v}(u,0,w) = u(x,z,t)\overrightarrow{i} + w(x,z,t)\overrightarrow{k}, \tag{2.43}$$

where $\overrightarrow{i}$ and $\overrightarrow{k}$ are the unit vectors along the horizontal and vertical directions, respectively.

The hypothesis that the flow is irrotational makes it possible to affirm the existence of a velocity potential such that, $\overrightarrow{v}(x,z,t) = \overrightarrow{\nabla}\phi(x,z,t)$, where the horizontal and vertical velocity components are given by $u = \phi_x$ and $w = \phi_z$. The symbol ∇ is the Nabla operator. It should be noted that, the Laplace equation $\nabla^2\phi = 0$ verifies the velocity potential throughout the area occupied by the fluid.

Since the fluid density $\rho = \rho_0 =$ constant, and using Newton's law for the rate of change of momentum, the Euler equation (2.18) can de rewritten in the following form

$$\frac{d\overrightarrow{v}}{dt} = \frac{\partial \overrightarrow{v}}{\partial t} + (\overrightarrow{v}.\overrightarrow{\nabla})\overrightarrow{v} = -\frac{1}{\rho}\overrightarrow{\nabla}P + g\overrightarrow{k}, \tag{2.44}$$

where $P = P(x, z, t)$ is the pressure at the point (x, z) and g is the acceleration due to gravity, which acts vertically downwards ($\overrightarrow{k}$ is the unit vector in the vertical direction). Using the assumption of irrotationality and thus the velocity potential as illustrated by equaton (2.13), in (2.44), we obtain after integration the dynamics condition related to the problem in the following form

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\overrightarrow{\nabla}\phi)^2 + \frac{p}{\rho} + gz = 0. \tag{2.45}$$

The two equations (2.44) and (2.45) for the velocity potential $\phi(x, z, t)$ of the fluid, must be completed by the relations imposed by the appropriate boundary conditions.

Taking into account the fact that:

- The horizontal bottom at $z = 0$ is hard and impermeable,
- The upper boundary $z = \eta(x, t)$ is a free surface.

As a result:

The vertical velocity cancels at $z = 0$

$$v(x, 0, t) = 0 \quad \text{which imply that} \quad \phi_z(x, 0, t) = 0. \tag{2.46}$$

Since the upper boundary of the liquid is free, we specify it by $z = h + \eta(x, t)$. Then at the point $x = x$, $z \equiv z(x, t)$, we can write:

$$\frac{dz}{dt} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{dx}{dt} = \eta_t + u\eta_x = w. \tag{2.47}$$

Given that $w = \phi_z$ and $u = \phi_x$, the last two parts of (2.47) can be rewritten as:

$$\phi_z = \eta_t + \eta_x \phi_x. \tag{2.48}$$

Similarly, at $z = h + \eta(x, t)$, the pressure $P = 0$. Then from (2.44), it follows that :

$$\phi_{xt} + \phi_x \phi_{xx} + \phi_z \phi_{zx} + g\eta_x = 0. \tag{2.49}$$

By integrating once along x, we obtain the dynamics condition as follows

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0. \tag{2.50}$$

Thus the problem to be solved is the Laplace equation with the kinematics, dynamics and bottom conditions, namely

$$\begin{cases} \phi_{2x} + \phi_{2z} = 0, \quad 0 \leqslant z \leqslant h + \eta(x,t), \\ \phi_z = 0, \quad z = 0, \\ \phi_z = \eta_t + \eta_x \phi_x, \quad z = h + \eta(x,t), \\ \phi_t + \dfrac{1}{2}\Big(\phi_x^2 + \phi_z^2\Big) + g\eta = 0, \quad z = h + \eta(x,t). \end{cases} \tag{2.51}$$

Since our study deals with small amplitude and long waves, it is preferable to scale our variables in order to avoid any ambiguity corresponding to a different physical situation. Thus, our variables will be scaled in such a way that

$$\widetilde{t} = \frac{t}{t_0}, \quad \widetilde{x} = \frac{x}{L}, \quad \widetilde{z} = \frac{z}{h}, \quad \widetilde{\eta} = \frac{\eta}{A}, \quad \widetilde{\phi} = \frac{\phi h}{LA\sqrt{gh}}, \tag{2.52}$$

where L is a typical wavelength of the surface waves thought the x-direction A is a typical amplitude of a surface wave η, and t_0 is a characteristic time, which will be used to measure time in the x-direction.

The Laplace equation and the boundary conditions at the free surface and at the bottom take the following form in which all the subscription "tilde" have been omitted:

$$\begin{cases} \phi_z = 0, \quad z = 0, \\ \beta\phi_{2x} + \phi_{2z} = 0, \quad 0 \leqslant z \leqslant 1 + \alpha\eta, \\ \eta_t + \alpha\eta_x\phi_x - \dfrac{1}{\beta}\,\phi_z = 0, \quad z = 1 + \alpha\eta, \\ \phi_t + \dfrac{\alpha}{2}\Big(\phi_x^2 + \dfrac{1}{\beta}\,\phi_z^2\Big) + \eta = 0, \quad z = 1 + \alpha\eta. \end{cases} \tag{2.53}$$

Our attention is particularly focused on low amplitude, weakly nonlinear waves in shallow and viscous water, so the amplitude parameters $\alpha = A/h$, which measures the ratio of wave amplitude to undisturbed fluid depth, the wavelength parameters $\beta = (h/L)^2$, which measures the square of the ratio of fluid depth to wavelength are considered as small.

2.2.2 Formulation of the Boussinesq system

The approximation of the velocity potential ϕ satisfying the Laplace's equation and the boundary condition at the bottom, can be written in the form of the following Taylor series

$$\phi(x,z,t) = \sum_{k=0}^{\infty} z^{2k} \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k} f(x,t)}{\partial x^{2k}}, \tag{2.54}$$

where the x-derivatives of the function $f(x,t)$ represent the values of the velocity potential at the bottom $z=0$. If the wave regime is considered for the classical Stokes number $S=\alpha/\beta$ is different from one, so the amplitude parameter α and the wavelength parameters β are treated in the same order but are not equal. By limiting the development of (2.54) to order 2, we obtain the following equation

$$\phi(x,z,t) = f(x,t) - \frac{1}{2}\,\beta z^2 \frac{\partial^2 f(x,t)}{\partial x^2} + \frac{1}{24}\,\beta^2 z^4 \frac{\partial^4 f(x,t)}{\partial x^4}. \tag{2.55}$$

Introducing equation (2.55) into the kinematics condition and using the relation $z = 1+\alpha\eta(x,t)$, we obtain the following equations in which $w(x,t) = \frac{\partial f(x,t)}{\partial x}$ and all terms greater than $O(\alpha)$, $O(\beta)$ and $O(\alpha\beta)$ have been neglected.

$$\eta_t + w_x + \alpha\eta_x w + \alpha\eta w_x - \frac{1}{6}\,\beta w_{3x} = 0. \tag{2.56}$$

Concerning the dynamics condition, a small transformation is necessary. We derive the equation with respect to x and assuming that $\frac{d(.)}{dx} = \frac{\partial(.)}{\partial x} + \alpha\eta_x\,\frac{\partial(.)}{\partial z}$ the dynamics condition can be written as follows

$$\phi_{xt} + \alpha\eta_x\phi_{zt} + \alpha(\phi_x\phi_{xx} + \frac{1}{\beta}\,\phi_z\phi_{xz}) + \eta_x = 0. \tag{2.57}$$

Proceeding in the same way as in the case of kinematics condition above, we obtain the following equation

$$\eta_x + w_t + \alpha w w_x + \frac{1}{2}\,\beta w_{2xt} = 0. \tag{2.58}$$

The equations (2.56) and (2.58) represent the system of equations governing the propagation of long waves in shallow water. These equations can be solved by a perturbative development to arrive at the KdV equation. We first solve the system (2.56) and (2.58) in $w(x,t)$ and $\eta(x,t)$, by a perturbative development [2]. At order 0, in α and β, the system (2.56) and (2.58) reduces to

$$\eta_x + w_t = 0 \quad \text{and} \quad \eta_t + w_x = 0. \tag{2.59}$$

The system of equation (2.59) admits the solution $w=\eta$, if $w_t + w_x = 0$. Since the parameters α and β are small, we can make a perturbative development of w in the form

$$w = \eta + \alpha A + \beta B, \tag{2.60}$$

where the coefficients $A \equiv A(x,t)$ and $B \equiv B(x,t)$ are arbitrary functions. These coefficients will subsequently be determined by introducing (2.60) (the Boussinesq system) into equations

(2.56) and (2.58), each time limited to the order of the small parameter corresponding to the coefficient to be determined, to which we impose the condition on f deduced from the order 0.

$$A_t + A_x = 0 + \theta(\alpha, \beta), \quad B_t + B_x = 0 + \theta(\alpha, \beta) \quad \text{and} \quad \eta_t + \eta_x = 0 + \theta(\alpha, \beta), \tag{2.61}$$

where $\theta(\alpha, \beta)$ represents the terms proportional to α and β.

Substituting (2.60) into (2.56) and (2.58), and neglecting the higher order terms of α and β, we obtain the following system

$$\eta_t + \eta_x + \alpha A_x + 2\alpha\eta\eta_x + \beta B_x - \frac{1}{6}\,\beta\eta_{3x} = 0, \tag{2.62}$$

$$\eta_t + \eta_x + \alpha A_t + \alpha\eta\eta_x + \beta B_t - \frac{1}{2}\,\beta\eta_{2xt} = 0. \tag{2.63}$$

Subtracting (2.62) from (2.63), we obtain

$$\alpha\left(A_x - A_t + \eta\eta_x\right) + \beta\left(B_x - B_t - \frac{1}{6}\,\eta_{3x} - \frac{1}{2}\,\eta_{2xt}\right) = 0. \tag{2.64}$$

Then applying the condition (2.61), we obtain the following equation

$$\alpha\left(2A_x + \eta\eta_x\right) + \beta\left(2B_x - \frac{2}{3}\,\eta_{3x}\right) = 0. \tag{2.65}$$

It is clearly observed that, the only condition that can satisfy the above equation is given by

$$2A_x + \eta\eta_x = 0 \quad \text{and} \quad 2B_x - \frac{2}{3}\,\eta_{3x} = 0, \tag{2.66}$$

for which the soluton is given by

$$A(x,t) = -\frac{1}{4}\,\eta^2 \quad \text{and} \quad B(x,t) = \frac{1}{3}\,\eta_{2x}. \tag{2.67}$$

After integration, of equation (2.67) into equation (2.60), we get the expression of w as a function of η as follows

$$w = \eta - \frac{\alpha}{4}\,\eta^2 + \frac{\beta}{3}\,\eta_{2x} + \alpha K_1(t) + \beta K_2(t), \tag{2.68}$$

where $K_1(t)$ and $K_2(t)$ are integration constants and are functions of time only. Finally, replacing equation (2.68) in equation (2.62), leads to the following equation

$$\eta_t + \eta_x + \frac{3}{2}\,\alpha\eta\eta_x + \frac{1}{6}\,\beta\eta_{xxx} = 0. \tag{2.69}$$

This nonlinear equation describes the unidirectional propagation of waves in shallow water. This last one takes a simpler and contemporary form if we make the following change of variables

$$t = \tau^{'} \qquad x = \xi + t. \tag{2.70}$$

Taking this into account, equation (2.69) can be rewritten as follows

$$\eta_{\tau'} + \frac{3}{2}\,\alpha\eta\eta_{\xi} + \frac{1}{6}\,\beta\eta_{\xi\xi\xi} = 0. \tag{2.71}$$

Then by introducing the new variables

$$u = \frac{\alpha\eta}{\beta}, \quad \text{and} \quad \tau = \frac{6\tau^{'}}{\beta}. \tag{2.72}$$

The equation (2.71) becomes

$$u_{\tau} + 6uu_{\xi} + u_{\xi\xi\xi} = 0. \tag{2.73}$$

By redefining the variables τ as t and ξ as x, for ease of notation, we arrive finally to the ubiquitous form of the equation KdV

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.74}$$

2.2.3 The properties of the classical KdV equation

The KdV equation is an equation that combines two fundamental properties namely non-linear and dispersion. In order to determine the effects of each terms in the equation, we are going to study the KdV equation as follows

$$u_t + uu_x + u_{xxx} = 0. \tag{2.75}$$

In the limit case where the nonlinear is dominant, it is the nonlinear term uu_x that dominates the equation

$$u_t + uu_x = 0. \tag{2.76}$$

The above equation has similarities with the Burger equation, which describes the propagation of a wave in a nonlinear system without dispersion in the sense that, a spatial component of the flow moves with a speed which depends only on the amplitude. The effect of nonlinearity tends to make the different parts of a wave propagate with different velocities. It therefore induces a tilt in the trailing part of the wave, generating a shock wave, as illustrates in Figure 2.3 (takes in []), in which it is clearly observed an initial profile localized in space will tend to straighten out to create a shock wave.

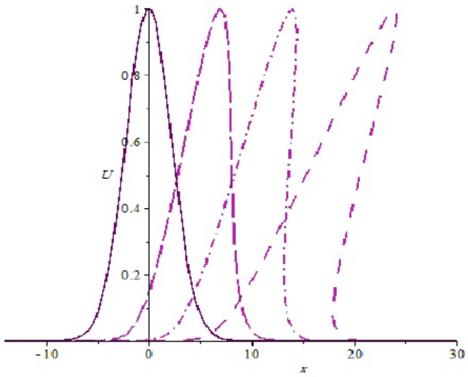


Figure 2.3: Formation of a shock wave governed by the Burgers equation

Let us now consider the limit case where the dispersion dominates, in other words, it is the term u_{xxx} of equation (2.75) that dominates. The equation then becomes a linear equation

$$u_t + u_{xxx} = 0. \tag{2.77}$$

By definition, a dispersive medium ensures that the different frequencies of a wave do not propagate at the same velocity (group velocity). This variance generates a of the wave as shown in figure 2.4 (takes in []). Equation (2.77) can be solved by assuming that the solution can be

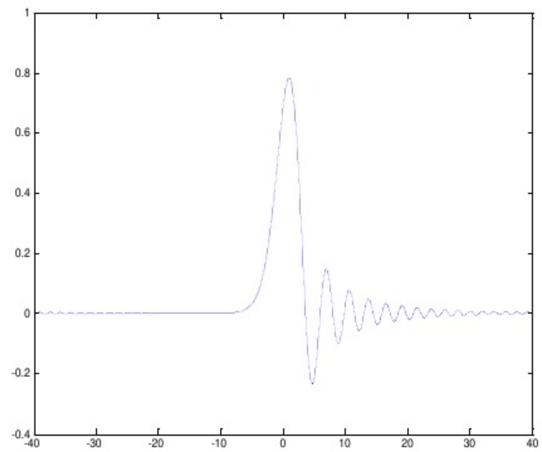


Figure 2.4: Effect of dispersion on the propagation of a wave in a medium governed by the linearized KdV equation

written as an ansatz namely $u(x,t) = e^{(i(kx-\omega t))}$, from which we obtain the so-called dispersion relation $\omega = -k^3$. In this case, the phase velocity $C_p = \omega/k = -k^2$, is totally dependent on the wavenumber k, and it is different from the group velocity $C_g = d\omega/dk = -3k^2$, which means that it is an equation of a dispersive medium.

The soliton can exist if and only if there is a total and delicate balance between the linear effect, in our case the dispersion, and the effect of the nonlinearity. For surface waves, the KdV equation shows that dispersion and nonlinearity are governed by the depth h of the water. The nonlinear term, proportional to $1/h$ increases for shallow water, while the dispersion, proportional to h^2 decreases. Although small fluctuations of h do not disturb the soliton, the equilibrium between the two can be reached if h is approximately constant.

2.2.4 The solution of the classical KdV equation

Over the past few decades, the construction of exact solutions for a large class of nonlinear equations including the KdV equation has been an extremely active area of research. Much of the literature in nonlinear equation theory uses the soliton solution model of the KdV equation as an example to introduce nonlinear theory. Many analytical methods for obtaining exact solutions of the KdV equation have been presented. Hirota's bilinear method, Backlund's transformation, tanh method, sin-cos method, Exponential function method, auxiliary equation method, Jacobi elliptic function expansion method and many others. In what follows, we will present some analytical methods for obtaining exact soliton-like solutions of the KdV equation.

2.2.4.1 The Hirota bilinear method

Hirota's bilinear method, also called the direct method, was developed in 1971 by Hirota to construct multi-soliton (N-soliton) solutions to nonlinear integrable evolution equations. Hirota demonstrated the effectiveness of the method by finding N-soliton solutions to the KdV equation [112]. The principle of Hirota's direct method is to transform nonlinear evolution equations into bilinear differential equations. To understand the method, we will take as an example the take as an example the equation KdV in its form (2.74). To convert equation (2.74) into a bilinear equation, Hirota used an approximation known as the Padé approximation, and that is by replacing $u(x,t)$ by G/F, where G and F are polynomials of exponential functions. By proceeding in this way, we obtain

$$\begin{aligned} u &= \frac{G}{F}, \\ u_t &= \frac{G_tF - GF_t}{F^2}, \\ u_x &= \frac{G_xF - GF_x}{F^2}, \end{aligned} \tag{2.78}$$

$$u_{xxx} = \frac{G_{xxx}}{F} - \frac{3G_{xx}F_x + 3G_xF_{xx} + GF_{xxx}}{F^2} + 6\frac{G_xF_x^2 + GF_{xx}F_x}{F^3} - \frac{GF_x^3}{F^4}.$$

Replacing the terms of (2.78) in equation (2.74), we obtain an extremely complicated equation as follows

$$\begin{aligned} u_t + 6uu_x + u_{xxx} =& \frac{G_tF - GF_t}{F^2} + 6\frac{GG_xF - G^2F_x}{F^3} + 6\frac{FG_xF_x^2 + FGF_{xx}F_x - GF_x^3}{F^4} \\ &+ \frac{F^2G_{xxx} - 3FG_{xx}F_x - 3FG_xF_{xx} - FGF_{xxx}}{F^3} = 0. \end{aligned} \tag{2.79}$$

To simplify the equation, and that after some modifications of equation (2.79), Hirota introduced a bilinear differential operator called the Hirota D-operator, and it is defined by Hietarinta [113] in the following forms

$$D_x^n f.g = (\partial x_1 - \partial x_2)^n f(x_1)g(x_2)|_{x_1=x_2=x}. \tag{2.80}$$

With this Hirota operator-D, equation (2.79) reduces to a quadratic equation, also called the Hirota bilinear form

$$u(x,t) = 2\frac{ff_{xx} - f_x^2}{f^2} = 2(\log f)_{xx}. \tag{2.81}$$

In other words, $G = 2(ff_{xx} - f_x^2)$ and $F = f^2$, where $f(x,t)$ is given by

$$f(x,t) = 1 + \sum_{k=1}^{\infty} \epsilon^n f_n(x,t). \tag{2.82}$$

The KdV equation can be written, then, in the Hirota bilinear form, as

$$(D_tD_x + D_x^4)f.f = 0. \tag{2.83}$$

For a solution to a single soliton, we set

$$f(x,t) = 1 + \epsilon f_1(x,t), \tag{2.84}$$

where ϵ is an expansion parameter. For a 2-soliton solution, we set

$$f(x,t) = 1 + \epsilon f_1(x,t) + \epsilon^2 f_2(x,t). \tag{2.85}$$

The N-soliton solution is obtained from:

$$f(x,t) = 1 + \epsilon f_1(x,t) + \epsilon^2 f_2(x,t) + ... + \epsilon^n f_n(x,t). \tag{2.86}$$

In the present work, we will limit ourselves to the search for the solution to a soliton. For that, we will take $f_1 = \exp(\theta_1)$, where $\theta_1 = k_1x + c_1t$, with k_1 and c_1 are constants, k_1 is called the wave number. Substituting $u(x,t) = \exp(k_1 x + c_1 t)$ into the KdV equation (2.74), the relationship between k_1 and c_1 namely the dispersion relation can be obtained as follows

$$c_1 = -k_1^3. \tag{2.87}$$

And in view of this result we obtain

$$\theta_1 = k_1 x - k_1^3 t. \tag{2.88}$$

Consequently, for the one-soliton solution, we set

$$f(x,t) = 1 + \epsilon f_1(x,t) = 1 + \epsilon \exp\left(k_1 x - k_1^3 t\right), \tag{2.89}$$

By setting $\epsilon = 1$, the one soliton solution is obtained by recalling that $u(x,t) = 2(\ln f)_{xx}$, therefore we obtain

$$u(x,t) = \frac{2k_1^2 \exp\left(k_1 x - k_1^3 t\right)}{(1 + \exp\left(k_1 x - k_1^3 t\right))^2}, \tag{2.90}$$

or equivalently

$$u(x,t) = \frac{k_1^2}{2} \operatorname{sech}^2\left[\frac{1}{2}\bigg(k_1 x - k_1^3 t\bigg)\right]. \tag{2.91}$$

This soliton solution describes the long, small-amplitude, unidirectional wave motion in shallow water.

2.2.5 The limits of the clssical KdV equation

The KdV equation is obtained by an approximate calculation (the multiple scale method consists in working on limited developments). For incompressible, irrotational and non-viscous fluids, the multiple scale method gives good results and has the advantage of being much easier to use and of finding applications in multiple fields of nonlinear sciences. But as soon as the physico-chemical and atmospheric parameters influencing the fluid flow such as surface tension, viscosity or wind effects are no longer neglected, the results obtained by the KdV equation do not accurately reflect the observations. Moreover, the KdV equation is obtained by a small parameter expansion: they correspond to the lowest nonlinear approximation. However, nowadays, experiments are becoming more and more sensitive and precise, allowing the observation of effects that could not be detected before. Moreover, nowadays, with climate change inducing a global temperature increase, it will be unwise in our humble opinion to continue to neglect parameters such as surface tension and viscosity (which vary with temperature) and wind effects in the modeling of physical phenomena.

2.3 Derivation of the Generalized KdV equation with viscosity and surface tension

In this section, we formulate new generalized shallow water wave equations both for unidirectional and bidirectional wave motion taking into account the effects of viscosity and surface tension. For this purpuse, we use the approximation of the velocity potential to formulate the Boussinesq system and derive the generalized KdV equations for unidirectional waves motion with the effects of surface tensions and viscosity. On the other hand, by using the non-uniqueness of the decompositions of this Boussinesq system, we derive generalized equations for bidirectional waves in shallow water, which also includes the surface tension and viscosity effects. The methodology which allows to obtain these equations is formulated as follows.

2.3.1 Mathematical formulation

This section deals with the mathematical formulation of the shallow water wave problems. We illustrate the correction of the kinematics, dynamics and boundary conditions by the viscosity in non-dimensional variables. For this, we consider a layer of an incompressible and viscous fluid, being above a horizontal plane located at altitude $z = 0$ with the mean depth the parameter h. We choose the coordinates such that the movement of the fluid is two-dimensional. The properties of the system are independent of the y coordinate and the component of the velocity along y is zero. Then we assume that the velocity field inside the fluid is given by $\overrightarrow{v}(u, 0, w)$, where $u \equiv u(x, z, t)$, $w \equiv w(x, z, t)$ and that the surface equation is given by $z(x, t) = h+\eta(x, t)$, where the function $\eta(x, t)$ is the surface elevation. The fluid is subjected to the action of forces having perpendicular and parallel components. The perpendicular components are the pressure P_a and the force $\overrightarrow{f}$. The pressure P_a is variable and uniform in space and it dues to the gas column above the fluid. The force $\overrightarrow{f} = \rho\overrightarrow{g}$ per unit volume is due to gravity, where ρ is the density of the fluid and g is the gravity field. For the parallel component, we consider the viscosity force $\overrightarrow{F} = \mu\Delta\overrightarrow{v}$ per unit volume due to the friction of the fluid slices against each other, where μ is the dynamics viscosity.

The Navier-Stokes equation governing the dynamics of the system can be written in the form

$$\begin{aligned} w_t + uw_x + ww_z &= -\frac{1}{\rho}P - g - 2\nu w_{zz}, \\ P &= \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}\,, \end{aligned} \tag{2.92}$$

where P is the pressure in the fluid, σ is the surface tension and $\nu = \frac{\mu}{\rho}$ is the kinematics viscosity. Equation (2.92) has been obtained using the fundamental principe of dynamics to a fluid particle, subject to its weight, the volumetric force of viscosity and the forces of pressure.

The boundary condition at the bottom indicates that the fluid is bounded underneath by a rigid surface such that

$$u = 0 \quad \text{and} \quad w = 0, \quad \text{for} \quad z = 0. \tag{2.93}$$

The procedure for establishing the kinematic boundary condition at the surface corrected by viscosity has been argued demonstrated in [13]. It is given by

$$w = \eta_t + u\eta_x - 2\nu\eta_{xx}, \quad \text{for} \quad z = h + \eta(x,t). \tag{2.94}$$

The physical condition at the surface is given by

$$P_a - P = 0, \quad \text{for} \quad z = h + \eta(x,t). \tag{2.95}$$

These equations must be completed by the mass conservation requirement

$$u_x + w_z = 0. \tag{2.96}$$

The hypothesis that the flow is irrotational makes it possible to affirm the existence of a velocity potential such that, $v(x,z,t) = \nabla\phi(x,z,t)$, where the horizontal and vertical velocity components are given by $u = \phi_x$ and $w = \phi_z$. The symbol ∇ is the Nabla operator. It should be noted that, the Laplace equation $\nabla^2\phi = 0$ verifies the velocity potential throughout the area occupied by the fluid.

The kinematics condition can then be written as :

$$\eta_t + \phi_x\eta_x - 2\nu\eta_{xx} - \phi_z = 0, \quad z = h + \eta(x,t). \tag{2.97}$$

Equation (2.92) can now be integrated to yield the dynamics boundary condition

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta + 2\nu\phi_{zz} - \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0, \quad z = h + \eta(x,t). \tag{2.98}$$

The characteristic time $t_0 = L/c_0$ used to measure the time is defined from the characteristics length L in the x direction and speed $c_0 = \sqrt{gh}$ of the high wavelength waves. Since the study deals with small amplitude and long waves, it is preferable to scale the variables in order to avoid any ambiguity corresponding to a different physical situation. Thus, the variables will be scaled in such a way that

$$\tilde{t} = \frac{t}{t_0}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{z} = \frac{z}{h}, \tilde{\eta} = \frac{\eta}{A}, \quad \tilde{\phi} = \frac{\phi h}{LA\sqrt{gh}}, \tag{2.99}$$

where A is a typical amplitude of a surface elevation $\tilde{\eta}$.

Introducing equation (2.99) in equations (2.93), (2.96), (2.97) and (2.98), the equations for a fluid are written in a non-dimensionalized form such that

$$\begin{cases} \beta\tilde{\phi}_{2\tilde{x}} + \tilde{\phi}_{2\tilde{z}} = 0, \quad 0 \leq \tilde{z} \leq 1 + \alpha\tilde{\eta}, \\ \tilde{\phi}_{\tilde{z}} = 0, \quad \tilde{z} = 0, \\ \tilde{\eta}_{\tilde{t}} + \alpha\tilde{\eta}_{\tilde{x}}\tilde{\phi}_{\tilde{x}} - \frac{1}{\beta}\tilde{\phi}_{\tilde{z}} - \beta\frac{2L\nu}{c_0h^2}\tilde{\eta}_{\tilde{x}\tilde{x}} = 0, \; \tilde{z} = 1 + \alpha\tilde{\eta}, \\ \tilde{\phi}_{\tilde{t}} + \frac{1}{2}\,\alpha\left(\tilde{\phi}^2_{2\tilde{x}} + \frac{1}{\beta}\,\tilde{\phi}^2_{2\tilde{z}}\right) - \frac{\beta\sigma\tilde{\eta}_{2\tilde{x}}}{\rho g h^2(1+\alpha^2\beta\tilde{\eta}^2_{\tilde{x}})^{\frac{3}{2}}} + \tilde{\eta} + \frac{2L\nu}{c_0h^2}\tilde{\phi}_{2\tilde{z}} = 0, \quad \tilde{z} = 1 + \alpha\tilde{\eta}, \end{cases} \tag{2.100}$$

where σ and ν are the surface tension and viscosity coefficients, respectively. The amplitude parameter $\alpha = \frac{A}{h}$, measures the ratio of wave amplitude to undisturbed fluid depth. The wavelength parameter $\beta = (\frac{h}{L})^2$, measures the square of the ratio of fluid depth to wave length. We focus our attention on low amplitude, weakly nonlinear waves in shallow and viscous water, then α and β can be considered as small parameters.

The parameters L, c_0, h, g and ρ being constant, we can assumed that the Bond number written as follows $\tau(\sigma) = \frac{\sigma}{\rho g h^2}$ represents our surface tension parameter and $\delta(\nu) = \frac{2L\nu}{c_0h^2}$ represents our viscisity parameter. Equation (2.100) can then be rewritten in the following forms

$$\begin{cases} \beta\phi_{2x} + \phi_{2z} = 0, \quad 0 \leq z \leq 1 + \alpha\eta, \\ \phi_z = 0, \quad z = 0, \\ \eta_t + \alpha\eta_x\phi_x - \frac{1}{\beta}\phi_z - \beta\delta\eta_{xx} = 0, \quad z = 1 + \alpha\eta, \\ \phi_t + \frac{1}{2}\,\alpha\left(\phi^2_{2x} + \frac{1}{\beta}\,\phi^2_{2z}\right) - \frac{\beta\tau\eta_{2x}}{(1+\alpha^2\beta\eta^2_x)^{\frac{3}{2}}} + \eta + \delta\phi_{2z} = 0, \quad z = 1 + \alpha\eta, \end{cases} \tag{2.101}$$

where the subscription ($\sim$) have been omitted for the reason of simplicity.

2.3.2 Formulation of the Boussinesq system

The standard procedure in shallow water theory has a serious advantage in the sense that, by writing the velocity potential function $\phi(x, z, t)$ in the form of power series of z, it does not compromise the requirements of satisfying both the field equation and the boundary conditions of the bottom and free surface. Thus, we start by setting the velocity potential ϕ as a formal expansion

$$\phi(x,z,t)=\sum_{i=0}^{\infty} f_i(x,t)z^i. \tag{2.102}$$

We assume that equation (2.102) formally satisfies the Laplace equation given by equation (first equation in (2.102)). We obtain the recurrent relation $(i+1)(i+2)f_{i+2}(x,t) = -\beta(f_i(x,t))_{2x}$. We set $g(x,t)=f_0(x,t)$, which indicates the velocity potential at the bottom $z=0$ and we obtain

$$f_{2k}(x,t)=\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}g(x,t)}{\partial x^{2k}}. \tag{2.103}$$

Using the second equation of the system (2.102), the power series expansion used for the approximation of the velocity potential ϕ is given by [12]

$$\phi(x,z,t)=\sum_{k=0}^{\infty} z^{2k}\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}g(x,t)}{\partial x^{2k}}. \tag{2.104}$$

In this work, the wave regime is considered for the classical Stokes number $S=\alpha/\beta=1$, so that the amplitude parameter α and the wavelength parameter β may be treated on an equal footing such that $\alpha=\beta=\varepsilon$.

By limiting the development of equation (2.104) to order 3, we obtain the following equation

$$\phi(x,z,t)=g(x,t)-\frac{1}{2}\ \beta z^2\frac{\partial^2 g(x,t)}{\partial x^2}+\frac{1}{24}\ \beta^2 z^4\frac{\partial^4 g(x,t)}{\partial x^4}-\frac{1}{720}\ \beta^3 z^6\frac{\partial^6 g(x,t)}{\partial x^6}. \tag{2.105}$$

By introducing equation (2.105) into the kinematics condition and using the relation $z=1+\alpha\eta(x,t)$, we obtain the following equation in which $w(x,t)=\dfrac{\partial g(x,t)}{\partial x}$ and all terms greater than $O(\varepsilon^3)$ have been neglected.

$$\begin{aligned}\eta_t+w_x+\varepsilon\left(\eta_x w+\eta w_x-\frac{1}{6}\ w_{3x}-\delta\eta_{2x}\right)+\varepsilon^2\left(-\frac{1}{2}\ \eta_x w_{2x}-\frac{1}{2}\ \eta w_{3x}+\frac{1}{120}\ w_{5x}\right)\\ +\varepsilon^3\left(\frac{1}{24}\ \eta_x w_{4x}-\frac{1}{2}\ \eta^2 w_{3x}-\eta\eta_x w_{2x}+\frac{1}{24}\eta w_{5x}-\frac{1}{5040}w_{7x}\right)=0.\end{aligned} \tag{2.106}$$

Concerning the dynamics condition, a small transformation is necessary. We derive the equation with respect to x and assuming that $\dfrac{d(.)}{dx}=\dfrac{\partial(.)}{\partial x}+\alpha\eta_x\dfrac{\partial(.)}{\partial z}$, the dynamics condition can be written as follows

$$\phi_{xt}+\alpha\eta_x\phi_{zt}+\alpha\left(\phi_x\phi_{xx}+\frac{1}{\beta}\ \phi_z\phi_{xz}\right)+\eta_x+\delta\phi_{xzz}+\alpha\delta\eta_x\phi_{zzz}-\beta\tau\eta_{xxx}=0. \tag{2.107}$$

Proceeding in the same way as in the case of kinematics condition above, we obtain the following equation

$$\eta_x+w_t+\varepsilon\bigg(ww_x-\frac{1}{2}\,w_{2xt}-\tau\eta_{3x}-\delta w_{2x}\bigg)+\varepsilon^2\bigg(-\eta_x w_{xt}+\frac{1}{2}\,w_x w_{2x}-\eta w_{2xt}-\frac{1}{2}\,ww_{3x}$$
$$+\frac{1}{24}\,w_{4xt}+\frac{1}{2}\,\delta w_{4x}\bigg)+\varepsilon^3\bigg(\eta_x w_x^2-\eta\eta_x w_{xt}-\eta_x ww_{2x}+\frac{1}{6}\,\eta_x w_{3xt}-\frac{1}{2}\,\eta^2 w_{2xt}$$
$$-\eta ww_{3x}+\frac{1}{12}\,w_{2x}w_{3x}-\frac{1}{8}\,w_x w_{4x}+\eta w_x w_{2x}+\frac{1}{6}\,\eta w_{4xt}+\frac{1}{24}\,ww_{5x}-\frac{1}{720}\,w_{6xt}$$
$$+\delta\eta w_{4x}-\frac{1}{24}\,\delta w_{6x}+\delta\eta_x w_{3x}\bigg)=0, \tag{2.108}$$

where the notation for the small parameters β and α have been changed to ε and $w(x,t)=\frac{\partial g(x,t)}{\partial x}$ is the scaled horizontal velocity at the bottom of the fluid.

It is worth noting that equations (2.106) and (2.108) constitute the generalized Boussinesq system. To the best of our knowledge, these coupled Boussinesq equations include both viscosity and surface tension effects are derived for the first time in the literature. When $\delta=0$, these equations can be similar to those recently obtained by [12] in the absence of viscosity effect.

2.3.3 Equations for unidirectional waves

In the context of shallow water, it has been shown that, the KdV equation has been first introduced as a model for the unidirectional propagation of long waves over shallow water [114]. To derive the equations for unidirectional waves, we reduce the two equations (2.106) and (2.108) into a single dependent equation of η. To do this, we follow the work of Whitham [114] and assume that the relationship between the horizontal velocity at the mean height w and the elevation η is given by $w=\eta+\varepsilon\psi(\eta)$ where ψ is a function which will be determined later. The right wave hypothesis imposes that at the lowest order of α and β, the Boussinesq system is reduced to $\eta_t+w_x=0$ and $w_t+\eta_x=0$. Starting from there, it shows that w and η satisfies the linear wave equation $\eta_{tt}-\eta_{xx}=0$, which describes waves moving in one direction. More precisely, for the lowest order $O(\varepsilon=0)$, $w=\eta$ and $\eta_x+\eta_t=0$. We assume w in the form

$$w=\eta+\varepsilon A(\eta)+\varepsilon^2 B(\eta)+\varepsilon^3 C(\eta), \tag{2.109}$$

where the coefficients $A(\eta)$, $B(\eta)$ and $C(\eta)$ are arbitrary functions. These coefficients will subsequently be determined by introducing equation (2.109) into equations (2.106) and (2.108) of the Boussinesq system, each time limited to the order of the small parameter corresponding to the coefficient to be determined.

To determine the coefficient $A(\eta)$ corresponding to the first order of the small parameter ε, equation (2.109) is first introduced into equations (2.106) and (2.108). Neglecting the terms of higher order than $O(\varepsilon)$ in each equation, we obtain the following system

$$\begin{aligned} \eta_t + \eta_x + \varepsilon\left(A_x + 2\eta\eta_x - \frac{1}{6}\,\eta_{3x} - \delta\eta_{2x}\right) + O(\varepsilon^2) = 0, \\ \eta_t + \eta_x + \varepsilon\left(A_t + \eta\eta_x - \frac{1}{6}\,\eta_{2xt} - \tau\eta_{3x} - \delta\eta_{2x}\right) + O(\varepsilon^2) = 0. \end{aligned} \tag{2.110}$$

We look for the function A as it corresponds to the two equations (2.110) up to the first order of ε. Using the lower order equation $\eta_t + \eta_x = 0$, it is easy to see that all the t-derivatives terms of η can be expressed in terms of the x-derivatives such that $\frac{\partial}{\partial t} = -\frac{\partial}{\partial x}$. This allows us to reduce the system (2.110) into a single equation as follows

$$A_x + 2\eta\eta_x - \frac{1}{6}\,\eta_{3x} - \delta\eta_{2x} = A_t + \eta\eta_x + \frac{1}{2}\,(1 - 2\tau)\eta_{3x} - \delta\eta_{2x}. \tag{2.111}$$

A common approach is to use the lowest order relation between the time and space derivatives ($A_t = -A_x$) in equation (2.111). After integration, we finally obtain

$$A(\eta) = -\frac{1}{4}\,\eta^2 + \frac{1}{6}\,(2 - 3\tau)\eta_{2x}, \tag{2.112}$$

$$w = \eta + \varepsilon\left[-\frac{1}{4}\,\eta^2 + \frac{1}{6}\left(2 - 3\tau\right)\eta_{2x}\right] + \varepsilon^2 B(\eta) + \varepsilon^3 C(\eta), \tag{2.113}$$

$$\eta_t + \eta_x + \varepsilon\left[\frac{3}{2}\,\eta\eta_x + \frac{1}{6}\left(1 - 3\tau\right)\eta_{3x} - \delta\eta_{2x}\right] = 0. \tag{2.114}$$

Setting $\delta = 0$, $X = \sqrt{\frac{3}{2}}(x - t)$ and $T = \frac{1}{4}\sqrt{\frac{3}{2}}(\varepsilon t)$ equation (2.114) can be reduced to the standard form of the KdV equation

$$\eta_T + 6\eta\eta_X + \eta_{3X} = 0. \tag{2.115}$$

To determine the coefficient $B(\eta)$ corresponding to the second order of the small parameter ε, equation (2.113) is introduced into equations (2.107) and (2.108). Neglecting the terms of higher order than $O(\varepsilon^2)$ in each equation, all the t-derivatives of η are replaced by their expressions through the x-derivatives using the lowest order equation, namely (2.114). Upon the substitution, we obtain the following equations

$$\eta_t+\eta_x+\varepsilon\left[\frac{3}{2}\,\eta\eta_x+\frac{1}{6}\left(1-3\tau\right)\eta_{3x}-\delta\eta_{2x}\right]+\varepsilon^2\left[B_x-\frac{3}{4}\,\eta^2\eta_x+\left(\frac{1}{12}-\frac{1}{2}\,\tau\right)\eta_x\eta_{2x}\right.$$
$$\left.-\left(\frac{1}{12}+\frac{1}{2}\,\tau\right)\eta\eta_{3x}+\left(-\frac{17}{360}+\frac{1}{12}\,\tau\right)\eta_{5x}\right]+O(\varepsilon^3)=0,$$

$$\eta_t+\eta_x+\varepsilon\left[\frac{3}{2}\,\eta\eta_x+\frac{1}{6}\left(1-3\tau\right)\eta_{3x}-\delta\eta_{2x}\right]+\varepsilon^2\left[B_t+\frac{1}{2}\,\delta\eta_x^2+\frac{1}{2}\,\delta\eta\eta_{2x}-\frac{1}{2}\,\delta\eta_{2x}\right.$$
$$\left.+\left(\frac{11}{6}+\frac{7}{4}\,\tau\right)\eta_x\eta_{2x}+\frac{11}{12}\,\eta\eta_{3x}+\frac{1}{4}\left(\frac{11}{18}-\tau-\tau^2\right)\eta_{5x}\right]+O(\varepsilon^3)=0. \tag{2.116}$$

The solution of the function $B(\eta)$ can be expressed in terms of η and its x-derivatives. As a result, we obtain

$$B(\eta)=\frac{1}{8}\,\eta^3-\frac{1}{4}\,\delta\eta_x+\frac{1}{16}\,(3+7\tau)\eta_x^2+\frac{1}{4}\,\delta\eta\eta_x+\frac{1}{4}\,(2+\tau)\,\eta\eta_{2x}+\frac{1}{120}\,(12-20\tau-15\tau^2)\eta_{4x}. \tag{2.117}$$

Inserting equation (2.117) into equation (2.113) and the first equation of equation (2.116), yields in second order $O(\varepsilon^2)$, the equation

$$w=\eta+\varepsilon\left[-\frac{1}{4}\,\eta^2+\frac{1}{6}\left(2-3\tau\right)\eta_{2x}\right]+\varepsilon^2\left[\frac{1}{8}\,\eta^3-\frac{1}{4}\,\delta\eta_x+\frac{1}{16}\,(3+7\tau)\eta_x^2\right.$$
$$\left.+\frac{1}{4}\,(2+\tau)\,\eta\eta_{2x}+\frac{1}{4}\,\delta\eta\eta_x+\frac{1}{120}\,(12-20\tau-15\tau^2)\eta_{4x}\right]+\varepsilon^3C(\eta), \tag{2.118}$$

$$\eta_t+\eta_x+\varepsilon\left[\frac{3}{2}\,\eta\eta_x+\frac{1}{6}\left(1-3\tau\right)\eta_{3x}-\delta\eta_{2x}\right]+\varepsilon^2\left[-\frac{3}{8}\,\eta^2\eta_x-\frac{1}{4}\,\delta\eta_{2x}+\frac{1}{4}\,\delta\eta\eta_{2x}\right.$$
$$\left.+\frac{1}{24}\left(23+15\tau\right)\eta_x\eta_{2x}+\frac{1}{12}\left(5-3\tau\right)+\frac{1}{4}\,\delta\eta_x^2+\frac{1}{360}\left(19-30\tau-45\tau^2\right)\eta_{5x}\right]=0. \tag{2.119}$$

Introducing equation (2.118) in equations (2.106) and (2.108) and proceeding as above, we obtain

$$w=\eta+\varepsilon\left[-\frac{1}{4}\,\eta^2+\frac{1}{6}\left(2-3\tau\right)\eta_{2x}\right]+\varepsilon^2\left[\frac{1}{8}\,\eta^3-\frac{1}{4}\,\delta\eta_x+\frac{1}{16}\,(3+7\tau)\eta_x^2+\frac{1}{4}\,\delta\eta\eta_x\right.$$
$$\left.+\frac{1}{4}\,(2+\tau)\,\eta\eta_{2x}+\frac{1}{120}\,(12-20\tau-15\tau^2)\eta_{4x}\right]+\varepsilon^3\left[-\frac{5}{64}\,\eta^4+\frac{1}{32}\left(3-21\tau\right)\eta\eta_x^2\right. \tag{2.120}$$
$$+\frac{1}{8}\,\delta\eta\eta_x+\left(\frac{7}{20}-\frac{1}{4}\,\tau+\frac{1}{16}\,\tau^2\right)\eta\eta_{4x}+\frac{1}{16}\left(2-3\tau\right)\eta^2\eta_{2x}-\frac{1}{16}\left(1+3\tau\right)\delta\eta_{3x}$$

$$+\left(\frac{1091}{1440}+\frac{1}{3}\,\tau+\frac{21}{32}\,\tau^2\right)\eta_x\eta_{3x}+\left(\frac{163}{360}+\frac{29}{48}\,\tau+\frac{7}{16}\,\tau^2\right)\eta_{2x}^2-\frac{1}{8}\left(1+2\tau\right)\delta\eta_x\eta_{2x}$$

$$+\frac{1}{48}\left(-11+15\tau\right)\delta\eta\eta_{3x}-\frac{11}{32}\,\delta\eta^2\eta_x+\left(\frac{61}{1890}-\frac{1}{20}\,\tau-\frac{1}{24}\,\tau^2-\frac{1}{16}\,\tau^3\right)\eta_{6x}$$

$$-\frac{1}{8}\,\delta^2\eta_x^2+\left(\frac{3}{16}-\frac{3}{16}\right)\int\eta_x^3dx+\frac{1}{16}\,\delta\int\eta_x^2dx+\frac{1}{16}\left(5-3\tau\right)\delta\int\eta\eta_{4x}dx$$

$$+\frac{7}{32}\,\delta\int\eta^2\eta_{2x}dx\Bigg],$$

$$\eta_t+\eta_x+\varepsilon\left[\frac{3}{2}\,\eta\eta_x+\frac{1}{6}\left(1-3\tau\right)\eta_{3x}-\delta\eta_{2x}\right]+\varepsilon^2\Bigg[-\frac{3}{8}\,\eta^2\eta_x+\frac{1}{24}\left(23+15\tau\right)\eta_x\eta_{2x}$$

$$-\frac{1}{4}\,\delta\eta_{2x}+\frac{1}{4}\,\delta\eta\eta_{2x}+\frac{1}{12}\left(5-3\tau\right)+\frac{1}{4}\,\delta\eta_x^2+\frac{1}{360}\left(19-30\tau-45\tau^2\right)\eta_{5x}\Bigg] \tag{2.121}$$

$$+\varepsilon^3\Bigg[\frac{3}{16}\,\eta^3\eta_x-\frac{1}{16}\,\delta\eta_x^2-\left(\frac{13}{32}+\frac{13}{32}\tau\right)\eta_x^3+\frac{1}{8}\left(4-\tau\right)\eta^2\eta_{3x}-\frac{3}{16}\,\delta\eta\eta_x^2$$

$$+\frac{1}{8}\,\delta\eta^2\eta_{2x}-\frac{1}{8}\,\delta\eta\eta_{2x}+\left(\frac{11}{16}+\frac{29}{6}\tau\right)\eta\eta_x\eta_{2x}+\left(\frac{1079}{1440}-\frac{5}{45}\,\tau+\frac{19}{32}\,\tau^2\right)\eta_x\eta_{4x}$$

$$-\frac{1}{48}\left(1+9\tau\right)\delta\eta_{4x}+\left(\frac{19}{80}-\frac{5}{24}\,\tau-\frac{1}{16}\,\tau^2\right)\eta\eta_{5x}-\frac{1}{4}(1+\tau)\delta\eta_{2x}^2+\left(\frac{1}{24}+\frac{1}{8}\,\tau\right)\delta\eta\eta_{4x}$$

$$-\frac{1}{4}\,\delta^2\eta_x\eta_{2x}+\left(-\frac{25}{48}+\frac{3}{48}\,\tau\right)\delta\,\eta_x\eta_{3x}+\left(\frac{377}{288}+\frac{15}{16}\,\tau+\frac{49}{32}\,\tau^2\right)\eta_{2x}\eta_{3x}$$

$$+\left(\frac{55}{3024}-\frac{19}{720}\,\tau-\frac{1}{48}\,\tau^2-\frac{1}{16}\,\tau^3\right)\eta_{7x}\Bigg]=0.$$

At the third order $O(\varepsilon^3)$, the correction function $C(\eta)$ is given by

$$C(\eta)=-\frac{5}{64}\,\eta^4+\left(\frac{7}{20}-\frac{1}{4}\,\tau+\frac{1}{16}\,\tau^2\right)\eta\eta_{4x}+\frac{1}{16}\,(2-3\tau)\eta^2\eta_{2x}+\frac{1}{32}\,(3-21\tau)\eta\eta_x^2$$

$$+\left(\frac{1091}{1440}+\frac{1}{3}\,\tau+\frac{21}{32}\,\tau^2\right)\eta_x\eta_{3x}+\left(\frac{163}{360}+\frac{29}{48}\,\tau+\frac{7}{16}\,\tau^2\right)\eta_{2x}^2-\frac{1}{16}\,(1+3\tau)\delta\eta_{3x}$$

$$+\frac{1}{8}\,\delta\eta\eta_x+\frac{1}{48}\,(-11+15\tau)\delta\eta\eta_{3x}-\frac{11}{32}\,\delta\eta^2\eta_x+\left(\frac{61}{1890}-\frac{1}{20}\,\tau-\frac{1}{24}\,\tau^2-\frac{1}{16}\,\tau^3\right)\eta_{6x} \tag{2.122}$$

$$-\frac{1}{8}\,(1+2\tau)\delta\eta_x\eta_{2x}-\frac{1}{8}\,\delta^2\eta_x^2+\left(\frac{3}{16}-\frac{3}{16}\right)\int\eta_x^3dx+\frac{1}{16}\,\delta\int\eta_x^2dx+\frac{7}{32}\,\delta\int\eta^2\eta_{2x}dx$$

$$+\frac{1}{16}\,(5-3\tau)\delta\int\eta\eta_{4x}dx.$$

2.3.4 Equations for bidirectional waves

The KdV equation and its extended versions including higher order corrections are commonly derived from the Boussinesq system by specializing to a wave moving to the right. In the present section, solutions of the Boussinesq system are considered, in which the surface elevation splits into two components u and $\xi(x,t)$ corresponding to the right- and left-moving waves (In a similar manner, Mattioli [115] have derived the first order system of coupled KdV equations for the right- and left-moving waves. Marchant [116], have considered the assumption that the amplitude of the left-moving wave is of $O(\varepsilon^3$ and obtained the interaction of solitary waves at fourth order). In the present work, it is assumed that the amplitude of the left-moving wave is of $O(\varepsilon))$ as compared with that of the right-moving wave. The procedure, similar to that commonly used for the unidirectional waves to derive the KdV equation and its extended high order versions, is applied to the Boussinesq equations to decompose them to a set of equations consisting of a relation expressing the fluid velocity through u and $\xi(x,t)$ and a system of two coupled equations for u and $\xi(x,t)$. It is shown that a non-uniqueness of such a decomposition can be used to derive a system of equations for u and $\xi(x,t)$, in which, to any order, one of the equations is dependent only on the surface elevation u for the main right-moving wave while the second equation includes both u and $\xi(x,t)$. In addition, there are freedoms in choosing the form of the equation for u, in particular, to any order, it can be put (at the expense of an appropriate choice of the second equation form) into the form having the same differential structure as the high order KdV equations arising in the unidirectional case but with arbitrary coefficients in the high order differential polynomials. Thus, the derivation procedure of the bidirectional wave equation in presence of viscosity, surface tension are clearly presented in annex.

2.4 Derivation of the Generalized KdV equation with viscosity, surface tension and wind effects

In this section, we simultaneous combine the dissipation due to viscosity, surface tension and wind effects and show that the model equation can lead to a generalized KdV equation that includes higher diffusion and instability effects. The combination of such effects has never been studied in the literature to the best of our knowledge. We show that such effects have considerable impacts on shallow water wave dynamics. The methodology which allows to obtain these equations is formulated as follows.

2.4.1 Mathematical formulation

It has been shown that, the KdV equation can be derived using Euler's equation for an incompressible and non viscous fluid, the bottom and surface boundary conditions and the assumption of irrotational flow. In the present work, contrary to this approach, we use the dynamics and kinematics boundary conditions both corrected by viscosity effect, the Navier-Stokes equation and the incompressible and irrotational flow assumptions to investigate the dynamics of viscous flowing shallow water waves. This approach has been used to better study the evolution of long shallow wave dynamics [14, 67]. The goal is to determine the shape and dynamics of the (1+1)-dimensional free surface flow of a fluid. We assume that the fluid is viscous, incompressible, and in irrational motion. Then, we choose the velocity field inside the fluid in the form $\overrightarrow{v}(u, 0, w)$ where $u \equiv u(x, z, t)$ and $w \equiv w(x, z, t)$. The function $z = h_0 + \eta(x, t)$ is the equation of the surface, where h_0 is the average depth of the channel containing the fluid and $\eta(x, t)$ is the elevation of the surface, where x is the horizontal coordinate, z the vertical coordinate and t is the time. The hypothesis that the flow is irrotational makes it possible to affirm the existence of a velocity potential such that, $v(x, z, t) = \nabla\phi(x, z, t)$, where the horizontal and vertical velocity components are given by $u = \phi_x$ and $w = \phi_z$. The symbol ∇ is the Nabla operator. It should be noted that, the velocity potential $\phi(x, z, t)$ satisfies the Laplace's equation which is solved in a domain bounded by the free surface, a horizontal solid bottom and two vertical solid walls located at the ends of the domain. Thus the problem to be solved is the Laplace equation with the kinematics, dynamics and bottom conditions, namely

$$\begin{cases} \phi_{2x} + \phi_{2z} = 0, \quad 0 \leqslant z \leqslant h_0 + \eta(x, t), \\ \phi_z = 0, \quad z = 0, \\ \phi_z = \eta_t + \eta_x\phi_x - 2\nu\eta_{xx}, \quad z = h_0 + \eta(x, t), \\ \phi_t + \dfrac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = -\dfrac{P}{\rho} - 2\nu\phi_{zz}, \quad z = h_0 + \eta(x, t), \end{cases} \tag{2.123}$$

where g is the acceleration due to the gravity, ν is the kinematic viscosity, P is the pressure in the fluid and ρ is the density of the fluid.

In this work, we assume that the dynamics free surface condition states that the total pressure at the surface $z = h_0 + \eta(x, t)$ can be split into two components P_1 and P_2 such as $P = P_1 + P_2$, where $P \equiv P(x, t)$, $P_1 \equiv P_1(x, t)$ is the Laplace pressure and $P_2 \equiv P_2(x, t)$ is the wind pressure, which will be expressed following the Miles model [19]. Thus, the atmospheric pressure due to the wind can be written as follows

$$P_2 = (\xi + i\mu)k\rho_a \frac{u_*}{\kappa}\,\eta, \tag{2.124}$$

where ξ and μ are two coefficients depending on both the wavenumber k and phase velocity, ρ_a is the density of air, u_* the friction velocity and κ is the Von Kàrmàn constant. The coefficient ξ that is in phase with the surface elevation $\eta(x,t)$, does not participate in the energy transfer between wind an wave and will therefore be neglected. The coefficient μ provides a term in phase with $i\eta$ and therefore in quadrature with the surface elevation $\eta(x,t)$, which induces its participation in the energy transfer. The possible values of parameter μ have been tabulated by Conte and Miles [78] by solving numerically the Rayleigh equation in the context of a logarithmic profile. These values have been used in deep water field by Kharif et al. [73] and it has been observed that for low frequency waves, a strong wind is needed to maintain the modulational instability and that, the wind needs less force to maintain this instability for higher frequency waves. From there, it is clearly seen that the wind effects are very essential and important in the water wave dynamics.

For a wave in one spatial dimension, we have $\eta(x,t) = Ae^{i(kx-ct)}$ (where c is the phase velocity and A is the amplitude) which implies that $\eta_x = ik\eta$. Thus, the wind pressure can be written in one spatial dimensions in the following forms

$$P_2 = \lambda\eta_x \quad \text{where} \quad \lambda = \frac{\mu\rho_a}{2\kappa^2}u_*^2. \tag{2.125}$$

The Laplace pressure in one spatial dimension can be written as follows

$$P_1 = \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}. \tag{2.126}$$

Since our study deals with small amplitude and long waves, it is preferable to scale our variables in order to avoid any ambiguity corresponding to a different physical situation. Thus, our variables will be scaled in such a way that

$$\widetilde{t} = \frac{t}{t_0}, \quad \widetilde{x} = \frac{x}{L}, \quad \widetilde{z} = \frac{z}{h_0}, \quad \widetilde{\eta} = \frac{\eta}{A}, \quad \widetilde{\phi} = \frac{\phi h_0}{LA\sqrt{gh_0}}, \tag{2.127}$$

where L is a typical wavelength of the surface waves thought the x-direction, A is a typical amplitude of a surface wave η, and t_0 is a characteristic time, which will be used to measure time in the x-direction.

