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EXISTENTIAL GENERALIZATION ON ATTRIBUTES IN FORMAL CONCEPT ANALYSIS

THESIS

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Dedication

This thesis is dedicated to my family

For their endless encouragement and support

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Abstract

In almost every domain of life, there are situations where big databases are used. In these situations, extracting and exploiting information from such databases become extremely difficult. In Formal Concept Analysis, such information is called "patterns". When these data are transformed into formal contexts, patterns can be extracted, mainly in two forms: i) formal concepts describing object sets together with their common attributes, and ii) association rules including implications between attributes or objects. More often, the number of these patterns appears very large in a context, making them difficult to be studied. Many authors have proposed several methods of reducing the number of these patterns, notably in [7, 26, 27, 39, **?**, 43, 44, 45]. Generalization of attributes is one of these methods.

Generalization on attributes in a formal context is a method of aggregation of attributes in order to form new attributes called generalized attributes. It was first mentioned in [34] where the authors consider a taxonomy on items to extract relevant information in the formal context of a transactions database in the form of association rules. However, in this study, the authors considered both the items of the leaves of the taxonomy and that of the others nodes, called generalized items. With the type of generalization described in [27], the attributes which are put together do not appear in the generalized context, and then the generalized context has a size less than the initial one. Depending of the way attributes are grouped in a formal context, there are three types of generalization (see [27]): the universal generalization denoted by ($\forall$-generalization), the alpha-generalization denoted by (α-generalization) and the existential generalization denoted by ($\exists$-generalization). By reducing the size of the context trough a generalization, one expects to also reduce the size of the concept lattice. But that is not always the case, especially with the $\exists$-generalization.

In this work, we have brought our contribution to the resolution of the following problems: i) The study of the size of concept lattices: by studying a special case, we have shown that in the $\exists$-generalization, the size of concept lattices can increase exponentially. Then, we have studied the worst case of increase one can get after an $\exists$-generalization on a pair of attributes in a given formal context; and to round up, we have presented some conditions for which the $\exists$-generalization stabilizes the size of concept

lattices. ii) The search of a method of grouping attributes: here, we have proposed a way of grouping attributes such that the size of the concept lattice does not increase after an ∃-generalization. By observing some existing similarity measures, we have found that they do not enhance a decision on wether the size of the concept lattice increases or no. This gave us enough reason to construct a new similarity measure compatible with the ∃-generalization, and such that putting together similar attributes do not leads to more new concepts than putting together non similar ones. iii) The study of the relation between implications of the initial formal context and that of the generalized formal context: here, we have mainly studied the variation of the size of the set of all informative implications between the initial formal context and the generalized formal context.

Key Words : Formal Concept Analysis, ∃-generalization, Attribute, Similarity Measure, Attributes Implication.

Résumé

Dans la quasitotalité des domaines de la vie, on rencontre des situations où de très grandes bases de données sont présentes. Dans ces situations, on a des difficultés d'extraction et d'exploitation d'information de ces bases de données. En Analyse des Concepts Formels, ces informations sont appelées "motifs". Lorsque ces bases de données sont transformées en contextes formels, des motifs peuvent être extraits, principalement sous deux formes: i) des concepts formels décrivant des ensembles d'objets avec leurs attributs communs, et ii) des règles d'association parmi lesquelles les implications entre des attributs. Très souvent, le nombre de ces motifs apparait très grand dans un contexte, les rendant ainsi difficile à étudier. Plusieurs auteurs ont proposé des méthodes de réduction du nombre de ces motifs, notamment dans [7, 26, 27, 39, ?, 43, 44, 45]. La généralisation est l'une de ces méthodes.

La généralisation en Analyse des Concepts Formels est une technique d'aggrégation d'attributs ou d'objets dans le but de former d'autres attributs appelés attributs généralisés. Il a été pour la première fois mentionné dans [34], dans lequel les auteurs ont utilisé une taxonomie sur les objets ou les attributs pour extraire d'importantes informations dans le contexte formel d'une base de données de transactions, notamment sous forme de règles d'association. Cependant, les auteurs de cette étude ont considéré aussi bien les attributs situés aux extremités de la taxonomie que ceux se trouvant sur d'autres noeuds, appelés attributs généralisés. Avec la généralisation décrite dans [27], les attributs qui sont généralisés n'apparaissent pas dans le contexte généralisé, donnant ainsi à celui-ci une taille automatiquement inférieure à celle du contexte initial. En fonction de la manière dont les attributs sont groupés dans un contexte formel, on distingue trois type de généralisation (voir [27]): la généralisation universelle notée ($\forall$-généralisation), la généralisation α notée (α-généralisation) et la généralisation existentielle notée ($\exists$-généralisation). En réduisant la taille du contexte formel à partir d'une généralisation, on s'attend aussi à une réduction de la taille du treillis des concepts. Cependant, ce n'est pas toujours le cas, notamment avec la généralisation existentielle.

Dans ce travail, nous avons apporté notre contribution dans la résolution des problèmes suivants: i) L'étude de la taille du treillis des concepts: en étudiant un cas précis, nous avons montré que dans la généralisation existentielle, la taille du treillis de concepts peut augmenter exponenciellement. Ensuite, nous avons étudier le pire des cas d'augmentation que l'on puisse obtenir après une généralisation existentielle d'une paire s'attributs dans un contexte formel donnée et enfin nous avons présenté quelques conditions pour lesquelles la taille du treillis se stabilise. ii) La recherche d'une méthode de regroupement des attributs: ici, nous avons proposé une méthode de regroupement d'attributs assurant la non augmentation de la taille du treillis de concepts après une généralisation existentielle. En examinant les mesures de similarité existantes, nous avons constaté qu'elles ne permettent pas de décider sur l'augmentation ou non de la taille du treillis des concepts après regroupement des attributs, ceci nous a motivé à construire une nouvelle mesure de similarité compatible avec la généralisation existentielle, de sorte que la mise en commun d'attributs similaires ne conduise pas à plus de concepts que la mise en commun d'attributs non similaires. iii) l'étude du lien entre les implications du contexte de départ et celles du contexte généralisé: ici, nous avons essentiellement étudié les variations en nombre des implications informatives entre le contexte de départ et le contexte généralisé.

Mots clés : Analyse des Concepts Formels, Généralisation Existentielle, Attribut, Mésure de Similarité, Implication sur les Attributs.

Contents

General Introduction

0.1 Motivations

In an economic environment where the population keeps growing, the demand for goods and services tends to follow the population growth. This is not only observed in developed or emerging countries where the standard of living is high, but is more and more a reality in almost all developing countries, especially those that are involved in the fight against poverty. Hence, in their attempt to increase production so as to meet the rising demand, enterprises often face the problem of management of big databases, which contain information about their customers or about the transactions made by their customers. Such information can cover different kinds of products sold, the characteristic of these products, their sources, their prices,... Most often, they are analysed in order to predict the customers choices.

In such big databases, enterprises often have enormous difficulties extracting useful information just by observation, and even when they succeed, the volume of information extracted is often too much that it makes it very difficult for them to select the relevant ones. Hence, the needed information can be contained into thousands of information from data. Hence, to efficiently extract relevant information from such big databases, experts often look for a means of reducing both the size of the databases and that of the volume of information that they contain. Formal Concept Analysis has so far proposed several technics ([27]) of extraction of useful information from big data; mainly in the form of clusters (or concepts), and association rules. The set of all the concepts forms a lattice called concept lattice.

To control the size of concept lattices, some authors have proposed the constraining of concept lattices by attributes dependencies in the form of attribute implications [40] or in the form of attributes-dependency formulas [6]. This method consisted of defining projection mappings on the lattice, such that the analysis is carried out only on the side of the lattice satisfying these attributes dependencies. Some other methods have been suggested: decomposition [43, 44, 45] where concept lattices of direct sums and direct products of formal contexts are respectively decomposed into direct products and tensor products for better viewing; iceberg lattices [39] which consist of considering only the concepts that the intents have a greater support than a user-specified minimum support (minsupp, with $0 \leqslant minsup \leqslant 1$); α-Galois lattices [**?**] in which extensional projections

$proj_\alpha$ (for $0 \leqslant \alpha \leqslant 1$) which reduce the initial concept lattice by modifying the notion of extents, are defined on the powerset of the entities of the context; Fault tolerant patterns [7] in which some extensions of formal concepts are often computed instead of formal concepts in other to reduce the number of patterns, notably when there are noises in the data set (and then the number of concepts tends to explode); closure or Kernel operators and approximation [26] which permit to obtain from a concept lattice, a new lattice of smaller size without a huge loss of information; and generalization [27]. In the present contribution, we are following the direction in [27], where some attributes are put together to define generalized attributes.

When some attributes are put together, the stake is to decide whether an object has this new combined attribute. Different scenarios have been discussed in [27], among which the ∃-generalization.

0.2 Objectives and results found

Generalization of attributes leads to the decrease in the size of formal contexts, and one expects the size of the corresponding concept lattices to decrease as well. But that is not always the case. If some generalizations reduce both the size of the formal context and that of the corresponding concept lattice, it is not always the case for others: in [27], a case of ∃-generalization was presented where the size of the concept lattice increases by 1. Natural questions are then raised: how far can the increase in the size of the concept lattice be after an ∃-generalization? and how can the attributes be put together to ensure the decrease in the size of the concept lattice?

The main aim of this PhD thesis is to determine to what extent the increase in size of a concept lattice can be, after an ∃-generalization on attributes and to propose a way of grouping attributes together by ∃-generalization such that both the size of the formal context and that of the corresponding concept lattice decrease. It has a great and positif impact on reducing the size of big data (big formal context) and the quantity of information (concept) in these data in order to make their analysis more easy. We have therefore contributed at three levels in generalizing attributes in Formal Concepts Analysis: At the first level, we have shown that the size of the lattice can increase exponentially after an ∃-generalization, and we have presented the worst case of increase one can get. At the second level, we have constructed a new similarity measure on attributes such that putting together similar attributes would not lead to more new concepts as grouping together non similar ones. Thirdly, we have studied the size of the set of informative attributes implications between the initial context and the generalized one, and discovered that an increase in size of the lattice after an ∃-generalization is not always compensated by a reduction of the basis of implications.

0.3 Organisation of the thesis

This work is divided into five (05) chapters.

In Chapter 1, some preliminaries are presented, notably those that can permit a good understanding of the problem discussed in the thesis. They are in relation with lattice theory, Formal Concept Analysis and Generalization.

In Chapter 2, one of the main problems is addressed: to what extent can the increase in the size of the concept lattice be after an $\exists$-generalization on a pair of attributes? Here, we study the effect of adding a new attribute to a formal context. After that, we present a family of formal contexts containing a special pair of attributes such that when put together, it leads to an exponential increase in the size of the concept lattice. Moreover, we expose a case where the $\exists$-generalization stabilizes the size of the lattice and discuss the maximal increase of size of a concept lattice after an $\exists$-generalization on a pair of attributes.

In Chapter 3, we answer the question of how to group attributes together so that their $\exists$-generalization does not lead to an increase in the size of the concept lattice. First, we introduce the notion of similarity measure and give a brief survey of the existing types of similarity measures. With specific examples, we show that the existing measures are not suitable for $\exists$-generalization. To round up, we propose a new similarity measure compatible with the $\exists$-generalization and test it on lexicographic data.

Chapter 4 examines the attribute implications while moving from an initial context to the generalized context. Here, we study the variation of informative implications while moving from a formal context to the corresponding generalized context, and present a case where the size of the canonical base of implications increases after the existential generalization.

Finally, the conclusion of this dissertation follows and proposes some future lines of research.

Chapter 1

Preliminaries

This section exposes some theoretical basic notions on lattices, Formal Concepts Analysis and Generalization. Most of the definitions and results of this section are from [10, 15, 27].

1.1 Some basic notions of lattice theory

There are two standard ways of defining a lattice: by means of algebraic structure and by means of order.

Definition 1.1.1. *[10] A nonempty set $\mathcal{L}$ together with two binary operations $\vee$ and $\wedge$ (read "join" and "meet" respectively) on $\mathcal{L}$ is called a lattice if it satisfies the following identities:*

i) $x \vee y \approx y \vee x$ *and* $x \wedge y \approx y \wedge x$ *(commutative laws);*

ii) $x \vee (y \vee z) \approx (x \vee y) \vee z$ *and* $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ *(associative laws);*

iii) $x \vee x \approx x$ *and* $x \wedge x \approx x$ *(idempotent laws);*

iv) $x \approx x \vee (x \wedge y)$ *and* $x \approx x \wedge (x \vee y)$ *(absorption laws).*

Example 1.1.1. *Let X be a set, and $\mathcal{P}(X)$ the powerset of X. Then $\mathcal{P}(X)$ together with $\cup$ and $\cap$ ("join" and "meet") is a lattice.*

Definition 1.1.2. *[10] A binary relation $\leqslant$ defined on a set A is a partial order on the set A if the following conditions hold identically in A:*

i) $a \leqslant a$ *(reflexivity);*

ii) $a \leqslant b$ *and* $b \leqslant a$ *imply* $a = b$ *(antisymmetry);*

iii) $a \leqslant b$ *and* $b \leqslant c$ *imply* $a \leqslant c$ *(transitivity).*

If in addition, $a \leqslant b$ or $b \leqslant a$ for every $a, b \in A$, then we say that $\leqslant$ is a ***total order*** *on A.*

Example 1.1.2. *Let X be a set, and $\mathcal{P}(X)$ the powerset of X. Then the relation $\subseteq$ is a partial order on $\mathcal{P}(X)$.*

Definition 1.1.3. *[10] Let A be a subset of a partially order set P. An element $p \in P$ is an **upper bound** for A if $a \leqslant p$ for every $a \in A$. An element $p \in P$ is the **least upper bound** of A or **supremum** of A (sup(A)) if p is an upper bound of A, and $a \leqslant b$ for every $a \in A$ implies $p \leqslant b$.*

*Similarly, we can define what it means for p to be a **lower bound** of A, and for p to be the **greatest lower bound** also called the **infimum** of A (inf(A)).*

We now give the second definition of a lattice.

Definition 1.1.4. *[10] A poset L is a lattice if for every $a, b \in L$ both $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$ exist in L.*

In lattices, some elements are often called join-irreducible.

Definition 1.1.5. *[1] Let $(L, <)$ be a complete lattice and v an element of L. One set $v_* = \bigvee\{x \in L; x < v\}$. v is join-irreducible if $v \neq v_*$.*

Some of the most studied classes of lattices are that of distributive lattices and complete lattices.

Definition 1.1.6. *[10] A **distributive lattice** is a lattice which satisfies both of the distributive laws:*

i) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;

ii) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$.

Definition 1.1.7. *[10] A partially ordered set (Poset) P is complete if for every subset A of P, both $Sup(A)$ (denoted by $\vee A$) and $Inf(A)$ (denoted by $\wedge A$) exist in P.*

*All complete Posets are lattices and a lattice L which is complete as a poset is a **complete lattice**.*

Theorem 1.1.1. *[10] Let P be a poset such that $\wedge A$ exists for every subset A, or $\vee A$ exists for every subset A. Then P is a complete lattice.*

Example 1.1.3. *Let X be a set and $\mathcal{P}(X)$ the powerset of X. Then $\mathcal{P}(X)$ together with the usual ordering $\subseteq$ is a complete lattice.*

Definition 1.1.8. *[10] A sublattice L' of a complete lattice L is called a **complete sublattice** of L if for every subset A of L' the elements $\vee A$ and $\wedge A$, as defined in L, are actually in L'.*

Complete lattices are sometimes produced or recognised through closure operator.

Definition 1.1.9. *[10] Let A be a set. A mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is called a **closure operator** on A if for $X, Y \subseteq A$, it satisfies:*

- $X \subseteq C(X)$ *(extensive);*

- $C^2(X) = C(X)$ *(idempotent);*

- $X \subseteq Y$ *implies* $C(X) \subseteq C(Y)$ *(monotone);*

Given a closure operator C on $\mathcal{P}(A)$, a subset X of A is called a **closed subset** if $C(X) = X$. That poset of closed subsets of A with set inclusion as the partial ordering is denoted by L_C.

Theorem 1.1.2. *[10] Let C be a closure operator on a set A. Then L_C is a complete lattice with*

$$\bigvee_{i\in I} C(A_i) = C(\bigcup_{i\in I} A_i)$$

and

$$\bigwedge_{i\in I} C(A_i) = \bigcap_{i\in I} C(A_i)$$

Theorem 1.1.3. *[10] Every complete lattice is isomorphic to the closed subsets of some set A with a closure operator C.*

The above basic notions on lattice theory are often used in Formal Concept Analysis.

1.2 Basis of Formal Concepts Analysis

Formal Concepts Analysis is often based on some important notions that we present in the following definitions, mostly from [18] and [4].

Definition 1.2.1. *A formal context is a triplet (G, M, I) where G and M are non-empty sets and I is a binary relation between G and M, i.e, $\mathrm{I} \subseteq G \times M$.*

For a formal context, elements g from G are called **objects** and elements m from M are called **attributes**. $(g, m) \in \mathrm{I}$ indicates that object g has attribute m.

For a given cross table with n rows and p columns, a corresponding formal context (G, M, I) consists of a set $G = \{g_1, ..., g_n\}$, a set $M = \{m_1, ..., m_p\}$, and a relation I defined by: $(g, m) \in \mathrm{I}$ if and only if the table entry corresponding to row i and column j contains $\times$.

Example 1.2.1. *The following table is a formal context describing several transactions in a shop.*

	cake	*bread*	*chocolate*	*butter*
T_1	×	×	×	×
T_2	×		×	×
T_3		×	×	×
T_4		×	×	×
T_5	×			

Every formal context induces a pair of operators, the so-called concept-forming operators.

Definition 1.2.2. *For a formal context* (G, M, I)*, operators* $(^{\mathrm{I}}) : \mathcal{P}(G) \to \mathcal{P}(M)$ *and* $(^{\mathrm{I}}) : \mathcal{P}(M) \to \mathcal{P}(G)$ *are defined for every* $A \subseteq G$ *and* $B \subseteq M$*, by*

$$A^{\mathrm{I}} = \{m \in M | \text{ for each } g \in A, (g, m) \in \mathrm{I}\},$$

and

$$B^{\mathrm{I}} = \{g \in G | \text{ for each } m \in B, (g, m) \in \mathrm{I}\}.$$

Example 1.2.2. *In the formal context of example* 1.2.1*, we have:* $\{T_2\}^{\mathrm{I}} = \{$*cake, chocolate, butter*$\}$, $\{T_2, T_3\}^{\mathrm{I}} = \{$*chocolate, butter*$\}$, $\{$*cake*$\}^{\mathrm{I}} = \{T_1, T_2, T_5\}$,$\{$*bread, chocolate*$\}^{\mathrm{I}} = \{T_1, T_3, T_4\}$.

The notion of formal concept is fundamental in FCA. Formal concepts are particular clusters in cross-tables, defined by means of attributes sharing.

Definition 1.2.3. *A* ***formal concept*** *in a context* (G, M, I) *is a pair* (A, B) *of* $A \subseteq G$ *and* $B \subseteq M$ *such that* $A^{\mathrm{I}} = B$ *and* $B^{\mathrm{I}} = A$.

Remark 1.2.1. *For a given formal context* (G, M, I) *and a given attribute* a*, we denote by* $a^{'}$ *the set of objects having the attribute* a.

For a formal concept (A, B) in (G, M, I), A and B are called the **extent** and **intent** of (A, B), respectively. Note the following verbal description of formal concepts: (A, B) is a formal concept if and only if A contains just objects sharing all attributes from B and B contains just attributes shared by all objects from A.

Example 1.2.3. *In the formal context of example* 1.2.1*,* $(A, B) = (\{T_1, T_2, T_3, T_4\}, \{$*chocolate, butter*$\})$ *is a formal concept.*

Definition 1.2.4. *For formal concepts* (A_1, B_1) *and* (A_2, B_2) *of* (G, M, I)*. One defines the binary relation* $(A_1, B_1) \leqslant (A_2, B_2)$ *iff* $A_1 \subseteq A_2$ *(iff* $B_2 \subseteq B_1$*). The concept* (A_1, B_1) *is called a* ***subconcept*** *of* (A_2, B_2) *and* (A_2, B_2) *is called a* ***superconcept*** *of* (A_1, B_1).

Example 1.2.4. *In the formal context of example* 1.2.1*, let* $(A_1, B_1) = (\{T_1, T_2\}, \{$*cake, chocolate, butter*$\})$ *and* $(A_2, B_2) = (\{T_1, T_2, T_3, T_4\}, \{$*chocolate, butter*$\})$ *be formal context. We have* $(A_1, B_1) \leqslant (A_2, B_2)$.

Definition 1.2.5. *Let denote by* $\mathfrak{B}(G, M, \mathrm{I})$ *the collection of all formal concepts of* (G, M, I)*, i.e,* $\mathfrak{B}(G, M, \mathrm{I}) = \{(A, B) \in \mathcal{P}(G) \times \mathcal{P}(M) \;/\; A^{\mathrm{I}} = B, \text{ and } B^{\mathrm{I}} = A\}$. $\mathfrak{B}(G, M, \mathrm{I})$ *equipped with the subconcept-superconcept ordering* $\leqslant$ *is called the concepts lattice of* (G, M, I).

In the above definition, if (A, B) and (A_1, B_1) are two concepts of a formal context $\mathbb{K}$, then
$\bigvee(\{(A, B), (A_1, B_1)\}) = ((A \cup A_1)^{\mathrm{II}}, B \cap B_1)$ and $\bigwedge(\{(A, B), (A_1, B_1)\}) = (A \cap A_1, (B \cup B_1)^{\mathrm{II}})$.

Using Lattice Miner, all concepts of a formal context can be automatically computed as shown in the following example with the context of example 1.2.1.

Example 1.2.5.

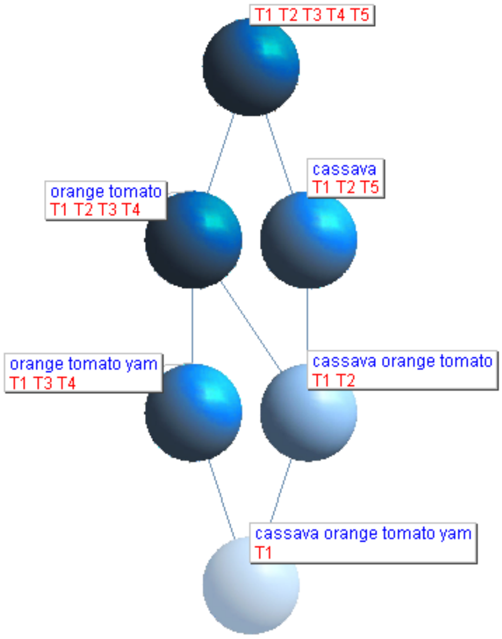


Figure 1.1: The concept lattice of example 1.2.1.

In the above Hasse diagram, the concept $(\{T_1, T_3, T_4\}, \{bread, butter, chocolate\})$ means that in transactions T_1, T_2 and T_3, bread is bought together with butter and chocolate. Other concepts are read in the same way.

Definition 1.2.6. *A formal context $\mathbb{K} := (G, M, \mathrm{I})$ is **clarified** if for any objects $g, h \in G$, from $g' = h'$, it always follows that $g = h$ and correspondingly, $m' = n'$ implies $m = n$ for all $m, n \in M$.*

Clarification can therefore be performed by removing identical rows and columns (only one of several identical rows/columns is left).

Example 1.2.6. *The following formal context is not clarified.*

$\mathbb{K}$	*cake*	*bread*	*chocolate*	*butter*
T_1	$\times$	$\times$	$\times$	$\times$
T_2	$\times$		$\times$	$\times$
T_3		$\times$	$\times$	$\times$
T_4		$\times$	$\times$	$\times$
T_5	$\times$			

The context obtained from the above formal context by clarification is the following:

	cake	*bread*	*chocolate*
T_1	×	×	×
T_2	×		×
T_3		×	×
T_5	×		

Theorem 1.2.1. *[18] If (G_1, M_1, I_1) is a clarified context resulting from (G_2, M_2, I_2) by clarification, then $\mathfrak{B}(G_1, M_1, \mathrm{I}_1)$ is isomorphic to $\mathfrak{B}(G_2, M_2, \mathrm{I}_2)$.*

Definition 1.2.7. *For a formal context (G, M, I), an attribute $m \in M$ is called **reducible** if there is $Z \subset M$ with $m \notin Z$ such that $m^{\mathrm{I}} = \bigcap_{z \in Z} z^{\mathrm{I}}$ i.e, the column corresponding to m is the meet of columns corresponding to zs from M. An object $g \in G$ is called **reducible** if there is $P \subset G$ with $g \notin P$ such that $g^{\mathrm{I}} = \bigcap_{z \in P} z^{\mathrm{I}}$ i.e, the row corresponding to g is the meet of rows corresponding to zs from P.*

Example 1.2.7. *A non reduced formal context. Here, the attribute "bread" is a reducible attribute.*

	cake	*bread*	*chocolate*
T_1			×
T_2	×	×	×
T_3	×		

The reduced formal context obtained from the above context by removing the reducible attribute "bread" is the following.

	cake	*chocolate*
T_1		×
T_2	×	×
T_3	×	

Definition 1.2.8. *Let (G, M, I) be a formal context. Then (G, M, I) is*

- *row reduced if no object $g \in G$ is reducible,*
- *column reduced if no attribute $m \in M$ is reducible,*
- *reduced if it is both row reduced and column reduced.*

Example 1.2.8. *The following context is a reduced context.*

	cake	*chocolate*
T_1		×
T_3	×	

Any context yields a concept lattice. Now we present the basic theorem of concept lattice.

Definition 1.2.9. *[18] A subset X of a complete lattice V is called* supremum-dense *in V, if every element from V can be represented as the supremum of a subset of X and, dually,* infimum-dense*, if every element from V can be represented as the infimum of a subset of X.*

Theorem 1.2.2. *[18](The basic Theorem on Concept Lattices) The concept lattice $\mathfrak{B}(G, M, \mathrm{I})$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t \in T}(A_t, B_t) = (\bigcap_{t\in T} A_t, (\bigcup_{t\in T} B_t)^{\mathrm{II}}),$$

and

$$\bigvee_{t \in T}(A_t, B_t) = ((\bigcup_{t\in T} A_t)^{\mathrm{II}}, \bigcap_{t\in T} B_t).$$

A complete lattice V is isomorphic to $\mathfrak{B}(G, M, I)$ if and only if there are mappings $\gamma : G \to V$ and $\mu : M \to V$ such that $\gamma(G)$ is supremum-dense in V, $\mu(M)$ is infimum-dense in V and gIm is equivalent to $\gamma(g) \leqslant \mu(m)$ for all $g \in G$ and all $m \in M$. In particular, $V \cong \mathfrak{B}(V, V, \leqslant)$.

Information can also be extracted from data in the form of associations rules, including implications on attributes. Let Y be a non empty set (of attributes).

Definition 1.2.10. *[4] An attribute implication over Y is an expression $A \Rightarrow B$, where $A \subseteq Y$ and $B \subseteq Y$.*

Example 1.2.9. *Let consider the set $Y = \{y_1, y_2, y_3, y_4\}$. Then $\{y_1, y_3\} \Rightarrow \{y_2, y_4\}$, and $\{y_2, y_3\} \Rightarrow \{y_1, y_2, y_4\}$ are implications over Y.*

Definition 1.2.11. *[4] An attribute implication $A \Rightarrow B$ is **true or valid** in a formal context $\mathbb{K} = (G, M, \mathrm{I})$ if and only if for each object g, if g has all attributes from A then g has all attributes from B.*

Definition 1.2.12. *[4] Let Y be a set of attributes.*

a) A theory (over Y) is any set T of implications (over Y),

b) A model of a theory T is any subset $M \subseteq Y$ such that every $A \Rightarrow B$ from T is true in M.

Example 1.2.10. *theories over $\{y_1, y_2, y_3\}$*

- $T_1 = \{\{y_3\} \Rightarrow \{y_1, y_2\}, \{y_1, y_3\} \Rightarrow \{y_2\}\}$,
- $T_2 = \{\{y_1, y_3\} \Rightarrow \{y_2\}\}$.

For a given theory T, $Mod(T)$ denotes the set of all the models of T.

Example 1.2.11. *Let consider the theories of the example above. then*

- *for* $T_1 = \{\{y_3\} \Rightarrow \{y_1, y_2\}, \{y_1, y_3\} \Rightarrow \{y_2\}\}$, $Mod(T_1) = \{\emptyset, \{y_1\}, \{y_2\}, \{y_1, y_2\}, \{y_1, y_2, y_3\}\}$,

- $T_2 = \{\{y_1, y_3\} \Rightarrow \{y_2\}\}$, $Mod(T_2) = \{\emptyset, \{y_1\}, \{y_2\}, \{y_1, y_2\}, \{y_1, y_2, y_3\}\}$.

Remark 1.2.2. *The study of attribute implications, theories and models are very important in Basket Market Analysis, because it permit to deduce which items are surely present in some transactions, knowing that some items are already contained in these transactions.*

Definition 1.2.13. *[4] An attribute implication $A \Rightarrow B$ **follows semantically** from a theory T (which is denoted by $T \models A \Rightarrow B$) if and only if $A \Rightarrow B$ is true in every model M of T.*

Example 1.2.12. *Let consider the set $Y = \{y_1, y_2, y_3\}$. We consider the theory $T_1 = \{\{y_3\} \Rightarrow \{y_1, y_2\}, \{y_1, y_3\} \Rightarrow \{y_2\}\}$ and $A \Rightarrow B = \{y_2, y_3\} \Rightarrow \{y_1\}$. As we have seen in the previous example, $Mod(T_1) = \{\emptyset, \{y_1\}, \{y_2\}, \{y_1, y_2\}, \{y_1, y_2, y_3\}\}$, and $\|\{y_2, y_3\} \Rightarrow \{y_1\}\|_{\emptyset} = 1$, $\|\{y_2, y_3\} \Rightarrow \{y_1\}\|_{\{y_1\}} = 1$, $\|\{y_2, y_3\} \Rightarrow \{y_1\}\|_{\{y_2\}} = 1$, $\|\{y_2, y_3\} \Rightarrow \{y_1\}\|_{\{y_1,y_2\}} = 1$, $\|\{y_2, y_3\} \Rightarrow \{y_1\}\|_{\{y_1,y_2,y_3\}} = 1$. Therefore, $T_1 \models \{y_2, y_3\} \Rightarrow \{y_1\}$.*

Definition 1.2.14. *[4] A set T of attribute implications over Y is called **non-redundant** if and only if for any $A \Rightarrow B \in T$, $T \backslash \{A \Rightarrow B\} \not\models A \Rightarrow B$.*

Implications can also be defined in formal contexts.

Definition 1.2.15. *[4] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context, T a set of attributes over M. T is called **complete** in $\mathbb{K}$ if and only if for any attributes implication $A \Rightarrow B$, $A \Rightarrow B$ is true in (G, M, I) if and only if $T \models A \Rightarrow B$.*

Theorem 1.2.3. *[4] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context. A set T of attributes implications over M is called a **basis** if and only if*

a) T is complete in (G, M, I),

b) T is non-redundant.

Remark 1.2.3. *Basically speaking, a basis of implications in a formal context $\mathbb{K} = (G, M, \mathrm{I})$ is a (relatively small) set of valid implications of $\mathbb{K}$ which can be used to generate all other valid implications in $\mathbb{K}$.*

Now we define what are called informative implications in a formal context. They are implications that provide interesting information compared to others types of attribute implications.

Definition 1.2.16. *[18] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context. An **informative** implication of $\mathbb{K}$ is an implication of the form $A \Rightarrow A^*$, where A is a set of attributes and*

$$A^* = A^{\mathrm{II}} \backslash [A \cup \bigcup_{x \in A} (A \backslash \{x\})^{\mathrm{II}}] \neq \emptyset$$

Definition 1.2.17. *[18] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context. A set $P \subseteq M$ is called a **pseudo-intent** of $\mathbb{K}$ if $P \neq P^{\mathrm{II}}$ and $Q^{\mathrm{II}} \subseteq P$ holds for every pseudo-intent $Q \subsetneq P$.*

Theorem 1.2.4. *[18] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context. The set T of attributes implications*

$$T = \{P \Rightarrow P^{\mathrm{II}},\ P \text{ is a Pseudo-intent of } \mathbb{K}\}$$

is non-redundant and complete.

Remark 1.2.4. *[18] In practice, the implications of T are not stated in the form $P \Rightarrow P^{\mathrm{II}}$, but in the form $P \Rightarrow P^{\mathrm{II}} \backslash P$. In this case, T is called the **Duquenne-Guigues-Basis** or the **canonical basis** or simply the **stem base** of all the valid implications in $\mathbb{K}$.*

Remark 1.2.5. *[18] The Duquenne-Guigues-Basis is the smallest basis of implications one can get in a given formal context.*

Definition 1.2.18. *[34] Given a set of Transactions, where each transaction is a set of items, an association rule is an expression $X \Rightarrow Y$, where X and Y are sets of items.*

The intuitive meaning of such a rule is that transactions in the database which contain the items in X tend to also contain the items in Y.

Definition 1.2.19. *[34]*

- *The support of the rule $X \Rightarrow Y$ is the percentage of transactions that contains both X and Y.*
- *The confidence of the rule $X \Rightarrow Y$ is the percentage of transactions that contains Y, knowing it already contains X.*

Example 1.2.13. *[34]*

- 70% *of people that purchase exercise books also by pens;*
- 95% *of transactions made in a bookshop contains both exercise books and pens.*

Here, if we set by X the set of exercise books and Y the set of pens in a bookshop, then 70% and 95% are respectively the confidence and the support of the rule $X \Rightarrow Y$.

In the rest of this thesis, the formal contexts are considered to be finite.

1.3 Generalization in Formal Concept Analysis

In this section, we define the notion of generalization, present the different kinds of generalization and their first properties.

Definition 1.3.1. *[27] In the field of data mining, generalized patterns are pieces of knowledge extracted from data when a taxonomy is used.*

Simply speaking, generalization in a formal context $\mathbb{K} = (G, M, \mathrm{I})$ is a way of grouping attributes together in order to form new others groups S of attributes. It can be observed in several domains of life:

- In basket market analysis, products can be generalized into product categories. For example, in a market, we consider the following items in sale: maize, yams, potato, tomato, guavas, orange, cassava, cocoa, coffee, vegetable, bananas, cabbage, rice, juice and mineral water. A seller dealing simultaneously in all these items can decide to evaluate the transactions concerning some of these products in order to make predictions on his future sales. Another seller, for the same raisons, can also decide to evaluate the transactions on some groups of items: **fruits** ({tomato, guavas, orange}), **tubers** ({cassava, yams, potato}) or **drinks** ({juice, mineral water}). The items **tubers**, **fruits** and **drinks** can be referred to as **generalized items**. The transactions concerning these groups of items provide the seller with new information which could not be easily identified if only the items themselves were considered. Moreover, customers in a market can also be grouped to form generalized group according to some specificities (level of education or income for example).

- In educational field, marks of students can be generalized or categorized in order to facilitate some analysis on results of exams. Subjects can also be categorized into modules in other to facilitate the computation of marks of students.

- In the field of health, patients can be generalized according to their reactions to treatments, to their age, etc.

To the best of our knowledge, generalized attributes were first considered by Srikant and Agrawal ([34]), as they discussed the problem of mining **generalized association rules**. In fact, they consider an "is-a" relationship (taxonomy) on items to extract relevant information in the form of association rules in transactions data. They called such rules generalized association rules. Note that using a taxonomy is equivalent to the **$\exists$-generalization** of some attributes. The following example from [34] shows the added-value of generalizing attributes. If a taxonomy is available and says for example that: Jacket is-a Outwear, Ski Pants is-a Outwear, Outwear is-a Clothes, etc, then generalizing rules that span different levels of the taxonomy could lead to discovering interesting information that were not possible without generalizing the attributes. A rule like "people who buy Outwear tend to buy Hiking Boots" may be inferred from the fact that people bought Jackets with Hiking Boots and Ski Pants with Hiking Boots.

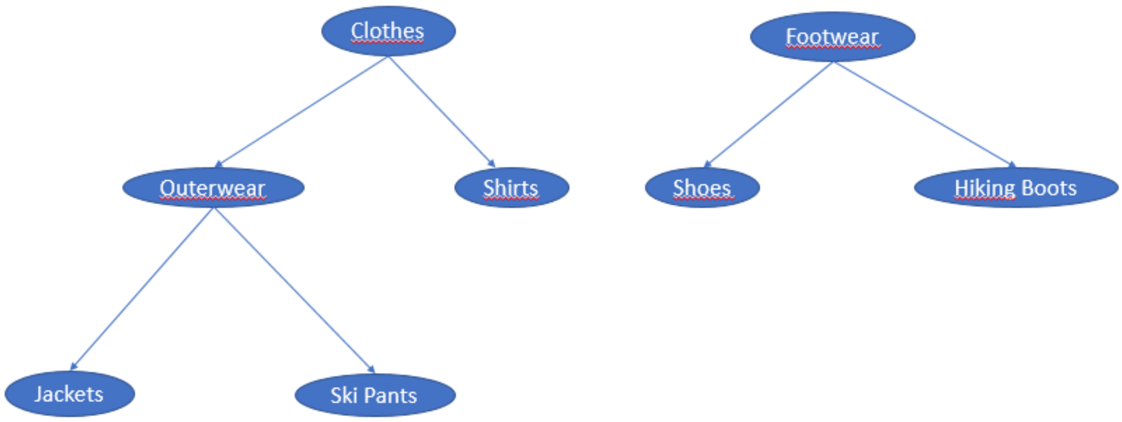


Figure 1.2: The taxonomy defined in [34]

Given a formal context $\mathbb{K} = (G, M, \mathrm{I})$, when attributes of $\mathbb{K}$ are grouped together, they form a new set S of attributes such that the set $\{m_s | s \in S\}$ covers the set M. Each attribute m_s is a group of attributes of the initial context $\mathbb{K}$, and is called the **generalized attribute**. More often, m_s is simply identify by the index s. A new formal context $\mathbb{K}_S = (G, S, \mathrm{J})$ is then constructed, where the binary relation J is still to be defined. That context is called **generalized formal context**. The definition of the binary relation J leads to different scenarios of generalization. More of this can be found in [27]. There exists principally three types of generalization according to the way the relation J is defined:

($\forall$)-**generalization**: The object should satisfy each of the attributes that were combined ($(g, s) \in \mathrm{J}$ if and only if for every $m \in s$, $(g, m) \in \mathrm{I}$). It is called the **Universal Generalization**.

Example 1.3.1. *Let recall the following context $\mathbb{K}$.*

	cake	*bread*	*chocolate*	*butter*
T_1	×	×	×	×
T_2	×		×	×
T_3		×	×	×
T_4		×	×	×
T_5	×			

Grouping the attributes "cake" and "bread" via universal generalization (s={cake, bread}) leads to the following generalized formal context:

	s	*chocolate*	*butter*
T_1	×	×	×
T_2		×	×
T_3		×	×
T_4		×	×
T_5			

(α)-generalization: The object should satisfy at least a certain proportion of the attributes that were combined ($(g, s) \in$ J if and only if $\frac{|\{m \in s | (g,m) \in I\}|}{|s|} \geqslant \alpha$). That is the **$\alpha$-generalization**.

Example 1.3.2. *Let consider the group constituted by attributes "cake", "bread" and "chocolate" of the context $\mathbb{K}$. setting $\alpha = \frac{2}{3}$, then the α-Generalization of that group of attributes (s={cake, bread, chocolate}) leads to the following generalized context:*

	s	*butter*
T_1	×	×
T_2	×	×
T_3	×	×
T_4	×	×
T_5		

($\exists$)-generalization: The object should satisfy at least one of the attributes that were combined ($(g, s) \in$ J if and only if there exists $m \in s$ such that $(g, m) \in$ I). That is the **$\exists$-generalization**.

Example 1.3.3. *From the $\exists$-Generalization of attributes "cake" and "bread" of the context $\mathbb{K}$ (s={cake, bread}), we obtain the following generalized formal context:*

	s	*chocolate*	*butter*
T_1	×	×	×
T_2	×	×	×
T_3	×	×	×
T_4	×	×	×
T_5	×		

Generalizing attributes in a formal context often leads to a variation of the size of the corresponding concepts lattice. Here we present the first properties on the size of concept lattices after generalizations

Theorem 1.3.1. *[27] The $\forall$-generalizations on attributes do not increase the size of the concept lattice.*

Theorem 1.3.2. *[27] The $\exists$-generalizations on distributive concept lattices whose contexts are object-reduced do not increase the size of the concept lattice.*

Let consider the following formal context $\mathbb{K}_{B_4}$. We denote by B_4 the corresponding concept lattice.

$\mathbb{K}_{B_4}$	m_1	m_2	m_3	m_4
a	×	×		
b		×		
c			×	×
d				×

For any attribute m, we denote by μm the concept $(m^{\mathrm{I}}, m^{\mathrm{II}})$.

Proposition 1.3.1. *[27]*

i) The lattice B_4 is the smallest lattice on which there is an $\exists$-generalization that increases the size of the initial concept lattice.

ii) If a context contains attributes m_1, m_2, m_3, m_4 such that $\mu m_1 \leqslant \mu m_2$, $\mu m_3 \leqslant \mu m_4$, $\mu m_2 \wedge \mu m_3 \leqslant \mu m_1$ and $\mu m_1 \wedge \mu m_4 \leqslant \mu m_3$; then there is an $\exists$-generalization that does not decrease the size of the concept lattice.

Chapter 2

On the size of the $\exists$-Generalized concept lattices

2.1 Introduction

Formal Concept Analysis (FCA) offers several tools for qualitative data analysis. One possibility is to group objects that share common attributes together and get a concept lattice that describes the data. Quite often the size of this concept lattice is very large. Many authors have investigated methods to reduce the size of this lattice. In [27], the authors consider putting together some attributes to reduce the size of the attributes sets. But this reduction does not always carry over the set of concepts. They have provided some counter examples where the size of the concept lattice increases by one after putting two attributes together and asked the following important question: "How many new concepts can be generated by an $\exists$-generalization on just two attributes?"

In [34, 28], it is presented some applications of generating attributes, but the discussion about the size of the information discovered is not stated. However, some works have directly or indirectly dealt with the size of concept lattices and its control:

In [8], the authors have discussed a virtual correlation observed between the size of concept lattices and that of all the pseudo-intents of randomly generated formal contexts.

To control the size of concept lattices, some authors have proposed to constrain the concept lattice by attributes dependencies in the form of attributes implications [40] or in the form of attributes-dependency formulas [6], which means to only select from the initial lattice the concepts that satisfy these attributes dependencies. In these two situations, the attributes dependencies are set in advance by an expert. The size of the constrained lattice is usually smaller than that of the initial lattice. However, these methods are limited, since the choice of the constraints is not well precise.

Other authors, working on pattern structures (a more general case of contexts), have applied extensional projection on pattern structures [17] to obtain projected pattern structures having a less number of patterns concepts than the initial patterns structures. However, the projected patterns concepts are subsets of the set of patterns concepts of the initial patterns structures.

In [28], a discussion is made on how to construct a concept lattice of a context resulting from another context with a known concept lattice by removing exactly one incidence. Removing that incidence leads to the construction of a closure operator and an interior operator such that if a concept of the initial context is a fixpoint of both operators, then it is duplicated into two different concepts in the new concept lattice, and if a concept of the initial context is not a fixpoint of any of the two operators, it vanishes from the new concept lattice. Hence, removing an incidence from a formal context sometimes leads to a reduction or to an increase of the size of the initial concept lattice. Moreover, generalization on attributes can be seen as adding many incidences to the initial context (adding the generalized attribute to the initial context) and removing other incidences in the initial context (removing the attributes which are generalized). In [27], an empirical test (simulation) related to the variation of size of concept lattices after an ∃-generalization was carried out on mushrooms data with 8126 mushrooms and 117 variables representing their characteristics. In this tests, 20 attributes were first randomly selected 5 times. In each group of twenty attributes, a pair of them were randomly chosen 10 times and ∃-generalized. Finally, 50 generalized contexts with 8126 objects (mushrooms) and 19 attributes were obtained, and the sizes of their concept lattices were computed and led to the following table:

Size decrease	Nb of tests
[-0,007, -0,004[	8
[-0,004, 0[	7
[0, 6[	17
[6, 13[	7
[13, 41[	11

Table 2.1: The variation of the size of the lattice after an ∃-generalization on a pair of attributes in the mushroom data, as indicated in [27]

Theses tests showed that an ∃-generalization on pairs of attributes led to the decrease of the size of the concept lattice on thirty five cases, and to an increase of the size of the lattice on fifteen cases. However, in situations where the size of the lattice increases, the increase was very low, as one can observe on table 2.1.

Based on the above empirical experience, one could suspect that an ∃-generalization on a pair of attributes will often lead to the decrease of size of the concept lattice, and even with the few cases where the size of the lattice increases, the increase will not be drastic.

The main question of this chapter is then : **To what extent can be the increase of size of a concept lattice after an ∃-generalization on a pair of attributes?**.

Generalizing two attributes m_1, m_2 to get a new attribute m_{12} can be done in two steps: (i): adding m_{12} to the initial context and (ii) removing m_1, m_2 from the context. Therefore we start this chapter by discussing in section 2.2 the effect of adding a new attribute in a context $\mathbb{K}$. The main result here says that the maximal number of new concepts is $|\mathfrak{B}(\mathbb{K})|$ and can be reached. This means that adding a new attribute to $\mathbb{K}$

can double the size of $\mathfrak{B}(\mathbb{K})$. In section 2.3 we also provides a family of contexts for which the size increases exponentially after putting two attributes together, and study in section 2.4 the maximum increase one can get after an ∃-generalization on solely two attributes. A case where the size of the concept lattice remain constant after the generalization is also presented in section 2.5, and the link between ∃-generalization and other fields of research such as granularity, factorisation of formal contexts and patterns structures is raised in section 2.6.