The Laplace equation, Navier Stokes equations and the boundary conditions at the free surface and at the bottom take the following form in which all the subscription ($\sim$) have been omitted

$$\begin{cases} \phi_z = 0, \quad z = 0, \\ \beta\phi_{2x} + \phi_{2z} = 0, \quad 0 \leqslant z \leqslant 1 + \alpha\eta, \\ \eta_t + \alpha\eta_x\phi_x - \beta\dfrac{2L\nu}{c_0h^2}\eta_{xx} - \dfrac{1}{\beta}\,\phi_z = 0, \quad z = 1 + \alpha\eta, \\ \phi_t + \dfrac{\alpha}{2}\left(\phi_x^2 + \dfrac{1}{\beta}\,\phi_z^2\right) + \dfrac{2L\nu}{c_0h^2}\phi_{2z} + \beta\dfrac{\lambda L}{gh_0}\eta_x + \eta - \beta\sigma\dfrac{\eta_{xx}}{\rho gh(1+\beta\alpha^2\eta_x^2)^{3/2}} = 0, \quad z = 1 + \alpha\eta, \end{cases} \tag{2.128}$$

where σ and ν are the surface tension and viscosity coefficients, respectively. The amplitude parameter $\alpha = \frac{A}{h}$, measures the ratio of wave amplitude to undisturbed fluid depth. The wavelength parameter $\beta = (\frac{h}{L})^2$, measures the square of the ratio of fluid depth to wave length. We focus our attention on low amplitude, weakly nonlinear waves in shallow and viscous water, then α and β can be considered as small parameters.

The parameters L, c_0, h, g and ρ being constant, we can assumed that the Bond number written as follows $\tau(\sigma) = \frac{\sigma}{\rho g h^2}$ represents our surface tension parameter, $\delta(\nu) = \frac{2L\nu}{c_0h^2}$ our viscisity parameter and $\chi(\lambda) = \frac{\lambda L}{gh_0}$ our wind parameter. Equation (2.128) can then be rewritten in the following forms

$$\begin{cases} \phi_z = 0, \quad z = 0, \\ \beta\phi_{2x} + \phi_{2z} = 0, \quad 0 \leqslant z \leqslant 1 + \alpha\eta, \\ \eta_t + \alpha\eta_x\phi_x - \beta\delta\eta_{xx} - \dfrac{1}{\beta}\,\phi_z = 0, \quad z = 1 + \alpha\eta, \\ \phi_t + \dfrac{\alpha}{2}\left(\phi_x^2 + \dfrac{1}{\beta}\,\phi_z^2\right) + \delta\phi_{2z} + \beta\chi\eta_x + \eta - \beta\tau\dfrac{\eta_{xx}}{(1+\beta\alpha^2\eta_x^2)^{3/2}} = 0, \quad z = 1 + \alpha\eta. \end{cases} \tag{2.129}$$

2.4.2 Formulation of the Boussinesq system

In this work, the wave regime is considered when the classical Stokes number ($S = \alpha/\beta$) is equal to one, so the amplitude parameter α and the wavelength parameters β in the x-directions are treated in the same order and on an equal footing, such that we have $\alpha = \beta = \epsilon$. The approximation of the velocity potential ϕ satisfying the Laplace's equation and the boundary condition at the bottom, can be written in the form of the following series

$$\phi(x,z,t) = \sum_{m=0}^{\infty} \frac{(-1)^m\beta^m}{(2m)!}\, z^{2m} \frac{\partial^{2m}}{\partial x^{2m}}\, g(x,t), \tag{2.130}$$

where the function $g(x,t)$ represent the values of the velocity potential at the bottom $z = 0$. Introducing (2.130) in the kinematic and dynamic boundary conditions, we obtain system of equations for $\eta(x,t)$ and $g(x,t)$ in the form of infinite series with respect to α or β. Then, we consider a truncated Taylor expansion to a finite number of terms with respect to β. Retaining only the terms of the third order in small parameters, that is, at order $O(\epsilon^3)$, we get the third-order Boussinesq system of equations including terms due to viscosity, surface tension and wind

$$\begin{aligned}\eta_t+w_x &+ \epsilon(\eta_x w + \eta w_x - \frac{1}{6}\,w_{3x} - \delta\eta_{2x}) + \epsilon^2(-\frac{1}{2}\,\eta_x w_{2x} - \frac{1}{2}\,\eta w_{3x} + \frac{1}{120}\,w_{5x}) \\ &+ \epsilon^3(\frac{1}{24}\,\eta_x w_{4x} - \frac{1}{2}\,\eta^2 w_{3x} - \eta\eta_x w_{2x} + \frac{1}{24}\,\eta w_{5x} - \frac{1}{5040}\,w_{7x}) = 0,\end{aligned} \tag{2.131}$$

$$\begin{aligned}\eta_x+w_t &+ \epsilon(ww_x - \frac{1}{2}\,w_{2xt} - \tau\eta_{3x} - \delta w_{2x} + \chi\eta_{2x}) + \epsilon^2(-\eta_x w_{xt} + \frac{1}{2}\,w_x w_{2x} - \eta w_{2xt} \\ &- \frac{1}{2}\,ww_{3x} + \frac{1}{24}\,w_{4xt} + \frac{1}{2}\,\delta w_{4x}) + \epsilon^2(\eta_x w_x^2 - \eta\eta_x w_{xt} - \eta_x ww_{2x} + \frac{1}{6}\,\eta_x w_{3xt} \\ &- \frac{1}{2}\eta^2 w_{2xt} - \eta ww_{3x} + \frac{1}{12}\,w_{2x}w_{3x} - \frac{1}{8}\,w_x w_{4x} + \eta w_x w_{2x} + \frac{1}{6}\,\eta w_{4xt} + \frac{1}{24}\,ww_{5x} \\ &- \frac{1}{720}\,w_{6xt} + \delta\eta w_{4x} - \frac{1}{24}\,\delta w_{4x} + \delta\eta_x w_{3x}) = 0,\end{aligned} \tag{2.132}$$

where the horizontal velocity at the bottom of the fluid component w represent the x-derivative of a function $g(x,t)$ such as $w(x,t) = \frac{\partial g(x,t)}{\partial x}$.

2.4.3 Unidirectional shallow water equations

In this section, along the lines of Whitham [47], we use special properties of solutions to lower-order equations for w and η. To derive the first-, second- and third-order corrections of the horizontal velocity components and the equations describing right-moving waves in higher orders in ϵ, we can, reduce the Boussinesq system to an asymptotically equivalent set of equations consisting of a relationship between the horizontal velocity w and the surface elevation η and an evolution equation for the elevation. Thus in the lower order, the Boussinesq system of equations (2.131) and (2.132) becomes

$$\eta_t + w_x = 0, \tag{2.133}$$

$$\eta_x + w_t = 0. \tag{2.134}$$

The right-moving wave assumption implies that the lowest order Boussinesq system (2.133) and (2.134) are satisfied if $w = \eta$ and $\eta_x + \eta_t = 0$. In the following order iteration, we assume that w can be represented by an expansion of the form

$$w = \eta + \epsilon A(\eta) + \epsilon^2 B(\eta) + \epsilon^3 C(\eta), \tag{2.135}$$

where w is the velocity corrected at first order and the coefficients $A(\eta)$, $B(\eta)$ and $C(\eta)$ are arbitrary functions. These coefficients will subsequently be determined by introducing equations (2.135) into equations (2.131) and (2.132) of the Boussinesq system, then retaining the terms up to $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$, respectively. By proceeding in this way for the first order of the small parameter ϵ, we obtain

$$\eta_t + \eta_x + \epsilon(A_x + 2\eta\eta_x - \delta\eta_{2x} - \frac{1}{6}\,\eta_{3x}) = 0, \tag{2.136}$$

$$\eta_t + \eta_x + \epsilon(A_t + \eta\eta_x - \delta\eta_{2x} + \chi\eta_{xx} + \frac{1}{2}\,\eta_{2xt} - \tau\eta_{3x}) = 0. \tag{2.137}$$

Then, we look for the function $A(\eta)$ as it corresponds to the two equations (2.136) and (2.137) up to the first order of ε. Using the lower order equation $\eta_t + \eta_x = 0$, it is easy to see that all the t-derivatives terms of η can be expressed in terms of the x-derivatives such that $\frac{\partial}{\partial t} = -\frac{\partial}{\partial x}$. This allows us to reduce the system (2.136) and (2.137) into a single equation as follows

$$A_x + 2\eta\eta_x - \delta\eta_{2x} - \frac{1}{6}\,\eta_{3x} = A_t + \eta\eta_x - \delta\eta_{2x} + \chi\eta_{xx} + \frac{1}{2}\,\eta_{3x} - \tau\eta_{3x}. \tag{2.138}$$

A common approach is used the lowest order relation between their time and space derivatives ($A_t = -A_x$) in equations (2.138). After integration, we finally obtain

$$A(\eta) = -\frac{1}{4}\,\eta^2 + \frac{1}{2}\,\chi\eta_x + \frac{1}{6}\,(2 - 3\tau)\,\eta_{2x}. \tag{2.139}$$

Thus, equation (2.135) can take a form corresponding to the first order of the horizontal component of the corrected velocity taking into account viscosity, surface tension and wind effects as follows

$$w = \eta + \epsilon\biggl(-\frac{1}{4}\,\eta^2 + \frac{1}{2}\,\chi\eta_x + \frac{1}{6}\,(2 - 3\tau)\eta_{2x}\biggr). \tag{2.140}$$

Substituting the velocity component given by equation (2.140) into equation (2.131), where, only terms at $O(\epsilon)$ are considered, we obtain finally a new (1+1)-dimensional third-order evolution equation including the terms due to viscosity, surface tension and wind effects

$$\eta_t + \eta_x + \epsilon\biggl(\frac{3}{2}\,\eta\eta_x + \frac{1}{2}\,(\chi - 2\delta)\eta_{2x} + \frac{1}{6}\,(1 - 3\tau)\eta_{3x}\biggr) = 0. \tag{2.141}$$

It is worth noting that, by setting $\delta = 0$, $\chi = 0$, $X = \sqrt{\frac{3}{2}}(x - t)$ and $T = \frac{1}{4}\sqrt{\frac{3}{2}}(\varepsilon t)$ equation (2.141) can be reduced to the standard form of the KdV equation $\eta_T + 6\eta\eta_X + \eta_{3X} = 0$. In the same line, if we choose $\chi = 0$ and $\delta \neq 0$, equation (2.141) reduces to the third-order generalized KdV equation, which have been described and established above.

In the next order iteration, we assume that the horizontal component of the corrected velocity w can be represented by an expansion having the form

$$w = \eta + \epsilon\left(-\frac{1}{4}\eta^2 + \frac{1}{2}\chi\eta_x + \frac{1}{6}(2-3\tau)\eta_{2x}\right) + \epsilon^2 B(\eta), \tag{2.142}$$

where $B(\eta)$ is unknown function of η and their derivatives. To determine these coefficients corresponding to the second order of the parameter ϵ, equation (2.142) is first introduced into equations (2.137) and (2.132) and neglecting the terms of higher order than $O(\epsilon^2)$ in each equation. Thus we obtain the following system in which all the t-derivatives terms of η can be expressed in terms of the x-derivatives through the equation (2.141)

$$\begin{aligned}\eta_t + \eta_x + \epsilon\left[\frac{3}{2}\eta\eta_x + \frac{1}{2}(\chi-2\delta)\eta_{2x} + \frac{1}{6}(1-3\tau)\eta_{3x}\right] + \epsilon^2\Bigg[B_x - \frac{3}{4}\eta^2\eta_x + \frac{1}{2}\chi\eta\eta_{2x} \\ + \frac{1}{2}\chi\eta_x^2 + \frac{1}{12}(1-6\tau)\eta_x\eta_{2x} - \frac{1}{12}(1+6\tau)\eta\eta_{3x} - \frac{1}{12}\chi\eta_{4x} - \left(\frac{17}{360} - \frac{1}{12}\tau\right)\eta_{5x}\Bigg] = 0\end{aligned} \tag{2.143}$$

$$\begin{aligned}\eta_t + \eta_x + \epsilon\left[\frac{3}{2}\eta\eta_x + \frac{1}{2}(\chi-2\delta)\eta_{2x} + \frac{1}{6}(1-3\tau)\eta_{3x}\right] + \epsilon^2\Bigg[B_t - \frac{1}{4}(\chi-2\delta)\eta_x^2 - \frac{1}{4}\chi^2\eta_{3x} \\ + \left(\frac{11}{6} + \frac{7}{4}\tau\right)\eta_x\eta_{2x} + \frac{11}{12}\eta\eta_{3x} + \frac{1}{4}\chi(1+2\tau)\eta_{4x} + \left(\frac{11}{72} - \frac{1}{4}\tau - \frac{1}{4}\tau^2\right)\eta_{5x}\Bigg] = 0\end{aligned} \tag{2.144}$$

Following the same procedure that we have used at first order of perturbation, we obtain

$$\begin{aligned}B(\eta) = \frac{1}{8}\eta^3 - \frac{1}{4}\chi\eta\eta_x + \frac{1}{4}(2+\tau)\eta\eta_{2x} - \frac{1}{8}\chi^2\eta_{2x} + \frac{1}{12}\chi(2+3\tau)\eta_{3x} \\ + \frac{1}{16}(3+7\tau)\eta_x^2 + \frac{1}{120}(12-20\tau-15\tau^2)\eta_{4x} - \frac{1}{8}(\chi-2\delta)\int\eta_x^2 dx.\end{aligned} \tag{2.145}$$

Thus, equation (2.135) can take a form corresponding to the second order of the horizontal component of the corrected velocity taking into account viscosity, surface tension and wind effects as follows

$$\begin{aligned}w = \eta + \epsilon\left[-\frac{1}{4}\eta^2 + \frac{1}{2}\chi\eta_x + \frac{1}{6}(2-3\tau)\eta_{2x}\right] + \epsilon^2\Bigg[\frac{1}{8}\eta^3 - \frac{1}{4}\chi\eta\eta_x + \frac{1}{4}(2+\tau)\eta\eta_{2x} \\ - \frac{1}{8}\chi^2\eta_{2x} + \frac{1}{16}(3+7\tau)\eta_x^2 + \frac{1}{12}\chi(2+3\tau)\eta_{3x} + \frac{1}{120}(12-20\tau-15\tau^2)\eta_{4x} \\ - \frac{1}{8}(\chi-2\delta)\int\eta_x^2 dx\Bigg].\end{aligned} \tag{2.146}$$

Substituting the velocity component given by equation (2.146) into equation (2.131), where, only terms at $O(\epsilon)$ and $O(\epsilon^2)$ are considered, we obtain finally a new (1+1)-dimensional fifth-order evolution equation including the terms due to viscosity, surface tension and wind effects

$$\begin{aligned}\eta_t+\eta_x+\epsilon\Big[\frac{3}{2}\,\eta\eta_x+\frac{1}{2}\,(\chi-2\delta)\eta_{2x}+\frac{1}{6}\,(1-3\tau)\eta_{3x}\Big]+\epsilon^2\Big[-\frac{3}{8}\,\eta^2\eta_x+\frac{1}{8}\,(\chi+2\delta)\eta_x^2\\+\frac{1}{4}\,\chi\eta\eta_{2x}+\frac{1}{24}\,(23+15\tau)\eta_x\eta_{2x}-\frac{1}{8}\,\chi^2\eta_{3x}+\frac{1}{12}\,(5-3\tau)\eta\eta_{3x}+\frac{1}{12}\,\chi(1+3\tau)\eta_{4x}\\+\frac{1}{360}\,(19-30\tau-45\tau^2)\eta_{5x}\Big]=0.\end{aligned} \tag{2.147}$$

In the next order iteration, we assume that the horizontal component of the corrected velocity w can be represented by an expansion having the form

$$\begin{aligned}w=\eta+\epsilon\Big[-\frac{1}{4}\,\eta^2+\frac{1}{2}\,\chi\eta_x+\frac{1}{6}\,(2-3\tau)\eta_{2x}\Big]+\epsilon^2\Big[\frac{1}{8}\,\eta^3-\frac{1}{4}\,\chi\eta\eta_x+\frac{1}{4}\,(2+\tau)\eta\eta_{2x}\\-\frac{1}{8}\,\chi^2\eta_{2x}+\frac{1}{16}\,(3+7\tau)\eta_x^2+\frac{1}{12}\,\chi(2+3\tau)\eta_{3x}+\frac{1}{120}\,(12-20\tau-15\tau^2)\eta_{4x}\\-\frac{1}{8}\,(\chi-2\delta)\int\eta_x^2dx\Big]+\epsilon^3C(\eta),\end{aligned} \tag{2.148}$$

where $C(\eta)$ is unknown function of η and their derivatives. To determine this coefficient corresponding to the third order of the parameter ϵ, equation (2.148) is first introduced into equations (2.131) and (2.132) and neglecting the terms of higher order than $O(\epsilon^3)$ in each equation. Thus we obtain the following system in which all the t-derivatives terms of η can be expressed in terms of the x-derivatives through the equation (2.147)

$$\begin{aligned}\eta_t+\eta_x+\epsilon\Big[\frac{3}{2}\,\eta\eta_x+\frac{1}{2}\,(\chi-2\delta)\eta_{2x}+\frac{1}{6}\,(1-3\tau)\eta_{3x}\Big]+\epsilon^2\Big[-\frac{3}{8}\,\eta^2\eta_x+\frac{1}{8}\,(\chi+2\delta)\eta_x^2\\+\frac{1}{4}\,\chi\eta\eta_{2x}+\frac{1}{24}\,(23+15\tau)\eta_x\eta_{2x}-\frac{1}{8}\,\chi^2\eta_{3x}+\frac{1}{12}\,(5-3\tau)\eta\eta_{3x}+\frac{1}{12}\,\chi(1+3\tau)\eta_{4x}\\+\frac{1}{360}\,(19-30\tau-45\tau^2)\eta_{5x}\Big]+\epsilon^3\Big[C_x+\frac{1}{2}\,\eta^3\eta_x+\frac{1}{8}\,(2\delta-5\chi)\eta\eta_x^2+\frac{1}{16}\,(5+7\tau)\eta_x^3\\-\frac{1}{4}\,\chi\eta^2\eta_{2x}-\frac{1}{8}\,\chi^2\eta_x\eta_{2x}+\frac{1}{8}\,(8+11\tau)\eta\eta_x\eta_{2x}-\frac{1}{12}\,(\delta-2\chi)\eta_{2x}^2+\frac{1}{16}(3+4\tau)\eta^2\eta_{3x}\\-\frac{1}{8}\,\chi^2\eta\eta_{3x}-\frac{1}{48}(4\delta-6\chi-12\tau\chi)\eta_x\eta_{3x}-\frac{1}{48}\,(27+29\tau)\eta_{2x}\eta_{3x}-\frac{1}{24}\,\chi(1+6\tau)\eta\eta_{4x}\\-\Big(\frac{43}{120}+\frac{3}{16}\,\tau+\frac{1}{8}\,\tau^2\Big)\eta_x\eta_{4x}+\Big(-\frac{9}{80}+\frac{1}{24}\,\tau-\frac{1}{8}\,\tau^2\Big)\eta\eta_{5x}-\frac{1}{720}\,\chi(17+30\tau)\eta_{6x}\end{aligned} \tag{2.149}$$

$$+\frac{1}{48}\,\chi^2\eta_{5x}+\bigg(-\frac{71}{5040}+\frac{17}{720}\,\tau+\frac{1}{48}\,\tau^2\bigg)\eta_{7x}-\frac{1}{8}\,(\chi-2\delta)\eta_x\int\eta_x^2dx\bigg]=0,$$

$$\eta_t+\eta_x+\epsilon\bigg[\frac{3}{2}\,\eta\eta_x+\frac{1}{2}\,(\chi-2\delta)\eta_{2x}+\frac{1}{6}\,(1-3\tau)\eta_{3x}\bigg]+\epsilon^2\bigg[-\frac{3}{8}\,\eta^2\eta_x+\frac{1}{8}\,(\chi+2\delta)\eta_x^2$$
$$+\frac{1}{4}\,\chi\eta\eta_{2x}+\frac{1}{24}\,(23+15\tau)\eta_x\eta_{2x}-\frac{1}{8}\,\chi^2\eta_{3x}+\frac{1}{12}\,(5-3\tau)\eta\eta_{3x}+\frac{1}{12}\,\chi(1+3\tau)\eta_{4x}$$
$$+\frac{1}{360}\,(19-30\tau-45\tau^2)\eta_{5x}\bigg]+\epsilon^3\bigg[C_t-\frac{1}{8}\,\eta^3\eta_x-\frac{1}{16}\,(6\delta-5\chi)\eta\eta_x^2+\frac{1}{8}\,(7-10\tau)\eta_x^3$$
$$+\frac{1}{8}\,\chi\eta^2\eta_{2x}+\frac{1}{16}\,(8\delta\chi-8\delta^2+9\chi^2)\eta_x\eta_{2x}-\bigg(\frac{13}{24}\,\delta+\frac{5}{8}\,\delta\tau+\frac{23}{48}\,\chi+\frac{25}{16}\,\tau\chi\bigg)\eta_{2x}^2$$
$$+\frac{1}{8}\,(15-16\tau)\eta\eta_x\eta_{2x}-\bigg(\frac{7}{6}\,\delta+\frac{1}{4}\,\delta\tau-\frac{1}{4}\,\chi+\frac{15}{8}\,\tau\chi\bigg)\eta_x\eta_{3x}+\frac{1}{16}\,(7-2\tau)\eta^2\eta_{3x}$$
$$+\bigg(\frac{199}{72}+\frac{119}{48}\,\tau+\frac{49}{16}\,\tau^2\bigg)\eta_{2x}\eta_{3x}-\frac{1}{24}\,(\delta-3\chi^3)\eta_{4x}+\bigg(\frac{1337}{720}-\frac{1}{48}\,\tau+\frac{21}{16}\,\tau^2\bigg)\eta_x\eta_{4x}$$
$$+\frac{1}{24}\,\chi(11-6\tau)\eta\eta_{4x}-\frac{1}{16}\,\chi^2(1+2\tau)\eta_{5x}+\bigg(\frac{1}{24}\,\delta+\frac{11}{144}\,\chi+\frac{1}{24}\,\tau\chi+\frac{1}{8}\,\tau^2\chi\bigg)\eta_{6x}$$
$$+\bigg(\frac{47}{80}-\frac{11}{24}\,\tau\bigg)\eta\eta_{5x}+\bigg(\frac{109}{2160}-\frac{11}{144}\,\tau-\frac{1}{16}\,\tau^2-\frac{1}{8}\,\tau^3\bigg)\eta_{7x}-\frac{1}{8}\,(\chi-2\delta)\eta_x\int\eta_x^2dx\bigg]=0. \tag{2.150}$$

Following the same procedure that we have used at first and second order of perturbations, we obtain

$$C(\eta)=-\frac{5}{64}\,\eta^4+\frac{1}{16}\,(2-3\tau)\eta^2\eta_{2x}+\frac{1}{32}\,(3-21\tau)\eta\eta_x^2+\bigg(\frac{163}{360}+\frac{29}{48}\,\tau+\frac{7}{16}\,\tau^2\bigg)\eta_{2x}^2$$
$$+\frac{3}{16}\,\chi\eta^2\eta_x-\bigg(\frac{11}{48}\,\delta+\frac{5}{16}\,\delta\tau+\frac{31}{96}\,\chi+\frac{25}{32}\,\tau\chi\bigg)\eta_x\eta_{2x}+\bigg(\frac{7}{20}-\frac{1}{4}\,\tau+\frac{1}{16}\,\tau^2\bigg)\eta\eta_{4x}$$
$$+\bigg(-\frac{5}{16}\,\delta+\frac{37}{96}\,\chi-\frac{9}{32}\,\tau\chi+\frac{3}{16}\,\tau\delta\bigg)\eta\eta_{3x}+\bigg(\frac{1}{48}\,\delta+\frac{1}{20}\,\chi+\frac{1}{24}\,\tau\chi+\frac{1}{16}\,\tau^2\chi\bigg)\eta_{5x}$$
$$+\frac{1}{64}\,\left(8\delta\chi-8\delta^2+11\chi^2\right)\eta_x^2-\frac{1}{48}\left(\delta-3\chi^3\right)\eta_{3x}+\frac{1}{16}\,\chi^2\int\eta\eta_{3x}dx-\frac{1}{48}\chi^2(2+3\tau)\eta_{4x}$$
$$+\frac{1}{16}\,(3-3\tau)\int\eta_x^3dx+\bigg(\frac{1091}{1440}+\frac{1}{3}\,\tau+\frac{21}{32}\,\tau^2\bigg)\eta_x\eta_{3x}+\frac{1}{32}\,(9\chi-10\delta)\int\eta\eta_x^2dx$$
$$+\bigg(\frac{61}{1890}-\frac{1}{20}\,\tau-\frac{1}{24}\,\tau^2-\frac{1}{16}\,\tau^3\bigg)\eta_{6x}+\bigg(\frac{5}{16}\,\delta-\frac{13}{96}\,\chi+\frac{1}{32}\,\tau\chi-\frac{3}{16}\,\tau\delta\bigg)\int\eta\eta_{4x}. \tag{2.151}$$

Thus, equation (2.135) can take a form corresponding to the third order of the horizontal component of the corrected velocity taking into account viscosity, surface tension and wind effects as follows

$$w = \eta + \epsilon\Big[-\frac{1}{4}\eta^2 + \frac{1}{2}\chi\eta_x + \frac{1}{6}(2-3\tau)\eta_{2x}\Big] + \epsilon^2\Big[\frac{1}{8}\eta^3 - \frac{1}{4}\chi\eta\eta_x + \frac{1}{4}(2+\tau)\eta\eta_{2x} - \frac{1}{8}\chi^2\eta_{2x}$$
$$+\frac{1}{16}(3+7\tau)\eta_x^2 + \frac{1}{12}\chi(2+3\tau)\eta_{3x} + \frac{1}{120}(12-20\tau-15\tau^2)\eta_{4x} - \frac{1}{8}(\chi-2\delta)\int\eta_x^2dx\Big]$$
$$+\epsilon^3\Big[-\frac{5}{64}\eta^4 + \frac{3}{16}\chi\eta^2\eta_x + \frac{1}{32}(3-21\tau)\eta\eta_x^2 + \Big(\frac{163}{360} + \frac{29}{48}\tau + \frac{7}{16}\tau^2\Big)\eta_{2x}^2 + \frac{1}{16}(2-3\tau)\eta^2\eta_{2x}$$
$$+\Big(-\frac{5}{16}\delta + \frac{37}{96}\chi - \frac{9}{32}\tau\chi + \frac{3}{16}\tau\delta\Big)\eta\eta_{3x} + \Big(\frac{1}{48}\delta + \frac{1}{20}\chi + \frac{1}{24}\tau\chi + \frac{1}{16}\tau^2\chi\Big)\eta_{5x}$$
$$+\frac{1}{64}\left(8\delta\chi - 8\delta^2 + 11\chi^2\right)\eta_x^2 - \frac{1}{48}\chi^2(2+3\tau)\eta_{4x} + \Big(\frac{1091}{1440} + \frac{1}{3}\tau + \frac{21}{32}\tau^2\Big)\eta_x\eta_{3x}$$
$$+\frac{1}{16}\chi^2\int\eta\eta_{3x}dx + \Big(\frac{61}{1890} - \frac{1}{20}\tau - \frac{1}{24}\tau^2 - \frac{1}{16}\tau^3\Big)\eta_{6x} + \frac{1}{16}(3-3\tau)\int\eta_x^3dx$$
$$+\frac{1}{32}(9\chi-10\delta)\int\eta\eta_x^2dx - \Big(\frac{11}{48}\delta + \frac{5}{16}\delta\tau + \frac{31}{96}\chi + \frac{25}{32}\tau\chi\Big)\eta_x\eta_{2x} - \frac{1}{48}\left(\delta - 3\chi^3\right)\eta_{3x}$$
$$+\Big(\frac{7}{20} - \frac{1}{4}\tau + \frac{1}{16}\tau^2\Big)\eta\eta_{4x} + \Big(\frac{5}{16}\delta - \frac{13}{96}\chi + \frac{1}{32}\tau\chi - \frac{3}{16}\tau\delta\Big)\int\eta\eta_{4x}.\Big]. \tag{2.152}$$

Substituting the velocity component given by equation (2.152) into equation (2.131), where, only terms at $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$ are considered, we obtain finally a new (1+1)-dimensional seventh-order evolution equation including the terms due to viscosity, surface tension and wind efects

$$\eta_t + \eta_x + \epsilon\Big[\frac{3}{2}\eta\eta_x + \frac{1}{2}(\chi-2\delta)\eta_{2x} + \frac{1}{6}(1-3\tau)\eta_{3x}\Big] + \epsilon^2\Big[-\frac{3}{8}\eta^2\eta_x + \frac{1}{8}(\chi+2\delta)\eta_x^2$$
$$+\frac{1}{4}\chi\eta\eta_{2x} + \frac{1}{24}(23+15\tau)\eta_x\eta_{2x} - \frac{1}{8}\chi^2\eta_{3x} + \frac{1}{12}(5-3\tau)\eta\eta_{3x} + \frac{1}{12}\chi(1+3\tau)\eta_{4x}$$
$$+\frac{1}{360}(19-30\tau-45\tau^2)\eta_{5x}\Big] + \epsilon^3\Big[\frac{3}{16}\eta^3\eta_x - \frac{1}{32}(2\delta+5\chi)\eta\eta_x^2 + \frac{1}{32}(19-13\tau)\eta_x^3 \tag{2.153}$$
$$-\frac{1}{16}\chi\eta^2\eta_{2x} - \frac{1}{32}(8\delta^2 - 8\delta\chi - 7\chi^2)\eta_x\eta_{2x} - \frac{1}{32}(10\delta + 10\delta\tau + 5\chi + 25\tau\chi)\eta_{2x}^2$$
$$+\frac{1}{16}(23-5\tau)\eta\eta_x\eta_{2x} + \frac{1}{16}(5+\tau)\eta^2\eta_{3x} + \Big(\frac{317}{288} + \frac{15}{16}\tau + \frac{49}{32}\tau^2\Big)\eta_{2x}\eta_{3x}$$

$$- \frac{1}{16}\,(10\delta + 2\delta\tau - 3\chi + 13\tau\chi)\eta_x\eta_{3x} + \left(\frac{19}{80} - \frac{5}{24}\,\tau - \frac{1}{16}\,\tau^2\right)\eta\eta_{5x} + \frac{5}{24}\,\chi\eta\eta_{4x}$$

$$- \frac{1}{16}\,\chi^2\eta\eta_{3x} + \left(\frac{1079}{1440} - \frac{5}{48}\,\tau + \frac{19}{32}\,\tau^2\right)\eta_x\eta_{4x} + \left(\frac{1}{48}\,\delta + \frac{19}{720}\,\chi + \frac{1}{16}\,\tau^2\chi\right)\eta_{6x}$$

$$- \frac{1}{48}\,\chi^2(1 + 3\tau)\eta_{5x} - \frac{1}{48}\,(\delta - 3\chi^3)\eta_{4x} + \left(\frac{55}{3024} - \frac{19}{720}\,\tau - \frac{1}{48}\,\tau^2 - \frac{1}{16}\,\tau^3\right)\eta_{7x}$$

$$- \frac{1}{8}\,(\chi - 2\delta)\eta_x \int \eta_x^2 dx\Bigg] = 0.$$

Therfore, this equation is a generalization of the KdV equation for the seventh-order nonlinear evolution. This equation taking into account the effects of viscosity, surface tension and wind can provide better results in modeling the movement of unidirectional waves.

Conclusion

In this chapter, we have presented the basic equations related to the physical and mathematical modeling of the equations describing the waves dynamics in shallow water. Thereafter, the methodology applied to reach our goals has been presented. In fact, with the help of linear approximation applying on the Navier-Stokes equations, we have obtained a system of equations for potential flow which includes viscosity, surface tension and wind. The correction due to the viscosity have been applied not only to the kinematics boundary condition on the surface, but also to the dynamics condition modeled by Bernoulli's equation. The effects of wind and surface tension are introduce in ours models through the Miles's model and the Laplace pressure, respectively. We have applied the perturbation theory to the Boussinesq system and derived higher third-, fifth- and seventh-order nonlinear evolution equations modelling unidirectional wave motion in (1+1)-dimensions for viscous shallow water waves with surface tension effects in first instance and the combination of surface tension and wind effects in second times.

Chapter III

Results and discussion

Introduction

This chapter presents the main results obtained throughout this thesis. We present the effects of some physical parameters such as viscosity, surface tension, and wind on hydrodynamic wave for shallow water. In fact, we investigate the soliton-like solutions of the new KdV-type equations obtained in Chapter II and clearly present the effects of such physical parameters on the phase and group velocities and on the solitons dynamics. We show that the viscosity, surface tension and wind effects can have a considerable impact on hydrodynamic waves and on the solitons dynamics. We also demonstrate that taking into account such parameters leads to some improved versions of the KdV equation and then can better describe the waves dynamics in shallow water.

3.1 The generalized KdV equation with viscosity and surface tension effects

Since the discovery of solitons, the shallow water wave theory has been the subject of very interesting works. The KdV equation, which describes the motion of small but finite amplitude waves that propagate in the positive x-direction [10], has been the subject of intensive works, and experiments. Several phenomena and studies in the field of nonlinear science are described with general coefficients or with higher-order nonlinear and dissipative terms allowing to observe new effects [12]. In that sense, several authors have improved the KdV equation by introducing high-order terms, leading generally to near partially integrable or integrable high-order equations with quasi-soliton solutions [90]. In these study, it has been shown that, the KdV equation can be derived using Euler's equation for an incompressible and non viscous fluid, the bottom and surface boundary conditions and the assumption of irrotational flow. In our work, instead of this approach, we have used the dynamics and kinematics boundary conditions both corrected by viscosity effect, the Navier-Stokes equation and the incompressible and irrotational flow assumptions to study the dynamics of viscous flowing shallow water waves. This approach has been shown to better describe the evolution of long shallow wave dynamics [14, 67]. We simultaneous combine the dissipation due to viscosity and surface tension effects and show that the obtained model equations can lead to some new generalized KdV equations that include higher diffusion and instability effects. Such effects have been shown to have an impact on the dynamics of shallow water waves [14, 67].

We go beyond the fifth order evolution equation for long wave dissipative solitons derived by Depassier and Letelier [14] (for dissipative wave supposed to have a lower amplitude $O(\varepsilon^2)$). We show that the dynamics of the wave amplitude for the unidirectional propagation of long waves over shallow water can have a lower amplitude $O(\varepsilon^3)$, and can be governed by new generalized KdV equations for a viscous flowing shallow water waves given by :

$$\begin{aligned} &u_t + u_x + \varepsilon(a_1uu_x + a_2u_{3x} + a_3u_{2x}) + \varepsilon^2(b_1u^2u_x + b_2u_{2x} + b_3u_xu_{2x} + b_4uu_{2x} + b_4u_x^2 \\ &+ b_5uu_{3x} + b_6u_{5x}) + \varepsilon^3(c_1u^3u_x + c_2u_x^2 + c_2u^2u_{2x} + c_3u_x^3 + c_4uu_x^2 + c_5uu_{2x} + c_6uu_xu_{2x} \\ &+ c_7u^2u_{3x} + c_8u_xu_{4x} + c_9uu_{5x} + c_{10}u_{2x}u_{3x} + c_{11}u_{4x} + c_{12}u_{2x}^2 + c_{13}u_xu_{3x} + c_{14}uu_{4x} \\ &+ c_{15}u_{7x}) = 0, \end{aligned} \tag{3.1}$$

where the coefficients $a_i(i = 1, 2, 3)$, $b_i(i = 1, ..., 6)$ and $c_i(i = 1, ..., 15)$ depend on some parameters. The parameter ε is a non-dimensional measure of the small wave amplitude. The subscripts of the form "nx" denote derivatives of the order n with respect to x and t, where x an t are the space and time variables.

It is worth noting that, after the degeneration of each coefficient of equation (3.1), this equation can be reduced to some well-known equations. For example, if $b_i(i = 1, ..., 6) = 0$, $c_i(i = 1, ..., 15) = 0$, $\varepsilon \neq 0$ and $a_i(i = 1, 2, 3) \neq 0$, equation (3.1) can be reduced to the generalized third-order KdV equations [117]. Especially, in the limit case where $\varepsilon = 1, a_1 = 6, a_2 = 1$ and $a_3 = 0$, this equation can be turned to the standard KdV equation [4]. If $c_i(i = 1, ..., 15) = 0$ and $a_i(i = 1, 2, 3)$, $b_i(i = 1, ..., 6)$ and ε are real and arbitrary parameters, equation (3.1) can be reduced to the fifth-order KdV equations, including the Lax equation [118], the Sawada-Kotera equation [119] and the Kaup-Kupershmidt equation [120], just to cite a few. Similarly, if $a_i(i = 1, 2, 3)$, $b_i(i = 1, ..., 6)$, $c_i(i = 1, ..., 15)$ and ε are real and arbitrary parameters, we obtain the seventh-order KdV equations such as the one introduced by Pomeau et al. [85]. It is therefore clear that equation (3.1) can be similar to many equations well-known to the scientific community and can describes many physical situations. This equation can therefore better describes the dynamics of nonlinear waves in shallow water.

Replacing the parameters $a_i(i = 1, 2, 3)$, $b_i(i = 1, ..., 6)$ and $c_i(i = 1, ..., 15)$ by their expressions for viscosity and surface tension, we obtain the equation to be solved given by equation (2.121) in Chapter 2 as follows

$$
\begin{aligned}
&\eta_t+\eta_x+\varepsilon\Big[\frac{3}{2}\,\eta\eta_x+\frac{1}{6}\Big(1-3\tau\Big)\eta_{3x}-\delta\eta_{2x}\Big]+\varepsilon^2\Big[-\frac{3}{8}\,\eta^2\eta_x+\frac{1}{24}\Big(23+15\tau\Big)\eta_x\eta_{2x}\\
&\quad-\frac{1}{4}\,\delta\eta_{2x}+\frac{1}{4}\,\delta\eta\eta_{2x}+\frac{1}{12}\Big(5-3\tau\Big)+\frac{1}{4}\,\delta\eta_x^2+\frac{1}{360}\Big(19-30\tau-45\tau^2\Big)\eta_{5x}\Big]\\
&\quad+\varepsilon^3\Big[\frac{3}{16}\,\eta^3\eta_x-\frac{1}{16}\,\delta\eta_x^2-\Big(\frac{13}{32}+\frac{13}{32}\tau\Big)\eta_x^3+\frac{1}{8}\Big(4-\tau\Big)\eta^2\eta_{3x}-\frac{3}{16}\,\delta\eta\eta_x^2\\
&\quad+\frac{1}{8}\,\delta\eta^2\eta_{2x}-\frac{1}{8}\,\delta\eta\eta_{2x}+\Big(\frac{11}{16}+\frac{29}{6}\tau\Big)\eta\eta_x\eta_{2x}+\Big(\frac{1079}{1440}-\frac{5}{45}\,\tau+\frac{19}{32}\,\tau^2\Big)\eta_x\eta_{4x}\\
&\quad-\frac{1}{48}\Big(1+9\tau\Big)\delta\eta_{4x}+\Big(\frac{19}{80}-\frac{5}{24}\,\tau-\frac{1}{16}\,\tau^2\Big)\eta\eta_{5x}-\frac{1}{4}(1+\tau)\delta\eta_{2x}^2+\Big(\frac{1}{24}+\frac{1}{8}\,\tau\Big)\delta\eta\eta_{4x}\\
&\quad-\frac{1}{4}\,\delta^2\eta_x\eta_{2x}+\Big(-\frac{25}{48}+\frac{3}{48}\,\tau\Big)\delta\;\eta_x\eta_{3x}+\Big(\frac{377}{288}+\frac{15}{16}\,\tau+\frac{49}{32}\,\tau^2\Big)\eta_{2x}\eta_{3x}\\
&\quad+\Big(\frac{55}{3024}-\frac{19}{720}\,\tau-\frac{1}{48}\,\tau^2-\frac{1}{16}\,\tau^3\Big)\eta_{7x}\Big]=0.
\end{aligned}
\tag{3.2}
$$

This equation is a new generalization of the KdV equation to seventh-order. We show in the rest of our work that it has soliton-like solutions and can be used to better describe the dynamics of waves in shallow water in the presence of surface tension and viscosity effects.

3.1.1 Soliton solutions of the generalized KdV equation with viscosity and surface tension

In this section, by using Hirota's bilinear method, we investigate the soliton solutions of the generalized KdV evolution equation (3.2) which describes the dynamics of shallow water waves. A variety of powerful methods for finding soliton solutions in complex physical systems have been developed, such as Darboux transformation, Bäcklund transformation, inverse scattering transformation, Hirota bilinear method, symmetry method, and similarity transformation, juste to cite a few. Among the different methods available to investigate soliton solutions in various equations, the Hirota bilinear method has several advantages according to other methods. The method obtains the results directly, quickly, and needs simple algorithms in programming. The method also allows testing if a certain equation satisfies the necessary requirements to admit soliton solutions [12]. Here, we apply Hirota's bilinear method to look for the soliton type solutions of equation (3.2). For this, we assume that the general form of solution is given by

$$\eta(x,t) = D\frac{\partial^2}{\partial x^2}\ln(f(x,t)) = D\frac{ff_{2x} - f_x^2}{f^2}. \tag{3.3}$$

The parameter D is a constant to be determined and $f(x,t)$ is the auxiliary function which can be written in the form of a traveling wave solution such that

$$f(x,t) = 1 + f_1(x,t) = 1 + \exp(\theta), \tag{3.4}$$

where $\theta = kx - \omega t + \xi_0$ and the parameters k, ω and ξ_0 are the wave number, the angular frequency and the phase shift, respectively.

First of all, the substitution of

$$\eta(x,t) = \exp(\theta_i) \quad \text{where} \;\; \theta_i = k_i x - \omega_i t \quad i = 1, 2, ..., N, \tag{3.5}$$

into the linear terms of (3.2) we obtain the dispersion relation between the wave number k_i and the pulsaton ω_i as follows

$$\begin{aligned}\omega = \frac{1}{15120}\Bigg[&15120k + \varepsilon\Big[(2520 - 7560\tau)k - 15120\delta\Big]k^2 + \varepsilon^2\Big[(798 - 1260\tau \\ &- 1890\tau^2)k^3 - 3780\delta\Big]k^2 + \varepsilon^3\Big[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 - 2835\tau\delta\Big]k^4\Bigg].\end{aligned} \tag{3.6}$$

To determine the constant D, we introduce equation (3.3) into equation (3.2) where, $f(x,t)$ is taken as

$$\begin{aligned}f(x,t) = 1 + \exp\Bigg\{kx - \frac{1}{15120}\Bigg[&15120k + \varepsilon\Big[(2520 - 7560\tau)k - 15120\delta\Big]k^2 \\ &+ \varepsilon^2\Big[(798 - 1260\tau - 1890\tau^2)k^3 - 3780\delta\Big]k^2 + \varepsilon^3\Big[(275 - 399\tau - 375\tau^2 \\ &- 945\tau^3)k^3 - 2835\tau\delta\Big]k^4\Bigg]t\Bigg\}.\end{aligned} \tag{3.7}$$

Equating the coefficients of the different powers of $\exp(\theta)$ to zero yields a system of polynomial equations, after solving this system with Mathematica, we find $D = 2$. This means that the soliton solution is given by

$$\eta(x,t) = \frac{2k^2\exp(kx - \omega t)}{(1 + \exp(kx - \omega t))^2} = \frac{k^2}{1 + \cosh(kx - \omega t)}. \tag{3.8}$$

Using equation (3.6) and after some transformations, we obtain

$$\begin{aligned}\eta(x,t) =& \frac{k^2}{2}\,\mathrm{sech}^2\Bigg\{\frac{k}{2}x - \frac{1}{30240}\Bigg[15120k + \varepsilon\Big[(2520-7560\tau)k - 15120\delta\Big]k^2 \\ &+ \varepsilon^2\Big[(798-1260\tau-1890\tau^2)k^3 - 3780\delta\Big]k^2 + \varepsilon^3\Big[(275-399\tau-375\tau^2 \\ &- 945\tau^3)k^3 - 2835\tau\delta\Big]k^4\Bigg]t\Bigg\}.\end{aligned} \tag{3.9}$$

This soliton solution describes the long, small-amplitude, unidirectional wave motion in shallow water with surface tension and viscosity effects.

3.1.2 Phase and group velocity

The Phase and group velocities are two important and related concepts in wave dynamics. In fact, the wave is a perturbation that propagates through a medium so we can associate such a wave two velocities including the ohase velocity and the group velocity which are mostly (but not always) different. To clearly understand the difference between phase and group velocities of waves, consider the following analogy. A group of people, say city marathon runners start from the starting at the same time. Initially it would appear that all of them are running at the same velocity. As the time passes, group spreads out (disperses) simply because each runner in the group is running with different velocity. If you think of phase velocity to be like the velocity of an individual runner, then the group velocity is the velocity of the entire group as a whole. Obviously and most often, individual runners can run faster than the group as a whole. to stretch this analogy, we note that the phase velocity v_{ph} of waves are typically larger than the group velicty v_{gr} of waves. However, this really depends on the properties of the medium. The media in which $v_{gr} = v_{ph}$ is called the non-dispersive medium. But the media in which $v_{gr} < v_{ph}$ is called normal dispersion. The media in which $v_{gr} > v_{ph}$ is called anomolous dispersive media. It must be emphasised that dispersion is a property of the medium in which a wave travels. It is not the property of the waves themselves.

The relation between phase and group velocities is given by

$$v_{gr} = \frac{d\omega}{dk} = v_{ph} - \lambda\frac{dv_{ph}}{d\lambda}, \tag{3.10}$$

where λ is the wavelength. Generally, $\omega(k)$ is called thye dispersion relation and indicates the dispersion properties of a medium. As (3.10) predicts, if the phase velocity does not depends on the wavelength of the propagating wave, then $v_{gr} = v_{ph}$. For example, sound waves are non-dispersive in air, i.e, all the individual components that make up the sound wave travel at same velocity. Phase velocity of sound waves is independent of the wavelength when it propagates in air. We propose to evaluate in the following the phase and group velocities of waves modeled by the obtained equation (3.2).

3.1.2.1 Phase velocity

The phase velocity of a wave is the rate at which the wave propagates in any medium. This is the velocity at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity given as follows

$$v_{ph} = \frac{\omega}{k}, \tag{3.11}$$

where ω is the angular frequency and k the wave number. Thus, from equation (3.6), we can easily deduce that the phase velocity of the wave is given by

$$\begin{aligned} v_{ph} = \frac{\omega}{k} = \frac{1}{15120}\,&(15120 + \varepsilon[(2520 - 7560\tau)\,k - 15120\delta]k + \varepsilon^2[(798 - 1260\tau \\ &- 1890\tau^2)k^3 - 3780\delta]k + \varepsilon^3[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 - 2835\tau\delta]k^3). \end{aligned} \tag{3.12}$$

We notice that the phase velocity is not constant, but depends on k, which implies that we are in a dispersive medium, so waves of different frequencies propagate at different velocities. If we consider a wave packet consisting of three waves of neighboring pulses initially in phase at a date t, we will find after a time Δt that the wave packet will have widened, and that it will have propagated at a velocity lower than the velocity of its fastest component.

3.1.2.2 Group velocity

The group velocity is the velocity with which the envelope of a pulse propagates in a medium, it corresponds to the velocity with which the information is transported in the wave and therefore cannot exceed the velocity of light. The expression of the group velocity can be obtained from the dispersion relation as follows

$$v_{gr} = \frac{\partial \omega}{\partial k}, \tag{3.13}$$

where ω is the angular frequency and k the wave number. Thus, from equation (3.6), we can easily deduce that the group velocity of the soliton can be written in the following form

$$
\begin{aligned}
v_{gr} = \frac{\partial \omega}{\partial k} = \frac{1}{15120}\,& (15120 + 2\varepsilon[(2520 - 7560\tau)\,k - 15120\delta]k + \varepsilon(2520 - 7560\tau)k^2 \\
&+ 3\varepsilon^2(798 - 1260\tau - 1890\tau^2)k^4 + 3\varepsilon^3(275 - 399\tau - 375\tau^2 - 945\tau^3)k^6 \\
&+ 2\varepsilon^2[(798 - 1260\tau - 1890\tau^2)k^3 - 3780\delta]k + 4\varepsilon^3[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 \\
&- 2835\delta\tau]k^3).
\end{aligned} \tag{3.14}
$$

We notice that the group velocity is not constant, but depends on k, which implies that we are in a dispersive medium, so waves of different frequencies propagate at different velocities.

By observing the equation (3.6), it is clearly observed that in the limit case where $\varepsilon = 0$, the pulsation is directly proportional to the wave number ($\omega = k$). This means that the phase velocity is independent of the pulsation. Moreover, the phase velocity is equal to the group velocity ($v_{gr} = v_{ph}$). In this case, the medium is called non-dispersive. This regime corresponds to the behavior of the oscillator chain when it is crossed by a wave of very long wavelength. The wave perceives the chain as a continuous medium, and does not distinguish the masses from each other.

3.1.3 Effects of viscosity and surface tension on the solitons dynamics, phase and group velocities

For physical application, it is important to discuss about the influence of surface tension and viscosity on the dynamics of the obtained soliton solutions. However, it is well known that in the context of shallow water, the average height of the fluid and the length of the channel must be considered in the order of meters (m) [121, 122]. Therefore, the possible values of surface tension (τ) and viscosity (δ) reducing to the order $10^{-7} - 10^{-6}$ are negligible. Thus, to study the effects of surface tension and viscosity on the dynamics of our solution, we choose our parameters τ and δ in the interval $[0.1 - 0.6]$ which implies that the height of the fluid and the length of the channel are of the order of millimeters (mm). Moreover, the choice of these parameters imposes that the forms of equilibrium taking by the fluid result from a competition between the forces of gravity (gravitational or inertial) and the forces of surface tension of the liquid, also called capillary forces [123]. Regarding the small perturbation parameters ε, the choice of the values of this parameter has been guided by previous analytical, numerical and experimental works. For example, Do-Carmos et al. [122] have investigated numerically the propagation of surface waves in shallow water by choosing $\varepsilon = 0.1$ and $\varepsilon = 0.25$. They confirmed their results both analytically and experimentally. Along the same line, Renouard et al. [121] have studied experimentally the generation, Dampening and reflection of a solitary wave by choosing ε in the range values $[0.114 - 0.485]$. Chambarel et al. [123] have numerically investigated the influence of wind on extreme wave events in shallow water by choosing $A = 0.20m$ and $h = 1m$ which implies that $\varepsilon = \frac{A}{h} = 0.20$. Just recently, Karczewska et al. [11] have studied the dynamics of solitons in shallow water beyond KdV by using ε in the range values $[0.1 - 0.3]$, just to cite a few.

3.1.3.1 Effects of viscosity and surface tension on the phase and group velocities

The k-dependence of the phase velocity is shown in figures 1-(a) and 1-(b). Both figures 1-(a) and 1-(b) show that, the phase velocity is an increasing function of the wave number. The figure 1-(a), shows the influence of the viscosity (parameter δ) on the phase velocity curve. Indeed, we plot the phase velocity as a function of the wave number for three values of δ. We see that when the value of the parameter δ increases, the phase velocity weakly decreases. We have also investigated the impact of the surface tension (parameter τ) on the phase velocity. We plot the phase velocity as a function of the wave number for three values of τ. The result illustrated in figure 1-(b) clearly shows that the phase velocity strongly decreases when the value of the parameter τ increases. In conclusion, both viscosity and surface tension have the same effect on the phase velocity. However, the effect of the surface tension is very greater than that of the viscosity.

The k-dependence of the group velocity is shown in figures 1-(c) and 1-(d). These figures show that, the group velocity is an increasing function of the wave number. The figure 1-(c), shows the influence of the viscosity (parameter δ) on the group velocity curve. Indeed, we plot the group velocity as a function of the wave number for three values of δ. We see that when the value of the parameter δ increases, the group velocity decreases very weakly. We have also investigated the impact of the surface tension (parameter τ) on the group velocity. We plot the group velocity as a function of the wave number for three values of τ. The result illustrated in figure 1-(d) clearly shows that the group velocity strongly decreases when the value of the parameter τ increases. In conclusion, both viscosity and surface tension have the same effect on the group velocity. However, the effect of the surface tension is very greater than that of the viscosity.

Furthermore, by carefully observing Figures 3.1-(a) and 3.1-(b) or Figures 3.1-(c) and 3.1-

(d), it can be seen that the magnitude of the group velocity is greater than that of the phase velocity ($v_{gr} > v_{ph}$). This fact could mean that, the dispersion is anomalous, the wave regresses inside a packet, from the head to the tail of this one.

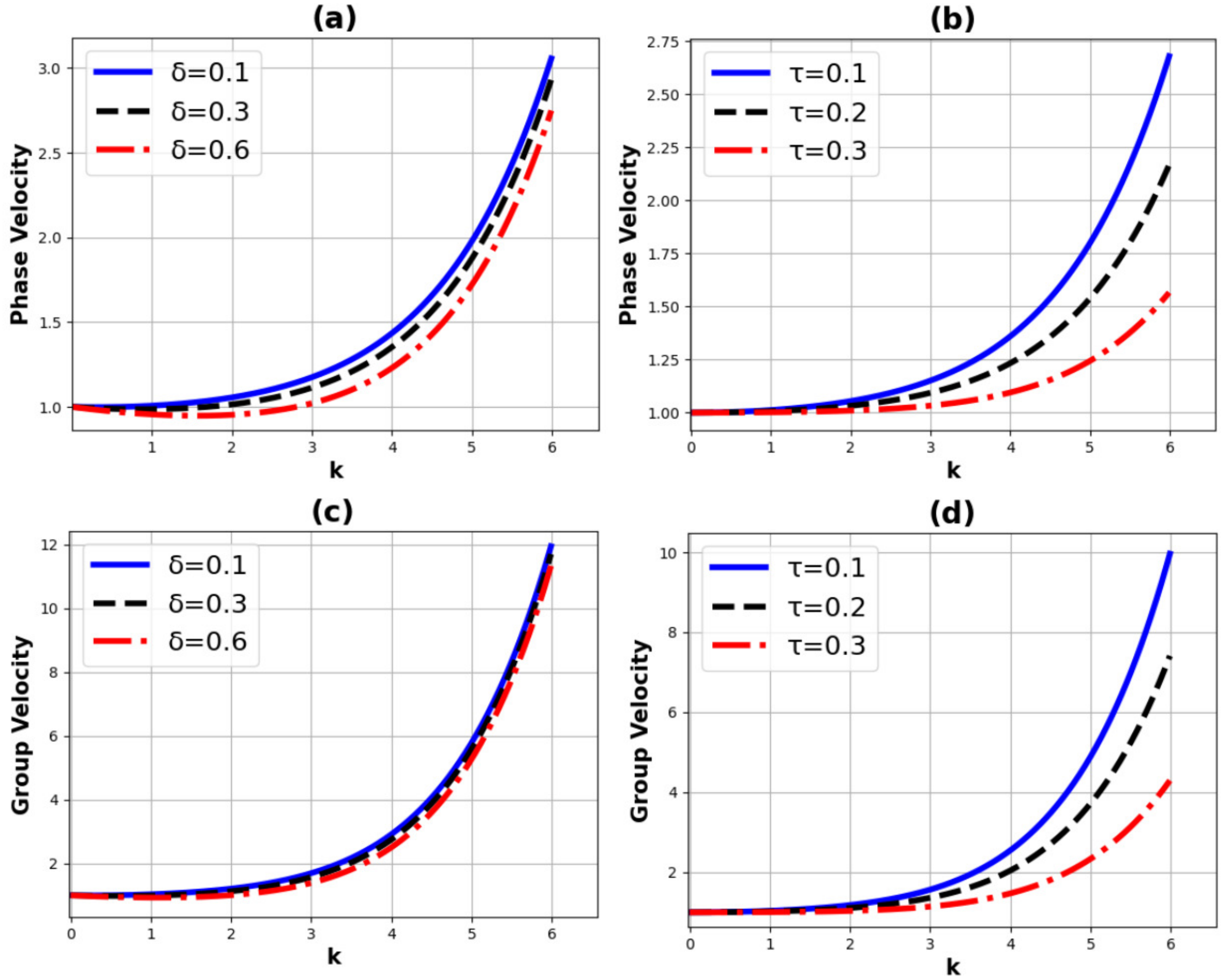


Figure 3.1: Variation of the phase and group velocities, with $\varepsilon = 0.1$ for different values of viscosity and surface tension parameters: The panels (a) and (b) illustrate the behavior of phase velocity under the effects of viscosity, (Blue line): $\delta = 0.1$, (Black dashed line): $\delta = 0.3$, (Red dashed line): $\delta = 0.6$ and surface tension, (Blue line): $\tau = 0.1$, (Black dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$, respectively. The panels (c) and (d) illustrate the behavior of group velocity under the effects of viscosity, (Blue line): $\delta = 0.1$, (Black dashed line): $\delta = 0.3$, (Red dashed line): $\delta = 0.6$ and surface tension, (Blue line): $\tau = 0.1$, (Black dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$, respectively.