2.2 Adding a new attribute into a context

When constructing concept lattices, the incremental methods [42] consists in starting with one attribute and adding the rest one after another. In this section we review the effect of adding one attribute. Let $\mathbb{K} := (G, M, \mathrm{I})$ be a context, and $a \notin M$ an attribute that can be shared by some elements of G. We set $M^a := M \cup \{a\}$, $a' := \{g \in G \mid g \text{ has the attribute } a\}$ and $\mathbb{K}^a := (G, M^a, \mathrm{I}^a)$ where $\mathrm{I}^a := \mathrm{I} \cup \{(g, a) \mid g \text{ has the new attribute } a\}$. To distinguish between the derivation on sets of objects in $\mathbb{K}$ and in $\mathbb{K}^a$, we will use the name of the relation instead of $'$. That said, we will write for $A \subseteq G$,

$$A^{\mathrm{I}} = \{m \in M \mid g \,\mathrm{I}\, m \text{ for all } g \in A\}$$

and

$$A^{\mathrm{I}^a} = \{m \in M \cup \{a\} \mid g \,\mathrm{I}^a\, m \text{ for all } g \in A\}.$$

If $a' = G$, then $|\mathfrak{B}(\mathbb{K}^a)| = |\mathfrak{B}(\mathbb{K})|$. Each concept (A, B) of $\mathbb{K}$ has a corresponding concept $(A, B \cup \{a\})$ in $\mathbb{K}^a$, and vice-versa. The above equality still holds even if $a' \neq G$, but $a' = B^{\mathrm{I}}$ for some $B \subseteq M$. The following result expresses the relation between $\mathbb{K}$ and $\mathbb{K}^a$.

Proposition 2.2.1. *Let $\mathbb{K}$ be a context and $\mathbb{K}^a$ be the context obtained by adding the attribute a. The map*

$$\begin{array}{rccl} \phi_a : & \mathfrak{B}(\mathbb{K}) & \longrightarrow & \mathfrak{B}(\mathbb{K}^a) \\ & (A, B) & \longmapsto & \begin{cases} (A, B \cup \{a\}) & \textit{if } A \subseteq a' \\ (A, B) & \textit{elsewhere} \end{cases} \end{array}$$

is an injective map.

Proof. The map ϕ_a is well defined. For a concept $(A, B) \in \mathfrak{B}(\mathbb{K})$ with $A \subseteq a'$, we have $(B \cup \{a\})^{\mathrm{I}^a} = B^{\mathrm{I}} \cap a' = A \cap a' = A$, and $A^{\mathrm{I}^a} = A^{\mathrm{I}} \cup \{a\} = B \cup \{a\}$. Thus $(A, B \cup \{a\})$ is a concept of $\mathbb{K}^a$. For a concept $(A, B) \in \mathfrak{B}(\mathbb{K})$ with $A \nsubseteq a'$, we have $B^{\mathrm{I}^a} = B^{\mathrm{I}} = A$, and $A^{\mathrm{I}^a} = A^{\mathrm{I}} = B$, since a is not in A^{I^a}. If two concepts (A_1, B_1) and (A_2, B_2) of $\mathbb{K}$ have the same image under ϕ_a, then A_1 and A_2 are both included in a' or both not included in a', and are therefore equal, hence ϕ_a is injective. ∎

After adding an attribute a to a context $\mathbb{K}$, we will identify $(A, B) \in \mathfrak{B}(\mathbb{K})$ with $\phi_a(A, B) \in \mathfrak{B}(\mathbb{K}^a)$, and write $(A, B) \equiv \phi_a(A, B)$. From Proposition 2.2.1 we get $|\mathfrak{B}(\mathbb{K})| \leqslant |\mathfrak{B}(\mathbb{K}^a)|$. Moreover, the increase due to adding a, which is the difference $|\mathfrak{B}(\mathbb{K}^a)| - |\mathfrak{B}(\mathbb{K})|$, can be computed as the number of concepts of $\mathbb{K}^a$ that cannot be identified (via ϕ_a) with any concept in $\mathfrak{B}(\mathbb{K})$.

We consider (A, B) in $\mathfrak{B}(\mathbb{K})$ with $A \not\subseteq a'$. It holds

$$\mathfrak{B}(\mathbb{K}^a) \ni (A, B) \equiv (A, B) \in \mathfrak{B}(\mathbb{K}), \text{ since } A \not\subseteq a'.$$

Moreover, $A \cap a'$ is an extent of $\mathbb{K}^a$. If $A \cap a'$ is also an extent of $\mathbb{K}$, then

$$\mathfrak{B}(\mathbb{K}^a) \ni \left(A \cap a', (A \cap a')^{\mathrm{I}^a}\right) \equiv \left(A \cap a', (A \cap a')^{\mathrm{I}}\right) \in \mathfrak{B}(\mathbb{K}) \quad \text{because } A \cap a' \subseteq a'.$$

Note that $(A \cap a')^{\mathrm{I}^a} = (A \cap a')^{\mathrm{I}} \cup \{a\}$ and $(A \cap a')^{\mathrm{I}} = (A \cap a')^{\mathrm{I}^a} \cap M$. Although (A, B) and $(A \cap a', (A \cap a')^{\mathrm{I}} \cup \{a\})$ are two different concepts of $\mathbb{K}^a$, they are equivalent to two concepts of $\mathbb{K}$ when $A \cap a'$ is an extent of $\mathbb{K}$. A concept (A, B) of $\mathbb{K}$ induces two concepts of $\mathbb{K}^a$ whenever $A \not\subseteq a'$. In the definition of ϕ_a in Proposition 2.2.1, from a concept (A, B) of $\mathbb{K}$, we went for (A, B) in $\mathbb{K}^a$ instead of $(A \cap a', B \cup \{a\})$. This choice is motivated by the injectivity of ϕ_a being straightforward. If $A \not\subseteq a'$ and $A \cap a'$ is an extent of $\mathbb{K}$, then the two concepts induced by (A, B) in $\mathbb{K}^a$ have their equivalent in $\mathfrak{B}(\mathbb{K})$. Then adding a to $\mathbb{K}$ will increase the size of the concept lattice only if there is $A \in \operatorname{Ext}(\mathbb{K})$ such that $A \cap a' \notin \operatorname{Ext}(\mathbb{K})$.

Each extent of $\mathbb{K}^a$ is an extent of $\mathbb{K}$ or an intersection of an extent of $\mathbb{K}$ with a'. The concepts of $\mathbb{K}^a$ that cannot be identified (via ϕ_a) to a concept of $\mathbb{K}$ are

$$\left\{\left(A \cap a', (A \cap a')^{\mathrm{I}} \cup \{a\}\right) \mid A \in \operatorname{Ext}(\mathbb{K}) \text{ and } A \cap a' \notin \operatorname{Ext}(\mathbb{K})\right\}.$$

Note that it is possible to have two different extents $A_1, A_2 \in \operatorname{Ext}(\mathbb{K})$ with $A_1 \cap a' = A_2 \cap a' \notin \operatorname{Ext}(\mathbb{K})$. In this case we say that the extents A_1 and A_2 coincide on a'. The increase is then less or equal to $|\mathfrak{B}(\mathbb{K})|$. We can now sum up the finding of the above discussion in the next proposition.

Proposition 2.2.2. *Let $\mathbb{K}^a$ be a context obtained by adding an attribute a to a context $\mathbb{K}$. Let*

$$\mathcal{H}(a) := \{A \cap a' \mid A \in \operatorname{Ext}(\mathbb{K}) \text{ and } A \cap a' \notin \operatorname{Ext}(\mathbb{K})\}. \tag{2.1}$$

1. *The increase in the number of concepts due to adding the attribute a to $\mathbb{K}$ is*

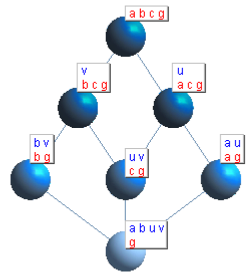
$$|\mathfrak{B}(\mathbb{K}^a)| - |\mathfrak{B}(\mathbb{K})| = |\mathcal{H}(a)| \leqslant |\mathfrak{B}(\mathbb{K})|$$

2. *The maximal increase $|\mathcal{H}(a)| = |\mathfrak{B}(\mathbb{K})|$ is reached when each $A \in \operatorname{Ext}(\mathbb{K})$ satisfies $A \cap a' \notin \operatorname{Ext}(\mathbb{K})$ and no pairs $A_1, A_2 \in \operatorname{Ext}(\mathbb{K})$ coincide on a'.*

Before we continue with the discussion on the maximal increase, let consider the following formal context (Table 2.2) and its concept lattice.

$\mathbb{K}$	v	u	a	b
a		×	×	
b	×			×
c	×	×		
g	×	×	×	×

Table 2.2: A formal context.

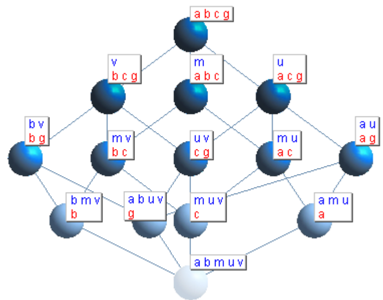


The corresponding concept lattice

Let us look at two examples, where an attribute m has being added to the context of the table above.

$\mathbb{K}$	v	u	a	b	m
a		×	×		×
b	×			×	×
c	×	×			×
g	×	×	×	×	

Table 2.3: A first table obtained by adding an attribute m to the context in Table 2.2



The corresponding concept lattice

$\mathbb{K}$	v	u	a	b	m
a		×	×		×
b	×			×	
c	×	×			×
g	×	×	×	×	

Table 2.4: A second table obtained by adding an attribute m to the context in Table 2.2

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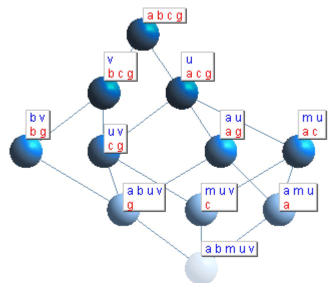


The corresponding concept lattice.

In the first case (Table 2.3) the concept lattice of Table 2.2 has been doubled and the maximal increase is reached. In the second case (Table 2.4), only the concepts in the interval $[\emptyset^{\mathrm{II}}; \{a,c\}^{\mathrm{II}}]$ of the concept lattice of Table 2.2 has been doubled. Note that in both cases, $g \in \emptyset^{\mathrm{II}} \neq \emptyset$.

Based on the examples in Table 2.3, Table 2.4 and Proposition 2.2.2, we can now discuss the maximal increase. First we have the following proposition.

Proposition 2.2.3. *Let $\mathbb{K}$ be a formal context and a be an attribute added to $\mathbb{K}$. The following are equivalent:*

(i) For every extent A of $\mathbb{K}$, $A \cap a'$ is not an extent of $\mathbb{K}$.

(ii) $\emptyset^{\mathrm{II}} \setminus a' \neq \emptyset$.

Proof. If $A \in \operatorname{Ext}(\mathbb{K})$ and $A \cap a' \notin \operatorname{Ext}(\mathbb{K})$, then $A \nsubseteq a'$. Moreover, if $A \nsubseteq a'$ for every extent A of $\mathbb{K}$, then in particular $\emptyset^{\mathrm{II}} \nsubseteq a'$. Thus there is $g \in \emptyset^{\mathrm{II}}$ such that $g \notin a'$. This element g is in every extent of $\mathbb{K}$, but is not in a'. Conversely, if an element g is in $\emptyset^{\mathrm{II}} \setminus a'$, then g is in every extent A of $\mathbb{K}$, and g is not in $A \cap a'$. Thus $g \in (A \cap a')^{\mathrm{II}}$ and $g \notin A \cap a'$, i.e. $A \cap a'$ is not closed in $\mathbb{K}$. Thus $A \cap a' \notin \operatorname{Ext}(\mathbb{K})$ for each $A \in \operatorname{Ext}(\mathbb{K})$. ■

Both contexts of Table 2.3 and Table 2.4 satisfy the above conditions (the added attribute a is m). Then each extent of $\mathbb{K}$ generates two extents of $\mathbb{K}^m$ and one of these cannot be identified (via ϕ_m) with an extent of $\mathbb{K}$. However, some of these new concepts can be equal. In fact if two extents coincide on m', then they generate the same new concept. To avoid coincidences on m', it is enough to have $m' = G \setminus \{g\}$.

Corollary 2.2.1. *Let $\mathbb{K}$ be a formal context such that $\emptyset^{\mathrm{II}} \neq \emptyset$ and $\mathbb{K}^a$ a context obtained by adding an attribute a to $\mathbb{K}$ such that $a' = G \setminus \emptyset^{\mathrm{II}}$. Then we have*

$$|\mathfrak{B}(\mathbb{K}^a)| = 2 \cdot |\mathfrak{B}(\mathbb{K})| .$$

Proof. It is the immediate consequence of Proposition 2.2.3. ■

Using these results we can now present some huge increases after generalizing only two attributes.

2.3 The number of concepts generated by an existential generalization

By putting together some attributes, we reduce the number of attributes in the context, and hope to also reduce the size of the concept lattice. This is true for ∀-generalization, but not always the case for ∃-generalization. In [27] some examples were presented where the size increased by one after an ∃-generalization. The main question here is whether the size of the concept lattice can increase exponentially after putting solely two attributes together. The present section gives a positive answer to this question. In fact, we provide a family of contexts where the increase is exponential in the size of the attribute set.

Let $\mathbb{K} := (G, M, \mathrm{I})$ be a formal context. We denote by S the set of generalized attributes of $\mathbb{K}$. Since the final goal is to reduce the size of the lattice, we will assume that S forms a partition of M [1]. Then at least the number of attributes is reduced. For an ∃-generalization, we recall that an object g has the generalized attribute s iff g has

[1] It is also possible to allow some attributes to appear in different groups. In this case the number of generalized attributes can be larger than in the initial context

at least one of the attributes in s; i.e. $s' = \bigcup\{a' \mid a \in s\}$. We obtain a relation J on $G \times S$ defined by:

$$g \, \mathrm{J} \, s \iff \exists m \in s \text{ such that } g \, \mathrm{I} \, m.$$

We look at a very simple case, where two attributes $a, b \in M$ are generalized to get a new one, say s. This means that from a context (G, M, I), we remove the attributes a and b from M and add an attribute $s \notin M$ to M, with $s' = a' \cup b'$. In particular we show that the number of concepts of $(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab})$ can be extremely larger than that of (G, M, I).

Recall that for any set E, the concepts lattice of the context $(E, E, \neq)$ is a Boolean algebra isomorphic to the powerset of E, and then has $2^{|E|}$ concepts. By S_n, we denote a set with n elements where $n \geqslant 2$, and we write for simplicity $S_n := \{1, 2, \cdots, n\}$. We define a context $\mathbb{K}^1_n$ by:

$$\mathbb{K}^1_n := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$$

with

$$g \, \mathrm{I} \, m :\iff \begin{cases} g, m \in S_n & \text{and } g \neq m, \text{ or} \\ g = g_1 & \text{and } m \in S_n, \text{ or} \\ g = 1 & \text{and } m = m_1, \text{ or} \\ g \in S_n \setminus \{1\} & \text{and } m = m_2. \end{cases} \tag{2.2}$$

We generalize the attributes m_1 and m_2 to get m_{12} and denote the resulting context by $\mathbb{K}^1_{nge} := (S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I})$ with $m'_{12} = m'_1 \cup m'_2$.

For the case $n = 2$, the contexts and their concepts lattices are displayed in Fig.2.1. Let precise that practically, that kind of formal contexts are still to be found. We want

$\mathbb{K}^1_2$	1	2	m_1	m_2
1		×	×	
2	×			×
g_1	×	×		

$\mathbb{K}^1_{2ge}$	1	2	m_{12}
1		×	×
2	×		×
g_1	×	×	

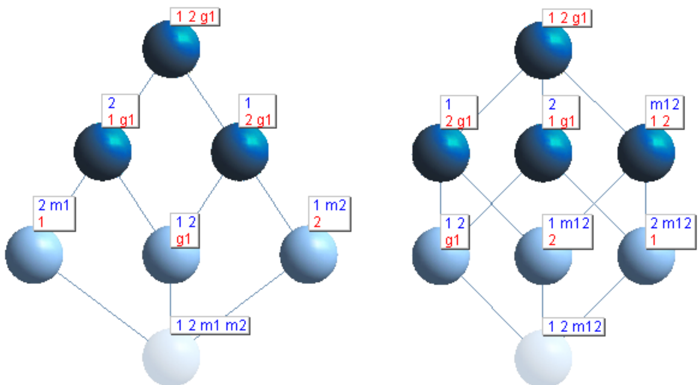


Figure 2.1: $\mathfrak{B}(\mathbb{K}^1_2)$ (left) and $\mathfrak{B}(\mathbb{K}^1_{2ge})$ (right), as defined by (2.2) with $n = 2$.

to compare the number of concepts of $\mathbb{K}^1_{nge}$ and that of $\mathbb{K}^1_n$ and their differences. With Lattice Miner, we have computed the table below which shows some of these numbers:

n	2	3	4	5	$\cdots$	10	$\cdots$	20	$\cdots$
$\lvert\mathfrak{B}(\mathbb{K}_n^1)\rvert$	7	13	25	49	$\cdots$	1537	$\cdots$	1572865	$\cdots$
$\lvert\mathfrak{B}(\mathbb{K}_{n\text{ge}}^1)\rvert$	8	16	32	64	$\cdots$	2048	$\cdots$	2097152	$\cdots$
$\lvert\mathfrak{B}(\mathbb{K}_{n\text{ge}}^1)\rvert - \lvert\mathfrak{B}(\mathbb{K}_n^1)\rvert$	1	3	7	15	$\cdots$	511	$\cdots$	524287	$\cdots$

Table 2.5: Examples of increase after an ∃-generalization.

Notations: Exceptionally, in this section we denote by I the restriction of the incidence relation of $\mathbb{K}_n^1$ on any subcontext of $\mathbb{K}_n^1$, and also by I the incidence relation in the generalized context $\mathbb{K}_{n\text{ge}}^1$. We set

$$\begin{aligned}
\mathbb{K}_{00} :=&(S_n \cup \{g_1\}, S_n, \mathrm{I}),\\
\mathbb{K}_{02} :=&(S_n \cup \{g_1\}, S_n \cup \{m_2\}, \mathrm{I}),\\
\mathbb{K}_{01} :=&(S_n \cup \{g_1\}, S_n \cup \{m_1\}, \mathrm{I}),\\
\mathbb{K}_{0s} :=&(S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I}) = \mathbb{K}_{n\text{ge}}^1,\\
\mathbb{K}_{12} :=&(S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I}) = \mathbb{K}_n^1.
\end{aligned}$$

However, for a single element $\{x\}$, we will often use just the symbol $'$ to express the objects that have the attribute x, instead of the name of the relation.

The context $\mathbb{K}_{00}$ has 2^n concepts because g_1 is a reducible object in $\mathbb{K}_{00}$ and the remaining context after removing g_1 is $(S_n, S_n, \neq)$. The context $\mathbb{K}_n^1$ is obtained by adding successively m_2 to $\mathbb{K}_{00}$ to get $\mathbb{K}_{02}$, and then m_1 to $\mathbb{K}_{02}$. The generalized context is obtained by adding $s = m_{12}$ to $\mathbb{K}_{00}$.

The following proposition hold:

Proposition 2.3.1. *Let $n \geqslant 2$. Let $\mathbb{K} = (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$ defined by (2.2). Then generalizing the attributes m_1 and m_2 increases the size of the concept lattice by $2^{n-1} - 1$.*

Proof. Adding the attribute m_2 to $\mathbb{K}_{00} = (S_n \cup \{g_1\}, S_n, \mathrm{I})$, we get the context $\mathbb{K}_{02}$. Every extent A of $\mathbb{K}_{00}$ is of the form $A = A_1 \cup \{g_1\}$ with $A_1 \subseteq S_n$ and satisfies $A \cap m_2' \notin \mathrm{Ext}\,(\mathbb{K}_{00})$, since $(g_1, m_2) \notin \mathrm{I}$. It therefore generates two concepts in $\mathbb{K}_{02}$. Let A be an extent of $\mathbb{K}_{00}$. Then A contains 1 or A does not contains 1. Extents which do not contains 1 are such that $A_1 \subseteq m_2' = \{2, \cdots n\}$, and do not coincide on m_2'. Therefore they generate 2^{n-1} concepts in $\mathbb{K}_{02}$ that cannot be identified (via ϕ_{m_2}) to any concept of $\mathbb{K}_{00}$. If A is an extent of $\mathbb{K}_{00}$ containing 1 then $A \setminus \{1\}$ is also an extent of $\mathbb{K}_{00}$, and both extents coincide on m_2'. Thus by Proposition 2.2.2 we get

$$|\mathfrak{B}\,(\mathbb{K}_{02})| = 2^n + 2^{n-1}. \tag{2.3}$$

Now adding m_1 to $\mathbb{K}_{02}$ will generate at most two concepts, since $m_1' = \{1\}$ and

$$\mathcal{H}(m_1) \subseteq \{A \cap m_1' \mid A \in \mathrm{Ext}\,(\mathbb{K}_{02})\} = \{\emptyset, m_1'\}$$

For every extent A which do not contain 1, $A \cap m_1' = \emptyset$ and $\emptyset \in \mathrm{Ext}(\mathbb{K}_{02})$. Hence, $\emptyset = A \cap m_1' \notin (H)(m_1)$. For every extent A containing 1, $A \cap m_1' = m_1' \notin \mathrm{Ext}(\mathbb{K}_{02})$. Hence, $m_1' \in (H)(m_1)$. Therefore

$$\mathcal{H}(m_1) = \{m_1'\} \text{ and } |\mathfrak{B}(S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})| = 2^n + 2^{n-1} + 1. \tag{2.4}$$

The context $(S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I})$ is isomorphic to $(S_{n+1}, S_{n+1}, \neq)$. The object g_1 is identified with $n+1$ and the attribute m_{12} with $n+1$. Thus generalizing m_1 and m_2 to m_{12} leads to a lattice with 2^{n+1} concepts. The increase is then

$$2^{n+1} - \left(2^n + 2^{n-1} + 1\right) = 2^{n-1} - 1$$

which is exponential in the number of attributes of the initial context. ■

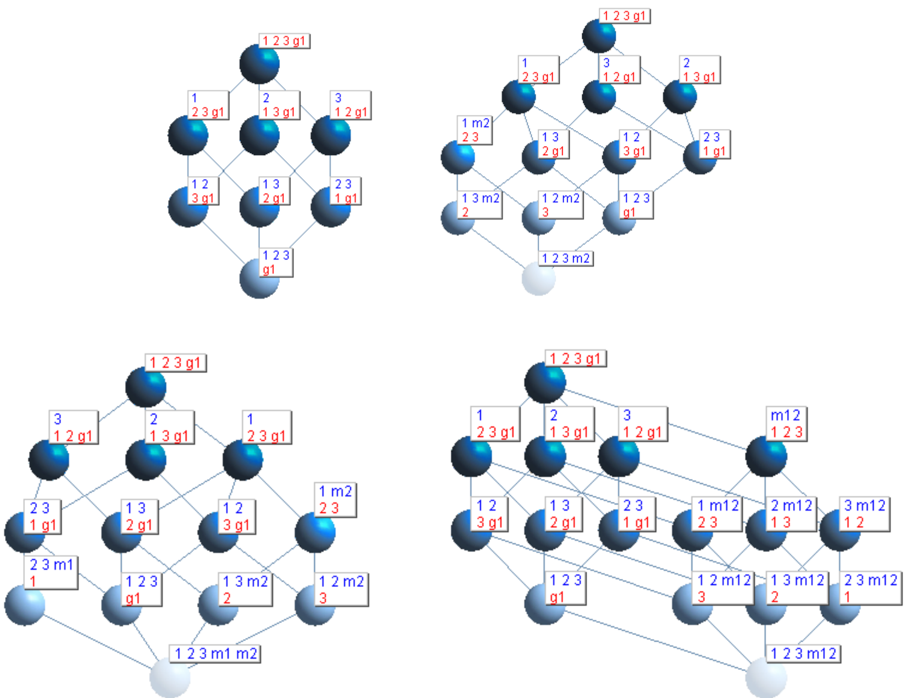


Figure 2.2: $\mathfrak{B}(\mathbb{K}_{00})$ (upper left); $\mathfrak{B}(\mathbb{K}_{02})$ (upper right); $\mathfrak{B}(\mathbb{K}_{12})$ (down left); and $\mathfrak{B}(\mathbb{K}_{0s})$ (down right); For $n = 3$.

In the above construction of $\mathbb{K}^1_{nge}$ the idea is to construct a context $(E, E, \neq)$ with $|E| = n+1$ from the initial context, via an ∃-generalization. The objects in S_n are split between m_1 and m_2 with no overlap. We can choose a split that assigns k objects of S_n to m_1 and the other $n-k$ to m_2. Let

$$\mathbb{K}^k_n := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$$

be such a context, where I is defined by

$$g\,\mathrm{I}\,m :\Longleftrightarrow \begin{cases} g, m \in S_n & \text{and } g \neq m, \text{ or} \\ g = g_1 & \text{and } m \in S_n \text{ or} \\ g \in \{1,2,...,k\} & \text{and } m = m_1 \text{ or} \\ g \in S_n \setminus \{1,2,...,k\} & \text{and } m = m_2 \end{cases} \tag{2.5}$$

Then the existential generalization of the attributes m_1 and m_2 to m_{12} leads to the generalized context $\mathbb{K}^k_{nge} := (S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I}) \cong (S_{n+1}, S_{n+1}, \neq)$. To get the cardinality of $\mathfrak{B}(\mathbb{K}^k_n)$, we observe that

(i) $\mathbb{K}_{00} := (S_n \cup \{g_1\}, S_n, \mathrm{I})$ has 2^n concepts. The extents of $\mathbb{K}_{00}$ are of the form $A \cup \{g_1\}$, $A \subseteq S_n$.

(ii) $\mathbb{K}_{02} := (S_n \cup \{g_1\}, S_n \cup \{m_2\}, \mathrm{I})$ has $2^n + 2^{n-k}$ concepts. They are of the form $(A \cup \{g_1\}, S_n \setminus A)$ with $A \subseteq S_n$ or the form $(A, (S_n \setminus A) \cup \{m_2\})$ with $A \subseteq m_2'$.

(iii) $\mathbb{K}_{01} := (S_n \cup \{g_1\}, S_n \cup \{m_1\}, \mathrm{I})$ has $2^n + 2^k$ concepts, which are of the form $(A \cup \{g_1\}, S_n \setminus A)$ with $A \subseteq S_n$ or the form $(A, (S_n \setminus A) \cup \{m_1\})$ with $A \subseteq m_1'$.

That leads to the following result:

Proposition 2.3.2. *Let $n \geqslant 2$, $1 \leqslant k < n$ and $\mathbb{K}^k_n$ defined by (2.5).*

a) The context $\mathbb{K}^k_n$ has $2^n + 2^{n-k} + 2^k - 1$ concepts.

b) Generalizing m_1 and m_2 increases the number of concepts by

$$2^n - 2^k - 2^{n-k} + 1.$$

Proof. $\mathbb{K}_{12} := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$ is obtained from $\mathbb{K}_{02}$ by adding m_1. Therefore we need to compute $\mathcal{H}(m_1)$ with respect to $\mathbb{K}_{02}$. Let $A \in \mathrm{Ext}(\mathbb{K}_{02})$. We distinguish two cases: $g_1 \notin A$ or $g_1 \in A$.

(i) If $g_1 \notin A$, then $A \subseteq m_2'$, and $A \cap m_1' = \emptyset$ is an extent of $\mathbb{K}_{02}$. No new concept is generated.

(ii) If $g_1 \in A$, then the extent A is of the form $A = A_1 \cup \{g_1\}$ with $A_1 \subseteq S_n$. Since $m_1' \cap m_2' = \emptyset$, we get

$$A_1 \cap m_1' \notin \mathrm{Ext}\,(\mathbb{K}_{02}) \iff A_1 \cap m_1' \nsubseteq m_2'$$

Thus the number of additional concept generated is

$$|\{A \cap m_1' | A \in \mathrm{Ext}\,(\mathbb{K}_{02}) \text{ and } A \cap m_1' \nsubseteq m_2'\}|$$

Among the extents of $\mathbb{K}_{02}$ with $A \cap m_1' \nsubseteq m_2'$, there are $2^k - 1$ that do not coincide on m_1', for example those with $\emptyset \neq A_1 \subseteq m_1'$. This means that adding m_1 to $\mathbb{K}_{02}$ will generate $2^k - 1$ new concepts that cannot be identified with concepts in $\mathbb{K}_{02}$. Therefore $\mathbb{K}^k_n$ has $2^n + 2^{n-k} + 2^k - 1$ concepts.

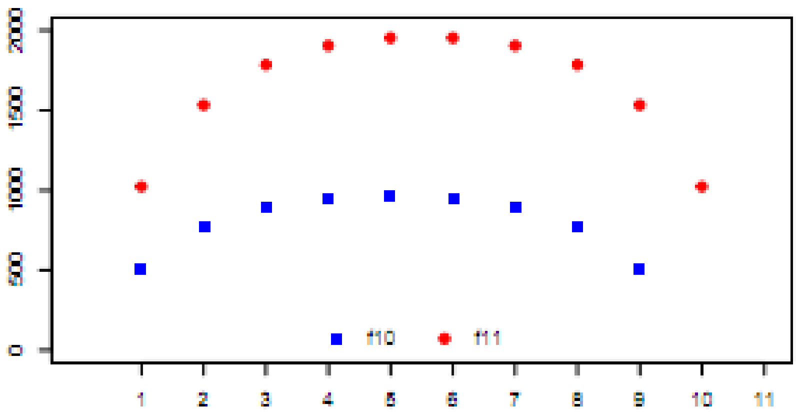
The increase after the generalization is given by $2^{n+1} - (2^n + 2^{n-k} + 2^k - 1) = 2^n - 2^{n-k} - 2^k + 1$ concepts. ■

The natural question here is: which $\mathbb{K}_n^k$ has a maximal increase? The increase by an ∃-generalization that puts m_1 and m_2 together in $\mathbb{K}_n^k$ is

$$f_n(k) := 2^n - 2^k - 2^{n-k} + 1.$$

This function is convex and its slope vanishes at $k = \frac{n}{2}$.

$$\begin{aligned} f_n^{'}(k) &= -\ln(2)2^k + \ln(2)2^{n-k} = 0 \iff n = 2k. \\ f_n^{''}(k) &= -\ln^2(2)2^k - \ln^2(2)2^{n-k} < 0. \end{aligned}$$



The maximum is reached when the objects are evenly split; i.e $k = \frac{n}{2}$ for n even, or $k \in \left\{\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor + 1\right\}$ for n odd, with $\lfloor\frac{n}{2}\rfloor$ the whole part of $\frac{n}{2}$. That is the case for the context

$$\mathbb{K}_n^{\lfloor\frac{n}{2}\rfloor} := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$$

with

$$g \,\mathrm{I}\, m \iff \begin{cases} g, m \in S_n & \text{and } g \neq m, \text{ or} \\ g = g_1 & \text{and } m \in S_n, \text{ or} \\ g \in \{1, \cdots, \lfloor\frac{n}{2}\rfloor\} & \text{and } m = m_1, \text{ or} \\ g \in S_n \setminus \{1, \cdots, \lfloor\frac{n}{2}\rfloor\} & \text{and } m = m_2. \end{cases}$$

Assuming $n = 2q$ the increase is $f_{2q}(q) = 2^{2q} - 2 \cdot 2^q + 1 = (2^q - 1)^2$.

We could allow overlap in constructing $\mathbb{K}_n^k$ by using any covering of S_n with two proper subsets m_1' and m_2'; this means

$$m_1' \cup m_2' = S_n \text{ with } \emptyset \subsetneq m_1', m_2' \subsetneq S_n.$$

An ∃-generalization that puts the attributes m_1 and m_2 together to get m_{12}, will also produce a generalized context with 2^{n+1} concepts. However the concept lattice of $\mathbb{K}_n$ will have more concepts when $m_1' \cap m_2' \neq \emptyset$ compared to when $m_1' \cap m_2' = \emptyset$.

Let $\mathbb{K}_{12} := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$ with $m_1' \cap m_2' \neq \emptyset$. If $m_1' \subseteq m_2'$ or $m_2' \subseteq m_1'$ then putting m_1 and m_2 together will not increase the size of the concepts lattice. Therefore, we assume that $m_1' \nsubseteq m_2'$ and $m_2' \nsubseteq m_1'$.

Proposition 2.3.3. *Let $n > 2$ and $\mathbb{K}_s$ the generalized context obtained from $\mathbb{K}_{12}$ by putting m_1 and m_2 together. Then:*

1. *The size of the concepts lattice of the context $\mathbb{K}_n$ is*

$$2^n + 2^{|m_2'|} + 2^{|m_1'|} - 2^{|m_2' \cap m_1'|}.$$

2. *After the generalization, the size of the initial lattice increases by*

$$2^n - 2^{|m_1'|} - 2^{|m_2'|} + 2^{|m_1' \cap m_2'|}.$$

Proof. The context $\mathbb{K}_{12}$ is obtained by adding m_1 to $\mathbb{K}_{00}$ to get $\mathbb{K}_{01}$ and then m_2 to $\mathbb{K}_{01}$. $m_1' \subseteq S_n$ and for all $A_1 \subseteq m_1'$, $A = A_1 \cup \{g_1\} \in \mathrm{Ext}\,(\mathbb{K}_{00})$ and $A \cap m_1' = A_1 \notin \mathrm{Ext}\,(\mathbb{K}_{00})$ because $g_1 \in A_1^{\mathrm{II}}$. Hence, $A_1 \in \mathcal{H}(m_1)$. Moreover, $\mathcal{H}(m_1) \subseteq \mathcal{P}(m_1')$. We conclude that $\mathcal{H}(m_1) = \mathcal{P}(m_1')$, and then $|\mathcal{H}(m_1)| = 2^{|m_1'|}$. Hence, $\mathbb{K}_{12} := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$ has $2^n + 2^{|m_1'|} + \mathcal{H}(m_2)$ concepts, where $\mathcal{H}(m_2)$ is to be determined with respect to $\mathbb{K}_{01} := (S_n \cup \{g_1\}, S_n \cup \{m_1\}, \mathrm{I})$. The concepts of $\mathbb{K}_{01}$ are of the form $(A_1 \cup \{g_1\}, S_n \setminus A_1)$ with $A_1 \subseteq S_n$ or of the form $(A_1, (S_n \setminus A_1) \cup \{m_1\})$ with $A_1 \subseteq m_1'$. Let $A \in \mathrm{Ext}\,(\mathbb{K}_{01})$. There are two cases, $g_1 \notin A$ or $g_1 \in A$.

- If $g_1 \notin A$ then $A \subseteq m_1'$ and $A \cap m_2'$ is a subset of m_1', and then an extent of $\mathbb{K}_{01}$. No new concept is generated.
- If $g_1 \in A$ then $A = A_1 \cup \{g_1\}$ with $A_1 \subseteq S_n$. Since all subsets of m_1' are extents of $\mathbb{K}_{01}$ and the object g_1 does not have the attribute m_2, we have

$$A \cap m_2' \notin \mathrm{Ext}\,(\mathbb{K}_{01}) \iff A \cap m_2' \in \mathcal{P}(m_2') \setminus \mathcal{P}(m_1').$$

Thus, adding m_2 to $\mathbb{K}_{01}$ will generate $2^{|m_2'|} - 2^{|m_1' \cap m_2'|}$ new concepts that cannot be identified (via ϕ_{m_2}) with concepts in $\mathbb{K}_{01}$. Then $\mathbb{K}_{12}$ has

$$2^n + 2^{|m_1'|} + 2^{|m_2'|} - 2^{|m_1' \cap m_2'|}$$

concepts. The increase of the size of the lattice is then

$$\begin{aligned}|\mathfrak{B}(\mathbb{K}_s)| - |\mathfrak{B}(\mathbb{K}_{12})| &= 2^{n+1} - \left(2^n + 2^{|m_1'|} + 2^{|m_2'|} - 2^{|m_1' \cap m_2'|}\right)\\ &= 2^n - 2^{|m_1'|} - 2^{|m_2'|} + 2^{|m_1' \cap m_2'|}\end{aligned}$$

■

Example 2.3.1. *This is a concrete case with $n = 3$. Its context is isomorphic to*

	1	2	3	m_1	m_2
1		×	×	×	
2	×		×	×	×
3	×	×			×
g_1	×	×	×		

and has $2^3 + 2^2 + 2^2 - 2^1 = 14$ concepts.

Remark 2.3.1. *Note that $n = |m_1' \cup m_2'|$ and the increase is*

$$2^{|m_1' \cup m_2'|} - 2^{|m_1'|} - 2^{|m_2'|} + 2^{|m_1' \cap m_2'|}$$

which is a general formula that holds, even if $m_1' \cup m_2' \neq S_n$. The starting context is $\mathbb{K}_{00} := (S_n \cup \{g_1\}, S_n, \mathrm{I})$ and has 2^n extents. After adding an attribute m_1 to $\mathbb{K}_{00}$, we increase the number of extents by $2^{|m_1'|}$. After adding m_2 to $\mathbb{K}_{00}$, we increase the number of extents by $2^{|m_2'|}$. After adding an attribute s with $s' = m_1' \cup m_2'$ to $\mathbb{K}_{00}$, we increase the number of extents by $2^{|m_1' \cup m_2'|}$. If we add an attribute t with $t' = m_1' \cap m_2'$ to $\mathbb{K}_{00}$ we will increase the number of extents by $2^{|m_1' \cap m_2'|}$. But these extents "appear" already when m_1 or m_2 is added to $\mathbb{K}_{00}$, and are therefore counted twice when both m_1 and m_2 are added to $\mathbb{K}_{00}$.

Remark 2.3.2. *The counting with $\mathbb{K}_{12}$ has been made easy by the fact that each "subset" of S_n identifies an extent of $\mathbb{K}_{00}$. If $m_1' \cap m_2'$ is not empty, then $\mathbb{K}_{12}$ has more concepts while the number of generalized concept remains the same. Then the condition $m_1' \cap m_2' = \emptyset$ is necessary (but not sufficient) to get the maximal increase. If n is even, then generalizing could increase the size of the concepts lattice up to $\left(2^{\lfloor \frac{n}{2} \rfloor} - 1\right)\left(2^{\lceil \frac{n}{2} \rceil} - 1\right)$ concepts. Is this the maximal increase for contexts of similar size?*

Remark 2.3.3. *Note that all contexts $\mathbb{K}_{12}$ constructed are reduced. Requiring the contexts to be reduced is a fair assumption. If not then we should first remove reducible attributes before processing with a generalization. This removal does not affect the size of the concept lattice. However putting together two reducible attributes will for sure not decrease the size, but probably increases it.*

Remark 2.3.4. *B_4 is the smallest lattice for which there are two attributes whose ∃-generalization increases the size of the concept lattice. All lattices presented in this section contain a labelled copy of B_4 (as subposet!).*

In this section we have found out that the size of the generalized concept lattice can be exponentially larger than that of the initial concept lattice after an existential generalization. In the next section we discuss the maximum of the increase after an ∃-generalization.

2.4 The maximum increase after an existential generalization

This section aims at finding the maximum of increase one can get after an existential generalization, especially the case where the size of the lattice increases after the generalization. Let consider a context $\mathbb{K} = (G, M, \mathrm{I})$ and two attributes a and b such that their existential generalization increases the size of the lattice. Let s be their existential generalized attribute.

In general, we get the context $\mathbb{K}$ by adding the attribute a to $\mathbb{K}_{ab}$ to get $\mathbb{K}_b$, and then by adding the attribute b to $\mathbb{K}_b$. Recall that if an attribute m is added to any context $\mathbb{K}$, then the number of concepts increases by

$$|\mathcal{H}(m)| = |\{A \cap m' | A \in \mathrm{Ext}(\mathbb{K}) \text{ and } A \cap m' \notin \mathrm{Ext}(\mathbb{K})\}|.$$

We denote by $a \cap b$ the attribute such that $(a \cap b)' := a' \cap b'$, and $a \cup b$ the attribute such that $(a \cup b)' := a' \cup b' = s'$. We start from $\mathbb{K}_{ab} = (G, M\backslash\{a, b\}, \mathrm{I})$.

Adding the attribute a to $\mathbb{K}_{ab}$ increases its number of concepts by

$$|\mathcal{H}(a)| = |\{A \cap a' | A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and } A \cap a' \notin \mathrm{Ext}(\mathbb{K}_{ab})\}| \leqslant 2^{|a'|}.$$

Adding the attribute b to $\mathbb{K}_{ab}$ increases its number of concepts by

$$|\mathcal{H}(b)| = |\{A \cap b' | A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and } A \cap b' \notin \mathrm{Ext}(\mathbb{K}_{ab})\}| \leqslant 2^{|b'|}.$$

Adding the attribute $a \cap b$ to $\mathbb{K}_{ab}$ increases its number of concepts by

$$|\mathcal{H}(a \cap b)| = |\{A \cap a' \cap b' | A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and } A \cap a' \cap b' \notin \mathrm{Ext}(\mathbb{K}_{ab})\}| \leqslant 2^{|a' \cap b'|}.$$

If $a' \cap b'$ is empty, then $|\mathcal{H}(a \cap b)| \leqslant 1$.

Adding the attribute $a \cup b$ to $\mathbb{K}_{ab}$ increases its number of concepts by

$$\begin{aligned}|\mathcal{H}(a \cup b)| = |\{A \cap (a' \cup b') | A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and}\\ A \cap (a' \cup b') \notin \mathrm{Ext}(\mathbb{K}_{ab})\}| \leqslant 2^{|a' \cup b'|} \leqslant 2^{|a'|+|b'|}.\end{aligned}$$

If $|\mathcal{H}(a \cup b)| = 2^{|a'|+|b'|}$, then $a' \cap b' = \emptyset$ and no subset of $a' \cup b'$ is an extent of $\mathbb{K}_{ab}$, but is the restriction of an extent of $\mathbb{K}_{ab}$ on $a' \cup b'$. In this case, $|\mathcal{H}(a)| = 2^{|a'|}$, $|\mathcal{H}(b)| = 2^{|b'|}$ and $|\mathcal{H}(a \cap b)| = 1$.

Let

$$d_1 = |\{A \subseteq a' | A \in \mathrm{Ext}(\mathbb{K}_{ab})\}|.$$

$$d_2 = |\{A \subseteq b' | A \in \mathrm{Ext}(\mathbb{K}_{ab})\}|.$$

and

$$d_0 = |\{A \subseteq a' \cup b' | A \in \mathrm{Ext}(\mathbb{K}_{ab})\}|.$$

Then $|\mathcal{H}(a)| = 2^{|a'|} - d_1$, $|\mathcal{H}(b)| = 2^{|b'|} - d_2$ and $|\mathcal{H}(s)| = 2^{|a' \cup b'|} - d_0$. Since $a' \cap b' = \emptyset$, the following holds for any extent $A \neq \emptyset$ of $\mathbb{K}_{ab}$:

$$A \subseteq a' \cup b' \Leftrightarrow A \subseteq a' \text{ xor } A \subseteq b' \text{ xor } A \subseteq a' \cup b' \text{ with } A \nsubseteq a \text{ and } A \nsubseteq b.$$

where *xor* denotes the exclusive or. Therefore $d_1 + d_2 - d_0 \leqslant 0$

To express the maximum increase of size of the concept lattice after an existential generalization, we have the following result.

Theorem 2.4.1. *Let* $\mathbb{K} := (G, M, \mathrm{I})$ *be an attribute reduced context with* $|G| \geqslant 3$ *and* $|M| > 3$. *Let* a *and* b *be two attributes such that their existential generalization* $s = a \cup b$ *increases the size of the concepts lattice. Then*

a) $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a, b)|$, *with*

$$|\mathcal{H}(a, b)| = |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|$$

b) The increase after the generalization is

$$|\mathcal{H}(a \cup b)| - |\mathcal{H}(a, b)| \leqslant 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1.$$

Proof. Let $\mathbb{K} := (G, M, \mathrm{I})$ be a such context and a, b two attributes of $\mathbb{K}$. One proceed to the ∃-generalization of attributes a and b. We set $\mathbb{K}_b = (G, M_b, \mathrm{I}_b)$. Then we have: $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_b)| + h^*(b) = |\mathfrak{B}(\mathbb{K}_{ab})| + h(a) + h^*(b)$ where $h^*(b) = |\mathcal{H}^*(b)| = |\{B \cap b'| B \in \mathrm{Ext}(\mathbb{K}_b),\ B \cap b' \notin \mathrm{Ext}(\mathbb{K}_b)\}|$. Note that $\mathcal{H}^*$ is determined with respect to $\mathbb{K}_b$.

Our aim is to express $h^*(b)$ as a function of $h(b)$ and $h(a \cap b)$. According to the elements above, $\mathrm{Ext}(\mathbb{K}_b) = \mathrm{Ext}(\mathbb{K}_{ab}) \cup \mathcal{H}(a)$. Hence, we have $\mathcal{H}^*(b) = \{B \cap b' | B \in \mathrm{Ext}(\mathbb{K}_b), B \cap b' \notin \mathrm{Ext}(\mathbb{K}_b)\} = \{B \cap b' | B \in \mathrm{Ext}(\mathbb{K}_{00}), B \cap b' \notin \mathrm{Ext}(\mathbb{K}_b)\} \cup \{B \cap b' | B \in \mathcal{H}(a), B \cap b' \notin \mathrm{Ext}(\mathbb{K}_b)\}$

Replacing $\mathrm{Ext}(\mathbb{K}_b)$ by $\mathrm{Ext}(\mathbb{K}_{ab}) \cup \mathcal{H}(a)$, we obtain $\{B \cap b' | B \in \mathrm{Ext}(\mathbb{K}_{ab}), B \cap b' \notin \mathrm{Ext}(\mathbb{K}_b)\} = \mathcal{H}(b) \backslash \mathcal{H}(a) \cap \mathcal{H}(b) = \mathcal{H}(b) \backslash \mathcal{H}(a)$, and $\{\mathrm{B} \cap b' | B \in \mathcal{H}(a), B \cap b' \notin \mathrm{Ext}(\mathbb{K}_b)\} = \mathcal{H}(a \cap b) \backslash (\mathcal{H}(b) \cup \mathcal{H}(a))$.

It comes that $h^*(b) = h(b) + h(a \cap b) - |\mathcal{H}(a) \cap \mathcal{H}(b)| + |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)|$. Hence, $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a)| + |\mathcal{H}(b)| + |\mathcal{H}(a \cap b)| + |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)| - |\mathcal{H}(a) \cap \mathcal{H}(b)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|$.

For the increase to be maximal, we need $a' \cap b' = \emptyset$. In that case, $|\mathcal{H}(a \cap b)| \in \{0, 1\}$.

- If $|\mathcal{H}(a \cap b)| = 0$, then $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a)| + |\mathcal{H}(b)|$.

- If $|\mathcal{H}(a \cap b)| = 1$, then we have two subcases:

  Firstly, we suppose that the only element of $\mathcal{H}(a \cap b)$ is not in $\mathcal{H}(a) \cup \mathcal{H}(b)$. Then, $|\mathcal{H}(a) \cap \mathcal{H}(b)| = |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)| = |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| = |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)| = 0$ and $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a)| + |\mathcal{H}(b)| + |\mathcal{H}(a \cap b)|$.

  Secondly, we suppose that the only element of $\mathcal{H}(a \cap b)$ belongs to either $\mathcal{H}(a)$ or $\mathcal{H}(b)$. Then $|\mathcal{H}(a \cap b)| + |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)| = 0$, $|\mathcal{H}(a) \cap \mathcal{H}(b)| \in \{0, 1\}$ and $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a)| + |\mathcal{H}(b)| + 1 - |\mathcal{H}(a) \cap \mathcal{H}(b)|$.

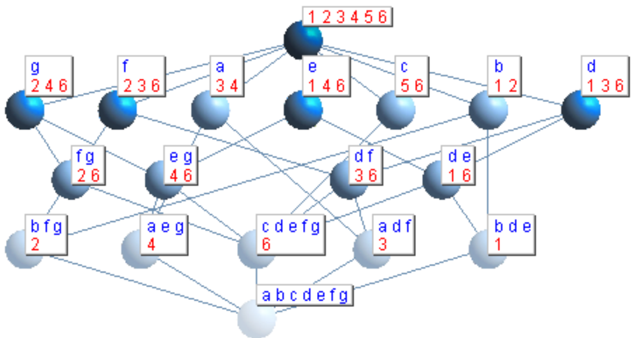
In all the above cases, considering that $|\mathfrak{B}(\mathbb{K}^s_{ab})| = |\mathfrak{B}(\mathbb{K}_{ab})| + |\mathcal{H}(a \cup b)|$, the increase after the generalization is $|\mathfrak{B}(\mathbb{K}^s_{ab})| - |\mathfrak{B}(\mathbb{K})| = |\mathcal{H}(a \cup b)| - |\mathcal{H}(a, b)| \leqslant 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + (d_1 + d_2 - d_0) \leqslant 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1$, since $d_1 + d_2 - d_0 \leqslant 0$. ■

Remark 2.4.1. *If generalizing a and b does not increase the size of the lattice, then the difference* $|\mathcal{H}(a \cup b)| - |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|$ *is at most zero, and will describe the reduction in the number of concepts.*

Remark 2.4.2. *In cases where the context does not contain a copy of* B_4 *as specified in [27], the increase of the size of a lattice after a generalization does not depend on whether the attributes are reducible or no. Both attributes can even be non reducible and the size of the lattice still increases or remains stable after an existential generalization, as specified in the following examples.*

Example 2.4.1.

$\mathbb{K}$	a	b	c	d	e	f	g
1		×		×	×		
2		×				×	×
3	×			×		×	
4	×				×		×
5			×				
6			×	×	×	×	×



$\mathbb{K}^s_{ab}$	c	d	e	f	g	s
1		×	×			×
2				×	×	×
3		×		×		×
4			×		×	×
5	×					
6	×	×	×	×	×	

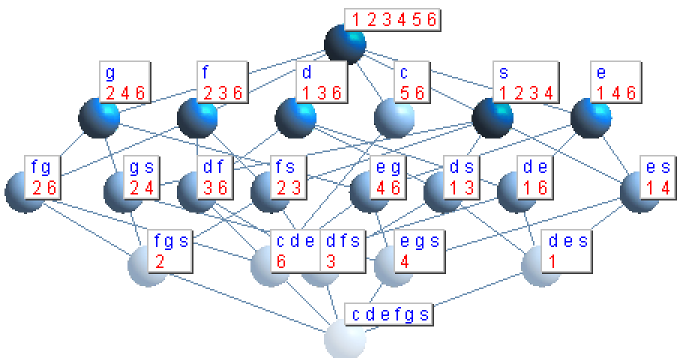
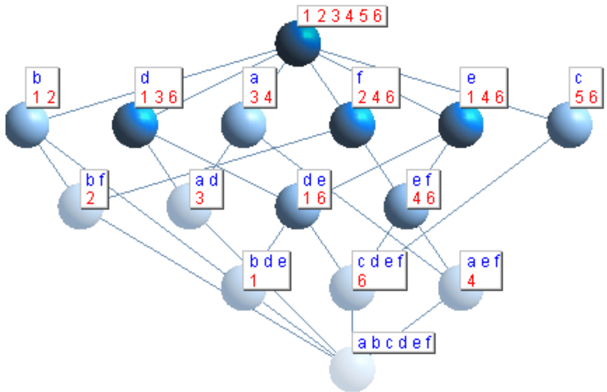


Figure 2.3: None of the attributes a and b is reducible, $|\mathfrak{B}(G, M, \mathrm{I})| = 18$ (up) and $|\mathfrak{B}(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab})| = 21$ (down)

$\mathbb{K}$	a	b	c	d	e	f
1		×		×	×	
2		×				×
3	×			×		
4	×				×	×
5			×			
6			×	×	×	×



$\mathbb{K}^s_{ab}$	c	d	e	f	s
1		×	×		×
2				×	×
3		×			×
4			×	×	×
5	×				
6	×	×	×	×	

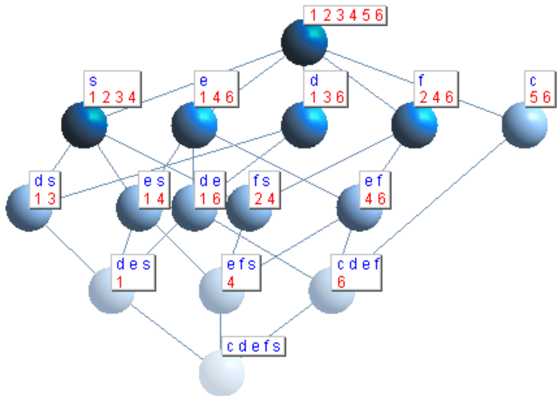


Figure 2.4: None of the attributes a and b is reducible and $|\mathfrak{B}(G, M, \mathrm{I})| = 15$ (up) and $|\mathfrak{B}(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab})| = 15$ (down)

2.5 A case of stability of the size of the lattice after an existential generalization

The ∃-generalization does not always leads to the strict increase or decrease of the size of the concept lattice. There are certainly some cases where the size of the lattice remain unchange after an ∃-generalization on two attributes of a reduced formal context. Here we present one of these cases were the existential generalization stabilises the size of the lattice. It is particularly important because an expert exploring and analysing a database can suspect that all information that will come from the data (as concepts) are relevant information, and can decide to reduce the size of the data[2] without reducing the quantities of these information. Since no expert wishes the size of the lattice to increase after a generalization, he will then try to keep stable the size of the lattice.

Let $\mathbb{K} = (G, M, \mathrm{I})$ be a context such that $|M| > 3$. For any triple of attributes (m_1, m_2, m_3), we consider the condition $C(m_1, m_2, m_3)$ defined by:

For every element A_0 of the set $\mathcal{P}(m_1^{\mathrm{I}} \cup m_2^{\mathrm{I}})$ verifying $m_1^{\mathrm{I}} \nsubseteq A_0$, $m_2^{\mathrm{I}} \nsubseteq A_0$ and $|A_0| = |m_2^{\mathrm{I}}| + |m_1^{\mathrm{I}}| - 2$, there exists a unique attribute z such that $z \notin \{m_1, m_2, m_3\}$, $A_0 = z^{\mathrm{I}}$ and $A_0^{\mathrm{I}} = \{z\}$.

The following result holds:

Proposition 2.5.1. *Let $\mathbb{K} = (G, M, \mathrm{I})$ be a reduced context such that $|M| > 3$, $|G| > 4$. We suppose that there exists attributes a, b and c of $\mathbb{K}$ such that:*

i) $a^{\mathrm{I}} \cup b^{\mathrm{I}} \subsetneq G$, $a^{\mathrm{I}} \cap b^{\mathrm{I}} = \emptyset$, $a^{\mathrm{I}} \cap c^{\mathrm{I}} = \emptyset$ and $b^{\mathrm{I}} \subseteq c^{\mathrm{I}}$,

ii) $|b^{\mathrm{I}}| > 1$, $|a^{\mathrm{I}}| > 1$,

iii) $C(a, b, c)$ holds and there exists a unique attribute z_0 such that $b^{\mathrm{I}} \subseteq z_0^{\mathrm{I}}$,

iv) for all $z \in M$, $a^{\mathrm{I}} \nsubseteq z^{\mathrm{I}}$,

then there exists an ∃-generalization that stabilizes the size of the initial lattice.

Proof. Let $\mathbb{K} = (G, M, \mathrm{I})$ be such context and s be the generalized attribute resulting from the ∃-generalization of a and b. Since $a^{\mathrm{I}} \cap b^{\mathrm{I}} = \emptyset$, $a^{\mathrm{I}} \cap c^{\mathrm{I}} = \emptyset$ and $b^{\mathrm{I}} \subseteq c^{\mathrm{I}}$, the attribute b is a reducible attribute in the context $\mathbb{K}^s = (G, M \cup \{s\}, \mathrm{I}^s)$ and then, considering the context $\mathbb{K}^s$, its elimination will keep stable the size of the lattice $\mathfrak{B}(\mathbb{K}^s)$. It means that

$$|\mathfrak{B}(G, M_b \cup \{s\}, \mathrm{I}_b^s)| = |\mathfrak{B}(G, M \cup \{s\}, \mathrm{I}^s)|.$$

Yet, we have that

$$|\mathfrak{B}(G, M \cup \{s\}, \mathrm{I}^s)| = |\mathfrak{B}(G, M, \mathrm{I})| + |\mathcal{H}(s)|,$$

and

$$|\mathfrak{B}(G, M_b \cup \{s\}, \mathrm{I}_b^s)| = |\mathfrak{B}(G, M_{ab} \cup \{s\}, \mathrm{I}_{ab}^s)| + |\mathcal{H}(a)|.$$

[2] he can reduce the number of attributes in the data

It then comes out that

$$|\mathfrak{B}(G, M, \mathrm{I})| + |\mathcal{H}(s)| = |\mathfrak{B}(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab})| + |\mathcal{H}(a)|.$$

We will now show that every subset A of $a^{\mathrm{I}} \cup b^{\mathrm{I}}$ (except $a^{\mathrm{I}} \cup b^{\mathrm{I}}$) is either an extent of $\mathbb{K}$ or cannot be written as an intersection of $a^{\mathrm{I}} \cup b^{\mathrm{I}}$ with an extent of $\mathbb{K}$.

- Let $T \in \mathrm{Ext}_{\mathbb{K}}$ such that $T \subsetneq G$, $T \nsubseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$ and $A = T \cap (a^{\mathrm{I}} \cup b^{\mathrm{I}}) \neq \emptyset$ for some A. Then $a \notin T^{\mathrm{I}}$ and $b \notin T^{\mathrm{I}}$, else $T \subseteq a^{\mathrm{I}} \subseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$ or $T \subseteq b^{\mathrm{I}} \subseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$, which is absurd. Moreover $A \subseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$ and $|A| < |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 1$. In fact, if $|A| \geqslant |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 1$, then $A = a^{\mathrm{I}} \cup b^{\mathrm{I}}$ or $|A| = |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 1$. In both cases, $a^{\mathrm{I}} \subseteq z^{\mathrm{I}}$ or $b^{\mathrm{I}} \subseteq z^{\mathrm{I}}$ for all $z \in T^{\mathrm{I}}$. Let z be an element of T^{I}. Then $z^{\mathrm{I}} \cap a^{\mathrm{I}} \neq \emptyset$, since $|A| \leqslant |T| \leqslant |z^{\mathrm{I}}|$. If $a^{\mathrm{I}} \subseteq z^{\mathrm{I}}$, then it is a contradiction to iv). If $b^{\mathrm{I}} \subseteq z^{\mathrm{I}}$, then it is a contradiction to iii) because $c^{\mathrm{I}} \cap a^{\mathrm{I}} \neq \emptyset$ and then $z \neq c$. Therefore, there is no extent T of $\mathbb{K}$ such that $T \subsetneq G$, $T \nsubseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$, $T \cap (a^{\mathrm{I}} \cup b^{\mathrm{I}}) \neq \emptyset$ and $|T \cap (a^{\mathrm{I}} \cup b^{\mathrm{I}})| \geqslant |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 1$.

- Since $a^{\mathrm{I}} \nsubseteq z^{\mathrm{I}}$ for all $z \in M$, $a^{\mathrm{II}} = \{a\}$. Since the attribute c is the only attribute of $\mathbb{K}$ such that $b^{\mathrm{I}} \subseteq c^{\mathrm{I}}$, then $b^{\mathrm{II}} = \{b, c\}$. Moreover, $\emptyset^{\mathrm{II}} = \emptyset$ because $a^{\mathrm{I}} \cap b^{\mathrm{I}} = \emptyset$. Hence, $a^{\mathrm{I}}, b^{\mathrm{I}}, \emptyset \in \mathrm{Ext}_{\mathbb{K}}$.

- Now we consider a proper subset A_0 of $a^{\mathrm{I}} \cup b^{\mathrm{I}}$ such that $A_0 \notin \{a^{\mathrm{I}}, b^{\mathrm{I}}\}$ and $1 \leqslant |A_0| \leqslant |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2$.

Then there are several cases:

*) If $a^{\mathrm{I}} \subseteq A_0$, then $A_0^{\mathrm{I}} \subseteq a^{\mathrm{II}} = \{a\}$, which implies that $A_0^{\mathrm{I}} = \emptyset$ or $A_0^{\mathrm{I}} = \{a\}$. In both cases, $A_0 \notin \mathrm{Ext}_{\mathbb{K}}$ because $A_0 \subseteq G$ and $A_0^{\mathrm{II}} = a^{\mathrm{I}}$ respectively.

**) If $b^{\mathrm{I}} \subseteq A_0$, then $A_0^{\mathrm{I}} \subseteq b^{\mathrm{II}} = \{b, c\}$, which implies that $A_0^{\mathrm{I}} = \emptyset$ or $A_0^{\mathrm{I}} = \{b\}$ or $A_0^{\mathrm{I}} = \{c\}$ or $A_0^{\mathrm{I}} = \{b, c\}$. But $A_0^{\mathrm{I}} \neq \{b\}$ (because $b^{\mathrm{I}} \subseteq c^{\mathrm{I}}$) and $A_0^{\mathrm{I}} \neq \{c\}$ (because there is no subset A_0 of $a^{\mathrm{I}} \cup b^{\mathrm{I}}$ such that $A_0^{\mathrm{I}} = \{c\}$). Hence, $A_0^{\mathrm{I}} = \emptyset$ or $A_0^{\mathrm{I}} = \{b, c\}$. In both cases, $A_0 \notin \mathrm{Ext}_{\mathbb{K}}$ because $A_0 \subseteq G$ and $A_0^{\mathrm{II}} = b^{\mathrm{I}}$ respectively.

However, in cases *) and **), if T is an extent of the context $\mathbb{K}$ such that $T \subsetneq G$, $T \nsubseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$ and $T \cap (a^{\mathrm{I}} \cup b^{\mathrm{I}}) = A_0$; and z is an element of T^{I}, then $a^{\mathrm{I}} \subseteq z^{\mathrm{I}}$ or $b^{\mathrm{I}} \subseteq z^{\mathrm{I}}$. If $a^{\mathrm{I}} \subseteq z^{\mathrm{I}}$, then it is a contradiction to iv). If $b^{\mathrm{I}} \subseteq z^{\mathrm{I}}$, then it is a contradiction to iii) because $z \neq c$. Therefore, there is no extent T of $\mathbb{K}$ such that $T \subsetneq G$, $T \nsubseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$ and $T \cap (a^{\mathrm{I}} \cup b^{\mathrm{I}}) = A_0$.

We suppose that $a^{\mathrm{I}} \nsubseteq A_0$ and $b^{\mathrm{I}} \nsubseteq A_0$.

*) If $|A_0| = |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2$, then there exists $z \in M$ such that $z \notin \{a, b, c\}$, $A_0 = z^{\mathrm{I}}_{A_0}$ and $A_0^{\mathrm{I}} = \{z\}$. Therefore $A_0 \in \mathrm{Ext}_{\mathbb{K}}$.

*) If $|A_0| < |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2$, then there is $z \in M$ and $B^z_{A_0} \subseteq a^{\mathrm{I}} \cup b^{\mathrm{I}}$ such that $z \notin \{a, b, c\}$, $|A_0 \cup B^z_{A_0}| = |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2$, $(A_0 \cup B^z_{A_0})^{\mathrm{I}} = \{z\}$ and $A_0 \cup B^z_{A_0} = z^{\mathrm{I}}$. Let precise that $a^{\mathrm{I}} \nsubseteq A_0 \cup B^z_{A_0}$, else $z = a$, which is a contradiction. Also, $b^{\mathrm{I}} \nsubseteq A_0 \cup B^z_{A_0}$, else $z \in \{b, c\}$, which is a again a contradiction. Therefore, $A_0 \subseteq z^{\mathrm{I}}$. Hence, we set $Y^{\mathbb{K}}_{A_0} = \{z \in M; A_0 \subseteq z^{\mathrm{I}}\}$ and $Y^{*\mathbb{K}}_{A_0} = \{z \in M; \exists B^z_{A_0}, |A_0 \cup B^z_{A_0}| = |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2, (A_0 \cup B^z_{A_0})^{\mathrm{I}} = \{z\}, A_0 \cup B^z_{A_0} = z^{\mathrm{I}}, A_0 \subseteq z^{\mathrm{I}}\}$. Then $A_0^{\mathrm{I}} = Y^{\mathbb{K}}_{A_0}$ and $A_0 \subseteq (Y^{\mathbb{K}}_{A_0})^{\mathrm{I}}$. It is still to be

shown that we do not have $A_0 \subsetneq (Y_{A_0}^{\mathbb{K}})^{\mathrm{I}}$.

To get there, we first show that for all $z \in Y_{A_0}^{*\mathbb{K}}$, for all $g \in B_{A_0}^z$, there exists $z' \in Y_{A_0}^{*\mathbb{K}}$ such that $g \notin B_{A_0}^{z'}$.

Let consider $z \in Y_{A_0}^{*\mathbb{K}}$ and $g \in B_{A_0}^z$. Since $z \in Y_{A_0}^{*\mathbb{K}}$, $(A_0 \cup B_{A_0}^z)^{\mathrm{I}} = \{z\}$ and $A_0 \cup B_{A_0}^z = z^{\mathrm{I}}$. We set $A' = A_0 \cup (B_{A_0}^z \backslash \{g\})$. Then $|A'| < |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2$ and then there exists $z' \in M$, and $B_{A'}^{z'} \subseteq G$ such that $[A_0 \cup (B_{A_0}^z \backslash \{g\})] \cup B_{A'}^{z'} = z'^{\mathrm{I}}$ and $[[A_0 \cup (B_{A_0}^z \backslash \{g\})] \cup B_{A'}^{z'}]^{\mathrm{I}} = \{z'\}$, by iii). Setting $B_{A_0}^{z'} = (B_{A_0}^z \backslash \{g\}) \cup B_{A'}^{z'}$, we get $(A_0 \cup B_{A_0}^{z'})^{\mathrm{I}} = \{z'\}$ and $A_0 \cup B_{A_0}^{z'} = z'^{\mathrm{I}}$ and $g \notin z'^{\mathrm{I}}$ (in fact, if $g \in z'^{\mathrm{I}}$, then $g \in B_{A'}^{z'}$, which means that $z^{\mathrm{I}} \subseteq z'^{\mathrm{I}}$, which is absurd).

We also show that $\bigcap_{z \in Y_{A_0}^{*\mathbb{K}}} B_{A_0}^z = \emptyset$.

Let suppose that $\bigcap_{z \in Y_{A_0}^{*\mathbb{K}}} B_{A_0}^z \neq \emptyset$. Then there exists $g_0 \in \bigcap_{z \in Y_{A_0}^{*\mathbb{K}}} B_{A_0}^z$. Since $|a^{\mathrm{I}}| > 1$ and $|b^{\mathrm{I}}| > 1$, we have $|Y_{A_0}^{*\mathbb{K}}| \geqslant 2$. Let $z \in Y_{A_0}^{*\mathbb{K}}$. Considering $B_{A_0}^z$, $g_0 \in B_{A_0}^z$. However, as we have shown before, there exists $z' \in Y_{A_0}^{*\mathbb{K}}$ such that $z \neq z'$ and $g_0 \notin z'$. This implies that $g_0 \notin B_{A_0}^{z'}$, which is absurd.

Hence, if $A_0 \subsetneq (Y_{A_0}^{\mathbb{K}})^{\mathrm{I}}$, then there exists an object $g_0 \in (Y_{A_0}^{\mathbb{K}})^{\mathrm{I}} \backslash A_0$. It means that $g_0 \in (Y_{A_0}^{\mathbb{K}})^{\mathrm{I}} \backslash A_0 = (\bigcap_{z \in Y_{A_0}^{\mathbb{K}}} z^{\mathrm{I}}) \backslash A_0 = \bigcap_{z \in Y_{A_0}^{\mathbb{K}}} (z^{\mathrm{I}} \backslash A_0) \subseteq \bigcap_{z \in Y_{A_0}^{*\mathbb{K}}} (z^{\mathrm{I}} \backslash A_0) = \bigcap_{z \in Y_{A_0}^{*\mathbb{K}}} B_{A_0}^z$, which is absurd because $\bigcap_{z \in Y_{A_0}^{*\mathbb{K}}} B_{A_0}^z = \emptyset$.

Therefore, $A_0 = (Y_{A_0}^{\mathbb{K}})^{\mathrm{I}}$, which implies that $A_0 \in \mathrm{Ext}_{\mathbb{K}}$.

Hence, for all proper subset A_0 of $a^{\mathrm{I}} \cup b^{\mathrm{I}}$ such that $A_0 \notin \{a^{\mathrm{I}}, b^{\mathrm{I}}\}$ and $|A_0| \geqslant 1$, $A_0 \in \mathrm{Ext}_{\mathbb{K}}$ or there is no extent T in $\mathbb{K}$ such that $A = T \cap (a^{\mathrm{I}} \cup b^{\mathrm{I}})$.

From iv) the context $\mathbb{K}$ has no attribute z such that $a^{\mathrm{I}} \cup b^{\mathrm{I}} \subseteq z^{\mathrm{I}}$; therefore, $a^{\mathrm{I}} \cup b^{\mathrm{I}}$ is not an extent of $\mathbb{K}$ (due to $a^{\mathrm{I}} \cap b^{\mathrm{I}} = \emptyset$). Also, G is an extent of $\mathbb{K}$, $a^{\mathrm{I}} \cup b^{\mathrm{I}} \subsetneq G$ and $a^{\mathrm{I}} \cup b^{\mathrm{I}} = G \cap s^{\mathrm{I}^s}$. Hence,

$$|\mathcal{H}(s)| = 1$$

Now we consider the generalized formal context $\mathbb{K}_{os} = (G, M_{00} \cup \{s\}, \mathrm{I}_{os}) = \mathbb{K}_{ab}^s = (G, M_{ab} \cup \{s\}, \mathrm{I}_{ab}^s)$.

We will show that every subset A of a^{I} (except a^{I}) is an extent of $\mathbb{K}_{os}$.

Let A be a subset of a^{I} such that $A \subsetneq a^{\mathrm{I}}$. There are two cases:

- If $A = \emptyset$, then $A \in \mathrm{Ext}_{\mathbb{K}_{os}}$. Else, there would exist $z \in Y_A^{*\mathbb{K}}$ such that $|z^{\mathrm{I}}| > |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2$, which would be absurd.

- If $A \neq \emptyset$, then $1 \leqslant A < |a^{\mathrm{I}}| \leqslant |a^{\mathrm{I}}| + |b^{\mathrm{I}}| - 2$ and as we have seen before, there exists $z \in M \backslash \{a, b, c\}$ and $B_A^z \subseteq G$ such that $A \cup B_A^z = z^{\mathrm{I}}$ and $(A \cup B_A^z)^{\mathrm{I}} = \{z\}$ in $\mathbb{K}$, by iii). But such z is an attribute of $\mathbb{K}_{00}$ and as we have shown previously with such set A, $\emptyset = \bigcap_{z \in Y_A^{*\mathbb{K}}} B_A^z = \bigcap_{z \in Y_A^{*\mathbb{K}_{00}}} B_A^z$ and then $\bigcap_{z \in Y_A^{*\mathbb{K}}} z^{\mathrm{I}} = \bigcap_{z \in Y_A^{*\mathbb{K}}} (A \cup B_A^z) = A \cup \bigcap_{z \in Y_A^{*\mathbb{K}}} B_A^z = A$, which means that $A = Y_A^{*\mathbb{K}\mathrm{I}} = Y_A^{*\mathbb{K}_{00}\mathrm{I}_{00}} \supseteq Y_A^{\mathbb{K}_{00}\mathrm{I}_{00}} \supseteq Y_A^{\mathbb{K}_{os}\mathrm{I}_{os}}$. Hence, $Y_A^{\mathbb{K}_{os}\mathrm{I}_{os}} \subseteq A$ and since $A \subseteq Y_A^{\mathbb{K}_{os}\mathrm{I}_{os}}$, we conclude that $A = Y_A^{\mathbb{K}_{os}\mathrm{I}_{os}}$. Also, we obviously have $A^{\mathrm{I}_{os}} = Y_A^{\mathbb{K}_{os}}$. Therefore, $A \in \mathrm{Ext}_{\mathbb{K}_{os}}$.

Also, because there is no attribute z of $\mathbb{K}$ such that $a^{\mathrm{I}} \subseteq z^{\mathrm{I}}$, the generalized attribute s is the only attribute of $\mathbb{K}^s_{ab}$ such that $a^{\mathrm{I}} \subseteq s^{\mathrm{I}^s_{ab}}$, and then a^{I} is not an extent of $\mathbb{K}^s_{ab}$ because a is not an attribute of $\mathbb{K}^s_{ab}$. Moreover, G is an extent of $\mathbb{K}^s_{ab}$, $a^{\mathrm{I}} \subsetneq G$ and $a^{\mathrm{I}} = G \cap a^{\mathrm{I}}$. We conclude that

$$|\mathcal{H}(a)| = 1$$

Hence,

$$|\mathfrak{B}(G, M, \mathrm{I})| + 1 = |\mathfrak{B}(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab})| + 1$$

which means that

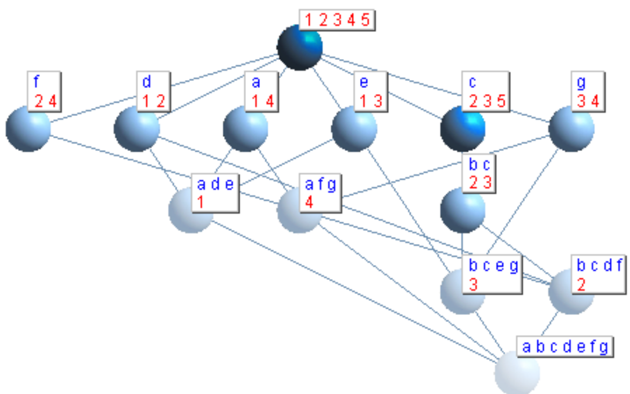
$$|\mathfrak{B}(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab})| = |\mathfrak{B}(G, M, \mathrm{I})|$$

■

Example 2.5.1.

Consider the following context $\mathbb{K}$ with de corresponding concept lattice

$\mathbb{K}$	a	b	c	d	e	f	g
1	×			×	×		
2		×	×	×		×	
3		×	×		×		×
4	×					×	×
5			×				



The ∃-generalization of attributes a and b in $\mathbb{K}$ leads to the following context $\mathbb{K}^s_{ab}$ with the corresponding concept lattice.

$\mathbb{K}^s_{ab}$	c	d	e	f	g	s
1		×	×			×
2	×	×		×		×
3	×		×		×	×
4				×	×	×
5	×					

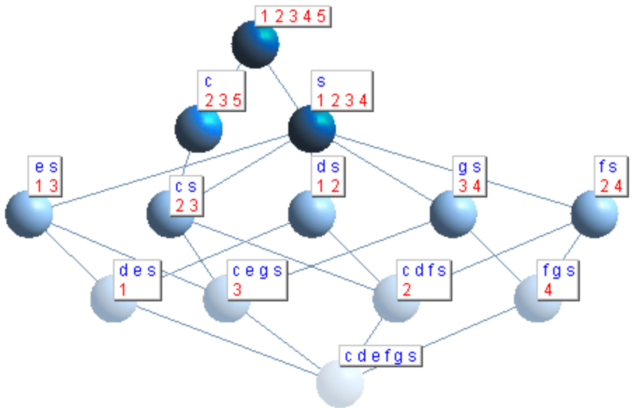


Figure 2.5: An example of stability after an ∃-generalization: $|\mathfrak{B}(G, M, \mathrm{I})| = 13$ and $|\mathfrak{B}(G, M_{ab} \cup \{s\}, I^{s}_{ab})| = 13$

The generalization can also be done in many attributes at the same time. For all non empty set X of attribute, we set s_X the attribute obtained by generalizing simultaneously all the attributes in X. Then the following result holds for ∃-generalization:

Theorem 2.5.1. *Let $\mathbb{K} =: (G, M, \mathrm{I})$ be a reduced formal context such that $|M| > 3$. Then the ∃-generalization in $\mathbb{K}$ is associative, ie*

$$\mathfrak{B}(G, M_{abc} \cup \{s_{abc}\}, \mathrm{I}^{s_{abc}}_{abc}) = \mathfrak{B}(G, (M_{bc} \cup \{s_{bc}\})_{as_{bc}} \cup \{a \cup s_{bc}\}, \mathrm{I}^{a\cup s_{bc}}_{as_{bc}}) = \mathfrak{B}(G, (M_{ab} \cup \{s_{ab}\})_{cs_{ab}}) \cup \{s_{ab} \cup c\}, \mathrm{I}^{s_{ab}\cup c}_{s_{ab}c})$$

for attributes a, b, c of $\mathbb{K}$.

Proof. After a ∃-generalization on attributes $\{a, b, c\}$ in the context $\mathbb{K}$, we get the generalized attribute s_{abc} such that $s'_{abc} = a' \cup b' \cup c'$. Hence, the corresponding generalized concepts lattice is $\mathfrak{B}(G, M_{abc} \cup \{s_{abc}\}, \mathrm{I}^{s_{abc}}_{abc})$.

Generalizing the attributes $\{b, c\}$ in the context $\mathbb{K}$, we get the generalized attribute s_{bc} such that $s'_{bc} = b' \cup c'$, and the following generalized concepts lattice: $\mathfrak{B}(G, M_{bc} \cup \{s_{bc}\}, \mathrm{I}^{s_{bc}}_{bc})$.

In the context $(G, M_{bc} \cup \{s_{bc}\}, \mathrm{I}^{s_{bc}}_{bc})$, we generalize the attributes a and s_{bc} to get the attribute $a \cup s_{bc}$ such that $(a \cup s_{bc})' = a' \cup s'_{\{b,c\}}$. The generalized concepts lattice is then given by $\mathfrak{B}(G, (M_{bc} \cup \{s_{bc}\})_{as_{bc}} \cup \{a \cup s_{bc}\}, \mathrm{I}^{a\cup s_{bc}}_{as_{bc}})$.

But $(a \cup s_{bc})' = a' \cup s'_{bc} = a' \cup (b' \cup c') = a' \cup b' \cup c' = s'_{abc}$, because $s'_{bc} = b' \cup c'$. Moreover, $(M_{bc} \cup \{s_{bc}\})_{as_{bc}} = M_{abc}$.

Therefore,

$$\mathfrak{B}(G, M_{abc} \cup \{s_{abc}\}, \mathrm{I}^{s_{abc}}_{abc}) = \mathfrak{B}(G, (M_{bc} \cup \{s_{bc}\})_{as_{bc}} \cup \{a \cup s_{bc}\}, \mathrm{I}^{a\cup s_{bc}}_{as_{bc}}).$$

In the same way, generalizing attribute $\{a, b\}$ in the context $\mathbb{K}$ to get the generalized attribute s_{ab} such that $s'_{ab} = a' \cup b'$ and then c and s_{ab} to get the attribute $s_{ab} \cup c$ such that $(s_{ab} \cup c)' = s'_{ab} \cup c'$ leads to:

$$\mathfrak{B}(G, M_{abc} \cup \{s_{abc}\}, \mathrm{I}_{abc}^{s_{abc}}) = \mathfrak{B}(G, (M_{ab} \cup \{s_{ab}\})_{cs_{ab}}) \cup \{s_{ab} \cup c\}, \mathrm{I}_{s_{ab}c}^{s_{ab} \cup c}).$$

■

Remark 2.5.1. *In the same way, generalizing several groups of attributes simultaneously is equivalent to generalizing each group one after another by applying associativity on each of them.*

2.6 The relationship between ∃-generalization and granularity, factorisation and pattern structure

This section looks at the link between ∃-generalization and other related notions, notably the granularity in formal context, the factorization of formal contexts and pattern structures.

2.6.1 The ∃-generalization in connection with granularity

Formal Concepts Analysis (FCA) plays a crucial role in various domains, especially in qualitative data analysis. Here, the number of extracted pieces of information can grow very fast. To control the number of concepts, [4] gave a method to control the structure of the concept lattice by specifying the level of granularity of attributes. In its method, some cases showed an increase of size of the concept lattice after the changing of level of granularity, and other cases revealed a decrease of size of the concept lattice. The authors said that these cases needed further examination as well as the possibility of informing the user who is choosing a new level of granularity, whether the size of the lattice will increase or no. The notion of granularity has a connection with ∃-generalization. First, we recall some definitions.

Definition 2.6.1. *[4]*

- *Granulation is a collection of pieces of information (granule) of the outer-world, which serve as the basic of reasoning.*

- *Let M be a set of attributes. A* ***granularity tree*** *(g-tree) of an attribute y is a rooted tree such that each node of the tree is labeled by a (unique) attribute name; the root is labeled by y; to each label z of a node, a set $z^{\downarrow}$ (which described the objects to which the attribute z is applied) is associated; and if the nodes labeled by $z_1, ..., z_n$ are successors of the node labeled by z, then $z_1^{\downarrow}, ..., z_n^{\downarrow}$ is a partition of $z^{\downarrow}$.*

- *A* ***cut*** *in a g-tree for an attribute y, also called a refinement of y is a set $\mathcal{C}_y$ of node labels of the g-tree such that for each leaf node u, there exists exactly one node v on the path from the root y to u such that the label of v belongs to $\mathcal{C}_y$.*

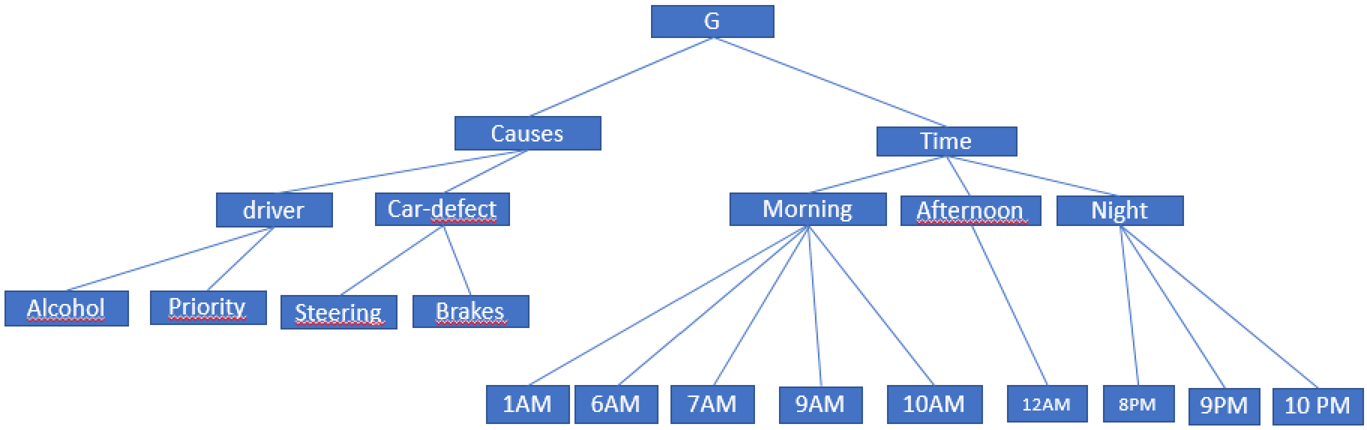
The following example of data concerning the causes of car accidents is from [5].

Example 2.6.1.

Car accidents.

Ref. no.	Driver name	Cause of accident	Time of accident
029	Lech	Steering	6AM
082	Schwartz	Priority	6AM
103	Zulle	Alcohol	10PM
105	Kiril	Brakes	9PM
109	Runde	Alcohol	12AM
189	Kohn	Steering	10PM
212	Mundici	Brakes	10PM
217	Mach	Steering	12AM
255	Brjusov	Alcohol	9PM
315	Nowak	Alcohol	8PM
460	Welich	Brakes	10AM
495	Pris	Steering	7AM
501	Pazdera	Priority	10AM
508	Brisville	Alcohol	6AM
622	Tiziano	Brakes	9AM
631	Rashad	Priority	9AM
640	Kulkarni	Priority	10AM
720	Pogonowski	Alcohol	1AM
731	Chen	Brakes	9AM
802	Serhat	Priority	9PM
930	Lyapkin	Brakes	9PM
977	Brycz	Priority	6AM

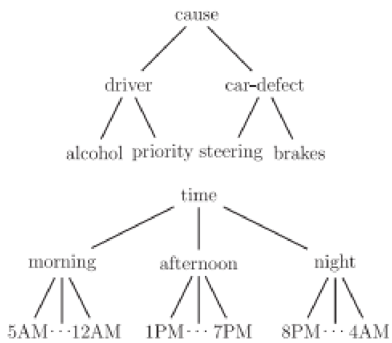
From the above data, we get the following tree:



The granule of the above tree is then given by: Granule={Cause, Time, Steering, Priority, Alcohol, Brakes, driver, car-defect, 1AM, 6AM, 7AM, 10AM, 12AM, 8PM, 9PM, 10PM, morning, afternoon, night}.

The following are some examples of the g-trees of attributes "cause" and "time" respectively.

The above g-trees leads to some examples of cuts of attributes "cause" and "time":



Example 2.6.2. $C^1_{time} = \{time\}$,
$\mathcal{C}^2_{time} = \{morning, afternoon, night\}$,
$\mathcal{C}^3_{time} = \{1AM, ..., 12AM, 8PM, ..., 10PM\}$,
$\mathcal{C}^1_{cause} = \{cause\}$,
$\mathcal{C}^2_{cause} = \{driver, car - defect\}$,
$\mathcal{C}^3_{cause} = \{alcohol, priority, steering, brakes\}$.

Definition 2.6.2. *[5] Now we consider a formal context $\mathbb{K} = (G, M, \mathrm{I})$. For each attribute y of $\mathbb{K}$, one suppose there exists a g-tree T_y.*

- *A* ***level of granularity*** *in the formal context $\mathbb{K} = (G, M, \mathrm{I})$ is characterized by a given family of cuts $\mathcal{Y} = \{\mathcal{C}_y, y \in M\}$.*
- *A binary relation on $\mathcal{C}_\mathcal{Y} = \cup_{y\in M}\mathcal{C}_y$ is defined as follow: $(g, z) \in \mathrm{I}_\mathcal{Y}$ if and only if $g \in z^{\downarrow}$. The formal context $(G, \mathcal{C}_\mathcal{Y}, \mathrm{I}_\mathcal{Y})$ results from the context (G, M, I) by replacing each attribute $y \in Y$ by the corresponding collection C_y of attributes.*

Definition 2.6.3. *[5] Let $y \in M$. We consider two cuts $\mathcal{C}^1_y = \{y_1, ..., y_n\}$ and $\mathcal{C}^2_y = \{z_1, ..., z_m\}$. Then it is defined a binary relation called the* ***refinement relation*** *as follow:*

$$\mathcal{C}^1_y \leqslant \mathcal{C}^2_y \text{ iff for any } y_i, \text{ there exists } z_j \text{ such that } y_i^{\downarrow} \subseteq z_j^{\downarrow}.$$

In the same way, On the set of all collection of cuts of g-trees of the formal context $\mathbb{K}$, it is defined a binary relation as follow: for all collections $\mathcal{Y}_1 = \{C^1_y; y \in M\}$ and $\mathcal{Y}_2 = \{C^2_y; y \in M\}$ of cuts,

$$\mathcal{Y}_1 \leqslant \mathcal{Y}_2 \text{ iff } \mathcal{C}^1_y \leqslant \mathcal{C}^2_y \text{ for each } y \in M.$$

Definition 2.6.4. *[5] Given two collection of cuts $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of different granularities in the context $\mathbb{K}$, such that $\mathcal{C}_{\mathcal{Y}_1} = \cup_{y\in M}\mathcal{C}^1_y$ and $\mathcal{C}_{\mathcal{Y}_2} = \cup_{y\in M}\mathcal{C}^2_y$.*

- *If $\mathcal{Y}_1 \leqslant \mathcal{Y}_2$, then we say that $\mathcal{Y}_1$ is* ***finer*** *than $\mathcal{Y}_2$ and $\mathcal{Y}_2$ is* ***coarser*** *than $\mathcal{Y}_1$.*

- *In some cases, one of the contexts* $(G, \mathcal{C}_{\mathcal{Y}_1}, \mathrm{I}_{\mathcal{C}_{\mathcal{Y}_1}})$ *and* $(G, \mathcal{C}_{\mathcal{Y}_2}, \mathrm{I}_{\mathcal{C}_{\mathcal{Y}_2}})$ *can be deduced from the other by* ***coarsening*** *or by* ***refinement****. In these cases, the finer (respectively the coarser) of the two corresponding contexts is obtained from the other by replacing each of its attributes by its corresponding finer attribute (respectively its coarser attributes). There can also be some cases where one of them can not be deduced from the other.*

- *Passing from the finer to the coarser collection of cuts is called a* ***reduction of the level of granularity****, and passing from the coarser to the finer collection of cuts is called an* ***increase of the level of granularity****.*

These are some examples of level of granularity: $\mathcal{Y}_1 = \{\mathcal{C}^3\text{cause}, \mathcal{C}^3\text{time}\}$, $\mathcal{Y}_2 = \{\mathcal{C}^2\text{cause}, \mathcal{C}^2\text{time}\}$, $\mathcal{Y}_3 = \{\mathcal{C}^3\text{cause}, \mathcal{C}^2\text{time}\}$. One can easily observe that $\mathcal{Y}_1 \leqslant \mathcal{Y}_2$, $\mathcal{Y}_1 \leqslant \mathcal{Y}_3$ and $\mathcal{Y}_3 \leqslant \mathcal{Y}_2$. With $\mathcal{Y}_2$ and $\mathcal{Y}_3$, these are their corresponding formal contexts:

K(Y2)	driver	car-defect	morning	Afternoon	night
Lech			x		
Schwartz	x		x		
Zulle	x				x
Kiril		x			x
Runde	x			x	
Kohn		x			x
Mundici		x			x
Mach		x		x	
Brjusov	x				x
Nowak	x				x
Welich		x	x		
Pris	x		x		
Pazdera		x	x		
Brisville	x		x		
Tiziano		x	x		
Rashad	x		x		
Kulkami	x		x		
Pogonowski	x		x		
Chen	x		x		
Serhat	x				x
Lyapkin		x			x
Brycz	x		x		

K(Y3)	alcohol	priority	steering	brakes	morning	Afternoon	night
Lech			x		x		
Schwartz		x			x		
Zulle	x			x			x
Kiril							x
Runde	x					x	
Kohn			x	x			x
Mundici							x
Mach			x			x	
Brjusov	x						x
Nowak	x			x			x
Welich					x		
Pris			x		x		
Pazdera		x			x		
Brisville	x			x	x		
Tiziano					x		
Rashad		x			x		
Kulkami		x			x		
Pogonowski	x			x	x		
Chen					x		
Serhat		x		x			x
Lyapkin							x
Brycz		x			x		

To appreciate the structure of the concept lattice, the level of granularity is changed from one level to another and the patterns obtained are appreciated. The concept lattice over such attributes may not contain interesting formal concepts because the selected attributes are too coarse, resulting in a low level of granularity of concepts. If one uses attributes with a higher level of granularity instead, such as "alcohol","night", "afternoon", etc., the concept lattice may reveal some interesting patterns, such as a formal concepts containing "alcohol" and "night" among its attributes. If such concept is applied to a large number of accidents, it may reveal interesting information. Such a concept may not be detectable when using coarser attributes. On the other hand,too high a level of granularity may result in overly specific formal concepts which may be of too little interest to the user.

The main concern here is to study the variation of size of concept lattices while one moves from a smaller level of granularity to a higher level of granularity and vice-versa.

In order to get there, we study the relation between the set of extents of a formal context and that of a subcontext having the same set of objects with the context. That relation is given in the following result:

Theorem 2.6.1. *Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context and $\{z_1, ..., z_n\} \subsetneq M$. One sets $M_{00} = M_{z_1,...,z_n} = M \backslash \{z_1, ..., z_n\}$, $\mathrm{I}_{00} = \mathrm{I} \cap (G \times M_{00})$ and $\mathbb{K}_{00} = (G, M_{00}, \mathrm{I}_{00})$. Then*

$$\mathrm{Ext}(\mathbb{K}) = \mathrm{Ext}(\mathbb{K}_{00}) \cup (\cup_{T \in \mathcal{P}(\{1,...,n\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i \in T} z_i))$$

where $\mathcal{H}_{\mathbb{K}_{00}}(z_i) := \{A \cap z_i' | A \in \mathrm{Ext}(\mathbb{K}_{00})$ and $A \cap z_i' \notin \mathrm{Ext}(\mathbb{K}_{00})\}$, for all $i \in \{1, ..., n\}$; and $\cap_{i \in T} z_i$ is a representative attribute such that $(\cap_{i \in T} z_i)' = \cap_{i \in T} z_i'$

Proof. We have $\mathrm{Ext}(\mathbb{K}_{00}) \subseteq \mathrm{Ext}(\mathbb{K})$.