To better investigate the effect of viscosity on the phase and group velocities, we plot both phase and group velocities as a function of the viscosity parameter δ. Figure 3.2 [(a), (b), (c), (d), (e) and (f)] shows that, both phase velocity (blue line) and group velocity (dashed red line) decrease when the viscosity increases. It is clearly shows the influence of the surface tension

τ on the group and phase velocities as a function of the viscosity curves. Thus, regarding the effect of surface tension, the results presented in Figure 3.2 are the same as those presented in Figure 3.1.

In fact, we see that in the interval of τ values [0.0, 0.2, 0.3, 0.4], the two velocities have the same sign (positve). This can mean that, the wave packet envelope and a given phase point both move to the right.

In the interval of τ values [0.0, 0.2, 0.3] and $\delta < 0.3$, the amplitude of the group velocity is greater than that of phase velocity. Thus, for these parameter values, the dispersion is anomalous. The wave packet moves faster than a given phase point and although going in the same direction (right) as the wave packet, a given phase point seems to regress within a packet and propagate from the head to the tail.

In figure 3.2-(c), for $\tau = 0.3$ and $\delta = 0.3$, the two velocities have same values, in other words, the envelope of the wave packet and a given phase point move at the same velocity. In this case, the medium is nondispersive regime, all given points of the phase within the wave packet, move as a block the wave propagates without changing its shape. However, for $\delta > 0.3$ and for τ values in the interval [0.3, 0.4, 0.5, 0.6], the phase velocity is greater than the group velocity ($v_{ph} > v_{gr}$), a given point of the phase moves faster than the wave packet. This fact could means that, the dispersion is normal and a given point of the phase progresses, within a wave packet, from the tail to the head.

In figure 3.2-(e), for $\tau = 0.5$ and $\delta \simeq 0.32$, the group velocity is zero ($v_{gr} = 0$), the wave packet does not propagate, which means that the wave is unable to carry energy. In this case, the superposition of two waves of the same pulsation and the same amplitude propagating in two opposite directions produces a stationnary wave. However, for $\tau = 0.5$ and $\delta > 0.32$ and figure 3.2-(f), the two velocities have different signs. In fact the group velocity changes sign and becomes negative while the phase velocity remains positive. In this case, the wave is said to be backpropagating, that is to say that, the wave packet and a given point of the phase propagate in the opposite direction. More clearly, the wave wave packet propagates along the negative x-direction and a given point of the phase propagates along the positive x-direction.

3.1.3.2 Effects of surface tension, viscosity on the soliton dynamics

The effect of viscosity or surface tension in the study of wave dynamics at the water surface is a very interesting subject which is nowadays the object of more and more interesting works. In fact, the study of one or the other of these parameters on the dynamics of the waves in various

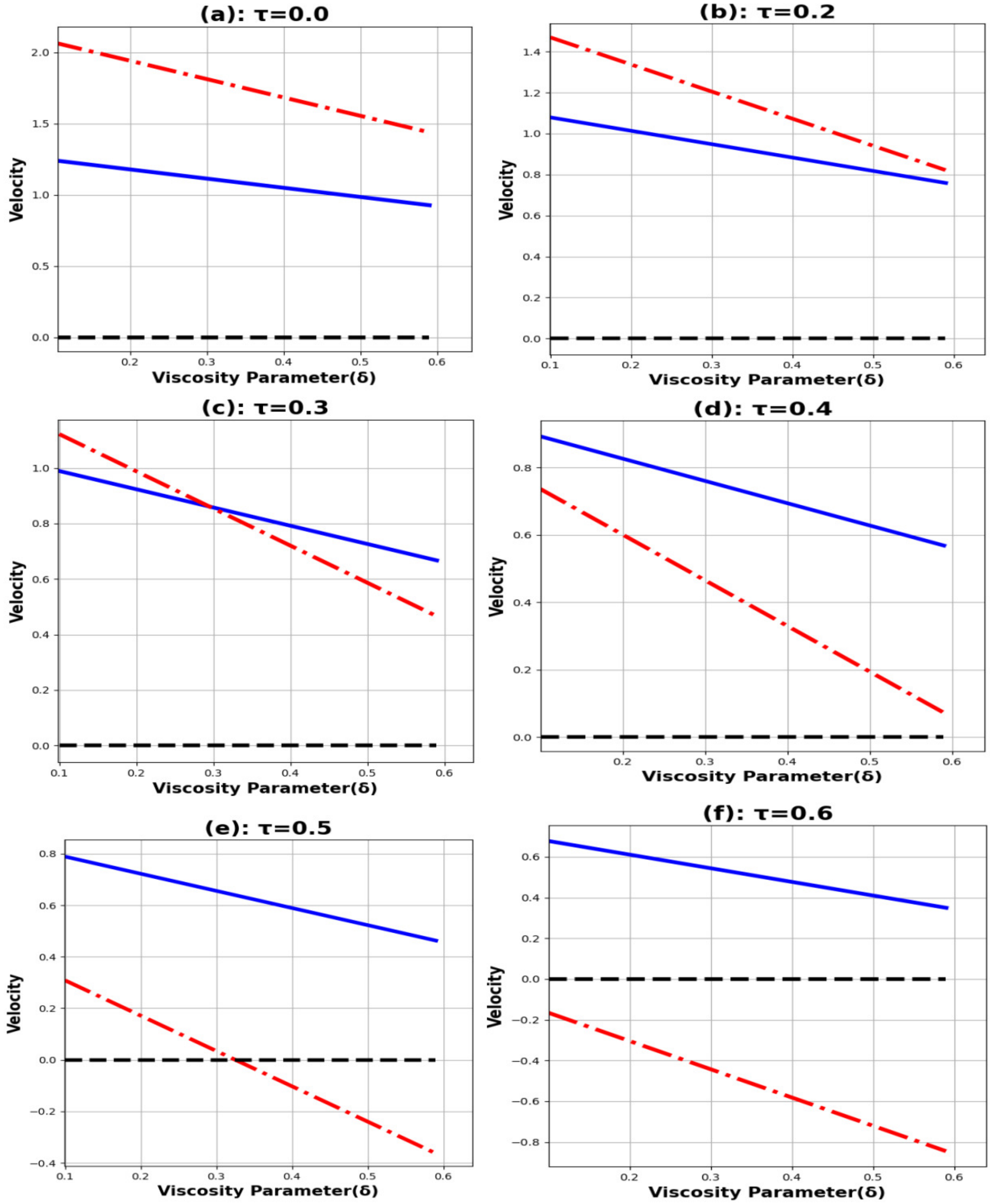


Figure 3.2: Phase (Blue line) and group (Red dashed line) velocities as a function of the viscosity parameter, for $\varepsilon = 0.3$ and different values of surface tension parameter: (a): $\tau = 0.0$, (b): $\tau = 0.2$, (c): $\tau = 0.3$, (d): $\tau = 0.4$, (e): $\tau = 0.5$, (f): $\tau = 0.6$.

fields of the nonlinear sciences were carried out and interesting results have been produces. For example, Li [124] studied the performance of centrifugal pumps using water and viscous oil as working fluids. The results show that, high viscosity leads to a rapid increase in disc friction losses on the outer faces of the impeller liner and hub, as well as hydraulic losses in the pump flow channels. It also shows that, the flow patterns near the impeller outlet are little affected by fluid viscosity, but those near the impeller inlet are strongly affected by viscosity. In the same sense, Gaver et al. [125] studied airway opening in a tabletop model designed to mimic bronchial walls held in apposition by airway lining fluid. They measured the relationship between airway opening velocity and applied airway opening pressure in thin-walled polyethylene tubes of different radii using lining fluids of different surface tensions and viscosities. The results show that when the capillary number is low, the opening pressure acts as an apparent "yield pressure" that must be exceeded before airway opening can begin. the capillary number is high (greater than 0.5), the viscous forces significantly increase the overall opening pressures. Based on these results, the authors made predictions about airway opening times. The results suggest that, airway closure can persist for a considerable portion of inspiration when the viscosity or surface tension of the lining fluid is high. Thus, along the same idea, we investigate in the followingthe effects of the viscosity, the surface tension and the amplitude parameter on the soliton dynamics.

The effects of the viscosity on the soliton dynamics are illustrated in figures 3.3, 3.4 and 3.5. It is shown in figure 3.3 $[(a), (b), (c)$ and $(d)]$ that, for the same value of the small parameter ε, the viscosity strongly impact the width of the soliton. Indeed, we plot the soliton solution given by equation (3.9) for four different values of viscosity parameter. The result show that, when the values of viscosity parameter δ increases, the width of the soliton also increases, but its amplitude remains constant. This result is in agreement with the dissipative character of viscosity. In this case, the wave obtained is a dissipative soliton and propagates by losing energy.

Figure 3.4 shows the 2-dimensional plot of the soliton under the effects of the viscosity. Indeed, in figure 3.5, by comparing the width of the soliton represented by the blue ($\delta = 0.6$), red ($\delta = 1.5$), purple ($\delta = 3.0$) and black ($\delta = 4.0$) lines, it is clear that the width of the soliton increases with the value of the viscosity parameter. This may therefore suggests that a soliton propagating to the right in a viscous medium tends to increase (respectively decrease) in width if the viscosity of medium increases (respectively decreases).

Figure 3.5 $[(a), (b), (c)$ and $(d)]$ illustrates the impact of the amplitude parameter ε on the soliton dynamics. It is revealed that, the growth of ε amplifies the effects of viscosity while keeping the amplitude of soliton constant.

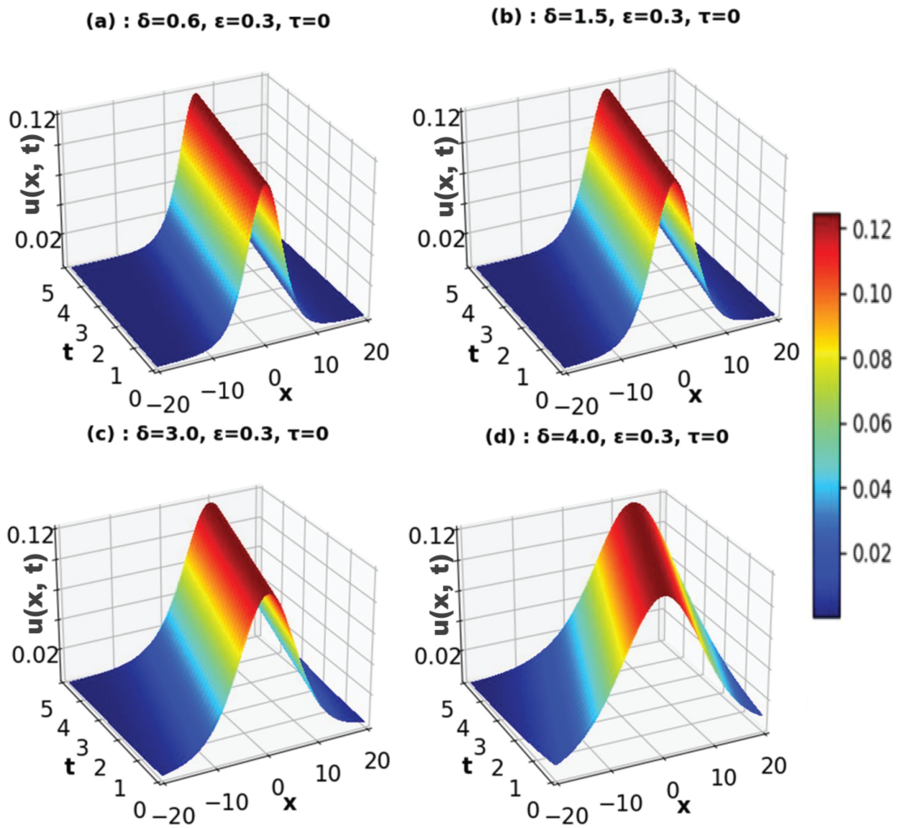


Figure 3.3: Effect of viscosity on the soliton solutions of the generalized inhomogeneous KdV equation (3.2) without effect of surface tension ($\tau = 0$), with $k = 0.5$, $\varepsilon = 0.3$ and different values of the viscosity parameter : **(a)**: $\delta = 0.6$; **(b)**: $\delta = 1.5$; **(c)**: $\delta = 3.0$ and **(d)**: $\delta = 4.0$.

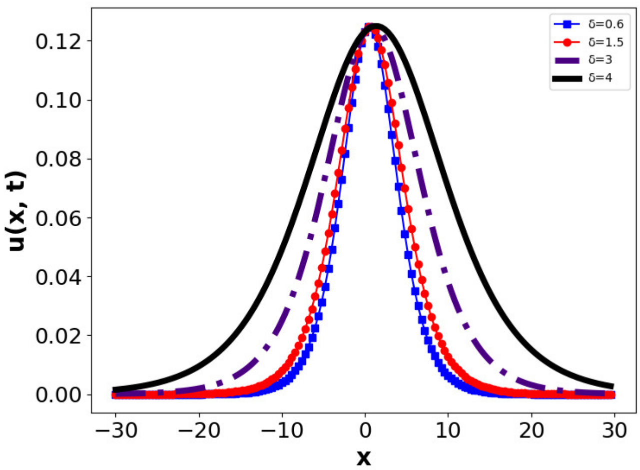


Figure 3.4: 2D-plot of the soliton solutions of the generalized inhomogeneous KdV equation (3.2) without effect of surface tension ($\tau = 0$), with $\varepsilon = 0.3$, $t = 0.5$, $k = 0.5$, for different values of viscosity parameter: (Blue line): $\delta = 0.6$, (Red line): $\delta = 1.5$, (Purple line): $\delta = 3.0$, (Black line): $\delta = 4.0$.

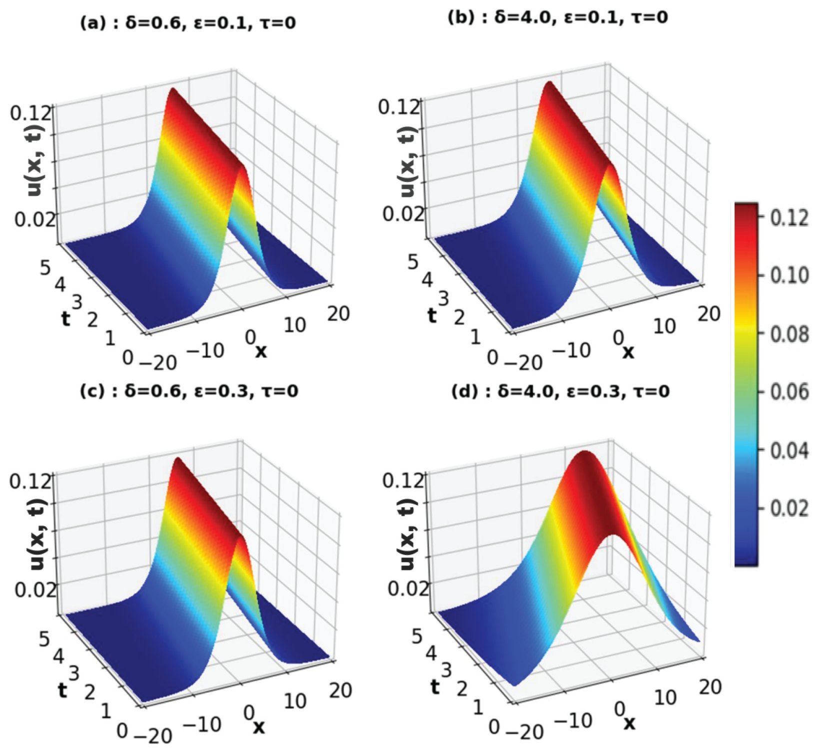


Figure 3.5: Effect of viscosity and the small parameter ε on the soliton solutions of the generalized inhomogeneous KdV equation (3.2) without effect of surface tension ($\tau = 0$), with $k = 0.5$, and different values of the viscosity parameter : **(a)**: $\varepsilon = 0.1,\ \ \delta = 0.6$; **(b)**: $\varepsilon = 0.1,\ \ \delta = 4.0$; **(c)**: $\varepsilon = 0.3,\ \ \delta = 0.6$ and **(d)**: $\varepsilon = 0.3,\ \ \delta = 4.0$.

Figures 3.6, 3.7 and 3.8 illustrate the behavior of the soliton solution of equation (3.2) under the effect of surface tension. In figure 3.6, we fix the value of the parameter ε and vary the value of τ. It is clearly shows that as the case of viscosity, the width of the soliton increases when the value of the surface tension parameter increases when the wave amplitude remains constant. However, the comparison between the effects of viscosity and surface tensionreveals that the effects of the viscosity on the width of soliton are greater important than that of surface tension.

In order the better illustrate the effect of surface tension, the 2-dimensional plot of the soliton under the effects of the surface tension is shows in figure 3.7. By comparing the width of the soliton represented by the blue ($\tau = 0.6$), red ($\tau = 2.5$), purple ($\tau = 4.0$) and black ($\tau = 6.0$) lines, we observe that, the width of the soliton is an increasing function of the surface tension parameter.

Figure 3.8 [(a), (b), (c) and (d)] shows the impact of the amplitude parameter ε on the soliton dynamics. For this, we plot the soliton solution for two values of parameter ε and surface tension parameter, the result reveals that, the growth of ε amplifies the effects of surface tension.

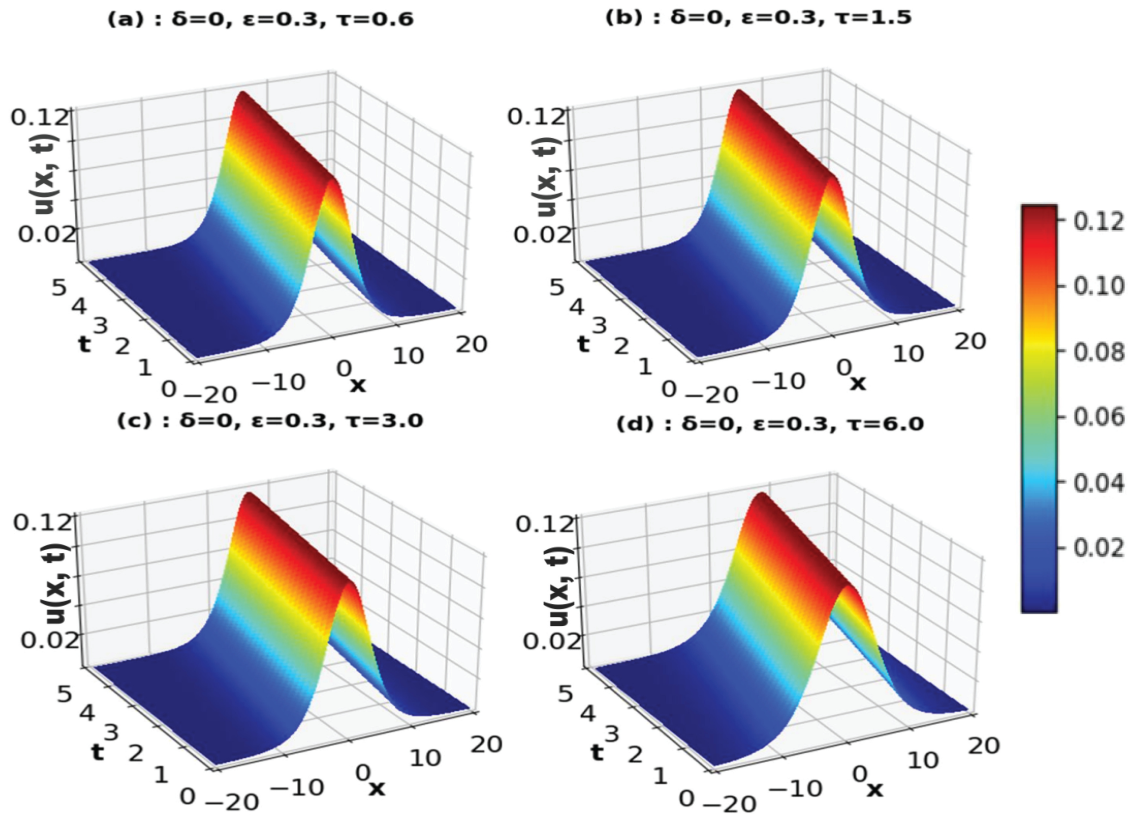


Figure 3.6: Effect of surface tension on the soliton solutions of the generalized inhomogeneous KdV equation (3.2) without effect of viscosity ($\delta = 0$), with $k = 0.5$, $\varepsilon = 0.3$ and different values of the tension surface parameter : **(a)**: $\tau = 0.6$; **(b)**: $\tau = 1.5$; **(c)**: $\tau = 3.0$ and **(d)**: $\tau = 6.0$.

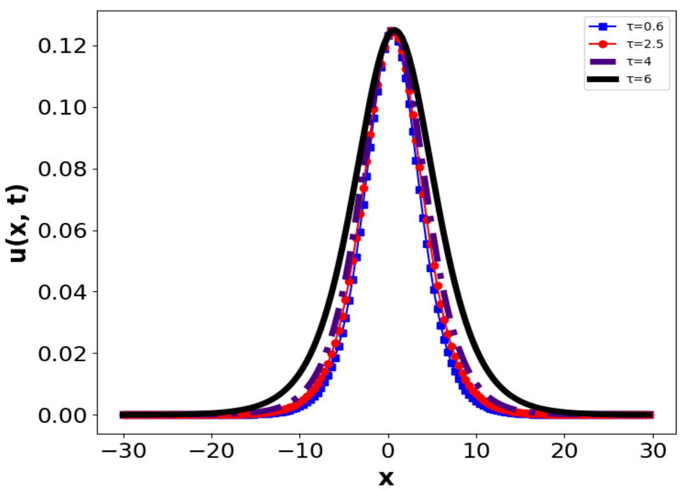


Figure 3.7: 2D-plot of the soliton solutions for the generalized inhomogeneous KdV equation (3.2) without effect of viscosity ($\delta = 0$), with $\varepsilon = 0.3$, $t = 0.5$, $k = 0.5$, for different values of surface tension parameter: (Blue line): $\tau = 0.6$, (Red line): $\tau = 2.5$, (Purple line): $\tau = 4.0$, (Black line): $\tau = 6.0$.

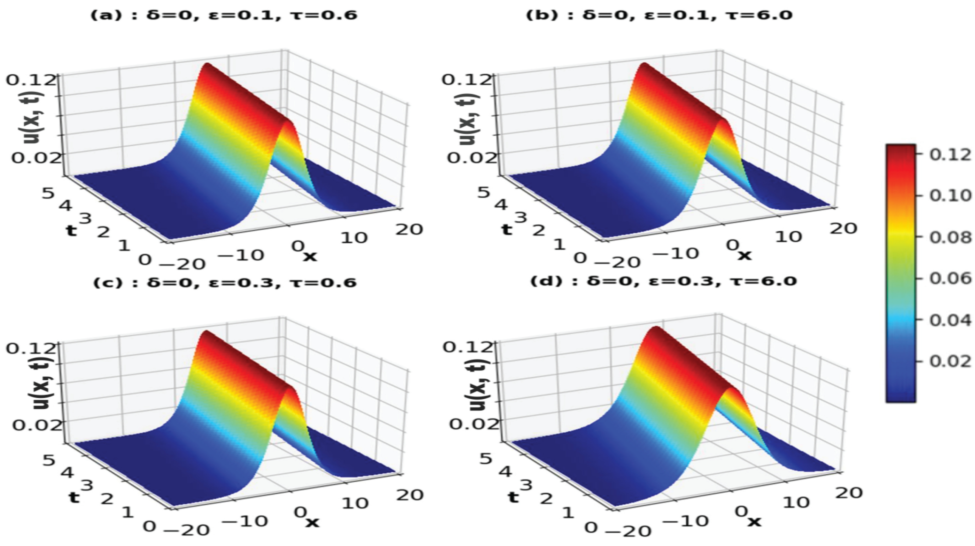


Figure 3.8: Effect of surface tension and the small parameter ε on the soliton solutions of the generalized inhomogeneous KdV equation (3.8) without effect of viscosity ($\delta = 0$), with $k = 0.5$, and different values of the surface tension parameter : **(a)**: $\varepsilon = 0.1,\ \ \tau = 0.6$; **(b)**: $\varepsilon = 0.1,\ \ \tau = 6.0$; **(c)**: $\varepsilon = 0.3,\ \ \tau = 0.6$ and **(d)**: $\varepsilon = 0.3,\ \ \tau = 6.0$.

In figures 3.9, 3.10 and 3.11, we take into account both the effects of viscosity and surface tension. It is seen that the effects of viscosity are strongly amplified by the effects of surface tension.

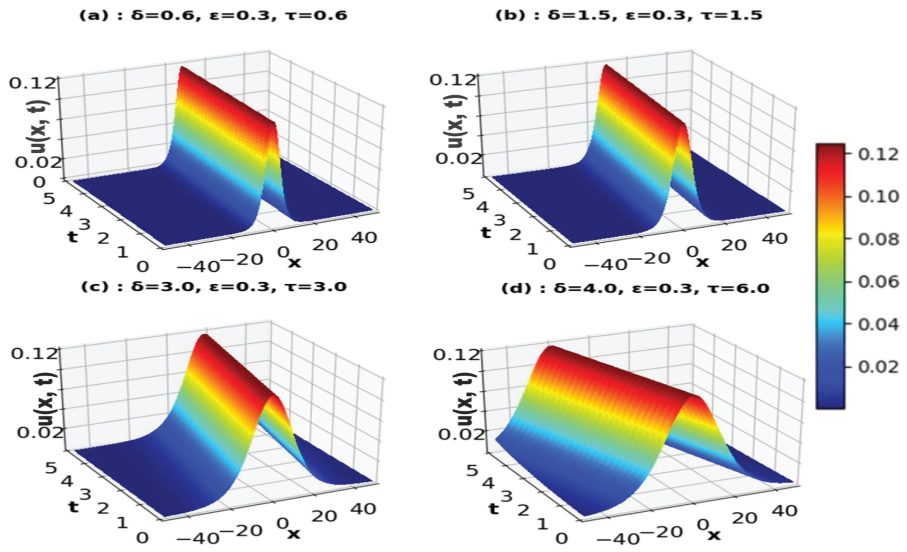


Figure 3.9: Combined effect of viscosity and surface tension on the soliton solutions for the generalized inhomogeneous KdV equation (3.2) with $k = 0.5$, $\varepsilon = 0.3$ and different values of surface tension and viscosity parameters : **(a)**: $\tau = 0.6,\ \ \delta = 0.6$; **(b)**: $\tau = 1.5,\ \ \delta = 1.5$; **(c)**: $\tau = 3.0,\ \ \delta = 3.0$ and **(d)**: $\tau = 6.0,\ \ \delta = 4.0$.

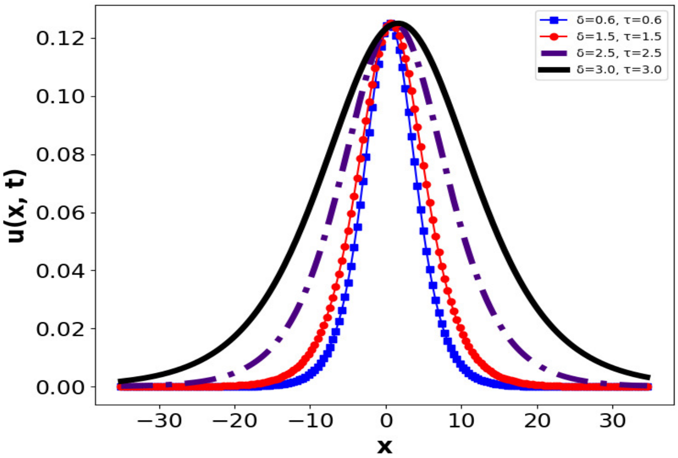


Figure 3.10: 2D-plot of the soliton solutions for the generalized inhomogeneous KdV equation (3.2) with $\varepsilon = 0.3$, $t = 0.5$, $k = 0.5$, for different values of surface tension and viscosity parameters: (Blue line): $\delta = 0.6,\ \tau = 0.6$, (Red line): $\delta = 1.5,\ \tau = 1.5$, (Purple line): $\delta = 2.5\ \tau = 2.5$, (Black line): $\delta = 3.0,\ \tau = 3.0$.

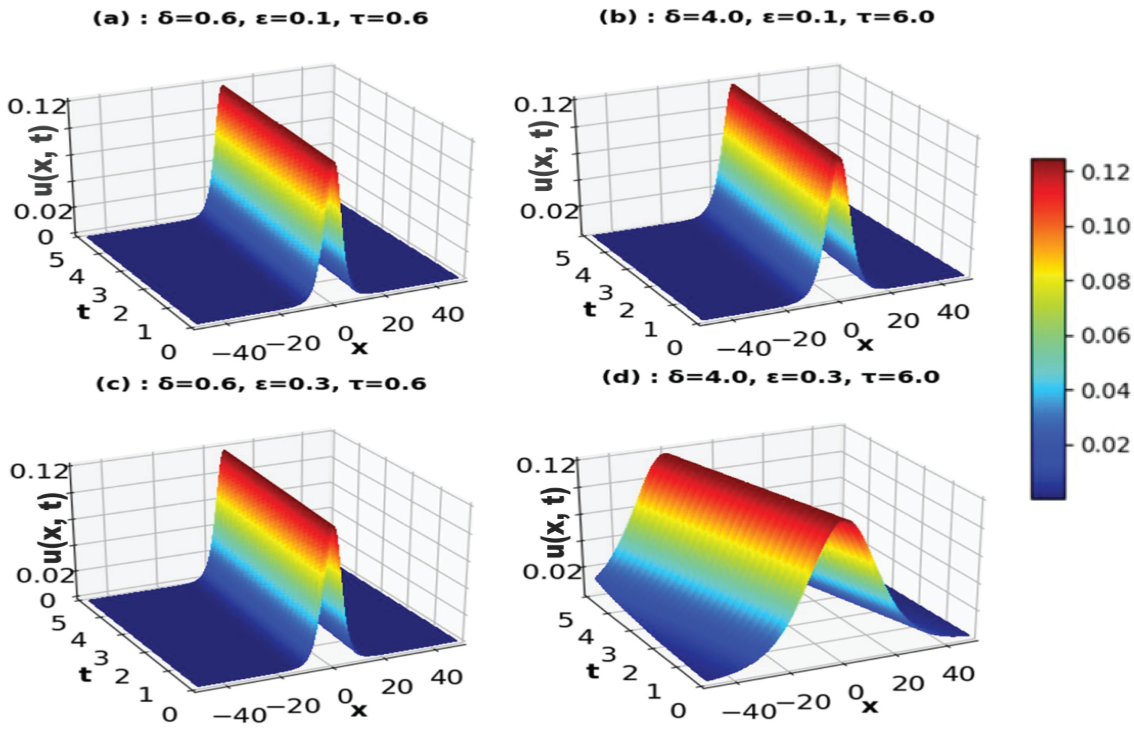


Figure 3.11: Combined effect of viscosity, surface tension and that of the small parameter ε on the soliton solutions of the generalized inhomogeneous KdV equation (3.2) with $k = 0.5$, and different values of the surface tension and viscosity parameters : **(a)**: $\varepsilon = 0.1,\ \tau = 0.6,\ \delta = 0.6$; **(b)**: $\varepsilon = 0.1$, $\tau = 6.0,\ \delta = 4.0$; **(c)**: $\varepsilon = 0.3,\ \tau = 0.6,\ \delta = 0.6$ and **(d)**: $\varepsilon = 0.3,\ \tau = 6.0,\ \delta = 4.0$.

3.2 The generalized KdV equation with viscosity, surface tension and wind effects

Several works have been devoted to study the impact of the wind on the dynamics of nonlinear waves. The Jeffreys model [16] assumes that, waves move in the opposite direction to the wind and thus constitute a barrier to the wind flow. In this model, the windward side of the wave receives the wind frontally, while the leeward side is sheltered. This creates a pressure difference and thus a force exerted on the wave [17]. The model developed by Phillips [18] assumes that, the fluid flow is potential and the air flow is turbulent. The turbulence of the air creates random pressure variations which create waves. The Miles model [19] is based on an analysis of the stability of parallel flows in both air and water. It consider air to be incompressible and non-viscous, with a logarithmic velocity profile [17].

During the 20th century, starting from one or the other of the overcited models, several authors have studied the impact of wind on the dynamics of extreme waves. For example, by using a pressure distribution over the steep crests given by the Jeffreys model, Kharif et al. [73] have investigated experimentally and numerically the influence of wind on extreme wave events in deep water. [40] have used the Jeffreys model and investigated experimentally without wind and in presence of wind the rogue wave formation due to the dispersive focusing mechanism. The authors conclude that, the duration of the rogue wave event increases with the wind velocity. In the same line, amplification of nonlinear surface waves by wind have been investigated by Leblanc [22] using the Miles' model. Despite all these interesting works, the question of the interaction between wind and waves remains an open subject. Indeed, all these works listed above have been carried out in order to study the effects of wind on the dynamics of extreme waves such as rogue waves. Most of these works deal only with the generalized nonlinear Schrödinger equations. Motivated by the obtained results, we extend the study to the generalized KdV equation. This equation have been shown to better describes the propagation of solitary waves in shallow water. We simultaneous combine the dissipation due to viscosity, surface tension and wind effects the combination of such effects has never been studied in the literature to the best of our knowledge. We show that such effects have considerable impacts on shallow water wave dynamics and that the model equation can lead to some generalized KdV equations that includes higher diffusion and instability effects given by

For the first-order expansion of the small parameter ε. This equation will be called the generalized third-KdV equation with viscosity, surface tension and wind effects in the rest of our work

$$\eta_t + \eta_x + \epsilon\left(\frac{3}{2}\,\eta\eta_x + \frac{1}{2}\,(\chi - 2\delta)\eta_{2x} + \frac{1}{6}\,(1 - 3\tau)\eta_{3x}\right) = 0. \tag{3.15}$$

And for the second-order expansion of the small parameter ε. This equation will be called the generalized fifth-KdV equation with viscosity, surface tension and wind effects in the rest of our work

$$\begin{aligned}\eta_t + \eta_x + \epsilon\left[\frac{3}{2}\,\eta\eta_x + \frac{1}{2}\,(\chi - 2\delta)\eta_{2x} + \frac{1}{6}\,(1 - 3\tau)\eta_{3x}\right] + \epsilon^2\Bigg[- \frac{3}{8}\,\eta^2\eta_x + \frac{1}{8}\,(\chi + 2\delta)\eta_x^2 \\ + \frac{1}{4}\,\chi\eta\eta_{2x} + \frac{1}{24}\,(23 + 15\tau)\eta_x\eta_{2x} - \frac{1}{8}\,\chi^2\eta_{3x} + \frac{1}{12}\,(5 - 3\tau)\eta\eta_{3x} + \frac{1}{12}\,\chi(1 + 3\tau)\eta_{4x} \\ + \frac{1}{360}\,(19 - 30\tau - 45\tau^2)\eta_{5x}\Bigg] = 0.\end{aligned} \tag{3.16}$$

These two equations are new generalizations of the KdV equation to the third- and fifth-order. We show in the rest of our work that they have soliton-like solutions and that they can be used to better describe the dynamics of solitary waves in shallow water in the presence of wind effets, surface tension and viscosity.

3.2.1 Soliton solutions of the generalized third-KdV equation with viscosity, surface tension and wind effects

In order to investigate the soliton solutions of the above equation, we apply Hirota's bilinear method. For this, we assume that the general form of solution is given by

$$\eta(x,t) = R\frac{\partial^2}{\partial x^2}\ln(f(x,t)) = R\frac{ff_{2x} - f_x^2}{f^2}, \tag{3.17}$$

where R is some function to be determined and $f(x,t)$ is the unknown function namely the auxiliary function. Usually, the function $f(x,t)$ has the form of a traveling wave given by the perturbation expansion

$$f(x,t) = 1 + \sum_{n=0}^{N} \upsilon^n f_n(x,t), \tag{3.18}$$

where υ is a bookkeeping nonsmall parameter and $f_n(x,t)$, $n = 1, 2, ...$, which are independent of υ, are unknown real functions to be determined. In the case of single soliton solution ($N = 1$), we assume that $f_1(x,t) = \exp{(\theta)}$ such a way that

$$f(x,t) = 1 + f_1(x,t) = 1 + \exp(\theta), \tag{3.19}$$

where $\theta = kx - \omega t$, the parameters k and ω are the wave number along the x-direction and the angular frequency, respectively.

Substituting $\eta(x,t)$ by $\exp(\theta)$ into the linear terms of (3.15) leads the dispersion relation given by

$$\omega = k + \epsilon\left(\frac{1}{2}\,k^2(\chi - 2\delta) + \frac{1}{6}\,k^3(1 - 3\tau)\right). \tag{3.20}$$

To determine the function R, we introduce equation (3.17) intoequation (3.15) where, the $f(x,t)$ is taken as

$$f(x,t) = 1 + \exp\left[kx - \left[k + \epsilon\left(\frac{1}{2}\,k^2(\chi - 2\delta) + \frac{1}{6}\,k^3(1 - 3\tau)\right)\right]t\right]. \tag{3.21}$$

Equating the coefficients of the different powers of $\exp(\theta)$ to zero yields a system of polynomial equations, after solving this system with Mathematica, we find the following two values of R

$$R_{11} = \frac{4(k - 2\delta - 3k\tau + \chi)}{3k} \quad \text{and} \quad R_{12} = \frac{2(2k + 2\delta - 6k\tau - \chi)}{3k}. \tag{3.22}$$

This means that the soliton solution is given by

$$\eta_{1i}(x,t) = \frac{R_{1i}k^2 \exp\left(kx - \left[k + \epsilon\left(\frac{1}{2}\,k^2(\chi - 2\delta) + \frac{1}{6}\,k^3(1 - 3\tau)\right)\right]t\right)}{\left(1 + \exp\left(kx - \left[k + \epsilon\left(\frac{1}{2}\,k^2(\chi - 2\delta) + \frac{1}{6}\,k^3(1 - 3\tau)\right)\right]t\right)\right)^2}. \tag{3.23}$$

After some transformations, we obtain

$$\eta_{1i} = \frac{k^2}{4}\,R_{1i}\text{sech}^2\left[\frac{1}{2}\left(kx - \left[k + \epsilon\left(\frac{1}{2}\,k^2(\chi - 2\delta) + \frac{1}{6}\,k^3(1 - 3\tau)\right)\right]t\right)\right], \tag{3.24}$$

where the different values of the parameter $R_{1i}(i = 1, 2)$ are given by equation (3.22).

This soliton solution describes the (1+1)-dimensional long, small-amplitude, unidirectional wave motion in shallow water with the effects of wind, surface tension and viscosity.

3.2.2 Phase and group velocity

The wave is a perturbation that moves through a medium. It is therefore possible to associate it with two wave velocities, namely the phase velocity and the group velocity, which are sometimes different. We propose to evaluate in the following the phase and group velocities of waves modeled by the obtained equation (3.15).

3.2.2.1 Phase velocity

The phase velocity of a wave is the velocity with which the phase of a wave moves. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity given as follows

$$v_{ph} = \frac{\omega}{k}, \tag{3.25}$$

where ω is the angular frequency and k the wave number. Thus, from equation (3.20), we can easily deduce that the phase velocity of the soliton is given by

$$v_{ph} = 1 + \epsilon\left(\frac{1}{2}\,k(\chi - 2\delta) + \frac{1}{6}\,k^2(1 - 3\tau)\right). \tag{3.26}$$

We notice that the phase velocity is not constant, but depends on k, which implies that we are in a dispersive medium, so waves of different frequencies propagate at different velocities. If we consider a wave packet consisting of three waves of neighboring pulses initially in phase at a date t, we will find after a time Δt that the wave packet will have widened, and that it will have propagated at a velocity lower than the velocity of its fastest component.

3.2.2.2 Group velocity

Waves can be in a group and such group are called waves packets, so the velocity with which a wave packet moves is called group velocity. The expression of the group velocity can be obtained from the following dispersion relation

$$v_{gr} = \frac{\partial \omega}{\partial k}, \tag{3.27}$$

where ω is the dispersion relation and k the wave number. Thus, from equation (3.20), we can easily deduce that the group velocity of the soliton can be written in the following form

$$v_{gr} = 1 + \epsilon\left(k(\chi - 2\delta) + \frac{1}{2}\,k^2(1 - 3\tau)\right). \tag{3.28}$$

We notice that the phase velocity is not constant, but depends on k, which implies that we are in a dispersive medium, so waves of different frequencies propagate at different velocities.

By observing the equation (3.26), it is clearly observed that in the limit case where $\varepsilon = 0$, the pulsation is directly proportional to the wave number ($\omega = k$). This means that the phase velocity is independent of the pulsation. Moreover, the phase velocity is equal to the group velocity. In this case, the medium is called non-dispersive regime. This regime corresponds to the behavior of the oscillator chain when it is crossed by a wave of very long wavelength. The wave perceives the chain as a continuous medium, and does not distinguish the masses from each other.

3.2.3 Effects of viscosity, surface tension and wind on the solitons dynamics, phase and group velocities

For physical application, it is important to discuss about the influence of surface tension and viscosity on the dynamics of the obtained soliton solutions. For that, it will be a question of giving different values to our parameters and deduce their effects on the studied dynamics, the justification of the choice of the range of values of our parameters τ and δ have been given above. However, given the lack of literature concerning the improvement of the KdV equation taking into account the effects of the wind in the context of shallow water, we have chosen the theorical values of the wind parameter χ so that they are be of the same order as those of the viscosity (δ) and surface tension (τ) parameters in order to balance the different forces acting on the fluid. In this way, the results discussed in the rest of this work with the used values of τ, δ and χ in the interval $[0.1 - 0.6]$ are applicable to gravity-capillary waves.

3.2.3.1 Effects of viscosity, surface tension and wind on the phase and group velocities

The k-dependence of the phase and group velocities are shown in figures-3.12. In this figure, it is clearly observed that the results are the same as those obtained above (the growth of these speeds when the wave number increases). The effects of viscosity and surface tension on the phase and group velocities are also similar to the results obtained above (the decrease of these velocities when viscosity or surface tension increases).

The figure 3.12-(c) shows the influence of the wind (parameter χ) on the phase velocity curve. Indeed, we plot the phase velocity as a function of the wave number for four values of χ. We see that when the value of the parameter χ increases, the phase velocity increases. Thus we can say that the presence of wind increases the phase velocity of the wave. This observation confirms the hypothesis that, the energy of the wind can be transferred to the wave.

In figure 3.12-(f), we observe the effects of wind (parameter χ) on the group velocity. Indeed, we plot the group velocity as a function of the wave number for four values of the parameter χ. It is shown that when the value of the parameter χ increases, the group velocity also increases. This could mean that, a wave packet propagating in a windy environment will receive additional energy due to the influence of the wind.

Moreover, by carefully observing Figures 3.12-(c) and 3.12-(f), it can be seen that the magnitude of the group velocity is greater than that of the phase velocity ($v_{gr} > v_{ph}$). This fact could mean that, the dispersion is anomalous, the wave regresses inside a packet, from the head to the tail of this one.

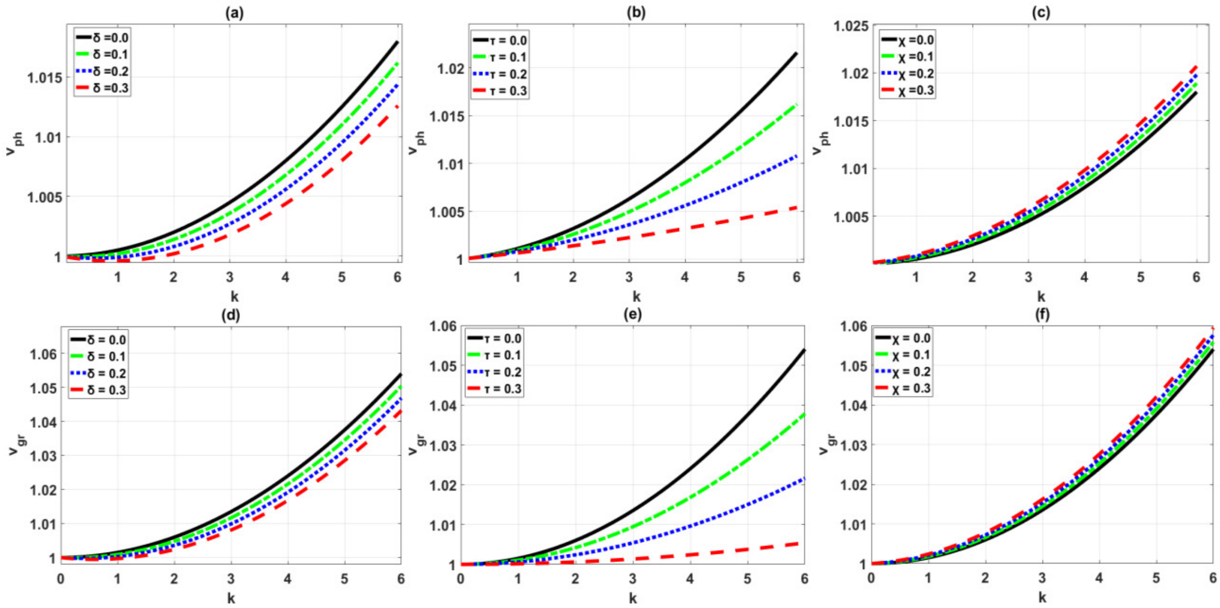


Figure 3.12: Phase and group velocities variation given by equations (3.26) and (3.28) with $\epsilon = 0.003$, for different values of viscosity, surface tension and wind parameters: The panels (a), (b) and (c) illustrate the behavior of phase velocity under the effects of viscosity, (Black line): $\delta = 0.0$, (Green dashed line): $\delta = 0.1$, (Blue dashed line): $\delta = 0.2$, (Red dashed line): $\delta = 0.3$, surface tension, (Black line): $\tau = 0.0$, (Green dashed line): $\tau = 0.1$, (Blue dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$ and wind, (Black line): $\chi = 0.0$, (Green dashed line): $\chi = 0.1$, (Blue dashed line): $\chi = 0.2$, (Red dashed line): $\chi = 0.3$, respectively. The panels (d), (e) and (f) illustrate the behavior of group velocity under the effects of viscosity, (Black line): $\delta = 0.0$, (Green dashed line): $\delta = 0.1$, (Blue dashed line): $\delta = 0.2$, (Red dashed line): $\delta = 0.3$, surface tension, (Black line): $\tau = 0.0$, (Green dashed line): $\tau = 0.1$, (Blue dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$ and wind, (Black line): $\chi = 0.0$, (Green dashed line): $\chi = 0.1$, (Blue dashed line): $\chi = 0.2$, (Red dashed line): $\chi = 0.3$, respectively.

3.2.3.2 Effects of viscosity, surface tension and wind on the soliton dynamics

The study of the effect of wind on the dynamics of waves is still an open subject and is not fully mastered. Numerous works on the effect of wind on wave dynamics in deep water have led to valuable and interesting results. Thus, taking into account the wind in the study of wave dynamics at the water surface is a very interesting subject. In the following, we describe the shape and motion of the soliton solution, from the expressions of equation (3.23), which have been explicitly constructed by using the Hirota's bilinear method. We also investigate the effects of wind, viscosity and surface tension on the soliton dynamics.

The effects of the viscosity on the soliton dynamics are illustrated in figures 3.13 and 3.14. In figure 3.13 [(a); (b); (c) and (d)], we observe that, for the same value of the small parameters ϵ, the viscosity strongly impacts the amplitude of the soliton. Indeed, when the values of viscosity parameter δ increases, the amplitude of the soliton decreases.

Figure 3.14 shows the 2-dimensional plot of the soliton under the effects of the viscosity. Thus, by comparing the amplitude of the soliton represented by the Black ($\delta = 0.0$), Green dotted ($\delta = 0.10$), Blue dashed ($\delta = 0.20$) and Red dashed ($\delta = 0.30$) lines, it is clear that the viscosity has an important effect on the amplitude of the soliton.

In figure 3.15 and 3.16, we graphically investigate the influence of the surface tension on the soliton dynamics. In figure 3.16 [(a), (b), (c) and (d)], we observe that the amplitude of the soliton is a decreasing function of surface tension parameter (parameter τ). Indeed, we plot the soliton solution with four values of parameter τ, it is shown that when the value of the parameter τ increases, the amplitude of the soliton decreases. The 2-dimensional plot of the soliton under the effects of the surface tension (Black ($\tau = 0.0$), Green dotted ($\tau = 0.10$), Blue dashed ($\tau = 0.20$) and Red dashed ($\tau = 0.30$) lines), is illustrated in figure 3.16. It is clear that, the amplitude of the soliton decreases with the value of the surface tension parameter, but its structure remains unchanged. The results obtained here are in agreement with the definition of surface tension.

The influence of wind on the soliton dynamics is presented in figures 3.17 and 3.18. We observe in figure 3.17 [(a), (b), (c) and (d)] that, the wind strongly impacts the amplitude of the soliton. It has been shown that when the value of the parameter χ increases, the amplitude of the soliton also increases. The 2-dimensional plot of the soliton under the effects of the wind (Black ($\chi = 0.0$), Green dotted ($\chi = 0.10$), Blue dashed ($\chi = 0.20$) and Red dashed ($\chi = 0.30$) lines), is illustrated in figure 3.18. It is clear that, the amplitude of the soliton is an increasing function of wind parameter χ, while its structure remains unchanged.

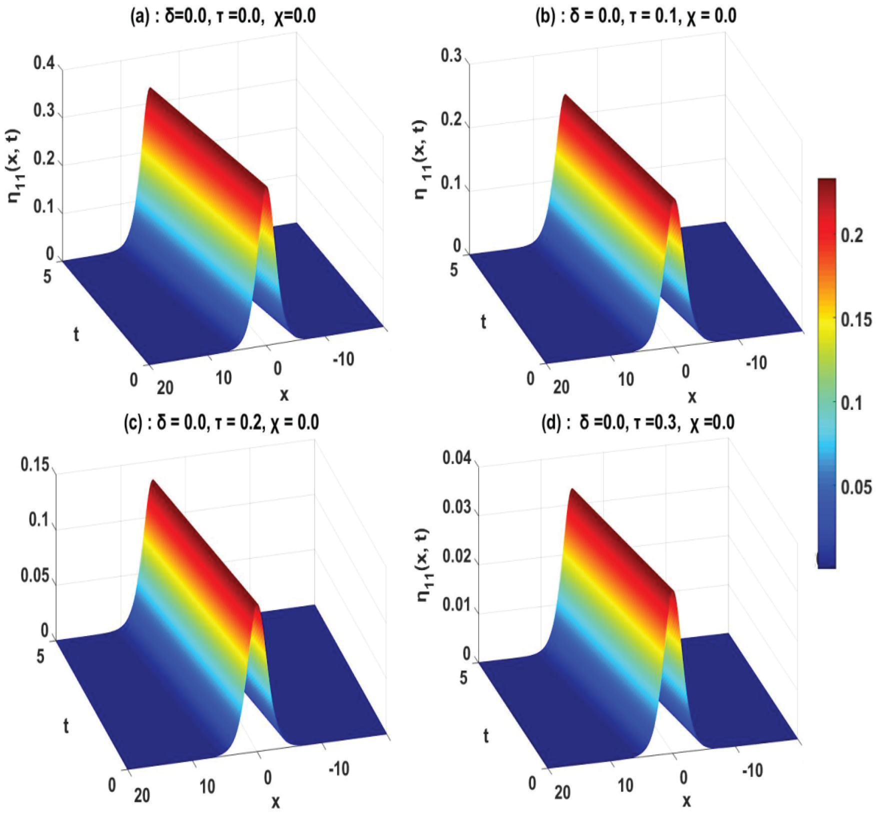


Figure 3.13: Effect of surface tension on the soliton solutions of the third-order generalized KdV equation (3.15) without viscosity ($\delta = 0$) and wind ($\chi = 0$) effects, with $k = 1$, $\epsilon = 0.003$ and different values of the surface tension parameter : (a): $\tau = 0.0$; (b): $\tau = 0.1$; (c): $\tau = 0.2$ and (d): $\tau = 0.3$.

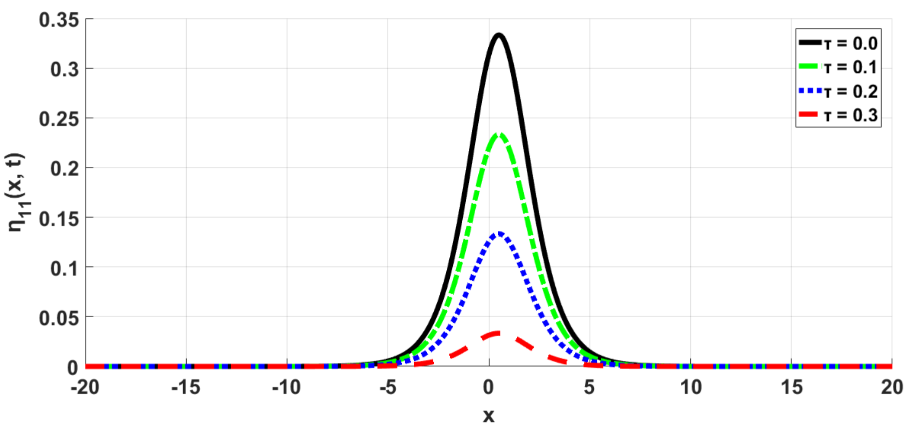


Figure 3.14: 2D-plot of the soliton solutions of the third-order generalized KdV equation (3.15) without surface tension ($\tau = 0$) and wind ($\chi = 0$) effects, with $k = 1$, $\epsilon = 0.003$ and different values of the viscosity parameter: (Black line): $\delta = 0.0$, (Green dashed line): $\delta = 0.1$, (Blue dashed line): $\delta = 0.2$, (Red dashed line): $\delta = 0.3$.