We consider $A \in \cup_{T\in\mathcal{P}(\{1,...,n\})\backslash\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}z_i)$. Then there exists $T \in \mathcal{P}(\{1,...,n\})\backslash\emptyset$ such that $A \in \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}z_i)$. Hence, there exists $B \in \mathrm{Ext}(\mathbb{K}_{00})$ such that $A = B \cap (\cap_{i\in T}z_i^{\mathrm{I}})$. Hence, $A^{\mathrm{I}\mathrm{I}} = (B^{\mathrm{I}_{00}\mathrm{I}_{00}} \cap (\cap_{i\in T}z_i^{\mathrm{I}}))^{\mathrm{I}\mathrm{I}} = (B^{\mathrm{I}_{00}\mathrm{I}} \cap \{z_i, i \in T\}^{\mathrm{I}})^{\mathrm{I}\mathrm{I}} = (B^{\mathrm{I}_{00}} \cup \{z_i, i \in T\})^{\mathrm{I}\mathrm{I}\mathrm{I}} = (B^{\mathrm{I}_{00}} \cup \{z_i, i \in T\})^{\mathrm{I}} = B^{\mathrm{I}_{00}\mathrm{I}} \cap (\cap_{i\in T}z_i^{\mathrm{I}}) = B^{\mathrm{I}_{00}\mathrm{I}_{00}} \cap (\cap_{i\in T}z_i^{\mathrm{I}}) = B \cap (\cap_{i\in T}z_i^{\mathrm{I}}) = A$. Hence, $A \in \mathrm{Ext}(\mathbb{K})$.

We conclude that

$$\mathrm{Ext}(\mathbb{K}_{00}) \cup (\cup_{T\in\mathcal{P}(\{1,...,n\})\backslash\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}z_i)) \subseteq \mathrm{Ext}(\mathbb{K}).$$

Now let $A \in \mathrm{Ext}(\mathbb{K})$. Since $\mathrm{Ext}(\mathbb{K}_{00}) \subseteq \mathrm{Ext}(\mathbb{K})$, there are two cases: $A \in \mathrm{Ext}(\mathbb{K}_{00})$ or $A \notin \mathrm{Ext}(\mathbb{K}_{00})$.

If $A \in \mathrm{Ext}(\mathbb{K}_{00})$, then it is over.

If $A \notin \mathrm{Ext}(\mathbb{K}_{00})$, then there exists at leats an index $i \in \{1, ..., n\}$ such that $z_i \notin M_{00}$ and $A \subseteq z_i^{\mathrm{I}}$. Let $T_A = \{i \in \{1,...,n\} | z_i \notin M_{00}, A \subseteq z_i^{\mathrm{I}}\}$. Then $A = A^{\mathrm{I}\mathrm{I}} = A^{\mathrm{I}_{00}\mathrm{I}} \cap (\cap_{i\in T_A}z_i^{\mathrm{I}}) = A^{\mathrm{I}_{00}\mathrm{I}_{00}} \cap (\cap_{i\in T_A}z_i^{\mathrm{I}}) = A^{\mathrm{I}_{00}\mathrm{I}_{00}} \cap (\cap_{i\in T_A}z_i)^{\mathrm{I}}$. Hence, $A \in \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T_A}z_i)$, meaning that $A \in \cup_{T\in\mathcal{P}(\{1,...,n\})\backslash\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}z_i)$.

We conclude that

$$\mathrm{Ext}(\mathbb{K}) \subseteq \mathrm{Ext}(\mathbb{K}_{00}) \cup (\cup_{T\in\mathcal{P}(\{1,...,n\})\backslash\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}z_i)).$$

■

Corollary 2.6.1. *Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context and $\{z_1, ..., z_n\} \subsetneq M$. Setting $M_{00} = M\backslash\{z_1, ..., z_n\}$, $\mathrm{I}_{00} = \mathrm{I} \cap (G \times M_{00})$ and $\mathbb{K}_{00} = (G, M_{00}, \mathrm{I}_{00})$. Then*

$$|\,\mathrm{Ext}(\mathbb{K})| = |\,\mathrm{Ext}(\mathbb{K}_{00})| + |\cup_{T\in\mathcal{P}(\{1,...,n\})\backslash\emptyset}\,\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}z_i)|$$

Proof. It is the immediate consequence of the above theorem. ■

In order to study the variations of the size of the concept lattice after a change of the level of granularity, we consider a situation where one moves from a coarser granularity characterized by a collection of cuts $\mathcal{C}_1$ to a finer granularity characterized by a collection of cuts $\mathcal{C}_2$. Let say

$$\bigcup \mathcal{C}_1 = \{x_1, ..., x_p, X_{p+1}, ..., X_{p+t}\}$$

where x_i are finest attributes for all $i \in \{1, ..., p\}$ and X_{p+j} are coarser attributes for all $j \in \{1, ..., t\}$, with $|X_{p+j}| \geqslant 2$. Since one moves from C_1 to C_2, some of the coarser attributes X_{p+j} are refined into less coarser attributes. Without any lost of generality, we suppose that

$$\bigcup \mathcal{C}_2 = \{x_1, ..., x_p, X_{p+1}, ..., X_{p+s}, Y_{p+s+1}, ..., Y_{p+q}\}$$

where for all $i \in \{1, ..., s\}$, X_{P+j} has not being refined and for all $j \in \{s + 1, ..., q\}$ ($q \geqslant t$), the attributes Y_{p+j} are less coarser attributes obtained from a refinement of X_{p+k}, with $k \in \{s + 1, ..., t\}$. Let precise that the attributes x_r ($r \in \{1, ..., p\}$) cannot be refined.

The following result presents the conditions under which the size of the lattice can increase while passing from the coarser to the finer granularity in a formal context:

Proposition 2.6.1. *Let* $\mathbb{K} = (G, M, \mathrm{I})$ *be a formal context. Let consider a coarser granularity level characterized by the collection of cuts* $\mathcal{C}_1$ *such that*

$$\bigcup \mathcal{C}_1 = \{x_1, ..., x_p, X_{p+1}, ..., X_{p+t}\}$$

and a finer granularity level characterized by the collection of cuts $\mathcal{C}_2$ *such that*

$$\bigcup \mathcal{C}_2 = \{x_1, ..., x_p, X_{p+1}, ..., X_{p+s}, Y_{p+s+1}, ..., Y_{p+q}\}\ (q>t),$$

where $(X_{p+i})_{i\in\{s+1,...,t\}}$ *are coarser attributes of* $\bigcup \mathcal{C}_1$ *that has being refined to form the less coarser attributes* $(Y_{p+j})_{j\in\{s+1,...,q\}}$ *of* $\bigcup \mathcal{C}_2$*. Then increasing the level of granularity from* $\mathcal{C}_1$ *to* $\mathcal{C}_2$ *will increase the size of the concept lattice corresponding to the collection of cuts* $\mathcal{C}_1$ *if and only if*

$$|\cup_{T\in\mathcal{P}(\{1,...,t\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T} X_{p+j})| - |\cup_{R\in\mathcal{P}(\{1,...,s\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} X_{p+j})| - |\cup_{T\in\mathcal{P}(\{s+1,...,q\})\setminus\emptyset} \mathcal{H}\mathcal{H}_{\mathbb{K}_{01}}(\cap_{i\in T} Y_{p+j})| < 0$$

Proof. We set $\mathbb{K}_{00} = (G, (\bigcup \mathcal{C}_1)\setminus \cup_{j\in\{1,...,t\}} X_{p+j}, \mathrm{I}_{00}) = (G, M_{00}, \mathrm{I}_{00})$. Then from theorem 2.6.1,

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_1})| = |\operatorname{Ext}(\mathbb{K}_{00})| + |\cup_{T\in\mathcal{P}(\{1,...,t\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T} X_{p+j})| \quad \text{(I)}.$$

where

$$\mathcal{H}_{\mathbb{K}_{00}}(X_{p+j}) := \{A \cap X'_{p+j} | A \in \operatorname{Ext}(\mathbb{K}_{00}) \text{ and } A \cap X'_{p+j} \notin \operatorname{Ext}(\mathbb{K}_{00})\}, \text{ for all } j \in \{1, ..., t\}.$$

In the same way, setting

$$\mathbb{K}_{01} = (G, (\bigcup \mathcal{C}_2)\setminus \cup_{j\in\{s+1,...,q\}} Y_{p+j}, \mathrm{I}_{01}) = (G, M_{00} \cup \{X_{p+1}, ..., X_{p+s}\}, \mathrm{I}_{01}),$$

and also from theorem 2.6.1, we have

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| = |\operatorname{Ext}(\mathbb{K}_{01})| + |\cup_{T\in\mathcal{P}(\{s+1,...,q\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{01}}(\cap_{j\in T} Y_{p+j})|$$

where

$$\mathcal{H}_{\mathbb{K}_{01}}(Y_{p+j}) := \{A \cap Y'_{p+j}; A \in \operatorname{Ext}(\mathbb{K}_{01}) \text{ and } A \cap Y'_{p+j} \notin \operatorname{Ext}(\mathbb{K}_{01})\},$$

for all $j \in \{s+1, ..., q\}$.

Since

$$|\operatorname{Ext}(\mathbb{K}_{01})| = |\operatorname{Ext}(\mathbb{K}_{00})| + |\cup_{R\in\mathcal{P}(\{1,...,s\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} X_{p+j})|,$$

we have

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| = |\operatorname{Ext}(\mathbb{K}_{00})| + |\cup_{R\in\mathcal{P}(\{1,...,s\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} X_{p+j})| + |\cup_{T\in\mathcal{P}(\{s+1,...,q\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{01}}(\cap_{i\in T} Y_{p+j})| \quad \text{(II)}$$

From (I) and (II), we get the following equality:

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_1})| - |\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| = |\operatorname{Ext}(\mathbb{K}_{00})| + |\cup_{T\in\mathcal{P}(\{1,...,t\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T} X_{p+j})| - |\operatorname{Ext}(\mathbb{K}_{00})| - |\cup_{R\in\mathcal{P}(\{1,...,s\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} X_{p+j})| - |\cup_{T\in\mathcal{P}(\{s+1,...,q\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{01}}(\cap_{i\in T} Y_{p+j})|,$$

which leads to

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_1})| - |\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| = |\cup_{T\in\mathcal{P}(\{1,\dots,t\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}X_{p+j})| - |\cup_{R\in\mathcal{P}(\{1,\dots,s\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R}X_{p+j})| - |\cup_{T\in\mathcal{P}(\{s+1,\dots,q\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{01}}(\cap_{i\in T}Y_{p+j})|,$$

Hence, we conclude that increasing the level of granularity from $\mathcal{C}_1$ to $\mathcal{C}_2$ will increase the size of the concept lattice corresponding to the collection of cuts $\mathcal{C}_1$ if and only if

$$|\cup_{T\in\mathcal{P}(\{1,\dots,t\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in T}X_{p+j})| - |\cup_{R\in\mathcal{P}(\{1,\dots,s\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R}X_{p+j})| - |\cup_{T\in\mathcal{P}(\{s+1,\dots,q\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{01}}(\cap_{i\in T}Y_{p+j})| < 0$$

■

Example 2.6.3. *Let consider the following formal context $\mathbb{K} = (G, M, \mathrm{I})$ where $G = \{1, 2, 3, 4, 5\}$ and $M = \{a, b, c, d\}$.*

$\mathbb{K}$	a	b	c	d
1	×			×
2	×		×	×
3		×	×	×
4	×	×		
5			×	×

We consider the following collection of cuts $\mathcal{C}_1$ such that $\bigcup\mathcal{C}_1 = \{b, c, a_1, a_2, d_1, d_2\}$, where the attributes a, d of $\mathbb{K}$ are split to respectively form the finer attributes a_1, a_2 and d_1, d_2, and which leads to the following formal context $\mathbb{K}^{\mathcal{C}_1}$

$\mathbb{K}^{\mathcal{C}_1}$	b	c	a_1	a_2	d_1	d_2
1			×		×	
2		×	×		×	
3	×	×				×
4	×			×		
5						×

Note that in this collection of cuts, $\{x_1, ..., x_p\} = \{b, c\}$, and

$$\{X_{p+1}, ..., X_{p+t}\} = \{a_1, a_2, d_1, d_2\}.$$

We consider the following collection of cuts $\mathcal{C}_2$ such that

$$\bigcup\mathcal{C}_2 = \{b, c, a_1, a_2, d_{11}, d_{12}, d_{21}, d_{22}\},$$

where the attributes $\{d_1\}$ and $\{d_2\}$ of $\mathbb{K}^{\mathcal{C}_1}$ are split to respectively form the finer attributes d_{11}, d_{12} and d_{21}, d_{22}, and which leads to the following formal context $\mathbb{K}^{\mathcal{C}_2}$.

$\mathbb{K}^{\mathcal{C}_2}$	b	c	a_1	a_2	d_{11}	d_{12}	d_{21}	d_{22}
1			×		×			
2		×	×			×		
3	×	×					×	
4	×			×				
5								×

Then, $\text{Ext}(\mathbb{K}_{00}) = \text{Ext}(G, \{b, c\}, \text{I}_{00}) = \{\{3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4, 5\}\}$ and then $|\text{Ext}(\mathbb{K}_{00})| = 4$.

Moreover $a_1' = d_1' = \{1, 2\}$, $a_2' = \{4\}$, $d_2' = \{3, 5\}$, $a_1' \cap d_1' = \{1, 2\}$, and for all $X \in \mathcal{P}(a_1, a_2, d_1, d_2) \backslash \{\emptyset, \{a_1\}, \{a_2\}, \{d_1\}, \{d_2\}, \{a_1, d_1\}\}$, $X^{\text{I}} = \emptyset$.

Then,

$$\cup_{T \in \mathcal{P}(\{1,\dots,t\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j \in T} X_{p+j}) = \mathcal{H}_{\mathbb{K}_{00}}(a_1) \cup \mathcal{H}_{\mathbb{K}_{00}}(a_2) \cup \mathcal{H}_{\mathbb{K}_{00}}(d_1) \cup \mathcal{H}_{\mathbb{K}_{00}}(d_2) \cup \mathcal{H}_{\mathbb{K}_{00}}(a_1 \cap d_1) = \{\emptyset, \{2\}, \{1, 2\}, \{4\}, \{3, 5\}\}$$

and then

$$|\cup_{T \in \mathcal{P}(\{1,\dots,t\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j \in T} X_{p+j})| = 5.$$

Note that in the collection of cuts $\mathcal{C}_1$, $\{X_{p+1}, \dots, X_{p+s}\} = \{a_1, a_2\}$ and $\{X_{p+s+1}, \dots, X_{p+q}\} = \{d_{11}, d_{11}, d_{21}, d_{22}\}$, $d_{11}' = \{1\}$, $d_{12}' = \{2\}$, $d_{21}' = \{3\}$, $d_{22}' = \{5\}$, and for all $X \in \mathcal{P}(d_{11}, d_{12}, d_{21}, d_{22}) \backslash \{\emptyset, \{d_{11}\}, \{d_{12}\}, \{d_{21}\}, \{d_{22}\}\}$, $X^{\text{I}} = \emptyset$.

Hence, $\cup_{T \in \mathcal{P}(\{1,\dots,s\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j \in T} X_{p+j}) = \mathcal{H}_{\mathbb{K}_{00}}(a_1) \cup \mathcal{H}_{\mathbb{K}_{00}}(a_2) \cup \mathcal{H}_{\mathbb{K}_{00}}(a_1 \cap a_2) = \{\emptyset, \{2\}, \{1, 2\}\} \cup \{\emptyset, \{4\}\} \cup \{\emptyset\} = \{\emptyset, \{2\}, \{1, 2\}, \{4\}\}$. Therefore

$$|\cup_{T \in \mathcal{P}(\{1,\dots,s\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j \in T} X_{p+j})| = 4.$$

But

$$\text{Ext}(\mathbb{K}_{01}) = \text{Ext}(\mathbb{K}_{00}) \cup (\cup_{T \in \mathcal{P}(\{1,\dots,s\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i \in T} X_{p+j})) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4, 5\}\}.$$

Then,

$$\cup_{T \in \mathcal{P}(\{s+1,\dots,q\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{01}}(\cap_{i \in T} Y_{p+j}) = \mathcal{H}_{\mathbb{K}_{01}}(d_{11}) \cup \mathcal{H}_{\mathbb{K}_{01}}(d_{12}) \cup \mathcal{H}_{\mathbb{K}_{01}}(d_{21}) \cup \mathcal{H}_{\mathbb{K}_{01}}(d_{22}) = \{\{1\}, \{5\}\}.$$

Hence,

$$|\cup_{T \in \mathcal{P}(\{s+1,\dots,q\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{01}}(\cap_{i \in T} Y_{p+j})| = 2.$$

It comes that:

$$|\cup_{T \in \mathcal{P}(\{1,\dots,t\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i \in T} X_{p+j})| - |\cup_{T \in \mathcal{P}(\{1,\dots,s\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i \in T} X_{p+j})| - |\cup_{T \in \mathcal{P}(\{s+1,\dots,q\}) \backslash \emptyset} \mathcal{H}_{\mathbb{K}_{01}}(\cap_{i \in T} Y_{p+j})| = 5 - 4 - 2 = -1.$$

Therefore, passing from the lower level of granularity characterized by the collection of cuts $\mathcal{C}_1$ to the higher level of granularity characterized by the collection of cuts $\mathcal{C}_2$ will reduce the size of the concept lattice $\mathfrak{B}(\mathbb{K}^{\mathcal{C}_1})$ by 1.

Also, we note that $|\mathfrak{B}(\mathbb{K}^{\mathcal{C}_2})| - |\mathfrak{B}(\mathbb{K}^{\mathcal{C}_1})| = 10 - 9 = 1$.

Remark 2.6.1. *They could be some cases where the collection of cuts with smaller level of granularity is not necessarily a refinement of the collection of cuts with the higher level of granularity. For instance, from a collection of cuts $\mathcal{C}_0$ such that*

$$\bigcup \mathcal{C}_0 = \{x_1, ..., x_p, X_{p+1}, ..., X_{p+s}, X_{p+s+1}, ..., X_{p+t}\}$$

(with $s = t$), one can decide to only refined attributes $X_{p+1}, ..., X_{p+s}$ into less coarser attributes $Y_{q_1}, ..., Y_{q_r}$, leading to the collection of cuts $\mathcal{C}_1$ such that

$$\bigcup \mathcal{C}_1 = \{x_1, ..., x_p, X_{p+s+1}, ..., X_{p+t}, Y_{q_1}, ..., Y_{q_r}\}$$

One can also decide to only refine the attributes $X_{p+s+1}, ..., X_{p+t}$ into less coarser attributes $Z_{s_1}, ..., Z_{s_v}$, leading to the collection of cuts $\mathcal{C}_2$ such that

$$\bigcup \mathcal{C}_2 = \{x_1, ..., x_p, X_{p+1}, ..., X_{p+s}, Z_{s_1}, ..., Z_{s_v}\}$$

such that $|\bigcup \mathcal{C}_2| \geqslant |\bigcup \mathcal{C}_1|$.

Then the collection of cuts $\mathcal{C}_2$ has a higher level of granularity than the collection of cuts $\mathcal{C}_1$.

Hence, moving from $\mathcal{C}_1$ to $\mathcal{C}_2$, and setting

$$\mathbb{K}_{01} = (G, (\bigcup \mathcal{C}_0)\backslash\{X_{p+1}, ..., X_{p+s}\}, \mathrm{I}_{01})$$

and

$$\mathbb{K}_{02} = (G, (\bigcup \mathcal{C}_0)\backslash\{X_{p+s+1}, ..., X_{p+t}\}, \mathrm{I}_{02})$$

We have

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_1})| = |\operatorname{Ext}(\mathbb{K}_{01})| + |\cup_{R\in\mathcal{P}(\{1,...,r\})\backslash\emptyset}\, \mathcal{H}_{\mathbb{K}_{01}}(\cap_{j\in R} Y_{q_j})|,$$

and

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| = |\operatorname{Ext}(\mathbb{K}_{02})| + |\cup_{T\in\mathcal{P}(\{1,...,v\})\backslash\emptyset}\, \mathcal{H}_{\mathbb{K}_{02}}(\cap_{j\in T} Z_{s_j})|.$$

Hence $|\operatorname{Ext}(\mathbb{K}_{\mathcal{C}_2})| - |\operatorname{Ext}(\mathbb{K}_{\mathcal{C}_1})| = (|\operatorname{Ext}(\mathbb{K}_{02})| + |\cup_{T\in\mathcal{P}(\{1,...,v\})\backslash\emptyset}\, \mathcal{H}_{\mathbb{K}_{02}}(\cap_{j\in T} Z_{s_j})|) - (|\operatorname{Ext}(\mathbb{K}_{01})| + |\cup_{R\in\mathcal{P}(\{1,...,r\})\backslash\emptyset}\, \mathcal{H}_{\mathbb{K}_{01}}(\cap_{j\in R} Y_{q_j})|)$, *which means that*

$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| > |\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_1})|$ *iff* $|\operatorname{Ext}(\mathbb{K}_{02})| - |\operatorname{Ext}(\mathbb{K}_{01})| > |\cup_{R\in\mathcal{P}(\{1,...,r\})\backslash\emptyset} \mathcal{H}_{\mathbb{K}_{01}}(\cap_{j\in R} Y_{q_j})| - |\cup_{T\in\mathcal{P}(\{1,...,v\})\backslash\emptyset}\, \mathcal{H}_{\mathbb{K}_{02}}(\cap_{j\in T} Z_{s_j})|$.

Now we consider the case in a formal context $\mathbb{K} = (G, M, \mathrm{I})$ where one moves from a finer granularity characterized by a collection of cuts $\mathcal{C}_1$ such that

$$\bigcup \mathcal{C}_1 = \{x_{i_1}, ..., x_{i_l}, x_{i_{l+1}}, ..., x_{i_{l+q}}, X_{j_1}, ..., X_{j_m}, X_{j_{m+1}}, ..., X_{j_{m+n}}\}$$

to a coarser granularity characterized by a collection of cuts $\mathcal{C}_2$ such that

$$\bigcup \mathcal{C}_2 = \{x_{i_1}, ..., x_{i_l}, X_{j_1}, ..., X_{j_m}, Y_{k_1}(x), ..., Y_{k_p}(x), Y_{e_1}(x, X), ..., Y_{e_s}(x, X), \\ \mathrm{Y}_{h_1}(X), ..., Y_{h_v}(X)\}$$

where the $Y_{k_j}(x)$ $(1 \leqslant i \leqslant p)$ are coarser attributes constituted from the attributes x_{ik} $(l+1 \leqslant k \leqslant l+q)$, $Y_{e_j}(x,X)$ $(1 \leqslant i \leqslant s)$ are coarser attributes constituted from the attributes x_{ik} $(l+1 \leqslant k \leqslant l+q)$ and X_{jk} $(m+1 \leqslant k \leqslant m+n)$ and the $Y_{k_j}(X)$ $(1 \leqslant j \leqslant v)$ are the coarser attributes constituted from the attributes X_{jk} $(m+1 \leqslant k \leqslant m+n)$. This case is equivalent to an ∃-generalization, and the generalized attributes are precisely the coarser attributes.

Let precise that the X_{jk} $(1 \leqslant k \leqslant m+n)$ are the coarser attributes from $\bigcup \mathcal{C}_1$, and the attributes $x_{i_1}, ..., x_{i_q}$ and the coarser attributes $X_{j_1}, ..., X_{j_m}$ from $\bigcup \mathcal{C}_1$ are not involved in the coarsening.

The following proposition holds:

Proposition 2.6.2. *Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context. Let consider a finer granularity level characterized by a collection of cuts $\mathcal{C}_1$ such that*

$$\bigcup \mathcal{C}_1 = \{x_{i_1}, ..., x_{i_l}, x_{i_{l+1}}, ..., x_{i_{l+q}}, X_{j_1}, ..., X_{j_m}, X_{j_{m+1}}, ..., X_{j_{m+n}}\}$$

and a coarser granularity level characterized by a collection of cuts $\mathcal{C}_2$ such that
$\bigcup \mathcal{C}_2 = \{x_{i_1}, ..., x_{i_l}, X_{j_1}, ..., X_{j_m}, Y_{k_1}(x), ..., Y_{k_p}(x), Y_{e_1}(x,X), ..., Y_{e_s}(x,X)$
$Y_{h_1}(X), ..., Y_{h_v}(X)\}$
$(q+n > p+s+v)$, where $(x_{j_{l+i}})_{i\in\{1,...,l+q\}}$ and $(X_{j_{m+i}})_{i\in\{m+1,...,m+n\}}$ are attributes of $\bigcup \mathcal{C}_1$ that has being put together to form the coarser attributes $(Y_{k_j})_{j\in\{1,...,p+s+v\}}$ of $\bigcup \mathcal{C}_2$. Then decreasing the level of granularity from $\mathcal{C}_1$ to $\mathcal{C}_2$ will increase the size of the concepts lattice corresponding to the collection of cuts C_1 if and only if

$$|\cup_{R\in\mathcal{P}(\{1,...,p+s+v\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R} Q_i)| - |\cup_{R\in\mathcal{P}(\{1,...,q+n\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} Z_j)| > 0$$

where $Q_i = Y_{k_i}(x)$ if $(1 \leqslant i \leqslant p)$, $Q_i = Y_{e_i}(x,X)$ if $(1 \leqslant i \leqslant s)$, and $Q_i = Y_{h_i}(X)$ if $(1 \leqslant i \leqslant v)$.

Proof. We set

$$\mathbb{K}_{00} = (G, (\bigcup \mathcal{C}_1)\setminus\{x_{i_{l+1}}, ..., x_{i_{l+q}}, X_{j_{m+1}}, ..., X_{j_{m+n}}\}, \mathrm{I}_{00})$$

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_1})| = |\operatorname{Ext}(\mathbb{K}_{00})| + |\cup_{R\in\mathcal{P}(\{1,...,q+n\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} Z_j)|,$$

where $Z_j = x_j$ if $(j_{l+1} \leqslant j \leqslant j_{l+q})$ and $Z_j = X_j$ if $(j_{m+1} \leqslant j \leqslant j_{m+n})$.

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| = |\operatorname{Ext}(\mathbb{K}_{00})| + |\cup_{R\in\mathcal{P}(\{1,...,p+s+v\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R} Q_i)|,$$

where $Q_i = Y_{k_i}(x)$ if $(1 \leqslant i \leqslant p)$, $Q_i = Y_{e_i}(x,X)$ if $(1 \leqslant i \leqslant s)$, and $Q_i = Y_{h_i}(X)$ if $(1 \leqslant i \leqslant v)$.

Hence,

$$|\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_2})| - |\operatorname{Ext}(\mathbb{K}^{\mathcal{C}_1})| = |\operatorname{Ext}(\mathbb{K}_{00})| + |\cup_{R\in\mathcal{P}(\{1,...,p+s+v\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R} Q_i)| - \\ |\operatorname{Ext}(\mathbb{K}_{00})| - |\cup_{R\in\mathcal{P}(\{1,...,q+n\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} Z_j)| = |\cup_{R\in\mathcal{P}(\{1,...,p+s+v\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R} Q_i)| - \\ |\cup_{R\in\mathcal{P}(\{1,...,q+n\})\setminus\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R} Z_j)|.$$

We conclude that decreasing the level of granularity from $\mathcal{C}_1$ to $\mathcal{C}_2$ will increase the size of the concept lattice corresponding to the collection of cuts $\mathcal{C}_1$ if and only if

$$|\cup_{R\in\mathcal{P}(\{1,...,p+s+v\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R}Q_i)| - |\cup_{R\in\mathcal{P}(\{1,...,q+n\})\setminus\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R}Z_j)| > 0$$

■

Example 2.6.4.

Let consider the following formal context $\mathbb{K} = (G, M, \mathrm{I})$ where $G = \{1, 2, 3, 4, 5\}$ and $M = \{a, b, c, d\}$.

$\mathbb{K}$	a	b	c	d
1	×			×
2	×		×	×
3		×	×	×
4	×	×		
5			×	×

We consider the following collection of cuts $\mathcal{C}_1$ such that

$$\bigcup \mathcal{C}_1 = \{b, c, a_1, a_2, d_{11}, d_{12}, d_{21}, d_{22}\},$$

and which leads to the following formal context $\mathbb{K}^{\mathcal{C}_1}$.

$\mathbb{K}^{\mathcal{C}_1}$	b	c	a_1	a_2	d_{11}	d_{12}	d_{21}	d_{22}
1			×		×			
2		×	×			×		
3	×	×					×	
4	×			×				
5								×

Note that in the collection of cuts $\mathcal{C}_1$, $\{x_{i1}, ..., x_{il}\} = \{b, c\}$, $\{x_{il+1}, ..., x_{il+q}\} = \emptyset$ and $\{X_{j_1}, ..., X_{j_m}\} = \{a_1, a_2\}$, $\{X_{j_{m+1}}, ..., X_{j_{m+n}}\} = \{d_{11}, d_{12}, d_{21}\}$, $\{d_{22}\}\}$.

The group of attributes d_{11}, d_{12} and d_{21}, d_{22} are coarsed to respectively form the coarser attributes d_1 and d_2, which are presented in $\bigcup \mathcal{C}_2 = \{b, c, a_1, a_2, d_1, d_2\}$, leading to the formal context $\mathbb{K}^{\mathcal{C}_2}$ below.

$\mathbb{K}^{\mathcal{C}_2}$	b	c	a_1	a_2	d_1	d_2
1			×		×	
2		×	×		×	
3	×	×				×
4	×			×		
5						×

Note that in this collection of cuts, $\{x_1, ..., x_p\} = \{b, c\}$, $\{Y_{k_1}(x), ..., Y_{k_p}(x)\} = \emptyset$, $\{Y_{e_1}(x, X), ...,$
$Y_{e_s}(x, X)\} = \emptyset$ and $\{Y_{h_1}(X), ..., Y_{h_v}(X)\} = \{d_1, d_2\}$.

Then, $\text{Ext}(\mathbb{K}_{00}) = \text{Ext}(G, \{b, c, a_1, a_2\}, \text{I}_{00}) = \{\emptyset, \{2\}, \{3\}, \{4\}\{2, 3\}, \{3, 4\},$
$\{1, 2\}\{1, 2, 3, 4, 5\}\}$ and then $|\text{Ext}(\mathbb{K}_{00})| = 8$.

Moreover $d_1' = \{1, 2\}$, $d_2' = \{3, 5\}$ and then

$$\cup_{R\in\mathcal{P}(\{1,...,v\})\backslash\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R}Q_i) = \mathcal{H}_{\mathbb{K}_{00}}(d_1) \cup \mathcal{H}_{\mathbb{K}_{00}}(d_2) \cup \mathcal{H}_{\mathbb{K}_{00}}(d_1 \cap d_2) = \{\{3, 5\}\}$$

and then

$$|\cup_{R\in\mathcal{P}(\{1,...,v\})\backslash\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R}Q_i)| = 1.$$

Note that in the collection of cuts $\mathcal{C}_1$,

$$\{x_{i1}, ..., x_{il}\} = \{b, c\}, \{x_{il+1}, ..., x_{il+q}\} = \emptyset, \{X_{j_1}, ..., X_{j_m}\} = \{a_1, a_2\},$$

and

$$\{X_{j_{m+1}}, ..., X_{j_{m+n}}\} = \{d_{11}, d_{12}, d_{21}, d_{22}\}.\ d_{11}' = \{1\},\ d_{12}' = \{2\},\ d_{21}' = \{3\},\ d_{22}' = \{5\},$$

and for all $X \in \mathcal{P}(\{d_{11}, d_{12}, d_{21}, d_{22}\})\backslash\{\emptyset, \{d_{11}\}, \{d_{12}\}, \{d_{21}\}, \{d_{22}\}\}$, $X^{\text{I}} = \emptyset$.

Hence,

$$\cup_{R\in\mathcal{P}(\{1,...,s\})\backslash\emptyset}\mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R}Z_j) = \mathcal{H}_{\mathbb{K}_{00}}(d_{11}) \cup \mathcal{H}_{\mathbb{K}_{00}}(d_{12}) \cup \mathcal{H}_{\mathbb{K}_{00}}(d_{21}) \cup \mathcal{H}_{\mathbb{K}_{00}}(d_{22}) = \{\{1\}, \{5\}\},$$

meaning that,

$$|\cup_{R\in\mathcal{P}(\{1,...,s\})\backslash\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R}Z_j)| = 2.$$

Therefore

$$|\cup_{R\in\mathcal{P}(\{1,...,v\})\backslash\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{i\in R}Q_i)| - |\cup_{R\in\mathcal{P}(\{1,...,s\})\backslash\emptyset} \mathcal{H}_{\mathbb{K}_{00}}(\cap_{j\in R}Z_j)| = 1 - 2 = -1.$$

Therefore, passing from the higher level of granularity characterized by the collection of cuts C_1 to the lower level of granularity characterized by the collection of cuts C_2 will decrease the size of the concept lattice $\mathfrak{B}(\mathbb{K}^{\mathcal{C}_1})$ by 1.

Remark 2.6.2. *Note that* $|\mathfrak{B}(\mathbb{K}^{\mathcal{C}_1})| = 10$ *and* $|\mathfrak{B}(\mathbb{K}^{\mathcal{C}_2})| = 9$.

2.6.2 The ∃-generalization in relation with factorisation of formal context

The ∃-generalization also has a link with factorisation of formal contexts. The main question to be answered here is under which conditions the attributes of a formal context (G, M, I) can be obtained from those of the other formal context (G, N, J) as disjunctions in the following sense: for all $m \in M$, there is a subset $S_m \subseteq N$ such that

$$(g, m) \in \mathrm{I} \Leftrightarrow (g, n) \in \mathrm{J} \text{ for some } n \in S_m,$$

or equivalently

$$m^{\mathrm{I}} = \cup_{n \in S_m} n^J$$

Note that this also allows 1-element disjunction, so that some elements of M can simply be copied to N. This problem was first set by **Ganter**.

Remark 2.6.3. *Its implies that the attributes of* (G, M, I) *are generalized attributes resulting from the ∃-generalization of groups of attributes of* (G, N, J).

Definition 2.6.5. *Let* (G, M, I) *a formal context. A formal context* (N, M, K) *is a left factorization in the boolean factorisation of* (G, M, I) *if there exists a formal context* (G, N, J) *such that for every object* g *of* G *and every attribute* m *of* M, $(g, m) \in \mathrm{I}$ *iff there is* $t \in N$*;* $(g, t) \in \mathrm{I}$ *and* $(t, m) \in \mathrm{I}$.

The following result initiated by **Ganter** and **Kwuida** holds:

Theorem 2.6.2. *Let* (G, M, I) *be a formal context.* (G, M, I) *can be obtained from* (G, N, J) *via attribute disjunction if and only if* (G, N, J) *is a left factorisation context in a boolean factorisation of* (G, M, I).

Proof. We consider two formal contexts $\mathbb{K}_M = (G, M, \mathrm{I})$ and $\mathbb{K}_N = (G, N, \mathrm{J})$. If (G, M, I) can be obtained from (G, N, J) by a left factorisation, that is there exists a formal context (N, M, K) such that $(G, M, \mathrm{I}) = (G, N, \mathrm{J}) \circ (N, M, K)$. Let $g \in G$ and $m \in M$ such that $(g, m) \in \mathrm{I}$. Then there is $n \in N$ such that $(g, n) \in \mathrm{J}$ and $(n, m) \in K$. We set $S_m = \{n \in N; (g, n) \in \mathrm{J}, (n, m) \in K\}$. If $g \in \cup_{n \in S_m} n^{\mathrm{J}}$, then there exists $n_0 \in S_m$ such that $g \in n_0^{\mathrm{J}}$. Since $n_0 \in S_m$, $(g, n_0) \in \mathrm{J}, (n_0, m) \in K$, which leads to the fact that $(g, m) \in \mathrm{I}$, and then $g \in m^{\mathrm{I}}$. In the same way, $g \in m^{\mathrm{I}}$ implies that $g \in \cup_{n \in S_m} n^{\mathrm{J}}$. Now, we suppose that the context (G, M, I) can be obtained from (G, N, J) by disjunction. Then for all $m \in M$, there exists $S_m \subseteq N$ such that $m^{\mathrm{I}} = \cup_{n \in S_m} n^J$. We then consider the binary relation K defined by: for all $n \in N$ and $m \in M$, $(n, m) \in K$ if and only if $n \in S_m$. Then we obtain a formal context (N, M, K). If $(g, m) \in \mathrm{I}$, then $g \in m^{\mathrm{I}}$, and there is $n \in S_m$ such that $g \in n^{\mathrm{J}}$. Hence, $(g, n) \in \mathrm{J}$ and $(n, m) \in K$, and then $(g, m) \in (G, N, \mathrm{J}) \circ (N, M, K)$. Conversely, if $(g, m) \in (G, N, \mathrm{J}) \circ (N, M, K)$, then there exists $n \in N$ such that $(g, n) \in \mathrm{J}$ and $(n, m) \in K$, which means that $g \in \cup_{n \in m^K} n^{\mathrm{J}}$, and then that $m^{\mathrm{I}} = \cup_{n \in m^K} n^{\mathrm{J}}$. ■

The following result relating ∃-generalization and factorisation is an immediate consequence of the above theorem.

Corollary 2.6.2. *Let* (G, M, I) *be a formal context and* $\{a, b\}$ *a pair of attributes of* M*,* s *the generalized attribute resulting from the ∃-generalization of* a *and* b *and* $(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab})$ *the generalized context. Then*

$$(G, M_{ab} \cup \{s\}, \mathrm{I}^s_{ab}) = (G, M, \mathrm{I}) \circ (M, M_{ab} \cup \{s\}, K)$$

where $(m, m_0) \in K$ *iff* $m^{\mathrm{I}} \subseteq m_0^{\mathrm{I}^s_{ab}}$.

Example 2.6.5.

$\mathbb{K}^1$	a	b	c	d
1	×			×
2	×		×	×
3			×	
4		×		×

$\mathbb{K}^2$	m_1	m_2	m_3	m_4	m_5	m_6
1	×					×
2		×		×		×
3					×	
4			×			×

Table 2.6: $\mathbb{K}^1 = (G, M, \mathrm{I})$ (left) and $\mathbb{K}^2 = (G, N, \mathrm{J})$ (right)

With $S_a = \{m_1, m_2\}$, $S_b = \{m_3\}$, $S_c = \{m_4, m_5\}$ and $S_d = \{m_6\}$, we define the binary relation K as $(n, m) \in K$ iff $n \in S_m$, and the context (N, M, K) is the following:

$\mathbb{K}^3$	a	b	c	d
m_1	×			
m_2	×			
m_3		×		
m_4			×	
m_5			×	
m_6				×

Hence, it is easy to see that $\mathbb{K}^3 \circ \mathbb{K}^2 = \mathbb{K}^1$.

Remark 2.6.4. *The notion of factorization is important because it can permit to determine the generalized context after an ∃-generalization on attributes in a formal context* (G, M, I)*. The generalized context can be obtained by constructing the formal context* (N, M, K) *and by multiplying it by* (G, M, I)*.*

2.6.3 The ∃-generalization in relation with patterns structures

The more general forms of formal contexts are patterns structures. Since one can define generalized pattern for formal contexts, the question to know whether an ∃-generalization can also be defined on pattern structures is straightforward. Before looking for a way of giving an answer to this question, we present some definitions.

Let G be a set.

Definition 2.6.6. *[17] A pattern structure is a triple $(G, \underline{D}, \delta)$ with $\underline{D} = (D, \sqcap)$ a meet-semilattice, $\delta : G \to D$ a mapping and $D_\delta = \delta(G) = \{\delta(g) | g \in G\}$ a complete subsemilattice $(D_\delta, \sqcap)$ of $(D, \sqcap)$.*

Remark 2.6.5. *[17] If $(G, \underline{D}, \delta)$ is a patterns structure, one define the derivation operators $(.)^\square$ as*

$$A^\square := \sqcap_{g \in A} \delta(g) \text{ for } A \in \mathcal{P}(G),$$

and

$$d^\square := \{g \in G | d \sqsubseteq \delta(g)\}, \text{ for } d \in D.$$

Remark 2.6.6. - *The elements of D are called **patterns**, and the order on them is given by $c \sqsubseteq d :\leftrightarrow c \sqcap d = c$,*

- *The operators $(.)^\square$ make a Galois connection between the power set of G and $(D, \sqsubseteq)$,*

- *The pairs (A, d) satisfying $A^\square = d$ and $A = d^\square$ with $A \subseteq G$ and $d \in D$, are called **patterns concepts**, with extent A and **patterns intents** d.*

Let $(G, \underline{D}, \delta)$ with $\underline{D} = (D, \sqcap)$ be a patterns structure. In [17], the following relation $\mathrm{I}_\sqsubseteq$ as being defined on $G \times D$ as follow:

$$(g, d) \in \mathrm{I}_\sqsubseteq \text{ if and only if } d \sqsubseteq \delta(g).$$

Definition 2.6.7. *[17] Let $(G, \underline{D}, \delta)$ be a patterns structure. Then the formal context $(G, D, \mathrm{I}_\sqsubseteq)$ is called a **representation** context of $(G, \underline{D}, \delta)$.*

From the patterns structure $(G, \underline{D}, \delta)$, we get the formal context $(G, D, \mathrm{I}_\sqsubseteq)$.

Definition 2.6.8. *[17] Let D be a set, $\psi : D \longrightarrow D$ be a map and $\sqsubseteq$ be a partial order. Then ψ is a projection if it is contractive ($\psi(x) \sqsubseteq x$ for every $x \in D$), monotone and idempotent.*

In some cases, some patterns in a given patterns structure are too difficult to be handled. To solve the problem, expert are often tempted to replace these patterns by other patterns more simple than the previous ones, even if this replacement can lead to some lost of information. That is done by means of projections $\psi : D \longrightarrow D$ such that the pattern structure $(G, \underline{D}, \delta)$ is replaced by $(G, \underline{D}, \psi \circ \delta)$.

Our aim is to show that this situation, with some conditions, is equivalent to an ∃-generalization. For every pattern $d \in D$, we set $\downarrow d = \{t \in D | t \sqsubseteq d\}$.

The following proposition holds:

Proposition 2.6.3. *Let $(G, \underline{D}, \delta)$ be a patterns structure, ψ be a projection on D which leads to $(G, \underline{D}, \psi \circ \delta)$ and $d \in D$. If $\psi^{-1}(d) \cap (\downarrow d) \neq \emptyset$, then d is the ∃-generalized attribute of all the attributes of $\psi^{-1}(d)$ in the formal context $(G, D, \mathrm{I}_{\sqsubseteq})$.*

Proof. Let $(G, \underline{D}, \delta)$ be a patterns structure and $d \in D$ be a pattern. Since ψ is a projection, $\psi^{-1}(d) \neq \emptyset$. We set $X = \psi^{-1}(d)$. Then $\psi(x) = d$ for every $x \in X$. Let $(G, D, \mathrm{I}_{\sqsubseteq})$ be the representation formal context associated to $(G, \underline{D}, \delta)$. Then for every $x \in X$ and $g \in G$, if $(g, x) \in \mathrm{I}_{\sqsubseteq}$, then $x \sqsubseteq \delta(g)$, and then $\psi(x) \sqsubseteq \psi(\delta(g)) \sqsubseteq \delta(g)$, because ψ is contractive. Hence, $x^{\mathrm{I}_{\sqsubseteq}} \subseteq (\psi(x))^{\mathrm{I}_{\sqsubseteq}}$. Since that is true for every $x \in X$, we conclude that $\cup_{x \in X} x^{\mathrm{I}_{\sqsubseteq}} \subseteq (\psi(x))^{\mathrm{I}_{\sqsubseteq}} = d^{\mathrm{I}_{\sqsubseteq}}$. Conversely, we suppose that $g \in d^{\mathrm{I}_{\sqsubseteq}}$. Let $x_0 \in (\downarrow d) \cap X$. Then $x_0 \sqsubseteq d = \psi(x_0)$. Hence, $x_0 \sqsubseteq \psi(x_0) = d \sqsubseteq \delta(g)$, meaning that $x_0 \sqsubseteq \delta(g)$. Therefore, $g \in x_0^{\mathrm{I}_{\sqsubseteq}} \subseteq \cup_{x \in X} x^{\mathrm{I}_{\sqsubseteq}}$. It comes that

$$d^{\mathrm{I}_{\sqsubseteq}} = \cup_{x \in X} x^{\mathrm{I}_{\sqsubseteq}}.$$

■

However the ∃-generalization here never leads to an increase of the size of the representation concept lattice because the generalized pattern $\psi(x)$ is also a pattern of D.

2.7 Conclusion

In this chapter, we have shown a family of formal contexts in which the existential generalization on a specific pair of attributes increases the size of the lattice exponentially. We have also found the maximal increase when two attributes are put together, and we have presented a case of stability of size of the lattice after an ∃-generalization. Our next direction of interest is to look at similarity measures that discriminate attributes if putting these together increases the number of concepts.

Chapter 3

A similarity measure for generalization

3.1 Introduction

In the previous chapter, we have studied the variation of the size of the concept lattice after an existential generalization on attributes. Haven not been able to characterize the formal contexts whose concept lattice decreases after an $\exists$-generalization, we therefore asked if there exists a formal way of putting attributes together and be assure that the size of the lattice do not increase. We suspect that putting together incompatible attributes would probably increase the size of the concept lattice more than compatible ones. Therefore it is necessary to consider and study the similarity measures on attributes.

From there comes the question: **Is there a similarity measure (possibly cheap and fast to compute), which is compatible with the changing of size of a concept lattice after an existential generalization?** i.e. if m_1, m_2 are more similar than m_3, m_4, then putting m_1, m_2 together should not lead to more concepts as putting m_3, m_4 together. This chapter is an attempt to answer this question. We first define the notion of similarity measure and present some existing similarity measures (the most used and the most known ones). Further, we test these similarity measures on attributes to be generalized and prove their incompatibility. Finally, we propose a new similarity measure compatible with $\exists$-generalization in a reduced formal context.

3.2 Some existing similarity measures

Some study has already been done on similarity measures. In [2, 13], the authors discuss similarity measures on concepts, and even on lattices. For our purpose, we need a measure of similarity on attributes such that if m_1, m_2 are more similar than m_3, m_4, then generalizing m_1, m_2 should not lead to more concepts as generalizing m_3, m_4. We say that such a similarity measure is **compatible with the generalization**. Before we move further, let recall some definitions:

Definition 3.2.1. *[2] Given a set M of attributes, a* ***similarity measure*** *on M is a function $S : M \times M \to \mathbb{R}$ such that for all m_1, m_2 in M,* *(i)* $S(m_1, m_2) \geqslant 0$, *positivity*

(ii) $S(m_1, m_2) = S(m_2, m_1)$ *symmetry*

(iii) $S(m_1, m_1) \geqslant S(m_1, m_2)$ *maximality*

Remark 3.2.1. *[2] If in addition,* $S(m_1, m_2) = S(m_1, m_1) \iff m_1 = m_2$ *and* $S(m_1, m_2)S(m_2, m_3) \leqslant (S(m_1, m_2) + S(m_2, m_3))S(m_1, m_3)$ $\forall m_1, m_2, m_3 \in M$*, then* S *is called a* ***metric similarity measure****.*

Remark 3.2.2. *The hypothesis of* ***symmetry*** *in the above definition has being contested by some authors, notably in [41]. However, the above definition remains the most considered and the most used by philosophers, physiologists, and scientists.*

Hence, similarity measures aim at quantifying to which extent two attributes resemble each other. They are normalized when their values are between 0 and 1. Getting a similarity measure compatible with $\exists$-generalization will be a valuable tools in preprocessing and will warn the analyser on possible lost or gain when generalizing.

Definition 3.2.2. *Let* S *be a normalized similarity measure on a set of attributes* M*.* a, b, c *tree attributes.*

- *The attributes* a *and* b *are* ***less similar*** *if* $S(a, b) < 0,5$*;*

- *The attributes* a *and* b *are* ***more similar*** *if* $S(a, b) \geqslant 0,5$*;*

- *The attribute* a *is more similar to attribute* b *than to attribute* c *if* $S(a, b) \geqslant S(a, c)$*;*

- *The attribute* a *is less similar to attribute* b *than to attribute* c *if* $S(a, b) \leqslant S(a, c)$*.*

In the literature, there are three main types of similarity measures, depending on the type of data on which they are used:

Correlation coefficients: They are often used in data to compare variables with qualitative characters subdivided in more than two states.

Distance similarity coefficients: They are generally used in data with pure quantitative variables. In most cases, for quantitative data, the similarity between two taxa is expressed as a function of their distance in a dimensional space whose coordinates are the characters.

Coefficients of association: They are often used in data with presence-absence characters or in data with individuals having qualitative characters subdivided into two states.

Now we present the different types of existing similarity measures.

3.2.1 Correlation coefficients

Correlation coefficients are among the most ancien coefficients of similarity. The most known one is the product moment correlation, constructed by **Pearson** in 1896, in order to compare two objets or variables with qualitative characters that have more than two states. Its formula is given by

$$P(X_1, X_2) = \frac{\sum_{i=1}^{k}(x_{i1} - \overline{x_1})(x_{i2} - \overline{x_2})}{\sqrt{\sum_{i=1}^{k}(x_{i1} - \overline{x_1})^2 \sum_{i=1}^{k}(x_{i2} - \overline{x_2})^2}}$$

where k is the number of individuals for which the two variables are being computed, x_{i1} is the state of the variable X_1 for the individual i, x_{i2} is the state of the variable X_2 for the individual i and $\overline{x_1}$ is the mean of the different states $x_{11}, ..., x_{k1}$ of the variable X_1. Other existing correlation coefficients include the **Spearman**'s rank-order correlation coefficient and the **Kindall**'s tau correlation coefficient. These coefficients were put in place in order to capture the association between two ordinal variables.

Ever since **Pearson** proposed a coefficient of correlation in 1896, numerous similarity measures and distance have been proposed in various fields.

3.2.2 Distance similarity coefficients

There are many existing similarity measures based on distance of a n dimensional space.

In order to measure the divergence between two populations having some given characters, **Klauber** suggested in 1940 to divide the difference of the means of the two population by the sum of those means. In 1952, with the objective of determining to what extent the simultaneous treatment of a number of characters on two populations will reproduce a well known relationship between them, **Clark** decided to extent **Klauber**'s idea to populations having several characters. Thus, moving from the first formula with two populations X_1 and X_2 ($\frac{\overline{X_1} - \overline{X_2}}{\overline{X_1} + \overline{X_2}} = \frac{\overline{X_1}}{\overline{X_1} + \overline{X_2}} - \frac{\overline{X_2}}{\overline{X_1} + \overline{X_2}} = x_1 - x_2$) which can only be represented in a one dimensional space to a formula applicable to a k dimensional space, where k is the number of characters shared by the two populations, with each character weighted in inverse proportion to the sum of the means of the characters in the two populations compared; he defined the coefficient of divergence as follow :

$$Clk(x_1, x_2) = \sqrt{\frac{1}{k} \sum w_i^2 (x_{i1} - x_{i2})^2}$$

where k is the number of characters, $w_i = \frac{1}{x_{i1} + x_{i2}}$ the weight of the $i^{th} character$.

Moving from the correlation coefficient of **Pearson** to distance, **Sokal** realised by discussions with taxonomists, that they understood the concept of distance more than that of the correlation. The concept of distance was more appealing to taxonomists than the concept of correlation and association. But at that time, most of the coefficients used to compare two taxa were coefficients of association and correlation coefficients, and many taxonomists found some difficulties using them, especially the **Pearson**'s correlation. He proposed a new distance standardly closed to the correlation of **Pearson** of racial likeness for standardized data, and that could therefore be easily used by taxonomists. That coefficient of distance were constructed by dividing the sum of the squared difference by the number of characters for which the two taxa are compared. It gives a relative evaluation of taxonomic similarity.

$$S(x_1, x_2) = \frac{1}{n} \sum (x_{i1} - x_{i2})^2$$

where x_1 and x_2 are the taxa being compared, n the number of characters, x_{i1} the state of taxon 1 for character i.

In 1971, most of the studies in biology and botanic were based on comparing samples of species between ecosystems or biotopes. Following the same vision, **Brinkhurst** and **Johnson**, looking for a way to determine the number of characters of macroinvertebrate associations in the Bay of Quinte and the area of lake Ontario, attempted to establish the degree of difference between the associations, to determine which of the mean taxonomic groups contributed towards this difference, and to examine the species diversity in these macroinvertebrate associations. In order to evaluate the difference between those associations, they used two coefficients index in order to be sure that the affinity of samples is not only due to the sharing of most species, but also to the occurrence of these species in the same proportion. The first index was that of **Jaccard** and the second was a new one called the percentage similarity of community, and defined as follow :

$$BJ_c(x_1, x_2) = \sum_{k=1}^{n} \min(x_{i1}, x_{i2})^2$$

where n is the number of species being compared, x_{i1} the percentage of specie i in the total animals in association X_1.

In the same year, studying the salinity and temperature of phytoplankton population in two transient beach ponds and nearby long island sound in New York, **Levandowsky** was confronted in comparing two samples collected from june to november from the two ponds. He then put to place in 1971, a new similarity measure called the modified **Jaccard**'s Index, because it could not only be used in binary data, but also in a general quantitative data. It was defined as follow:

$$L(x_1, x_2) = 1 - \frac{\sum_i \min(x_{i1}, x_{i2})}{\sum_i \max(x_{i1}, x_{i2})}$$

where x_{i1} is the weight of taxon i in the sample x_1 , and x_{i2} is the weight of taxon i in the sample x_2.

Looking through many existing similarity coefficients, **Gower** realised that they were made either for dichotomous characters, or for qualitative characters, or for quantitative characters. He also found out that most of those measures were not included in computer programs. Without questioning how each similarity coefficient should be used in different circumstances, he realised that there was a way of putting in place a new and more general similarity coefficient that could include several existing ones as special cases and could be used under different circumstances. Then he defined a measure that could also be used in different kinds of data (dichotomous, qualitative and quantitative) and could be particularly and easily included in computer programs. This general measure was defined as follow: for two given individuals i and j,

$$G_{ij} = 1 - \frac{\sum_k S_{ijk}}{\sum_k \delta_{ijk}}$$

where S_{ijk} is the score assigned to the two individuals on a given character k, and δ_{ijk} is the representation of the possibility of making comparison between the two individuals i and j on a given character k.

$$\delta_{ijk} \in \{0,1\}$$

$$\begin{array}{lll} S_{ijk} & \in \{0,1\} & \text{if the character } k \text{ is qualitative} \\ S_{ijk} & = 1 - \frac{abs(x_i - x_j)}{R_k} & \text{if the character } k \text{ is quantitative with} \\ & & x_1, ..., x_n \text{ values and } R_k \text{ is the range of character } k. \end{array}$$

This measure is sometime presented as a weighted similarity coefficient as follow:

$$G_{ij} = \frac{\sum_k \delta_{ijk}.S_{ijk}}{\sum_k \delta_{ijk}}$$

Until 1976, looking for a successful way of comparing the level of pollution between two stations, **Pinkham** and **Pearson** found out that the analytical technics of the existing measures (**Jaccard**'s index, **Sokal and Michener**'s simple matching coefficient, the **Chutter**'s biotic index, the **Brinkhurst and Johnson**'s similarity index, the **Pearson**'s product moment correlation,...) could not truthfully and accurately reflect the extent to which these two levels of pollution are similar or dissimilar, especially when they were applied on quantitative data. Some of them only took into account the occurrences or the structure of the samples related to the two stations, ignoring their abundance. It is the case with association index. Some of them only consider the relative abundance of the two samples. It is the case of the **Brinkhurst and Johnson**'s similarity index, and the **Pearson**'s product moment correlation. They decided to put into place a new similarity index that could consider all those insufficiencies and simultaneously compare both abundance and occurrence of the two species. That measure was therefore defined as follow :

$$PP(x_1, x_2) = \frac{1}{k} \sum_{i=1}^{k} \frac{\min(x_{i1}, x_{i2})}{\max(x_{i1}, x_{i2})}$$

where k is the number of taxa being compared.

3.2.3 Coefficients of association

There are two subsets of coefficients of association: those that only depend on characteristics present in at least one of the taxa compared, but are independent of the attributes absent in both taxa (denoted by type 1), and those that also take into account the attributes absent in both taxa (denoted by type 2). Those measures use

- a as the number of cases where the two variables occur together in a sample,
- d as the number of cases where none of the two attributes occur in a sample,
- b as the number of cases in which only the first variable occur, and
- c as the number of cases where only the second variable occur.

One of the most important similarity measure of type 1 is the **Jaccard measure** defined by $\frac{a}{a+b+c}$, proposed in order to classify ecological species. Also in the ecological field, the **Dice coefficient of association** defined by $\frac{2a}{2a+b+c}$ aims at quantifying the extent to which two different species are associated in a biotope, the **Sorensen coefficient of association** given by $\frac{4a}{4a+b+c}$ and the **Anderberg coefficient of association** defined by $\frac{8a}{8a+b+c}$ are of the same type. The **Sneath and Sokal 2** similarity coefficient defined by $\frac{\frac{1}{2}a}{\frac{1}{2}a+b+c}$, put in place in order to compare organisms in numerical taxonomy, the **Kulczynski similarity** measure given by $\frac{1}{2}(\frac{a}{a+b}+\frac{a}{a+c})$ and the **Ochiai similarity** measure determined by $\frac{a}{\sqrt{(a+b)(a+c)}}$ are also from this first type.

The most used similarity coefficient of the second type is the **Sokal and Michener** coefficient of association defined by $\frac{a+d}{a+d+b+c}$, also called the **simple matching coefficient**, put in place to express the similarity between two species of bees. Moreover, the **Rogers and Tanimoto similarity measure** given by $\frac{\frac{1}{2}(a+d)}{\frac{1}{2}(a+d)+b+c}$ whose aim was to compare species of plants in the ecological field, the **Sokal and Sneath 1** similarity coefficient defined by $\frac{2(a+d)}{2(a+d)+b+c}$ to make comparison in numerical taxonomy and the **Russels and Rao** similarity measure given by $\frac{a}{a+d+b+c}$, who was put in place with the aim of showing resemblance between species of *anopheline larvae,* are included in this type. Same is the **Yule and Kendall** similarity coefficients defined by $\frac{ad}{ad+bc}$, and often used in the statistical field. Some of the above similarity measures can be found in [13], which also define the similarity measure on concept lattices.

in the following, we test the compatibility of existing similarity measures to $\exists$-generalization.

3.3 Test of existing similarity measures

Similarity and dissimilarity measures play a key role in pattern analysis problems such as classification, clustering, etc. Regarding the definitions of the above kinds of similarity measures, only the coefficients of association can be applied to formal contexts, since formal contexts are data with binary characters.

Now, we study the impact of the above coefficients of association on a special pair of attributes in some formal contexts. The objective is to show that these similarity measures does not always permit us to conclude whether their $\exists$-generalization increases the size of the lattice or no.

To start, we consider any formal context $\mathbb{K}$ containing two attributes x, y such that $x' \subseteq y'$ and $|x'| = 1$. Then $|x' \setminus y'| = 0$ and the generalization of the attributes x and

y does not increase the size of the lattice. The case $|y' \setminus x'| = 20$ and $|G \setminus y'| = 1$ yields $a = |x'| = 1$, $b = |x' \setminus y'| = 0$, $c = |y' \setminus x'| = 20$ and $d = |G \setminus y'| = 1$.

Using the following coefficients of association of type 1: Jaccard (Jc), Dice (Di), Sorensen (So), Anderberg (An), Sneath and Sokal 2 (SS_2), Kulczynski (Ku) and Orchiai (Orch), and the following coefficients of association of type 2: Sokal and Michener (SM), Rogers and Tanimoto (RT), Sneath and Sokal 1 (SS_1) and Russel and Rao (RR), we get the table below for $s(x, y)$:

Jc	Di	So	An	SS_2	Ku	Orch	SM	RT	SS1	RR
0,05	0,09	0,17	0,29	0,02	0,52	0,22	0,09	0,05	0,17	0,05

Table 3.1: The similarity between x and y for coefficients of association of type 1 and 2

The table above shows that the previous similarity measures between the attributes x and y are very low, despite the fact that their generalization does not increase the size of the lattice.

Our second example is the formal context $\mathbb{K}_6 := (S_6 \cup \{g_1\}, S_6 \cup \{m_1, m_2\}, \mathrm{I})$ below, with $S_6 = \{1, 2, 3, 4, 5, 6\}$.

$\mathbb{K}_6$	1	2	3	4	5	6	m_1	m_2
1		×	×	×	×	×	×	
2	×		×	×	×	×	×	×
3	×	×		×	×	×	×	×
4	×	×	×		×	×	×	×
5	×	×	×	×		×	×	×
6	×	×	×	×	×			×
g_1	×	×	×	×	×	×		

We observe that $|m_1' \cap m_2'| = 4$, $|m_1' \setminus m_2'| = 1$ and $|m_2' \setminus m_1'| = 1$. Putting together the attributes m_1 and m_2 by a $\exists$-generalization increases the size of the lattice by 16. The following table gives the measures of type 1 and type 2 between the attribute m_1 and any other attribute i.

	Jc	Di	So	An	SS_2	Ku	Orch	SM	RT	SS1	RR
$i \in S_5$	0,57	0,80	0,89	0,94	0,50	0,80	0,80	0,71	0,56	0,83	0,57
$i = 6$	0,83	0,91	0,95	0,97	0,71	0,92	0,91	0,75	0,75	0,92	0,71
$i = m_2$	0,67	0,80	0,89	0,94	0,50	0,80	0,80	0,71	0,56	0,83	0,57

Table 3.2: The similarity between m_1 and m_2 for coefficients of association of type 1 and 2

According to Table 3.2, every similarity measure of the two types shows that the attribute m_1 is more similar to m_2 than to any other attribute $i \in S_6$ (apart from

$i = 6$); But putting m_1 and m_2 together increases the size of the lattice. We can conclude that these similarity measures are not compatible with the $\exists$-generalization. We are actually looking for a measure on attributes that will flag pairs of attributes as **less similar** when putting these together increases the size of the concept lattice.

3.4 A similarity measure compatible with $\exists$-generalization

In this section we define a similarity measure on attributes which is compatible with the existential generalization. This generalization means that from an attribute reduced context $\mathbb{K} := (G, M, \mathrm{I})$, two attributes a, b are removed and replaced with an attribute s defined by $s' = a' \cup b'$. We recall that $M_{ab} := M \setminus \{a, b\}$ and

$$\begin{aligned} \mathbb{K}_{ab} :=& (G, M_{ab}, \mathrm{I} \cap (G \times M_{ab})), && \text{(removing } a, b \text{ from } \mathbb{K}) \\ \mathbb{K}_{ab}^s :=& (G, M_{ab} \cup \{s\}, I_{ab}^s), && \text{(adding } s \text{ to } \mathbb{K}_{ab}) \end{aligned}$$

where $I_{ab}^s := (\mathrm{I} \cap (G \times M_{ab})) \cup \{(g, s) \mid g \,\mathrm{I}\, b \text{ or } g \,\mathrm{I}\, a\}$. Furthermore we denote the set of extents of $\mathbb{K}_{ab}$ by $\mathrm{Ext}(\mathbb{K}_{ab})$. We also recall the following notations:

$$\begin{aligned} \mathcal{H}(a) &:= \{A \cap a' \mid A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and } A \cap a' \notin \mathrm{Ext}(\mathbb{K}_{ab})\}, \\ \mathcal{H}(b) &:= \{A \cap b' \mid A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and } A \cap b' \notin \mathrm{Ext}(\mathbb{K}_{ab})\}, \\ \mathcal{H}(a \cup b) &:= \{A \cap (a' \cup b') \mid A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and } A \cap (a' \cup b') \notin \mathrm{Ext}(\mathbb{K}_{ab})\}, \\ \mathcal{H}(a \cap b) &:= \{A \cap (a' \cap b') \mid A \in \mathrm{Ext}(\mathbb{K}_{ab}) \text{ and } A \cap (a' \cap b') \notin \mathrm{Ext}(\mathbb{K}_{ab})\}. \end{aligned}$$

We will often write $h(x)$ for $|\mathcal{H}(x)|$, for any $x \in \{a, b, a \cap b, a \cup b\}$.

Now, we define the following gain function:

$$\begin{aligned} \psi : M \times M &\longrightarrow \mathbb{Z} \\ (a, b) &\longmapsto \psi(a, b) = |\mathcal{H}(a \cup b)| - |\mathcal{H}(a, b)| \end{aligned}$$

Note that $\mathcal{H}(a \cup b) = \mathcal{H}(b \cup a)$, and $\mathcal{H}(a, b) = \mathcal{H}(b, a)$ because the order of adding the attributes a and b does not matter. Therefore $\psi(a, b) = \psi(b, a)$. By definition, $\psi(a, a) = 0$. Further, we define the map δ as follow:

$$\begin{array}{rccl} \delta: & M \times M & \longrightarrow & \mathbb{R} \\ & (a, b) & \longmapsto & \begin{cases} 1 & \text{if } \psi(a, b) \leqslant 0 \\ 0 & \text{else} \end{cases} \end{array}$$

Since $\mathbb{K}$ is a finite context, there is a pair of attributes a_0, b_0 in M such that

$$|a_0'| + |b_0'| = \max_{a, b \in M} (|a'| + |b'|).$$

We set $n_0 = 2^{|a_0'| + |b_0'|} - 2^{|a_0'|} - 2^{|b_0'|} + 1$. Then $n_0 \geqslant 2^{|a'| + |b'|} - 2^{|a'|} - 2^{|b'|} + 1$ for all pairs $\{a, b\} \subseteq M$. With the function δ, we construct the following map:

$$\begin{array}{rccl} S_{\text{gen}}: & M \times M & \longrightarrow & \mathbb{R} \\ & (a, b) & \longmapsto & \frac{1 + \delta(a, b)}{2} - \frac{|\psi(a, b)|}{2n_0} \end{array}$$

where $|\psi(a, b)|$ is the absolute value of $\psi(a, b)$. That leads to the following results.

Proposition 3.4.1. *Let (G, M, I) be a reduced context with $|G| \geqslant 3$ and $|M| > 3$. Then S_{gen} is a normalized similarity measure on M.*

Proof. Let a, b be two attributes of (G, M, I). Since $|\psi(a,b)| \leqslant n_0$, we can easily check that $0 \leqslant S_{\text{gen}}(a,b) = S_{\text{gen}}(b,a) \leqslant S_{\text{gen}}(a,a) = 1$. ■

S_{gen} also has the following properties:

Proposition 3.4.2. *Let (G, M, I) be a reduced context with $|G| \geqslant 3$ and $|M| > 3$. Let $a, b, c, d \in M$. It holds:*

a) $S_{gen}(a,b) \geqslant \frac{1}{2}$ if and only if $\psi(a,b) \leqslant 0$.

b) If $\psi(a,b) \leqslant 0 < \psi(d,c)$, then $S_{gen}(d,c) < S_{gen}(a,b)$.

c) If $0 < \psi(a,b) \leqslant \psi(d,c)$, then $S_{gen}(d,c) \leqslant S_{gen}(a,b)$.

d) If $\psi(a,b) \leqslant \psi(d,c) \leqslant 0$, then $S_{gen}(a,b) \leqslant S_{gen}(d,c)$.

Proof. Let $\mathbb{K} = (G, M, \mathrm{I})$ be a context and $a, b, c, d \in M$.

a) If $\psi(a,b) \leqslant 0$, then $\delta(a,b) = 1$ and

$$S_{\text{gen}}(a,b) = \frac{1+\delta(a,b)}{2} - \frac{|\psi(a,b)|}{2n_0} = \frac{1}{2}\left(2 + \frac{\psi(a,b)}{n_0}\right) \geqslant \frac{1}{2}.$$

Now, $S_{\text{gen}}(a,b) \geqslant \frac{1}{2}$ implies $\frac{1+\delta(a,b)}{2} - \frac{|\psi(a,b)|}{2n_0} \geqslant \frac{1}{2}$ and $|\psi(a,b)| \leqslant n_0\delta(a,b)$. If $\delta(a,b) = 0$, then $|\psi(a,b)| = 0$. If $\delta(a,b) = 1$, then $\psi(a,b) \leqslant 0$ by definition of δ.

Hence, $S_{\text{gen}}(a,b) \geqslant \frac{1}{2}$ if and only if $\psi(a,b) \leqslant 0$.

b) If $\psi(a,b) \leqslant 0 < \psi(d,c)$, then $S_{\text{gen}}(d,c) < \frac{1}{2} \leqslant S_{\text{gen}}(a,b)$.

c) If $0 < \psi(a,b) \leqslant \psi(d,c)$, then $\delta(a,b) = \delta(d,c) = 0$, and

$$S_{\text{gen}}(d,c) = \frac{1}{2} - \frac{\psi(d,c)}{2n_0} \leqslant \frac{1}{2} - \frac{\psi(a,b)}{2n_0} = S_{\text{gen}}(a,b).$$

d) If $\psi(a,b) \leqslant \psi(d,c) \leqslant 0$, then $\delta(a,b) = \delta(d,c) = 1$, and

$$S_{\text{gen}}(a,b) = 1 + \frac{\psi(a,b)}{2n_0} \leqslant 1 + \frac{\psi(d,c)}{2n_0} = S_{\text{gen}}(d,c).$$

■

Proposition 3.4.3. *Let (G, M, I) be a reduced context and $a, b \in M$. The following assertions are equivalent:*

(i) $\delta(a,b) = 1$.

(ii) $\psi(a,b) \leqslant 0$.

(iii) $S_{gen}(a,b) \geqslant \frac{1}{2}$.

(iv) *An $\exists$-generalization of a and b does not increase the size of the concepts lattice.*

Proof. (i) $\iff$ (ii) follows from the definition of δ.
(ii) $\iff$ (iii) is Proposition 3.4.2 a).
(ii) $\iff$ (iv) follows from the fact that $\psi(a,b) = |\mathcal{H}(a \cup b)| - |\mathcal{H}(a,b)|$ is actually the difference $|\mathfrak{B}(G, M_{ab} \cup \{s\}, I^s_{ab})| - |\mathfrak{B}(G,M,I)|$ between the number of concepts before and after generalizing a,b to s with $s' = a' \cup b'$. ■

Therefore, generalizing two attributes a,b in a reduced context (G,M,I) increases the size of the lattice if and only if $S_{\mathrm{gen}}(a,b) < \frac{1}{2}$. The threshold $\frac{1}{2}$ is just a consequence of the way S_{gen} has been defined.

To test our results we have designed a naive algorithm (see Algorithm 1) that computes S_{gen} on all pairs of attributes a,b of $\mathbb{K}$.

Algorithm 1: Computing a similarity measure

Data: An attribute reduced context (G,M,I)
Result: ψ and S_{gen} on $M \times M$

Choose x,y in M, $x \neq y$ with $|x'| + |y'|$ maximal;
$n_0 \leftarrow 2^{|x'|+|y'|} - 2^{|x'|} - 2^{|y'|} + 1$;
$T \leftarrow \emptyset$;
foreach a *in* M **do**
  $T \leftarrow T \cup \{a\}$;
  foreach b *in* $M \setminus T$ **do**
    $\mathrm{Ext}_0 \leftarrow \mathrm{Ext}(G, M_{ab}, \mathrm{I}_{ab})$;
    foreach x *in* $\{a, b, a \cup b, a \cap b\}$ **do** $\mathcal{H}(x) \leftarrow \emptyset$;
    foreach A *in* Ext_0 **do**
      foreach x *in* $\{a, b, a \cup b, a \cap b\}$ **do**
        if $A \cap x' \notin \mathrm{Ext}_0$ **then** $\mathcal{H}(x) \leftarrow \mathcal{H}(x) \cup \{A \cap x'\}$;
      end
    end
  end
  $\psi(a,b) \leftarrow |\mathcal{H}(a \cup b)| - |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|$; $\psi(b,a) \leftarrow \psi(a,b)$;
  if $\psi(a,b) \leqslant 0$ **then**
    $\delta(a,b) \leftarrow 1$
  else
    $\delta(a,b) \leftarrow 0$
  end
  $S_{\mathrm{gen}(a,b)} \leftarrow \dfrac{1+\delta(a,b)}{2} - \dfrac{|\psi(a,b)|}{2n_0}$
end

If the set of attributes M is considered as a vector, then for any attribute $a \in M$, we set $T(a)$ the set of all attributes coming before a in M.

The time complexity of our algorithm is given by

$$\sum_{a\in M}(1+\sum_{b\in M\setminus T(a)}((q(a,b)+4)[4(q(a,b)+1)+4]+3),$$

which is equal to

$$|M|+\sum_{a\in M}\sum_{b\in M\setminus T(a)}(4q^2(a,b)+24q(a,b)+35), \quad \text{with } q(a,b)=|\operatorname{Ext}(\mathbb{K}_{ab})|.$$

Now we analyse the complexity of the given algorithm.

3.5 Analysing the complexity of the above algorithm

We have constructed a similarity measure compatible with the change of size of the lattice after an $\exists$-generalization of a pair of attributes in a formal context. In order to give the complexity of our algorithm, we recall the following tools and definitions which help to understand the notion of complexity before determining the complexity of the above algorithm.

3.5.1 Some definitions

The determination of the complexity will use the notions of subcontext, contranominal context and minimal generators.

Definition 3.5.1. *[18] Let $\mathbb{K}=(G,M,\mathrm{I})$ be a formal context, $H\subseteq G$ and $N\subseteq M$. Then the formal context $(H,N,\mathrm{I}\cap(H\times N))$ is called a* ***subcontext*** *of $\mathbb{K}$.*

Definition 3.5.2. *[1] A contranominal scale is a formal context of the form $(S,S,\neq)$ where S is a set.*

When the set S is finite with k elements, then the context $(S,S,\neq)$ is denoted by $\mathbb{N}^c(k)$.

Definition 3.5.3. *[1]*

- *A formal context $\mathbb{K}$ is $\mathbb{N}^c(k)$-free if there does not exist a subcontext $\mathbb{K}_1$ of $\mathbb{K}$ such that $\mathbb{K}_1\cong\mathbb{N}^c(k)$.*
- *In the same way, a lattice L is $\mathfrak{B}(\mathbb{N}^c(k))$-free whenever the lattice $\mathfrak{B}(\mathbb{N}^c(k))$ does not order embed into L.*

Proposition 3.5.1. *[1] Let $\mathbb{K}$ be a context and k an element of $\mathbb{N}$ such that $\mathfrak{B}(\mathbb{N}^c(k))$ embeds into $\mathfrak{B}(\mathbb{K})$. Then $\mathbb{N}^c(k)$ is a subcontext of $\mathbb{K}$.*

The following definition says what minimal generators are:

Definition 3.5.4. *[1] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context. A set $S \subseteq G$ is said to be a **minimal generator** if $T^{\mathrm{II}} \neq S^{\mathrm{II}}$ for every proper subset $T \subsetneq S$.*
The set of all minimal generators of a context $\mathbb{K}$ will be denoted by $MINGEN(\mathbb{K})$.

The following result characterizes minimal generators:

Proposition 3.5.2. *[1] Let (G, M, I) be a formal context. A set $S \subsetneq G$ is a minimal generator if and only if for every $g \in S$, it holds that $(S \backslash \{g\})^{\mathrm{I}} \neq S^{\mathrm{I}}$.*

The next theorem gives the link between minimal generators and contranominal scales.

Theorem 3.5.1. *[1] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a context and $A \subsetneq G$. There exists a contranominal scale $\mathbb{K}_1$ (a subcontext of $\mathbb{K}$) having A as its object set iff A is a minimal generator. In particular, if G is finite:* $\max\{|A|;\ A \text{ is a minimal generator}\} = \max\{k \in \mathbb{N} : \mathbb{N}^c(k) \leqslant \mathbb{K}\}$.

Theorem 3.5.2. *[1] Let $\mathbb{K} = (G, M, \mathrm{I})$ be a $\mathbb{N}^c(k)$-free context. Then*

$$|\mathfrak{B}(\mathbb{K})| \leqslant (|G|.|M|)^{k-1} + 1$$

Notation 3.5.1. *The above upper bound has recently being ameliorated, and the ameliorated upper bound is $\sum_{i=0}^{k-1} C^i_{|G|}$ and is denoted by $f(n, k)$.*

The notions of $\mathbb{N}^c(k)$-free contexts and minimal generators have led us to an upper bound of the size of concept lattices of $\mathbb{N}^c(k)$-free contexts. That upper bound will be of help for constructing the complexity of the given algorithm. However, others important notions are still to be defined, notably the notions directly in relation with the complexity of an algorithm. These notions concern steps in algorithms and time complexity, and are presented in the following definitions.

Definition 3.5.5. *[25] A step in an algorithm is any arithmetic operation, access to arrays or comparisons.*

Definition 3.5.6. *[25] The theory of complexity studies the time and memory space an algorithm needs as a function of the size of the input data.*

More often, complexity is used to make comparison among several algorithms solving the same problem.

Definition 3.5.7. *The time complexity of an algorithm A is a function $f : \mathbb{N} \longrightarrow \mathbb{N}$, where $f(n)$ is the maximal number of steps A needs to solve a problem instance having input data of length n.*

Remark 3.5.1. *The complexity is always measured for the worst possible case for a given length of the input, a situation which does not always traduce reality, in the sense that an algorithm can have an exponential complexity, but work very fast in practice.*

Remark 3.5.2. *In most cases it is not possible to exactly obtain the complexity $f(n)$ of an algorithm. In these cases, it is often given an estimation of how fast $f(n)$ can grows.*

Let consider two maps $f : \mathbb{N} \longrightarrow \mathbb{R}^+$ and $g : \mathbb{N} \longrightarrow \mathbb{R}^+$.

Definition 3.5.8. *[25]*

- *$f(n) = O(g(n))$ if there is a constant $c > 0$ such that $f(n) \leqslant cg(n)$ for all sufficiently large n;*

- *$f(n) = \Omega(g(n))$ if there is a constant $c > 0$ such that $f(n) \geqslant cg(n)$ for all sufficiently large n;*

- *$f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.*

Definition 3.5.9. *[25] Let A be an algorithm solving a problem P.*

- *If $f(n) = O(g(n))$, we say that f has* ***at most rate of growth*** *$g(n)$, the algorithm A having $f(n)$ as the number of steps has complexity $O(g(n))$ and the problem P which is solved by the algorithm A has the complexity $O(g(n))$;*

- *If $f(n) = \Omega(g(n))$, we say that f has* ***at least rate of growth*** *$g(n)$, the algorithm A having $f(n)$ as the number of steps has complexity $\Omega(g(n))$ and the problem P which is solved by the algorithm A has the complexity $\Omega(g(n))$;*

- *If $f(n) = \Theta(g(n))$, we say that f has* ***rate of growth*** *$g(n)$, the algorithm A having $f(n)$ as the number of steps has complexity $\Theta(g(n))$ and the problem P which is solved by the algorithm A has the complexity $\Theta(g(n))$.*

3.5.2 Determination of the complexity of the algorithm for generalization

The time complexity of our algorithm is given by

$$\sum_{a\in M}(1 + \sum_{b\in M\backslash T(a)}((q(a,b)+4)[4(q(a,b)+1)+4]+3),$$

which is equal to

$$|M| + \sum_{a\in M}\sum_{b\in M\backslash T(a)}(4q^2(a,b)+24q(a,b)+35),$$

with q(a,b)=|Ext($\mathbb{K}_{ab}$)|.

By Theorem 3.5.1, for every formal $\mathbb{K} = (G, M, \mathrm{I})$, there is an element k_0 of $\mathbb{N}$ such that $\mathbb{K}$ is $N_c^{k_0}$-free. It is sufficient to choose k_0 such that

$$k_0 = \max\{|A|;\ \text{A is a minimal generator}\} + 1$$

Hence, by Theorem 3.5.2, the upper bound for the number of formal concept of the formal context $\mathbb{K} = (G, M, \mathrm{I})$ is given by

$$|\mathfrak{B}(\mathbb{K})| \leqslant (|G|.|M|)^{k_0} + 1$$

We set $n = |G||M|$. Then

$$|\mathfrak{B}(\mathbb{K})| \leqslant n^{k_0} + 1$$

Hence, for every pair $\{a, b\}$ of M, considering the context $\mathbb{K}_{ab} = (G, M_{ab}, \mathrm{I}_{ab})$, we have

$$q(a, b) = |\operatorname{Ext}(\mathbb{K}_{ab})| = |\mathfrak{B}(\mathbb{K}_{ab})| \leqslant |\mathfrak{B}(\mathbb{K})| \leqslant n^{k_0} + 1$$

Hence,

$$q(a, b)^2 \leqslant n^{2.k_0} + 2.n^{k_0} + 1$$

Therefore,

$$4.q(a, b)^2 \leqslant 4n^{2.k_0} + 8.n^{k_0} + 4$$

which means that

$$4.q(a, b)^2 + 24.q(a, b) \leqslant 4n^{2.k_0} + 8.n^{k_0} + 4 + 24.n^{k_0} + 24$$

and then that

$$4.q(a, b)^2 + 24.q(a, b) + 35 \leqslant 4n^{2.k_0} + 8.n^{k_0} + 4 + 24.n^{k_0} + 24 + 35$$

Therefore,

$$4.q(a, b)^2 + 24.q(a, b) + 35 \leqslant 4n^{2.k_0} + 8.n^{k_0} + 4 + 24.n^{k_0} + 24 + 35 \leqslant 4n^{2.k_0} + 32.n^{k_0} + 63$$

Thus $f(n) \leqslant n + n^2.(4n^{2.k_0} + 32.n^{k_0} + 63) \leqslant n + 4n^{2.k_0+2} + 32n^{k_0+2} + 63.n^2$, because $|M| \leqslant |G|.|M| = n$.

Then, $f(n) \leqslant n + 4n^{2.k_0+2} + 32n^{k_0+2} + 63.n^2 \leqslant 100.n^{2.k_0+2}$,

Hence,

$$f(n) \leqslant 100.n^{2.k_0+2},$$

and

$$f(n) = \mathcal{O}(n^{2.k_0+2}),$$

We conclude that the above algorithm has the complexity $\mathcal{O}(n^{2.k_0+2})$.

3.6 Validation and experimentation on data

After the construction of a new similarity measure compatible with a ∃-generalization on attributes in a formal context, this section aims at validating the new similarity measure on data. But before we continue, let precise that when an expert is about to analyse a data set and make a generalization of some attributes or items, he first has a subjective idea about the group of attributes that are about to be put together, based on his rationality and knowledge of the data. However, these attributes put together by ∃-generalization may not always meet his expectation. For example, he can be expecting the size of the concept lattice to reduce after a rational grouping of some attributes, and the size of the lattice increases. To avoid such a situation, an efficient analyses of the database by an expert should be base by combining both a rational and objective grouping of attributes, based on effectively measuring the similarity of the attributes he rationally suspects that should be generalized. The data used in this section are mostly from [31] and [15]. We first present the data before starting the analysis.

3.6.1 A case from lexicographic data

Formal Concepts Analysis has been applied to compare lexical databases. In [31] Uta Priss proposes an example in where the information channel is "*building*". With respect to this, the main difference between English and German is that in English, the word "house" only refers to small residential buildings whereas in German even small office buildings and large residential buildings can be called "Haus", and only factories would normally not be called "Haus". Moreover, "building" in English refers to either a factory, an office or even a big residential house. But only a factory can be called "*Gebäude*" in German. She presented in the figure below the information channel of the word "building"[1] in both English and German.

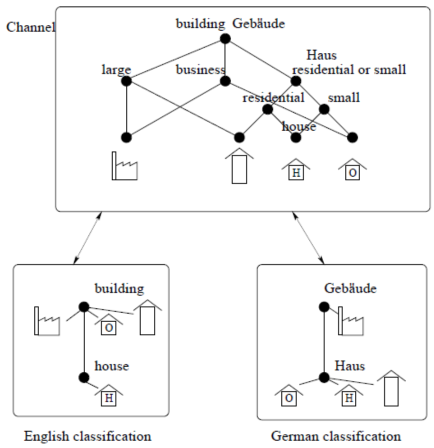


Figure 3.1: The information channel of the word "building" in English and German

.

[1] in the sense of Barwise and Seligman [6]

With the above information channel we can construct a formal context as follows: The objects are different kinds of buildings: small house ("h"), office ("o"), factory ("f") and large residential house ("l"). The attributes are different names of these objects in both languages: English and German. These are "building", "house", "Haus", "Gebäude", "large building" (short: "large"), "business building" (short: "business"), "residential house" (short: "residential"), and "small house" (short: "small"). Thus $G = \{$h, o, f, l$\}$ and $M = \{$building, house, Haus, Gebäude, large, business, residential, small$\}$. In the following, a set of objects will be denoted as a concatenation of those objects. For example we will write ho or oh for the set $\{$h, o$\}$. The English and German classifications of the word "building" are then presented in the following formal context:

	building	house	Haus	Gebäude	large	business	residential	small
factory	×			×	×	×		
office	×		×			×		×
house		×	×				×	×
large	×		×		×		×	

With the above formal context, $n_0 = 2^{3+3} - 2^3 - 2^3 + 1 = 49$.

Now, we test the similarity measure S_{gen} on all the pairs of attributes of the above formal context and produce some analysis. First, we present the results of two specific pairs of attributes, and extent them to all pairs of attributes using the algorithm presented in the previous chapter.

Let consider the pair of attributes $\{house, Gebäude\}$. We set $a := house$ and $b := Gebäude$. Then $a' \cup b' = \{f, h\}$ and $a' \cap b' = \emptyset$. With this pair of attributes, the formal context $\mathbb{K}_{ab}$ is given as follow:

	building	Haus	large	business	residential	small
factory	×		×	×		
office	×	×		×		×
house		×			×	×
large	×	×	×		×	

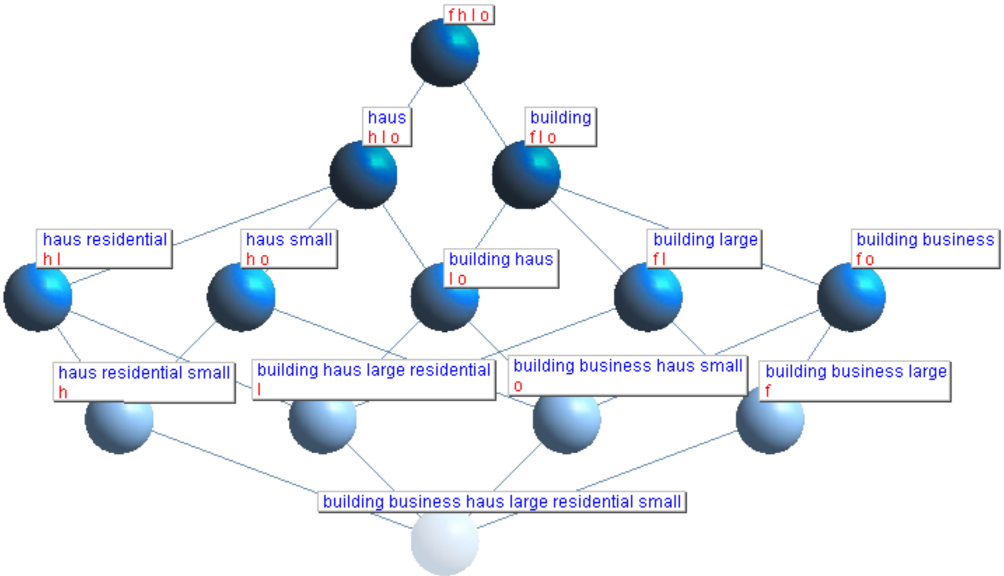


Figure 3.2: The context $\mathbb{K}_{ab}$ with the corresponding lattice, with a = *house* and b = *Gebude*

.

From the above figure, it comes that:

$$\mathrm{Ext}(\mathbb{K}_{ab}) = \{fohl, fol, ohl, fo, fl, ol, oh, hl, f, o, h, l, \emptyset\}, \text{ and}$$

$\mathcal{H}(a) = \mathcal{H}(b) = \mathcal{H}(a \cap b) = \emptyset$ and $\mathcal{H}(a \cup b) = \{fohl\}$. Therefore, $\psi(a, b) = 1$ and $S_{gen}(a, b) = \frac{1}{2} - \frac{1}{98} \approx 0.49$.

Now we consider the pair of attributes {haus, building}. We set a := *haus* and b := *building*. Then $a' \cup b' = \{f, o, h, l\}$ and $a' \cap b' = \emptyset$. With this pair of attributes, the formal context $\mathbb{K}_{ab}$ and the corresponding lattice are given in the following:

	house	Gebäude	large	business	residential	small
factory		×	×	×		
office				×		×
house	×				×	×
large			×		×	

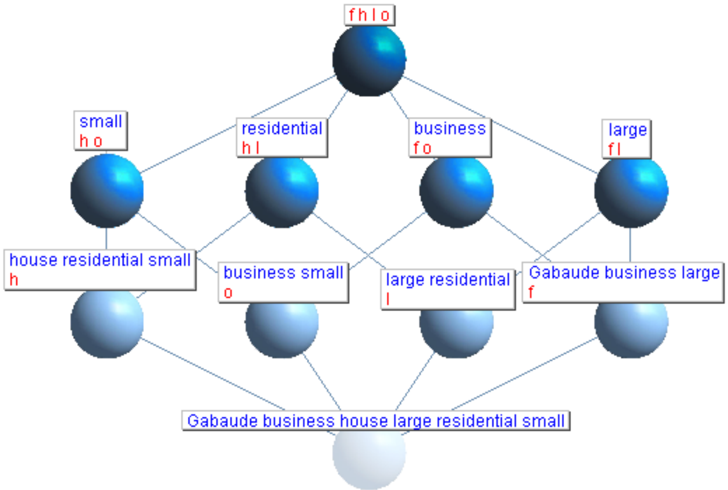


Figure 3.3: The concept lattice of context $\mathbb{K}_{ab}$ with $a = haus$ and $b = building$

.

From the above figure, it comes that:

$$\text{Ext}(\mathbb{K}_{ab}) = \{fohl, ho, hl, fo, fl, h, o, l, f, \emptyset\}, \text{ and}$$

$\mathcal{H}(a) = \{fol\}$, $\mathcal{H}(b) = \{ohl\}$, $\mathcal{H}(a \cap b) = \{ol\}$, and $\mathcal{H}(a \cup b) = \emptyset$. Therefore, $\psi(a, b) = -3$ and $S_{gen}(a, b) = 1 - \frac{3}{98} \approx 0.97$.

Using the above algorithm, we compute $\psi(a, b)$ and $S_{gen}(a, b)$ for all pairs $a, b \in M$. The table below gives $\psi(a, b)$ below the diagonal, and $S_{gen}(a, b)$ on the rest.

	building	house	Haus	Gebäude	large	business	residential	small
building	1.00	0.98	0.97	1.00	0.99	0.98	0.97	0.97
house	-2	1.00	1.00	0.49	0.49	0.49	1.00	1.00
Haus	-3	0	1.00	0.98	0.97	0.97	0.99	0.99
Gebäude	0	1	-2	1.00	1.00	1.00	0.49	0.49
large	-1	1	-3	0	1.00	0.98	0.49	0.97
business	-2	1	-3	0	-2	1.00	0.98	0.49
residential	-3	0	-1	1	1	-2	1.00	0.98
small	-3	0	-1	1	-3	1	-2	1.00

Table 3.3: The computed $\psi(a, b)$ and $S_{gen}(a, b)$ for all pairs $a, b \in M$

.

From the above table, the attributes "house" and "Gebäude" are less similar. It reflects the fact that the words "Gebäude" (in German) and "house" (in English) do not have the same meaning. It is also the case for the attributes "house" and "business buildings" as well as "Gebäude" and "residential building". Hence, putting together each of the above pairs of attributes will increase the size of the lattice. On the contrary, the attributes "large" and "Haus", "building" and "Haus" are more similar through S_{gen}. It is because the word "Haus" which designates a house, a business office or simply large building in German, often coincides with the words "building" or "large building" in English. For these pairs, the existential generalization will not increase the size of the lattice.

3.6.2 Other data

We consider some formal contexts from [18], notably the context for an educational film, "living beings and water" which explains the capacity of some living beings to live in water, and the context for "triangles" which presents the different types of triangles with their main characteristics.

The context for "living beings and water" is described in the following table:

	a	b	c	d	e	f	g	h	i
Leech	×	×					×		
Bream	×	×					×	×	
Frog	×	×	×			×			
Dog	×		×				×	×	×
Spike-weed	×	×		×		×			
Reed	×	×	×	×		×			
Bean	×		×	×	×				
Maize	×		×	×		×			

where a:=needs water to live, b:= lives in water, c:= lives on land, d:= needs chlorophyll to produce food, e:= two seed leaves, f:= one seed leaf, g:= can move around, h:= has limbs, and i:=suckles its offspring.