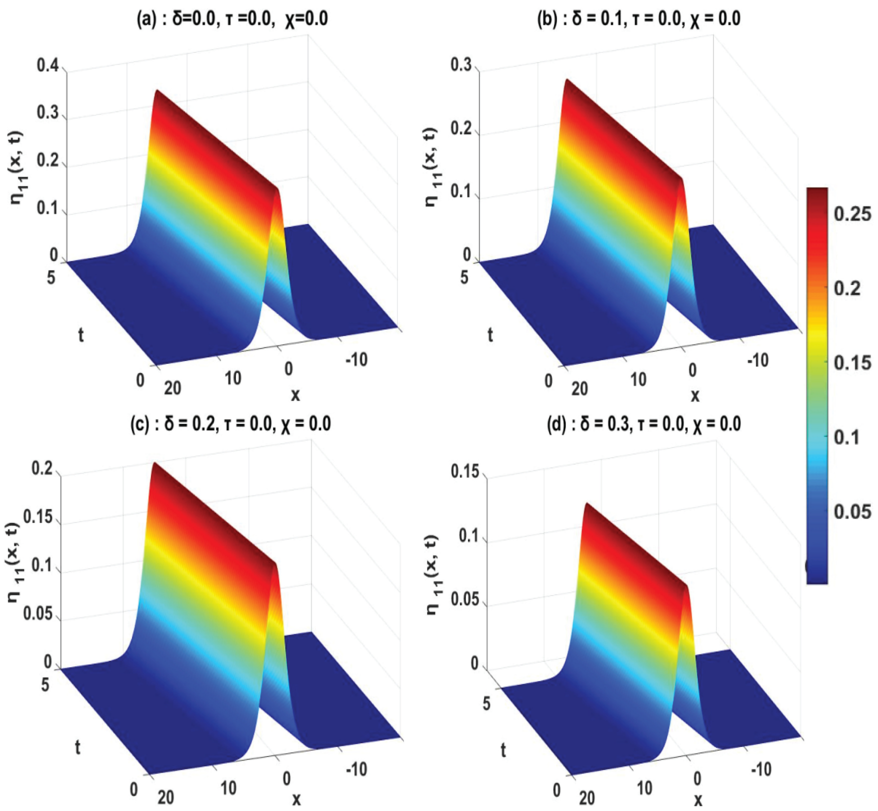


Figure 3.15: Effect of viscosity on the soliton solutions of the third-order generalized KdV equation (3.15) without surface tension ($\tau = 0$) and wind ($\chi = 0$) effects, with $k = 1$, $\epsilon = 0.003$ and different values of the viscosity parameter : (a): $\delta = 0.0$; (b): $\delta = 0.1$; (c): $\delta = 0.2$ and (d): $\delta = 0.3$.

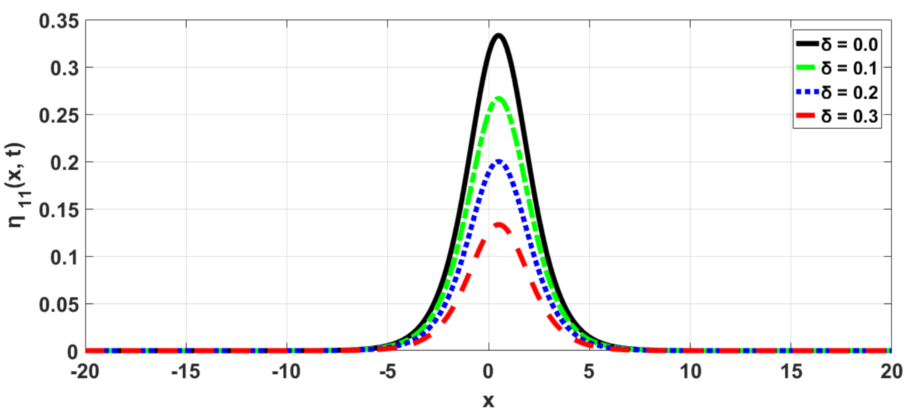


Figure 3.16: 2D-plot of the soliton solutions of the third-order generalized KdV equation (3.15) without surface tension ($\tau = 0$) and wind ($\chi = 0$) effects, with $k = 1$, $\epsilon = 0.003$ and different values of the viscosity parameter: (Black line): $\delta = 0.0$, (Green dashed line): $\delta = 0.1$, (Blue dashed line): $\delta = 0.2$, (Red dashed line): $\delta = 0.3$.

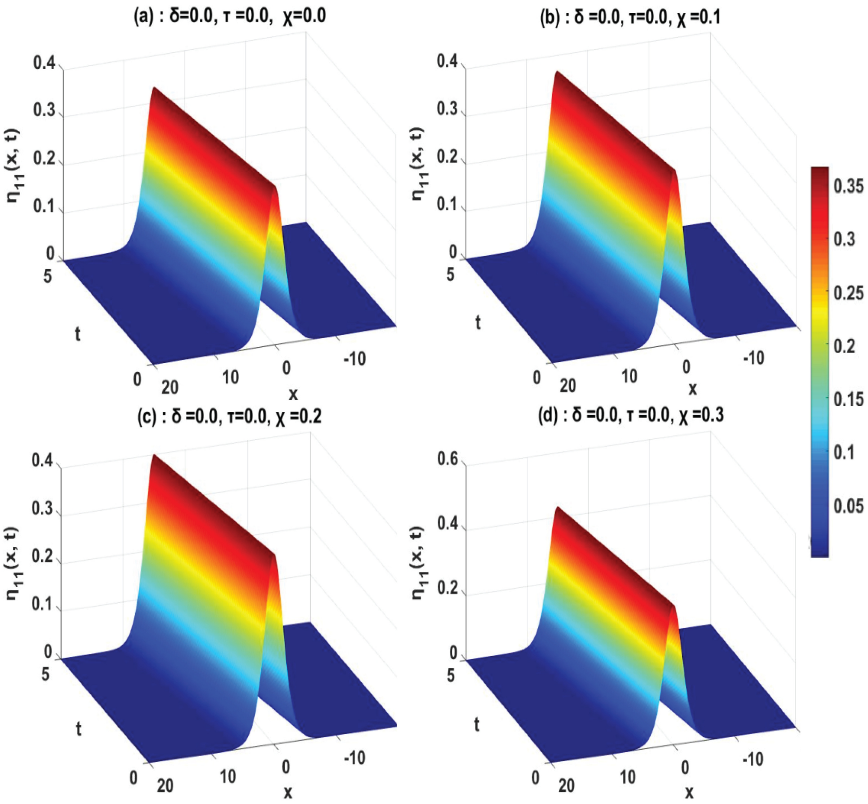


Figure 3.17: Effect of wind on the soliton solutions of the third-order generalized KdV equation (3.15) without viscosity ($\delta = 0.0$) and surface tension ($\tau = 0.0$) effects, with $k = 1$, $\epsilon = 0.003$ and different values of the wind parameter : (a): $\chi = 0.0$; (b): $\chi = 0.1$; (c): $\chi = 0.2$ and (d): $\chi = 0.3$.

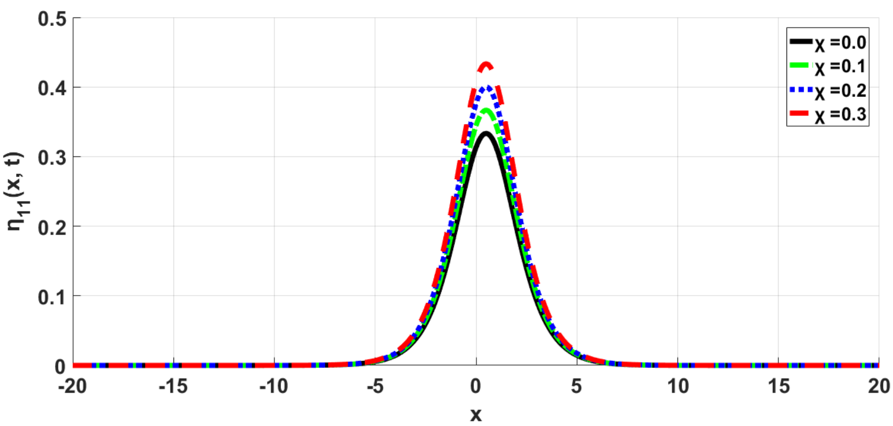


Figure 3.18: 2D-plot of the soliton solutions of the third-order generalized KdV equation (3.15) without viscosity ($\delta = 0$) and wind ($\chi = 0$) effects, with $k = 1$, $\epsilon = 0.003$ and different values of the surface tension parameter: (Black line): $\tau = 0.0$, (Green dashed line): $\tau = 0.1$, (Blue dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$.

3.2.4 Soliton solutions of the generalized fifth-KdV equation with viscosity, surface tension and wind effects

Substituting $\eta(x,t)$ by $\exp(kx-\omega t)$ into the linear terms of (3.16) leads the dispersion relation given by

$$\omega = k + \epsilon\bigg(\frac{1}{2}\,k^2(\chi-2\delta) + \frac{1}{6}\,k^3(1-3\tau)\bigg) + \epsilon^2\bigg(-\frac{1}{8}\,k^3\chi^2 + \frac{1}{12}\,k^4\chi(1+3\tau) + \frac{1}{360}\,k^5(19-30\tau-45\tau^2)\bigg). \tag{3.29}$$

To determine the function R, we introduce equation (3.17) into equation (3.16) where, the $f(x,t)$ is taken as

$$f(x,t) = 1 + \exp\Bigg\{kx - \bigg[k + \epsilon\bigg(\frac{1}{2}\,k^2(\chi-2\delta) + \frac{1}{6}\,k^3(1-3\tau)\bigg) + \epsilon^2\bigg(-\frac{1}{8}\,k^3\chi^2 + \frac{1}{12}\,k^4\chi(1+3\tau) + \frac{1}{360}\,k^5(19-30\tau-45\tau^2)\bigg)\bigg]t\Bigg\}. \tag{3.30}$$

Equating the coefficients of the different powers of $\exp(kx-\omega t)$ to zero yields a system of polynomial equations, after solving this system with Mathematica, we find the following four values of R

$$R_{21} = -\frac{4(-12k + 24\delta - 19k^3\epsilon + 36k\tau - 30k^3\epsilon\tau + 45k^3\epsilon\tau^2 - 12\chi - 14k^2\epsilon\chi - 42k^2\epsilon\tau\chi + 9k\epsilon\chi^2)}{3k(12 + 11k^2\epsilon + 2k\delta\epsilon + 3k2\epsilon\tau + 3k\epsilon\chi)},$$

$$R_{22} = -\frac{4(-12k + 12\delta - 19k^3\epsilon + 36k\tau + 30k^3\epsilon\tau + 45k^3\epsilon\tau^2 - 6\chi + 11k^2\epsilon\chi + 33k^2\epsilon\tau\chi + 9k\epsilon\chi^2)}{3k(12 + 11k^2\epsilon + 2k\delta\epsilon + 3k2\epsilon\tau + 3k\epsilon\chi)},$$

$$R_{23} = -\frac{1}{18k^3\epsilon}\,(-36k + 225k^3\epsilon + 6k^2\delta\epsilon + 9k^3\epsilon\tau + 21k^2\epsilon\chi \pm A), \tag{3.31}$$

$$R_{23} = -\frac{1}{18k^3\epsilon}\,(36k - 225k^3\epsilon + 6k^2\delta\epsilon - 9k^3\epsilon\tau + 21k^2\epsilon\chi \pm B),$$

where the coefficients A and B are given by

$$A = \Bigg[(36k - 225k^3\epsilon - 6k^2\delta\epsilon - 9k^3\epsilon\tau - 21k^2\epsilon\chi)^2 + 36k^3\epsilon(-48k + 240\delta + 380k^3\epsilon + 144k\tau - 600k^3\epsilon\tau - 900k^3\epsilon\tau^2 - 120\chi + 76k^2\epsilon\chi + 228k^2\epsilon\tau\chi + 36k\epsilon\chi^2)\Bigg]^{1/2},$$

$$B = \Bigg[(-36k + 225k^3\epsilon - 6k^2\delta\epsilon + 9k^3\epsilon\tau - 21k^2\epsilon\chi)^2 - 36k^3\epsilon(48k + 144\delta - 380k^3\epsilon - 144k\tau + 600k^3\epsilon\tau + 900k^3\epsilon\tau^2 - 72\chi + 84k^2\epsilon\chi + 252k^2\epsilon\tau\chi - 36k\epsilon\chi^2)\Bigg]^{1/2}.$$

This means that the soliton solution is given by

$$\eta_{2i}(x,t) = \frac{R_{2i}k^2 \exp(kx - \omega t)}{(1 + \exp(kx - \omega t))^2}. \tag{3.32}$$

After some transformations, we obtain

$$\eta_{2i} = \frac{k^2\ R_{2i}}{4} \operatorname{sech}^2\left[\frac{1}{2}(kx - \omega t)\right], \tag{3.33}$$

where the parameters $R_{2i}(i = 1, 2, 3, 4)$ are given by equation (3.31) and the parameter ω is given by equation (3.29).

This soliton solution describes a (1+1)-dimensional unidirectional motion of long, low amplitude waves in shallow water, taking into account the effects of wind, surface tension and viscosity.

3.2.5 Phase and group velocity

The wave is a perturbation that moves through a medium. It is therefore possible to associate it with two wave velocities, namely the phase velocity and the group velocity, which are sometimes different. We propose to evaluate in the following the phase and group velocities of waves modeled by the obtained equation (3.16)

3.2.5.1 Phase velocity

Starting from equation (3.29), it is possible to obtain the phase velocity of the soliton as follows

$$\begin{aligned} v_{ph} = \frac{\omega}{k} &= 1 + \epsilon\left(\frac{1}{2}k(\chi - 2\delta) + \frac{1}{6}k^2(1 - 3\tau)\right) \\ &+ \epsilon^2\left(-\frac{1}{8}k^2\chi^2 + \frac{1}{12}k^3\chi(1 + 3\tau) + \frac{1}{360}k^4(19 - 30\tau - 45\tau^2)\right). \end{aligned} \tag{3.34}$$

We notice that the phase velocity is not constant, but depends on k, which implies that we are in a dispersive medium.

3.2.5.2 Group velocity

Starting from equation (3.29), it is possible to obtain the group velocity of the soliton as follows

$$\begin{aligned} v_{gr} = \frac{\partial \omega}{\partial k} &= 1 + \epsilon\biggl(k(\chi - 2\delta) + \frac{1}{2}\,k^2(1 - 3\tau)\biggr) \\ &+ \epsilon^2\biggl(- \frac{3}{8}\,k^2\chi^2 + \frac{1}{3}\,k^3\chi(1 + 3\tau) + \frac{1}{72}\,k^4(19 - 30\tau - 45\tau^2)\biggr). \end{aligned} \tag{3.35}$$

3.2.6 Effects of viscosity, surface tension and wind on the solitons dynamics, phase and group velocities

The k-dependence of the phase and group velocities is shown in figures-3.19. These figures show that, the behavior of the phase and group velocities for the fifth-order generalized KdV equation under the effects of viscosity, surface tension and wind are identical to that of the phase velocity for the third-order generalized KdV equation.

Similarly, the effects of viscosity, surface tension, and wind on the soliton solution of the fifth-order generalized KdV equation are the same as those for the third-order generalized KdV equation. Therefore, we have deemed it necessary not to present their graphical illustration.

Conclusion

In this chapter, we have presented the different soliton solutions of the obtained new KdV equations improved by the effects of viscosity, surface tension and wind. We have investigated the effects of viscosity, surface tension and wind on the phase and group velocities of the waves and on the solitons dynamics, by varying the values of each corresponding parameter. The results have been revealed that such parameters can strongly influence the dynamics of hydrodynamic waves in shallow water.

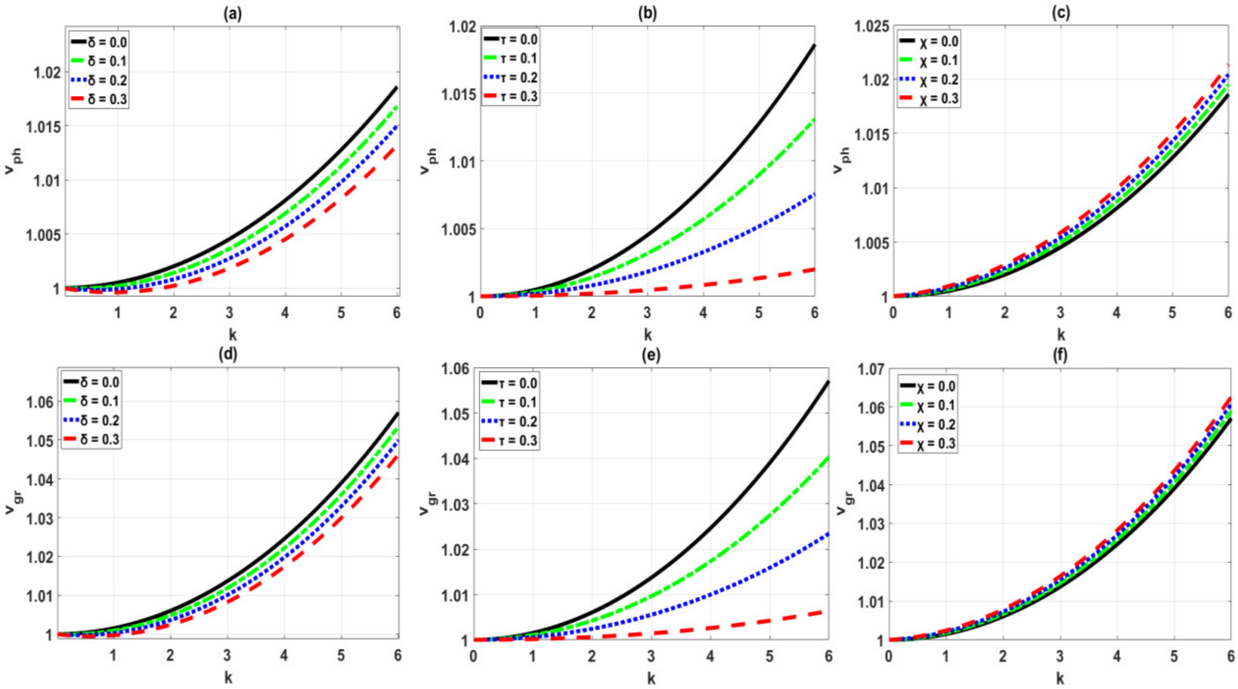


Figure 3.19: Phase and group velocities variation given by equations (3.34) and (3.35), respectively, with $\epsilon = 0.003$, for different values of viscosity, surface tension and wind parameters: The panels (a), (b) and (c) illustrate the behavior of phase velocity under the effects of viscosity, (Black line): $\delta = 0.0$, (Green dashed line): $\delta = 0.1$, (Blue dashed line): $\delta = 0.2$, (Red dashed line): $\delta = 0.3$, surface tension, (Black line): $\tau = 0.0$, (Green dashed line): $\tau = 0.1$, (Blue dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$ and wind, (Black line): $\chi = 0.0$, (Green dashed line): $\chi = 0.1$, (Blue dashed line): $\chi = 0.2$, (Red dashed line): $\chi = 0.3$, respectively. The panels (d), (e) and (f) illustrate the behavior of group velocity under the effects of viscosity, (Black line): $\delta = 0.0$, (Green dashed line): $\delta = 0.1$, (Blue dashed line): $\delta = 0.2$, (Red dashed line): $\delta = 0.3$, surface tension, (Black line): $\tau = 0.0$, (Green dashed line): $\tau = 0.1$, (Blue dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$ and wind, (Black line): $\chi = 0.0$, (Green dashed line): $\chi = 0.1$, (Blue dashed line): $\chi = 0.2$, (Red dashed line): $\chi = 0.3$, respectively.

General Conclusion and Perspectives

In this thesis, we have studied the effects of viscosity, surface tension and wind on hydrodynamic waves for shallow water. The thesis have been organized in three parts. In Chapter I, we have presented a literature review on the hydrodynamic waves. Indeed, the history, main characteristics, conditions of existence and mechanisms of generation of waves encountered in shallow water have been presented. In Chapter II, we have presented the basic equations related to the physical modeling of the equations describing the wave dynamics in shallow water. Thereafter, the methodology applied to reach our goals has been presented. In Chapter III, the results of our investigations have been presented. The soliton-like solutions of the obtained equations and the impact of viscosity, surface tension and wind on the phase and group velocities of the waves and on the soliton dynamics in general have been investigated.

The results have shown that the approximation of the Navier-Stocks equation, the correction of the kinematics boundary condition at the surface and bottom by the viscosity, the Laplace pressure, the Mile model and the assumption of an incompressible and irrotational flow, lead to the formulation of a new Boussinesq system which takes into account the effects of the physical parameters mentioned above. The results of the perturbation theory applied to Boussinesq system have led to new generalized KdV-type equations depending on the order of the perturbation theory. The results have also revealed that a non-uniqueness of the decomposition of the Boussinesq system can be used to derive a system of equations for the surface elevation for right- and left-moving waves. One of the equations depends only on the surface elevation for the main right-moving wave while the second equation includes both surface elevation for right- and left-moving waves.

The solitons solutions of the obtained equations have been constructed and the effects of viscosity, surface tension and wind on the solitons dynamics have been investigated. The results have shown that, when the values of the viscosity parameter δ and the surface tension parameter τ increase, the width of the solitons increases. This results may therefore suggests that, solitons propagating in a viscous medium tends to increase (respectively decrease) in width if the viscosity of medium increases (respectively decreases). It have been also shown that, when the value of the wind parameter χ increases, the amplitude of the solitons also increases. This suggests that, during the propagation of solitons, if a strong wind occurs its energy can be transmitted to the solitons.

The effects of these physical parameters on the phase and group velocities of the waves have been also investigated. It has been revealed that, when the values of the viscosity parameter δ and the surface tension parameter τ increase, the phase and group velocities decrease. However, the effect of the surface tension has been found very greater important than that of the viscosity. When the value of the wind parameter χ increases, the amplitudes of the phase and group velocities increase.

The results obtained in this thesis predict a behavior of nonlinear waves that propagate in shallow water. These results can confirm the idea that, physical parameters such as viscosity, surface tension and wind are very important to improve the understanding and modeling of wave dynamics in shallow water.

Perspectives

Several points related to this topic remain unsolved or unexplored, and then may be the subject for future investigations.

■ In this work, the perturbation theory applied to the Boussinesq system have been done in (1+1)-dimension. However, several phenomena and studies in the field of nonlinear physics are described in higher dimensions. Thus, our work can be extended and investigated in (2+1)-dimensions.

■ In general, the KdV equation is derived under the assumption of flat bottom of the channel. This assumption is not realistic for most of the situations in the real world, bottoms of rivers, seas or oceans are non-flat. In the future, the assumption of the non-flat bottom of the channel will be considered.

■ The study of the effects of variable coefficients on the waves dynamics has continuously attracted considerable attention over the last two decades. For future, the effects of space and time variable coefficients for the KdV models will be investigated.

Annex

Derivation of equations for bidirectional waves

In the field of shallow water, through the non-uniqueness of the Boussinesq decomposition, it has been shown that waves also propagate in two directions [90]. To derive the equations for bi-directional waves, the approach is to split the surface elevation $\eta(x,t)$ into two components, namely $u(x,t)$ and $\xi(x,t)$ corresponding to the right- and left-moving waves respectively. The left-moving wave is of the order $O(\varepsilon)$ smaller than that of the right-moving wave. Thus, one can assume:

$$\begin{aligned} \eta(x,t) &= u(x,t) + \varepsilon\xi(x,t), \\ w(x,t) &= w^{+}(x,t) + \varepsilon w^{-}(x,t), \end{aligned} \tag{3.36}$$

where $w^{+}(x,t) = u(x,t) + \varepsilon R(x,t)$ and $w^{-}(x,t) = \varepsilon S(x,t)$ are the scaled horizontal velocity at the bottom of the fluid to the right- and left-moving respectively. As for the equations for unidirectional waves, we applied the same procedure. Then, at the lowest order we imposes that the Boussinesq system is reduced to $u_t + w_x^{+} = 0$ and $u_x + w_t^{+} = 0$, which are satisfied by the right-moving wave $u_x + u_t = 0$ and the corresponding equation for the left-moving wave $\xi_x - \xi_t = 0$. We express w in the form

$$w = u + \varepsilon R + \varepsilon S = u + \varepsilon Q, \tag{3.37}$$

where $Q = (R + S)$ is arbitrary function of x and t. This function will subsequently be determined by introducing equation (3.37) and the first equation of equation (3.36) into equations (2.106) and (2.108). Neglecting the terms of higher order than $O(\varepsilon)$ in each equation, we obtain the following system

$$\begin{aligned} &u_x + u_t + \varepsilon(Q_x + \xi_x + 2uu_x - \delta u_{2x} - \frac{1}{6}\,u_{3x}) = 0, \\ &u_x + u_t + \varepsilon(Q_t + \xi_x + uu_x - \delta u_{2x} + \frac{1}{2}\,u_{3x} - \tau u_{3x}) = 0, \end{aligned} \tag{3.38}$$

Proceeding in the same way as in [90], we assume that the two equations in (3.37) are identical and can take the form $u_x + u_t + \varepsilon(k_1 u u_x + k_2 u_{2x} + k_3 u_{3x}) = 0$. Thus, we obtain

$$\begin{aligned} Q_x &= -\left(\xi_x + 2uu_x - \delta u_{2x} - \frac{1}{6}\, u_{3x}\right) + k_1 uu_x + k_2 u_{2x} + k_3 u_{3x}, \\ Q_t &= -\left(\xi_x + uu_x - \delta u_{2x} + \frac{1}{2}\, u_{3x} - \tau u_{3x}\right) + k_1 uu_x + k_2 u_{2x} + k_3 u_{3x}. \end{aligned} \tag{3.39}$$

We perform a partial derivation of these two equations with respect to t and x respectively. We then substitute u_t by $-u_x$ and ξ_t by ξ_x, and subtract the equations, we obtain

$$(3 - 2k_1)\ (u_x^2 + uu_{2x}) - 2(k_2 + \delta)u_{3x} + \left(\frac{1}{3}\ (1 - 3\tau) - 2k_3\right) u_{4x} = 0. \tag{3.40}$$

The equation (3.40) is satisfied by $k_1 = 3/2$, $k_2 = -\delta$ and $k_3 = (1/6)(1 - 3\tau)$. Thus, both equations in (3.38) take the form

$$u_x + u_t + \varepsilon\left(\frac{3}{2}\, uu_x - \delta u_{2x} + \frac{1}{6}(1 - 3\tau)u_{3x}\right) = 0, \tag{3.41}$$

which coincides with the same order equation for $\eta(x, t)$ in the pure right-moving wave case.

Substituting the coefficients $k_i (i = 1, 2, 3)$ by their expressions in the first equation of (3.39), and integrating once with respect to x leads to

$$Q = -\xi - \frac{1}{4}\, u^2 + \frac{1}{6}\ (2 - 3\tau)u_{2x}. \tag{3.42}$$

To determine the equation corresponding to the second order $O(\epsilon^2)$, we assume that the equations for the horizontal velocity w and the left-moving wave ξ, can be written in the form

$$\begin{aligned} w &= u + \varepsilon\left(-\xi - \frac{1}{4}\, u^2 + \frac{1}{6}\ (2 - 3\tau)u_{2x}\right) + \varepsilon^2 M, \\ \xi_t &= \xi_x + \varepsilon H_x, \end{aligned} \tag{3.43}$$

where $H \equiv H(x, t)$ and $M \equiv M(x, t)$ are free functions. To determine the functions $H(x, t)$ and $M(x, t)$, equation (3.43) is introduced into equations (2.106) and (2.108). Neglecting the terms of higher order than $O(\varepsilon^2)$ in each equation, all the t-derivatives of u and ξ are replaced by their expressions through the x-derivatives using the lowest order equation, namely equation (3.41) and the second equation of equation (3.43). Upon these substitutions, we obtain the following equations

$$\begin{aligned}
&u_x+u_t+\varepsilon\left(\frac{3}{2}\,uu_x-\delta u_{2x}+\frac{1}{6}(1-3\tau)u_{3x}\right)+\varepsilon^2\bigg(M_x+H_x-\delta\xi_{2x}+\frac{1}{6}\,\xi_{3x}-\frac{3}{4}\,u^2u_x\\
&\quad+\frac{1}{12}(1-6\tau)u_xu_{2x}-\frac{1}{12}(1+6\tau)uu_{3x}-\frac{1}{360}\,(17-30\tau)u_{5x}\bigg)=0,\\
&u_x+u_t+\varepsilon\left(\frac{3}{2}\,uu_x-\delta u_{2x}+\frac{1}{6}(1-3\tau)u_{3x}\right)+\varepsilon^2\bigg(M_t-H_x-\xi u_x-u\xi_x+\delta\xi_{2x}+\frac{1}{2}\,\delta u_x^2\\
&\quad+\frac{1}{2}\,(1-2\tau)\xi_{3x}+\frac{11}{12}\,uu_{3x}+\left(\frac{11}{6}+\frac{7}{4}\tau\right)u_xu_{2x}+\left(\frac{11}{72}-\frac{1}{4}\tau-\frac{1}{4}\tau^2\right)u_{5x}\bigg)=0.
\end{aligned} \tag{3.44}$$

The next step consists to assume from the requirement that the both equations (3.44) can take the form

$$u_x+u_t+\varepsilon\left(\frac{3}{2}\,uu_x-\delta u_{2x}+\frac{1}{6}\,(1-3\tau)u_{3x}\right)+\varepsilon^2(b_1u_{5x}+b_2uu_{3x}+b_3u_xu_{2x}+b_4u^2u_x+b_5u_x^2)=0, \tag{3.45}$$

which implies that

$$\begin{aligned}
M_x&=-H_x+\delta\xi_{2x}-\frac{1}{6}\,\xi_{3x}+\frac{1}{4}(3+4b_4)u^2u_x+\frac{1}{12}(1+6\tau+12b_2)uu_{3x}\\
&\quad-\frac{1}{12}(1-6\tau-12b_3)u_xu_{2x}+\frac{1}{360}(17-30\tau+360b_1)u_{5x}+b_5u_x^2,\\
M_t&=H_x+\xi u_x+u\xi_x-\delta\xi_{2x}-\frac{1}{2}(1-2\tau)\xi_{3x}+\frac{1}{2}(2b_5-\delta)u_x^2-\left(\frac{11}{6}+\frac{7}{4}\tau-b_3\right)u_xu_{2x}\\
&\quad+b_4u^2u_x-\frac{1}{12}(11-12b_2)uu_{3x}-\left(\frac{11}{72}-\frac{1}{4}\,\tau-\frac{1}{4}\,\tau^2-b_1\right)u_{5x}.
\end{aligned} \tag{3.46}$$

Integrating the first equation of equation (3.46) with respect to x leads to

$$\begin{aligned}
M&=-H+\delta\xi_x-\frac{1}{6}\,\xi_{2x}+\frac{1}{12}(3+4b_4)u^3+\frac{1}{12}(1+6\tau+12b_2)uu_{2x}\\
&\quad-\frac{1}{12}(1+2b_2-6b_3)u_x^2+b_5\int u_x^2+\frac{1}{360}(17-30\tau+360b_1)u_{5x}.
\end{aligned} \tag{3.47}$$

Substituting equation (3.47) into the second equation of equation (3.46) yields

$$\begin{aligned}
H_x+H_t&=-\xi u_x-u\xi_x+2\delta\xi_{2x}+\frac{1}{3}(1-3\tau)\xi_{3x}-2\left(b_4+\frac{3}{8}\right)u^2u_x+\frac{1}{2}(\delta-4b_5)u_x^2\\
&\quad-2\bigg[+\left(b_3-\frac{5}{8}\,\tau-\frac{23}{24}\right)u_xu_{2x}+\left(b_1-\frac{19}{360}+\frac{1}{12}\,\tau+\frac{1}{8}\,\tau^2\right)u_{5x}\\
&\quad+\left(b_2-\frac{5}{12}+\frac{1}{4}\,\tau\right)uu_{3x}\bigg].
\end{aligned} \tag{3.48}$$

To find the solution $H(x,t)$ of equation (3.48), we sum its derivatives with respect to x and t and obtain

$$H = -\frac{1}{2}(\pi u)_x + \delta\pi_{2x} + \frac{1}{6}(1-3\tau)\pi_{3x} + G. \tag{3.49}$$

By replacing H by its expression in (3.49) we obtain

$$\begin{aligned} M =& -G + \frac{1}{2}\,u\xi + \frac{1}{2}\,u_x\pi - \frac{1}{6}(2-3\tau)\,\xi_{2x} + \frac{1}{12}(1+6\tau+12b_2)uu_{2x} + \frac{1}{12}(3+4b_4)u^3 \\ &+ b_5\int u_x^2 dx - \frac{1}{12}(1+2b_2-6b_3)u_x^2 + \frac{1}{360}(17-30\tau+360b_1)u_{5x}, \end{aligned} \tag{3.50}$$

such that $\xi = \pi_x$ and G satisfies the following equation

$$\begin{aligned} G_x + G_t = -2\Bigg[&\left(b_4 + \frac{3}{8}\right)u^2u_x + \left(b_2 - \frac{5}{12} + \frac{1}{4}\,\tau\right)uu_{3x} - \frac{1}{4}(\delta - 4b_5)u_x^2 \\ &+ \left(b_3 - \frac{5}{8}\,\tau\frac{23}{24}\right)u_xu_{2x} + \left(b_1 - \frac{19}{360} + \frac{1}{12}\,\tau + \frac{1}{8}\,\tau^2\right)u_{5x}\Bigg]. \end{aligned} \tag{3.51}$$

Introducing equation (3.49) into the second equation of equation (3.43) and integrating with respect to x leads to equation in terms of the left-moving wave for the first order $O(\varepsilon)$ as follows

$$\pi_t - \pi_x + \varepsilon\left(\frac{1}{2}(\pi u)_x - \delta\pi_{2x} - \frac{1}{6}(1-3\tau)\pi_{3x} - G\right) = 0. \tag{3.52}$$

Introducing equations (3.49) and (3.50) into equation (3.44), we obtain

$$\begin{aligned} u_x + u_t + \varepsilon\left(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\right) + \varepsilon^2\Bigg(& -G_x + \xi u_x + \frac{1}{2}\,u\xi_x + \frac{1}{2}\,\delta u_x^2 - \delta\xi_{2x} \\ + \frac{1}{2}\,\pi u_{2x} - \frac{1}{6}(1-3\tau)\xi_{3x} + b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2\Bigg) &= 0. \end{aligned} \tag{3.53}$$

To establish the equation corresponding to the third order $O(\varepsilon^3)$, we assume that the equations for the horizontal velocity ω and the left-moving wave ξ can be written in the form

$$\begin{aligned} w = u + \varepsilon&\left(-\xi - \frac{1}{4}\,u^2 + \frac{1}{6}\,(2-3\tau)u_{2x}\right) + \varepsilon^2\Bigg(-G + \frac{1}{2}\,u\xi + \frac{1}{2}\,u_x\pi - \frac{1}{6}(2-3\tau)\,\xi_{2x} \\ &+ \frac{1}{12}(1+6\tau+12b_2)uu_{2x} + \frac{1}{12}(3+4b_4)u^3 - \frac{1}{12}(1+2b_2-6b_3)u_x^2 + b_5\int u_x^2 \\ &+ \frac{1}{360}(17-30\tau+360b_1)u_{5x}\Bigg) + \varepsilon^3 P, \\ \xi_t =& \ \xi_x + \varepsilon H_x + \varepsilon^2 L_x, \end{aligned} \tag{3.54}$$

where the free functions $L \equiv L(x,t)$ and $P \equiv P(x,t)$ will be determined later. Introducing equation (3.54) into equations (2.106) and (2.108), we obtain the following equations

$$
\begin{aligned}
&u_x + u_t + \varepsilon\Big(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big) + \varepsilon^2\Big(- G_x + \xi u_x + \frac{1}{2}\,u\xi_x + \frac{1}{2}\,\delta u_x^2 - \delta\xi_{2x} \\
&- \frac{1}{6}(1-3\tau)\xi_{3x} + b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2\Big) + \varepsilon^3\Big[P_x + L_x - uG_x \\
&- Gu_x + \xi uu_x - 2\xi\xi_x + \frac{1}{2}\,\pi u_x^2 + \frac{1}{4}\,\xi_xu^2 + \frac{1}{2}\,\pi uu_{2x} - \Big(\frac{2}{3} + \frac{1}{2}\,\tau\Big)\xi_xu_{2x} + \frac{1}{6}\,G_{3x} \\
&- \frac{1}{6}(1-3\tau)\xi_{2x}u_x - \frac{1}{2}(1+\tau)\xi u_{3x} - \frac{1}{12}\,\pi u_{4x} + \frac{1}{12}(1+6\tau)u\xi_{3x} + \Big(\frac{17}{360} - \frac{1}{12}\,\tau\Big)\xi_{5x} \\
&+ \frac{1}{2}(\delta + 2b_5)uu_x^2 - \Big(\frac{1}{12} + \frac{1}{2}\,b_2 - \frac{1}{2}\,b_3 + \frac{1}{3}\,b_4\Big)u_x^3 + \frac{1}{3}(3+4b_4)u^3u_x - \Big(\frac{1}{3}\,b_5 + \frac{1}{6}\,\delta\Big)u_{2x}^2 \\
&- \Big(\frac{1}{3}\,b_5 + \frac{1}{6}\,\delta\Big)u_xu_{3x} - \Big(\frac{3}{4} - b_2 - b_3 + b_4 - \tau\Big)uu_xu_{2x} - \Big(\frac{7}{24} - b_2 + \frac{1}{6}\,b_4 - \frac{1}{2}\,\tau\Big)u^2u_{3x} \\
&- \Big(\frac{1}{72} + \frac{1}{6}\,b_2 + \frac{1}{2}\,b_3 + \frac{1}{3}\,\tau\Big)u_{2x}u_{3x} - \Big(\frac{9}{80} - b_1 + \frac{1}{3}\,b_2 + \frac{1}{6}\,b_3 + \frac{1}{12}\,\tau\Big)u_xu_{4x} \\
&+ \Big(- \frac{23}{240} + b_1 - \frac{1}{6}\,b_2 + \frac{1}{12}\,\tau\Big)uu_{5x} + \frac{1}{2}(\delta + 2b_5)u_x\int u_x^2dx - \Big(\frac{1}{189} + \frac{1}{6}\,b_1 - \frac{7}{720}\,\tau\Big)u_{7x}\Big] = 0,
\end{aligned}
\tag{3.55}
$$

$$
\begin{aligned}
&u_x + u_t + \varepsilon\Big(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big) + \varepsilon^2\Big(- G_x + \xi u_x + \frac{1}{2}\,u\xi_x + \frac{1}{2}\,\delta u_x^2 - \delta\xi_{2x} + \frac{1}{2}\,\pi u_{2x} \\
&- \frac{1}{6}(1-3\tau)\xi_{3x} + b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2\Big) + \varepsilon^3\Big[P_t - L_x - \frac{1}{2}uG_x - Gu_x \\
&+ \frac{3}{4}\,\xi uu_x - \frac{1}{4}\,\pi u_x^2 + \frac{1}{2}u^2\xi_x + \xi\xi_x - \frac{3}{2}\,\delta\xi_xu_x + \delta G_{2x} - \frac{1}{2}\,\pi uu_{2x} - \delta\xi u_{2x} - \Big(\frac{1}{3} + \tau\Big)\xi_xu_{2x} \\
&- \Big(\frac{2}{3} + \frac{1}{2}\,\tau\Big)u_x\xi_{2x} + \Big(\frac{5}{4} - \frac{1}{4}\,\tau\Big)\xi u_{3x} + \frac{11}{12}\,u\xi_{3x} + \frac{1}{12}\,\pi u_{4x} + \Big(\frac{11}{72} - \frac{1}{4}\,\tau - \frac{1}{4}\,\tau^2\Big)\xi_{5x} - \frac{2}{3}\,b_4u^3u_x \\
&- \frac{1}{6}\,(2-3\tau)G_{3x} + \Big(\frac{7}{6} + b_2 - b_3 - \frac{4}{3}\,b_4 - b_4\tau\Big)u_x^3 + \Big(\frac{1}{2}\,b_5 - \delta - 2b_4\delta\Big)uu_x^2 - (2b_5\delta + \delta^2)u_xu_{2x} \\
&+ \frac{1}{2}(\delta + 2b_5)u_x\int u_x^2dx + \Big(\frac{1}{4} - b_2 - \frac{5}{6}\,b_4 + \frac{1}{4}\,\tau\Big)u^2u_{3x} + \Big(\frac{191}{240} - b_1 - \frac{5}{6}\,b_2 - \tau + \frac{1}{4}\,\tau^2\Big)uu_{5x} \\
&+ \Big(\frac{85}{18} - 15b_1 - \frac{5}{6}\,b_2 - 2b_3 - \frac{1}{8}\,\tau - \frac{3}{2}\,b_3\tau + \frac{1}{2}\,\tau^2\Big)u_{2x}u_{3x} - \Big(\frac{4}{3}\,b_5 - \frac{2}{3}\,\delta - b_2\delta + b_3\delta + b_5\tau\Big)u_{2x}^2
\end{aligned}
\tag{3.56}
$$

$$+\left(\frac{231}{80}-\frac{13}{2}\,b_1-\frac{7}{6}\,b_2-\frac{5}{6}\,b_3-\frac{37}{24}\,\tau-\frac{3}{2}\,b_2\tau\right)u_xu_{4x}-\left(\frac{4}{3}\,b_5-\frac{1}{3}\,\delta+2b_2\delta+b_5\tau+\delta\tau\right)u_xu_{3x}$$

$$+\left(\frac{31}{12}-\frac{7}{2}\,b_2-b_3-4b_4-\frac{7}{4}\,\tau-3b_4\,\tau\right)uu_xu_{2x}+\left(\frac{83}{1080}-\frac{5}{6}\,b_1-\frac{41}{240}\,\tau-\frac{1}{24}\,\tau^2\right)u_{7x}\Bigg]=0.$$

In the next step, we assume the requirement that both equations (3.55) can be written in the following forms

$$\begin{aligned}&u_x+u_t+\varepsilon\left[\frac{3}{2}\,uu_x-\delta u_{2x}+\frac{1}{6}(1-3\tau)u_{3x}\right]+\varepsilon^2\Bigg[-G_x+\xi u_x+\frac{1}{2}\,u\xi_x+\frac{1}{2}\,\delta u_x^2-\delta\xi_{2x}\\&+\frac{1}{2}\,\pi u_{2x}-\frac{1}{6}(1-3\tau)\xi_{3x}+b_1u_{5x}+b_2uu_{3x}+b_3u_xu_{2x}+b_4u^2u_x+b_5u_x^2\Bigg]+\varepsilon^3\Bigg[c_1u_{7x}\\&+c_2uu_{5x}+c_3u_xu_{4x}+c_4u^2u_{3x}+c_5u_{2x}u_{3x}+c_6uu_xu_{2x}+c_7u_xu_{3x}+c_8u_x^3+c_9u_x\int u_x^2\\&+c_{10}u^3u_x+c_{11}uu_x^2+c_{12}u_{2x}^2+c_{13}u_xu_{2x}\Bigg]=0.\end{aligned}\tag{3.57}$$

The expressions for the x- and t-derivatives of P must be then written as

$$\begin{aligned}P_x=&-L_x+uG_x+Gu_x-\xi uu_x+2\xi\xi_x-\frac{1}{2}\,\pi u_x^2-\frac{1}{4}\,\xi_xu^2-\frac{1}{2}\,\pi uu_{2x}+\left(\frac{2}{3}+\frac{1}{2}\,\tau\right)\xi_xu_{2x}\\&+\frac{1}{6}(1-3\tau)\xi_{2x}u_x-\frac{1}{6}\,G_{3x}+\frac{1}{2}(1+\tau)\xi u_{3x}+\frac{1}{12}\,\pi u_{4x}-\frac{1}{12}(1+6\tau)u\xi_{3x}-\left(\frac{17}{360}-\frac{1}{12}\,\tau\right)\xi_{5x}\\&-\frac{1}{2}(\delta+2b_5-2c_{11})uu_x^2-\frac{1}{3}(3+4b_4-3c_{10})u^3u_x-\frac{1}{2}(\delta+2b_5-2c_9)u_x\int u_x^2dx\\&+\left(\frac{1}{3}\,b_5+\frac{1}{6}\,\delta+c_{12}\right)u_{2x}^2+\left(\frac{1}{3}\,b_5+\frac{1}{6}\,\delta+c_7\right)u_xu_{3x}+\left(\frac{3}{4}-b_2-b_3+b_4-\tau+c_6\right)uu_xu_{2x}\\&+c_{13}u_xu_{2x}+\left(\frac{7}{24}-b_2+\frac{1}{6}\,b_4-\frac{1}{2}\,\tau+c_4\right)u^2u_{3x}+\left(\frac{1}{72}+\frac{1}{6}\,b_2+\frac{1}{2}\,b_3+\frac{1}{3}\,\tau+c_5\right)u_{2x}u_{3x}\\&+\left(\frac{9}{80}-b_1+\frac{1}{3}\,b_2+\frac{1}{6}\,b_3+\frac{1}{12}\,\tau+c_3\right)u_xu_{4x}+\left(\frac{1}{12}+\frac{1}{2}\,b_2-\frac{1}{2}\,b_3+\frac{1}{3}\,b_4+c_8\right)u_x^3\\&-\left(-\frac{23}{240}+b_1-\frac{1}{6}\,b_2+\frac{1}{12}\,\tau-c_2\right)uu_{5x}+\left(\frac{1}{189}+\frac{1}{6}\,b_1-\frac{7}{720}\,\tau+c_1\right)u_{7x},\end{aligned}\tag{3.58}$$

$$\begin{aligned}P_t=&\,L_x+\frac{1}{2}\,uG_x+Gu_x-\frac{3}{4}\xi uu_x+\frac{1}{4}\,\pi u_x^2-\frac{1}{2}u^2\xi_x-\xi\xi_x+\frac{3}{2}\,\delta\xi_xu_x-\delta G_{2x}+\frac{1}{2}\,\pi uu_{2x}+\delta\xi u_{2x}\\&+\left(\frac{1}{3}+\tau\right)\xi_xu_{2x}+\left(\frac{2}{3}+\frac{1}{2}\,\tau\right)u_x\xi_{2x}-\left(\frac{5}{4}-\frac{1}{4}\,\tau\right)\xi u_{3x}-\frac{11}{12}\,u\xi_{3x}-\frac{1}{12}\,\pi u_{4x}+\frac{1}{6}(2-3\tau)G_{3x}\end{aligned}$$

$$-\left(\frac{11}{72}-\frac{1}{4}\,\tau-\frac{1}{4}\,\tau^2\right)\xi_{5x}-\left(\frac{7}{6}+b_2-b_3-\frac{4}{3}\,b_4-b_4\tau-c_8\right)u_x^3+\left(\frac{2}{3}\,b_4+c_{10}\right)u^3u_x$$
$$-\frac{1}{2}(\delta+2b_5-c_9)u_x\int u_x^2dx-\left(\frac{1}{2}\,b_5-\delta-2b_4\delta-c_{11}\right)uu_x^2+(2b_5\delta+\delta^2+c_{13})u_xu_{2x}$$
$$-\left(\frac{31}{12}-\frac{7}{2}\,b_2-b_3-4b_4-\frac{7}{4}\,\tau-3b_4\,\tau-c_6\right)uu_xu_{2x}-\left(\frac{191}{240}-b_1-\frac{5}{6}\,b_2-\tau+\frac{1}{4}\,\tau^2-c_2\right)uu_{5x}$$
$$+\left(\frac{4}{3}\,b_5-\frac{1}{3}\,\delta+2b_2\delta+b_5\tau+\delta\tau+c_7\right)u_xu_{3x}+\left(\frac{4}{3}\,b_5-\frac{2}{3}\,\delta-b_2\delta+b_3\delta+b_5\tau+c_{12}\right)u_{2x}^2$$
$$-\left(\frac{1}{4}-b_2-\frac{5}{6}\,b_4+\frac{1}{4}\,\tau-c_4\right)u^2u_{3x}-\left(\frac{85}{18}-15b_1-\frac{5}{6}\,b_2-2b_3-\frac{1}{8}\,\tau-\frac{3}{2}\,b_3\tau+\frac{1}{2}\,\tau^2-c_5\right)u_{2x}u_{3x}$$
$$-\left(\frac{231}{80}-\frac{13}{2}\,b_1-\frac{7}{6}\,b_2-\frac{5}{6}\,b_3-\frac{37}{24}\,\tau-\frac{3}{2}\,b_2\tau-c_3\right)u_xu_{4x}-\left(\frac{83}{1080}-\frac{5}{6}\,b_1-\frac{41}{240}\,\tau-\frac{1}{24}\,\tau^2-c_1\right)u_{7x}. \tag{3.59}$$

Integrating equation (3.57) with respect to x, we obtain

$$P=-L+uG-\frac{1}{2}\,\pi uu_x-\frac{1}{4}\,\pi_xu^2+\pi_x^2-\frac{1}{6}\,G_{3x}-\frac{1}{12}(1+6\tau)\pi_{3x}u+\frac{1}{4}\,\pi_{2x}u_x+\frac{1}{12}(5+6\tau)\pi_xu_{2x}$$
$$+\frac{1}{12}\,\pi u_{3x}-\left(\frac{17}{360}-\frac{1}{12}\,\tau\right)\pi_{5x}+\frac{1}{12}(3c_{10}-4b_4-3)u^4+\left(\frac{23}{240}-b_1+\frac{b_2}{6}-\frac{\tau}{12}+c_2\right)uu_{4x}$$
$$+\left(\frac{b_1}{3}+\frac{\delta}{6}+c_7\right)u_xu_{2x}+\left(c_1+\frac{b_1}{6}+\frac{1}{189}-\frac{7}{720}\,\tau\right)u_{6x}+\left(c_{12}-c_7\right)\int u_{2x}^2dx$$
$$+\left(c_{11}-b_5-\frac{\delta}{2}\right)\int uu_x^2dx+\left(c_8+c_4-\frac{c_6}{2}\right)\int u_x^3dx+\left(\frac{7}{24}-b_2+\frac{b_4}{6}-\frac{\tau}{2}+c_4\right)u^2u_{2x}$$
$$+\frac{1}{12}\,c_{13}u_x^2+\left(\frac{b_2}{6}+\frac{b_3}{6}+\frac{\tau}{6}+c_3-c_2+\frac{1}{60}\right)u_xu_{3x}+\left(\frac{b_3}{6}+\frac{\tau}{12}+\frac{c_2}{2}+\frac{c_5}{2}-\frac{c_3}{2}-\frac{1}{720}\right)u_{2x}^2$$
$$+\left(\frac{b_2}{2}+\frac{b_4}{3}+\frac{c_6}{2}-\frac{b_3}{2}-c_4+\frac{1}{12}\right)uu_x^2+\left(c_9-b_5-\frac{\delta}{2}\right)\int\left(u_x\int u_x^2dx\right)dx. \tag{3.60}$$

Introducing equation (3.59) into the equation (3.58) yields

$$L_x+L_t=\frac{3}{4}\,uu_x\pi_x+\frac{1}{4}\,u_x^2\pi+\frac{1}{4}\,u^2\pi_{2x}+3\pi_x\pi_{2x}-\frac{3}{2}\,\delta u_x\pi_{2x}-\delta u_{2x}\pi_x-\frac{1}{6}(1+3\tau)u_{2x}\pi_{2x}$$
$$-\frac{1}{3}\,u_x\pi_{3x}+\frac{1}{12}(11-9\tau)u_{3x}\pi_x+\frac{1}{6}(5-3\tau)u\pi_{4x}-\frac{3}{2}uG_x-2Gu_x+\delta G_{2x}-\frac{1}{6}(1-3\tau)G_{3x}$$
$$+\left(\frac{19}{180}-\frac{\tau}{6}-\frac{\tau^2}{4}\right)\pi_{6x}+\left(\frac{9}{2}-\frac{5}{2}\,b_2-2b_3-3b_4-2c_6+\frac{\tau}{2}-3b_4\tau\right)uu_xu_{2x}+\frac{1}{2}(2b_5+\delta-2c_9)\mu$$

$$
\begin{aligned}
&-\left(\frac{b_1}{3}+\frac{2}{3}\,b_5+2c_7+2b_2\delta+b_5\tau+\delta\tau\right)u_xu_{3x}+\left(\frac{1}{4}-\frac{4}{3}\,b_4-2c_{10}\right)u^3u_x+\frac{1}{2}(2b_5+\delta-2c_9)\lambda\\
&+\left(\frac{98}{45}-\frac{11}{2}\,b_1-\frac{5}{6}\,b_2-\frac{2}{3}\,b_3-2c_3-\frac{5}{3}\,\tau-\frac{3}{2}\,b_2\tau\right)u_xu_{4x}-\left(\frac{c_{13}}{6}+2b_5\delta+c_{13}\delta+\delta^2\right)u_xu_{2x}\\
&-\left(\frac{1}{3}\,b_1+\frac{2}{3}\,b_5+2c_{12}-\frac{1}{3}\,\delta-b_2\delta+b_3\delta+b_5\tau\right)u_{2x}^2+\left(\frac{11}{12}-2b_2-\frac{2}{3}\,b_4-2c_4+\frac{1}{4}\,\tau\right)u^2u_{3x}\\
&-\left(\frac{1}{2}\,b_5+2c_{11}+2b_4\delta\right)uu_x^2+\frac{1}{2}(2b_5+\delta-2c_9)\nu+\left(\frac{4}{3}+\frac{1}{2}\,b_2-\frac{1}{2}\,b_3-b_4-2c_8-b_4\tau\right)u_x^3\\
&+\left(\frac{2}{3}-2b_1-\frac{2}{3}\,b_2-2c_2-\tau\right)uu_{5x}+\left(\frac{17}{315}-\frac{2}{3}\,b_1-2c_1-\frac{2}{15}\,\tau\right)u_{7x}\\
&+\left(\frac{65}{18}-15b_1-\frac{2}{3}\,b_2-\frac{3}{2}\,b_3-2c_5-\tau-\frac{3}{2}\,b_3\tau+\frac{\tau^2}{2}\right)u_{2x}u_{3x},
\end{aligned}
\tag{3.61}
$$

where

$$
\nu=\int(u_{2x}\int u_x^2dx)dx,\ \ \lambda=u_x\int u_x^2dx,\ \ \mu=\int u_x^3dx.
$$

To find the solution $L(x,t)$ of equation (3.61), we sum its derivatives with respect to x and t and obtain

$$
\begin{aligned}
L=&\ \frac{3}{8}\,uu_x\pi+\frac{1}{8}\,u^2\pi_x-\frac{1}{6}\,u_x\pi_{2x}-\frac{3}{4}\,\delta u_x\pi_x+\frac{3}{4}\,\pi_x^2-\frac{\delta}{2}\,u_{2x}\pi+\frac{1}{24}(11-9\tau)u_{3x}\pi\\
&+\frac{1}{12}(5-3\tau)u\pi_{3x}-\frac{1}{12}(1+3\tau)u_{2x}\pi_x+\frac{1}{8}u_x^2\int\pi dx+\frac{1}{360}(19-30\tau-45\tau^2)\pi_{5x}+K,
\end{aligned}
\tag{3.62}
$$

where the function K satisfies the following equation

$$
\begin{aligned}
K_x+K_t=&-\frac{3}{2}\,uG_x-2Gu_x+\delta G_{2x}-\frac{1}{6}(1-3\tau)G_{3x}-\left(\frac{c_{13}}{6}+2b_5\delta+c_{13}\delta+\delta^2\right)u_xu_{2x}\\
&+\frac{1}{2}(2b_5+\delta-2c_9)\mu+\left(\frac{98}{45}-\frac{11}{2}\,b_1-\frac{5}{6}\,b_2-\frac{2}{3}\,b_3-2c_3-\frac{5}{3}\,\tau-\frac{3}{2}\,b_2\tau\right)u_xu_{4x}\\
&+\left(\frac{9}{2}-\frac{5}{2}\,b_2-2b_3-3b_4-2c_6+\frac{\tau}{2}-3b_4\tau\right)uu_xu_{2x}-\left(\frac{b_1}{3}+\frac{2}{3}\,b_5+2c_7+2b_2\delta+b_5\tau+\delta\tau\right)u_xu_{3x}\\
&+\left(\frac{1}{4}-\frac{4}{3}\,b_4-2c_{10}\right)u^3u_x+\frac{1}{2}(2b_5+\delta-2c_9)\lambda-\left(\frac{1}{2}\,b_5+2c_{11}+2b_4\delta\right)uu_x^2+\frac{1}{2}(2b_5+\delta-2c_9)\nu\\
&-\left(\frac{1}{3}\,b_1+\frac{2}{3}\,b_5+2c_{12}-\frac{1}{3}\,\delta-b_2\delta+b_3\delta+b_5\tau\right)u_{2x}^2+\left(\frac{4}{3}+\frac{1}{2}\,b_2-\frac{1}{2}\,b_3-b_4-2c_8-b_4\tau\right)u_x^3
\end{aligned}
\tag{3.63}
$$

$$+\left(\frac{11}{12}-2b_2-\frac{2}{3}\,b_4-2c_4+\frac{1}{4}\,\tau\right)u^2u_{3x}+\left(\frac{2}{3}-2b_1-\frac{2}{3}\,b_2-2c_2-\tau\right)uu_{5x}$$
$$+\left(\frac{65}{18}-15b_1-\frac{2}{3}\,b_2-\frac{3}{2}\,b_3-2c_5-\tau-\frac{3}{2}\,b_3\tau+\frac{\tau^2}{2}\right)u_{2x}u_{3x}+\left(\frac{17}{315}-\frac{2}{3}\,b_1-2c_1-\frac{2}{15}\,\tau\right)u_{7x}.$$

By replacing L by its expression in (3.60), we obtain

$$\begin{aligned}
P=&-K+uG-\frac{1}{6}\,G_{2x}-\frac{3}{8}\,u^2\pi_x-\frac{7}{8}\,\pi uu_x+\frac{3}{4}\,\delta u_x\pi_x+\frac{1}{4}\,\pi_x^2+\frac{1}{12}\,c_{13}u_x^2+\frac{5}{12}\,u_x\pi_{2x}+\frac{1}{2}\,\delta u_{2x}\pi\\
&-\frac{1}{4}(2+\tau)u\pi_{3x}+\frac{1}{4}(2+3\tau)u_{2x}\pi_x+\frac{1}{2}(2c_{11}-2b_5-\delta)\int uu_x^2dx-\frac{3}{8}(1+\tau)\pi u_{3x}\\
&-\left(\frac{1}{10}-\frac{1}{6}\,\tau-\frac{1}{8}\,\tau^2\right)\pi_{5x}+\left(c_4+c_8-\frac{c_6}{8}\right)\mu-\left(\frac{1}{4}-\frac{b_4}{3}-\frac{c_{10}}{4}\right)u^4+(c_{12}-c_7)\int u_{2x}^2dx\\
&-\frac{1}{8}\,u_x^2\int\pi dx+\left(\frac{1}{12}-\frac{b_3}{2}+\frac{b_4}{3}-c_4+\frac{c_6}{2}\right)uu_x^2+\left(-\frac{1}{720}+\frac{b_3}{6}+\frac{c_2}{2}-\frac{c_3}{2}+\frac{c_5}{2}+\frac{\tau}{12}\right)u_{2x}^2\\
&+\left(\frac{7}{24}-b_2+\frac{b_4}{6}+c_4-\frac{\tau}{2}\right)u^2u_{2x}+\left(\frac{1}{60}+\frac{b_2}{6}+\frac{b_3}{6}-c_2+c_3+\frac{\tau}{6}\right)u_xu_{3x}\\
&+\left(\frac{23}{240}-b_1+\frac{b_2}{6}+c_2-\frac{\tau}{12}\right)uu_{4x}+\frac{1}{2}(2c_9-2b_5-\delta)\int u_x\left(\int u_x^2\right)dx\\
&+\left(\frac{b_1}{3}+c_7+\frac{\delta}{6}\right)u_xu_{2x}+\left(\frac{1}{189}+\frac{b_1}{6}+c_1-\frac{7}{720}\tau\right)u_{6x}.
\end{aligned}\tag{3.64}$$

Introducing equation (3.62) in the second equation of equation (3.54) and integrating with respect to x leads to the equation of the right- and left-moving wave for the second order $O(\varepsilon^2)$ as follows

$$\begin{aligned}
&\pi_t-\pi_x+\varepsilon\left[\frac{1}{2}(\pi u)_x-\delta\pi_{2x}-\frac{1}{6}(1-3\tau)\pi_{3x}-G\right]+\varepsilon^2\left[-\frac{3}{8}\,uu_x\pi-\frac{1}{8}\,u^2\pi_x+\frac{1}{6}\,u_x\pi_{2x}\right.\\
&\quad+\frac{3}{4}\,\delta u_x\pi_x+\frac{\delta}{2}\,u_{2x}\pi-\frac{1}{12}(5-3\tau)u\pi_{3x}-\frac{1}{24}(11-9\tau)u_{3x}\pi-\frac{3}{4}\,\pi_x^2+\frac{1}{12}(1+3\tau)u_{2x}\pi_x\\
&\quad\left.-\frac{1}{8}\,u_x^2\int\pi dx-\frac{1}{360}(19-30\tau-45\tau^2)\pi_{5x}-K\right]=0,
\end{aligned}\tag{3.65}$$

where the functions G and K satisfy equations (3.51) and (3.63) respectively. The function

$u(x,t)$ corresponds to the pure right-moving wave equation and can be written as

$$\begin{aligned} u_x + u_t + \varepsilon \left[\frac{3}{2}\, u u_x - \delta u_{2x} + \frac{1}{6}(1-3\tau) u_{3x} \right] + \varepsilon^2 \Big[b_1 u_{5x} + b_2 u u_{3x} + b_3 u_x u_{2x} + b_4 u^2 u_x \\ + b_5 u_x^2 \Big] + \varepsilon^3 \Big[c_1 u_{7x} + c_2 u u_{5x} + c_3 u_x u_{4x} + c_4 u^2 u_{3x} + c_5 u_{2x} u_{3x} + c_6 u u_x u_{2x} + c_7 u_x u_{3x} \\ + c_8 u_x^3 + c_9 u_x \int u_x^2 + c_{10} u^3 u_x + c_{11} u u_x^2 + c_{12} u_{2x}^2 + c_{13} u_x u_{2x} \Big] = 0. \end{aligned} \tag{3.66}$$

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List of Publications

1. Mouassom, L. Fernand, Alain Mvogo, and C. Biouele Mbane. Rogue wave solutions of the chiral nonlinear Schrödinger equation with modulated coefficients. *Pramana* 94, no. 1 (2020): 1-8. **Impact Factor 2.689**.