The following table of similarity measures of the context "Living beings and water" shows dissimilarities between the attributes of the following pairs "{two seed leaves, has limbs}", "{two seed leaves, suckles its offsprings} and {one seed leaf, suckles its offsprings}, since $S_{gen}(e, h) = 0.499994 = S_{gen}(e, i) = S_{gen}(f, i)$. Hence, the $\exists$-generalization of these pairs of attributes increases the size of the concept lattice. These situations may come from the fact that living beings that has either two seeds leaves or one seed leaf are not animals, and therefore, they cannot suck their offsprings.

	a	b	c	d	e	f	g	h	i
a	1.00000	0.99968	0.99968	0.99981	0.99994	0.99981	0.99981	0.99981	0.99994
b	−5	1.00000	0.99930	0.99975	1.00000	0.99981	0.99968	0.99968	0.99981
c	−5	−11	1.00000	0.99968	0.99994	0.99968	0.99975	0.99981	1.00000
d	−3	−4	−5	1.00000	0.99994	0.99987	0.99962	0.99981	1.00000
e	−1	0	−1	−1	1.00000	0.99987	1.00000	0.49994	0.49994
f	−3	−3	−5	−2	−2	1.00000	0.99981	0.99994	0.49994
g	−3	−5	−4	−6	0	−3	1.00000	0.99987	0.99994
h	−3	−5	−3	−3	1	−1	−2	1.00000	0.99994
i	−1	−3	−1	0	1	1	−1	−1	1.00000

Let consider the context of "triangles" bellow, as presented in [18] :

	a	b	c	d	e	f	g
$T_1((0,0),(6,0),(3,1))$		×	×	×		×	
$T_2((0,0),(1,0),(0,1))$		×	×				×
$T_3((0,0),(4,0),(1,2))$		×		×	×		
$T_4((0,0),(2,0),(1,\sqrt{3}))$	×		×	×	×		
$T_5((0,0),(2,0),(5,1))$		×		×		×	
$T_6((0,0),(2,0),(1,3))$		×	×	×	×		
$T_7((0,0),(2,0),(0,1))$		×					×

with a:=equilateral, b:= not equilateral, c:= isosceles, d:= oblique, e:= acute, f:= obtuse, g:= right; and for every $i \in \{1, ..., 7\}$, T_i is a set of tree points in an orthonormal mark, describing a triangle.

Similarly, from the table of similarity measures of the above context, one can observe that the attribute a is neither similar to the attribute f, nor to the attribute g, since S_{gen}(equilateral, obtuse) = 0.49987 = S_{gen}(equilateral, right), as we can see in the following table. Hence, the ∃-generalization of these pairs of attribute increases the size of the concept lattice. These situations certainly come from the fact that equilateral triangles have neither obtuse angles, nor right angles.

	a	b	c	d	e	f	g
a	1.00000	0.99937	0.99987	0.99987	0.99987	0.49987	0.49987
b	−5	1.00000	0.99861	0.99912	0.99912	0.99975	0.99975
c	−1	−11	1.00000	0.99924	0.99950	0.99975	0.99975
d	−1	−7	−6	1.00000	099962	0.99974	0.99937
e	−1	−7	−4	−3	1.00000	0.99962	0.99987
f	1	−2	−2	−2	−3	1.00000	0.99987
g	1	−2	−2	−5	−1	−1	1.00000

3.7 Conclusion

The constructed similarity measure is effectively compatible with the ∃-generalization as seen in the above examples. After the study of the size of the concept lattice after an ∃-generalization, it would be also interesting to study the attribute implications which are among relevant information that can be extracted from data.

Chapter 4

On the size of the generalized implications

4.1 Introduction

As we have mentioned in the second chapter, relevant information are often obtained from data in two main forms: formal concepts and association rules. Implications are one of the most important and interesting type of association rules. As we will discuss below, the number of attribute implications in formal contexts can also be very large, making it difficult for an expert to analyse them for the purpose of decisions. Hence, generalization can also be useful as a way of reducing their size in order to make the analysis of data more easy. Let precise that attribute implications consist of deducing which attributes are satisfied by some objects, knowing that other specific attributes are already satisfied by these objects. They are of great importance in data analysis because they are often of help to experts in their decision making, especially when predictions are made in data. We have studied the variations of the size of the concept lattice after an $\exists$-generalization and we have discovered that the size of the concept lattice can increase drastically. The following questions then hold: **what is the variation in the size of the set of attributes informative implications after an $\exists$-generalization? Does the increase in the size of the concept lattice after an $\exists$-generalization leads to a reduction in size of the base of implications**?

The rest of the chapter is organized as follow: In section 4.2, we evaluate the variations in the number of informative attributes implications using the effect of adding a new attribute to a formal context. In section 4.3, we compare the size of the set of informative attributes implications of a formal context to that of the corresponding generalized context after an $\exists$-generalization on attributes. Section 4.4 studies the variation of the canonical base of implications after the $\exists$-generalization, and Section 4.5 concludes the chapter.

4.2 On the number of informative implications

In this section, we investigate the variations in the number of informative attributes implications using the effect of adding a new attribute to a formal context.

Let $\mathbb{K} := (G, M, \mathrm{I})$ be a formal context and a an element not belonging to M. We consider the context $\mathbb{K}^a := (G, M \cup \{a\}, \mathrm{I}^a)$ obtained by adding attribute a to $\mathbb{K}$. For every subset $A \subseteq M$, we set

$$A^*_{\mathbb{K}} = A^{\mathrm{II}} \backslash (A \cup \bigcup_{x\in A} (A\backslash\{x\})^{\mathrm{II}})$$

and for every subset $A \subseteq M \cup \{a\}$, we set

$$A^*_{\mathbb{K}^a} = A^{\mathrm{I}^a\mathrm{I}^a} \backslash (A \cup \bigcup_{x\in A} (A\backslash\{x\})^{\mathrm{I}^a\mathrm{I}^a}).$$

We also set $Impl_{\mathbb{K}} = \{A \subseteq M | A^*_{\mathbb{K}} \neq \emptyset\}$, $Impl_{\mathbb{K}^a} = \{A \subseteq M \cup \{a\} | A^*_{\mathbb{K}^a} \neq \emptyset\}$ and $E_{\{M,a\}} = \{T \subseteq M; (T\backslash\{x\})^{\mathrm{I}} \nsubseteq a' \ \forall\, x \in T\}$, where $a' = \{g \in G | g$ has the new attribute $a\}$.

Then the following result holds

Lemma 4.2.1. *Let $\mathbb{K} := (G, M, \mathrm{I})$ be a formal context and a an element not belonging to M. We consider the context $\mathbb{K}^a := (G, M \cup \{a\}, \mathrm{I}^a)$ obtained by adding attribute a to $\mathbb{K}$. If $A \subseteq M$, then the following statements are equivalent:*

i) $A^*_{\mathbb{K}^a} \neq \emptyset$,

ii) $A^*_{\mathbb{K}} \neq \emptyset$ *or* $A^{\mathrm{I}} \subseteq a'$ *with* $A^*_{\mathbb{K}} = \emptyset$ *and* $A \in E_{\{M,a\}}$.

Proof. Let $\mathbb{K} := (G, M, \mathrm{I})$ be such context and $\mathbb{K}^a := (G, M \cup \{a\}, \mathrm{I}^a)$ obtained by adding attribute a to $\mathbb{K}$.

We know that

$$\begin{aligned} A^*_{\mathbb{K}^a} &= A^{\mathrm{I}^a\mathrm{I}^a} \backslash (A \cup \bigcup_{x\in A} (A\backslash\{x\})^{\mathrm{I}^a\mathrm{I}^a}) \\ &= A^{\mathrm{I}\mathrm{I}^a} \backslash (A \cup \bigcup_{x\in A} (A\backslash\{x\})^{\mathrm{I}\mathrm{I}^a}) \end{aligned}$$

ii)$\to$ *i)* If $A^*_{\mathbb{K}} \neq \emptyset$, then there is an element $x \in A^*_{\mathbb{K}} = A^{\mathrm{II}} \backslash (A \cup \bigcup_{x\in A} (A\backslash\{x\})^{\mathrm{II}})$. Therefore, $x \neq a$ because $x \in M$. However, $x \in A^{\mathrm{II}} \subseteq A^{\mathrm{II}^a} = A^{\mathrm{I}^a\mathrm{I}^a}$. Let suppose that there exists $y \in A$ such that $x \in (A\backslash\{y\})^{\mathrm{I}^a\mathrm{I}^a} = (A\backslash\{y\})^{\mathrm{II}^a}$. Then $x \in (A\backslash\{y\})^{\mathrm{II}}$ since $x \neq a$, which is absurd. It then comes that $x \notin \bigcup_{y\in A} (A\backslash\{y\})^{\mathrm{I}^a\mathrm{I}^a}$, which implies that $x \in A^*_{\mathbb{K}^a}$. Since that is true for every $x \in A^*_{\mathbb{K}}$, we conclude that $A^*_{\mathbb{K}} \subseteq A^*_{\mathbb{K}^a}$, and then that $A^*_{\mathbb{K}^a} \neq \emptyset$.

If $A^{\mathrm{I}} \subseteq a'$ with $A^*_{\mathbb{K}} = \emptyset$ and $A \in E_{\{M,a\}}$. Since $A^{\mathrm{I}} \subseteq a'$, then $A^{\mathrm{I}^a} \subseteq a^{I^a}$, which means that $a \in A^{\mathrm{I}^a\mathrm{I}^a}$. Moreover, $a \notin A \cup \bigcup_{x\in A} (A\backslash\{x\})^{\mathrm{I}^a\mathrm{I}^a}$ because $A \subseteq M$ and $A \in E_{\{M,a\}}$. Hence, $a \in A^{\mathrm{I}^a\mathrm{I}^a} \backslash (A \cup \bigcup_{x\in A} (A\backslash\{x\})^{\mathrm{I}^a\mathrm{I}^a}) = A^*_{\mathbb{K}^a}$. In fact, $A^*_{\mathbb{K}^a} = \{a\}$ since $A^*_{\mathbb{K}} = \emptyset$. It means that $A^*_{\mathbb{K}^a} \neq \emptyset$.

i)$\rightarrow$ *ii*) Let suppose that $A^*_{\mathbb{K}^a} \neq \emptyset$. We have $a \notin A^{\mathrm{I}\mathrm{I}^a}$ or $a \in A^{\mathrm{I}\mathrm{I}^a}$. If $a \notin A^{\mathrm{I}\mathrm{I}^a}$, then $\emptyset \neq A^*_{\mathbb{K}^a} = A^*_{\mathbb{K}}$. If $a \in A^{\mathrm{I}\mathrm{I}^a}$, then $A^{\mathrm{I}} \subseteq a'$. In this case, if $a \in (A\backslash\{x\})^{\mathrm{I}\mathrm{I}^a}$ for some $x \in A$, then $\emptyset \neq A^*_{\mathbb{K}^a} = A^*_{\mathbb{K}}$. Else, $a \notin (A\backslash\{x\})^{\mathrm{I}\mathrm{I}^a}$ for every $x \in A$, which implies that $A \in E_{\{M,a\}}$. In this last subcase, $A^*_{\mathbb{K}^a} = A^*_{\mathbb{K}} \cup \{a\}$ where one can have $A^*_{\mathbb{K}} \neq \emptyset$ or $A^*_{\mathbb{K}} = \emptyset$. Finally, we only have the two following situations, $A^*_{\mathbb{K}} \neq \emptyset$ or $A^{\mathrm{I}} \subseteq a'$ with $A^*_{\mathbb{K}} = \emptyset$ and $A \in E_{\{M,a\}}$.

■

This other result also holds

Lemma 4.2.2. *Let* $\mathbb{K} := (G, M, \mathrm{I})$ *be a formal context and* a *an element not belonging to* M*. We consider the context* $\mathbb{K}^a := (G, M \cup \{a\}, \mathrm{I}^a)$ *obtained by adding the attribute* a *to* $\mathbb{K}$*. If* $A \nsubseteq M$*, then the following statements are equivalent:*

**)* $(A \cup \{a\})^*_{\mathbb{K}^a} \neq \emptyset$ *in the context* $\mathbb{K}^a$*;*

***)* $(A^{\mathrm{I}} \cap a')^{\mathrm{I}} \backslash (A^{\mathrm{I}\mathrm{I}} \cup \bigcup_{x \in A}((A\backslash\{x\})^{\mathrm{I}} \cap a')^{\mathrm{I}}) \neq \emptyset$ *in the context* $\mathbb{K}$.

Proof. Let $\mathbb{K} := (G, M, \mathrm{I})$ be such context and $\mathbb{K}^a := (G, M \cup \{a\}, \mathrm{I}^a)$ obtained by adding attribute a to $\mathbb{K}$. We suppose that $A \nsubseteq M$. Then

$$\begin{aligned}
(A \cup \{a\})^*_{\mathbb{K}^a} &= (A \cup \{a\})^{\mathrm{I}^a \mathrm{I}^a} \backslash ((A \cup \{a\}) \cup \bigcup_{x \in A \cup \{a\}} ((A \cup \{a\}) \backslash \{x\})^{\mathrm{I}^a \mathrm{I}^a}) \\
&= (A^{\mathrm{I}^a} \cap a')^{\mathrm{I}^a} \backslash ((A \cup \{a\}) \cup A^{\mathrm{I}^a \mathrm{I}^a} \cup \bigcup_{x \in A} ((A\backslash\{x\})^{\mathrm{I}} \cap a')^{\mathrm{I}^a}) \\
. &= (A^{\mathrm{I}} \cap a')^{\mathrm{I}} \cup \{a\} \backslash ((A \cup \{a\}) \cup (A^{\mathrm{I}\mathrm{I}} \cup \{a\}) \cup \\
&\bigcup_{x \in A} ((A\backslash\{x\})^{\mathrm{I}} \cap a')^{\mathrm{I}} \cup \{a\}) \\
&= (A^{\mathrm{I}} \cap a')^{\mathrm{I}} \backslash (A^{\mathrm{I}\mathrm{I}} \cup \bigcup_{x \in A} ((A\backslash\{x\})^{\mathrm{I}} \cap a')^{\mathrm{I}}).
\end{aligned}$$

Hence, $(A \cup \{a\})^*_{\mathbb{K}^a} \neq \emptyset$ in the context $\mathbb{K}^a$ if and only if $(A^{\mathrm{I}} \cap a')^{\mathrm{I}} \backslash (A^{\mathrm{I}\mathrm{I}} \cup \bigcup_{x \in A}((A\backslash\{x\})^{\mathrm{I}} \cap a')^{\mathrm{I}}) \neq \emptyset$ in the context $\mathbb{K}$. ■

Thus, setting

$$\mathcal{I}_{\mathbb{K}}(a) = \{A \subseteq M : A^{\mathrm{I}} \subseteq a', A \in E_{\{M,a\}}, A^*_{\mathbb{K}} = \emptyset\} \cup \{A \cup \{a\} : A \subseteq M, (A^{\mathrm{I}} \cap a')^{\mathrm{I}} \backslash (A^{\mathrm{I}\mathrm{I}} \cup \textstyle\bigcup_{x \in A}((A\backslash\{x\})^{\mathrm{I}} \cap a')^{\mathrm{I}}) \neq \emptyset\},$$

we have the following proposition

Proposition 4.2.1. *Let* $\mathbb{K} := (G, M, \mathrm{I})$ *be a formal context and* a *an element not belonging to* M*. The addition of attribute* a *to the context* $\mathbb{K}$ *gives the context* $\mathbb{K}^a := (G, M \cup \{a\}, \mathrm{I}^a)$*, and*

$$Impl_{\mathbb{K}^a} = Impl_{\mathbb{K}} \cup \mathcal{I}_{\mathbb{K}}(a)$$

Proof. This the immediate consequence of lemma 4.2.1 and lemma 4.2.2. ∎

Now we look at what happen when several new attributes are added simultaneously to a formal context. We consider a formal context $\mathbb{K} = (G, M, \mathrm{I})$ and a nonempty set X of attributes not belonging to M. We denote by $\mathbb{K}^X$ the formal context $(G, M \cup X, \mathrm{I}^X)$. For any subset Y of X and any subset A of M, we denote by X_Y the set $X \backslash Y$ and by $(A \cup Y)^*_{/\mathbb{K}}$ the set

$$(A^{\mathrm{I}} \cap Y')^{\mathrm{I}} \backslash [A \cup \bigcup_{y \in Y} (A^{\mathrm{I}} \cap (Y \backslash \{y\})')^{\mathrm{I}} \cup \bigcup_{x \in A} ((A \backslash \{x\})^{\mathrm{I}} \cap Y')^{\mathrm{I}}],$$

where $Y' = \bigcap_{y \in Y} y'$, with y' the set of objects having the attribute y.

For every subset Q of X, we set

$$\mathcal{F}_A(Q) = \{v \in X_Q | A^{\mathrm{I}} \cap Q' \subseteq v'\}$$

and

$$\mathcal{R}_A(Q) = \{v \in \mathcal{F}_A(Q) | ((A \backslash \{x\})^{\mathrm{I}} \cap Q') \nsubseteq v', (A^{\mathrm{I}} \cap (Q \backslash \{y\})') \nsubseteq v' \ \forall\, x \in A, \forall\, y \in Q\}.$$

Let $Y \in \mathcal{P}(X)$. For every $Q \in \mathcal{P}(Y)$, we set

$$Y(Q) = \{A \cup Q | A \subseteq M, \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(Q) \neq \emptyset, (A \cup Q)^*_{/\mathbb{K}} = \emptyset\} \cup \{A \cup Q | A \subseteq M, (A \cup Q)^*_{/\mathbb{K}} \neq \emptyset\}.$$

and

$$\mathcal{J}_{\mathbb{K}}(Y) = \bigcup_{Q \in \mathcal{P}(Y)} Y(Q)$$

For example, if $Y = \emptyset$ (as a subset of a set X), then we have:

$$\begin{aligned}\mathcal{J}_{\mathbb{K}}(\emptyset) &= \{A \subseteq M : \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}} = \emptyset\} \cup \{A \subseteq M : A^*_{\mathbb{K}} \neq \emptyset\} \\ &= \{\mathrm{A} \subseteq M : A^*_{\mathbb{K}} = \emptyset \text{ and } \exists v \in X A^{\mathrm{I}} \subseteq v' \text{ and } \forall x \in A, (A \backslash \{x\})^{\mathrm{I}} \nsubseteq v'\} \\ &\cup \{A \subseteq M : A^*_{\mathbb{K}} \neq \emptyset\}\end{aligned}$$

if $X = \{a\}$, then

$$\begin{aligned}
\mathcal{J}_{\mathbb{K}}(\{a\}) &= Y(\emptyset) \cup Y(\{a\}) \\
&= [\{A \subseteq M : A^{\mathrm{I}} \subseteq a', \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}} = \emptyset\} \cup \{A \subseteq M : A^*_{\mathbb{K}} \neq \emptyset\}] \\
&\cup [\{A \cup \{a\} | A \subseteq M, (A \cup \{a\})^*_{/\mathbb{K}} \neq \emptyset\}] \\
&= [\{A \subseteq M : A^*_{\mathbb{K}} = \emptyset, A^{\mathrm{I}} \subseteq a' and \forall x \in A, (A\backslash\{x\})^{\mathrm{I}} \nsubseteq a'\} \cup \{A \subseteq M : A^*_{/\mathbb{K}} \neq \emptyset\}] \\
&\cup [\{A \cup \{a\} | A \subseteq M, (A \cup \{a\})^*_{/\mathbb{K}} \neq \emptyset\}] \\
&= \mathcal{J}_{\mathbb{K}}(\emptyset) \cup \{A \cup \{a\} | A \subseteq M, (A \cup \{a\})^*_{/\mathbb{K}} \neq \emptyset\} \\
&= Impl_{\mathbb{K}} \cup \mathcal{I}_{\mathbb{K}}(a).
\end{aligned}$$

The following proposition holds:

Proposition 4.2.2. *Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context, X be a nonempty set of attributes not belonging to M and $\mathbb{K}^X = (G, M \cup X, \mathrm{I}^X)$. We have*

$$\{A \in \mathcal{P}(M \cup X) : A^*_{\mathbb{K}^X} \neq \emptyset\} = \mathcal{J}_{\mathbb{K}}(X)$$

Proof. Let A be an element of $\mathcal{P}(M)$ such that $A \cup Q \in \mathcal{J}_{\mathbb{K}}(X)$ for some $Q \in \mathcal{P}(X)$.

If $(A \cup Q)^*_{/\mathbb{K}} \neq \emptyset$, then $A \cup Q \in \mathcal{P}(M \cup X)$ and $(A \cup Q)^*_{\mathbb{K}^X} = (A \cup Q)^{\mathrm{I}^X \mathrm{I}^X} \backslash [A \cup Q \cup \bigcup_{y \in Q} (A^{\mathrm{I}} \cap (Q\backslash\{y\})')^{\mathrm{I}^X} \cup \bigcup_{x \in A} ((A\backslash\{x\})^{\mathrm{I}} \cap Q')^{\mathrm{I}^X}]$. Let $u \in (A \cup Q)^*_{/\mathbb{K}}$. Then $u \in (A^{\mathrm{I}} \cap Q')^{\mathrm{I}} \subseteq (A \cup Q)^{\mathrm{I}^X \mathrm{I}^X}$. Moreover, $u \notin A \cup Q$. Let suppose that there exists $y \in Q$ such that $u \in (A^{\mathrm{I}} \cap (Q\backslash\{y\})')^{\mathrm{I}^X}$. Then $(A^{\mathrm{I}} \cap (Q\backslash\{y\})') \subseteq u^{\mathrm{I}^X} = u^{\mathrm{I}}$, meaning that $u \in (A^{\mathrm{I}} \cap (Q\backslash\{y\})')^{\mathrm{I}}$, which is absurd. In the same way, if there is $x \in A$ such that $u \in ((A\backslash\{x\})^{\mathrm{I}} \cap Y')^{\mathrm{I}^X}$, then $u \in ((A\backslash\{x\})^{\mathrm{I}} \cap Y')^{\mathrm{I}}$ which is absurd. Therefore, $u \in (A \cup Q)^*_{\mathbb{K}^X}$, which means that $(A \cup Q)^*_{\mathbb{K}^X} \neq \emptyset$.

If $(A \cup Q)^*_{/\mathbb{K}} = \emptyset$, then $\mathcal{R}_A(Q) \neq \emptyset$. Therefore, $(A \cup Q)^*_{\mathbb{K}^X} = \mathcal{R}_A(Q) \neq \emptyset$.

Therefore, $\mathcal{J}_{\mathbb{K}}(X) \subseteq \{A \in \mathcal{P}(M \cup X) : A^*_{\mathbb{K}^X} \neq \emptyset\}$.

Let A be an element of $\mathcal{P}(M \cup X)$ such that $A^*_{\mathbb{K}^X} \neq \emptyset$. Then there exists $B \subseteq M$ and $Q \subseteq X$ such that $A = B \cup Q$, and $(B \cup Q)^*_{\mathbb{K}^X} \neq \emptyset$. If $(B \cup Q)^*_{/\mathbb{K}} \neq \emptyset$, then $A \in \{A \cup Q | A \subseteq M, Q \in \mathcal{P}(X), (A \cup Q)^*_{/\mathbb{K}} \neq \emptyset\} \subseteq \mathcal{J}_{\mathbb{K}}(X)$. If $(B \cup Q)^*_{/\mathbb{K}} = \emptyset$, then $\mathcal{R}_B(Q) \neq \emptyset$ and $\mathcal{F}_B(X) \neq \emptyset$, meaning that $B \cup Q \in \{A \cup Q | A \subseteq M, Q \in \mathcal{P}(Y), \mathcal{F}_A(Q) \neq \emptyset, \mathcal{R}_A(Q) \neq \emptyset, (A \cup Q)^*_{/\mathbb{K}} = \emptyset\} \subseteq \mathcal{J}_{\mathbb{K}}(X)$.

Therefore, $\{A \in \mathcal{P}(M \cup X) : A^*_{\mathbb{K}^X} \neq \emptyset\} \subseteq \mathcal{J}_{\mathbb{K}}(X)$ ■

Remark 4.2.1. *One can observe that for a formal context $\mathbb{K} = (G, M, \mathrm{I})$ and a nonempty set X of attributes not belonging to M,*
$\mathcal{J}_{\mathbb{K}}(X) = Impl_{\mathbb{K}} \cup [\bigcup_{Q \in \mathcal{P}(X)\backslash\{\emptyset, X\}} \{A \cup Q | A^{\mathrm{I}} \cap Q' \subseteq X'_Q, \mathcal{R}_A(Q) \neq \emptyset, (A \cup Q)_{/\mathbb{K}} = \emptyset\} \cup \{A \cup X | A \subseteq M, (A \cup X)^*_{/\mathbb{K}} \neq \emptyset\}] \cup [\bigcup_{Q \in \mathcal{P}(X)\backslash\{\emptyset, X\}} \{A; A^{\mathrm{I}} \subseteq X', \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}} = \emptyset\} \cup \{A \cup Q | A \subseteq M, (A \cup Q)^*_{/\mathbb{K}} \neq \emptyset\}]$.

That leads us to the following result:

Corollary 4.2.1. *Let* $\mathbb{K} = (G, M, \mathrm{I})$ *be a formal context and* X *a nonempty set of attributes not belonging to* M*. We set* $\mathbb{K}^X = (G, M \cup X, \mathrm{I}^X)$*,* $X' = \bigcap_{x \in X} x'$ *and* $\mathcal{J}^*_{\mathbb{K}}(X) = \{A \subseteq M : A^{\mathrm{I}} \subseteq X'$ *and* $\mathcal{R}_A(\emptyset) \neq \emptyset$ *and* $A^*_{\mathbb{K}} = \emptyset\} \cup [\mathcal{J}_{\mathbb{K}}(X) \backslash \mathcal{J}_{\mathbb{K}}(\emptyset)]$*. We have also*

$$Impl_{\mathbb{K}^X} = Impl_{\mathbb{K}} \cup \mathcal{J}^*_{\mathbb{K}}(X)$$

4.3 From the initial implications to the ∃-generalized implications

Let consider a formal context $\mathbb{K} = (G, M, \mathrm{I})$, two attributes a and b of $\mathbb{K}$ and $s = a \cup b$ the generalized attribute obtained from the ∃-generalization of a and b. Let $\mathbb{K}^s$ be the context $(G, M \cup \{s\}, \mathrm{I}^s)$ and $\mathbb{K}^s_{ab} = (G, (M \setminus \{a, b\}) \cup \{s\}, \mathrm{I}^s_{ab})$ be the generalized context. Our aim is to compare the sizes of $Impl(\mathbb{K})$ and $Impl(\mathbb{K}^s_{ab})$.

We set $\mathbb{K}_{ab} := (G, M_{ab}, \mathrm{I}_{ab})$, where $M_{ab} = M \backslash \{a, b\}$.

Then from proposition 4.2.1,

$$Impl_{\mathbb{K}^s_{ab}} = Impl_{\mathbb{K}_{ab}} \cup \mathcal{I}_{\mathbb{K}_{ab}}(s) = Impl_{\mathbb{K}_{ab}} \cup \mathcal{I}_{\mathbb{K}_{ab}}(a \cup b)$$

Moreover, also from proposition 4.2.1, we have $Impl_{\mathbb{K}} = Impl_{\mathbb{K}_b} \cup \mathcal{I}_{\mathbb{K}_b}(b) = Impl_{\mathbb{K}_{ab}} \cup \mathcal{I}_{\mathbb{K}_{ab}}(a) \cup \mathcal{I}_{\mathbb{K}_b}(b)$. Now we express $\mathcal{I}_{\mathbb{K}_b}(b)$ as a function of $\mathcal{I}_{\mathbb{K}_{ab}}(b)$.

From the definition of $\mathcal{I}_{\mathbb{K}}(b)$ with $\mathbb{K} = \mathbb{K}_b$, we have

$\mathcal{I}_{\mathbb{K}_b}(b) = \{A \subseteq M_{ab} \cup \{a\} | A^{\mathrm{I}^a_{ab}} \subseteq b', A \in E_{\{M_{ab} \cup \{a\}, b\}}, A^*_{\mathbb{K}^a_{ab}} = \emptyset\} \cup \{A \cup \{b\} | A \subseteq M_{ab} \cup \{a\}, (A^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}} \backslash (A^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}} \cup \bigcup_{x \in A} ((A \backslash \{x\})^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}}) \neq \emptyset\} = \mathcal{F}_1(b) \cup \mathcal{F}_2(b)$.

We express each member $\mathcal{F}_1(b)$ and $\mathcal{F}_2(b)$ of the above union as a function of I_{ab}, a' and b'.

But the fact that $A^*_{\mathbb{K}^a_{ab}} = \emptyset$ implies that $a \in A^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}}$ if and only if $\exists x \in A$ such that $a \in (A \backslash \{x\})^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}}$. In fact, let suppose that $A^*_{\mathbb{K}^a_{ab}} = \emptyset$. If $\exists x \in A$ such that $a \in (A \backslash \{x\})^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}}$, we obviously have $a \in A^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}}$. Now we suppose that $a \in A^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}}$. If $\forall x \in A$, $a \notin (A \backslash \{x\})^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}}$, then $A^*_{\mathbb{K}^a_{ab}} = A^*_{\mathbb{K}_{ab}} \cup \{a\}$, meaning that $a \in A^*_{\mathbb{K}^a_{ab}}$, which is absurd. Therefore, $\exists x \in A$, such that $a \in (A \backslash \{x\})^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}}$. From there, we conclude that $A^*_{\mathbb{K}^a_{ab}} = \emptyset$ implies that $A^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}} \backslash (A \cup \bigcup_{x \in A} (A \backslash \{x\})^{\mathrm{I}^a_{ab} \mathrm{I}^a_{ab}} = A^{\mathrm{I}_{ab} \mathrm{I}_{ab}} \backslash (A \cup \bigcup_{x \in A} (A \backslash \{x\})^{\mathrm{I}_{ab} \mathrm{I}_{ab}}$.

Hence, since $A \subseteq M_{ab} \cup \{a\}$ implies that $a \in A$ (meaning that $A \subseteq M_{ab}$) or $a \notin A$ (meaning that $A = B \cup \{a\}$ with $B \subseteq M_{ab}$), we have

$$\begin{aligned}
\mathcal{F}_1(b) &= \{A \subseteq M_{ab} \cup \{a\} | A^{\mathrm{I}^a_{ab}} \subseteq b', A \in E_{\{M_{ab} \cup \{a\}, b\}}, A^*_{\mathbb{K}^a_{ab}} = \emptyset\} \\
&= \{A \subseteq M_{ab} | A^{\mathrm{I}^a_{ab}} \subseteq b', A \in E_{\{M_{ab}, b\}}, A^*_{\mathbb{K}^a_{ab}} = \emptyset\} \\
&\cup \{B \cup \{a\} | B \subseteq M_{ab}, (B \cup \{a\})^{\mathrm{I}^a_{ab}} \subseteq b', B \cup \{a\} \in E_{\{M_{ab} \cup \{a\}, b\}}, (B \cup \{a\})^*_{\mathbb{K}^a_{ab}} = \emptyset\}.
\end{aligned}$$

Each member of the above union in $\mathcal{F}_1(b)$ is expressed as a function of I_{ab}, a' and b'. First, the development of $\{A \subseteq M_{ab} | A^{\mathrm{I}^a_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^*_{\mathbb{K}^a_{ab}} = \emptyset\}$ gives the set $\{A \subseteq M_{ab} | A^{\mathrm{I}_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^*_{\mathbb{K}_{ab}} = \emptyset\}$ as expressed below:

$\{A \subseteq M_{ab} | A^{\mathrm{I}^a_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^*_{\mathbb{K}^a_{ab}} = \emptyset\}$
$= \{A \subseteq M_{ab} | A^{\mathrm{I}^a_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \backslash (A \cup \bigcup_{x\in A}(A\backslash\{x\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}}) = \emptyset\}$
(because $A^*_{\mathbb{K}^a_{ab}} = A^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \backslash (A \cup \bigcup_{x\in A}(A\backslash\{x\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}})$
$=\{A \subseteq M_{ab} | A^{\mathrm{I}_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^{\mathrm{I}_{ab}\mathrm{I}_{ab}} \backslash (A \cup \bigcup_{x\in A}(A\backslash\{x\})^{\mathrm{I}_{ab}\mathrm{I}_{ab}}) = \emptyset\}$
(Since $A^*_{\mathbb{K}^a_{ab}} = \emptyset$ and then $A^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \backslash (A \cup \bigcup_{x\in A}(A\backslash\{x\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}}) = A^{\mathrm{I}_{ab}\mathrm{I}_{ab}} \backslash (A \cup \bigcup_{x\in A}(A\backslash\{x\})^{\mathrm{I}_{ab}\mathrm{I}_{ab}})$
$=\{A \subseteq M_{ab} | A^{\mathrm{I}_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^*_{\mathbb{K}_{ab}} = \emptyset\}$.

In the same way, from $\{B \cup \{a\} | B \subseteq M_{ab}, (B \cup \{a\})^{\mathrm{I}^a_{ab}} \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B \cup \{a\})^*_{\mathbb{K}^a_{ab}} = \emptyset\}$, we obtain the following set $\{B \cup \{a\} | B \subseteq M_{ab}, B^{\mathrm{I}_{ab}} \cap a' \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}_{ab}} \backslash [B^{\mathrm{I}_{ab}\mathrm{I}_{ab}} \cup \bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}_{ab}}] = \emptyset\}$ as developed bellow:

$\{B \cup \{a\} | B \subseteq M_{ab}, (B \cup \{a\})^{\mathrm{I}^a_{ab}} \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B \cup \{a\})^*_{\mathbb{K}^a_{ab}} = \emptyset\}$
$= \{B \cup \{a\} | B \subseteq M_{ab}, B^{\mathrm{I}_{ab}} \cap a' \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B \cup \{a\})^*_{\mathbb{K}^a_{ab}} = \emptyset\}$
(because $(B \cup \{a\})^{\mathrm{I}^a_{ab}} = B^{\mathrm{I}_{ab}} \cap a^{\mathrm{I}^a_{ab}} = B^{\mathrm{I}_{ab}} \cap a'$).
$=\{B \cup \{a\} | B \subseteq M_{ab}, B^{\mathrm{I}_{ab}} \cap a' \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B \cup \{a\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \backslash [(B \cup \{a\}) \cup \bigcup_{x\in B\cup\{a\}}(B \cup \{a\}\backslash\{x\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}}] = \emptyset\}$
(because $(B \cup \{a\})^*_{\mathbb{K}^a_{ab}} = (B \cup \{a\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \backslash [(B \cup \{a\}) \cup \bigcup_{x\in B\cup\{a\}}(B \cup \{a\}\backslash\{x\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}}]$).
$=\{B \cup \{a\} | B \subseteq M_{ab}, B^{\mathrm{I}_{ab}} \cap a' \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}^a_{ab}} \backslash [(B^{\mathrm{I}_{ab}\mathrm{I}_{ab}} \cup \{a\}) \cup \bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}^a_{ab}}] = \emptyset\}$
(since $(B \cup \{a\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \backslash [(B \cup \{a\}) \cup \bigcup_{x\in B\cup\{a\}}(B \cup \{a\}\backslash\{x\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}}] = (B^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}^a_{ab}} \backslash [(B^{\mathrm{I}_{ab}\mathrm{I}_{ab}} \cup \{a\}) \cup \bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}^a_{ab}}]$)
$=\{B \cup \{a\} | B \subseteq M_{ab}, B^{\mathrm{I}_{ab}} \cap a' \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}_{ab}} \backslash [B^{\mathrm{I}_{ab}\mathrm{I}_{ab}} \cup \bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}_{ab}}] = \emptyset\}$.
(since $a \in (B^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}^a_{ab}}$ and $a \in \bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}^a_{ab}}$).

Hence, the two sets above lead us to the following:

$$\mathcal{F}_1(b) = \{A \subseteq M_{ab} | A^{\mathrm{I}_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^*_{\mathbb{K}_{ab}} = \emptyset\} \cup \{B \cup \{a\} | B \subseteq M_{ab}, B^{\mathrm{I}_{ab}} \cap a' \subseteq b', B \cup \{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}_{ab}} \backslash [B^{\mathrm{I}_{ab}\mathrm{I}_{ab}} \cup \bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}} \cap a')^{\mathrm{I}_{ab}}] = \emptyset\}.$$

In the same way, $\mathcal{F}_2(b)$ is also developed as a function of I_{ab}, a' and b', as follow:

$\mathcal{F}_2(b) = \{A \cup \{b\} | A \subseteq M_{ab} \cup \{a\}, (A^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}} \backslash (A^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \cup \bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}}) \neq \emptyset\} = \{A \cup \{b\} | A \subseteq M_{ab}, (A^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}} \backslash (A^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \cup \bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}}) \neq \emptyset\} \cup \{B \cup$

$\{a,b\}|B \subseteq M_{ab}, ((B\cup\{a\})^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}} \backslash ((B\cup\{a\})^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}} \cup \bigcup_{x\in B\cup\{a\}}(((B\cup\{a\})\backslash\{x\})^{\mathrm{I}^a_{ab}} \cap b')^{\mathrm{I}^a_{ab}}) \neq \emptyset\}$. Since $A \subseteq M_{ab}\cup\{a\}$, the two members of the above union in $\mathcal{F}_2(b)$ represent the case where $a \in A$ and the case where $a \notin A$.

Each member of the above union constituted by $\mathcal{F}_2(b)$ is expressed as a function of I_{ab}, a' and b'. The case where $a \in A$ gives
$\{A\cup\{b\}|A \subseteq M_{ab}, (A^{\mathrm{I}^a_{ab}}\cap b')^{\mathrm{I}^a_{ab}}\backslash(A^{\mathrm{I}^a_{ab}\mathrm{I}^a_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}^a_{ab}}\cap b')^{\mathrm{I}^a_{ab}}) \neq \emptyset\} = \{A\cup\{b\}|A \subseteq M_{ab}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) \neq \emptyset\}\cup\{A\cup\{b\}|A\subseteq M_{ab}, A^{\mathrm{I}_{ab}}\cap b' \subseteq a', A\cup\{b\} \in E_{\{M_{ab}\cup\{b\},a\}}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) = \emptyset\}$.

Expressing the second member ($a \notin A$) of the union in $\mathcal{F}_2(b)$ as a function of I_{ab}, a' and b', we obtain:

$\{B\cup\{a,b\}|B \subseteq M_{ab}, (B^{\mathrm{I}^a_{ab}}\cap a'\cap b')^{\mathrm{I}^a_{ab}}\backslash((B^{\mathrm{I}^a_{ab}}\cap a')^{\mathrm{I}^a_{ab}}\cup(B^{\mathrm{I}^a_{ab}}\cap b')^{\mathrm{I}^a_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}^a_{ab}}\cap a'\cap b')^{\mathrm{I}^a_{ab}} \neq \emptyset\} = \{B\cup\{a,b\}|B \subseteq M_{ab}, (B^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}}\backslash((B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\cup(B^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}} \neq \emptyset\}$, because $a,b \in (B^{\mathrm{I}^a_{ab}}\cap a'\cap b')^{\mathrm{I}^a_{ab}}$ and $a,b \in \bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}^a_{ab}}\cap a'\cap b')^{\mathrm{I}^a_{ab}}\}$.

Therefore,

$\mathcal{F}_2(b) = \{A\cup\{b\}|A \subseteq M_{ab}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) \neq \emptyset\}\cup\{A\cup\{b\}|A\subseteq M_{ab}, A^{\mathrm{I}_{ab}}\cap b' \subseteq a', A\cup\{b\} \in E_{\{M_{ab}\cup\{b\},a\}}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) = \emptyset\}\cup\{B\cup\{a,b\}|B \subseteq M_{ab}, (B^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}}\backslash((B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\cup(B^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}} \neq \emptyset\}$.

From $\mathcal{F}_1(b)$ and $\mathcal{F}_2(b)$, $\mathcal{I}_{\mathbb{K}_b}(b)$ is deduced as follow:

$\mathcal{I}_{\mathbb{K}_b}(b) = \mathcal{F}_1(b)\cup\mathcal{F}_2(b) = [\{A \subseteq M_{ab}|A^{\mathrm{I}_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^*_{\mathbb{K}_{ab}} = \emptyset\}\cup\{B\cup\{a\}|B \subseteq M_{ab}, B^{\mathrm{I}_{ab}}\cap a' \subseteq b', B\cup\{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\backslash[B^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}] = \emptyset\}]\cup[\{A\cup\{b\}|A \subseteq M_{ab}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) \neq \emptyset\}\cup\{A\cup\{b\}|A\subseteq M_{ab}, A^{\mathrm{I}_{ab}}\cap b' \subseteq a', A\cup\{b\} \in E_{\{M_{ab}\cup\{b\},a\}}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) = \emptyset\}\cup\{B\cup\{a,b\}|B \subseteq M_{ab}, (B^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}}\backslash((B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\cup(B^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}} \neq \emptyset\}]$,

and because $\mathcal{I}_{\mathbb{K}}(b) = \{A \subseteq M_{ab}|A^{\mathrm{I}_{ab}} \subseteq b', A \in E_{\{M_{ab},b\}}, A^*_{\mathbb{K}_{ab}} = \emptyset\}\cup\{A\cup\{b\}|A \subseteq M_{ab}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) \neq \emptyset\}$, we have

$\mathcal{I}_{\mathbb{K}_b}(b) = \mathcal{I}_{\mathbb{K}}(b)\cup$
$\{B\cup\{a\}|B \subseteq M_{ab}, B^{\mathrm{I}_{ab}}\cap a' \subseteq b', B\cup\{a\} \in E_{\{M_{ab}\cup\{a\},b\}}, (B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\backslash[B^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}] = \emptyset\}\cup$
$\{A\cup\{b\}|A\subseteq M_{ab}, A^{\mathrm{I}_{ab}}\cap b' \subseteq a', A\cup\{b\} \in E_{\{M_{ab}\cup\{b\},a\}}, (A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}) = \emptyset\}\cup$

$\{B\cup\{a,b\}|B\subseteq M_{ab},(B^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}}\backslash((B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\cup(B^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}}\neq\emptyset\}$,

which leads to:

$$\mathcal{I}_{\mathbb{K}_b}(b)=\mathcal{I}_{\mathbb{K}}(b)\cup\mathcal{R}_{\mathbb{K}}(a,b),$$

where
$\mathcal{R}_{\mathbb{K}}(a,b)=\{B\cup\{a\}|B\subseteq M_{ab},B^{\mathrm{I}_{ab}}\cap a'\subseteq b',B\cup\{a\}\in E_{\{M_{ab}\cup\{a\},b\}},(B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\backslash[B^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}]=\emptyset\}\cup$
$\{A\cup\{b\}|A\subseteq M_{ab},A^{\mathrm{I}_{ab}}\cap b'\subseteq a',A\cup\{b\}\in E_{\{M_{ab}\cup\{b\},a\}},(A^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\backslash(A^{\mathrm{I}_{ab}\mathrm{I}_{ab}}\cup\bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}})=\emptyset\}\cup$
$\{B\cup\{a,b\}|B\subseteq M_{ab},(B^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}}\backslash((B^{\mathrm{I}_{ab}}\cap a')^{\mathrm{I}_{ab}}\cup(B^{\mathrm{I}_{ab}}\cap b')^{\mathrm{I}_{ab}}\cup\bigcup_{x\in B}((B\backslash\{x\})^{\mathrm{I}_{ab}}\cap a'\cap b')^{\mathrm{I}_{ab}}\neq\emptyset\}$.

We conclude that

$$Impl_{\mathbb{K}}=Impl_{\mathbb{K}_{ab}}\cup\mathcal{I}_{\mathbb{K}_{ab}}(a)\cup\mathcal{I}_{\mathbb{K}_b}(b)=Impl_{\mathbb{K}_{ab}}\cup\mathcal{I}_{\mathbb{K}_{ab}}(a)\cup\mathcal{I}_{\mathbb{K}_{ab}}(b)\cup\mathcal{R}_{\mathbb{K}_{ab}}(a,b)$$

and the following proposition holds:

Proposition 4.3.1. *Let $\mathbb{K}=(G,M,\mathrm{I})$ be a formal context and a and b two attributes of $\mathbb{K}$. Let $\mathbb{K}^s_{ab}=(G,(M\setminus\{a,b\})\cup\{s\},\mathrm{I}^s_{ab})$ be the generalized context after the $\exists$-generalization of a and b, with $s=a\cup b$. Then the increase in the number of informative attributes implications after the $\exists$-generalization of a and b is given by:*

$$|Impl(\mathbb{K}^s_{ab})|-|Impl(\mathbb{K})|=|\mathcal{I}_{\mathbb{K}_{ab}}(a\cup b)|-|\mathcal{I}_{\mathbb{K}_{ab}}(a)\cup\mathcal{I}_{\mathbb{K}_{ab}}(b)\cup\mathcal{R}_{\mathbb{K}_{ab}}(a,b)|$$

Example 4.3.1. *We consider the following formal context, in which a and c are put together through $\exists$-generalization. The aim of this example is to determine the variation of informative attribute implications between the initial context and the generalized context without constructing the generalized context.*

$\mathbb{K}_{\mathcal{B}_4}$	a	b	c	d
1	×	×		
2		×		
3			×	×
4				×

Removing the attributes a and c been generalized from $\mathbb{K}_{\mathcal{B}_4}$ yields the context $\mathbb{K}_{00}=\mathbb{K}_{ac}=(G,M\setminus\{a,c\},\mathrm{I}_{ac})$.

$\mathbb{K}_{ac}=\mathbb{K}_{00}$	b	d
1	×	
2	×	
3		×
4		×

We have $\{A \subseteq M_{ac} | A^{\mathrm{I}_{ac}} \subseteq a', A \in E_{\{M_{ac},a\}}, A^*_{\mathbb{K}_{00}} = \emptyset\} = \{\{b,d\}\}$*; and* $\{A \cup \{a\} | A \subseteq M_{ac}, (A^{\mathrm{I}_{ac}} \cap a')^{\mathrm{I}_{ac}} \backslash (A^{\mathrm{I}_{ac}\mathrm{I}_{ac}} \cup \bigcup_{x \in A}((A \backslash \{x\})^{\mathrm{I}_{ac}} \cap a')^{\mathrm{I}_{ac}}) \neq \emptyset\} = \{\{a\}\}$.