2. Alain Mvogo, **Mouassom, L. Fernand**, FM Enyegue A. Nyam, and C. Bioule Mbane. Exact solitary waves for the 2D Sasa-Satsuma equation. *Chaos, Solitons & Fractals* 133 (2020): 109657. **Impact Factor 10.41**.

3. Mouassom, L. Fernand, T. Nkoa Nkomom, Alain Mvogo, and Cesar Biouele Mbane. Effects of viscosity and surface tension on soliton dynamics in the generalized KdV equation for shallow water waves. *Communications in Nonlinear Science and Numerical Simulation* 102 (2021): 105942. **Impact Factor 4.552**.

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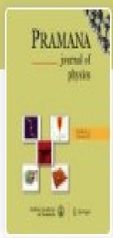

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Rogue wave solutions of the chiral nonlinear Schrödinger equation with modulated coefficients

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Abstract. The research of rogue wave solutions of the nonlinear Schrödinger (NLS) equations is still an open topic. NLS equations have received particular attention for describing nonlinear waves in optical fibres, photonics, plasmas, Bose–Einstein condensates and deep ocean. This work deals with rogue wave solutions of the chiral NLS equation. We introduce an inhomogeneous one-dimensional version, and using the similarity transformation and direct ansatz, we solve the equation in the presence of dispersive and nonlinear coupling which are modulated in time and space. As a result, we show how a simple choice of some free functions can display a lot of interesting rogue wave structures and the interaction of quantum rogue waves. The results obtained may give the possibility of conducting relevant experiments in quantum mechanics and achieving potential applications.

Keywords. Rogue waves; chiral nonlinear Schrödinger equation; modulated coefficients; quantum mechanics.

PACS Nos 05.45.Yv; 47.35.Fg; 02.30.Jr

1. Introduction

Nowadays, the study of rogue waves attracts a great deal of interest from scientific community, especially in nonlinear sciences. Rogue waves are giant single waves that may suddenly appear in oceans [1]. These are also known as monstrous waves, deadly waves or extreme waves. Their appearance can be quite unexpected and their origin is mysterious. The rogue wave phenomenon becomes more and more popular for describing nonlinear waves in other fields such as in nonlinear optics [2–4] and atmosphere [5], just to cite a few. Contrary to the dispersive behaviour adopted by traditional waves of low amplitude, rogue waves are self-reinforcing packets of solitary waves. The rogue wave phenomenon is not just a spectacular event accessible to routine observations and satellite images but also a combination of complex physical processes that occur under the accuracy conditions [6]. This complex combination of physical processes can be well described by the nonlinear Schrödinger (NLS) equations.

NLS equations are prototypical dispersive nonlinear partial differential equations which have been derived and analysed in various branches of physics such as nonlinear optics [7], photonics [8], Bose–Einstein condensates [9] and nonlinear oceanography [10,11]. It has been found that rogue waves are analytical solutions of some integrable NLS equations [12–15], Benjamin–Ono equation [16] and Hirota equation [17].

In recent years, it has been shown that the nonuniform physical systems are more suitable candidates for achieving the required controlling mechanisms [18]. The physical medium with defects or inhomogeneities always has some irregular changes, which are related to the realistic understanding of the nonlinear phenomena [19]. Along the same idea, experiment has shown that the inhomogeneity of a medium plays an important role in the generation of optical rogue waves, and the inhomogeneity is usually described by nonlinear differential equations with variable or inhomogeneity coefficients [20].

Many works have been devoted to the construction of analytical solutions of the NLS equations with inhomogeneity coefficients [21–26], and the results revealed that the generation of rogue waves can be well controlled through the management and the choice of variable coefficients [23–26].

Although significant progress has been made in that sense, it remains really difficult to relate all the characteristics of rogue waves to a specific NLS model equation. Motivated by these considerations, our aim in this work is to present a systematic theoretical study

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for considering inhomogeneity coefficients in the chiral NLS equation and explore the effect of inhomogeneities of the underlying dynamics. The chiral NLS equation plays a fundamental role in developing quantum mechanics, especially in the field of quantum Hall effect, where chiral excitations are known to appear. In this paper, we consider the inhomogeneous chiral NLS equation of the form

$$\frac{\partial\psi}{\partial t}+\beta(t,x)\frac{\partial^2\psi}{\partial x^2}-i\sigma(t,x)\times\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)\psi=0, \tag{1}$$

where $\psi\equiv\psi(t,x)$ is the complex function of space x and time t. The parameter $\beta(t,x)$ is the dispersion coefficient and $\sigma(t,x)$ is the nonlinear coupling coefficient. The symbol * refers to the complex conjugate. The particular case, when $\beta(t,x)$ and $\sigma(t,x)$ are constant coefficients, was studied in [27–30] where both bright and dark soliton solutions were investigated. In this work, we investigate the analytical rogue wave solutions of the inhomogeneous chiral NLS equation (1).

The rest of the paper is organised as follows: In §2, we investigate the first-order and second-order rational-like solutions as the rogue wave solutions of eq. (1) using direct ansatz and by applying similarity transformations [23,25,31]. In §3, the paper ends with the summary of the results achieved.

2. Similarity transformations and rogue wave solutions

We consider the inhomogeneous chiral NLS equation (1). In order to investigate its analytical rogue wave solutions, we first assume the function $\psi(t,x)$ in the following form [23,25,31]:

$$\psi(t,x)=[\phi_{\rm R}(t,x)+i\phi_{\rm I}(t,x)]{\rm e}^{i\varphi(t,x)}, \tag{2}$$

where $\phi_{\rm R}(t,x)$ and $\phi_{\rm I}(t,x)$ are the real part and the imaginary part of $\psi(t,x)$, respectively. The function $\varphi(t,x)$ is the phase. Introducing eq. (2) into eq. (1), we obtain the following system of equations:

$$\phi_{{\rm R},t}-\varphi_t\phi_{\rm I}+\frac{\beta(t,x)}{2}[\phi_{{\rm I},xx}+2\varphi_x\phi_{{\rm R},x}-\varphi_x^2\phi_{\rm I}+\varphi_{xx}\phi_{\rm R}]+2\sigma(t,x)[\phi_{\rm I}\phi_{\rm R}\phi_{{\rm I},x}+\varphi_x\phi_{\rm I}^3-\phi_{\rm I}^2\phi_{{\rm R},x}+\varphi_x\phi_{\rm I}\phi_{\rm R}^2]=0, \tag{3a}$$

$$-\phi_{{\rm I},t}-\varphi_t\phi_{\rm R}+\frac{\beta(t,x)}{2}[\phi_{{\rm R},xx}-2\varphi_x\phi_{{\rm I},x}-\varphi_x^2\phi_{\rm R}-\varphi_{xx}\phi_{\rm I}]+2\sigma(t,x)[-\phi_{\rm I}\phi_{\rm R}\phi_{{\rm R},x}+\varphi_x\phi_{\rm R}^3+\phi_{\rm R}^2\phi_{{\rm I},x}+\varphi_x\phi_{\rm R}\phi_{\rm I}^2]=0. \tag{3b}$$

Introducing the new variables $\eta(t,x)$ and $\tau(t,x)$, we further utilise the following similarity transformations for the real functions $\phi_{\rm R}(t,x)$, $\phi_{\rm I}(t,x)$ and the phase $\varphi(t,x)$:

$$\begin{aligned}\phi_{\rm R}(t,x)&=A(t)+B(t)P(\eta(t,x),\tau(t)),\\ \phi_{\rm I}(t,x)&=C(t)Q(\eta(t,x),\tau(t)),\\ \varphi(t,x)&=\chi(t,x)+\mu\tau(t),\end{aligned} \tag{4}$$

where μ is a constant and the real variables $\eta(t,x)$, $\chi(t,x)$, $P(\eta,\tau)$, $Q(\eta,\tau)$, $\tau(t)$, $A(t)$, $B(t)$ and $C(t)$ will be determined later. Introducing (4) into (3), we obtain

$$A_t+B_tP+B\eta_tP_\eta+B\tau_tP_\tau-(\chi_t+\mu\tau_t)CQ+\frac{\beta(t,x)}{2}[C\eta_{xx}Q_\eta+C\eta_x^2Q_{\eta\eta}+2\chi_x\eta_xBP_\eta-\chi_x^2CQ+\chi_{xx}(A+PB)]+2\sigma(t,x)([(A+PB)Q_\eta-BQP_\eta]C^2Q\eta_x+\chi_xCQ[(CQ)^2+(A+PB)^2])=0, \tag{5a}$$

$$-C_tQ-C\eta_tQ_\eta-C\tau_tQ_\tau-(\chi_t+\mu\tau_t)(A+PB)+\frac{\beta(t,x)}{2}[B\eta_{xx}P_\eta+B\eta_x^2P_{\eta\eta}-2\chi_x\eta_xCQ_\eta-\chi_x^2(A+PB)-\chi_{xx}(CQ)]+2\sigma(t,x)([(A+PB)Q_\eta-BQP_\eta]C(A+PB)\eta_x+\chi_x(A+PB)[(CQ)^2+(A+PB)^2])=0. \tag{5b}$$

From (5), we obtain the following similarity reductions:

$$\eta_{xx}=0, \tag{6a}$$

$$2\chi_t+\beta(t,x)\chi_x^2=0, \tag{6b}$$

$$\eta_t+\beta(t,x)\eta_x\chi_x=0, \tag{6c}$$

$$2\theta_t+\beta(t,x)\chi_{xx}\theta=0,\quad \theta=A,B,C, \tag{6d}$$

$$(A+PB)Q_\eta-BQP_\eta=0, \tag{6e}$$

$$B\tau_tP_\tau-\mu\tau_tCQ+\frac{\beta(t,x)}{2}C\eta_x^2Q_{\eta\eta}+2\sigma(t,x)\chi_xCQ[(CQ)^2+(A+PB)^2]=0, \tag{6f}$$

$$-C\tau_tQ_\tau-\mu\tau_t(A+PB)+\frac{\beta(t,x)}{2}B\eta_x^2P_{\eta\eta}+2\sigma(t,x)\chi_x(A+PB)[(CQ)^2+(A+PB)^2]=0. \tag{6g}$$

After some calculation, we obtain from eqs (6a)–(6d) the following equations:

$$\eta(t,x)=\alpha(t)x+\delta(t),\quad A(t)=a_0\sqrt{|\alpha(t)|},$$
$$\chi(t,x)=-\frac{\alpha_t}{2\beta(t,x)\alpha(t)}x^2-\frac{\delta_t}{2\beta(t,x)\alpha(t)}x+\chi_0(t),$$
$$B(t)=bA(t),\quad C(t)=cA(t), \tag{7}$$

where a_0, b and c are arbitrary constants, $\alpha(t)$ is the inverse of the wave width, and $\delta(t)$ and $\chi_0(t)$ are free functions of time t.

Proceeding to a more advanced reduction of eqs (6e)–(6g), we end up with a system of partial differential equations with constant coefficients and for that, we require the conditions

$$\tau_t=\frac{\beta(t,x)}{2}\eta_x^2\quad\text{and}\quad\sigma(t,x)=\frac{\beta(t,x)}{4}\frac{\eta_x^2}{\chi_x}GA^{(-2)}.$$

This can generate the constraints for the variable $\tau(t)$ and the nonlinearity coefficient $\sigma(t,x)$:

$$\tau(t)=\frac{1}{2}\int_0^t\alpha^2(u)\beta(u)\,\mathrm{d}u,$$
$$\sigma(t,x)=-\frac{G\beta^2(t,x)\alpha^2(t)}{4a_0^2(\alpha_t x+\delta_t)}. \tag{8}$$

Replacing the expressions of τ_t and $\sigma(t,x)$ given in (8), we obtain from eqs (6f) and (6g) the following coupled system of differential equations with constant coefficients:

$$bP_\tau-\mu cQ+cQ_{\eta\eta}$$
$$+cGQ[(cQ)^2+(1+bP)^2]=0, \tag{9a}$$
$$-cQ_\tau-\mu(1+pB)+bP_{\eta\eta}$$
$$+G(1+bP)[(cQ)^2+(1+bP)^2]=0. \tag{9b}$$

2.1 *First-order rational-like solution*

The first-order rational solution of (9) is obtained following [12,13,23,25]. It follows from (9) that we have the solutions

$$P(\eta,\tau)=-\frac{4}{bH_1(\eta,\tau)}$$

and

$$Q(\eta,\tau)=-\frac{8\tau}{cH_1(\eta,\tau)}$$

with $H_1(\eta,\tau)=1+2\eta^2+4\tau^4$ for $\mu=G=1$. Then, the first-order rational-like solution of (1) can be written as

$$\psi_1(t,x)=a_0\sqrt{|\alpha(t)|}\times\left[1-\frac{4+8i\tau(t)}{1+2[\alpha(t)x+\delta(t)]^2+4\tau^2}\right]\times e^{i[\chi(t,x)+\tau(t)]}, \tag{10}$$

where $\chi(t,x)$ and $\tau(t)$ are given by (7) and (8) respectively. Its intensity is given by the relation

$$|\psi_1(t,x)|^2=a_0^2|\alpha(t)|\frac{\{2[\alpha(t)x+\delta(t)]^2+4\tau^2-3\}^2+64\tau^2}{\{1+2[\alpha(t)x+\delta(t)]^2+4\tau^2\}^2}. \tag{11}$$

Let us choose some free functions of time to exhibit the obtained rational-like solution (10) for the fixed parameters $\alpha_0=1$ and $\chi_0(t)=0$. For simplicity, we choose the functions β and σ as modulated only in time or both in time and space depending on the expected behaviour.

(i) The free functions are chosen as polynomials of time t, i.e. $\alpha(t)=1$, $\beta(t)=t^2/2$. Figures 1a–1d present the behaviour of rational solution (10) for different terms $\delta(t)=t,t^2$, respectively, for which the coefficient $\sigma(t,x)\equiv\sigma(t)$ in eq. (1) is given by

$$\sigma(t)=-\frac{t^6}{16}\quad\text{for }\delta(t)=t, \tag{12a}$$

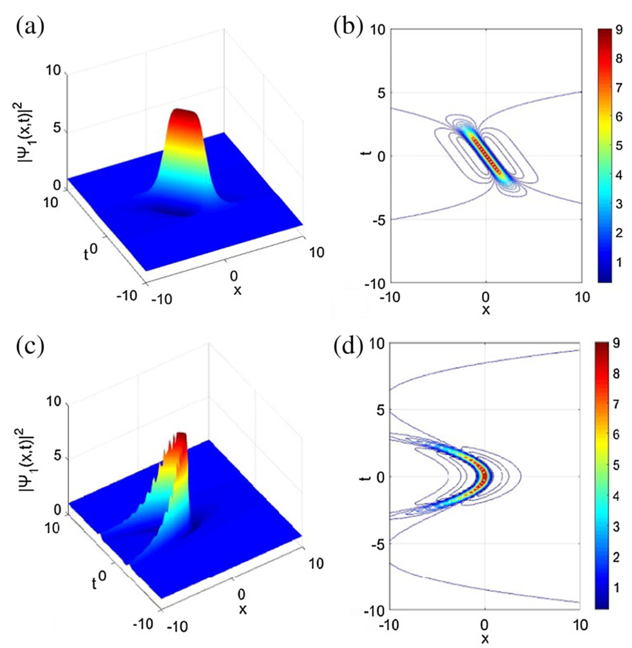

Figure 1. Rogue wave propagations (left column) with dromion structure (**a**), chirped structure (**c**) and the corresponding contour plots (right column), respectively, for the intensity $|\psi_1(t,x)|^2$ of the first-order rational solution (10) for $a_0=1$, $\alpha(t)=1$, $\beta(t)=0.5t^2$. (**a**) and (**b**) $\delta(t)=t$ and (**c**) and (**d**) $\delta(t)=t^2$.

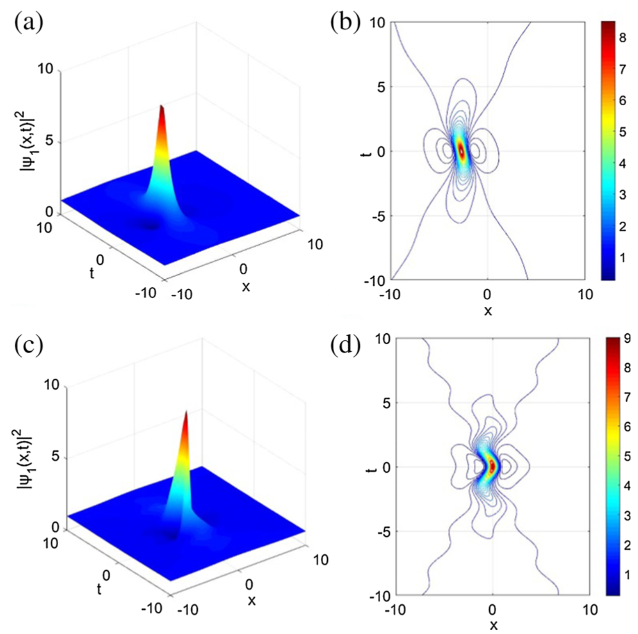

Figure 2. Rogue wave propagations (left column) with dromion structure and contour plots (right column), for the intensity $|\psi_1(t,x)|^2$ of the first-order rational solution (10) for $a_0 = 1$, $\alpha(t) = 1$, $\beta(t) = \cos^2(0.02t)$. (**a**) and (**b**) $\delta(t) = e^{(1+0.1\ \sin(t))}$ and (**c**) and (**d**) $\delta(t) = \sin^2(t)$.

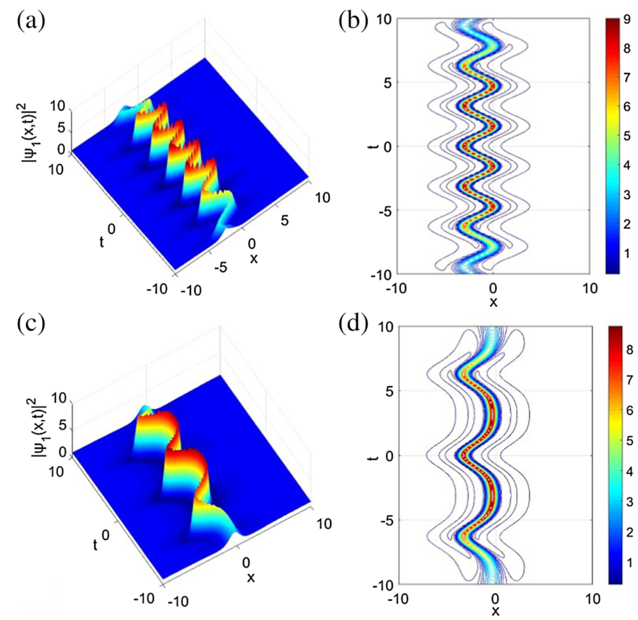

Figure 3. Rogue wave propagations with chirped structure and snaking behaviour (left column) and contour plots (right column), for the intensity $|\psi_1(t,x)|^2$ of the first-order rational solution (10) for $a_0 = 1.0$, $\alpha(t) = 1$, $\beta(t) = \sin^2(0, 08t)$. (**a**) and (**b**) $\delta(t) = 3\cos^2(t)$ and (**c**) and (**d**) $\delta(t) = e^{1.2\ \cos(t)}$.

$$\sigma(t) = -\frac{t^5}{32} \quad \text{for } \delta(t) = t^2. \tag{12b}$$

For $\alpha(t) = 1$, $\delta(t) = t$ and $\beta(t) = t^2/2$, figures 1a and 1b show a rogue wave with the dromion structure. When $\alpha(t) = 1$, $\delta(t) = t^2$ and $\beta(t) = t^2/2$, we have the propagation of a rogue wave with chirped structure in the same symmetric time interval (figures 1c and 1d).

(ii) The free functions are chosen as periodic functions of time such as trigonometric functions, i.e. $\alpha(t) = 1$, $\beta(t) = \cos^2(0.02t)$. Figures 2a–2d show the behaviour of rational solution (12) for different terms $\delta(t) = e^{(1+0.1\sin(t))}$, $\sin^2(t)$, respectively, for which the coefficient $\sigma(t)$ in eq. (1) is given by

$$\sigma(t) = -10\frac{\cos^4(t/50)}{\cos(t)e^{(1+0.1\ \sin(t))}} \quad \text{for } \delta(t) = e^{(1+0.1\sin(t))}, \tag{13a}$$

$$\sigma(t) = -\frac{\cos^4(t/50)}{8\cos(t)\sin(t)} \quad \text{for } \delta(t) = \sin^2(t). \tag{13b}$$

For $\alpha(t) = 1$, $\delta(t) = e^{(1+0.1\sin(t))}$ or $\delta(t) = \sin^2(t)$ and $\beta(t) = \cos^2(0.02t)$, figures 2a–2d show the rogue wave propagations with the dromion structures.

(iii) The free functions are chosen as periodic functions of time such as trigonometric functions, i.e. $\alpha = 1$ and $\beta(t) = \sin^2(0.08t)$. Figures 3a–3d show the behaviour of rational solution (12) for different terms $\delta(t) = 3\cos^2(t)$, $e^{(1.2\cos(t))}$, respectively, for which the coefficient $\sigma(t)$ in eq. (1) is given by

$$\sigma(t) = \frac{\sin^4(2t/25)}{24\sin(t)\cos(t)} \quad \text{for } \delta(t) = 3\cos^2(t), \tag{14a}$$

$$\sigma(t) = \frac{\sin^4(2t/25)}{4.8\sin(t)e^{1.2\cos(t)}} \quad \text{for } \delta(t) = e^{(1.2\cos(t))}. \tag{14b}$$

For $\alpha(t) = 1$, $\delta(t) = 3\cos^2(t)$ or $\delta(t) = e^{(1.2\cos(t))}$ and $\beta(t) = \sin^2(0.08t)$, we have chirping rogue structures with snaking behaviour propagating along the t-axis (figures 3a–3d).

(iv) The free functions are chosen as periodic functions of time such as Jacobian elliptic functions, i.e. $\alpha(t){=}dn(t,m)$ and $\beta(t){=}cn(t,m)$, where m is the module. Figures 4a–4d show the behaviour of rational solution (10) for different terms $\delta(t) = sn(t,m)$, $cn(t,m)$, respectively, for which the coefficient $\sigma(t,x)$ in eq. (1) is given by

$$\sigma(t,x) = -\frac{cn(t,m)dn^2(t,m)}{4\left[-xm^2 sn(t,m) + dn(t,m)\right]} \quad \text{for } \delta(t) = sn(t,m), \tag{15a}$$

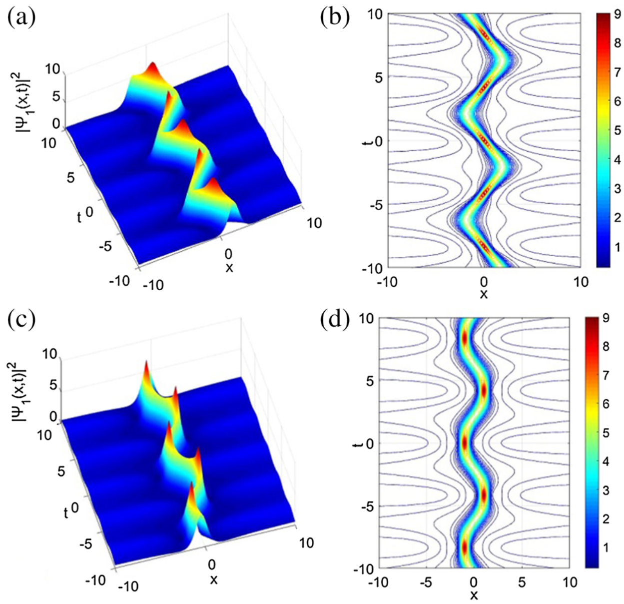

Figure 4. Rogue wave propagations with chirped structure and snaking behaviour (left column) and contour plots (right column), for the intensity $|\psi_1(t,x)|^2$ of the first-order rational solution (10) for $a_0 = 1.0, \alpha(t) = dn(t,m), \beta(t) = cn(t,m)$. (**a**) and (**b**) $\delta(t) = sn(t,m)$ and (**c**) and (**d**) $\delta(t) = cn(t,m)$, with $m = 0.707$.

$$\sigma(t,x) = \frac{cn^2(t,m)dn^2(t,m)}{4sn(t,m)[xm^2cn(t,m)+dn(t,m)]} \quad \text{for } \delta(t) = cn(t,m). \tag{15b}$$

For $\alpha(t)=dn(t,m)$, $\delta(t)=sn(t,m)$ or $\delta(t)=cn(t,m)$ and $\beta(t) = cn(t,m)$, where $m = 0.707$, we have chirping rogue structures with snaking behaviour propagating along the t-axis (figures 4a–4d).

(v) The free functions are chosen as periodic functions of time such as Jacobian elliptic functions, i.e. $\alpha(t) = cn(t,m)$ and $\beta(t) = dn(t,m)$. Figures 5a–5d give the behaviour of rational solution (10) for different terms $\delta(t) = sn(t,m), dn(t,m)$, respectively, for which the coefficient $\sigma(t,x)$ in eq. (1) is given by

$$\sigma(t,x) = -\frac{dn(t,m)cn^2(t,m)}{4[-xsn(t,m)+cn(t,m)]} \quad \text{for } \delta(t) = sn(t,m), \tag{16a}$$

$$\sigma(t,x) = \frac{cn^2(t,m)dn^2(t,m)}{4sn(t,m)[xdn(t,m)+m^2cn(t,m)]} \quad \text{for } \delta(t) = dn(t,m). \tag{16b}$$

For $\alpha(t)=cn(t,m)$, $\delta(t)=sn(t,m)$ or $\delta(t)=dn(t,m)$ and $\beta(t)=dn(t,m)$, where m=0.707, figures 5a–5d show periodic rogue waves with chirped structures.

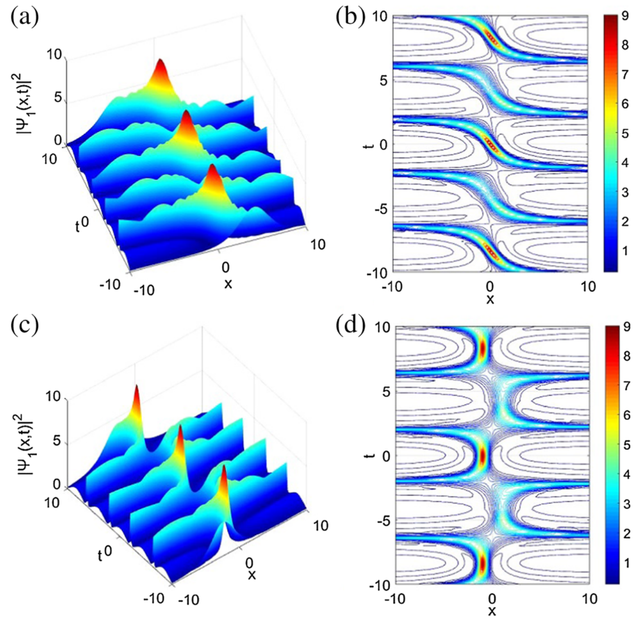

Figure 5. Periodic rogue wave propagations with chirped structure (left column) and contour plots (right column), for the intensity $|\psi_1(t,x)|^2$ of the first-order rational solution (10) for $a_0 = 1.0$, $\alpha(t) = cn(t,m)$, $\beta(t) = dn(t,m)$. (**a**) and (**b**) $\delta(t) = sn(t,m)$ and (**c**) and (**d**) $\delta(t) = dn(t,m)$, with $m = 0.707$.

2.2 *Second-order rational-like solution*

In this case, we have [12,13,23,25]

$$P(\eta,\tau) = \frac{R_2(\eta,\tau)}{bH_2(\eta,\tau)}$$

and

$$Q(\eta,\tau) = \frac{B_2(\eta,\tau)}{cH_2(\eta,\tau)},$$

where

$$\begin{aligned} H_2(\eta,\tau) &= \frac{1}{12}\eta^6 + \frac{1}{2}\eta^4\tau^2 + \eta^2\tau^4 + \frac{2}{3}\tau^6 + \frac{1}{8}\eta^4 \\ &\quad + \frac{9}{2}\tau^4 - \frac{3}{2}\eta^2\tau^2\frac{9}{16}\eta^2 + \frac{33}{8}\tau^2 + \frac{3}{32}, \\ R_2(\eta,\tau) &= -\frac{1}{2}\eta^4 - 6\eta^2\tau^2 - 10\tau^4 \\ &\quad - \frac{3}{2}\eta^2 - 9\tau^2 + \frac{3}{8}, \\ B_2(\eta,\tau) &= -\tau\left[\eta^4 + 4\eta^2\tau^2 + 4\tau^4 - 3\eta^2 + 2\tau^2 - \frac{15}{4}\right]. \end{aligned} \tag{17}$$

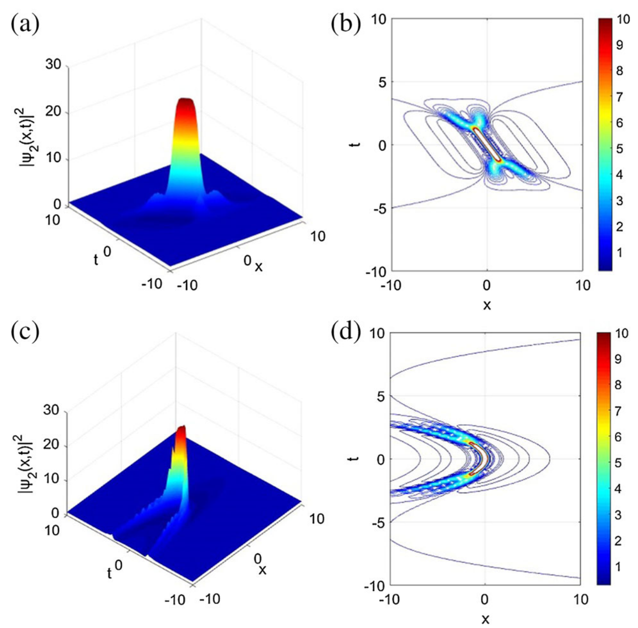

Figure 6. Rogue wave propagations (left column) with dromion structure (**a**), chirped structure (**c**) and the corresponding contour plots (right column), respectively, for the intensity $|\psi_2(t,x)|^2$ of the second-order rational solution (18) for $a_0 = 1.0$, $\alpha(t) = 1$, $\beta(t) = 0.5t^2$. (**a**) and (**b**) $\delta(t) = t$ and (**c**) and (**d**) $\delta(t) = t^2$.

On the basis of similarity transformations, we obtain the second-order rational solution of eq. (1) in the following form:

$$\psi_2(t,x) = a_0\sqrt{|\alpha(t)|}\left[1 - \frac{R_2(\eta,\tau) + i\,B_2(\eta,\tau)}{H_2(\eta,\tau)}\right] \times \mathrm{e}^{i[\chi(t,x)+\tau(t)]}, \tag{18}$$

where the intensity is given by

$$|\psi_2(t,x)|^2 = a_0^2|\alpha(t)|\frac{[H_2(\eta,\tau) + R_2(\eta,\tau)]^2 + B_2^2(\eta,\tau)}{H_2^2(\eta,\tau)}. \tag{19}$$

Here $\eta \equiv \eta(t,x)$, $\tau \equiv \tau(t)$ and $\chi \equiv \chi(t,x)$.

The corresponding behaviour of second-order rational solution is illustrated in figures 6–10 for the fixed parameters $\alpha_0 = 1$ and $\chi_0(t) = 0$. We choose: (i) $\alpha(t) = 1$, $\beta(t) = t^2/2$and $\delta(t) = t, t^2$ in figures 6a–6d, (ii) $\alpha(t) = 1$, $\beta(t) = \cos^2(0.02t)$ and $\delta(t) = \mathrm{e}^{(1+0.1\sin(t))}$, $\sin^2(t)$ in figures 7a–7d, (iii) $\alpha(t) = 1$, $\beta(t) = \sin^2(0.02t)$ and $\delta(t) = 3\cos^2(t)$, $\mathrm{e}^{1.2\cos(t)}$ in figures 8a–8d, (iv) $\alpha(t) = dn(t,m)$, $\beta(t) = cn(t,m)$ and $\delta(t) = sn(t,m)$, $cn(t,m)$ in figures 9a–9d and (v) $\alpha(t) = cn(t,m)$, $\beta(t) = dn(t,m)$, and $\delta(t) = sn(t,m)$, $dn(t,m)$ in figures 10a–10d.

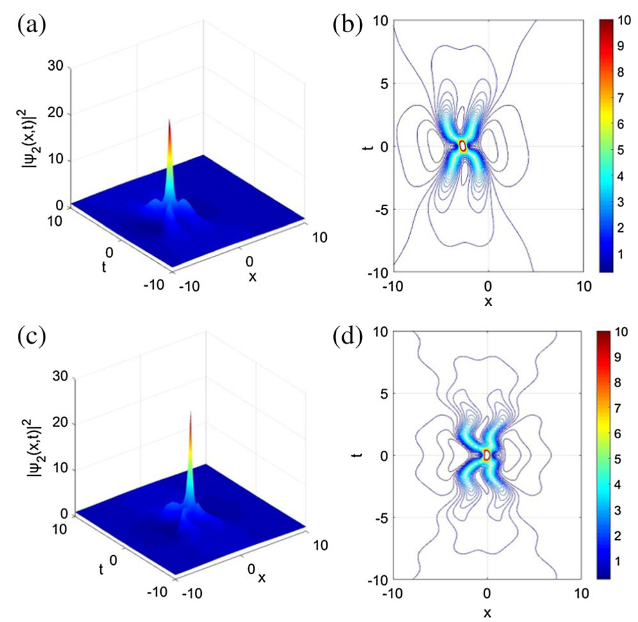

Figure 7. Rogue wave propagations (left column) with dromion structure and contour plots (right column), for the intensity $|\psi_2(t,x)|^2$ of the second-order rational solution (18) for $a_0 = 1.0$, $\alpha(t) = 1$, $\beta(t) = \cos^2(0.02t)$. (**a**) and (**b**) $\delta(t) = \mathrm{e}^{(1+0.1\,\sin(t))}$ and (**c**) and (**d**) $\delta(t) = \sin^2(t)$.

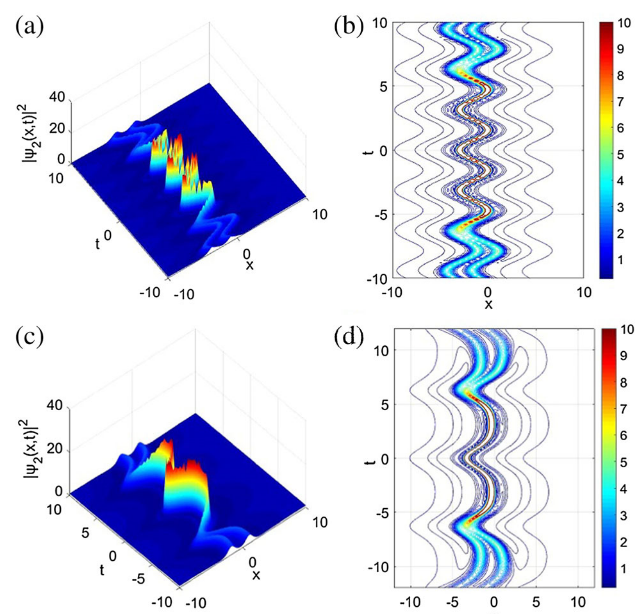

Figure 8. Rogue wave propagations with chirped structure and snaking behaviour (left column) and contour plots (right column), for the intensity $|\psi_2(t,x)|^2$ of the second-order rational solution (18) for $a_0 = 1.0$, $\alpha(t) = 1$, $\beta(t) = \sin^2(0,08t)$. (**a**) and (**b**) $\delta(t) = 3\cos^2(t)$ and (**c**) and (**d**) $\delta(t) = \mathrm{e}^{1.2\,\cos(t)}$.

Figures 6–10 show that although the same functions, as those in figures 1–5, are chosen respectively, the waves have double structures compared to the

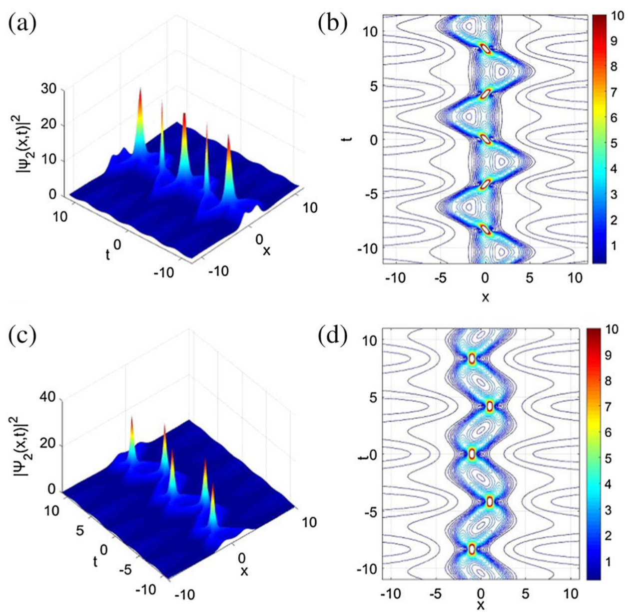

Figure 9. Rogue wave propagations with dromion structure and snaking behaviour (left column) and contour plots (right column), for the intensity $|\psi_2(t,x)|^2$ of the second-order rational solution (18) for $a_0 = 1.0$, $\alpha(t) = dn(t,m)$, $\beta(t) = cn(t,m)$. (**a**) and (**b**) $\delta(t) = sn(t,m)$ and (**c**) and (**d**) $\delta(t) = cn(t,m)$, with $m = 0.707$.

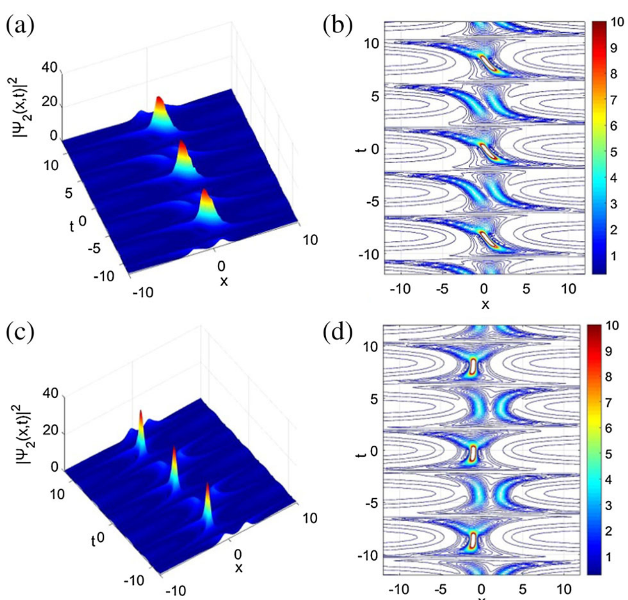

Figure 10. Periodic rogue waves with dromion structure (left column) and contour plots (right column), for the intensity $|\psi_2(t,x)|^2$ of the second-order rational solution (18) for $a_0 = 1.0$, $\alpha(t) = cn(t,m)$, $\beta(t) = dn(t,m)$. (**a**) and (**b**) $\delta(t) = sn(t,m)$ and (**c**) and (**d**) $\delta(t) = dn(t,m)$, with $m = 0.707$.

former single ones, and now with higher amplitude. Similar behaviours have been found in [12]. Furthermore, figures 6–10 depict the fascinating dynamical interactions of modified rogue waves known as rogons [23].

3. Conclusion

In this work, we presented the analytical solutions in terms of rational-like functions of the chiral nonlinear Schrödinger equation with inhomogeneous coefficients using a similarity transformation and a direct ansatz. We obtained the first and second order of rational solutions. In particular, by choosing some free functions of time, we showed that the system can exhibit the snake propagation traces and the fascinating interactions of modified rogue waves known as rogons [23]. The results obtained in this paper are similar to those found on the possible formation mechanisms for the optical rogue wave phenomenon in optical fibres, the oceanic rogue wave phenomenon in the deep ocean and the matter rogue wave phenomenon in Bose–Einstein condensates. To date, chiral effects have been almost completely ignored in efforts to explain the possible formation mechanism of rogue waves in some inhomogeneous NLS equation. The results presented in this paper may be important to describe the possible formation mechanisms for rogue wave phenomenon in the development of quantum mechanics.

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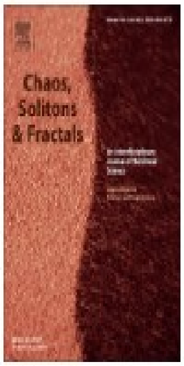

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Exact solitary waves for the 2D Sasa-Satsuma equation

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ABSTRACT

In this work, we investigate exact solitary wave solutions for the 2D Sasa-Satsuma equation by using the envelope transform and Jacobi elliptic function expansion method. We obtain breather, bright and dark solitary wave solutions. It is shown that 2D Sasa-Satsuma equation has very rich dynamical behavior and can describe the propagation of nonlinear waves in many fields of physics.

1. Introduction

In this work, we consider (2+1)-dimensional Sasa-Satsuma equation (SSE)

$$iu_t + \frac{1}{2}(u_{xx} + u_{yy}) + |u|^2u + i\epsilon[u_{xxx} + u_{yyy} + 3((|u|^2)_x u + (|u|^2)_y u) + 6(|u|^2u_x + |u|^2u_y)] = 0, \quad (1)$$

where $u \equiv u(t, x, y)$ is a complex function and t, x and y are three independent variables. The indications in indices represent partial derivatives. The parameter ϵ is a real positive multiplying the higher terms. In (2+1)-dimensions, the term $6\big(|u|^2u_x + |u|^2u_y\big)$ represents the self-steepening, the term related to the self-frequency shift, the term $3\big((|u|^2)_x u + (|u|^2)_y u\big)$ and the 3rd order dispersion term is given by the term $(u_{xxx} + u_{yyy})$. When $\epsilon = 0$, Eq. (1) is reduced to the classical two-dimensional nonlinear Schrödinger equation (NLS) equation. In (1+1)-dimensions, $|u|^2u_y = (|u|^2)_y u = u_{yyy} = 0$. For the (1+1)-dimensional case, the generalized NLS equations with higher-order dispersive and nonlinear terms were presented and their novel solutions were found [1,2]. The effects of higher-order terms have been also considered in [3–8].

The SSE was established by Sasa and Satsuma in 1991 [5]. As one of the multiple extended form of the nonlinear Schrödinger equation, this equation contains additional terms explaining the third-order dispersion, the self-steepening and the self-frequency shift as often found in many fields of physics, for example, ultra-short pulse propagation in optical fibers [9,10] and dynamics of deep water waves [11].

In recent years, the SSE has been the subject of intensive works [12–15]. Various methods such as Darboux transformation [16,17], extended trial equation method and generalized Kudryashov method [18], Riemann problem method [13], inverse scattering transform [5,12], Bäcklund Transformation [19], new auxiliary equation method [20] and unified transform method [21] have been used to investigate the analytical solutions of SSE in one dimension. However, several phenomena and studies in the field of nonlinear science are described in higher dimensions [22]. Malomed et al. [22] reported an interesting study on spatiotemporal solitons related with experimental and theoretical results of higher-dimensional physical systems. Along the same idea, Wang et al. [23] used a new source generalization procedure to obtain determinant solutions of the (2+1)-dimensional SSE. By using a modified direct method, exact solutions of (2+1)-dimensional generalized SSE have been derived [24]. Variable separation solution for 2D dimensional generalised SSE was obtained by using truncated painlevé analysis [25]. The Jacobi elliptic function expansion method [26–35] has several advantages according to other traditional methods. The envelope transform and Jacobi elliptic function expansion method had been well extended to the higher-dimensional generalized NLS equations to investigate localized and periodic wave solutions [36], and rogue wave solutions [37]. It is worthwhile to mention that the proposed method is reliable and effective, and gives more solutions. In this work, we implement the method and obtain exact solitary wave solutions of the 2D SSE. The solutions include bright-dark and breather solitary wave solutions.

The rest of this paper is organized as follows. In Section 2, the general form of solution is obtained by using the method. In Section 3, some solutions are obtained. Finally, some conclusions are given in Section 4.

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2. The general form of solution

To solve the governing model (1), we first consider the following envelope transform of the function $u(t, x, y)$ in the form [16] :

$$u(t,x,y)=\varphi(\eta)e^{i(\xi+\upsilon(\eta))}, \tag{2}$$

where $\upsilon=\upsilon(\eta)$, and $\varphi=\varphi(\eta)$ are real functions, $\xi=kx+hy+lt+\xi_0$, $\eta=ax+by+ct+\eta_0$, and $a, b, c, k, h, l, \epsilon_0, \eta_0$ are constant parameters.

We introduce (2) into (1) and then separate the imaginary and real parts. We then obtain the following system of equations:

$$\begin{aligned}&2(a^3+b^3)\varphi'''+\left[(a^2+b^2)-4\epsilon(ka^2+hb^2)\right]\varphi\upsilon''+2\left[(a^2+b^2)-2\epsilon(ka^2+hb^2)\right]\varphi'\upsilon'+24\epsilon(a+b)\varphi^2\varphi'\\&+2\left[(ka+hb+c+6\epsilon(ka^2+hb^2)\right]\varphi'-2\epsilon(a^3+b^3)\varphi'(\upsilon')^2-4\epsilon(a^3+b^3)\varphi'\upsilon'\upsilon''=0,\end{aligned} \tag{3a}$$

$$\begin{aligned}&\left[(a^2+b^2)-4\epsilon(ka^2+hb^2)\right]\varphi''-2(ka+hb+c)\varphi\upsilon'-(k^2+h^2+2l)\varphi-(a^2+b^2)\varphi(\upsilon')^2-12\epsilon(a+b)\varphi^3\upsilon'\\&+2(1-6\epsilon(k+h))\varphi^3-4\epsilon(a^3+b^3)\varphi''\upsilon'-4\epsilon(a^3+b^3)\varphi'\upsilon''-2\epsilon(a^3+b^3)\varphi\upsilon'''=0,\end{aligned} \tag{3b}$$

where $\varphi'=\frac{d\varphi}{d\eta}$ and $\upsilon'=\frac{d\upsilon}{d\eta}$.

According to Eq. (2), we can write $\upsilon(\eta)=q_0\ln(|\varphi(\eta)|)$, where q_0 is constant to be determined. Replacing $\upsilon(\eta)$ by its expression in each equation of the system (3) above, we obtain the following sytem:

$$\begin{aligned}&2(a^3+b^3)\varphi^3\varphi'''+q_o[(a^2+b^2)-4\epsilon(ka^2+hb^2)]\varphi^3\varphi''-4q_o^2\epsilon(a^3+b^3)\varphi(\varphi')^2\varphi''+q_o(a^2+b^2)\varphi^2(\varphi')^2\\&-2q_o^2\epsilon(a^3+b^3)\varphi(\varphi')^3+24\epsilon(a+b)\varphi^5\varphi'+4q_o^2\epsilon(a^3+b^3)(\varphi')^4+2[ka+hb+c+6\epsilon(ka^2+b^2)]\varphi^3\varphi'=0,\end{aligned} \tag{4a}$$

$$\begin{aligned}&[(a^2+b^2)-4\epsilon(ka^2+hb^2)]\varphi^2\varphi''-[8q_o\epsilon(a^3+b^3)-6q_o\epsilon]\varphi\varphi'\varphi''-q_o^2(a^2+b^2)\varphi(\varphi')^2-12q_o\epsilon(a+b)\varphi^4\varphi'\\&-2q_o(ka+hb+c+\epsilon(a^3+b^3))\varphi^2\varphi'-(k^2+h^2+2l)\varphi^3+2(1-6\epsilon(k+h))\varphi^5=0.\end{aligned} \tag{4b}$$

By means of Jacobi elliptic function expansion method, we introduce an auxiliary function

$$\varphi(\eta)=\sum_{k=0}^{n}a_kF^k(\eta), \tag{5}$$

where a_k, $(k=0,1,2,\dots,n)$ are constants to be determined and $F(\eta)$ satisfies the following differential equation:

$$(F'(\eta))^2=PF^4(\eta)+QF^2(\eta)+R, \tag{6}$$

where $F'=\frac{dF}{d\eta}$. By balancing the highest-order derivative terms and the highest-order nonlinear term in (4b), we obtain $n=1$. Then, (5) has the following form:

$$\varphi(\eta)=a_0+a_1F(\eta). \tag{7}$$

Introducing the expression $\upsilon(\eta)=q_0ln(|\varphi(\eta)|)$ in (2), we have the general form of solution of Eq. (1) in the form:

$$u=e^{i(kx+hy+lt+\mu_0)}\left(\left|\sum_{k=0}^{2}a_kF^k(ax+by+ct+\eta_0)\right|\right)^{iq_0}\left(\sum_{k=0}^{2}a_kF^k(ax+by+ct+\eta_0)\right). \tag{8}$$

It should be noted that, the expression of the function F depends directly on the choice of the value of the parameters P, Q, R. For example, for certain values of these parameters the expressions of the function F are grouped in the following Table 1.

In the limit $r\to0$ or $r\to1$, the expression of the function F is reduced to either a trigonometric function or a hyperbolic function. They are grouped in the following Table 2.