Therefore,

$$\mathcal{I}_{\mathbb{K}_{ac}}(a) = \{\{b,d\}, \{a\}\}$$

Moreover, $\{A \subseteq M_{ac} | A^{\mathrm{I}_{ac}} \subseteq c', A \in E_{\{M_{ac},c\}}, A^*_{\mathbb{K}_{00}} = \emptyset\} = \{\{b,d\}\}$*; and* $\{A \cup \{c\} | A \subseteq M_{ac}, (A^{\mathrm{I}_{ac}} \cap c')^{\mathrm{I}_{ac}} \backslash (A^{\mathrm{I}_{ac}\mathrm{I}_{ac}} \cup \bigcup_{x \in A}((A \backslash \{x\})^{\mathrm{I}_{ac}} \cap c')^{\mathrm{I}_{ac}}) \neq \emptyset\} = \{\{c\}\}$.

Hence,

$$\mathcal{I}_{\mathbb{K}_{ac}}(c) = \{\{b,d\}, \{c\}\}.$$

$\{A \subseteq M_{ac} | A^{\mathrm{I}_{ac}} \subseteq a' \cup c', A \in E_{\{M_{ac},a \cup c\}}, A^*_{\mathbb{K}_{00}} = \emptyset\} = \{\{b,d\}\}$*; and* $\{A \cup \{a \cup c\} | A \subseteq M_{ac}, (A^{\mathrm{I}_{ac}} \cap (a' \cup c'))^{\mathrm{I}_{ac}} \backslash (A^{\mathrm{I}_{ac}\mathrm{I}_{ac}} \cup \bigcup_{x \in A}((A \backslash \{x\})^{\mathrm{I}_{ac}} \cap a' \cup c')^{\mathrm{I}_{ac}}) \neq \emptyset\} = \emptyset$.

Therefore,

$$\mathcal{I}_{\mathbb{K}_{ac}}(a \cup c) = \{\{b,d\}\}.$$

Also,

$$\mathcal{R}_{\mathbb{K}_{ac}}(a,c) = \{\{d,a\}, \{b,c\}\}.$$

Thus, $\mathcal{I}_{\mathbb{K}_{ac}}(a) \cup \mathcal{I}_{\mathbb{K}_{ac}}(c) \cup \mathcal{R}_{\mathbb{K}_{ac}}(a,c) = \{\{b,d\},\{a\}\} \cup \{\{b,d\},\{c\}\} \cup \{\{d,a\},\{b,c\},\}$ $= \{\{a\},\{c\},\{b,d\},\{d,a\},\{b,c\}\}$.

Hence,

$$|\mathcal{I}_{\mathbb{K}_{ac}}(a) \cup \mathcal{I}_{\mathbb{K}_{ac}}(c) \cup \mathcal{R}_{\mathbb{K}_{ac}}(a,c)| = 5, \text{ and } |\mathcal{I}_{\mathbb{K}_{ac}}(a \cup c)| = 1$$

and then

$$|\mathcal{I}_{\mathbb{K}_{ac}}(a \cup c)| - |\mathcal{I}_{\mathbb{K}_{ac}}(a) \cup \mathcal{I}_{\mathbb{K}_{ac}}(c) \cup \mathcal{R}_{\mathbb{K}_{ac}}(a,c)| = 1 - 5 = -4.$$

Therefore, generalizing attributes a and c reduces the size of the set of informative attributes implications in the initial context $\mathbb{K}_{\mathcal{B}_4}$ *by* 4.

In general, if one or several ($n \geqslant 2$) groups of attributes $(X_i)_{i\in\{1,\ldots,n\}}$ of $\mathbb{K}$ are simultaneously put together through $\exists$-generalization, and if we set $X = \cup_{i\in\{1,\ldots,n\}} X_i$ with X_i been seen as a generalized attribute for all i, then we have the following result:

Proposition 4.3.2. *Let* $\mathbb{K} = (G, M, \mathrm{I})$ *be a formal context and* X *a set of attributes of* $\mathbb{K}$. *Let* $\mathbb{K}_X^{\{X_1,\ldots,X_n\}} = (G, (M \setminus X) \cup \{X_1, \ldots, X_n\}, \mathrm{I}_X^{\{X_1,\ldots,X_n\}})$ *be the generalized context after the simultaneous* $\exists$*-generalization of groups of attributes* $(X_i)_{i\in\{1,\ldots,n\}}$ *of* $\mathbb{K}$, *with* $X_i = \cup_{x \in X_i} x$ *been seen as a generalized attribute for all* $i \in \{1, \ldots, n\}$ *and* $X = \bigcup_{i\in\{1,\ldots,n\}} X_i$. *Then the increase in the number of informative implications is given by*

$$|Impl(\mathbb{K}_X^{\{X_1,\ldots,X_n\}})| - |Impl(\mathbb{K})| = |\mathcal{J}^*_{\mathbb{K}_X}(\{X_1, \ldots, X_n\})| - |\mathcal{J}^*_{\mathbb{K}_X}(X)|$$

Proof. This is the immediate consequence of the corollary 4.2.1. ■

In [34], an algorithm is given, and permit to extract generalized association rules that have a user-specified minimum support. These generalized rules included both the generalized informative and non informative association rules. That is not the case for generalized implications as studied here, because no minimum support is specified, and only informative generalized implications do matter.

The following naive algorithm can be of help to extract informative generalized attributes implications (contained in $\mathcal{J}^*_{\mathbb{K}}(\{Y_1, ..., Y_n\})$) from a formal context.

Algorithm 2: Computing generalized informative implications

Data: An attribute reduced context (G, M, I)

$X :=$ the set of all the attributes being generalized
$Y :=$ the set of all the generalized attributes
$M_{00} := M \backslash X$
$\mathbb{K}_{00} := (G, M_{00}, \mathrm{I}_{00})$
$\mathrm{J}_0 := \emptyset$,
$\mathrm{J}_1 := \emptyset$,
$\mathrm{J}_2 := \emptyset$
foreach $A \in \mathcal{P}(M_{00})$ **do**
  foreach $Q \in \mathcal{P}(Y)$ **do**
    $L := (A \cup Q)^{*}_{/\mathbb{K}}$
    if $L \neq \emptyset$ **then**
    $\mathrm{J}_1 := \mathrm{J}_1 \cup \{A \cup Q\}$
    ;
    if $L = \emptyset$ **then**
    $\mathcal{F} := \emptyset$
    $\mathcal{R} := \emptyset$
    $T := Y \backslash Q$
    foreach $v \in T$ **do**
      if $A^{\mathrm{I}} \cap Q' \subseteq v'$ **then**
      $\mathcal{F} := \mathcal{F} \cup \{v\}$
      $r := 0$
      $s := 0$
      foreach $x \in A$ **do**
        if $(A \backslash \{x\})^{\mathrm{I}} \cap Q' \nsubseteq v'$ **then**
        $r := r + 1$;
      end
      foreach $y \in Q$ **do**
        if $A^{\mathrm{I}} \cap (Q \backslash \{y\})' \nsubseteq v'$ **then**
        $s := s + 1$;
      end
      if $r = |A|$ *and* $s = |Q|$ **then**
      $\mathcal{R} := \mathcal{R} \cup \{v\}$;
      ;
    end
    if $\mathcal{R} \neq \emptyset$ **then**
    $\mathrm{J}_2 := \mathrm{J}_2 \cup \{A \cup Q\}$;
    ;
  end
end
$\mathrm{J}_0 := \mathrm{J}_1 \cup \mathrm{J}_2$

The time complexity of our algorithm is given by

$$f = [(|A|+|Q|+3+1)|T|+1+3+1+1]2^{|Y|}2^{|M_{00}|}+8 = [(|A|+|Q|+4)|T|+6]2^{|Y|+|M_{00}|}+8$$

If one supposes that f can be expressed as a function of $n = |G||M|$, then we can write $f = f(n)$.

However, $[(|A|+|Q|+4)|T|+6]2^{|Y|+|M_{00}|}+8 \leqslant [(2^{|M_{00}|}+2^{|Y|}+4)2^{|Y|}+6]2^{|Y|+|M_{00}|}+8$
$\leqslant (2^{|Y|+|M_{00}|} + 2^{2|Y|} + 4.2^{|Y|} + 6)2^{|Y|+|M_{00}|} + 8$
$\leqslant 2^{2|Y|+2|M_{00}|} + 2^{3|Y|+|M_{00}|} + 4.2^{2|Y|+|M_{00}|} + 14.2^{2|Y|+|M_{00}|}$
$\leqslant 2^{3|Y|+2|M_{00}|} + 2^{3|Y|+2|M_{00}|} + 4.2^{3|Y|+2|M_{00}|} + 14.2^{3|Y|+2|M_{00}|}$
$\leqslant 20.2^{3|Y|+2|M_{00}|}$

There exists $k_0 \in \mathbb{N}^*\backslash\{1\}$ such that $3|Y| + 2|M_{00}| \leqslant k_0.|M_{00}|$.

Hence,

$$3|Y| + 2|M_{00}| \leqslant k_0.|M_{00}| \leqslant k_0.|M| \leqslant k_0.|G||M| \leqslant k_0.n.$$

with $n = |G||M|$. Thus

$$f(n) \leqslant 20.2^{3|Y|+2|M_{00}|} \leqslant 20.2^{k_0.n},$$

and then

$$f(n) = \mathcal{O}(2^{k_0.n}),$$

Example 4.3.2. *Let reconsider the previous example with the formal context,*

$K_{\mathcal{B}_4}$	a	b	c	d
1	×	×		
2		×		
3			×	×
4				×

which yielded the following context $\mathbb{K}_{ac} = \mathbb{K}_{00}$ after the $\exists$-generalization of attributes a and c.

$\mathbb{K}_{ac} = \mathbb{K}_{00}$	b	d
1	×	
2	×	
3		×
4		×

Note that $X = \{a, c\}$, $M_{00} = M_{ac} = \{b, d\}$, $\mathcal{P}(M_{ac}) = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$ and $Y = \{a \cup c\}$. Moreover, $a' = \{1\}$, $c' = \{3\}$, $a' \cup c' = \{1, 3\}$ and $X' = 1' \cap 3' = \emptyset$.

$$\mathcal{J}^*_{\mathbb{K}_{00}}(X) = \{A | A \subseteq M_{00}, \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} \cup [\mathcal{J}_{\mathbb{K}_{00}}(X) \backslash \mathcal{J}_{\mathbb{K}_{00}}(\emptyset)]$$

Hence,
$\mathcal{J}^*_{\mathbb{K}_{00}}(\{a, c\}) = \{A | A \subseteq \{b, d\}, \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} \cup [\mathcal{J}_{\mathbb{K}_{00}}(\{a, c\}) \backslash \mathcal{J}_{\mathbb{K}_{00}}(\emptyset)]$.

But, $\{A|A \subseteq \{b,d\}, \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} = \{A|A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} = \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\}$*. Moreover,* $(\emptyset)' = G \neq \emptyset$*,* $b' = \{1,2\} \neq \emptyset$ *and* $d' = \{3,4\} \neq \emptyset$*. Therefore,* $\{b,d\}$ *is the only subset* T *of* M_{00} *such that* $T' = \emptyset$*.* $\mathcal{F}_{\{b,d\}}(\emptyset) = \{v \in \{a,c\}|(\{b,d\})^{\mathrm{I}_{00}} \subseteq v'\} = \{a,c\}$ *and* $\mathcal{R}_{\{b,d\}}(\emptyset) = \{v \in \mathcal{F}_{\{b,d\}}(\emptyset)|b^{\mathrm{I}_{00}} \nsubseteq v'$ *and* $d^{\mathrm{I}_{00}} \nsubseteq v'\} = \{a,c\} \neq \emptyset$*. Also,* $(\{b,d\})^*_{\mathbb{K}_{00}} = \emptyset$ *because* $(\{b,d\})^{\mathrm{I}_{00}\mathrm{I}_{00}} = \{b,d\}$*.*

Therefore $\{A|A \subseteq \{b,d\}, \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} = \{b,d\}$*.*

Moreover,
$\mathcal{J}_{\mathbb{K}_{00}}(\{a,c\}) \backslash \mathcal{J}_{\mathbb{K}_{00}}(\emptyset) = [\{A \cup \{a\}|A \subseteq \{b,d\}, \mathcal{F}_A(\{a\}) \neq \emptyset, \mathcal{R}_A(\{a\}) \neq \emptyset, (A \cup \{a\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a\}|A \subseteq \{b,d\}, (A \cup \{a\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}] \bigcup [\{A \cup \{c\}|A \subseteq \{b,d\}, \mathcal{F}_A(\{c\}) \neq \emptyset, \mathcal{R}_A(\{c\}) \neq \emptyset, (A \cup \{c\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{c\}|A \subseteq \{b,d\}, (A \cup \{c\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}] \bigcup [\{A \cup \{a,c\}|A \subseteq \{b,d\}, \mathcal{F}_A(\{a,c\}) \neq \emptyset, \mathcal{R}_A(\{a,c\}) \neq \emptyset, (A \cup \{a,c\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a,c\}|A \subseteq \{b,d\}, (A \cup \{a,c\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}]$*.*

But each of the three blocs of the above equality is developed as follow:

$[\{A \cup \{a\}|A \subseteq \{b,d\}, \mathcal{F}_A(\{a\}) \neq \emptyset, \mathcal{R}_A(\{a\}) \neq \emptyset, (A \cup \{a\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a\}|A \subseteq \{b,d\}, (A \cup \{a\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}] = [\{A \cup \{a\}|A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} \cap a' \subseteq c', \mathcal{R}_A(\{a\}) \neq \emptyset, (A \cup \{a\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a\}|A \subseteq \{b,d\}, (A^{\mathrm{I}_{00}} \cap a')^{\mathrm{I}_{00}} \backslash (A^{\mathrm{I}_{00}\mathrm{I}_{00}} \cup \bigcup_{x \in A}((A \backslash \{x\})^{\mathrm{I}_{00}} \cap a')^{\mathrm{I}_{00}}) \neq \emptyset\}] = \{\{d,a\}\} \cup \{\{a\}\} = \{\{d,a\},\{a\}\}$*, because* $\mathcal{F}_\emptyset(\{a\}) = \emptyset$*,* $\mathcal{F}_{\{b\}}(\{a\}) = \emptyset$*,* $\mathcal{F}_{\{d\}}(\{a\}) = \{c\}$*,* $\mathcal{F}_{\{b,d\}}(\{a\}) = \{c\}$*,* $\mathcal{R}_{\{d\}}(\{a\}) = \{c\}$*,* $\mathcal{R}_{\{b,d\}}(\{a\}) = \emptyset$ *and* $(\emptyset \cup \{a\})^*_{/\mathbb{K}_{00}} = \{b\} \neq \emptyset$*;*

$[\{A \cup \{c\}|A \subseteq \{b,d\}, \mathcal{F}_A(\{c\}) \neq \emptyset, \mathcal{R}_A(\{c\}) \neq \emptyset, (A \cup \{c\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{c\}|A \subseteq \{b,d\}, (A \cup \{c\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}] = [\{A \cup \{c\}|A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} \cap c' \subseteq a', \mathcal{R}_A(\{c\}) \neq \emptyset, (A \cup \{c\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{c\}|A \subseteq \{b,d\}, (A^{\mathrm{I}_{00}} \cap c')^{\mathrm{I}_{00}} \backslash (A^{\mathrm{I}_{00}\mathrm{I}_{00}} \cup \bigcup_{x \in A}((A \backslash \{x\})^{\mathrm{I}_{00}} \cap c')^{\mathrm{I}_{00}}) \neq \emptyset\}] = \{\{b,c\}\} \cup \{\{c\}\} = \{\{b,c\},\{c\}\}$*, because* $\mathcal{F}_\emptyset(\{c\}) = \emptyset$*,* $\mathcal{F}_{\{b\}}(\{c\}) = \{a\}$*,* $\mathcal{F}_{\{d\}}(\{c\}) = \emptyset$*,* $\mathcal{F}_{\{b,d\}}(\{c\}) = \{a\}$*,* $\mathcal{R}_{\{b\}}(\{c\}) = \{a\}$*,* $\mathcal{R}_{\{b,d\}}(\{c\}) = \emptyset$ *and* $(\emptyset \cup \{c\})^*_{/\mathbb{K}_{00}} = \{d\} \neq \emptyset$*;*
and

$[\{A \cup \{a,c\}|A \subseteq \{b,d\}, \mathcal{F}_A(\{a,c\}) \neq \emptyset, \mathcal{R}_A(\{a,c\}) \neq \emptyset, (A \cup \{a,c\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a,c\}|A \subseteq \{b,d\}, (A \cup \{a,c\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}] = [\{A \cup \{a,c\}|A \subseteq \{b,d\}, \mathcal{F}_A(\{a,c\}) \neq \emptyset, \mathcal{R}_A(\{a,c\}) \neq \emptyset, (A \cup \{a,c\})^*_{/\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a,c\}|A \subseteq \{b,d\}, (A^{\mathrm{I}_{00}} \cap a' \cap c')^{\mathrm{I}_{00}} \backslash (A^{\mathrm{I}_{00}} \cap a')^{\mathrm{I}_{00}} \cup (A^{\mathrm{I}_{00}} \cap b')^{\mathrm{I}_{00}} \cup \bigcup_{x \in A}((A \backslash \{x\})^{\mathrm{I}_{00}} \cap c')^{\mathrm{I}_{00}}) \neq \emptyset\}] = \emptyset$*, because* $\mathcal{F}_A(\{a,c\}) = \emptyset$ *for every* $A \subseteq \{b,d\}$*.*

Hence, the set of informative attribute implications of the initial context is identified to:

$$\mathcal{J}^*_{\mathbb{K}_{00}}(\{a,c\}) := \{A|A \subseteq \{b,d\}, \mathcal{F}_A(\emptyset) \neq \emptyset, , \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} \cup [\mathcal{J}_{\mathbb{K}_{00}}(\{a,c\}) \backslash \mathcal{J}_{\mathbb{K}_{00}}(\emptyset)] = \{\{a\},\{c\},\{d,a\},\{b,c\},\{b,d\}\}$$

In the same way, setting $X = \{a \cup c\}$ *with* $X' = a' \cup c' = \{1,3\}$, *it comes that:*
$\{A|A \subseteq \{b,d\}, \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} = \{A|A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} \subseteq \{1,3\}, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\}$. *Moreover,* $(\emptyset)' = G \nsubseteq \{1,3\}$, $b' = \{1,2\} \nsubseteq \{1,3\}$ *and* $d' = \{3,4\} \nsubseteq \{1,3\}$. *Therefore,* $\{b,d\}$ *is the only subset* T *of* M_{00} *such that* $T' \subseteq \{1,3\}$. *Also,* $\mathcal{F}_{\{b,d\}}(\emptyset) = \{v \in \{a \cup c\} | \{b,d\}^{\mathrm{I}_{00}} \subseteq v'\} = \{a \cup c\}$ *and* $\mathcal{R}_{\{b,d\}}(\emptyset) = \{v \in \mathcal{F}_{\{b,d\}}(\emptyset) | b^{\mathrm{I}_{00}} \nsubseteq v'$ *and* $d^{\mathrm{I}_{00}} \nsubseteq v'\} = \{a \cup c\} \neq \emptyset$. *Also,* $(\{b,d\})^*_{\mathbb{K}_{00}} = \emptyset$ *because* $(\{b,d\})^{\mathrm{I}_{00}\mathrm{I}_{00}} = \{b,d\}$.
Therefore

$$\{A : A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} \subseteq a' \cup c', \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} = \{b,d\}$$

Moreover, $\mathcal{J}_{\mathbb{K}_{00}}(\emptyset) = \{A|A \subseteq \{b,d\}, \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} \cup \{A|A \subseteq \{b,d\}, A^*_{/\mathbb{K}_{00}} \neq \emptyset\}$.

The fact that
$\mathcal{J}_{\mathbb{K}_{00}}(\{a \cup c\}) = \mathcal{J}_{\mathbb{K}_{00}}(\emptyset) \cup \{A \cup \{a \cup c\} | A \subseteq \{b,d\}, (A \cup \{a \cup c\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}$ *leads to*

For $a \cup c$, *the set of attribute that added to the context* $\mathbb{K}_{ac} = (G, M \backslash \{a,c\}, \mathrm{I}_{ac})$ *is the one element set* $X = \{a \cup b\}$. *Hence,*
$\mathcal{J}^*_{\mathbb{K}_{00}}(\{a \cup c\}) = \{A|A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} \subseteq a' \cup c', \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} \cup [\mathcal{J}_{\mathbb{K}_{00}}(\{a \cup c\}) \backslash \mathcal{J}_{\mathbb{K}_{00}}(\emptyset)]$
$= \{A|A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} \subseteq a' \cup c', \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a \cup c\} | A \subseteq \{b,d\}, (A \cup \{a \cup c\})^*_{/\mathbb{K}_{00}} \neq \emptyset\}$
$= \{A|A \subseteq \{b,d\}, A^{\mathrm{I}_{00}} \subseteq a' \cup c', \mathcal{F}_A(\emptyset) \neq \emptyset, \mathcal{R}_A(\emptyset) \neq \emptyset, A^*_{\mathbb{K}_{00}} = \emptyset\} \cup \{A \cup \{a \cup c\} | A \subseteq \{b,d\}, (A^{\mathrm{I}_{00}} \cap (a' \cup c'))^{\mathrm{I}_{00}} \backslash (A^{\mathrm{I}_{00}\mathrm{I}_{00}} \cup \bigcup_{x \in A}((A \backslash \{x\})^{\mathrm{I}_{00}} \cap a' \cup c')^{\mathrm{I}_{00}}) \neq \emptyset\}$
$= \{\{b,d\}\} \cup \emptyset = \{\{b,d\}\}$, *because* $\mathcal{F}_A(\{a \cup c\}) = \emptyset$ *for every* $A \subseteq \{b,d\}$.

Hence, the set of informative attribute implications of the generalized context is identified to:

$$\mathcal{J}^*_{\mathbb{K}_{00}}(\{a \cup c\}) = \{\{b,d\}\}$$

To the best of our knowledge, generalized association rules were first studied in [34]. There, using taxonomy, the authors reveal that generalizing attributes could leads to a discovery of some association rules that could not have been found if the rules were restricted only to the leaves of the taxonomy. However, their generalization included both the generalized attributes and the attributes been generalized, and then the generalized association rules as described by them always led to an increase in the number of association rules. In our method, generalized implications do not include the attributes that are generalized and therefore, they can be a reduction of the size of the set of generalized implications. As in the case of [34], some of the generalized attribute implications can constitute relevant information that might not have been discovered if generalization were not done. It can particularly happen when several groups of attributes are simultaneously put together through ∃-generalization to form generalized attributes implications having sets of generalized attributes as premise and conclusion. In the following, we look at some conditions under which one can obtained some of these kinds of implications. The following result holds:

Proposition 4.3.3. *Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context, and $(a_i)_{i\in[n]}$, $(b_j)_{j\in[m]}$ and $(c_k)_{k\in[p]}$ some groups of attributes that are put together through $\exists$-generalization to form the generalized attributes $\tilde{a} = \cup_{i\in[n]}a_i$, $\tilde{b} = \cup_{j\in[m]}b_j$ and $\tilde{c} = \cup_{k\in[p]}c_k$. Then a generalized implication of the form*
$\{\cup_{i\in[n]}a_i, \cup_{j\in[m]}b_j\} \Rightarrow \{\cup_{k\in[p]}c_k\}$ is an informative implication in the generalized context iff the following conditions hold:

- $a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \subseteq \cup_{k\in[p]}c_k^{\mathrm{I}}$ $\forall i \in [n]$, $\forall j \in [m]$*;*

- $\exists i \in [n]$ *such that* $a_i^{\mathrm{I}} \not\subseteq \cup_{k\in[p]}c_k^{\mathrm{I}}$ *and*

- $\exists j \in [m]$ *such that* $b_j^{\mathrm{I}} \not\subseteq \cup_{k\in[p]}c_k^{\mathrm{I}}$.

Proof. Let $\mathbb{K} = (G, M, \mathrm{I})$ be such formal context, and $(a_i)_{i\in[n]}$, $(b_j)_{j\in[m]}$ and $(c_k)_{k\in[p]}$ (with $[u] = \{1, ..., u\} \cap \mathbb{N}$ for every $u \in \{n, m, p\}$) some groups of attributes that are put together through $\exists$-generalization to form several generalized attributes $\tilde{a} = \cup_{i\in[n]}a_i$, $\tilde{b} = \cup_{j\in[m]}b_j$ and $\tilde{c} = \cup_{k\in[p]}c_k$. Let consider a generalized implication of the form $A \Rightarrow B$ where $A = \{\cup_{i\in[n]}a_i, \cup_{j\in[m]}b_j\}$ and $B = \{\cup_{k\in[p]}c_k\}$.

Then $A \Rightarrow B$ is an informative generalized implication if and only if $B = A^*_{\mathbb{K}_{os}} = (A^{\mathrm{I}_{os}\mathrm{I}_{os}})\backslash(A \cup \bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{os}\mathrm{I}_{os}})$, with $\mathbb{K}_{os}$ the generalized formal context and I_{os} its binary relation. However, $B = A^*_{\mathbb{K}_{os}} = (A^{\mathrm{I}_{os}\mathrm{I}_{os}})\backslash(A \cup \bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{os}\mathrm{I}_{os}})$, if and only if for every $x \in A$ and for every $y \in B$, $y \notin (\{\cup_{i\in[n]}a_i, \cup_{j\in[m]}b_j\}\backslash\{x\})^{\mathrm{I}_{os}\mathrm{I}_{os}}$, and $y \in A^{\mathrm{I}_{os}\mathrm{I}_{os}} = ((\cup_{i\in[n]}a_i^{\mathrm{I}}) \cap (\cup_{j\in[n]}b_j^{\mathrm{I}}))^{\mathrm{I}_{os}} = (\cup_{i\in[n]} \cup_{j\in[m]} a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}}$; That is true if and only if

$$\cup_{k\in[p]}c_k \in (\cup_{i\in[n]} \cup_{j\in[m]} a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}}, \text{ and}$$

$$\cup_{k\in[p]}c_k \notin (\{\cup_{i\in[n]}a_i\})^{\mathrm{I}_{os}\mathrm{I}_{os}} \cup (\{\cup_{j\in[m]}b_j\})^{\mathrm{I}_{os}\mathrm{I}_{os}}.$$

if and only if

$$\tilde{c} = \cup_{k\in[p]}c_k \in (\cup_{i\in[n]} \cup_{j\in[m]} a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}},$$

$$\tilde{c} = \cup_{k\in[p]}c_k \in (\cup_{i\in[n]} \cup_{j\in[m]} a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}}, \text{ and}$$

$$\tilde{c} = \cup_{k\in[p]}c_k \notin (\cup_{j\in[m]}b_j^{\mathrm{I}})^{\mathrm{I}_{os}};$$

if and only if

$$a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \subseteq \cup_{k\in[p]}c_k^{\mathrm{I}} \ \forall i \in [n], \forall j \in [m],$$

$$\exists i \in [n] \text{ such that } a_i^{\mathrm{I}} \not\subseteq \cup_{k\in[p]}c_k^{\mathrm{I}}, \text{ and}$$

$\exists j \in [m]$ such that $b_j^{\mathrm{I}} \nsubseteq \cup_{k\in[p]} c_k^{\mathrm{I}}$;

■

Example 4.3.3. *Let consider the following formal context*

$\mathbb{K}$	a_0	c_1	c_2	a_1	a_2	b_1	b_2
1		×		×	×	×	
2			×	×		×	×
3		×	×	×			×
4			×		×	×	×
5		×			×		×
6	×					×	×
7	×			×	×		

Then we proceed to the simultaneous $\exists$*-generalization of groups of attributes* $\{a_1, a_2\}$, $\{b_1, b_2\}$ *and* $\{c_1, c_2\}$, *which gives the generalized attribute* $a_{12} = a_1 \cup a_2$, $b_{12} = b_1 \cup b_2$ *and* $c_{12} = c_1 \cup c_2$.

One can easily observe that $c_1^{\mathrm{I}} \cup c_2^{\mathrm{I}} = \{1, 2, 3, 4, 5\}$. *Moreover,*
$a_1^{\mathrm{I}} \cap b_1^{\mathrm{I}} = \{1, 2\} \subseteq \{1, 2, 3, 4, 5\} = c_1^{\mathrm{I}} \cup c_2^{\mathrm{I}}$,
$a_1^{\mathrm{I}} \cap b_2^{\mathrm{I}} = \{1, 3\} \subseteq \{1, 2, 3, 4, 5\} = c_1^{\mathrm{I}} \cup c_2^{\mathrm{I}}$,
$a_2^{\mathrm{I}} \cap b_1^{\mathrm{I}} = \{1, 4\} \subseteq \{1, 2, 3, 4, 5\} = c_1^{\mathrm{I}} \cup c_2^{\mathrm{I}}$,
$a_2^{\mathrm{I}} \cap b_2^{\mathrm{I}} = \{4, 5\} \subseteq \{1, 2, 3, 4, 5\} = c_1^{\mathrm{I}} \cup c_2^{\mathrm{I}}$,
$a_1^{\mathrm{I}} = \{1, 2, 3, 7\} \nsubseteq \{1, 2, 3, 4, 5\} = c_1^{\mathrm{I}} \cup c_2^{\mathrm{I}}$,
$b_1^{\mathrm{I}} = \{1, 2, 4, 6\} \nsubseteq \{1, 2, 3, 4, 5\} = c_1^{\mathrm{I}} \cup c_2^{\mathrm{I}}$,

Therefore, in the generalized formal context, the set $\{a_{12}, b_{12}\}$ *yields an informative generalized implication.*

Remark 4.3.1. *Concerning the above example, note that* $\{a_{12}, b_{12}\}^{*}_{\mathbb{K}_{os}} = \{c_{12}\}$ *and the yielded informative generalized implication is given by* $\{a_{12}, b_{12}\} \Rightarrow \{c_{12}\}$.

A more general form of the above result is given by the following proposition:

Proposition 4.3.4. *Let* $\mathbb{K} = (G, M, \mathrm{I})$ *be a formal context, and* $(a_i)_{i\in[n]}$, $(b_j)_{j\in[m]}$ *and* $(c_k)_{k\in[p]}$ *some groups of attributes that are put together through* $\exists$*-generalization to form the generalized attributes* $\cup_{i\in[n]} a_i$, $\cup_{j\in[m]} b_j$ *and* $\cup_{k\in[p]} c_k$. *Then a generalized implication of the form* $A_0 \cup \{\cup_{i\in[n]} a_i, \cup_{j\in[m]} b_j\} \Rightarrow B_0 \cup \{\cup_{k\in[p]} c_k\}$ *is an informative implication in the generalized context iff the following conditions hold:*

- $A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \subseteq (\cup_{k\in[p]} c_k^{\mathrm{I}}) \cap B_0^{\mathrm{I}}$ $\forall i \in [n]$, $\forall j \in [m]$;

- $\forall x \in A_0$, $\exists i \in [n]$, $\exists j \in [m]$ *such that* $(A_0 \backslash \{x\})^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \nsubseteq \cup_{k\in[p]} c_k^{\mathrm{I}}$ *and* $\forall y \in B_0$, $\exists i \in [n]$, $\exists j \in [m]$ *such that* $(A_0 \backslash \{x\})^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \nsubseteq y^{\mathrm{I}}$;

- $\exists i \in [n]$ *such that* $A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}} \nsubseteq \cup_{k\in[p]} c_k^{\mathrm{I}}$ *and* $\forall y \in B_0, \exists i \in [n]$ *such that* $A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}} \nsubseteq y^{\mathrm{I}}$*;*

- $\exists j \in [m]$ *such that* $A_0^{\mathrm{I}} \cap b_j^{\mathrm{I}} \nsubseteq \cup_{k\in[p]} c_k^{\mathrm{I}}$ *and* $\forall y \in B_0, \exists j \in [m]$ *such that* $A_0^{\mathrm{I}} \cap b_j^{\mathrm{I}} \nsubseteq y^{\mathrm{I}}$*.*

Proof. Let $\mathbb{K} = (G, M, \mathrm{I})$ be a formal context, and $(a_i)_{i\in[n]}$, $(b_j)_{j\in[m]}$ and $(c_k)_{k\in[p]}$ some groups of attributes that are put together through $\exists$-generalization to form several generalized attributes $\cup_{i\in[n]} a_i$, $\cup_{j\in[m]} b_j$ and $\cup_{k\in[p]} c_k$. Let consider a generalized implication of the form $A \Rightarrow B$ where $A = A_0 \cup \{\cup_{i\in[n]} a_i, \cup_{j\in[m]} b_j\}$ and $B = B_0 \cup \{\cup_{k\in[p]} c_k\}$.

$A \Rightarrow B$ is an informative generalized implication if and only if $B = A^{*}_{\mathbb{K}_{os}} = (A^{\mathrm{I}_{os}\mathrm{I}_{os}}) \backslash (A \cup \bigcup_{x\in A}((A\backslash\{x\})^{\mathrm{I}_{os}\mathrm{I}_{os}})$, with $\mathbb{K}_{os}$ the generalized formal context; if and only if $B_0 \cup \{\cup_{k\in[p]} c_k\} \subseteq A^{\mathrm{I}_{os}\mathrm{I}_{os}} = (A_0^{\mathrm{I}} \cap (\cup_{i\in[n]} a_i^{\mathrm{I}}) \cap (\cup_{j\in[n]} b_j^{\mathrm{I}}))^{\mathrm{I}_{os}} = (\cup_{i\in[n]} A^{\mathrm{I}} \cap \cup_{j\in[m]} a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}}$.

That is true if and only if

$$\cup_{i\in[n]}(A_0^{\mathrm{I}} \cap \cup_{j\in[m]} a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}}) \subseteq (\cup_{k\in[p]} c_k^{\mathrm{I}}) \cap B_0^{\mathrm{I}}$$

and for every $x \in A$ and for every $y \in B$,
$y \notin (A_0 \cup \{\cup_{i\in[n]} a_i, \cup_{j\in[m]} b_j\} \backslash \{x\})^{\mathrm{I}_{os}\mathrm{I}_{os}} = (A_0 \cup \{\cup_{i\in[n]} a_i\})^{\mathrm{I}_{os}\mathrm{I}_{os}} \cup (A_0 \cup \{\cup_{j\in[m]} b_j\})^{\mathrm{I}_{os}\mathrm{I}_{os}} \cup ((A_0\backslash\{x\})^{\mathrm{I}} \cap (\cup_{i\in[n]} a_i') \cap (\cup_{j\in[m]} b_j'\})^{\mathrm{I}_{os}}$,

That is true if and only if

$\cup_{i\in[n]}(A_0^{\mathrm{I}} \cap \cup_{j\in[m]} a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}}) \subseteq (\cup_{k\in[p]} c_k^{\mathrm{I}}) \cap B_0^{\mathrm{I}}$;

$\cup_{k\in[p]} c_k \notin (\cup_{i\in[n]} \cup_{j\in[m]} (A_0\backslash\{x\})^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}}$, and
$y \notin (\cup_{i\in[n]} \cup_{j\in[m]} (A_0\backslash\{x\})^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}}$ $\forall y \in B_0$;

$\cup_{k\in[p]} c_k \notin (\cup_{i\in[n]} A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}})^{\mathrm{I}_{os}}$, and
$y \notin (\cup_{i\in[n]} A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}})^{\mathrm{I}_{os}}$ $\forall y \in B_0$;

$\cup_{k\in[p]} c_k \notin (\cup_{j\in[m]} A_0^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}}$, and
$y \notin (\cup_{j\in[m]} A_0^{\mathrm{I}} \cap b_j^{\mathrm{I}})^{\mathrm{I}_{os}} \forall y \in B_0$.

if and only if

$A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \subseteq (\cup_{k\in[p]} c_k^{\mathrm{I}}) \cap B_0^{\mathrm{I}}$ $\forall i \in [n]$, $\forall j \in [m]$;

$\forall x \in A_0$, $\exists i \in [n]$, $\exists j \in [m]$ such that $(A_0\backslash\{x\})^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \nsubseteq \cup_{k\in[p]} c_k^{\mathrm{I}}$ and
$\forall y \in B_0, \exists i \in [n]$, $\exists j \in [m]$ such that $(A_0\backslash\{x\})^{\mathrm{I}} \cap a_i^{\mathrm{I}} \cap b_j^{\mathrm{I}} \nsubseteq y^{\mathrm{I}}$;

$\exists i \in [n]$ such that $A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}} \nsubseteq \cup_{k\in[p]} c_k^{\mathrm{I}}$ and
$\forall y \in B_0, \exists i \in [n]$ such that $A_0^{\mathrm{I}} \cap a_i^{\mathrm{I}} \nsubseteq y^{\mathrm{I}}$;

$\exists j \in [m]$ such that $A_0^{\mathrm{I}} \cap b_j^{\mathrm{I}} \not\subseteq \cup_{k \in [p]} c_k^{\mathrm{I}}$ and
$\forall y \in B_0, \exists j \in [m]$ such that $A_0^{\mathrm{I}} \cap b_j^{\mathrm{I}} \not\subseteq y^{\mathrm{I}}$.

■

Now we investigate the impact of the ∃-generalization on the basis of implications, notably the Duquenne-Guigues basis. The main result here shows that ∃-generalizations on attributes in a formal context does not always reduce the size of the canonical basis of implications.

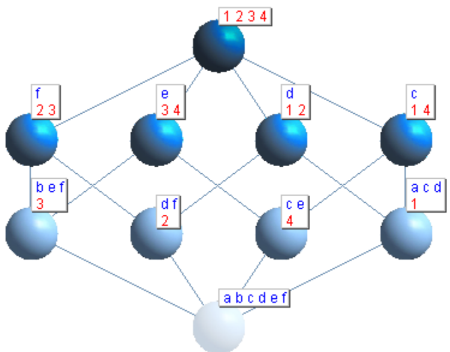
4.4 The study of the canonical base after an ∃-generalization

The last section shows that the ∃-generalization on attributes can lead to the reduction in the size of the set of all the valid informative implications in a formal context. The next question we ask is whether there are cases where the ∃-generalization increases the size of the canonical basis of implications? This section is an attempt to answer that question.

let consider the following formal context $\mathbb{K}$ with 6 attributes $\{a, b, c, d, e, f\}$ and 4 objects $\{1, 2, 3, 4\}$.

$\mathbb{K}$	a	c	d	e	f	b
1	×	×	×			
2			×		×	
3				×	×	×
4		×		×		

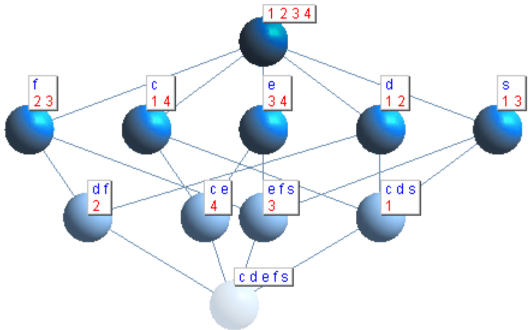
The concept lattice corresponding to the above formal context is given below:



In the above context, one realizes an ∃-generalization of the pair of attributes $\{a, b\}$. Let s be the corresponding generalized attribute. One then obtains the following generalized formal context $\mathbb{K}_{ab}^{s}$.

$\mathbb{K}^s_{ab}$	c	d	e	f	s
1	×	×			×
2		×		×	
3			×	×	×
4	×		×		

With the above generalized context, one obtains the following concept lattice:



One can see that generalizing the attributes a and b increases the size of the concept lattice. That situation could also be justify by the fact that the context $\mathbb{K}$ has a labelled copy of B_4 (as indicated in [27]).

Now one analyzes the variation of pseudo-intents in the two formal contexts above. To get there, we set by $\mathcal{P}_{\mathbb{K}}$ the set of pseudo-intents in $\mathbb{K}$ and $\mathcal{P}_{\mathbb{K}_{os}}$ the set of pseudo-intents of $\mathbb{K}_{os}$. The following table explained how to obtained the pseudo-intent in the formal context $\mathbb{K}$.

A	A^{II}	A$\in \mathcal{P}_{\mathbb{K}}$?	Why?	A	A^{II}	A$\in \mathcal{P}_{\mathbb{K}}$?	Why
$\emptyset$	$\emptyset$	no	$\emptyset^{II} = \emptyset$	cdf	abcdef	no	$cd^{II} \not\subseteq acd$
a	acd	yes		cdb	abcdef	no	$cd^{II} \not\subseteq cdb$
b	efb	yes		def	abcdef	no	$de^{II} \not\subseteq def$
c	c	no	$c^{II} = c$	deb	abcdef	no	$de^{II} \not\subseteq deb$
d	d	no	$d^{II} = d$	efb	efb	no	$efb^{II} = efb$
e	e	no	$e^{II} = e$	cef	abcdef	no	$cf^{II} \not\subseteq cef$
f	f	no	$f^{II} = f$	cfb	abcdef	no	$cf^{II} \not\subseteq cfb$
ab	abcdef	no	$a^{II} \not\subseteq ab$	dfb	abcdef	no	$b^{II} \not\subseteq dfb$
ac	acd	no	$a^{II} \not\subseteq ac$	afb	abcdef	no	$b^{II} \not\subseteq afb$
ad	acd	no	$a^{II} \not\subseteq ad$	ceb	abcdef	no	$b^{II} \not\subseteq ceb$
ae	abcdef	no	$a^{II} \not\subseteq ae$	acde	abcdef	no	$de^{II} \not\subseteq acde$
af	abcdef	no	$a^{II} \not\subseteq af$	acdf	abcdef	no	$cf^{II} \not\subseteq acdf$
cd	acd	yes		acdb	abcdef	no	$b^{II} \not\subseteq acdb$
ce	ce	no	$ce^{II} = ce$	adef	abcdef	no	$a^{II} \not\subseteq adef$
cf	abcdef	yes		adeb	abcdef	no	$a^{II} \not\subseteq adeb$
cb	abcdef	no	$b^{II} \not\subseteq cb$	aefb	abcdef	no	$a^{II} \not\subseteq aefb$
de	abcdef	yes		cdef	abcdef	no	$cf^{II} \not\subseteq cdef$
df	df	no	$df^{II} = df$	cdeb	abcdef	no	$b^{II} \not\subseteq cdeb$
db	abcdef	no	$b^{II} \not\subseteq db$	cdfb	abcdef	no	$b^{II} \not\subseteq cdfb$
ef	efb	yes		defb	abcdef	no	$de^{II} \not\subseteq defb$
eb	ebf	no	$b^{II} \not\subseteq eb$	adfb	abcdef	no	$a^{II} \not\subseteq adfb$
fb	ebf	no	$b^{II} \not\subseteq fb$	acfb	abcdef	no	$a^{II} \not\subseteq acfb$
acd	acd	no	$acd^{II} = acd$	acef	abcdef	no	$a^{II} \not\subseteq acef$
ace	abcdef	no	$a^{II} \not\subseteq ace$	acdef	abcdef	no	$de^{II} \not\subseteq acdef$
acf	abcdef	no	$a^{II} \not\subseteq acf$	acdeb	abcdef	no	$de^{II} \not\subseteq acdef$
acb	abcdef	no	$a^{II} \not\subseteq acb$	acdeb	abcdef	no	$de^{II} \not\subseteq acdeb$
ade	abcdef	no	$de^{II} \not\subseteq ade$	cdefb	abcdef	no	$de^{II} \not\subseteq cdefb$
adf	abcdef	no	$a^{II} \not\subseteq adf$	acefb	abcdef	no	$a^{II} \not\subseteq acefb$
adb	abcdef	no	$a^{II} \not\subseteq adb$	adefb	abcdef	no	$a^{II} \not\subseteq adefb$
aef	abcdef	no	$a^{II} \not\subseteq aef$	abcdef	abcdef	no	$abcdef^{II} = abcdef$
aeb	abcdef	no	$a^{II} \not\subseteq aeb$				
cde	abcdef	no	$de^{II} \not\subseteq cde$				

Hence, according to the above table, the pseudo-intents of the context $\mathbb{K}$ is given by $\mathcal{P}_{\mathbb{K}} = \{a, b, cd, cf, de, ef\}$ and then the canonical base of implications for $\mathbb{K}$ is given by the following set

$$\{a \Rightarrow cd, b \Rightarrow ef, cd \Rightarrow a, cf \Rightarrow abde, de \Rightarrow abcf, ef \Rightarrow b\}$$

which has 6 elements.

Now we consider the generalized formal context in which we analyse the pseudo-intents. The following table indicates how these pseudo-intents are obtained:

A	$A^{I_{os}I_{os}}$	A$\in \mathcal{P}_{\mathbb{K}_{os}}$?	Why?	A	$A^{I_{os}I_{os}}$	A$\in \mathcal{P}_{\mathbb{K}_{os}}$?	Why
$\emptyset$	$\emptyset$	no	$\emptyset^{I_{os}I_{os}} = \emptyset$	scd	scd	no	$\mathrm{scd}^{I_{os}I_{os}} = scd$
s	s	no	$\mathrm{s}^{I_{os}I_{os}} = s$	sce	abcdef	no	$\mathrm{cd}^{I_{os}I_{os}} \not\subseteq cdb$
c	c	no	$\mathrm{c}^{I_{os}I_{os}} = c$	scf	abcdef	no	$\mathrm{de}^{I_{os}I_{os}} \not\subseteq def$
d	d	no	$\mathrm{d}^{I_{os}I_{os}} = d$	sde	abcdef	no	$\mathrm{de}^{I_{os}I_{os}} \not\subseteq deb$
e	e	no	$\mathrm{e}^{I_{os}I_{os}} = e$	sdf	efb	no	$\mathrm{efb}^{I_{os}I_{os}} = efb$
f	f	no	$\mathrm{f}^{I_{os}I_{os}} = f$	sef	abcdef	no	$\mathrm{cf}^{I_{os}I_{os}} \not\subseteq cef$
sc	scd	yes		cde	abcdef	no	$\mathrm{cd}^{I_{os}I_{os}} \not\subseteq cde$
sd	scd	yes		cdf	abcdef	no	$\mathrm{cd}^{I_{os}I_{os}} \not\subseteq cdf$
se	sef	yes		cef	abcdef	no	$\mathrm{cf}^{I_{os}I_{os}} \not\subseteq cef$
sf	sef	yes		def	abcdef	no	$\mathrm{ef}^{I_{os}I_{os}} \not\subseteq def$
cd	scd	yes		scde	abcdef	no	$\mathrm{de}^{I_{os}I_{os}} \not\subseteq scde$
ce	ce	no	$\mathrm{ce}^{I_{os}I_{os}} = ce$	scdf	abcdef	no	$\mathrm{sf}^{I_{os}I_{os}} \not\subseteq scdf$
cf	scdef	yes		cdef	abcdef	no	$\mathrm{de}^{I_{os}I_{os}} \not\subseteq cdef$
de	scdef	yes		scef	abcdef	no	$\mathrm{cf}^{I_{os}I_{os}} \not\subseteq scef$
df	df	no	$\mathrm{df}^{I_{os}I_{os}} = df$	sdef	scdef	no	$\mathrm{de}^{I_{os}I_{os}} \not\subseteq sdef$
ef	sef	yes		scdef	scdef	no	$\mathrm{b}^{I_{os}I_{os}} = scdef$

The pseudo-intents of the context $\mathcal{P}_{\mathbb{K}^s_{ab}}$ is given by

$$\mathcal{P}_{\mathbb{K}^s_{ab}} = \{sc, sd, se, sf, cd, cf, de, ef\}.$$

That leads us to the following canonical base of implications for $\mathcal{P}_{\mathbb{K}^s_{ab}}$

$$\{sc \Rightarrow d, sd \Rightarrow c, se \Rightarrow f, sf \Rightarrow e, cd \Rightarrow s, cf \Rightarrow cde, de \Rightarrow scf, ef \Rightarrow s\},$$

with 8 elements.