3. Exact solitary wave solutions for the 2D SSE

To obtain solitary wave solutions of equation (1), we substitute (9) and (10) into (7) and we collect the terms with identical powers of $F^k(\eta)$ and $F^k(\eta)\sqrt{(PF^4+QF^2+R)}$. We then equate each coefficient to zero, and obtain a set of algebraic equations for $a, b, c, k, h, l, q_0, a_0, a_1$, namely:

$$4a_1^4P^2q_0^2\epsilon(a^3+b^3)=0,$$

$$8a_0a_1^3P^2q_0^2\epsilon(a^3+b^3)=0,$$

$$12a_1^4P(a^3+b^3)-2a_1^4Pq_0^2\epsilon(a^3+b^3)+24a_1^6\epsilon(a+b)=0,$$

$$36a_0a_1^3P(a^3+b^3)-2a_0a_1^3Pq_0^2\epsilon(a^3+b^3)+120a_0a_1^5\epsilon(a+b)=0,$$

$$a_1^4q_0(a^2+b^2)-4a_1^4PQq_0^2\epsilon(a^3+b^3)+2a_1^4Pq_0[(a^2+b^2)-4\epsilon(ka^2+hb^2)]=0,$$

$$2a_0a_1^3Pq_0(a^2+b^2)-12a_0a_1^3PQq_0^2\epsilon(a^3+b^3)+6a_0a_1^3Pq_0[(a^2+b^2)-4\epsilon(ka^2+hb^2)]=0,$$

$$6a_0^2a_1^2Q(a^3+b^3)-2a_1^4Rq_0^2\epsilon(a^3+b^3)+120a_1^2a_0^4\epsilon(a+b)+6a_0^2a_1^2[ka+hb+c+6\epsilon(ka^2+hb^2)]=0,$$

$$2a_1a_0^3Q(a^3+b^3)-2a_0a_1^3Rq_0^2\epsilon(a^3+b^3)+24a_0^5a_1\epsilon(a+b)+2a_0^3a_1[ka+hb+c+6\epsilon(ka^2+hb^2)]=0,$$

$$a_0^2a_1^2Pq_0(a^2+b^2)+a_1^4Qq_0(a^2+b^2)+a_1^4Qq_0[(a^2+b^2)-4\epsilon(ka^2+hb^2)]+6a_0^2a_1^2Pq_0[(a^2+b^2)-4\epsilon(ka^2+hb^2)]=0,$$

$$2a_1^4Q(a^3+b^3)+36a_0^2a_1^2P(a^3+b^3)-2a_1^4Qq_0^2\epsilon(a^3+b^3)+240a_0^2a_1^4\epsilon(a+b)+2a_1^4[ka+hb+c+6\epsilon(ka^2+hb^2)]=0,$$

$$6a_0a_1Q(a^3+b^3)+12a_0^3a_1P(a^3+b^3)-2a_0a_1^3Qq_0^2\epsilon(a^3+b^3)+240a_0^3a_1^3\epsilon(a+b)+6a_0a_1^3[ka+hb+c+6\epsilon(ka^2+hb^2)]=0,$$

Table 1
Expression of F(η) for the special chosen values of *P*, *Q* and *R* [38,39].

P	*Q*	*R*	*F*
r^2	$-(1+r^2)$	1	*sn*, *cd*
$-r^2$	$2r^2-1$	$1-r^2$	*cn*
-1	$2-r^2$	r^2-1	*dn*
1	$-(1+r^2)$	r^2	*ns*, *dc*
$1-r^2$	$2r^2-1$	$-r^2$	*nc*
r^2-1	$2-r^2$	-1	*nd*
$1-r^2$	$2-r^2$	1	*sc*
$-r^2(1-r^2)$	$2r^2-1$	1	*cd*
1	$2-r^2$	$1-r^2$	*cs*
1	$2r^2-1$	$-r^2(1-r^2)$	*ds*
$-\frac{1}{4}$	$\frac{r^2+1}{2}$	$-\frac{(1-r^2)^2}{4}$	$r\,cn \pm dn$
$\frac{1}{4}$	$\frac{-2r^2+1}{2}$	$\frac{1}{4}$	$ns \pm cs$
$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1-r^2}{4}$	$nc \pm sc$
$\frac{1}{4}$	$\frac{r^2-2}{2}$	$\frac{r^4}{4}$	$ns \pm ds$
$\frac{r^2}{4}$	$\frac{r^2-2}{2}$	$\frac{r^2}{4}$	$sn \pm i\,cn,\ \frac{dn}{\sqrt{1-r^2}\,sn \pm cn}$
$\frac{1}{4}$	$\frac{1-2r^2}{2}$	$\frac{1}{4}$	$r\,cn \pm i\,dn,\ \frac{sn}{1 \pm cn}$
$\frac{r^2}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{sn}{1 \pm dn}$
$\frac{r^2-1}{4}$	$\frac{r^2+1}{2}$	$\frac{r^2-1}{4}$	$\frac{dn}{1 \pm r\,sn}$
$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1-r^2}{4}$	$\frac{cn}{1 \pm sn}$
$\frac{(1-r^2)^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1}{4}$	$\frac{sn}{dn \pm cn}$
$\frac{r^4}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{cn}{\sqrt{1-r^2} \pm dn}$

Table 2
Jacobi elliptic functions in the limit $r \to 0$ and $r \to 1$ [38,39].

	$r \to 0$	$r \to 1$		$r \to 0$	$r \to 1$
$sn(\varepsilon, r)$	$\sin(\varepsilon)$	$\tanh(\varepsilon)$	$dc(\varepsilon, r)$	$\sec(\varepsilon)$	1
$cn(\varepsilon, r)$	$\cos(\varepsilon)$	$sech(\varepsilon)$	$nc(\varepsilon, r)$	$\sec(\varepsilon)$	$\cosh(\varepsilon)$
$dn(\varepsilon, r)$	1	$sech(\varepsilon)$	$sc(\varepsilon, r)$	$\tan(\varepsilon)$	$\sinh(\varepsilon)$
$cd(\varepsilon, r)$	$\cos(\varepsilon)$	1	$ns(\varepsilon, r)$	$\csc(\varepsilon)$	$\coth(\varepsilon)$
$sd(\varepsilon, r)$	$\sin(\varepsilon)$	$\sinh(\varepsilon)$	$ds(\varepsilon, r)$	$\csc(\varepsilon)$	$csch(\varepsilon)$
$nd(\varepsilon, r)$	1	$\cosh(\varepsilon)$	$cs(\varepsilon, r)$	$\cot(\varepsilon)$	$csch(\varepsilon)$

$$2a_0a_1^3Qq_0(a^2+b^2) - 8a_0a_1^3PRq_0^2\epsilon(a^3+b^3) - 4a_0a_1^3Q^2q_0^2\epsilon(a^3+b^3)$$
$$+ 2a_1a_0^3Pq_0[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] + 3a_0a_1^3Qq_0[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0,$$
$$4a_1^4QRq_0^2\epsilon(a^3+b^3) + a_0^2a_1^2Qq_0(a^2+b^2) + a_1^4Rq_0(a^2+b^2) + 3a_0^2a_1^2Qq_0[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0,$$
$$2a_0a_1^2Rq_0(a^2+b^2) - 4a_0a_1^3QRq_0^2\epsilon(a^3+b^3) + a_1a_0^3Qq_0[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0,$$
$$a_0^2a_1^2Rq_0(a^2+b^2) + 4a_1^4R^2q_0^2\epsilon(a^3+b^3) = 0,$$
$$2a_1^5(1-6\epsilon(k+h)) - a_1^3Pq_0^2(a^2+b^2) + 2a_1^3P[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0,$$
$$10a_0a_1^4(1-6\epsilon(k+h)) - a_0a_1^2Pq_0^2(a^2+b^2) + 4a_0a_1^2P[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0,$$
$$72a_0^2a_1^3q_0\epsilon(a+b) + 2a_1^3q_0[ka+hb+c+\epsilon(a^3+b^3)] + a_1^3Q[8q_0\epsilon(a^3+b^3) - 6q_0\epsilon] = 0,$$
$$10a_1a_0^4(1-6\epsilon(k+h)) - a_1^3Rq_0^2(a^2+b^2) - 3a_1a_0^2(k^2+h^2+2l) + a_1a_0^2Q[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0,$$

$$20a_0^2a_1^3(1-6\epsilon(k+h)) - a_1^3(k^2+h^2+2l) - a_1^3Qq_0^2(a^2+b^2) + a_1^3Q[(a^2+b^2) - 4\epsilon(ka^2+hb^2)]$$
$$+ 2a_1a_0^2P[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0,$$
$$2a_0^5(1-6\epsilon(k+h)) - a_0a_1^2Rq_0^2(a^2+b^2) - a_0^3(k^2+h^2+2l) = 0,$$
$$12a_1^5q_0\epsilon(a+b) + 2a_1^3P[8q_0\epsilon(a^3+b^3) - 6q_0\epsilon] = 0,$$
$$48a_0a_1^4q_0\epsilon(a+b) + 2a_0a_1^2P[8q_0\epsilon(a^3+b^3) - 6q_0\epsilon] = 0,$$
$$48a_1^2a_0^3q_0\epsilon(a+b) + 4a_0a_1^2q_0[ka+hb+c+\epsilon(a^3+b^3)] + a_0a_1^2Q[8q_0\epsilon(a^3+b^3) - 6q_0\epsilon] = 0,$$
$$12a_1a_0^4q_0\epsilon(a+b) + 2a_1a_0^2q_0[ka+hb+c+\epsilon(a^3+b^3) = 0,$$
$$20a_1^2a_0^3(1-6\epsilon(k+h)) - a_0a_1^2Qq_0^2(a^2+b^2) - 3a_0a_1^2(k^2+h^2+2l) + 2a_0a_1^2Q[(a^2+b^2) - 4\epsilon(ka^2+hb^2)] = 0.$$

Solving the system above with Maple gives the following solutions :

Case 1:

$$
\begin{aligned}
& a=-b,\, c=-\frac{b(12Pb^3\epsilon-8Pb^2\epsilon k+6a_1^2b\epsilon-12a_1^2\epsilon k+2Pb^2+a_1^2)}{2\epsilon(2Pb^2+3a_1^2)},\\
& h=-\frac{4Pb^2\epsilon k+6a_1^2\epsilon k-2Pb^2-a_1^2}{2\epsilon(2Pb^2+3a_1^2)},\, q_0=0,\, a_0=0,\\
& l=-(32P^2b^4\epsilon^2k^2-32PQa_1^2b^4\epsilon^2+96Pa_1^2b^2\epsilon^2k^2-48Qa_1^4b^2\epsilon^2-16P^2b^4\epsilon k+72a_1^4\epsilon^2k^2\\
& \quad -32Pa_1^2b^2\epsilon k+4P^2b^4-12a_1^4\epsilon k+4Pa_1^2b^2+a_1^4)/\left(8\epsilon^2(2Pb^2+3a_1^2)^2\right),
\end{aligned}
\tag{9}
$$

where b, k, and a_1 are some real free constants.

Case 2:

$$
\begin{aligned}
& h=\frac{8a^2\epsilon^2k-6a^2\epsilon k+6ab\epsilon k-6b^2\epsilon k-2a^2\epsilon-2b^2\epsilon+a^2-ab+b^2}{2\epsilon(-4b^2\epsilon+3a^2-3ab+3b^2)},\\
& a_1=\pm\sqrt{-\frac{P}{2\epsilon}\,(a^2-ab+b^2)},\, q_0=0,\, a_0=0,\\
& c=-\,(-8Qa^3b^2\epsilon^2-8Qb^5\epsilon^2+6Qa^5\epsilon-6Qa^4b\epsilon+6Qa^3b^2\epsilon+6Qa^2b^3-6Qab^4\epsilon+6Qb^5\epsilon\\
& \quad +36a^4\epsilon^2k-36a^3b\epsilon^2k+36ab^3\epsilon^2k-36b^4\epsilon^2k-12a^2b^2\epsilon^2+8a^2b\epsilon^2k-8ab^2\epsilon^2k-12b^4\epsilon^2\\
& \quad +6a^2\epsilon k+6a^2b^2\epsilon-12a^2b\epsilon k-6ab^3\epsilon+12ab^2\epsilon k+6b^4\epsilon-6b^3\epsilon k-2a^2b\epsilon-2b^3\epsilon\\
& \quad +a^2b-ab^2+b^3)/(2\epsilon(-4b^2\epsilon+3a^2-3ab+3b^2)),\\
& k=-\,(-192Qa^4b^2\epsilon^4k+192Qa^3b^3\epsilon^4k-192Qab^5\epsilon^4k+192Qb^6\epsilon^4k+144Qa^6\epsilon^3k-288Qa^5b\epsilon^3k\\
& \quad +288Qa^4b^2\epsilon^3k-288Qa^2b^4\epsilon^3k+288Qab^5\epsilon^3k-144Qb^6\epsilon^3k+48Qa^4b^2\epsilon^3-48Qa^3b^3\epsilon^3\\
& \quad +64Qa^2b^4\epsilon^3-16Qab^5\epsilon^3+16Qb^6\epsilon^3+64Qa^4b\epsilon^4k^2+64Qb^4\epsilon^4k^2-36Qa^6\epsilon^2+75Qa^5b\epsilon^2\\
& \quad -120a^4b^2\epsilon^2+96Qa^3b^3\epsilon^2-72Qa^2b^4\epsilon^2+24Qab^5\epsilon^2-12Qb^6\epsilon^2-96a^4\epsilon^3k^2+96a^3b\epsilon^3k^2\\
& \quad -192a^2b^2\epsilon^3k^2+96ab^3\epsilon^3k^2-96b^4\epsilon^3k^2-32a^4\epsilon^3k+72a^4\epsilon^2k^2-144a^3b\epsilon^2k^2-32a^2b^2\epsilon^3k\\
& \quad +216a^2b^2\epsilon^2k^2+-144ab^3\epsilon^2k^2+72b^4\epsilon^2k^2+40a^4\epsilon^2k-40a^3b\epsilon^2k+64a^2b^2\epsilon^2k-24ab^3\epsilon^2k\\
& \quad +24b^4\epsilon^2k+4a^4\epsilon^2-12a^4\epsilon k+24a^3b\epsilon k+8a^2b^2\epsilon^2-36a^2b^2\epsilon k+24ab^3\epsilon k+4b^4\epsilon^2\\
& \quad -12b^4\epsilon k-4a^4\epsilon+4a^3b\epsilon-8a^2b^2\epsilon+4ab^3\epsilon-4b^4\epsilon+a^4-2a^3b\\
& \quad +3a^2b^2-2ab^3+b^4)/(8\epsilon^2(-4b^2\epsilon+3a^2-3ab+3b^2)^2),
\end{aligned}
\tag{10}
$$

where $P<0$, $(a^2-ab+b^2)>0$ and a, b, k are some real free constants.

Case 3:

$$
\begin{aligned}
& a=Ab,\, a_0=0,\, q_0=0,\, a_1=\pm\sqrt{-\frac{2}{3}\,P},\, k=\frac{B+4}{4B\epsilon},\\
& c=-\,b(64AQb^2\epsilon^3+128Qb^2\epsilon^3+216Ab\epsilon^2h-144Qb^2\epsilon^2+288b\epsilon^3h+144Ab\epsilon^2\\
& \quad +96b\epsilon^3-432b\epsilon^2h-36Ab\epsilon+36A\epsilon h-48b\epsilon^2+48\epsilon^2h+12A\epsilon-18b\epsilon-72\epsilon h\\
& \quad +3A+12\epsilon-9)/(12B\epsilon),\\
& l=-\,(4608AQb^2\epsilon^4h+3072Qb^2\epsilon^5h-5184AQb^2\epsilon^3h-6912Qb^2\epsilon^4h+384AQb^2\epsilon^3+256Qb^2\epsilon^4\\
& \quad +5184Qb^2\epsilon^3h+144AQb^2\epsilon^2+1152A\epsilon^3h^2+192Qb^2\epsilon^3+768\epsilon^4h^2-1296A\epsilon^2h^2\\
& \quad -720Qb^2\epsilon^2-1728\epsilon^3h^2+1296\epsilon^2h^2+72A\epsilon+48\epsilon^2-9A\\
& \quad -12\epsilon-15)/(96(\epsilon^2(24A\epsilon+16\epsilon^2-27A-36\epsilon+27))),
\end{aligned}
\tag{11}
$$

where $P<0$, $A=\left(\frac{1}{2}\pm\frac{1}{6}\sqrt{-27+48\epsilon}\right)$, $B=(3A+4\epsilon-6)$, $\epsilon>\frac{27}{48}$ and b, h are some real free constants.

Combining the values of the parameters mentioned above with (8), we can have the exact solitary wave solutions of Eq. (1) for all cases.

In the case 1.

* For $P=r^2$, $Q=-(1+r^2)$, R=1 and $F(\eta,r)=sn(\eta,r)$ from Table 1, we have the general solution of the form

$u(t,x,y)=a_1\ sn(ax+by+ct+\eta_0)\ e^{i(kx+hy+lt+\xi_0)}$.

In the limit case $r\to 1$, from Table 2, the solitary wave solution is given by

$$u_{1.1}(t,x,y)=a_1\ \tanh(ax+by+ct+\eta_0)\ e^{i(kx+hy+lt+\xi_0)},\tag{12}$$

where $c=-\frac{b(c_1+c_2)}{c_3}$, $h=-\frac{4b^2\epsilon k+6a_1^2\epsilon k-2b^2-a_1^2}{2\epsilon(2b^2+3a_1^2)}$, $a=-b$, $l=-\frac{l_1+l_2+l_3}{l_4}$, with $c_1=12b^3\epsilon-8b^2\epsilon k+6a_1^2b\epsilon$, $c_2=-12a_1^2\epsilon k+2b^2+a_1^2$, $c_3=2\epsilon(2b^2+3a_1^2)$,

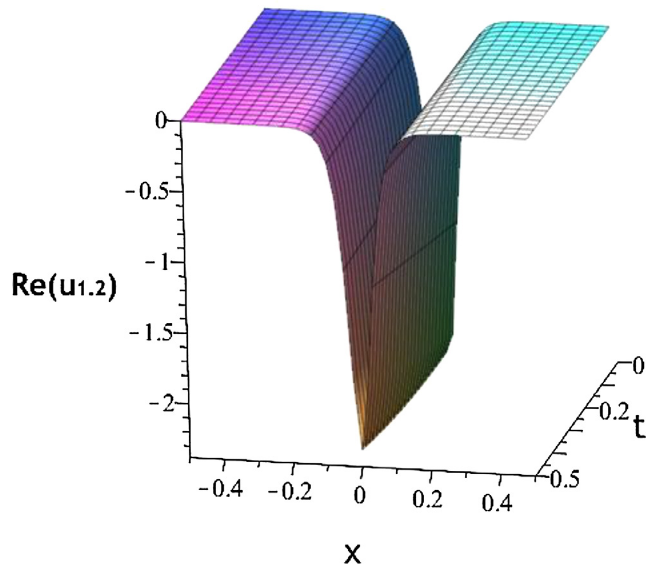

Fig. 1. Behavior of $Re(u_{1.2})$ for the parameter values: $a = k = 0.5$, $b = -0.5$, $a_1 = 0.25$, $\epsilon = 0.01$, $y = 0.5$. The figure shows the solitary wave propagation with a dark structure.

$l_1 = 32b^4\epsilon^2k^2 + 64a_1^2b^4\epsilon 2 + 96a_1^2b^2\epsilon 2k^2$, $l_2 = 96a_1^4b^2\epsilon 2 - 16b^4\epsilon k + 72a_1^4\epsilon 2k2 - 32a_1^2b^2\epsilon k$,
$l_3 = 4b^4 - 12a_1^4\epsilon k + 4a_1^2b^2 + a_1^4$, $l_4 = 8\epsilon^2(2b^2 + 3a_1^2)^2$.

* For $P = -r^2$, $Q = 2r^2 - 1$, $R = 1 - r^2$ and $F(\eta, r) = cn(\eta, r)$ from Table 1, we have the general solution as follows :

$$u(t, x, y) = a_1\ cn(ax + by + ct + \eta_0)\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$, from Table 2, the solitary wave solution is given by

$$u_{1.2}(t, x, y) = a_1\ sech(ax + by + ct + \eta_0)\ e^{i(kx+hy+lt+\xi_0)}, \tag{13}$$

where $c = -\dfrac{b(c_1 + c_2)}{c_3}$, $h = \dfrac{4b^2\epsilon k - 6a_1^2\epsilon k - 2b^2 + a_1^2}{2\epsilon(-2b^2 + 3a_1^2)}$, $a = -b$, $l = -\dfrac{l_1 + l_2 + l_3}{l_4}$, with $c_1 = -12b^3\epsilon + 8b^2\epsilon k + 6a_1^2b\epsilon$, $c_2 = -12a_1^2\epsilon k - 2b^2 + a_1^2$, $c_3 = 2\epsilon(-2b^2 + 3a_1^2)$,

$l_1 = 32b^4\epsilon^2k^2 + 32a_1^2b^4\epsilon 2 - 96a_1^2b^2\epsilon 2k^2$, $l_2 = -48a_1^4b^2\epsilon 2 - 16b^4\epsilon k + 72a_1^4\epsilon 2k2 + 32a_1^2b^2\epsilon k$,
$l_3 = +4b^4 - 12a_1^4\epsilon k - 4a_1^2b^2 + a_1^4$, $l_4 = 8\epsilon^2(-2b^2 + 3a_1^2)^2$.

In Fig. 1, we plot $\mathrm{Re}(u_{1.2})$ for the parameter values: $a = k = 0.5$, $b = -0.5$, $a_1 = 0.25$, $\epsilon = 0.01$, $y = 0.5$. The figure shows the solitary wave propagation with a dark structure. It is important to mention that the dark solitons have no internal freedom and are solutions of the SSE for both defocusing and focusing cases [40].

* For $P = 1$, $Q = -(1 + r^2)$, $R = r^2$ and $F(\eta, r) = ns(\eta, r)$ from Table 1, we have the general solution of the form

$$u(t, x, y) = a_1\ ns(ax + by + ct + \eta_0)\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$, from Table 2, the solitary wave solution is given by

$$u_{1.3}(t, x, y) = a_1\ \coth(ax + by + ct + \eta_0)\ e^{i(kx+hy+lt+\xi_0)}, \tag{14}$$

where $c = -\dfrac{b(c_1 + c_2)}{c_3}$, $h = -\dfrac{4b^2\epsilon k + 6a_1^2\epsilon k - 2b^2 - a_1^2}{2\epsilon(2b^2 + 3a_1^2)}$, $a = -b$, $l = -\dfrac{l_1 + l_2 + l_3}{l_4}$, with $c_1 = 12b^3\epsilon - 8b^2\epsilon k + 6a_1^2b\epsilon$, $c_2 = -12a_1^2\epsilon k + 2b^2 + a_1^2$, $c_3 = 2\epsilon(2b^2 + 3a_1^2)$,

$l_1 = 32b^4\epsilon^2k^2 + 64a_1^2b^4\epsilon 2 + 96a_1^2b^2\epsilon 2k^2$, $l_2 = 96a_1^4b^2\epsilon 2 - 16b^4\epsilon k + 72a_1^4\epsilon 2k2 - 32a_1^2b^2\epsilon k$,
$l_3 = 4b^4 - 12a_1^4\epsilon k + 4a_1^2b^2 + a_1^4$, $l_4 = 8\epsilon^2(2b^2 + 3a_1^2)^2$.

In Fig. 2, we plot $\mathrm{Re}(u_{1.3})$ for the parameter values: $a = -0.5$, $k = b = a_1 = 0.5$, $\epsilon = 1$, $y = 0.01$. The figure shows the wave propagation with a periodic breather structure. This breather is localized in space and oscillates in time. The breather solution of the SSE has been also investigated by Xu et al. [17] using Darboux transformation, just to cite a few.

* For $P = 1$, $Q = (2 - r^2)$, $R = 1 - r^2$ and $F(\eta, r) = cs(\eta, r)$ from Table 1, we have the general solution of the form

$$u(t, x, y) = a_1\ cs(ax + by + ct + \eta_0)\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$, from Table 2, the solitary wave solution is given by

$$u_{1.4}(t, x, y) = a_1\ csch(ax + by + ct + \eta_0)\ e^{i(kx+hy+lt+\xi_0)}, \tag{15}$$

where $c = -\dfrac{b(c_1 + c_2)}{c_3}$, $h = -\dfrac{4b^2\epsilon k + 6a_1^2\epsilon k + 2b^2 - a_1^2}{2\epsilon(2b^2 + 3a_1^2)}$, $a = -b$, $l = -\dfrac{l_1 + l_2 + l_3}{l_4}$, with $c_1 = 12b^3\epsilon - 8b^2\epsilon k + 6a_1^2b\epsilon$, $c_2 = -12a_1^2\epsilon k + 2b^2 + a_1^2$, $c_3 = 2\epsilon(2b^2 + 3a_1^2)$,

$l_1 = 32b^4\epsilon^2k^2 - 32a_1^2b^4\epsilon 2 + 96a_1^2b^2\epsilon 2k^2$, $l_2 = -48a_1^4b^2\epsilon 2 - 16b^4\epsilon k + 72a_1^4\epsilon 2k2 - 32a_1^2b^2\epsilon k$,
$l_3 = 4b^4 - 12a_1^4\epsilon k + 4a_1^2b^2 + a_1^4$, $l_4 = 8\epsilon^2(2b^2 + 3a_1^2)^2$.

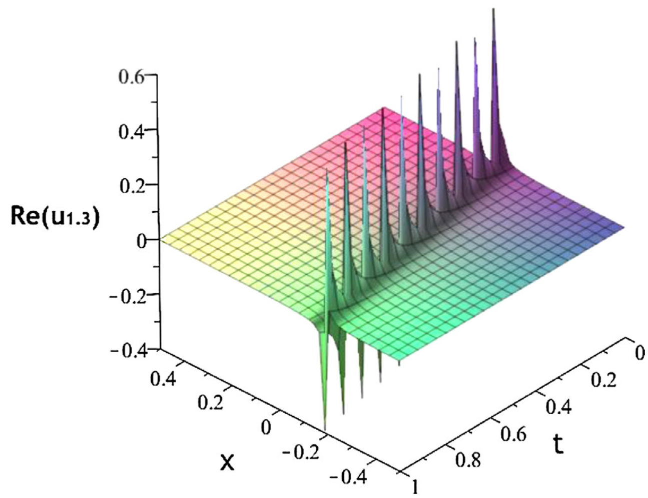

Fig. 2. Behavior of $Re(u_{1.3})$ for the parameter values: $a = -0.5,\ k = b = a_1 = 0.5,\ \epsilon = 1,\ y = 0.01$. The figure shows the wave propagation with a periodic breather structure.

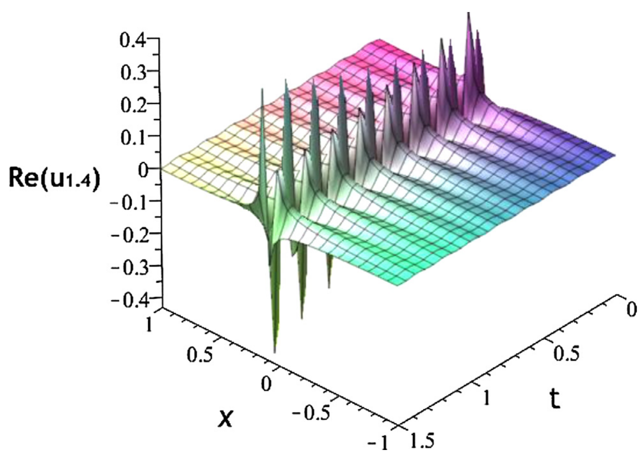

Fig. 3. Behavior of $Re(u_{1.4})$ for the parameter values: $a = k = 2,\ b = -2,\ a_1 = 0.1,\ \epsilon = 0.05,\ y = 0.1$. The figure shows the wave propagation with a breather structure whose amplitude varies in time.

In Fig. 3, we plot $\mathrm{Re}(u_{1.4})$ for the parameter values: $a = k = 2,\ b = -2,\ a_1 = 0.1,\ \epsilon = 0.05,\ y = 0.1$. The figure shows the wave propagation with a breather structure whose amplitude varies in time (standing breather).

* For $P = \frac{1}{4},\ Q = \frac{2r^2+1}{2},\ R = \frac{1}{4}$ and $F(\eta, r) = (ns(\eta, r) \pm cs(\eta, r))$ from Table 1, we have the general solution of the form

$$u(t, x, y) = a_1\ [ns(ax+by+ct+\eta_0) \pm cs(ax+by+ct+\eta_0)]\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$ from Table 2, the solitary wave solution is given by

$$u_{1.5}(t, x, y) = a_1\ [coth(ax+by+ct+\eta_0) \pm csch(ax+by+ct+\eta_0)]\ e^{i(kx+hy+lt+\xi_0)}, \tag{16}$$

where $c = -\frac{b(c_1+c_2)}{c_3},\ h = -\frac{b^2\epsilon k + 6a_1^2\epsilon k - \frac{1}{2}b^2 - a_1^2}{2\epsilon(\frac{1}{2}b^2 + 3a_1^2)},\ a = -b,\ l = -\frac{l_1+l_2+l_3}{l_4}$, with $c_1 = 3b^3\epsilon - 2b^2\epsilon k + 6a_1^2 b\epsilon,\ c_2 = -12a_1^2\epsilon k + \frac{1}{2}b^2 + a_1^2$, $c_3 = 2\epsilon(\frac{1}{2}b^2 + 3a_1^2)$,

$l_1 = 2b^4\epsilon^2k^2 + 4a_1^2b^4\epsilon 2 + 24a_1^2b^2\epsilon 2k^2,\ l_2 = 24a_1^4b^2\epsilon 2 - b^4\epsilon k + 72a_1^4\epsilon 2k2 - 8a_1^2b^2\epsilon k,$

$l_3 = \frac{1}{4}\,b^4 - 12a_1^4\epsilon k + a_1^2b^2 + a_1^4,\ l_4 = 8\epsilon^2\left(\frac{1}{2}\,b^2 + 3a_1^2\right)^2.$

* For $P = \frac{r^2}{4},\ Q = \frac{r^2-2}{2},\ R = \frac{r^2}{4}$ and $F(\eta, r) = (sn(\eta, r) \pm i\ cn(\eta, r))$ from Table 1, we have the general solution as follows :

$$u(t, x, y) = a_1\ [sn(ax+by+ct+\eta_0) \pm i\ cn(ax+by+ct+\eta_0)]\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$, from Table 2, the solitary wave solution is given by

$$u_{1.6}(t, x, y) = a_1\ [\tanh(ax+by+ct+\eta_0) \pm i\ sech(ax+by+ct+\eta_0)]\ e^{i(kx+hy+lt+\xi_0)}, \tag{17}$$

where the coefficients a, c, h and l are the same as in (16).

* For $P = \frac{1}{4},\ Q = \frac{1-2r^2}{2},\ R = \frac{1}{4}$ and $F = \frac{sn(\eta,r)}{1 \pm cn(\eta,r)}$ from Table 1, we have the general solution of the form

$$u(t, x, y) = a_1\ \left(\frac{sn(ax+by+ct+\eta_0)}{1 \pm cn(ax+by+ct+\eta_0)}\right)\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$ from Table 2, the solitary wave solution is given by

$$u_{1.7}(t,x,y) = a_1 \left(\frac{\tanh(ax+by+ct+\eta_0)}{1 \pm sech(ax+by+ct+\eta_0)} \right) e^{i(kx+hy+lt+\xi_0)}, \tag{18}$$

where the coefficients a, c, h and l are the same as in (16).

In the case 2.

$a_1 = \pm\sqrt{-\frac{P}{2\epsilon}\,(a^2-ab+b^2)}$, $(a^2-ab+b^2) > 0$ and $P < 0$.

* For $P=-r^2$, $Q=2r^2-1$, $R=1-r^2$ and $F=cn(\eta,r)$ from Table 1, we have the general solution of the form

$$u(t,x,y) = \pm\sqrt{-r^2\ \frac{(-a^2+ab-b^2)}{2\epsilon}}\ cn(ax+by+ct+\eta_0)\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$ from Table 2, the solitary wave solution is given by

$$u_{2.1} = \pm\sqrt{\frac{(-a^2+ab-b^2)}{2\epsilon}}\ sech(ax+by+ct+\eta_0)\ e^{i(kx+hy+lt+\xi_0)}, \tag{19}$$

where

$$\begin{aligned} c = &-(-8a^3b^2\epsilon^2 - 8b^5\epsilon^2 + 6a^2\epsilon - 6a^4b\epsilon + 6a^3b^2\epsilon + 6a^2b^3\epsilon - 6ab^4\epsilon + 6b^5\epsilon + 36a^4\epsilon^2k - 36a^3b\epsilon^3k \\ &+ 36ab^3\epsilon^2k - 36b^4\epsilon^2k - 12a^2b^2\epsilon^2 + 8a^2b\epsilon^2k - 8ab^2\epsilon^2k - 12b^4\epsilon^2 6a^2\epsilon k + 6a^2b^2\epsilon - 12a^2b\epsilon k \\ &- ab^3\epsilon + 12ab^2\epsilon k + 6b^4\epsilon - 6b^3\epsilon k - 2a^2b\epsilon - 2b^3\epsilon + a^2b - ab^2 + b^3)/(2\epsilon(-4b^2\epsilon + 3a^2 - 3ab + 3b^2)), \end{aligned}$$

$$h = \frac{8a^2\epsilon^2k - 6a^2\epsilon k + 6ab\epsilon k - 6b^2\epsilon k - 2a^2\epsilon - 2b^2\epsilon + a^2 - ab + b^2}{2\epsilon(-4b^2\epsilon + 3a^2 - 3ab + 3b^2)},$$

$$\begin{aligned} l = &-(-192a^4b^2\epsilon^4k + 192a^3b^3\epsilon^4k - 192ab^5\epsilon^4k + 192b^6\epsilon^4k + 144a^6\epsilon^3k - 288a^5b\epsilon^3k + 288a^4b^2\epsilon^3k \\ &- 288a^2b^4\epsilon^3k + 288ab^5\epsilon^3k - 144b^6\epsilon^3k + 48a^4b^2\epsilon^3 - 48a^3b^3\epsilon^3 + 64a^2b^4\epsilon^3 - 16ab^5\epsilon^3 + 16b^6\epsilon^3 \\ &+ 64a^4\epsilon^4k^2 + 64b^4\epsilon^4k^2 - 36a^6\epsilon^2 + 72a^5b\epsilon^2 - 120a^4b^2\epsilon^2 + 96a^3b^3\epsilon^2 - 72a^2b^4\epsilon^2 + 24ab^5\epsilon^2 \\ &- 12b^6\epsilon^2 - 96a^4\epsilon^3k^2 + 96a^3b\epsilon^3k^2 - 192a^2b^2\epsilon^3k^2 + 96ab^3\epsilon^3k^2 - 96b^4\epsilon^3k^2 - 32a^4\epsilon^3k + 72a^4\epsilon^2k^2 \\ &- 144a^3b\epsilon^2k^2 - 32a^2b^2\epsilon^3k + 216a^2b^2\epsilon^2k^2 - 144ab^3\epsilon^2k^2 + 72b^4\epsilon^2k^2 + 40a^4\epsilon^2k - 40a^3b\epsilon^2k \\ &+ 64a^2b^2\epsilon^2k - 24ab^3\epsilon^2k + 24b^4\epsilon^2k + 4a^4\epsilon^2 - 12b^4\epsilon k - 4a^4\epsilon + 4a^3b\epsilon - 8a^2b^2\epsilon + 4ab^3\epsilon \\ &- 4b^4\epsilon + a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4)/(8\epsilon^2(-4b^2\epsilon + 3a^2 - 3ab + 3b^2)^2). \end{aligned}$$

* For $P=-1$, $Q=2-r^2$, $R=r^2-1$ and $F=dn(\eta,r)$ or $P=-\frac{1}{4}$, $Q=\frac{r^2+1}{2}$, $R=\frac{(1-r^2)^2}{4}$ and $F=r\ cn(\eta,r) \pm dn(\eta,r)$ from Table 1, it is clearly seen that in the limit case $r \to 1$, the solitary wave solution are the same as in (19).

In the case 3.

$a_1 = \pm\sqrt{-\frac{2}{3}P}$ and $P < 0$.

* For $P=-r^2$, $Q=2r^2-1$, $R=1-r^2$ and $F=cn(\eta,r)$ from Table 1, we have the general solution of the form

$$u(t,x,y) = \pm\sqrt{\frac{2}{3}r^2}\ cn(ax+by+ct+\eta_0)\ e^{i(kx+hy+lt+\xi_0)}.$$

In the limit case $r \to 1$, from Table 2, the solitary wave solution is given by

$$u_3 = \pm\sqrt{\frac{2}{3}r^2}\ sech(ax+by+ct+\eta_0)\ e^{i(kx+hy+lt+\xi_0)}, \tag{20}$$

where

$$a = Ab,\quad k = \frac{B+4}{4B\epsilon},\quad A = \left(\frac{1}{2} \pm \frac{1}{6}\sqrt{-27+48\epsilon}\right),\quad B = (3A + 4\epsilon - 6),$$

$$\begin{aligned} c = &-(b(64Ab^2\epsilon^3 + 128b^2\epsilon^3 + 126Ab\epsilon^2h - 144b^2\epsilon^2 + 288b\epsilon^3h + 144Ab\epsilon^2 + 96b\epsilon^3 - 432b\epsilon^2h \\ &- 36Ab\epsilon + 36A\epsilon h - 48b\epsilon^2 + 48\epsilon^2h + 12A\epsilon - 18b\epsilon - 72\epsilon h + 3A + 12\epsilon - 9))/(12B\epsilon), \end{aligned}$$

$$\begin{aligned} l = &-(4608Ab^2\epsilon^4h + 3072b^2\epsilon^5h - 5184Ab^2\epsilon^3h - 6912b^2\epsilon^4h + 384Ab^2\epsilon^3 + 256b^2\epsilon^4 \\ &+ 5184b^2\epsilon^3h + 144Ab^2\epsilon^2 + 1152A\epsilon^3h^2 + 192b^2\epsilon^3 + 768\epsilon^4h^2 - 1296A\epsilon^2h^2 - 720b^2\epsilon^2 \\ &- 1728\epsilon^3h^2 + 1296\epsilon^2h^2 + 72A\epsilon + 48\epsilon^2 - 9A - 12\epsilon - 15)/(96\epsilon^2(24A\epsilon + 16\epsilon^2 - 27A - 36\epsilon + 27)), \end{aligned}$$

In Fig. 4, we plot $\mathrm{Re}(u_3)$ for the parameter values: $b=h=0.5$, $\epsilon=10$, $y=1$. The figure shows the wave propagation with a bright structure. Bright solitary waves are the most common solitonic solutions of the SSE and their has been investigated in one-dimension by various authors [15,17,20].

* For $P=-1$, $Q=2-r^2$, $R=r^2-1$ and $F=dn(\eta,r)$ or $P=-\frac{1}{4}$, $Q=\frac{r^2+1}{2}$, $R=\frac{(1-r^2)^2}{4}$ and $F=r\ cn(\eta,r) \pm dn(\eta,r)$. From Table 1, it is clearly seen that in the limit case $r \to 1$, the solitary wave solutions are the same as in (20).

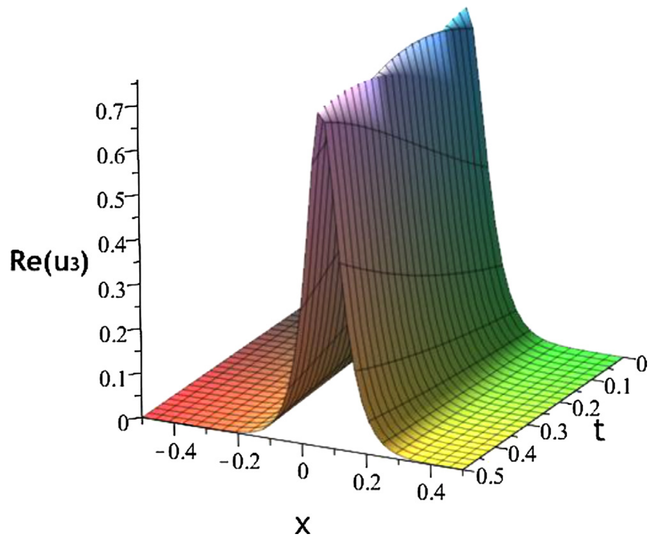

Fig. 4. Behavior of $Re(u_3)$ for the parameter values: $b = h = 0.5,\ \epsilon = 10,\ y = 1$. The figure shows the wave propagation with a bright structure.

4. Conclusion

We have presented the exact solitary wave solutions of the (2+1)-dimensional Sasa-Satsuma equation by using an envelope transform and Jacobi elliptic function expansion method. The method obtains the results directly, quickly, and needs simple algorithms in programming. As result, we obtained a series of solitary-wave solutions including bright-dark and breather solitary wave solutions. The results found in this work may be important in explaining the physical meaning of some (2+1)-dimensional nonlinear equations arising in nonlinear sciences. For physical applications, it is important to discuss about the stability of solutions. The stability procedure can be applied to all solutions of Eq. (1). However, one has to check the stability of each solution step by step since the functions $\rho(\eta)$ are different. In physical situation, usually the envelope of soliton should be slowly varying, thus it is also important to check that the carrier wave frequency is sufficiently large. We can expect that the solutions found in this work might be stable. These subjects are left for future studies.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Alain Mvogo: Conceptualization, Investigation, Methodology, Writing - review & editing. **L. Fernand Mouassom:** Formal analysis, Investigation, Writing - original draft, Writing - review & editing. **F. M. Enyegue A Nyam:** Validation, Writing - original draft. **C. Bioule Mbane:** Supervision, Validation.

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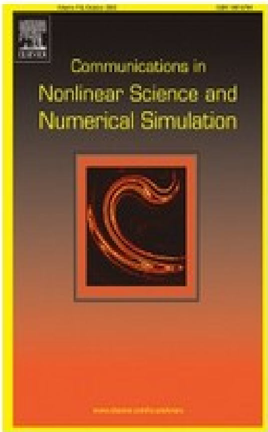

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Short communication

Effects of viscosity and surface tension on soliton dynamics in the generalized KdV equation for shallow water waves

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ABSTRACT

Various theories have been formulated for the study of weakly damped free-surface flows. These theories have been essentially focused on the forces relatively perpendicular to the fluid particle such as pressure forces, while neglecting forces relatively parallel to the fluid particle such as viscosity forces. In this work, with the help of linear approximation applying on the Navier-Stokes equations, we obtain a system of equations for potential flow which includes dissipative effect due to viscosity. The correction due to the viscosity is applied not only to the kinematics boundary condition on the surface, but also to the dynamics condition modeled by Bernoulli's equation. We show that, in the context of wave motion in shallow water, an expansion of the Boussinesq system can be decomposed into a set of coupled equations. The first equation depends only on the surface elevation for the right-moving, while the other equation depends simultaneous on the surface elevation for the right- and left-moving waves. The wave equation corresponding to the pure right-moving has the form of a generalized inhomogeneous Korteweg de Vries (KdV) equation with higher-order nonlinear and dissipative terms. We then investigate the soliton solutions of this equation by using the Hirota's bilinear method. The results show that, both group and phase velocities are a decreasing functions of the viscosity and surface tension parameters, δ and τ, respectively. The width of the soliton increases with the parameters δ and τ. The effects of viscosity on the soliton dynamics are more pronounced and are amplified by the surface tension effects.

1. Introduction

Solitary waves (or solitons) uncovered by the mathematician and naval engineering John Scott Russell, can be defined as localized waves that propagate without change of their amplitude, shape and velocity properties [1] and can be stable against mutual collisions [2]. Nowadays, the solitary wave concept is an intriguing problem which has a great deal of interest in many fields of nonlinear sciences such as optical fibers [3–7], biophysics [8–13], atmosphere [14,15], fluid mechanics [16,17] and shallow water [18–20]. Since the discovery of solitons, the shallow water wave theory has been the subject of very interesting works. For example, the problem of correspondence between symmetries and conservation laws for one-layer shallow water wave systems in the plane flow, axisymmetric flow and dispersive waves have been investigated by Yaşar

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et al. [21]. In the same line, the symmetry groups, symmetry reductions, optimal system, conservation laws and invariant solutions of the shallow water wave equation with nonlocal term have been studied by Rezvan et al. [22]. The mathematical models that describe the formation of solitary waves are based on the simplest nonlinear partial differential equations such as Boussinesq Eq. [23], Kadomtsev-Petviashvili Eq. [24], nonlinear Schrödinger (NLS) Eq. [25] and Korteweg de Vries Eq. [26], just to cite a few.

Among the above equations, one of the most important is the KdV equation because, the study of their solitary wave solutions provide a significant help to explain the physical mechanism of some complex phenomena occurring in many areas of physics [27–34]. Especially in the area of shallow water waves, the KdV equation, which describes the motion of small but finite amplitude waves that propagate in the positive x-direction [35], has been the subject of intensive works [35–38], and experiments [39].

Several phenomena and studies in the field of nonlinear science are described with general coefficients [40] or with higher-order nonlinear and dissipative terms allowing to observe new effects [39]. In that sense, several authors have improved the KdV equation by introducing high-order terms, leading generally to near partially integrable or integrable high-order equations with quasi-soliton solutions [41–45]. For example, in the context of shallow water wave dynamics, Fokou et al. [39,46–48], with the help of the Boussinesq perturbation expansion, derived a higher-order KdV equation. This equation contains many nonlinear and nonlocal terms that describe well the long, small-amplitude, unidirectional wave motion in shallow water with surface tension.

In the same line, the modification of the KdV equation by viscosity has been also investigated [49–54]. In general, the introduction of a dissipative term due to viscosity is done strategically [55]. Chester [49] was the first to introduce an attempt to study the effects of dissipation and dispersion. However, the first true formulation of the KdV equation with viscosity effect has been presented by Ott and Sudan [50]. In that direction, Lundgren [51] and Dias et al. [52] have established a set of equations in the context of both linearity and nonlinearity governing flow taking into account dissipation due to viscosity.

It has been shown that, the KdV equation can be derived using Euler's equation for an incompressible and non viscous fluid, the bottom and surface boundary conditions and the assumption of irrotational flow. In the present work, instead of this approach, we use the dynamics and kinematics boundary conditions both corrected by viscosity effect, the Navier-Stokes equation and the incompressible and irrotational flow assumptions to study the dynamics of viscous flowing shallow water waves. This approach has been shown to better describe the evolution of long shallow wave dynamics [36,56]. We simultaneous combine the dissipation due to viscosity and surface tension effects and show that the model equation can lead to a generalized inhomogeneous KdV equation that includes higher diffusion and instability effects. Such effects have been shown to have an impact on the dynamics of shallow water waves [36,42,56,57].

We go beyond the fifth order evolution equation for long wave dissipative solitons derived by Depassier et al. [56] (for dissipative wave supposed to have a lower amplitude $O(\varepsilon^2)$). We show that the dynamics of the wave amplitude for the unidirectional propagation of long waves over shallow water can have a lower amplitude $O(\varepsilon^3)$, and can be governed by a new generalized inhomogeneous KdV equation for a viscous flowing shallow water waves given by :

$$\begin{aligned} &u_t + u_x + \varepsilon(a_1 uu_x + a_2 u_{3x} + a_3 u_{2x}) + \varepsilon^2(b_1 u^2 u_x + b_2 u_{2x} + b_3 u_x u_{2x} + b_4 uu_{2x} + b_4 u_x^2 + b_5 uu_{3x} \\ &\quad + b_6 u_{5x}) + \varepsilon^3(c_1 u^3 u_x + c_2 u_x^2 + c_2 u^2 u_{2x} + c_3 u_x^3 + c_4 uu_x^2 + c_5 uu_{2x} + c_6 uu_x u_{2x} + c_7 u^2 u_{3x} \\ &\quad + c_8 u_x u_{4x} + c_9 uu_{5x} + c_{10} u_{2x} u_{3x} + c_{11} u_{4x} + c_{12} u_{2x}^2 + c_{13} u_x u_{3x} + c_{14} uu_{4x} + c_{15} u_{7x}) = 0, \end{aligned} \tag{1}$$

where the coefficients $a_i (i = 1, 2, 3)$, $b_i (i = 1, \dots, 6)$ and $c_i (i = 1, \dots, 15)$ depend on some parameters. The parameter ε is a non-dimensional measure of the small wave amplitude. The subscripts of the form $''nx''$ denote derivatives of the order n with respect to x and t, where x an t are the space and time variables.

It is worth noting that, after the degeneration of each coefficient of Eq. (1), this equation can be reduced to some well-known equations. For example, if $b_i (i = 1, \dots, 6) = 0$, $c_i (i = 1, \dots, 15) = 0$, $\varepsilon \neq 0$ and $a_i (i = 1, 2, 3) \neq 0$, Eq. (1) can be reduced to the generalized third-order KdV equations [58–61]. Especially, in the limit case where $\varepsilon = 1, a_1 = 6, a_2 = 1$ and $a_3 = 0$, this equation can be turned to the standard KdV equation [26]. If $c_i (i = 1, \dots, 15) = 0$ and $a_i (i = 1, 2, 3)$, $b_i (i = 1, \dots, 6)$ and ε are real and arbitrary parameters, Eq. (1) can be reduced to the fifth-order KdV equations, including the Lax equation [62], the Sawada-Kotera equation [63,64] and the Kaup-Kupershmidt equation [65], just to cite a few. Similarly, if $a_i (i = 1, 2, 3)$, $b_i (i = 1, \dots, 6)$, $c_i (i = 1, \dots, 15)$ and ε are real and arbitrary parameters, we obtain the seventh-order KdV equations such as the one introduced by Pomeau et al. [66]. It is therefore clear that Eq. (1) can be similar to many equations well-known to the scientific community and can describes many physical situations. This equation can therefore better describes the dynamics of nonlinear waves in shallow water.

The rest of our work is organized as follows. In Section 2, we formulate the shallow water problem in non-dimensional variables. The correction of the kinematics and boundary conditions by the viscosity is illustrated. In Section 3, the Boussinesq equations and the linear approximation of the Navier-Stokes equations are briefly described, from which two types of equations will be constructed. The first is the generalized inhomogeneous KdV equation depending only on the surface elevation for the right-moving viscous shallow water waves and containing higher-order nonlinear and dissipative terms corrections. The second is an equation that depends simultaneously on the surface elevation for the right- and left-moving wave. In Section 4, the soliton solutions are investigated by using the Hirota's bilinear method and a discussion on the effects of some parameters on the soliton dynamics are shown. Section 5 concludes the work.

2. Mathematical formulation

This section deals with the mathematical formulation of the shallow water wave problems. We illustrate the correction of the kinematics, dynamics and boundary conditions by the viscosity in non-dimensional variables. For this, we consider a layer of an incompressible and viscous fluid, being above a horizontal plane located at altitude $z = 0$ with the mean depth the parameter h. We choose the coordinates such that the movement of the fluid is two-dimensional. The properties of the system are independent of the y coordinate and the component of the velocity along y is zero. Then we assume that the velocity field inside the fluid is given by $\overrightarrow{v}(u, 0, w)$, where $u \equiv u(x, z, t)$, $w \equiv w(x, z, t)$ and that the surface equation is given by $z(x, t) = h + \eta(x, t)$, where the function $\eta(x, t)$ is the surface elevation. The fluid is subjected to the action of forces having perpendicular and parallel components. The perpendicular components are the pressure P_a and the force $\overrightarrow{f}$. The pressure P_a is variable and uniform in space and it dues to the gas column above the fluid. The force $\overrightarrow{f} = \rho\overrightarrow{g}$ per unit volume is due to gravity, where ρ is the density of the fluid and g is the gravity field. For the parallel component, we consider the viscosity force $\overrightarrow{F} = \mu\Delta\overrightarrow{v}$ per unit volume due to the friction of the fluid slices against each other, where μ is the dynamics viscosity.

The Navier-Stokes equation governing the dynamics of the system can be written in the form

$$w_t + uw_x + ww_z = -\frac{1}{\rho}P - g - 2\nu w_{zz},$$
$$P = \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} \tag{2}$$

where P is the pressure in the fluid, σ is the surface tension and $\nu = \frac{\mu}{\rho}$ is the kinematics viscosity. Eq. (2) has been obtained using the fundamental principe of dynamics to a fluid particle, subject to its weight, the volumetric force of viscosity and the forces of pressure.

The boundary condition at the bottom indicates that the fluid is bounded underneath by a rigid surface such that

$$u = 0 \quad \text{and} \quad w = 0, \quad \text{for} \quad z = 0. \tag{3}$$

The procedure for establishing the kinematic boundary condition at the surface corrected by viscosity has been argued demonstrated in [52]. It is given by

$$w = \eta_t + u\eta_x - 2\nu\eta_{xx}, \quad \text{for} \quad z = h + \eta(x, t). \tag{4}$$

The physical condition at the surface is given by

$$P_a - P = 0, \quad \text{for} \quad z = h + \eta(x, t). \tag{5}$$

These equations must be completed by the mass conservation requirement

$$u_x + w_z = 0. \tag{6}$$

The hypothesis that the flow is irrotational makes it possible to affirm the existence of a velocity potential such that, $v(x, z, t) = \nabla\phi(x, z, t)$, where the horizontal and vertical velocity components are given by $u = \phi_x$ and $w = \phi_z$. The symbol ∇ is the Nabla operator. It should be noted that, the Laplace equation $\nabla^2\phi = 0$ verifies the velocity potential throughout the area occupied by the fluid.

The kinematics condition can then be written as:

$$\eta_t + \phi_x\eta_x - 2\nu\eta_{xx} - \phi_z = 0, \quad z = h + \eta(x, t). \tag{7}$$

Eq. (2) can now be integrated to yield the dynamics boundary condition

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta + 2\nu\phi_{zz} - \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0, \quad z = h + \eta(x, t). \tag{8}$$

The characteristic time $t_0 = L/c_0$ used to measure the time is defined from the characteristics length L in the x direction and speed $c_0 = \sqrt{gh}$ of the high wavelength waves. Since the study deals with small amplitude and long waves, it is preferable to scale the variables in order to avoid any ambiguity corresponding to a different physical situation. Thus, the variables will be scaled in such a way that

$$t' = \frac{t}{t_0}, \quad x' = \frac{x}{L}, \quad z' = \frac{z}{h}, \eta' = \frac{\eta}{A}, \quad \phi' = \frac{\phi h}{LA\sqrt{gh}}, \tag{9}$$

where A is a typical amplitude of a surface elevation η'.

Introducing Eq. (9) in Eqs. (3), (6), (7) and (8), the equations for a fluid are written in a non-dimensionalized form such that

$$\beta\phi'_{2x'} + \phi'_{2z'} = 0, \quad 0 \le z' \le 1 + \alpha\eta', \tag{10}$$

$$\phi'_{z'} = 0, \quad z' = 0, \tag{11}$$

$$\eta'_{t'} + \alpha\eta'_{x'}\phi'_{x'} - \frac{1}{\beta}\phi'_{z'} - \beta\delta\eta'_{x'x'} = 0, \; z' = 1 + \alpha\eta', \tag{12}$$

$$\phi'_{t'} + \frac{1}{2}\alpha\left(\phi'^2_{2x'} + \frac{1}{\beta}\phi'^2_{2z'}\right) - \frac{\beta\tau\eta'_{2x'}}{(1+\alpha^2\beta\eta'^2_{x'})^{\frac{3}{2}}} + \eta' + \delta\phi'_{2z'} = 0, \quad z' = 1 + \alpha\eta'. \tag{13}$$

The parameter $\tau = \frac{\sigma}{\rho g h^2}$ is the Bond number and $\delta = \frac{2L\nu}{c_0 h^2}$, where σ and ν are the surface tension and viscosity coefficients, respectively. The amplitude parameter $\alpha = \frac{A}{h}$, measures the ratio of wave amplitude to undisturbed fluid depth. The wavelength parameter $\beta = (\frac{h}{L})^2$, measures the square of the ratio of fluid depth to wave length. We focus our attention on low amplitude, weakly nonlinear waves in shallow and viscous water, then α and β can be considered as small.

Eqs. (10) -(13) can be rewritten in the following forms

$$\beta\phi_{2x} + \phi_{2z} = 0, \quad 0 \le z \le 1 + \alpha\eta, \tag{14}$$

$$\phi_z = 0, \quad z = 0, \tag{15}$$

$$\eta_t + \alpha\eta_x\phi_x - \frac{1}{\beta}\phi_z - \beta\delta\eta_{xx} = 0, \quad z = 1 + \alpha\eta, \tag{16}$$

$$\phi_t + \frac{1}{2}\alpha\left(\phi^2_{2x} + \frac{1}{\beta}\phi^2_{2z}\right) - \frac{\beta\tau\eta_{2x}}{(1+\alpha^2\beta\eta^2_x)^{\frac{3}{2}}} + \eta + \delta\phi_{2z} = 0, \quad z = 1 + \alpha\eta, \tag{17}$$

where the subscription ($'$) have been omitted for the reason of simplicity.

3. Derivation of equations for shallow water waves

In this section, we formulate new generalized shallow water wave equations both for unidirectional and bidirectional wave motion taking into account the effects of viscosity and surface tension. For this purpuse, we use the approximation of the velocity potential to formulate the Boussinesq system and derive the generalized KdV equations for unidirectional waves motion with the effects of surface tensions and viscosity. On the other hand, by using the non-uniqueness of the decompositions of this Boussinesq system, we derive generalized equations for bidirectional waves in shallow water, which also includes the surface tension and viscosity effects.

3.1. Formulation of the Boussinesq system

The standard procedure in shallow water theory has a serious advantage in the sense that, by writing the velocity potential function $\phi(x, z, t)$ in the form of power series of z, it does not compromise the requirements of satisfying both the field equation and the boundary conditions of the bottom and free surface [44,45]. Thus, we start by setting the velocity potential ϕ as a formal expansion

$$\phi(x, z, t) = \sum_{i=0}^{\infty} f_i(x, t)z^i. \tag{18}$$

We assume that Eq. (18) formally satisfies the Laplace equation given by Eq. (14). We obtain the recurrent relation $(i+1)(i+2)f_{i+2}(x, t) = -\beta(f_i(x, t))_{2x}$. We set $g(x, t) = f_0(x, t)$, which indicates the velocity potential at the bottom $z = 0$ and we obtain

$$f_{2k}(x, t) = \frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}g(x, t)}{\partial x^{2k}}. \tag{19}$$

Using Eq. (15), the power series expansion used for the approximation of the velocity potential ϕ is given by [39,44,45]

$$\phi(x, z, t) = \sum_{k=0}^{\infty} z^{2k}\frac{(-1)^k\beta^k}{(2k)!}\frac{\partial^{2k}g(x, t)}{\partial x^{2k}}. \tag{20}$$

In this work, the wave regime is considered for the classical Stokes number $S = \alpha/\beta = 1$, so that the amplitude parameter α and the wavelength parameter β may be treated on an equal footing $\alpha = \beta = O(\varepsilon)$.

By limiting the development of Eq. (20) to order 3, we obtain the following equation

$$\phi(x,z,t) = g(x,t) - \frac{1}{2}\ \beta z^2 \frac{\partial^2 g(x,t)}{\partial x^2} + \frac{1}{24}\ \beta^2 z^4 \frac{\partial^4 g(x,t)}{\partial x^4} - \frac{1}{720}\ \beta^3 z^6 \frac{\partial^6 g(x,t)}{\partial x^6}. \tag{21}$$

By introducing Eq. (21) into the kinematics condition and using the relation $z = 1 + \alpha\eta(x,t)$, we obtain the following equation in which $w(x,t) = \dfrac{\partial g(x,t)}{\partial x}$ and all terms greater than $O(\varepsilon^3)$ have been neglected.

$$\begin{aligned}&\eta_t + w_x + \varepsilon\left(\eta_x w + \eta w_x - \frac{1}{6}\ w_{3x} - \delta\eta_{2x}\right) + \varepsilon^2\left(-\frac{1}{2}\ \eta_x w_{2x} - \frac{1}{2}\ \eta w_{3x} + \frac{1}{120}\ w_{5x}\right)\\ &+\varepsilon^3\left(\frac{1}{24}\ \eta_x w_{4x} - \frac{1}{2}\ \eta^2 w_{3x} - \eta\eta_x w_{2x} + \frac{1}{24}\eta w_{5x} - \frac{1}{5040} w_{7x}\right) = 0.\end{aligned} \tag{22}$$

Concerning the dynamics condition, a small transformation is necessary. We derive the equation with respect to x and assuming that $\dfrac{d(.)}{dx} = \dfrac{\partial(.)}{\partial x} + \alpha\eta_x \dfrac{\partial(.)}{\partial z}$, the dynamics condition can be written as follows

$$\phi_{xt} + \alpha\eta_x\phi_{zt} + \alpha\left(\phi_x\phi_{xx} + \frac{1}{\beta}\ \phi_z\phi_{xz}\right) + \eta_x + \delta\phi_{xzz} + \alpha\delta\eta_x\phi_{zzz} - \beta\tau\eta_{xxx} = 0. \tag{23}$$

Proceeding in the same way as in the case of kinematics condition above, we obtain the following equation

$$\begin{aligned}&\eta_x + w_t + \varepsilon\left(ww_x - \frac{1}{2}\ w_{2xt} - \tau\eta_{3x} - \delta w_{2x}\right) + \varepsilon^2\left(-\eta_x w_{xt} + \frac{1}{2}\ w_x w_{2x} - \eta w_{2xt} - \frac{1}{2}\ ww_{3x} + \frac{1}{24}\ w_{4xt} + \frac{1}{2}\ \delta w_{4x}\right)\\ &+\varepsilon^3\left(\eta_x w_x^2 - \eta\eta_x w_{xt} - \eta_x ww_{2x} + \frac{1}{6}\ \eta_x w_{3xt} - \frac{1}{2}\ \eta^2 w_{2xt} - \eta ww_{3x} + \frac{1}{12}\ w_{2x}w_{3x}\right.\\ &\left.-\frac{1}{8}\ w_x w_{4x} + \eta w_x w_{2x} + \frac{1}{6}\ \eta w_{4xt} + \frac{1}{24}\ ww_{5x} - \frac{1}{720}\ w_{6xt} + \delta\eta w_{4x} - \frac{1}{24}\ \delta w_{6x} + \delta\eta_x w_{3x}\right) = 0,\end{aligned} \tag{24}$$

where the notation for the small parameters β and α have been changed to ε and $w(x,t) = \frac{\partial g(x,t)}{\partial x}$ is the scaled horizontal velocity at the bottom of the fluid.

It is worth noting that Eqs. (22) and (24) constitute the generalized Boussinesq system. To the best of our knowledge, these coupled Boussinesq equations include both viscosity and surface tension effects are derived for the first time in the literature. When $\delta = 0$, these equations can be similar to those recently obtained by Fokou et al. [39] in the absence of viscosity effect.

3.2. Equations for unidirectional waves

In the context of shallow water, it has been shown that, the KdV equation has been first introduced as a model for the unidirectional propagation of long waves over shallow water [67]. To derive the equations for unidirectional waves, we reduce the two Eqs. (22) and (24) into a single dependent equation of η. To do this, we follow the work by Whitham [67] and assume that the relationship between the horizontal velocity at the mean height w and the elevation η is given by $w = \eta + \varepsilon\psi(\eta)$ where ψ is a function which will be determined later. The right wave hypothesis imposes that at the lowest order of α and β, the Boussinesq system is reduced to $\eta_t + w_x = 0$ and $w_t + \eta_x = 0$. Starting from there, it shows that w and η satisfies the linear wave equation $\eta_{tt} - \eta_{xx} = 0$, which describes waves moving in one direction. More precisely, for the lowest order $O(\varepsilon = 0)$, $w = \eta$ and $\eta_x + \eta_t = 0$. We assume w in the form

$$w = \eta + \varepsilon A(\eta) + \varepsilon^2 B(\eta) + \varepsilon^3 C(\eta), \tag{25}$$

where the coefficients $A(\eta)$, $B(\eta)$ and $C(\eta)$ are arbitrary functions. These coefficients will subsequently be determined by introducing Eq. (25) into equations (22) and (24) of the Boussinesq system, each time limited to the order of the small parameter corresponding to the coefficient to be determined.

To determine the coefficient $A(\eta)$ corresponding to the first order of the small parameter ε, Eq. (25) is first introduced into Eqs. (22) and (24). Neglecting the terms of higher order than $O(\varepsilon)$ in each equation, we obtain the following system

$$\begin{aligned}&\eta_t + \eta_x + \varepsilon\left(A_x + 2\eta\eta_x - \frac{1}{6}\ \eta_{3x} - \delta\eta_{2x}\right) + O(\varepsilon^2) = 0,\\ &\eta_t + \eta_x + \varepsilon\left(A_t + \eta\eta_x - \frac{1}{6}\ \eta_{2xt} - \tau\eta_{3x} - \delta\eta_{2x}\right) + O(\varepsilon^2) = 0.\end{aligned} \tag{26}$$

We look for the function A as it corresponds to the two equations (26) up to the first order of ε. Using the lower order equation $\eta_t + \eta_x = 0$, it is easy to see that all the t-derivatives terms of η can be expressed in terms of the x-derivatives such that $\frac{\partial}{\partial t} = -\frac{\partial}{\partial x}$. This allows us to reduce the system (26) into a single equation as follows

$$A_x + 2\eta\eta_x - \frac{1}{6}\ \eta_{3x} - \delta\eta_{2x} = A_t + \eta\eta_x + \frac{1}{2}\ (1-2\tau)\eta_{3x} - \delta\eta_{2x}. \tag{27}$$

A common approach is to use the lowest order relation between the time and space derivatives ($A_t = -A_x$) in Eq. (27). After integration, we finally obtain

$$A(\eta) = -\frac{1}{4}\,\eta^2 + \frac{1}{6}\,(2-3\tau)\eta_{2x}, \tag{28}$$

$$w = \eta + \varepsilon\Big[-\frac{1}{4}\,\eta^2 + \frac{1}{6}\,\Big(2-3\tau\Big)\eta_{2x}\Big] + \varepsilon^2 B(\eta) + \varepsilon^3 C(\eta), \tag{29}$$

$$\eta_t + \eta_x + \varepsilon\Big[\frac{3}{2}\,\eta\eta_x + \frac{1}{6}\,\Big(1-3\tau\Big)\eta_{3x} - \delta\eta_{2x}\Big] = 0. \tag{30}$$

Setting $\delta = 0$, $X = \sqrt{\frac{3}{2}}(x-t)$ and $T = \frac{1}{4}\sqrt{\frac{3}{2}}(\varepsilon t)$ Eq. (30) can be reduced to the standard form of the KdV equation

$$\eta_T + 6\eta\eta_X + \eta_{3X} = 0. \tag{31}$$

To determine the coefficient $B(\eta)$ corresponding to the second order of the small parameter ε, Eq. (29) is introduced into Eqs. (22) and (24). Neglecting the terms of higher order than $O(\varepsilon^2)$ in each equation, all the t-derivatives of η are replaced by their expressions through the x-derivatives using the lowest order equation, namely (30). Upon the substitution, we obtain the following equations

$$\begin{aligned}
&\eta_t + \eta_x + \varepsilon\Big[\frac{3}{2}\,\eta\eta_x + \frac{1}{6}\,\Big(1-3\tau\Big)\eta_{3x} - \delta\eta_{2x}\Big] + \varepsilon^2\Big[B_x - \frac{3}{4}\,\eta^2\eta_x + \Big(\frac{1}{12} - \frac{1}{2}\,\tau\Big)\eta_x\eta_{2x} - \Big(\frac{1}{12} + \frac{1}{2}\,\tau\Big)\eta\eta_{3x}\\
&+\Big(-\frac{17}{360} + \frac{1}{12}\,\tau\Big)\eta_{5x}\Big] + O(\varepsilon^3) = 0,\\
&\eta_t + \eta_x + \varepsilon\Big[\frac{3}{2}\,\eta\eta_x + \frac{1}{6}\,\Big(1-3\tau\Big)\eta_{3x} - \delta\eta_{2x}\Big] + \varepsilon^2\Big[B_t + \frac{1}{2}\,\delta\eta_x^2 + \frac{1}{2}\,\delta\eta\eta_{2x} - \frac{1}{2}\,\delta\eta_{2x} + \Big(\frac{11}{6} + \frac{7}{4}\,\tau\Big)\eta_x\eta_{2x}\\
&+\frac{11}{12}\,\eta\eta_{3x} + \frac{1}{4}\Big(\frac{11}{18} - \tau - \tau^2\Big)\eta_{5x}\Big] + O(\varepsilon^3) = 0.
\end{aligned} \tag{32}$$

The solution of the function $B(\eta)$ can be expressed in terms of η and its x-derivatives. As a result, we obtain

$$B(\eta) = \frac{1}{8}\,\eta^3 - \frac{1}{4}\,\delta\eta_x + \frac{1}{16}\,(3+7\tau)\eta_x^2 + \frac{1}{4}\,\delta\eta\eta_x + \frac{1}{4}\,(2+\tau)\ \eta\eta_{2x} + \frac{1}{120}\,(12-20\tau-15\tau^2)\eta_{4x}. \tag{33}$$

Inserting Eq. (33) into Eq. (29) and the first equation of Eq. (32), yields in second order $O(\varepsilon^2)$, the equation

$$\begin{aligned}
w = \eta &+ \varepsilon\Big[-\frac{1}{4}\,\eta^2 + \frac{1}{6}\,\Big(2-3\tau\Big)\eta_{2x}\Big] + \varepsilon^2\Big[\frac{1}{8}\,\eta^3 - \frac{1}{4}\,\delta\eta_x + \frac{1}{16}\,(3+7\tau)\eta_x^2 + \frac{1}{4}\,(2+\tau)\ \eta\eta_{2x}\\
&+\frac{1}{4}\,\delta\eta\eta_x + \frac{1}{120}\,(12-20\tau-15\tau^2)\eta_{4x}\Big] + \varepsilon^3 C(\eta),
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\eta_t + \eta_x + \varepsilon\Big[\frac{3}{2}\,\eta\eta_x + \frac{1}{6}\,\Big(1-3\tau\Big)\eta_{3x} - \delta\eta_{2x}\Big] + \varepsilon^2\Big[-\frac{3}{8}\,\eta^2\eta_x - \frac{1}{4}\,\delta\eta_{2x} + \frac{1}{24}\,\Big(23+15\tau\Big)\eta_x\eta_{2x}\\
&+\frac{1}{4}\,\delta\eta\eta_{2x} + \frac{1}{12}\,\Big(5-3\tau\Big) + \frac{1}{4}\,\delta\eta_x^2 + \frac{1}{360}\,\Big(19-30\tau-45\tau^2\Big)\eta_{5x}\Big] = 0.
\end{aligned} \tag{35}$$

By identifying Eqs. (1) and (35), we find the expression of the coefficients $a_i(1,2,3)$ and $b_i(1,\ldots,15)$, which depend on τ and δ.

Introducing Eq. (34) in Eqs. (22) and (24) and proceeding as above, we obtain

$$\begin{aligned}
w = \eta &+ \varepsilon\Big[-\frac{1}{4}\,\eta^2 + \frac{1}{6}\,\Big(2-3\tau\Big)\eta_{2x}\Big] + \varepsilon^2\Big[\frac{1}{8}\,\eta^3 - \frac{1}{4}\,\delta\eta_x + \frac{1}{16}\,(3+7\tau)\eta_x^2 + \frac{1}{4}\,\delta\eta\eta_x + \frac{1}{4}\,(2+\tau)\ \eta\eta_{2x}\\
&+\frac{1}{120}\,(12-20\tau-15\tau^2)\eta_{4x}\Big] + \varepsilon^3\Big[-\frac{5}{64}\,\eta^4 + \frac{1}{8}\,\delta\eta\eta_x + \Big(\frac{7}{20} - \frac{1}{4}\,\tau + \frac{1}{16}\,\tau^2\Big)\eta\eta_{4x} + \frac{1}{16}\,\Big(2-3\tau\Big)\eta^2\eta_{2x}\\
&+\Big(\frac{1091}{1440} + \frac{1}{3}\,\tau + \frac{21}{32}\,\tau^2\Big)\eta_x\eta_{3x} + \frac{1}{32}\Big(3-21\tau\Big)\eta\eta_x^2 + \Big(\frac{163}{360} + \frac{29}{48}\,\tau + \frac{7}{16}\,\tau^2\Big)\eta_{2x}^2 - \frac{1}{16}\,\Big(1+3\tau\Big)\delta\eta_{3x}\\
&-\frac{1}{8}\,\Big(1+2\tau\Big)\delta\eta_x\eta_{2x} + \frac{1}{48}\,\Big(-11+15\tau\Big)\delta\eta\eta_{3x} - \frac{11}{32}\,\delta\eta^2\eta_x + \Big(\frac{61}{1890} - \frac{1}{20}\,\tau - \frac{1}{24}\,\tau^2 - \frac{1}{16}\,\tau^3\Big)\eta_{6x}\\
&-\frac{1}{8}\,\delta^2\eta_x^2 + \Big(\frac{3}{16} - \frac{3}{16}\Big)\int\eta_x^3 dx + \frac{1}{16}\,\delta\int\eta_x^2 dx + \frac{1}{16}\,\Big(5-3\tau\Big)\delta\int\eta\eta_{4x}dx + \frac{7}{32}\,\delta\int\eta^2\eta_{2x}dx\Big],
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\eta_t+\eta_x+\varepsilon\Big[\frac{3}{2}\,\eta\eta_x+\frac{1}{6}\Big(1-3\tau\Big)\eta_{3x}-\delta\eta_{2x}\Big]+\varepsilon^2\Big[-\frac{3}{8}\,\eta^2\eta_x-\frac{1}{4}\,\delta\eta_{2x}+\frac{1}{24}\Big(23+15\tau\Big)\eta_x\eta_{2x}\\
&+\frac{1}{4}\,\delta\eta\eta_{2x}+\frac{1}{12}\Big(5-3\tau\Big)+\frac{1}{4}\,\delta\eta_x^2+\frac{1}{360}\Big(19-30\tau-45\tau^2\Big)\eta_{5x}\Big]+\varepsilon^3\Big[\frac{3}{16}\,\eta^3\eta_x-\frac{1}{16}\,\delta\eta_x^2\\
&-\Big(\frac{13}{32}+\frac{13}{32}\tau\Big)\eta_x^3+\frac{1}{8}\Big(4-\tau\Big)\eta^2\eta_{3x}+\Big(\frac{11}{16}+\frac{29}{6}\tau\Big)\eta\eta_x\eta_{2x}-\frac{3}{16}\,\delta\eta\eta_x^2+\frac{1}{8}\,\delta\eta^2\eta_{2x}-\frac{1}{8}\,\delta\eta\eta_{2x}\\
&+\Big(\frac{1079}{1440}-\frac{5}{45}\,\tau+\frac{19}{32}\,\tau^2\Big)\eta_x\eta_{4x}-\frac{1}{48}\Big(1+9\tau\Big)\delta\eta_{4x}+\Big(\frac{19}{80}-\frac{5}{24}\,\tau-\frac{1}{16}\,\tau^2\Big)\eta\eta_{5x}-\frac{1}{4}(1+\tau)\delta\eta_{2x}^2\\
&+\Big(\frac{377}{288}+\frac{15}{16}\,\tau+\frac{49}{32}\,\tau^2\Big)\eta_{2x}\eta_{3x}+\Big(-\frac{25}{48}+\frac{3}{48}\,\tau\Big)\delta\,\eta_x\eta_{3x}+\Big(\frac{1}{24}+\frac{1}{8}\,\tau\Big)\delta\eta\eta_{4x}-\frac{1}{4}\,\delta^2\eta_x\eta_{2x}\\
&+\Big(\frac{55}{3024}-\frac{19}{720}\,\tau-\frac{1}{48}\,\tau^2-\frac{1}{16}\,\tau^3\Big)\eta_{7x}\Big]=0.
\end{aligned}\tag{37}$$

At the third order $O(\varepsilon^3)$, the correction function $C(\eta)$ is given by

$$\begin{aligned}
C(\eta)=&-\frac{5}{64}\,\eta^4+\Big(\frac{7}{20}-\frac{1}{4}\,\tau+\frac{1}{16}\,\tau^2\Big)\eta\eta_{4x}+\frac{1}{16}\,(2-3\tau)\eta^2\eta_{2x}+\frac{1}{32}\,(3-21\tau)\eta\eta_x^2\\
&+\Big(\frac{1091}{1440}+\frac{1}{3}\,\tau+\frac{21}{32}\,\tau^2\Big)\eta_x\eta_{3x}+\Big(\frac{163}{360}+\frac{29}{48}\,\tau+\frac{7}{16}\,\tau^2\Big)\eta_{2x}^2-\frac{1}{16}\,(1+3\tau)\delta\eta_{3x}+\frac{1}{8}\,\delta\eta\eta_x\\
&+\frac{1}{48}\,(-11+15\tau)\delta\eta\eta_{3x}-\frac{11}{32}\,\delta\eta^2\eta_x+\Big(\frac{61}{1890}-\frac{1}{20}\,\tau-\frac{1}{24}\,\tau^2-\frac{1}{16}\,\tau^3\Big)\eta_{6x}-\frac{1}{8}\,(1+2\tau)\delta\eta_x\eta_{2x}\\
&-\frac{1}{8}\,\delta^2\eta_x^2+\Big(\frac{3}{16}-\frac{3}{16}\Big)\int\eta_x^3dx+\frac{1}{16}\,\delta\int\eta_x^2dx+\frac{1}{16}\,(5-3\tau)\delta\int\eta\eta_{4x}dx+\frac{7}{32}\,\delta\int\eta^2\eta_{2x}dx.
\end{aligned}\tag{38}$$

It should be noted that, the Eqs. (37) and (1) are the same where all the coefficients $a_i (i = 1, 2, 3)$, $b_i (i = 1, \dots, 6)$ and $c_i (i = 1, \dots, 15)$ strongly depend on the viscosity and the surface tension parameters.

3.3. Equations for bi-directional waves

In the field of shallow water, through the non-uniqueness of the Boussinesq decomposition, it has been shown that waves also propagate in two directions [43,68,69]. To derive the equations for bi-directional waves, the approach is to split the surface elevation $\eta(x,t)$ into two components, namely $u(x,t)$ and $\xi(x,t)$ corresponding to the right- and left-moving waves respectively. The left-moving wave is of the order $O(\varepsilon)$ smaller than that of the right-moving wave. Thus, one can assume:

$$\begin{aligned}
&\eta(x,t)=u(x,t)+\varepsilon\xi(x,t),\\
&w(x,t)=w^+(x,t)+\varepsilon w^-(x,t),
\end{aligned}\tag{39}$$

where $w^+(x,t) = u(x,t) + \varepsilon R(x,t)$ and $w^-(x,t) = \varepsilon S(x,t)$ are the scaled horizontal velocity at the bottom of the fluid to the right- and left-moving respectively. As for the equations for unidirectional waves, we applied the same procedure. Then, at the lowest order we imposes that the Boussinesq system is reduced to $u_t + w_x^+ = 0$ and $u_x + w_t^+ = 0$, which are satisfied by the right-moving wave $u_x + u_t = 0$ and the corresponding equation for the left-moving wave $\xi_x - \xi_t = 0$. We express w in the form

$$w=u+\varepsilon R+\varepsilon S=u+\varepsilon Q,\tag{40}$$

where $Q = (R + S)$ is arbitrary function of x and t. This function will subsequently be determined by introducing Eq. (40) and the first equation of Eq. (39) into Eqs. (22) and (24). Neglecting the terms of higher order than $O(\varepsilon)$ in each equation, we obtain the following system

$$\begin{aligned}
&u_x+u_t+\varepsilon(Q_x+\xi_x+2uu_x-\delta u_{2x}-\frac{1}{6}\,u_{3x})=0,\\
&u_x+u_t+\varepsilon(Q_t+\xi_x+uu_x-\delta u_{2x}+\frac{1}{2}\,u_{3x}-\tau u_{3x})=0,
\end{aligned}\tag{41}$$

Proceeding in the same way as in [43], we assume that the two equations in (40) are identical and can take the form $u_x + u_t + \varepsilon(k_1uu_x + k_2u_{2x} + k_3u_{3x}) = 0$. Thus, we obtain

$$\begin{aligned}
&Q_x=-(\xi_x+2uu_x-\delta u_{2x}-\frac{1}{6}\,u_{3x})+k_1uu_x+k_2u_{2x}+k_3u_{3x},\\
&Q_t=-(\xi_x+uu_x-\delta u_{2x}+\frac{1}{2}\,u_{3x}-\tau u_{3x})+k_1uu_x+k_2u_{2x}+k_3u_{3x}.
\end{aligned}\tag{42}$$

We perform a partial derivation of these two equations with respect to t and x respectively. We then substitute u_t by $-u_x$ and ξ_t by ξ_x, and subtract the equations, we obtain

$$(3-2k_1)\qquad(u_x^2+uu_{2x})-2(k_2+\delta)u_{3x}+\Big(\tfrac{1}{3}\,(1-3\tau)-2k_3\Big)u_{4x}=0.\tag{43}$$

The Eq. (43) is satisfied by $k_1 = 3/2$, $k_2 = -\delta$ and $k_3 = (1/6)(1-3\tau)$. Thus, both equations in (39) take the form

$$u_x + u_t + \varepsilon\left(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\right) = 0, \tag{44}$$

which coincides with the same order equation for $\eta(x,t)$ in the pure right-moving wave case.

Substituting the coefficients $k_i (i = 1, 2, 3)$ by their expressions in the first equation of (42), and integrating once with respect to x leads to

$$Q = -\xi - \frac{1}{4}\,u^2 + \frac{1}{6}\,(2-3\tau)u_{2x}. \tag{45}$$

To determine the equation corresponding to the second order $O(\epsilon^2)$, we assume that the equations for the horizontal velocity w and the left-moving wave ξ, can be written in the form

$$\begin{aligned} &w = u + \varepsilon\Big(-\xi - \frac{1}{4}\,u^2 + \frac{1}{6}\,(2-3\tau)u_{2x}\Big) + \varepsilon^2 M,\\ &\xi_t = \xi_x + \varepsilon H_x, \end{aligned} \tag{46}$$

where $H \equiv H(x,t)$ and $M \equiv M(x,t)$ are free functions. To determine the functions $H(x,t)$ and $M(x,t)$, Eq. (46) is introduced into Eqs. (22) and (24). Neglecting the terms of higher order than $O(\varepsilon^2)$ in each equation, all the t-derivatives of u and ξ are replaced by their expressions through the x-derivatives using the lowest order equation, namely Eq. (44) and the second equation of Eq. (46). Upon these substitutions, we obtain the following equations

$$\begin{aligned} &u_x + u_t + \varepsilon\Big(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big) + \varepsilon^2\Big(M_x + H_x - \delta\xi_{2x} + \frac{1}{6}\,\xi_{3x} - \frac{3}{4}\,u^2u_x\\ &\quad + \frac{1}{12}(1-6\tau)u_xu_{2x} - \frac{1}{12}(1+6\tau)uu_{3x} - \frac{1}{360}\,(17-30\tau)u_{5x}\Big) = 0,\\ &u_x + u_t + \varepsilon\Big(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big) + \varepsilon^2\Big(M_t - H_x - \xi u_x - u\xi_x + \delta\xi_{2x} + \frac{1}{2}\,(1-2\tau)\xi_{3x}\\ &\quad + \frac{1}{2}\,\delta u_x^2 + \frac{11}{12}\,uu_{3x} + (\frac{11}{6} + \frac{7}{4}\tau)u_xu_{2x} + (\frac{11}{72} - \frac{1}{4}\tau - \frac{1}{4}\tau^2)u_{5x}\Big) = 0. \end{aligned} \tag{47}$$

The next step consists to assume from the requirement that the both Eqs. (47) can take the form

$$u_x + u_t + \varepsilon\Big(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}\,(1-3\tau)u_{3x}\Big) + \varepsilon^2(b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2) = 0, \tag{48}$$

which implies that

$$\begin{aligned} M_x &= -H_x + \delta\xi_{2x} - \frac{1}{6}\,\xi_{3x} + \frac{1}{4}(3+4b_4)u^2u_x + \frac{1}{12}(1+6\tau+12b_2)uu_{3x} - \frac{1}{12}(1-6\tau-12b_3)u_xu_{2x}\\ &\quad + \frac{1}{360}(17-30\tau+360b_1)u_{5x} + b_5u_x^2,\\ M_t &= H_x + \xi u_x + u\xi_x - \delta\xi_{2x} - \frac{1}{2}(1-2\tau)\xi_{3x} + \frac{1}{2}(2b_5-\delta)u_x^2 - \Big(\frac{11}{6} + \frac{7}{4}\tau - b_3\Big)u_xu_{2x}\\ &\quad + b_4u^2u_x - \frac{1}{12}(11-12b_2)uu_{3x} - \Big(\frac{11}{72} - \frac{1}{4}\,\tau - \frac{1}{4}\,\tau^2 - b_1\Big)u_{5x}. \end{aligned} \tag{49}$$

Integrating the first equation of Eq. (49) with respect to x leads to

$$\begin{aligned} M &= -H + \delta\xi_x - \frac{1}{6}\,\xi_{2x} + \frac{1}{12}(3+4b_4)u^3 + \frac{1}{12}(1+6\tau+12b_2)uu_{2x} - \frac{1}{12}(1+2b_2-6b_3)u_x^2\\ &\quad + b_5\int u_x^2 + \frac{1}{360}(17-30\tau+360b_1)u_{5x}. \end{aligned} \tag{50}$$

Substituting Eq. (50) into the second equation of Eq. (49) yields

$$\begin{aligned} H_x + H_t &= -\xi u_x - u\xi_x + 2\delta\xi_{2x} + \frac{1}{3}(1-3\tau)\xi_{3x} - 2\Big[\Big(b_4 + \frac{3}{8}\Big)u^2u_x + \Big(b_3 - \frac{5}{8}\,\tau - \frac{23}{24}\Big)u_xu_{2x}\\ &\quad + \Big(b_2 - \frac{5}{12} + \frac{1}{4}\,\tau\Big)uu_{3x} - \frac{1}{4}(\delta-4b_5)u_x^2 + \Big(b_1 - \frac{19}{360} + \frac{1}{12}\,\tau + \frac{1}{8}\,\tau^2\Big)u_{5x}\Big]. \end{aligned} \tag{51}$$

To find the solution $H(x,t)$ of Eq. (51), we sum its derivatives with respect to x and t and obtain

$$H = -\frac{1}{2}(\pi u)_x + \delta\pi_{2x} + \frac{1}{6}(1-3\tau)\pi_{3x} + G. \tag{52}$$

By replacing H by its expression in (50) we obtain

$$\begin{aligned} M = & -G + \frac{1}{2}\, u\xi + \frac{1}{2}\, u_x\pi - \frac{1}{6}(2-3\tau)\ \xi_{2x} + \frac{1}{12}(1+6\tau+12b_2)uu_{2x} + \frac{1}{12}(3+4b_4)u^3 + b_5\int u_x^2 dx \\ & -\frac{1}{12}(1+2b_2-6b_3)u_x^2 + \frac{1}{360}(17-30\tau+360b_1)u_{5x}, \end{aligned} \tag{53}$$

such that $\xi = \pi_x$ and G satisfies the following equation

$$\begin{aligned} G_x + G_t = & -2\Big[\Big(b_4 + \frac{3}{8}\Big)u^2u_x + \Big(b_2 - \frac{5}{12} + \frac{1}{4}tau\Big)uu_{3x} \\ & -\frac{1}{4}(\delta - 4b_5)u_x^2 + \Big(b_3 - \frac{5}{8}tau\frac{23}{24}\Big)u_xu_{2x} \\ & +\Big(b_1 - \frac{19}{360} + \frac{1}{12}\ \tau + \frac{1}{8}tau^2\Big)u_{5x}\Big]. \end{aligned} \tag{54}$$

Introducing Eq. (52) into the second equation of Eq. (46) and integrating with respect to x leads to equation in terms of the left-moving wave for the first order $O(\varepsilon)$ as follows

$$\pi_t - \pi_x + \varepsilon\Big(\frac{1}{2}(\pi u)_x - \delta\pi_{2x} - \frac{1}{6}(1-3\tau)\pi_{3x} - G\Big) = 0. \tag{55}$$

Introducing Eqs. (52) and (53) into Eq. (47), we obtain

$$\begin{aligned} u_x + u_t + \varepsilon\Big(\frac{3}{2}\, uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big) + \varepsilon^2\Big(- G_x + \xi u_x + \frac{1}{2}\, u\xi_x + \frac{1}{2}\, \delta u_x^2 - \delta\xi_{2x} + \frac{1}{2}\, \pi u_{2x} \\ -\frac{1}{6}(1-3\tau)\xi_{3x} + b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2\Big) = 0. \end{aligned} \tag{56}$$

To establish the equation corresponding to the third order $O(\varepsilon^3)$, we assume that the equations for the horizontal velocity ω and the left-moving wave ξ can be written in the form

$$\begin{aligned} & w = u + \varepsilon\Big(- \xi - \frac{1}{4}\, u^2 + \frac{1}{6}\ (2-3\tau)u_{2x}\Big) + \varepsilon^2\Big(- G + \frac{1}{2}\, u\xi + \frac{1}{2}\, u_x\pi - \frac{1}{6}(2-3\tau)\ \xi_{2x} + b_5\int u_x^2 \\ & +\frac{1}{12}(1+6\tau+12b_2)uu_{2x} + \frac{1}{12}(3+4b_4)u^3 - \frac{1}{12}(1+2b_2-6b_3)u_x^2 + \frac{1}{360}(17-30\tau+360b_1)u_{5x}\Big) + \varepsilon^3 P, \\ & \xi_t = \xi_x + \varepsilon H_x + \varepsilon^2 L_x, \end{aligned} \tag{57}$$

where the free functions $L \equiv L(x,t)$ and $P \equiv P(x,t)$ will be determined later. Introducing Eq. (57) into Eqs. (22) and (24), we obtain the following equations

$$
\begin{aligned}
&u_x + u_t + \varepsilon\Big(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big) + \varepsilon^2\Big(- G_x + \xi u_x + \frac{1}{2}\,u\xi_x + \frac{1}{2}\,\delta u_x^2 - \delta\xi_{2x} + \frac{1}{2}\,\pi u_{2x}\\
&-\frac{1}{6}(1-3\tau)\xi_{3x} + b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2\Big) + \varepsilon^3\Big[P_x + L_x - uG_x - Gu_x + \xi uu_x\\
&-2\xi\xi_x + \frac{1}{2}\,\pi u_x^2 + \frac{1}{4}\,\xi_xu^2 + \frac{1}{2}\,\pi uu_{2x} - \Big(\frac{2}{3} + \frac{1}{2}\,\tau\Big)\xi_xu_{2x} - \frac{1}{6}(1-3\tau)\xi_{2x}u_x + \frac{1}{6}\,G_{3x} - \frac{1}{2}(1+\tau)\xi u_{3x}\\
&-\frac{1}{12}\,\pi u_{4x} + \frac{1}{12}(1+6\tau)u\xi_{3x} + \Big(\frac{17}{360} - \frac{1}{12}\,\tau\Big)\xi_{5x} + \frac{1}{2}(\delta+2b_5)uu_x^2 - \Big(\frac{1}{12} + \frac{1}{2}\,b_2 - \frac{1}{2}\,b_3 + \frac{1}{3}\,b_4\Big)u_x^3\\
&+\frac{1}{3}(3+4b_4)u^3u_x - \Big(\frac{3}{4} - b_2 - b_3 + b_4 - \tau\Big)uu_xu_{2x} - \Big(\frac{1}{3}\,b_5 + \frac{1}{6}\,\delta\Big)u_{2x}^2 - \Big(\frac{1}{3}\,b_5 + \frac{1}{6}\,\delta\Big)u_xu_{3x}\\
&-\Big(\frac{7}{24} - b_2 + \frac{1}{6}\,b_4 - \frac{1}{2}\,\tau\Big)u^2u_{3x} - \Big(\frac{1}{72} + \frac{1}{6}\,b_2 + \frac{1}{2}\,b_3 + \frac{1}{3}\,\tau\Big)u_{2x}u_{3x} + \frac{1}{2}(\delta+2b_5)u_x\int u_x^2dx\\
&-\Big(\frac{9}{80} - b_1 + \frac{1}{3}\,b_2 + \frac{1}{6}\,b_3 + \frac{1}{12}\,\tau\Big)u_xu_{4x} + \Big(- \frac{23}{240} + b_1 - \frac{1}{6}\,b_2 + \frac{1}{12}\,\tau\Big)uu_{5x} - \Big(\frac{1}{189} + \frac{1}{6}\,b_1 - \frac{7}{720}\,\tau\Big)u_{7x}\Big] = 0,\\
&u_x + u_t + \varepsilon\Big(\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big) + \varepsilon^2\Big(- G_x + \xi u_x + \frac{1}{2}\,u\xi_x + \frac{1}{2}\,\delta u_x^2 - \delta\xi_{2x} + \frac{1}{2}\,\pi u_{2x}\\
&-\frac{1}{6}(1-3\tau)\xi_{3x} + b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2\Big) + \varepsilon^3\Big[P_t - L_x - \frac{1}{2}uG_x - Gu_x\\
&+\frac{3}{4}\,\xi uu_x - \frac{1}{4}\,\pi u_x^2 + \frac{1}{2}u^2\xi_x + \xi\xi_x - \frac{3}{2}\,\delta\xi_xu_x + \delta G_{2x} - \frac{1}{2}\,\pi uu_{2x} - \delta\xi u_{2x} - \Big(\frac{1}{3} + \tau\Big)\xi_xu_{2x}\\
&-\Big(\frac{2}{3} + \frac{1}{2}\,\tau\Big)u_x\xi_{2x} + \Big(\frac{5}{4} - \frac{1}{4}\,\tau\Big)\xi u_{3x} + \frac{11}{12}\,u\xi_{3x} + \frac{1}{12}\,\pi u_{4x} + \Big(\frac{11}{72} - \frac{1}{4}\,\tau - \frac{1}{4}\,\tau^2\Big)\xi_{5x} - \frac{2}{3}\,b_4u^3u_x\\
&-\frac{1}{6}\,(2-3\tau)G_{3x} + \Big(\frac{7}{6} + b_2 - b_3 - \frac{4}{3}\,b_4 - b_4\tau\Big)u_x^3 + \Big(\frac{1}{2}\,b_5 - \delta - 2b_4\delta\Big)uu_x^2 - (2b_5\delta + \delta^2)u_xu_{2x}\\
&+\frac{1}{2}(\delta+2b_5)u_x\int u_x^2dx + \Big(\frac{1}{4} - b_2 - \frac{5}{6}\,b_4 + \frac{1}{4}\,\tau\Big)u^2u_{3x} + \Big(\frac{191}{240} - b_1 - \frac{5}{6}\,b_2 - \tau + \frac{1}{4}\,\tau^2\Big)uu_{5x}\\
&+\Big(\frac{85}{18} - 15b_1 - \frac{5}{6}\,b_2 - 2b_3 - \frac{1}{8}\,\tau - \frac{3}{2}\,b_3\tau + \frac{1}{2}\,\tau^2\Big)u_{2x}u_{3x} - \Big(\frac{4}{3}\,b_5 - \frac{2}{3}\,\delta - b_2\delta + b_3\delta + b_5\tau\Big)u_{2x}^2\\
&+\Big(\frac{231}{80} - \frac{13}{2}\,b_1 - \frac{7}{6}\,b_2 - \frac{5}{6}\,b_3 - \frac{37}{24}\,\tau - \frac{3}{2}\,b_2\tau\Big)u_xu_{4x} - \Big(\frac{4}{3}\,b_5 - \frac{1}{3}\,\delta + 2b_2\delta + b_5\tau + \delta\tau\Big)u_xu_{3x}\\
&+\Big(\frac{31}{12} - \frac{7}{2}\,b_2 - b_3 - 4b_4 - \frac{7}{4}\,\tau - 3b_4\,\tau\Big)uu_xu_{2x} + \Big(\frac{83}{1080} - \frac{5}{6}\,b_1 - \frac{41}{240}\,\tau - \frac{1}{24}\,\tau^2\Big)u_{7x}\Big] = 0.
\end{aligned}
\tag{58}
$$

In the next step, we assume the requirement that both equations (58) can be written in the following forms

$$
\begin{aligned}
&u_x + u_t + \varepsilon\Big[\frac{3}{2}\,uu_x - \delta u_{2x} + \frac{1}{6}(1-3\tau)u_{3x}\Big] + \varepsilon^2\Big[- G_x + \xi u_x + \frac{1}{2}\,u\xi_x + \frac{1}{2}\,\delta u_x^2 - \delta\xi_{2x} + \frac{1}{2}\,\pi u_{2x}\\
&-\frac{1}{6}(1-3\tau)\xi_{3x} + b_1u_{5x} + b_2uu_{3x} + b_3u_xu_{2x} + b_4u^2u_x + b_5u_x^2\Big] + \varepsilon^3\Big[c_1u_{7x} + c_2uu_{5x} + c_3u_xu_{4x}\\
&+c_4u^2u_{3x} + c_5u_{2x}u_{3x} + c_6uu_xu_{2x} + c_7u_xu_{3x} + c_8u_x^3 + c_9u_x\int u_x^2 + c_{10}u^3u_x + c_{11}uu_x^2 + c_{12}u_{2x}^2 + c_{13}u_xu_{2x}\Big] = 0.
\end{aligned}
\tag{59}
$$

The expressions for the x- and t-derivatives of P must be then written as

$$\begin{aligned}
&P_x=-L_x+uG_x+Gu_x-\xi uu_x+2\xi\xi_x-\frac{1}{2}\,\pi u_x^2-\frac{1}{4}\,\xi_x u^2-\frac{1}{2}\,\pi uu_{2x}+\left(\frac{2}{3}+\frac{1}{2}\,\tau\right)\xi_x u_{2x}\\
&+\frac{1}{6}(1-3\tau)\xi_{2x}u_x-\frac{1}{6}\,G_{3x}+\frac{1}{2}(1+\tau)\xi u_{3x}+\frac{1}{12}\,\pi u_{4x}-\frac{1}{12}(1+6\tau)u\xi_{3x}-\left(\frac{17}{360}-\frac{1}{12}\,\tau\right)\xi_{5x}\\
&-\frac{1}{2}(\delta+2b_5-2c_{11})uu_x^2-\frac{1}{3}(3+4b_4-3c_{10})u^3u_x-\frac{1}{2}(\delta+2b_5-2c_9)u_x\int u_x^2dx\\
&+\left(\frac{1}{3}\,b_5+\frac{1}{6}\,\delta+c_{12}\right)u_{2x}^2+\left(\frac{1}{3}\,b_5+\frac{1}{6}\,\delta+c_7\right)u_xu_{3x}+\left(\frac{3}{4}-b_2-b_3+b_4-\tau+c_6\right)uu_xu_{2x}\\
&+c_{13}u_xu_{2x}+\left(\frac{7}{24}-b_2+\frac{1}{6}\,b_4-\frac{1}{2}\,\tau+c_4\right)u^2u_{3x}+\left(\frac{1}{72}+\frac{1}{6}\,b_2+\frac{1}{2}\,b_3+\frac{1}{3}\,\tau+c_5\right)u_{2x}u_{3x}\\
&+\left(\frac{9}{80}-b_1+\frac{1}{3}\,b_2+\frac{1}{6}\,b_3+\frac{1}{12}\,\tau+c_3\right)u_xu_{4x}+\left(\frac{1}{12}+\frac{1}{2}\,b_2-\frac{1}{2}\,b_3+\frac{1}{3}\,b_4+c_8\right)u_x^3\\
&-\left(-\frac{23}{240}+b_1-\frac{1}{6}\,b_2+\frac{1}{12}\,\tau-c_2\right)uu_{5x}+\left(\frac{1}{189}+\frac{1}{6}\,b_1-\frac{7}{720}\,\tau+c_1\right)u_{7x}\\
&P_t=L_x+\frac{1}{2}\,uG_x+Gu_x-\frac{3}{4}\xi uu_x+\frac{1}{4}\,\pi u_x^2-\frac{1}{2}u^2\xi_x-\xi\xi_x+\frac{3}{2}\,\delta\xi_xu_x-\delta G_{2x}+\frac{1}{2}\,\pi uu_{2x}+\delta\xi u_{2x}\\
&+\left(\frac{1}{3}+\tau\right)\xi_xu_{2x}+\left(\frac{2}{3}+\frac{1}{2}\,\tau\right)u_x\xi_{2x}-\left(\frac{5}{4}-\frac{1}{4}\,\tau\right)\xi u_{3x}-\frac{11}{12}\,u\xi_{3x}-\frac{1}{12}\,\pi u_{4x}+\frac{1}{6}(2-3\tau)G_{3x}\\
&-\left(\frac{11}{72}-\frac{1}{4}\,\tau-\frac{1}{4}\,\tau^2\right)\xi_{5x}-\left(\frac{7}{6}+b_2-b_3-\frac{4}{3}\,b_4-b_4\tau-c_8\right)u_x^3+\left(\frac{2}{3}\,b_4+c_{10}\right)u^3u_x\\
&-\frac{1}{2}(\delta+2b_5-c_9)u_x\int u_x^2dx-\left(\frac{1}{2}\,b_5-\delta-2b_4\delta-c_{11}\right)uu_x^2+(2b_5\delta+\delta^2+c_{13})u_xu_{2x}\\
&-\left(\frac{31}{12}-\frac{7}{2}\,b_2-b_3-4b_4-\frac{7}{4}\,\tau-3b_4\,\tau-c_6\right)uu_xu_{2x}-\left(\frac{191}{240}-b_1-\frac{5}{6}\,b_2-\tau+\frac{1}{4}\,\tau^2-c_2\right)uu_{5x}\\
&+\left(\frac{4}{3}\,b_5-\frac{1}{3}\,\delta+2b_2\delta+b_5\tau+\delta\tau+c_7\right)u_xu_{3x}+\left(\frac{4}{3}\,b_5-\frac{2}{3}\,\delta-b_2\delta+b_3\delta+b_5\tau+c_{12}\right)u_{2x}^2\\
&-\left(\frac{1}{4}-b_2-\frac{5}{6}\,b_4+\frac{1}{4}\,\tau-c_4\right)u^2u_{3x}-\left(\frac{85}{18}-15b_1-\frac{5}{6}\,b_2-2b_3-\frac{1}{8}\,\tau-\frac{3}{2}\,b_3\tau+\frac{1}{2}\,\tau^2-c_5\right)u_{2x}u_{3x}\\
&-\left(\frac{231}{80}-\frac{13}{2}\,b_1-\frac{7}{6}\,b_2-\frac{5}{6}\,b_3-\frac{37}{24}\,\tau-\frac{3}{2}\,b_2\tau-c_3\right)u_xu_{4x}-\left(\frac{83}{1080}-\frac{5}{6}\,b_1-\frac{41}{240}\,\tau-\frac{1}{24}\,\tau^2-c_1\right)u_{7x}.
\end{aligned}\tag{60}$$

Integrating the first equation of Eq. (60) with respect to x, we obtain

$$\begin{aligned}
P\quad&=-L+uG-\frac{1}{2}\,\pi uu_x-\frac{1}{4}\,\pi_xu^2+\pi_x^2-\frac{1}{6}\,G_{3x}-\frac{1}{12}(1+6\tau)\pi_{3x}u+\frac{1}{4}\,\pi_{2x}u_x+\frac{1}{12}(5+6\tau)\pi_xu_{2x}\\
&+\frac{1}{12}\,\pi u_{3x}-\left(\frac{17}{360}-\frac{1}{12}\,\tau\right)\pi_{5x}+\frac{1}{12}(3c_{10}-4b_4-3)u^4+\left(\frac{23}{240}-b_1+\frac{b_2}{6}-\frac{\tau}{12}+c_2\right)uu_{4x}\\
&+\left(\frac{b_1}{3}+\frac{\delta}{6}+c_7\right)u_xu_{2x}+\left(c_1+\frac{b_1}{6}+\frac{1}{189}-\frac{7}{720}\,\tau\right)u_{6x}+\left(c_{12}-c_7\right)\int u_{2x}^2dx\\
&+\left(c_{11}-b_5-\frac{\delta}{2}\right)\int uu_x^2dx+\left(c_8+c_4-\frac{c_6}{2}\right)\int u_x^3dx+\left(\frac{7}{24}-b_2+\frac{b_4}{6}-\frac{\tau}{2}+c_4\right)u^2u_{2x}\\
&+\frac{1}{12}\,c_{13}u_x^2+\left(\frac{b_2}{6}+\frac{b_3}{6}+\frac{\tau}{6}+c_3-c_2+\frac{1}{60}\right)u_xu_{3x}+\left(\frac{b_3}{6}+\frac{\tau}{12}+\frac{c_2}{2}+\frac{c_5}{2}-\frac{c_3}{2}-\frac{1}{720}\right)u_{2x}^2\\
&+\left(\frac{b_2}{2}+\frac{b_4}{3}+\frac{c_6}{2}-\frac{b_3}{2}-c_4+\frac{1}{12}\right)uu_x^2+\left(c_9-b_5-\frac{\delta}{2}\right)\int\left(u_x\int u_x^2dx\right)dx.
\end{aligned}\tag{61}$$

Introducing Eq. (61) into the second equation of Eq. (60) yields

$$
\begin{aligned}
L_x + L_t =\; & \frac{3}{4}\, uu_x\pi_x + \frac{1}{4}\, u_x^2\pi + \frac{1}{4}\, u^2\pi_{2x} + 3\pi_x\pi_{2x} - \frac{3}{2}\,\delta u_x\pi_{2x} - \delta u_{2x}\pi_x - \frac{1}{6}(1+3\tau)u_{2x}\pi_{2x} \\
& - \frac{1}{3}\, u_x\pi_{3x} + \frac{1}{12}(11-9\tau)u_{3x}\pi_x + \frac{1}{6}(5-3\tau)u\pi_{4x} - \frac{3}{2}uG_x - 2Gu_x + \delta G_{2x} - \frac{1}{6}(1-3\tau)G_{3x} \\
& + \Big(\frac{19}{180} - \frac{\tau}{6} - \frac{\tau^2}{4}\Big)\pi_{6x} + \Big(\frac{9}{2} - \frac{5}{2}\, b_2 - 2b_3 - 3b_4 - 2c_6 + \frac{\tau}{2} - 3b_4\tau\Big)uu_xu_{2x} + \frac{1}{2}(2b_5 + \delta - 2c_9)\mu \\
& - \Big(\frac{b_1}{3} + \frac{2}{3}\, b_5 + 2c_7 + 2b_2\delta + b_5\tau + \delta\tau\Big)u_xu_{3x} + \Big(\frac{1}{4} - \frac{4}{3}\, b_4 - 2c_{10}\Big)u^3u_x + \frac{1}{2}(2b_5 + \delta - 2c_9)\lambda \\
& + \Big(\frac{98}{45} - \frac{11}{2}\, b_1 - \frac{5}{6}\, b_2 - \frac{2}{3}\, b_3 - 2c_3 - \frac{5}{3}\,\tau - \frac{3}{2}\, b_2\tau\Big)u_xu_{4x} - \Big(\frac{c_{13}}{6} + 2b_5\delta + c_{13}\delta + \delta^2\Big)u_xu_{2x} \\
& - \Big(\frac{1}{3}\, b_1 + \frac{2}{3}\, b_5 + 2c_{12} - \frac{1}{3}\,\delta - b_2\delta + b_3\delta + b_5\tau\Big)u_{2x}^2 + \Big(\frac{11}{12} - 2b_2 - \frac{2}{3}\, b_4 - 2c_4 + \frac{1}{4}\,\tau\Big)u^2u_{3x} \\
& - \Big(\frac{1}{2}\, b_5 + 2c_{11} + 2b_4\delta\Big)uu_x^2 + \frac{1}{2}(2b_5 + \delta - 2c_9)\nu + \Big(\frac{4}{3} + \frac{1}{2}\, b_2 - \frac{1}{2}\, b_3 - b_4 - 2c_8 - b_4\tau\Big)u_x^3 \\
& + \Big(\frac{2}{3} - 2b_1 - \frac{2}{3}\, b_2 - 2c_2 - \tau\Big)uu_{5x} + \Big(\frac{17}{315} - \frac{2}{3}\, b_1 - 2c_1 - \frac{2}{15}\,\tau\Big)u_{7x} \\
& + \Big(\frac{65}{18} - 15b_1 - \frac{2}{3}\, b_2 - \frac{3}{2}\, b_3 - 2c_5 - \tau - \frac{3}{2}\, b_3\tau + \frac{\tau^2}{2}\Big)u_{2x}u_{3x},
\end{aligned}
\tag{62}
$$

where

$$\nu = \int (u_{2x} \int u_x^2 dx)dx,\;\; \lambda = u_x \int u_x^2 dx,\;\; \mu = \int u_x^3 dx.$$

To find the solution $L(x,t)$ of Eq. (62), we sum its derivatives with respect to x and t and obtain

$$
\begin{aligned}
L =\; & \frac{3}{8}\, uu_x\pi + \frac{1}{8}\, u^2\pi_x - \frac{1}{6}\, u_x\pi_{2x} - \frac{3}{4}\,\delta u_x\pi_x + \frac{3}{4}\,\pi_x^2 - \frac{\delta}{2}\, u_{2x}\pi + \frac{1}{24}(11-9\tau)u_{3x}\pi + \frac{1}{12}(5-3\tau)u\pi_{3x} \\
& - \frac{1}{12}(1+3\tau)u_{2x}\pi_x + \frac{1}{8}u_x^2 \int \pi\, dx + \frac{1}{360}(19 - 30\tau - 45\tau^2)\pi_{5x} + K,
\end{aligned}
\tag{63}
$$

where the function K satisfies the following equation

$$
\begin{aligned}
K_x + K_t =\; & -\frac{3}{2}\, uG_x - 2Gu_x + \delta G_{2x} - \frac{1}{6}(1-3\tau)G_{3x} - \Big(\frac{c_{13}}{6} + 2b_5\delta + c_{13}\delta + \delta^2\Big)u_xu_{2x} \\
& + \frac{1}{2}(2b_5 + \delta - 2c_9)\mu + \Big(\frac{98}{45} - \frac{11}{2}\, b_1 - \frac{5}{6}\, b_2 - \frac{2}{3}\, b_3 - 2c_3 - \frac{5}{3}\,\tau - \frac{3}{2}\, b_2\tau\Big)u_xu_{4x} \\
& + \Big(\frac{9}{2} - \frac{5}{2}\, b_2 - 2b_3 - 3b_4 - 2c_6 + \frac{\tau}{2} - 3b_4\tau\Big)uu_xu_{2x} - \Big(\frac{b_1}{3} + \frac{2}{3}\, b_5 + 2c_7 + 2b_2\delta + b_5\tau + \delta\tau\Big)u_xu_{3x} \\
& + \Big(\frac{1}{4} - \frac{4}{3}\, b_4 - 2c_{10}\Big)u^3u_x + \frac{1}{2}(2b_5 + \delta - 2c_9)\lambda - \Big(\frac{1}{2}\, b_5 + 2c_{11} + 2b_4\delta\Big)uu_x^2 + \frac{1}{2}(2b_5 + \delta - 2c_9)\nu \\
& - \Big(\frac{1}{3}\, b_1 + \frac{2}{3}\, b_5 + 2c_{12} - \frac{1}{3}\,\delta - b_2\delta + b_3\delta + b_5\tau\Big)u_{2x}^2 + \Big(\frac{4}{3} + \frac{1}{2}\, b_2 - \frac{1}{2}\, b_3 - b_4 - 2c_8 - b_4\tau\Big)u_x^3 \\
& + \Big(\frac{11}{12} - 2b_2 - \frac{2}{3}\, b_4 - 2c_4 + \frac{1}{4}\,\tau\Big)u^2u_{3x} + \Big(\frac{2}{3} - 2b_1 - \frac{2}{3}\, b_2 - 2c_2 - \tau\Big)uu_{5x} \\
& + \Big(\frac{65}{18} - 15b_1 - \frac{2}{3}\, b_2 - \frac{3}{2}\, b_3 - 2c_5 - \tau - \frac{3}{2}\, b_3\tau + \frac{\tau^2}{2}\Big)u_{2x}u_{3x} + \Big(\frac{17}{315} - \frac{2}{3}\, b_1 - 2c_1 - \frac{2}{15}\,\tau\Big)u_{7x}.
\end{aligned}
\tag{64}
$$

By replacing L by its expression in (61), we obtain

$$
\begin{aligned}
P =\; & -K + uG - \frac{1}{6}\, G_{2x} - \frac{3}{8}\, u^2\pi_x - \frac{7}{8}\,\pi uu_x + \frac{3}{4}\,\delta u_x\pi_x + \frac{1}{4}\,\pi_x^2 + \frac{1}{12}\, c_{13}u_x^2 + \frac{5}{12}\, u_x\pi_{2x} + \frac{1}{2}\,\delta u_{2x}\pi \\
& - \frac{1}{4}(2+\tau)u\pi_{3x} + \frac{1}{4}(2+3\tau)u_{2x}\pi_x + \frac{1}{2}(2c_{11} - 2b_5 - \delta)\int uu_x^2 dx - \frac{3}{8}(1+\tau)\pi u_{3x} \\
& - \Big(\frac{1}{10} - \frac{1}{6}\,\tau - \frac{1}{8}\,\tau^2\Big)\pi_{5x} + \Big(c_4 + c_8 - \frac{c_6}{8}\Big)\mu - \Big(\frac{1}{4} - \frac{b_4}{3} - \frac{c_{10}}{4}\Big)u^4 + (c_{12} - c_7)\int u_{2x}^2 dx \\
& - \frac{1}{8}\, u_x^2 \int \pi\, dx + \Big(\frac{1}{12} - \frac{b_3}{2} + \frac{b_4}{3} - c_4 + \frac{c_6}{2}\Big)uu_x^2 + \Big(-\frac{1}{720} + \frac{b_3}{6} + \frac{c_2}{2} - \frac{c_3}{2} + \frac{c_5}{2} + \frac{\tau}{12}\Big)u_{2x}^2 \\
& + \Big(\frac{7}{24} - b_2 + \frac{b_4}{6} + c_4 - \frac{\tau}{2}\Big)u^2u_{2x} + \Big(\frac{1}{60} + \frac{b_2}{6} + \frac{b_3}{6} - c_2 + c_3 + \frac{\tau}{6}\Big)u_xu_{3x} \\
& + \Big(\frac{23}{240} - b_1 + \frac{b_2}{6} + c_2 - \frac{\tau}{12}\Big)uu_{4x} + \frac{1}{2}(2c_9 - 2b_5 - \delta)\int u_x\Big(\int u_x^2\Big)dx \\
& + \Big(\frac{b_1}{3} + c_7 + \frac{\delta}{6}\Big)u_xu_{2x} + \Big(\frac{1}{189} + \frac{b_1}{6} + c_1 - \frac{7}{720}\tau\Big)u_{6x}.
\end{aligned}
\tag{65}
$$

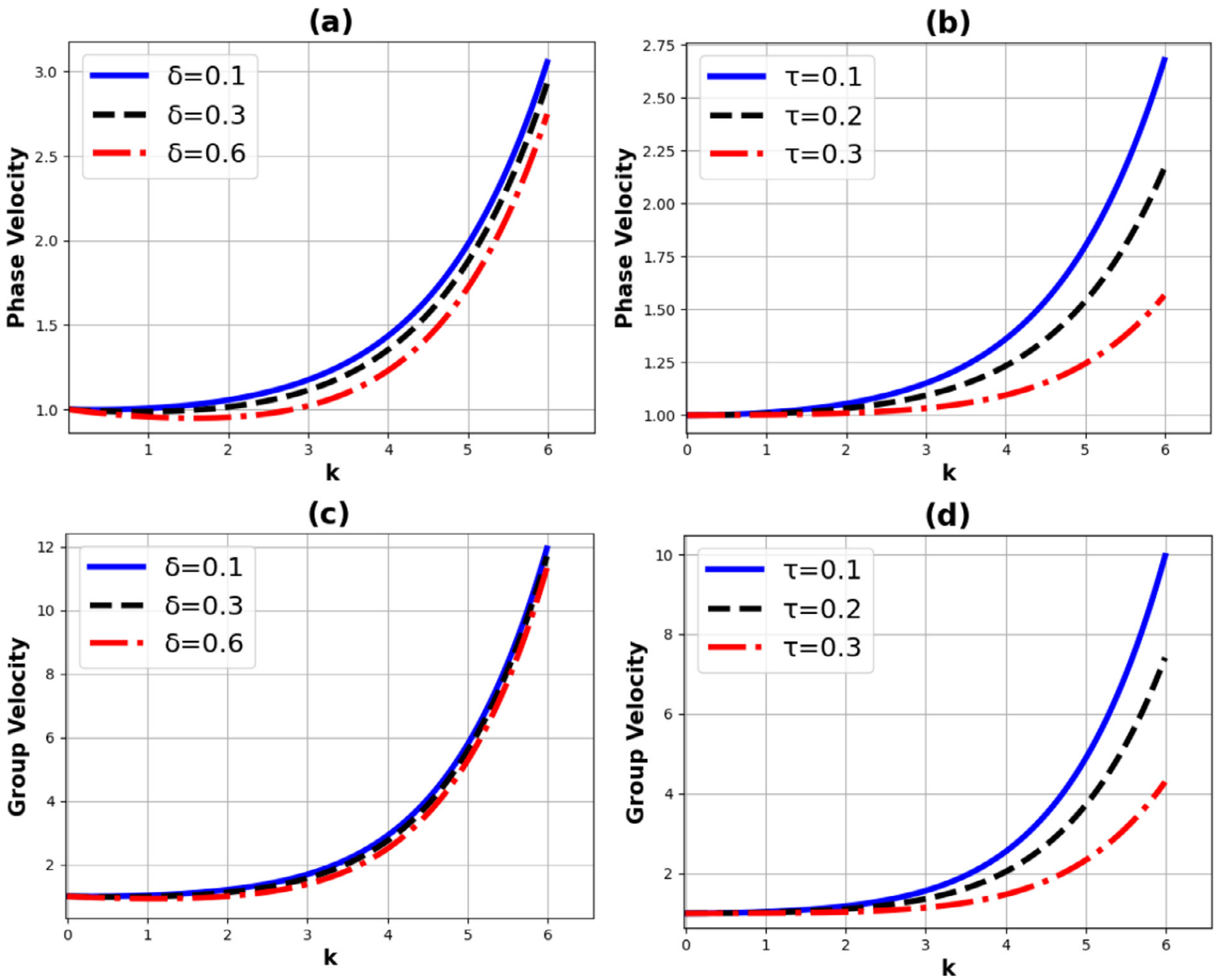

Fig. 1. Variation of the phase and group velocities, with $\varepsilon = 0.1$ for different values of viscosity and surface tension parameters: The panels (a) and (b) illustrate the behavior of phase velocity under the effects of viscosity, (Blue line): $\delta = 0.1$, (Black dashed line): $\delta = 0.3$, (Red dashed line): $\delta = 0.6$ and surface tension, (Blue line): $\tau = 0.1$, (Black dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$, respectively. The panels (c) and (d) illustrate the behavior of group velocity under the effects of viscosity, (Blue line): $\delta = 0.1$, (Black dashed line): $\delta = 0.3$, (Red dashed line): $\delta = 0.6$ and surface tension, (Blue line): $\tau = 0.1$, (Black dashed line): $\tau = 0.2$, (Red dashed line): $\tau = 0.3$, respectively.

Introducing Eq. (63) in the second equation of Eq. (57) and integrating with respect to x leads to the equation of the left-moving wave for the second order $O(\varepsilon^2)$ as follows

$$\begin{aligned}&\pi_t-\pi_x+\varepsilon\Big[\frac{1}{2}(\pi u)_x-\delta\pi_{2x}-\frac{1}{6}(1-3\tau)\pi_{3x}-G\Big]+\varepsilon^2\Big[-\frac{3}{8}\,uu_x\pi-\frac{1}{8}\,u^2\pi_x+\frac{1}{6}\,u_x\pi_{2x}+\frac{3}{4}\,\delta u_x\pi_x\\&+\frac{\delta}{2}\,u_{2x}\pi-\frac{1}{12}(5-3\tau)u\pi_{3x}-\frac{1}{24}(11-9\tau)u_{3x}\pi-\frac{3}{4}\,\pi_x^2+\frac{1}{12}(1+3\tau)u_{2x}\pi_x-\frac{1}{8}\,u_x^2\int\pi\,dx\\&-\frac{1}{360}(19-30\tau-45\tau^2)\pi_{5x}-K\Big]=0,\end{aligned}\tag{66}$$

where the functions G and K satisfy Eqs. (54) and (64) respectively. The function $u(x,t)$ corresponds to the pure right-moving wave equation and can be written as

$$\begin{aligned}&u_x+u_t+\varepsilon\Big[\frac{3}{2}\,uu_x-\delta u_{2x}+\frac{1}{6}(1-3\tau)u_{3x}\Big]+\varepsilon^2\Big[b_1u_{5x}+b_2uu_{3x}+b_3u_xu_{2x}+b_4u^2u_x+b_5u_x^2\Big]\\&+\varepsilon^3\Big[c_1u_{7x}+c_2uu_{5x}+c_3u_xu_{4x}+c_4u^2u_{3x}+c_5u_{2x}u_{3x}+c_6uu_xu_{2x}+c_7u_xu_{3x}+c_8u_x^3\\&+c_9u_x\int u_x^2+c_{10}u^3u_x+c_{11}uu_x^2+c_{12}u_{2x}^2+c_{13}u_xu_{2x}\Big]=0.\end{aligned}\tag{67}$$

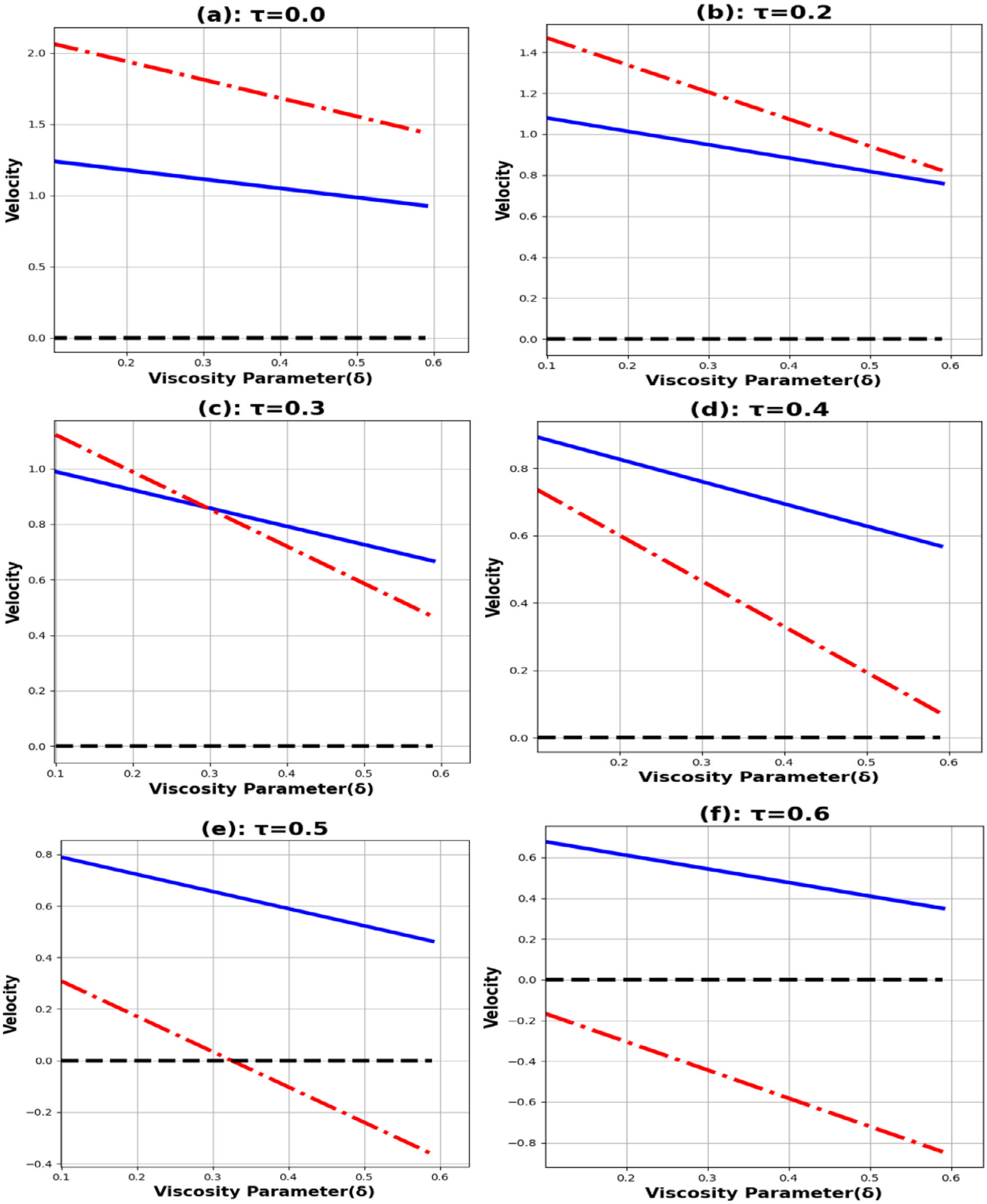

Fig. 2. Phase (Blue line) and group (Red dashed line) velocities as a function of the viscosity parameter, for $\varepsilon = 0.3$ and different values of surface tension parameter: (a): $\tau = 0.0$, (b): $\tau = 0.2$, (c): $\tau = 0.3$, (d): $\tau = 0.4$, (e): $\tau = 0.5$, (f): $\tau = 0.6$.

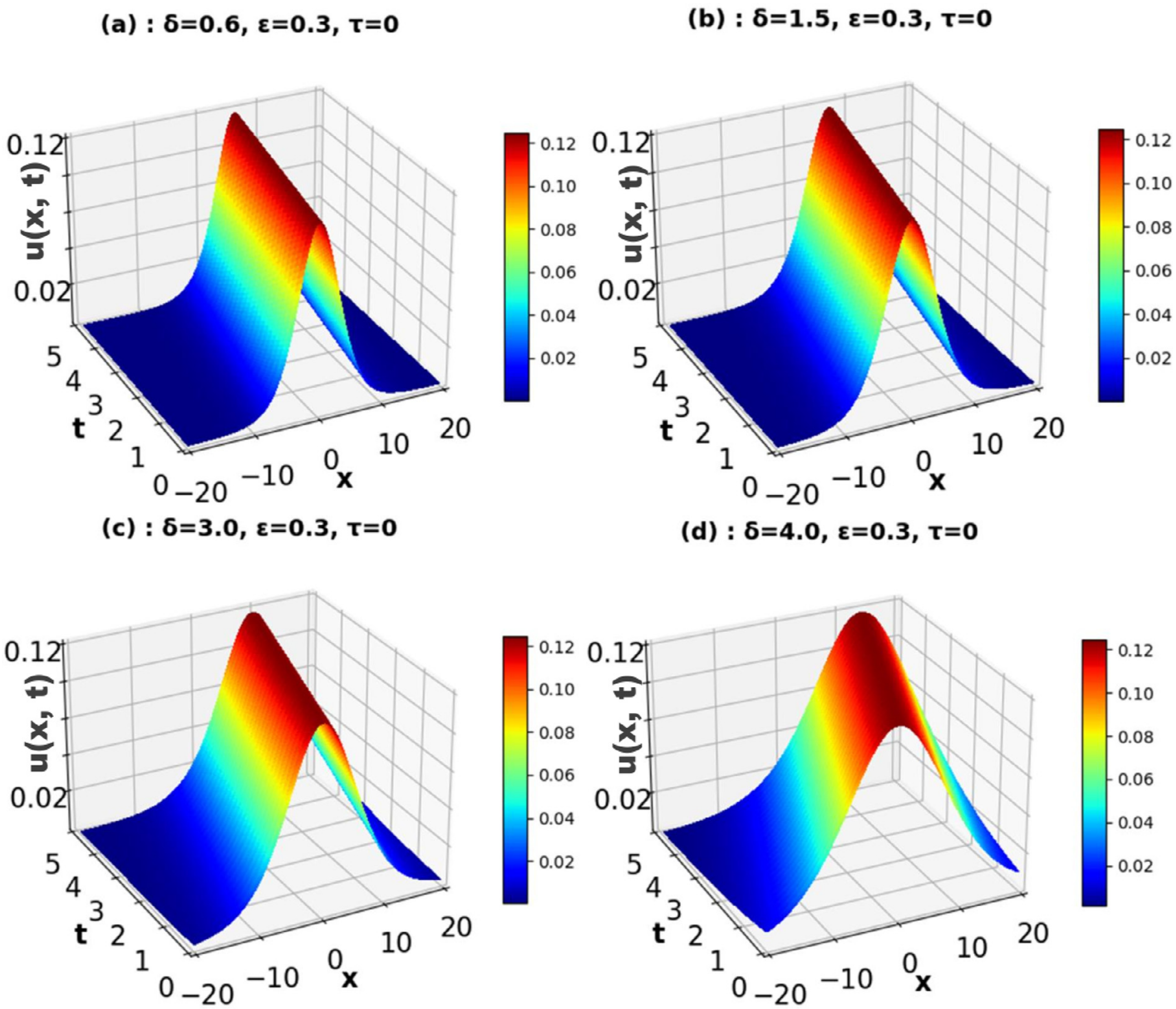

Fig. 3. Soliton solutions of the generalized inhomogeneous KdV Eq. (1) without effect of surface tension ($\tau = 0$), with $k = 0.5$, $\varepsilon = 0.3$ and different values of the viscosity parameter: **(a)**: $\delta = 0.6$; **(b)**: $\delta = 1.5$; **(c)**: $\delta = 3.0$ and **(d)**: $\delta = 4.0$. Here, the effect of viscosity on the soliton is illustrated.

In the next section by using Hirota's bilinear method, we investigate the soliton solutions of the generalized Kdv evolution Eq. (1) which describes the dynamics of shallow water waves. The effects of viscosity and surface tension are also investigated.

4. Soliton solutions of the generalized shallow water wave equations

In this section, we investigate the analytical soliton solutions of the obtained generalized shallow water wave equations by using the Hirota's bilinear method. We show that these equation can well describe the dynamics of unidirectional and bi-directional shallow water waves. We also discuss on the effects of surface tension and viscosity on the phase, group velocity and the soliton dynamics. Analytical solutions of the bi-directional evolution equation given by Eq. (66), would be desirable to define more concretely the dynamics, and the nature of right- and left-moving waves. We have not addressed this problem here, because the different decompositions of the Boussinesq system into a system of equations for right- and left-moving waves reflect several different physical situations [68]. In addition, solving such an equation requires a method allowing to obtain reliable results describing a specific physical situation. This requires a lot of space [43] and very long algorithms that are not simple in programming. Thus, in this section we focus our attention only on the soliton solutions for the equation of the right-moving wave.

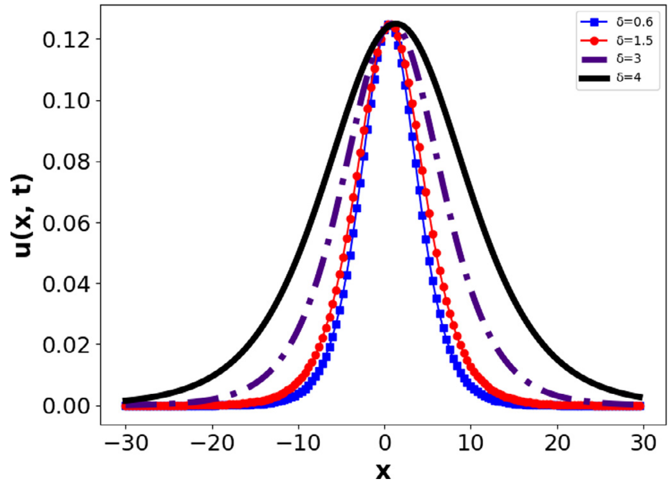

Fig. 4. Soliton solutions of the generalized inhomogeneous KdV Eq. (1) without effect of surface tension ($\tau = 0$), with $k = 0.5$, and different values of the viscosity parameter: (**a**): $\varepsilon = 0.1$, $\delta = 0.6$; (**b**): $\varepsilon = 0.1$, $\delta = 4.0$; (**c**): $\varepsilon = 0.3$, $\delta = 0.6$ and (**d**): $\varepsilon = 0.3$, $\delta = 4.0$. Here, the effect of viscosity and the small parameter ε on the soliton are illustrated.

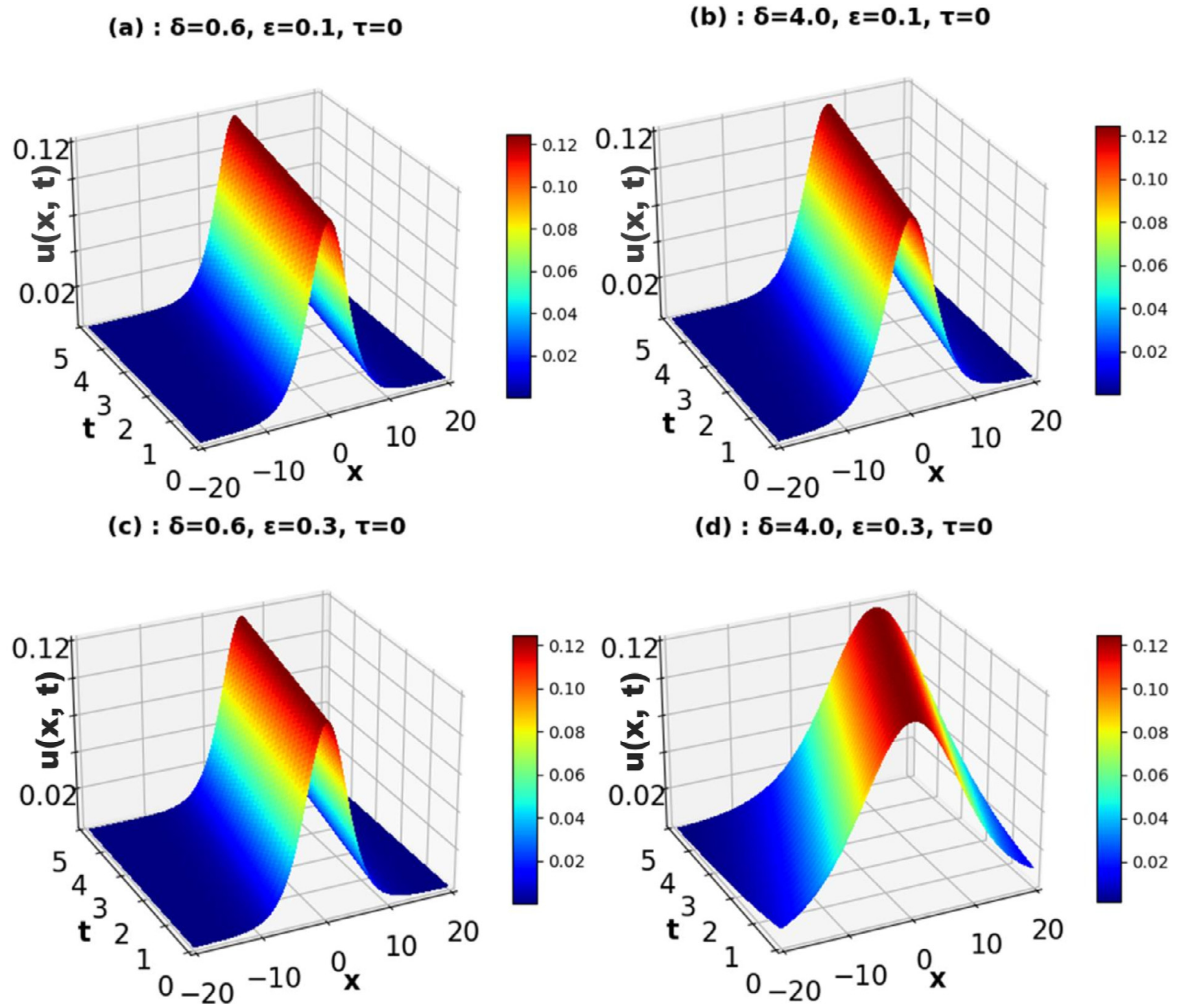

Fig. 5. (2D)-Plot of the soliton solutions of the generalized inhomogeneous KdV Eq. (1) without effect of surface tension ($\tau = 0$), with $\varepsilon = 0.3$, $t = 0.5$, $k = 0.5$, for different values of viscosity parameter: (Blue line): $\delta = 0.6$, (Red line): $\delta = 1.5$, (Purple line): $\delta = 3.0$, (Black line): $\delta = 4.0$.

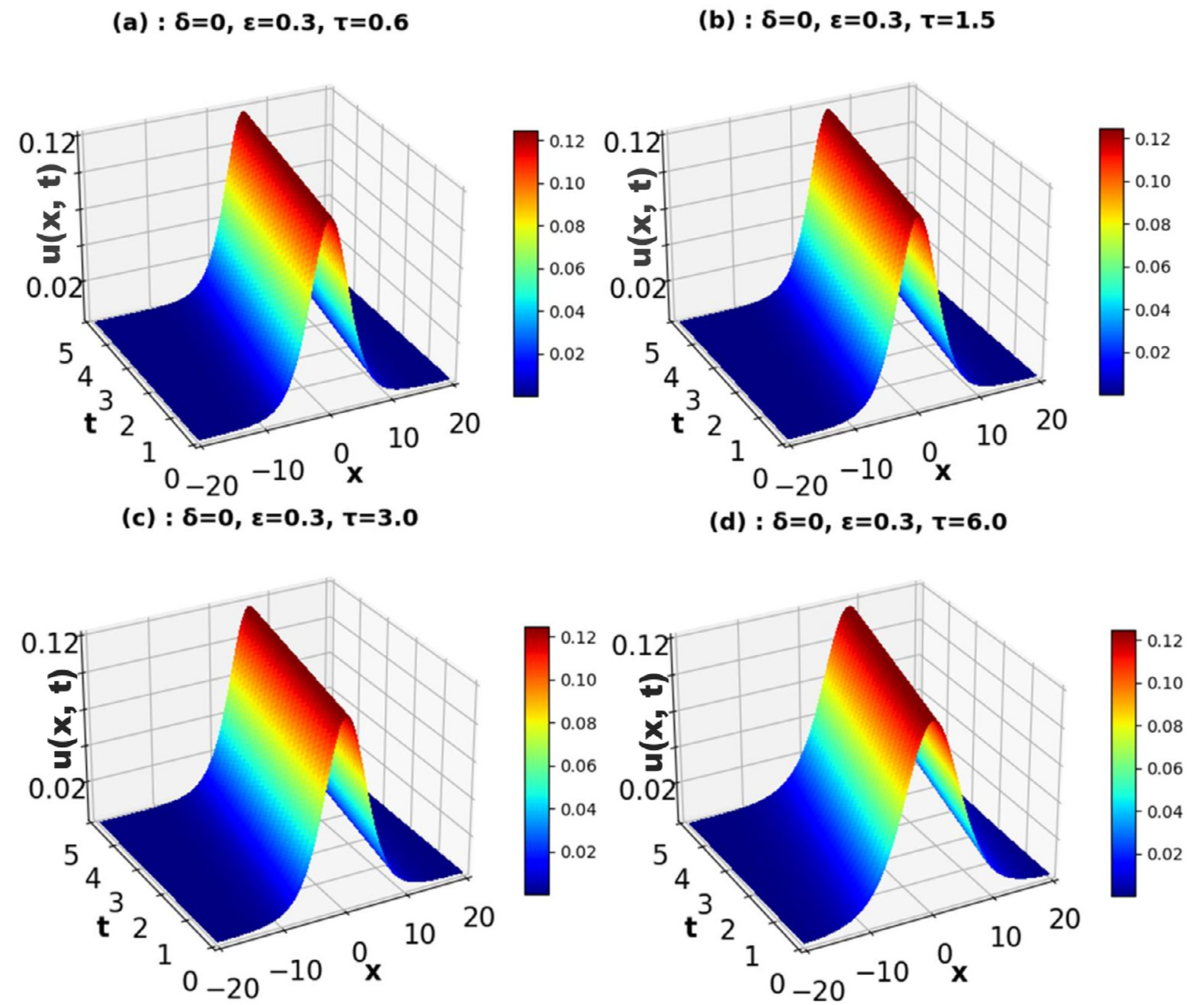

Fig. 6. Soliton solutions of the generalized inhomogeneous KdV Eq. (1) without effect of viscosity ($\delta = 0$), with $k = 0.5$, $\varepsilon = 0.3$ and different values of the tension surface parameter: (**a**): $\tau = 0.6$; (**b**): $\tau = 1.5$; (**c**): $\tau = 3.0$ and (**d**): $\tau = 6.0$. Here, the effect of surface tension on the soliton is illustrated.

4.1. Soliton solutions for the equation of the right-moving wave

A variety of powerful methods for finding soliton solutions in complex physical systems have been developed, such as Darboux transformation [70] Bäcklund transformation [71], inverse scattering transformation [72], Hirota bilinear method [73], symmetry method [74] and similarity transformation [75,76], juste to cite a few. Among the different methods available to investigate soliton solutions in various equations, the Hirota bilinear method has several advantages according to other methods. The method obtains the results directly, quickly, and needs simple algorithms in programming. The method also allows testing if a certain equation satisfies the necessary requirements to admit soliton solutions [39]. In this section, we apply Hirota's bilinear method to look for the soliton type solutions of Eq. (1). For this, we assume that the general form of solution is given by

$$u(x,t) = D\frac{\partial^2}{\partial x^2}\ln(f(x,t)) = D\frac{ff_{2x} - f_x^2}{f^2}. \tag{68}$$

The parameter D is a constant to be determined and $f(x,t)$ is the auxiliary function which can be written in the form of a traveling wave solution such that

$$f(x,t) = 1 + f_1(x,t) = 1 + \exp(\theta), \tag{69}$$

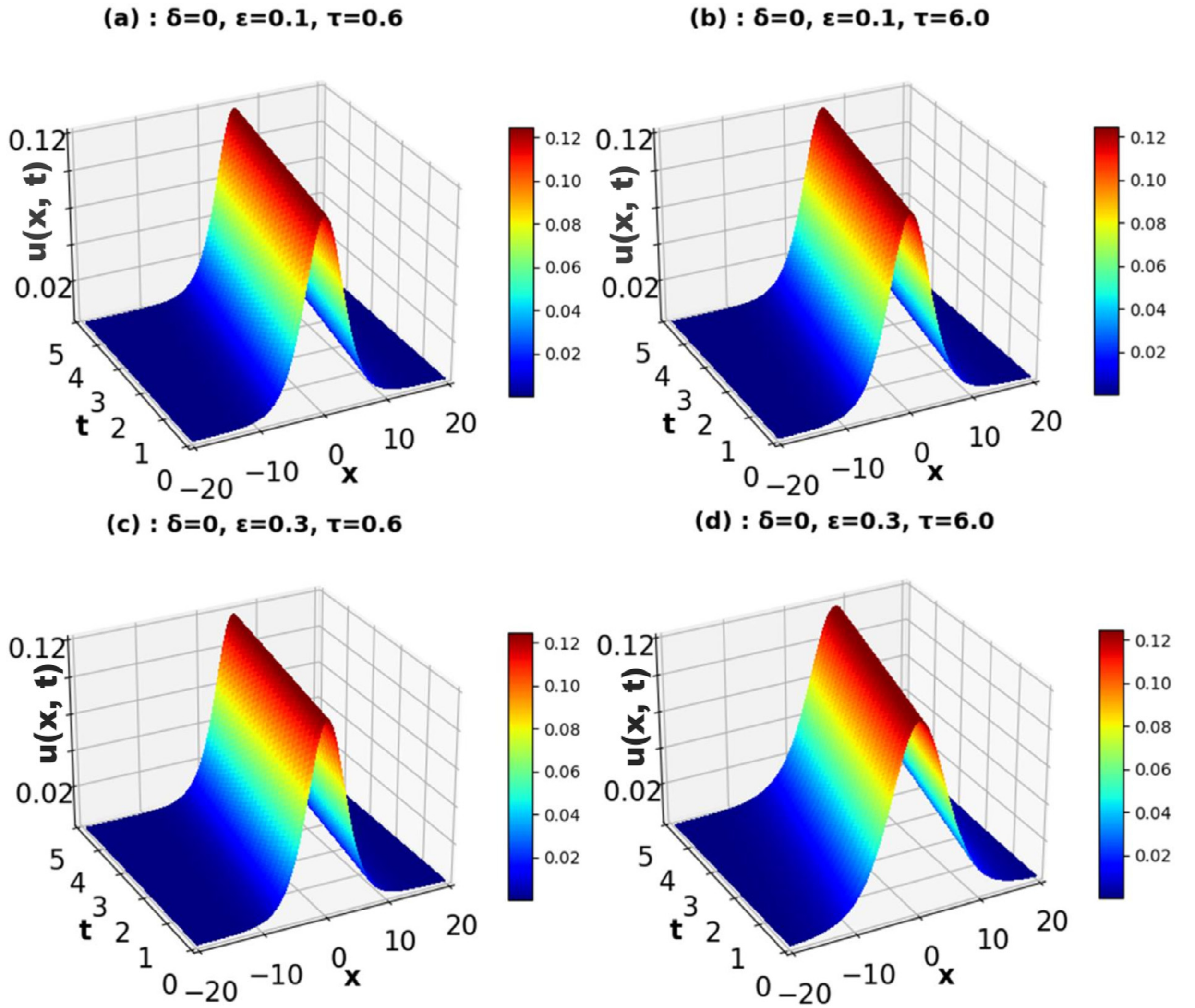

Fig. 7. Soliton solutions of the generalized inhomogeneous KdV Eq. (1) without effect of viscosity ($\delta = 0$), with $k = 0.5$, and different values of the surface tension parameter: (**a**): $\varepsilon = 0.1$, $\tau = 0.6$; (**b**): $\varepsilon = 0.1$, $\tau = 6.0$; (**c**): $\varepsilon = 0.3$, $\tau = 0.6$ and (**d**): $\varepsilon = 0.3$, $\tau = 6.0$. Here, the effect of surface tension and the small parameter ε on the soliton are illustrated.

where $\theta = kx - \omega t + \xi_0$ and the parameters k, ω and ξ_0 are the wave number, the angular frequency and the phase shift, respectively. Substituting $u(x,t) = \exp(\theta)$ into the linear terms of (1) we obtain the dispersion relation

$$\omega = \frac{1}{15120}\Bigg[15120k + \varepsilon\Big[(2520 - 7560\tau)k - 15120\delta\Big]k^2 + \varepsilon^2\Big[(798 - 1260\tau - 1890\tau^2)k^3 - 3780\delta\Big]k^2 + \varepsilon^3\Big[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 - 2835\tau\delta\Big]k^4\Bigg]. \tag{70}$$

To determine the constant D, we introduce Eq. (68) into Eq. (1) where, $f(x,t)$ is taken as

$$f(x,t) = 1 + \exp\Bigg\{kx - \frac{1}{15120}\Bigg[15120k + \varepsilon\Big[(2520 - 7560\tau)k - 15120\delta\Big]k^2 + \varepsilon^2\Big[(798 - 1260\tau - 1890\tau^2)k^3 - 3780\delta\Big]k^2 + \varepsilon^3\Big[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 - 2835\tau\delta\Big]k^4\Bigg]t\Bigg\}. \tag{71}$$

Equating the coefficients of the different powers of $\exp(\theta)$ to zero yields a system of polynomial equations, after solving this system with Mathematica, we find $D = 2$. This means that the soliton solution is given by

$$u(x,t) = \frac{2k^2 \exp(kx - \omega t)}{(1 + \exp(kx - \omega t))^2} = \frac{k^2}{1 + \cosh(kx - \omega t)}. \tag{72}$$

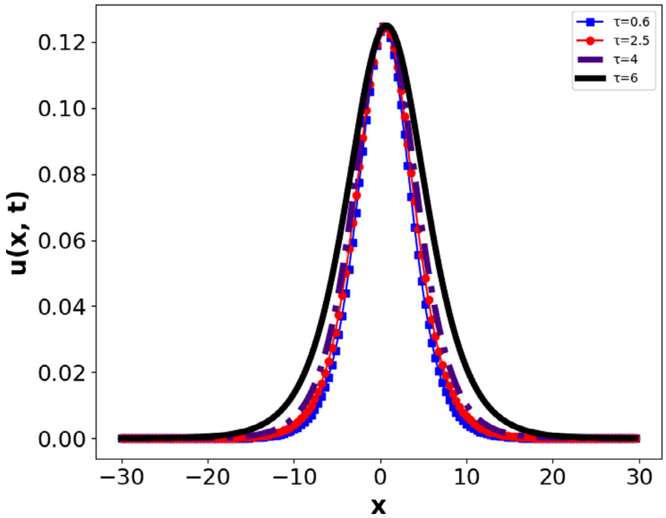

Fig. 8. (2D)-Plot of the soliton solutions for the generalized inhomogeneous KdV Eq. (1) without effect of viscosity ($\delta = 0$), with $\varepsilon = 0.3$, $t = 0.5$, $k = 0.5$, for different values of surface tension parameter: (Blue line): $\tau = 0.6$, (Red line): $\tau = 2.5$, (Purple line): $\tau = 4.0$, (Black line): $\tau = 6.0$.

Using Eq. (70) and after some transformations, we obtain

$$u(x,t) = \frac{k^2}{2}\,\mathrm{sech}^2\left\{\frac{k}{2}x - \frac{1}{30240}\left[15120k + \varepsilon\left[(2520 - 7560\tau)k - 15120\delta\right]k^2 + \varepsilon^2\left[(798 - 1260\tau - 1890\tau^2)k^3 - 3780\delta\right]k^2 + \varepsilon^3\left[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 - 2835\tau\delta\right]k^4\right]t\right\}. \tag{73}$$

This soliton solution describes the long, small-amplitude, unidirectional wave motion in shallow water with surface tension and viscosity effects.

4.2. Phase and group velocity

Soliton is a perturbation that moves through a medium. It is therefore possible to associate it with two wave velocities, namely the phase velocity and the group velocity, which are sometimes non equal. Thus, from Eq. (70), we can easily deduce that the phase velocity of the soliton is given by

$$v_{ph} = \frac{\omega}{k} = \frac{1}{15120}\Big(15120 + \varepsilon[(2520 - 7560\tau)k - 15120\delta]k + \varepsilon^2[(798 - 1260\tau - 1890\tau^2)k^3 - 3780\delta]k + \varepsilon^3[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 - 2835\tau\delta]k^3\Big). \tag{74}$$

The k-dependence of the phase velocity is shown in Figs. 1-(a) and 1-(b). Both Figs. 1-(a) and 1-(b) show that, the phase velocity is an increase function of the wave number. The Fig. 1-(a), shows the influence of the viscosity (parameter δ) on the phase velocity curve. Indeed, we plot the phase velocity as a function of the wave number for three values of δ. We see that when the value of the parameter δ increases, the amplitude of the phase velocity decreases very weakly. We have also investigated the impact of the surface tension (parameter τ) on the phase velocity. We plot the phase velocity as a function of the wave number for three values of τ. The result illustrated in Fig. 1-(b) clearly shows that, the amplitude of the phase velocity strongly decreases when the value of the parameter τ increases. In conclusion, both viscosity and surface tension have the same effect on the phase velocity. However, the effect of the surface tension is very greater than that of the viscosity.

The group velocity calculated from Eq. (70) can be written in the following form

$$v_{gr} = \frac{\partial\omega}{\partial k} = \frac{1}{15120}\Big(15120 + 2\varepsilon[(2520 - 7560\tau)k - 15120\delta]k + \varepsilon(2520 - 7560\tau)k^2 + 3\varepsilon^2(798 - 1260\tau - 1890\tau^2)k^4 + 3\varepsilon^3(275 - 399\tau - 375\tau^2 - 945\tau^3)k^6 + 2\varepsilon^2[(798 - 1260\tau - 1890\tau^2)k^3 - 3780\delta]k + 4\varepsilon^3[(275 - 399\tau - 375\tau^2 - 945\tau^3)k^3 - 2835\delta\tau]k^3\Big). \tag{75}$$

The k-dependence of the group velocity is shown in Figs. 1-(c) and 1-(d). These figures show that, the group velocity is an increasing function of the wave number. In order to illustrate the effects of viscosity and surface tension on the group

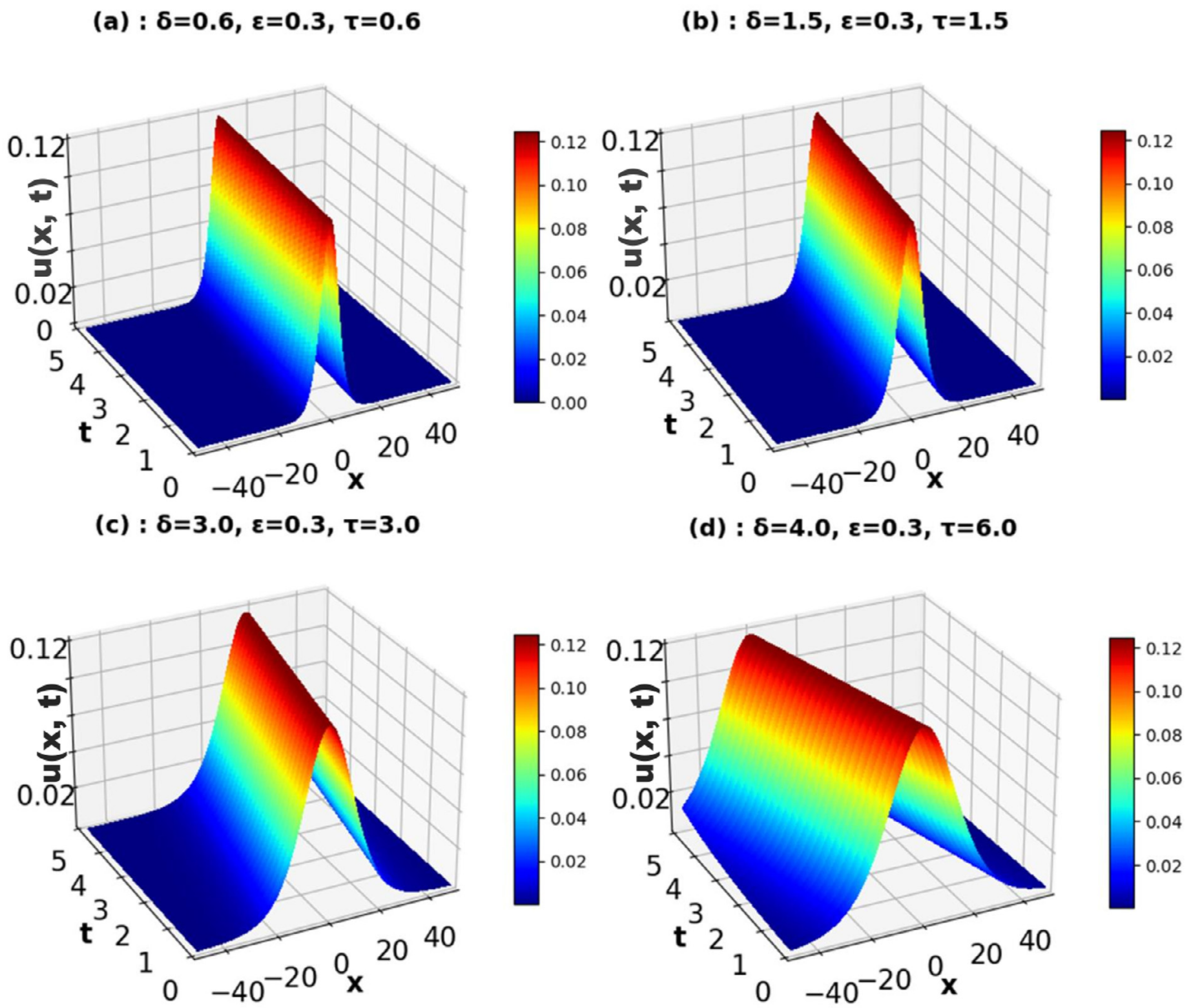

Fig. 9. Soliton solutions for the generalized inhomogeneous KdV Eq. (1) with $k = 0.5$, $\varepsilon = 0.3$ and different values of surface tension and viscosity parameters: **(a)**: $\tau = 0.6$, $\delta = 0.6$; **(b)**: $\tau = 1.5$, $\delta = 1.5$; **(c)**: $\tau = 3.0$, $\delta = 3.0$ and **(d)**: $\tau = 6.0$, $\delta = 4.0$. Here, the combined effect of surface tension and viscosity on the soliton is illustrated.

velocity, we plot the group velocity for three values of parameters δ and τ, respectively. The result shows that, both effects of viscosity and surface tension on the group velocity are the same as those for phase velocity. However, by a carefully observation of Figs. 1-(a) and 1-(b) or Figs. 1-(c) and 1-(d), we see that the amplitude of group velocity is greater than that of phase velocity.

To better investigate the effect of viscosity on the phase and group velocities, we plot both phase and group velocities as a function of the viscosity parameter δ.

Fig. 2 [(a), (b), (c), (d), (e) and (f)] shows that, both phase velocity (blue line) and group velocity (dashed red line) decrease when the viscosity increases. It is clearly shows the influence of the surface tension τ on the group and phase velocities as a function of the viscosity curves.

In fact, we see that in the interval of τ values [0.0, 0.2, 0.3, 0.4], the two velocities have the same sign. This can mean that, the wave packet envelope and the phase both move to the right.

In the interval of τ values [0.0, 0.2], the amplitude of the group velocity is greater than that of phase velocity, meaning that, the envelope of the wave packet moves faster than the phase.

In Fig. 2-(c), for $\tau = 0.3$ and $\delta = 0.3$, the two velocities have same values, in other words, the envelope of the wave packet and the phase move at the same speed. However, for $\tau = 0.3$ and $\delta > 0.3$ and for τ values in the interval [0.4, 0.5, 0.6], the amplitude of the phase velocity is greater than that of group velocity. Thus, the phase moves faster than the envelope of the wave packet.

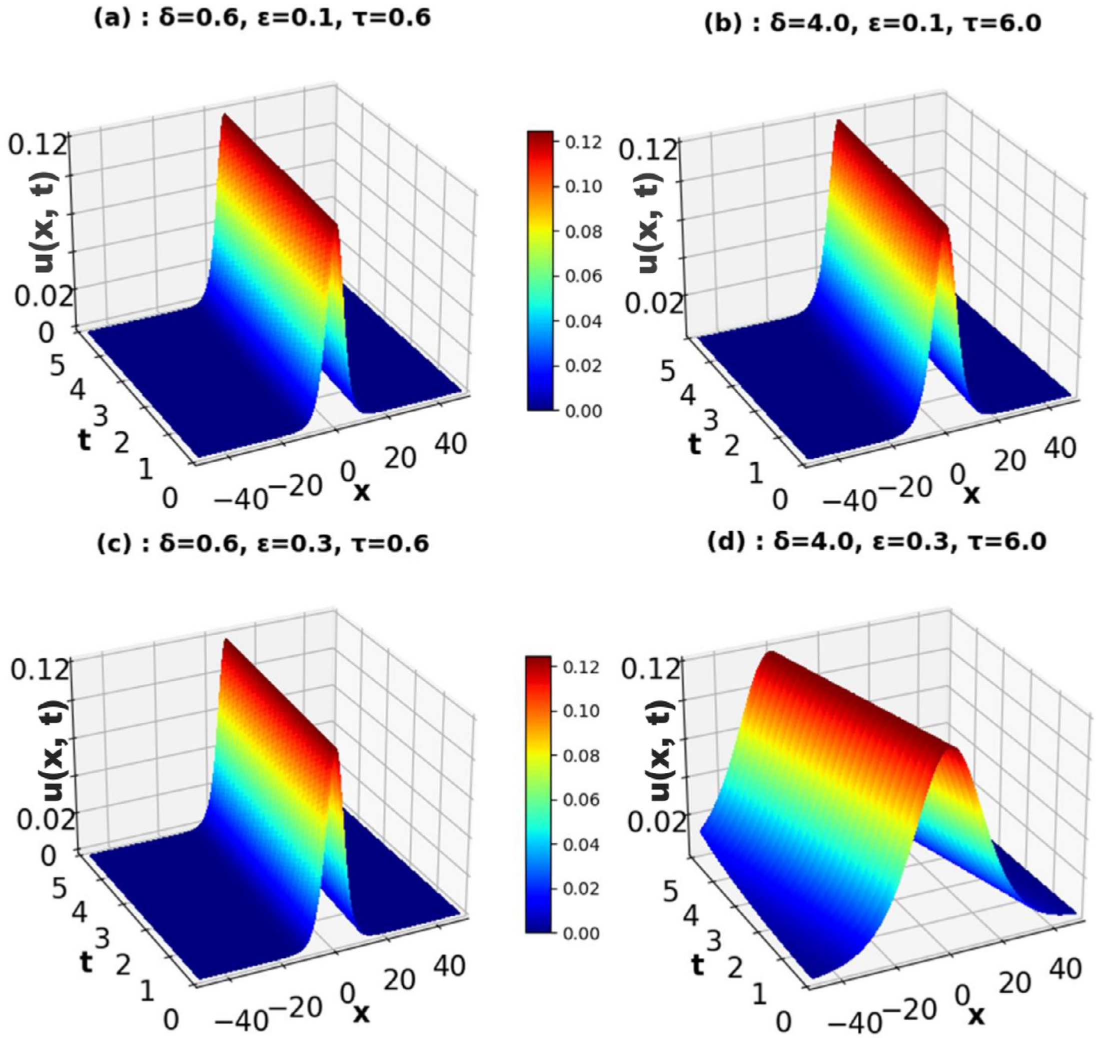

Fig. 10. Soliton solutions of the generalized inhomogeneous KdV Eq. (1) with $k = 0.5$, and different values of the surface tension and viscosity parameters: **(a)**: $\varepsilon = 0.1$, $\tau = 0.6$, $\delta = 0.6$; **(b)**: $\varepsilon = 0.1$, $\tau = 6.0$, $\delta = 4.0$; **(c)**: $\varepsilon = 0.3$, $\tau = 0.6$, $\delta = 0.6$ and **(d)**: $\varepsilon = 0.3$, $\tau = 6.0$, $\delta = 4.0$. Here, the effect of small parameter ε and the combined effect of surface tension and viscosity on the soliton are illustrated.

In Fig. 2-(e), for $\tau = 0.5$ and $\delta \simeq 0.32$, the group velocity is zero, meaning that the wave propagates without energy transfer. However, for $\tau = 0.5$ and $\delta > 0.32$ and Fig. 2-(f), the two velocities have different signs. This leads to the fact that, when the wave packet envelope moves to the left, the phase moves to the right.

4.3. Effects of surface tension, viscosity and amplitude parameter on the soliton dynamics

In the following, we investigate the effects of the viscosity, the surface tension and the amplitude parameter on the soliton dynamics. It has been revealed that such effects can have an impact in the dynamics of shallow water waves [36,56,57].

The effects of the viscosity on the soliton dynamics are illustrated in Figs. 3, 4 and 5. It is shown in Fig. 3 $[(a), (b), (c)$ and $(d)]$ that, for the same value of the small parameter ε, the viscosity strongly impact the width of the soliton. Indeed, when the values of viscosity paramer δ increases, the width of the soliton also increases, but its amplitude remains constant.

Fig. 4 $[(a), (b), (c)$ and $(d)]$ illustrates the impact of the amplitude parameter ε on the soliton dynamics. It is revealed that, the growth of ε amplifies the effects of viscosity while keeping the amplitude of soliton constant.

Fig. 6 shows the 2-dimensional plot of the soliton under the effects of the viscosity. Indeed, in Fig. 6, by comparing the width of the soliton represented by the blue ($\delta = 0.6$), red ($\delta = 1.5$), purple ($\delta = 3.0$) and black ($\delta = 4.0$) lines, it is clear that the width of the soliton increases with the value of the viscosity parameter. This may therefore suggests that a soliton propagating to the right in a viscous medium tends to increase (respectively decrease) in width if the viscosity of medium increases (respectively decreases).

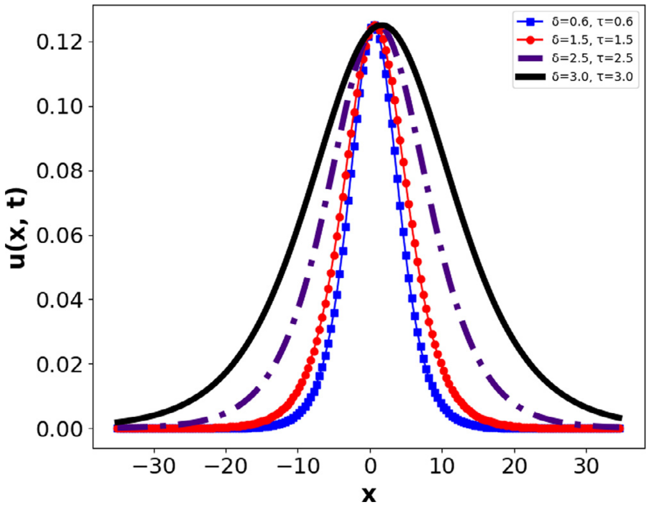

Fig. 11. (2D)-Plot of the soliton solutions for the generalized inhomogeneous KdV Eq. (1) with $\varepsilon = 0.3$, $t = 0.5$, $k = 0.5$, for different values of surface tension and viscosity parameters: (Blue line): $\delta = 0.6,\ \tau = 0.6$, (Red line): $\delta = 1.5,\ \tau = 1.5$, (Purple line): $\delta = 2.5\ \tau = 2.5$, (Black line): $\delta = 3.0,\ \tau = 3.0$.

Figs. 6, 7 and 8 illustrate the behavior of the soliton solution of Eq. (1) under the effect of surface tension. We observe that, the effects of surface tension are the same as those of viscosity. However, the comparison between Fig. 5 and Fig. 8 reveals that the effects of the viscosity on the width of soliton are greater important than that of surface tension.

Fig. 7 shows that, the growth of small parameter ε also amplifies the effects of surface tension while keeping the amplitude of the soliton unchanged.

In Figs. 9, 10 and 11, we take into account both the effects of viscosity and surface tension. It is seen that the effects of viscosity are strongly amplified by the effects of surface tension.

5. Conclusion

The aim of this work was to study the dynamics of solitons in an incompressible and viscous fluid, flowing in right- and left-directions in a shallow channel. We have used the Boussinesq perturbation expansion and a linear approximation applied to the Navier-Stokes equations to derive a system of coupled equations for the scaled horizontal velocity w and the surface elevation η. The assumption of a one directional (right-moving) wave and a relationship between horizontal velocity at mean height and elevation were used to decouple the system. The results have led to a generalized KdV equation including viscosity and surface tension effects. We have developed a procedure that reveals the non-uniqueness of the decomposition of the Boussinesq system expansion into a system of coupled equations for the surface elevation associated with right- and left-moving waves. The right-moving wave evolution equation found here is applicable not only to the present problem, but also to other problems of long-wave oscillatory instability in dissipative systems, such as surface waves in convection fluids [36], waves in plasmas [77] and others. The soliton solutions obtained by the Hirota bilinear method have been plotted for some parameter values. We have shown that these parameters can strongly impact the phase velocity, the group velocity and the width of the soliton. The results clearly revealed that, both surface tension and viscosity have strong effects on the dynamics of wave in shallow water. Our investigation can be extended to several physical interesting situations, such as (3+1)-dimensional water wave problems [47]. Work in this direction is under consideration and will be published in the future.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

L. Fernand Mouassom: Conceptualization, Investigation, Formal analysis, Methodology, Writing - original draft, Validation, Writing - review & editing. **T. Nkoa Nkomom:** Formal analysis, Investigation, Methodology, Writing - original draft. **Alain Mvogo:** Conceptualization, Validation, Project administration, Writing - review & editing. **Cesar Bioule Mbane:** Supervision, Validation, Project administration.

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