4.5 Conclusion

In this chapter, we have studied the variation of the number of informative implications after an $\exists$-generalization on attributes in a formal context. The results show that there are cases where the size of the set of generalized informative implications reduces after the generalization. Other results revealed some conditions under which some new informative generalized implications could be found. Also, a case of formal context where both the size of the Duquenne-Guigues basis and that of the concept lattice increase after the generalization is presented. A reduction of size of the set of informative implications could reduce the difficulties in analysing information in data.

General Conclusion

The study of generalization have been very important, notably while extracting certain patterns from formal data. In so doing, several types of generalization has being defined ($\forall$, α and $\exists$) and applied on attributes or objects of formal contexts, or on both of them. Some results has being revealed on $\forall$-generalization.

In this work, we were focused on $\exists$-generalization of attributes in finite formal contexts. We have studied the effect of adding a new attribute to a formal context and we have presented some corresponding results. This enabled us to construct a family of formal contexts showing that the $\exists$-generalization can lead to the exponential increase of size of the concept lattice. This family of formal contexts came as an answer to the experimentation realized in [27] which permitted the authors to suggest that $\exists$-generalization may increase the size of the lattice on very few cases, and even with these cases, the increase of size of the concept lattice could not be drastic.

We have studied the maximum increase one can obtain after an $\exists$-generalization on a pair of attributes in a finite formal context and presented some conditions under which the size of the lattice stabilizes.

Showing the associativity of $\exists$-generalization of attributes in formal contexts reveals that generalizing more than two attribute can be done in several steps: i) randomly choosing two attributes in the set X of the attributes to be generalized in the formal context, ii) generalizing the two attributes and continue to generalize the resulting attribute with an attribute of X different from the attributes previously chosen for generalization. This is repeated until all the attributes of X are taken into consideration in the generalizing process.

After presenting the different types of similarity measures, we have shown that the existing coefficients of associations are not compatible with the $\exists$-generalization. A new similarity measure on attributes is then constructed using the concept lattice of the formal context obtained from the initial context by removing the pair of attributes being generalized. The constructed similarity measure is such that the less the attributes are similar, the more their generalization increases the size of the concept lattice.

Studying the implications while generalizing attributes, we have study the variation in the number of informative implications in a formal context after an $\exists$-generalization.

Moreover, we have discovered a case of formal context in which an $\exists$-generalization increases both the size of the concept lattice and that of the set of the canonical basis of implications.

Our future research will be focussed on the following aspects:

- Give the Characterization of the formal contexts with an $\exists$-generalization decreasing the size of the concept lattice,
- Continue the search of other similarity measures more efficient and compatible with the $\exists$-generalization,
- Extend the study of attribute implications to bases of implication;
- Study the generalization on other types of formal context, especially the fuzzy formal contexts.

List of publications during the thesis

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A Similarity Measure to Generalize Attributes

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Abstract. Formal Concept Analysis (FCA) plays a crucial role in various domains, especially in qualitative data analysis. Here knowledge are extracted from an information system in form of clusters (forming a concept lattice) or in form of rules (implications basis). The number of extracted pieces of information can grow very fast. To control the number of cluster, one possibility is to put some attributes together to get a new attribute called a generalized attribute. However, generalizing does not always lead to the expected results: the number of concepts can even exponentially increase after generalizing two attributes [7,8]. A natural question is whether there is a similarity measure, (possibly cheap and fast to compute), that is compatible with generalizing attributes: i.e. if m_1, m_2 are **more similar** than m_3, m_4, then putting m_1, m_2 together should not lead to more concepts as putting m_3, m_4 together. This paper is an attempt to answer this question.

Keywords: Formal Concept Analysis; Generalizing Attributes; Similarity Measures.

1 Introduction

In Formal Concept Analysis (FCA), a **formal context** is a binary relation (G, M, I) that models an elementary information system, whereby G is the set of objects, M the set of attributes and $I \subseteq G \times M$ the incidence relation. To extract knowledge from such an elementary information system, one possibility is to get clusters of objects and/or attributes by grouping together those sharing the same characteristics. These pairs, called **concepts**, were formalized by Rudolf Wille [16]. For $A \subseteq G$ and $B \subseteq M$ we set

$$A' = \{m \in M \mid g \,\mathrm{I}\, m \text{ for all } g \in A\} \text{ and}$$
$$B' = \{g \in M \mid g \,\mathrm{I}\, m \text{ for all } m \in B\}.$$

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A **concept** is a pair (A, B) such that $A' = B$ and $B' = A$. A is called **extent** and B **intent** of the concept (A, B). The set of concepts of a context $\mathbb{K} := (G, M, \mathrm{I})$ is ordered by the relation $(A, B) \leq (C, D) :\Longleftrightarrow A \subseteq C$, and forms a lattice, denoted by $\mathfrak{B}(\mathbb{K})$ and called **concept lattice** of $\mathbb{K}$. To control the size of concept lattices, many methods have been suggested: decomposition [18,19,17], iceberg lattices [14] α-Galois lattices [15], fault tolerant patterns [3], closure or kernel operators and/or approximation [6]. In [7] the authors consider putting together some attributes to get a generalized attribute. Doing this one has to decide when an object satisfies a (new) generalized attribute. They discuss several scenarios among which the following, called $\exists$-generalization:

> an object $g \in G$ satisfies a generalized attribute $s \subseteq M$ if g satisfies at least one of the attributes in s. i.e. $s' = \bigcup\{m' \mid m \in s\}$.

In the rest of this contribution, we will simply say **generalization** to mean $\exists$-generalization. By generalizing (i.e putting together some attributes) we reduce the number of attributes and hope to also reduce the size of the concept lattice. Unfortunately this is not always the case. In [8] the authors provide some examples where the size increases exponentially after generalizing two attributes and also give the maximal increase.

In [1,5], the authors discuss similarity measures on concepts, and even on lattices. For our purpose, we need a measure of similarity on attributes such that if m_1, m_2 are more similar than m_3, m_4, then generalizing m_1, m_2 should not lead to more concepts as generalizing m_3, m_4. We say that such a similarity measure is **compatible with the generalization**. Given a set M of attributes, a **similarity measure** on M is defined as a function $S : M \times M \to \mathbb{R}$ such that for all m_1, m_2 in M,

(i) $S(m_1, m_2) \geq 0$, **positivity**
(ii) $S(m_1, m_2) = S(m_2, m_1)$ **symmetry**
(iii) $S(m_1, m_1) \geq S(m_1, m_2)$ **maximality**

If in addition $S(m_1, m_2) \leq 1$, we say that S is **normalized**. Similarity measures aim at quantifying to which extent two attributes resemble each other. Getting a similarity measure compatible with the generalization will be a valuable tool in preprocessing and will warn the data analyst on possible lost or gain when generalizing.

The rest of the paper is organized as follows: In Section 2, we investigate the existing similarity measures that we found in the literature. In Section 3, we give a new similarity measure that characterize the pairs of attributes which can increase the size of the concept lattice after generalizing. Section 4 exposes an example on lexicographic data and Section 5 concludes the paper.

2 Test of Existing Similarity Measures in ∃-Generalization

Similarity and dissimilarity measures play a key role in pattern analysis problems such as classification, clustering, etc. Ever since *Pearson* proposed a coefficient of correlation in 1896, numerous similarity measures and distance have been proposed in various fields. These measures can be grouped into tree main types, depending of the data on which they are used:

Correlation coefficients: They are often used in data to compare variables with qualitative characters subdivided in more than two states.

Distance similarity coefficients: They are generally used in data with pure quantitative variables. In most cases, for quantitative data, the similarity between two taxa is expressed as a function of their distance in a dimensional space whose coordinates are the characters.

Coefficients of association: They are often used in data with presence-absence characters or in data with individuals having qualitative characters subdivided into two states.

There are two subsets of coefficients of association: those that only depend on characteristics present in at least one of the taxa compared, but are independent of the attributes absent in both taxa (denoted by type 1), and those that also take into account the attributes absent in both taxa (denoted by type 2). Those measures use

- a as the number of cases where the two variables occur together in a sample,
- d as the number of cases where none of the two attributes occur in a sample,
- b as the number of cases in which only the first variable occur, and
- c as the number of cases where only the second variable occur.

One of the most important similarity measure of type 1 is the **Jaccard measure** $\left(\frac{a}{a+b+c}\right)$, proposed in order to classify ecological species. Also in the ecological field, the **Dice coefficient of association** $\left(\frac{2a}{2a+b+c}\right)$ aims at quantifying the extent to which two different species are associated in a biotope, the **Sorensen coefficient of association** $\left(\frac{4a}{4a+b+c}\right)$ and the **Anderberg coefficient of association** $\left(\frac{8a}{8a+b+c}\right)$ are of the same type. The **Sneath and Sokal 2** similarity coefficient $\left(\frac{\frac{1}{2}a}{\frac{1}{2}a+b+c}\right)$, put in place in order to compare organisms in numerical taxonomy, the **Kulczynski similarity** measure $\left(\frac{1}{2}\left(\frac{a}{a+b}+\frac{a}{a+c}\right)\right)$ and the **Ochiai similarity** measure $\left(\frac{a}{\sqrt{(a+b)(a+c)}}\right)$ are also from this first type.

The most used similarity coefficient of the second type is the **Sokal and Michener** coefficient of association $\left(\frac{a+d}{a+d+b+c}\right)$, also called the **simple matching coefficient**, put in place to express the similarity between two species of bees. Moreover, the **Rogers and Tanimoto similarity measure** $\left(\frac{\frac{1}{2}(a+d)}{\frac{1}{2}(a+d)+b+c}\right)$ whose aim was to compare species of plants in the ecological field, the **Sokal and Sneath 1** similarity coefficient $\left(\frac{2(a+d)}{2(a+d)+b+c}\right)$ was defined to make comparison in numerical taxonomy and the **Russels and Rao** similarity measure $\left(\frac{a}{a+d+b+c}\right)$ put in place with the aim of showing resemblance between species of *anopheline* *larvae,* are included in this type. Same are the **Yule and Kendall similarity coefficients** $\left(\frac{ad}{ad+bc}\right)$, often used in the statistical field. Some of the above similarity measures can be found in [5].

Regarding the definitions of the above kinds of similarity measures, only the coefficients of association suitable to formal contexts, since formal contexts are data with presence-absence characters. We will investigate the impact of these coefficients of association on a special pair of attributes in some formal contexts. The objective is to show that these similarity measures are not helpful in finding whether their generalization increases the size of the lattice or not.

Our first example is an arbitrary formal context (G, M, I) containing two attributes $x, y \in M$ such that $x' \subseteq y'$ and $|x' \cap y'| = 1$. Then $|x' \setminus y'| = 0$ and the generalization of the attributes x and y does not increase the size of the lattice. Choosing $|y' \setminus x'| = 20$ and $|G \setminus (x' \cup y')| = 1$ yields $a = |x' \cap y'| = 1$, $b = |x' \setminus y'| = 0$, $c = |y' \setminus x'| = 20$ and $d = |G \setminus (x' \cup y')| = 1$. For the coefficient of association of type 1 with Jaccard (Jc), Dice (Di), Sorensen (So), Anderberg (An), Sneath and Sokal 2 (SS_2), Kulczynski (Ku) and Orchiai (Orch), and the coefficient of association of type 2 with Sokal and Michener (SM), Rogers and Tanimoto (RT), Sneath and Sokal 1 (SS_1) and Russel and Rao (RR), we get the table below for $s(x, y)$:

Jc	Di	So	An	SS_2	Ku	Orch	SM	RT	SS1	RR
0,05	0,09	0,17	0,29	0,02	0,52	0,22	0,09	0,05	0,17	0,05

The table above shows that with almost all these measures, the similarity measured between the attributes x and y is very low, despite the fact that their generalization does not increase the size of the lattice.

Our second example is the formal context $\mathbb{K}_6 := (S_6 \cup \{g_1\}, S_6 \cup \{m_1, m_2\}, \mathrm{I})$ below, with $S_6 = \{1, 2, 3, 4, 5, 6\}$.

$\mathbb{K}_6$	1	2	3	4	5	6	m_1	m_2
1		×	×	×	×	×	×	
2	×		×	×	×	×	×	×
3	×	×		×	×	×	×	×
4	×	×	×		×	×	×	×
5	×	×	×	×		×	×	×
6	×	×	×	×	×			×
g_1	×	×	×	×	×	×		

We observe that $|m_1' \cap m_2'| = 4$, $|m_1' \backslash m_2'| = 1$ and $|m_2' \backslash m_1'| = 1$. Putting together the attributes m_1 and m_2 by a $\exists$-generalization increases the size of the lattice by 16. The following table shows the measures of type 1 and type 2 between the attribute m_1 and any other attribute i. All the similarity measures of the

	Jc	Di	So	An	SS_2	Ku	Orch	SM	RT	SS1	RR
$i \in S_5$	0,57	0,80	0,89	0,94	0,50	0,80	0,80	0,71	0,56	0,83	0,57
$i = 6$	0,83	0,91	0,95	0,97	0,71	0,92	0,91	0,75	0,75	0,92	0,71
$i = m_2$	0,67	0,80	0,89	0,94	0,50	0,80	0,80	0,71	0,56	0,83	0,57

two types show that the attribute m_1 is more similar to m_2 than to any other attribute $i \in S_6$ (apart from $i = 6$); But putting m_1 and m_2 together increases the size of the lattice. We can conclude that these similarity measures are not compatible with the $\exists$-generalization. We are actually looking for a measure on attributes that will flag pairs of attributes as **less similar** when putting these together increases the size of the concept lattice.

3 A Similarity Measure Compatible with $\exists$-Generalization

In this section we define a similarity measure on attributes which is compatible with the existential generalization. This generalization means that from an attribute reduced context $\mathbb{K} := (G, M, \mathrm{I})$, two attributes a, b are removed and replaced with an attribute s defined by $s' = a' \cup b'$. We set $M_0 := M \setminus \{a, b\}$ and

$$\begin{aligned}
\mathbb{K}_{00} &:= (G, M_0, \mathrm{I} \cap (G \times M_0)), && \text{(removing } a, b \text{ from } \mathbb{K}) \\
\mathbb{K}_{0s} &:= (G, M_0 \,\dot\cup \{s\}, I_0^s), && \text{(adding } s \text{ to } \mathbb{K}_{00})
\end{aligned}$$

where $I_0^s := (\mathrm{I} \cap (G \times M_0)) \cup \{(g, s) \mid g \,\mathrm{I}\, b \text{ or } g \,\mathrm{I}\, a\}$. Furthermore we denote the set of extents of $\mathbb{K}_{00}$ by $\mathrm{Ext}(\mathbb{K}_{00})$. We also set

$$\begin{aligned}
\mathcal{H}(a) &:= \{A \cap a' \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap a' \notin \mathrm{Ext}(\mathbb{K}_{00})\}\,, \\
\mathcal{H}(b) &:= \{A \cap b' \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap b' \notin \mathrm{Ext}(\mathbb{K}_{00})\}\,, \\
\mathcal{H}(a \cup b) &:= \{A \cap (a' \cup b') \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap (a' \cup b') \notin \mathrm{Ext}(\mathbb{K}_{00})\}\,, \\
\mathcal{H}(a \cap b) &:= \{A \cap (a' \cap b') \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap (a' \cap b') \notin \mathrm{Ext}(\mathbb{K}_{00})\}\,.
\end{aligned}$$

We will often write $h(x)$ for $|\mathcal{H}(x)|$, for any $x \in \{a, b, a \cap b, a \cup b\}$. Before we start the construction, let us recall the following result partly proved in [8]:

Theorem 1. *Let $\mathbb{K} := (G, M, \mathrm{I})$ be an attribute reduced context with $|G| \geq 3$ and $|M| > 3$. Let a and b be two attributes such that their existential generalization $s = a \cup b$ increases the size of the concept lattice. Then*

a) $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a, b)|$, *with* $|\mathcal{H}(a, b)| = |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|$.
b) The increase is $|\mathcal{H}(a \cup b)| - |\mathcal{H}(a, b)| \leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1$.

Proof. Let $\mathbb{K} := (G, M, \mathrm{I})$ be such context and a, b two attributes of $\mathbb{K}$. One proceeds to the $\exists$-generalization of attributes a and b.

a) We set $\mathbb{K}^a = (G, M \setminus \{b\}, \mathrm{I})$. It holds:

$$|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}^a)| + h^*(b) = |\mathfrak{B}(\mathbb{K}_{00})| + h(a) + h^*(b)$$

where $h^*(b) = |\{B \cap b';\ B \in \text{Ext}(\mathbb{K}^a),\ B \cap b' \notin \text{Ext}(\mathbb{K}^a)\}|$. Our aim is to express $h^*(b)$ as a function of $h(b)$ and $h(a \cap b)$. According to [8], $\text{Ext}(\mathbb{K}^a) = \text{Ext}(\mathbb{K}_{00}) \cup \mathcal{H}(a)$. Hence,

$$\begin{aligned}\mathcal{H}^*(b) &= \{B \cap b' \mid B \in \text{Ext}(\mathbb{K}^a), B \cap b' \notin \text{Ext}(\mathbb{K}^a)\}\\ &= \{B \cap b' \mid B \in \text{Ext}(\mathbb{K}_{00}) \text{ and } B \cap b' \notin \text{Ext}(\mathbb{K}^a)\}\\ &\qquad \cup \{B \cap b' \mid B \in \mathcal{H}(a) \text{ and } B \cap b' \notin \text{Ext}(\mathbb{K}^a)\}\end{aligned}$$

Replacing $\text{Ext}(\mathbb{K}^a)$ by $\text{Ext}(\mathbb{K}_{00}) \cup \mathcal{H}(a)$, we get

$$\{B \cap b' \mid B \in \text{Ext}(\mathbb{K}_{00}) \text{ and } B \cap b' \notin \text{Ext}(\mathbb{K}^a)\} = \mathcal{H}(b) \setminus \mathcal{H}(a) \quad \text{and}$$

$$\{B \cap b' \mid B \in \mathcal{H}(a) \text{ and } B \cap b' \notin \text{Ext}(\mathbb{K}^a)\} = \mathcal{H}(a \cap b) \setminus (\mathcal{H}(b) \cup \mathcal{H}(a)).$$

$$\begin{aligned}\text{Thus,} \quad h^*(b) &= h(b) + h(a \cap b) - |\mathcal{H}(a) \cap \mathcal{H}(b)| + |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)|\\ &\quad - |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)|.\end{aligned}$$

Hence,

$$\begin{aligned}|\mathfrak{B}(\mathbb{K})| &= |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a)| + |\mathcal{H}(b)| + |\mathcal{H}(a \cap b)| + |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)|\\ &\quad - |\mathcal{H}(a) \cap \mathcal{H}(b)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)|\\ &= |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|.\end{aligned}$$

b) Although b) was proved in [8], we can now get it from a). To maximize the increase $a' \cap b'$ should be $\emptyset$; i.e. $|\mathcal{H}(a \cap b)| \in \{0, 1\}$.
   - If $|\mathcal{H}(a \cap b)| = 0$, then

$$\begin{aligned}|\mathfrak{B}(\mathbb{K})| &= |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|\\ &= |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a)| + |\mathcal{H}(b)|.\end{aligned}$$

   - If $|\mathcal{H}(a \cap b)| = 1$, then we consider two subcases:
     – The only element of $\mathcal{H}(a \cap b)$ is not in $\mathcal{H}(a) \cup \mathcal{H}(b)$. Then,

$$\begin{aligned}|\mathcal{H}(a) \cap \mathcal{H}(b)| &= |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)|\\ &= |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| = |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)| = 0\end{aligned}$$

     and $|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a)| + |\mathcal{H}(b)| + |\mathcal{H}(a \cap b)|$.
     – The only element of $\mathcal{H}(a \cap b)$ is either in $\mathcal{H}(a)$ or $\mathcal{H}(b)$. Then

$$|\mathcal{H}(a \cap b)| + |\mathcal{H}(a \cap b) \cap \mathcal{H}(a) \cap \mathcal{H}(b)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(a)| - |\mathcal{H}(a \cap b) \cap \mathcal{H}(b)|$$

     is equal to zero and $|\mathcal{H}(a) \cap \mathcal{H}(b)| \in \{0, 1\}$. Thus

$$|\mathfrak{B}(\mathbb{K})| = |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a)| + |\mathcal{H}(b)| + 1 - |\mathcal{H}(a) \cap \mathcal{H}(b)|.$$

In all these subcases, considering that $|\mathfrak{B}(\mathbb{K}_{0s})| = |\mathfrak{B}(\mathbb{K}_{00})| + |\mathcal{H}(a \cup b)|$, the increase after the generalization is

$$\begin{aligned}
|\mathfrak{B}(\mathbb{K}_{0s})| - |\mathfrak{B}(\mathbb{K})| &= |\mathcal{H}(a \cup b)| - |\mathcal{H}(a, b)| \\
&\leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + (d_1 + d_2 - d_0) \\
&\leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1, \text{ since } d_1 + d_2 - d_0 \leq 0,
\end{aligned}$$

with $d_1 = |\{A \subseteq a' \mid A \in \text{Ext}(\mathbb{K}_{00})\}|$, $d_2 = |\{A \subseteq b' \mid A \in \text{Ext}(\mathbb{K}_{00})\}|$ and $d_0 = |\{A \subseteq a' \cup b' \mid A \in \text{Ext}(\mathbb{K}_{00})\}|$. □

Now, we define the following gain function:

$$\begin{aligned}
\psi : M \times M &\longrightarrow \mathbb{Z} \\
(a, b) &\longmapsto \psi(a, b) = |\mathcal{H}(a \cup b)| - |\mathcal{H}(a, b)|
\end{aligned}$$

Note that $\mathcal{H}(a \cup b) = \mathcal{H}(b \cup a)$, and $\mathcal{H}(a, b) = \mathcal{H}(b, a)$ because the order of adding the attributes a and b does not matter. Therefore $\psi(a, b) = \psi(b, a)$. By definition, $\psi(a, a) = 0$. Further, we define the map δ as followed:

$$\begin{aligned}
\delta : M \times M &\longrightarrow \mathbb{R} \\
(a, b) &\longmapsto \begin{cases} 1 & \text{if } \psi(a, b) \leq 0 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Since $\mathbb{K}$ is a finite context, there is a pair of attributes a_0, b_0 in M such that

$$|a_0'| + |b_0'| = \max_{a,b \in M} (|a'| + |b'|).$$

We set $n_0 = 2^{|a_0'|+|b_0'|} - 2^{|a_0'|} - 2^{|b_0'|} + 1$. Then $n_0 \geq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1$ for all pairs $\{a, b\} \subseteq M$. With the function δ, we construct the following map:

$$\begin{aligned}
S_{\text{gen}} : M \times M &\longrightarrow \mathbb{R} \\
(a, b) &\longmapsto S_{\text{gen}}(a, b) = \tfrac{1+\delta(a,b)}{2} - \tfrac{|\psi(a,b)|}{2n_0}
\end{aligned}$$

where $|\psi(a, b)|$ is the absolute value of $\psi(a, b)$. That leads to the following results.

Proposition 1. *Let (G, M, I) be a reduced context with $|G| \geq 3$ and $|M| > 3$. Then S_{gen} is a normalized similarity measure on M.*

Proof. Let a, b two attributes of (G, M, I). Since $|\psi(a, b)| \leq n_0$ we can easily check that $0 \leq S_{\text{gen}}(a, b) = S_{\text{gen}}(b, a) \leq S_{\text{gen}}(a, a) = 1$ holds. □

S_{gen} also has the following properties:

Proposition 2. *Let (G, M, I) be a reduced context with $|G| \geq 3$ and $|M| > 3$. Let $a, b, c, d \in M$. It holds:*

a) $S_{gen}(a, b) \geq \frac{1}{2}$ *if and only if* $\psi(a, b) \leq 0$.

b) If $\psi(a,b) \leq 0 < \psi(d,c)$ *then* $S_{gen}(d,c) < S_{gen}(a,b)$.
c) If $0 < \psi(a,b) \leq \psi(d,c)$ *then* $S_{gen}(d,c) \leq S_{gen}(a,b)$.
d) If $\psi(a,b) \leq \psi(d,c) \leq 0$ *then* $S_{gen}(a,b) \leq S_{gen}(d,c)$.

Proof. Let $\mathbb{K} = (G, M, I)$ be such a context and $a, b, c, d \in M$.

a) If $\psi(a,b) \leq 0$ then $\delta(a,b) = 1$ and

$$S_{\text{gen}}(a,b) = \frac{1 + \delta(a,b)}{2} - \frac{|\psi(a,b)|}{2n_0} = \frac{1}{2}\left(2 + \frac{\psi(a,b)}{n_0}\right) \geq \frac{1}{2}.$$

Now, $S_{\text{gen}}(a,b) \geq \frac{1}{2}$ implies $\frac{1+\delta(a,b)}{2} - \frac{|\psi(a,b)|}{2n_0} \geq \frac{1}{2}$ and $|\psi(a,b)| \leq n_0\delta(a,b)$. If $\delta(a,b) = 0$ then $|\psi(a,b)| = 0$. If $\delta(a,b) = 1$ then $\psi(a,b) \leq 0$ by definition of δ. Hence, $S_{\text{gen}}(a,b) \geq \frac{1}{2}$ if and only if $\psi(a,b) \leq 0$.

b) If $\psi(a,b) \leq 0 < \psi(d,c)$ then $S_{\text{gen}}(d,c) < \frac{1}{2} \leq S_{\text{gen}}(a,b)$.

c) If $0 < \psi(a,b) \leq \psi(d,c)$ then $\delta(a,b) = \delta(d,c) = 0$, and

$$S_{\text{gen}}(d,c) = \frac{1}{2} - \frac{\psi(d,c)}{2n_0} \leq \frac{1}{2} - \frac{\psi(a,b)}{2n_0} = S_{\text{gen}}(a,b).$$

d) If $\psi(a,b) \leq \psi(d,c) \leq 0$ then $\delta(a,b) = \delta(d,c) = 1$, and

$$S_{\text{gen}}(a,b) = 1 + \frac{\psi(a,b)}{2n_0} \leq 1 + \frac{\psi(d,c)}{2n_0} = S_{\text{gen}}(d,c).$$

□

Proposition 3. *Let* (G, M, I) *be a reduced context and* $a, b \in M$. *The following assertions are equivalent:*

(i) $\delta(a,b) = 1$.
(ii) $\psi(a,b) \leq 0$.
(iii) $S_{gen}(a,b) \geq \frac{1}{2}$.
(iv) *A* $\exists$*-generalization of* a *and* b *does not increase the size of the concept lattice.*

Proof. (i) $\iff$ (ii) follows from the definition of δ. (ii) $\iff$ (iii) is Proposition 2 a). (ii) $\iff$ (iv) follows from the fact that $\psi(a,b) = |\mathcal{H}(a \cup b)| - |\mathcal{H}(a,b)|$ is actually the difference $|\mathfrak{B}(G, M \cup \{s\} \setminus \{a,b\}, I)| - |\mathfrak{B}(G, M, I)|$ between the number of concepts before and after generalizing a, b to s with $s' = a' \cup b'$.

Therefore, generalizing two attributes a, b in a reduced context (G, M, I) increases the size of the lattice if and only if $S_{\text{gen}}(a,b) < \frac{1}{2}$. The threshold $\frac{1}{2}$ is just a consequence of the way S_{gen} has been defined.

To test our results we have designed a naive algorithm (see Algorithm 1) that computes S_{gen} on all pairs of attributes a, b of $\mathbb{K}$. If the set of attributes M is considered as a vector, then for any attribute $a \in M$, we set T(a) the set of all attributes coming before a in M. The complexity of our algorithm is given by

$$\sum_{a \in M}(1 + \sum_{b \in M \setminus T(a)}((q(a,b) + 4)[4(q(a,b) + 1) + 4] + 3),$$

which is equal to

$$|M| + \sum_{a \in M} \sum_{b \in M \setminus T(a)} (4q^2(a,b) + 24q(a,b) + 35), \qquad \text{with } q(a,b) = |\operatorname{Ext}(\mathbb{K}_{00})|.$$

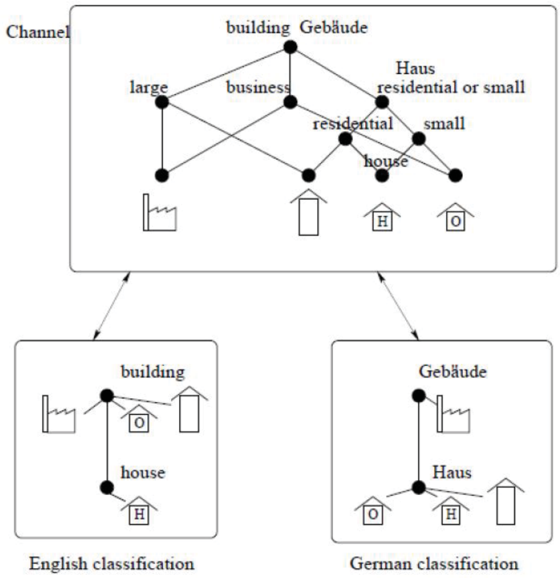
Algorithm 1: Computing a similarity measure

Data: An attribute reduced context (G, M, I)
Result: ψ and S_{gen} on $M \times M$

Choose x, y in M, $x \neq y$ with $|x'| + |y'|$ maximal;
$n_0 \leftarrow 2^{|x'|+|y'|} - 2^{|x'|} - 2^{|y'|} + 1$;
$T \leftarrow \emptyset$;
foreach a *in* M **do**
    $T \leftarrow T \cup \{a\}$;
    foreach b *in* $M \setminus T$ **do**
        $\mathrm{Ext}_0 \leftarrow \mathrm{Ext}(G, M \setminus \{a, b\}, I)$;
        foreach x *in* $\{a, b, a \cup b, a \cap b\}$ **do** $\mathcal{H}(x) \leftarrow \emptyset$;
        foreach A *in* Ext_0 **do**
            foreach x *in* $\{a, b, a \cup b, a \cap b\}$ **do**
                if $A \cap x' \notin \mathrm{Ext}_0$ **then** $\mathcal{H}(x) \leftarrow \mathcal{H}(x) \cup \{A \cap x'\}$;
            end
        end
    end
    $\psi(a,b) \leftarrow |\mathcal{H}(a \cup b)| - |\mathcal{H}(a) \cup \mathcal{H}(b) \cup \mathcal{H}(a \cap b)|$; $\quad \psi(b,a) \leftarrow \psi(a,b)$;
    if $\psi(a,b) \leq 0$ **then**
        $\delta(a,b) \leftarrow 1$
    else
        $\delta(a,b) \leftarrow 0$
    end
    $S_{\text{gen}(a,b)} \leftarrow \dfrac{1 + \delta(a,b)}{2} - \dfrac{|\psi(a,b)|}{2n_0}$
end

4 An Example from Lexicographic Data

Formal Concept Analysis has been applied to compare lexical databases. In [11] Uta Priss proposes an example in where the information channel is "*building*". With respect to this, the main difference between English and German is that in English, the word "house" only refers to small residential buildings whereas in German even small office buildings and large residential buildings can be called "Haus", and only factories would normally not be called "Haus". Moreover, "building" in English refers to either a factory, an office or even a big residential house. But only a factory can be called "Gebäude" in German. She presented in the figure below the information channel of the word "building" in the sense of Barwise and Seligman [2] in both English and German.



With the above information channel we can construct a formal context as follows: The objects are different kinds of buildings: small house (”h”), office (”o”), factory (”f”) and large residential house (”l”). The attributes are different names of these objects in both languages: English and German. These are ”building”, ”house”, ”Haus”, ”Gebäude”, ”large building” (short: ”large”), ”business building” (short: ”business”), ”residential house” (short: ”residential”), and ”small house” (short: ”small”). Thus $G = \{$h, o, f, l$\}$ and $M = \{$”building”, ”house”, ”Haus”, ”Gebäude”, ”large”, ”business”, ”residential”, ”small”$\}$. In the following, a set of objects will be denoted as a concatenation of those objects. For example we will write ho or oh for the set $\{$h, o$\}$. The English and German classifications of the word ”building” are then presented in the following formal context:

	building	house	Haus	Gebäude	large	business	residential	small
factory	×			×	×	×		
office	×		×			×		×
house		×	×				×	×
large	×		×		×		×	

For this formal context, $n_0 = 2^{3+3} - 2^3 - 2^3 + 1 = 49$. Let consider the attributes $a :=$ house and $b :=$ Gebäude. Then $a' \cup b' = \{f, h\}$ and $a' \cap b' = \emptyset$. We have

$$\text{Ext}(\mathbb{K}_{00}) = \{fohl, fol, ohl, fo, fl, ol, oh, hl, f, o, h, l, \emptyset\}, \text{ and}$$

$\mathcal{H}(a) = \mathcal{H}(b) = \mathcal{H}(a \cap b) = \emptyset$ and $\mathcal{H}(a \cup b) = \{fohl\}$. Therefore, $\psi(a, b) = 1$ and $S_{\text{gen}}(a, b) = \frac{1}{2} - \frac{1}{98} \approx 0.49$. Using our algorithm, we compute $\psi(a, b)$ and

$S_{\text{gen}}(a,b)$ for all pairs $a,b \in M$. The table below show $\psi(a,b)$ below the diagonal, and $S_{\text{gen}}(a,b)$ on the rest.

	building	house	Haus	Gebäude	large	business	residential	small
building	1.00	0.98	0.97	1.00	0.99	0.98	0.97	0.97
house	−2	1.00	1.00	0.49	0.49	0.49	1.00	1.00
Haus	−3	0	1.00	0.98	0.97	0.97	0.99	0.99
Gebäude	0	1	−2	1.00	1.00	1.00	0.49	0.49
large	−1	1	−3	0	1.00	0.98	0.49	0.97
business	−2	1	−3	0	−2	1.00	0.98	0.49
residential	−3	0	−1	1	1	−2	1.00	0.98
small	−3	0	−1	1	−3	1	−2	1.00

From the above table, the attributes "house" and "Gebäude" are less similar. It reflects the fact that these words "Gebäude" (in German) and "house" (in English) do not have the same meaning. It is also the case for the attributes "house" and "business buildings" as well as "Gebäude" and "residential building". Hence, putting together each of the above pairs of attributes will increase the size of the lattice. On the contrary, the attributes "large" and "Haus", "building" and "Haus" are more similar through S_{gen}. It is because the word "Haus" which designates a house, a business office or simply large building in German, often coincides with the words "building" or "large building" in English. For these pairs, the existential generalization will not increase the size of the lattice.

5 Conclusion

We have constructed a similarity measure compatible with the change in the size of the lattice after a generalization of a pair of attributes in a formal context. That measure should send a warning when grouping two attributes. Also, it enables us to characterize contexts where generalizing two attributes increases the size of the concept lattice. Our next step is to look at the implication between generalized attributes. We suspect that the number of implications decreases if the number of concepts increases.

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On the size of ∃-generalized concept lattices

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Formal Concept Analysis (FCA) offers several tools for qualitative data analysis. One possibility is to group objects that share common attributes together and get a concept lattice that describes the data. Quite often the size of this concept lattice is very large. Many authors have investigated methods to reduce the size of this lattice. In Kwuida et al. (2014) the authors consider putting together some attributes to reduce the size of the attribute sets. But this reduction does not always carry over to the set of concepts. They provided some counter examples where the size of the concept lattice increases by one after putting two attributes together, and asked the following question: "How many new concepts can be generated by an ∃-generalization on just two attributes?" The present paper provides a family of contexts for which the size increases on more than one concept after putting solely two attributes together.

1. Introduction

An elementary information system can be represented by a set G of objects or entities, a set M of attributes or characteristics together with an incidence relation I that encodes whether an object $g \in G$ has an attribute $m \in M$. For such a system we write $(g, m) \in \mathrm{I}$ or $g\mathrm{I}m$ to mean that the object g has the attribute m. The binary relation $\mathbb{K} := (G, M, \mathrm{I})$ is called a **formal context**.

To extract knowledge from such information systems, one possibility is to get clusters of objects and/or attributes by grouping together those sharing the same characteristics. These pairs, called **concepts**, were formalized by Rudolf Wille [19]. Traditional philosophers consider a concept as defined by two parts: an extent and an intent. The **extent** contains all entities belonging to the concept and the **intent** is the set of all attributes common to all entities in the concept. To formalize the notion of concept the operator $'$ (known as **derivation** in formal contexts) is defined:

$$A' := \{m \in M \mid (g, m) \in \mathrm{I} \text{ for all } g \in A\}, \text{ whenever } A \subseteq G$$
$$B' := \{g \in M \mid (g, m) \in \mathrm{I} \text{ for all } m \in B\}, \text{ whenever } B \subseteq M.$$

A' contains all attributes shared by the objects in A, and B' all objects having all attributes in B. A concept is a pair (A, B) with $A' = B$ and $B' = A$. The extents are then subsets A of G with $A'' = A$, and intents subsets B of M with $B'' = B$. For a single object or attribute x we write x' for $\{x\}'$. The map $X \mapsto X''$ is a **closure operator** (on $\mathcal{P}(G)$ or $\mathcal{P}(M)$, where $\mathcal{P}(*)$ denotes the set of subsets of $*$). X'' is called the closure of X in $\mathbb{K}$. Subsets X with $X'' = X$ (i.e. extents and intents) are closed subsets.

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Table 1
A formal context.

$\mathbb{K}$	v	u	a	b
a		×	×	
b	×			×
c	×	×		
g	×	×	×	×

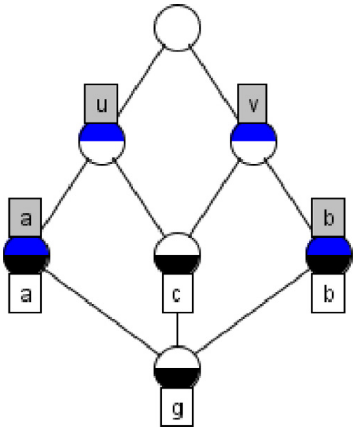

Fig. 1. A concept lattice.

We will denote the set of formal concepts of a context $\mathbb{K}$ by $\mathfrak{B}(\mathbb{K})$ and the set of its extents by $\text{Ext}(\mathbb{K})$. A concept $c_2 := (A_2, B_2)$ is said to be more general than a concept $c_1 := (A_1, B_1)$ if c_2 contains all objects of c_1. In that case each attribute satisfied by all objects of c_2 is also satisfied by all objects of c_1.

$$(A_1, B_1) \leqslant (A_2, B_2) :\iff A_1 \subseteq A_2, \quad (\text{or equivalently } B_1 \supseteq B_2).$$

The above defined relation $\leqslant$ is an order relation on concepts and models the **concept hierarchy**. Each subset of $\mathfrak{B}(\mathbb{K})$ has a supremum and an infimum with respect to $\leqslant$. Therefore $(\mathfrak{B}(\mathbb{K}), \leqslant)$ is a **complete lattice** called the **concept lattice** of the context $\mathbb{K}$. An extensive introduction to Formal Concept Analysis can be found on [8].

Fig. 1 shows the concept lattice of the formal context in Table 1. Concepts are nodes. The extent of a node contains all objects below this node, and its intent all attributes above it. The node at the center of this figure represents the concept $(\{c, g\}, \{u, v\})$.

The size of concept lattices can be very large, even exponential with respect to the size of the context. For example the formal context $(E, E, \neq)$ has $2^{|E|}$ concepts, where E is any set and $|E|$ denotes the number of elements in E. In fact $A' = E \setminus A$ and $A'' = A$ for any subset A of E. Thus all pairs $(A, E \setminus A)$ are concepts of the formal context $(E, E, \neq)$. To control the size of concept lattices several methods have been suggested: decomposition [20–22], iceberg lattices [15], α-Galois lattices [18], fault tolerant patterns [6], similarity measures [2], closure or kernel operators and/or approximation [12], generalized attributes [13]. In the present contribution we follow the direction in [13], where some attributes can be put together to define a new attribute called a **generalized attribute**.

When some attributes are put together, the main issue is to decide when an object has this new combined attribute. Different scenarios have been discussed in [13]:

($\forall$): The object should satisfy each of the attributes that were combined.
(α): The object should satisfy at least a certain proportion of the attributes that were combined.
($\exists$): The object should satisfy at least one of the attributes that were combined.

By putting together some attributes we reduce the number of attributes and hope to also reduce the size of the concept lattice. This is true for $\forall$-generalizations, but is not always the case for $\exists$-generalizations. In [13] some examples are presented where the size increases by one after a $\exists$-generalization. Then it was asked whether the size can increase by more than one element after putting solely two attributes together. The present paper gives answers to this question. It provides a family of contexts where the increase is exponential in the size of the attribute set. It also gives the maximum of increases that arise when two attributes are put together with the $\exists$-generalization.

Generalizing two attributes m_1, m_2 to get a new attribute m_{12} can be done in two steps: (i): adding m_{12} to the initial context and (ii) removing m_1, m_2 from the context. In Section 2 we discuss the effect of adding a new attribute in a context $\mathbb{K}$. The main result here says that the maximum of the number of new concepts is $|\mathfrak{B}(\mathbb{K})|$. This means that adding a new attribute to $\mathbb{K}$ can double the size of $\mathfrak{B}(\mathbb{K})$. In Section 3 we present a family of contexts where the size increases by more than one after putting two attributes together, and by then answer the question raised in [13]. Finally, we show

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in Section 4 that the increase from Section 3 is the maximum. The last section concludes the work and presents further directions to be investigated.

2. Adding a new attribute into a context

When constructing concept lattices the incremental methods [9,17] consist in starting with one object (or attribute) and adding the rest, one after another. In this section we study the effect of adding one attribute. Let $\mathbb{K} := (G, M, \mathrm{I})$ be a context, and $a \notin M$ an attribute that can be shared by some elements of G. We set $M_a := M \cup \{a\}$ and $\mathbb{K}_a := (G, M_a, \mathrm{I}_a)$ where

$$\mathrm{I}_a := \mathrm{I} \cup \{(g, a) \mid g \text{ has the new attribute } a\}.$$

We call $\mathbb{K}_a$ the context obtained by adding the attribute a to $\mathbb{K}$. To distinguish between the derivation on sets of objects in $\mathbb{K}$ and in $\mathbb{K}_a$ we will use the name of the relation instead of $'$. This is not necessary for the derivation on sets of attributes, unless we are looking for their closures. That said we will write for $A \subseteq G$

$$A^{\mathrm{I}} = \{m \in M \mid g\mathrm{I}m,\ \forall g \in A\} \quad \text{and} \quad A^{\mathrm{I}_a} = \{m \in M \cup \{a\} \mid g\mathrm{I}m,\ \forall g \in A\}.$$

If $a' = G$, then $|\mathfrak{B}(\mathbb{K}_a)| = |\mathfrak{B}(\mathbb{K})|$. Each concept (A, B) of $\mathbb{K}$ has a corresponding concept $(A, B \cup \{a\})$ in $\mathbb{K}_a$, and vice-versa. The above equality still holds even if $a' \neq G$, but $a' = B'$ for some $B \subseteq M$.

Proposition 1. *Let $\mathbb{K}$ be a formal context and $\mathbb{K}_a$ the formal context obtained by adding the attribute a to $\mathbb{K}$. Then the map ϕ_a is injective.*

$$\begin{aligned} \phi_a : \mathfrak{B}(\mathbb{K}) &\longrightarrow \mathfrak{B}(\mathbb{K}_a) \\ (A, B) &\longmapsto \begin{cases} (A, B \cup \{a\}) & \text{if } A \subseteq a' \\ (A, B) & \text{else.} \end{cases} \end{aligned}$$

Proof. The map ϕ_a is well defined. In fact, for a concept $(A, B) \in \mathfrak{B}(\mathbb{K})$ with $A \subseteq a'$, we have $(B \cup \{a\})' = B' \cap a' = A \cap a' = A$, and $A^{\mathrm{I}_a} = A^{\mathrm{I}} \cup \{a\} = B \cup \{a\}$. Thus $(A, B \cup \{a\})$ is a concept of $\mathbb{K}_a$. For a concept $(A, B) \in \mathfrak{B}(\mathbb{K})$ with $A \nsubseteq a'$, we have $B' = A$, and $A^{\mathrm{I}_a} = A^{\mathrm{I}} = B$, since a is not in A^{I_a}. The injectivity of ϕ_a is trivial. If two concepts (A_1, B_1) and (A_2, B_2) of $\mathbb{K}$ have the same image under ϕ_a, then A_1 and A_2 are both included in a' or both not included in a', and should therefore be equal. □

After adding an attribute a to a context $\mathbb{K}$, we will identify $(A, B) \in \mathfrak{B}(\mathbb{K})$ with $\phi_a(A, B) \in \mathfrak{B}(\mathbb{K}_a)$, and write $(A, B) \equiv \phi_a(A, B)$. From Proposition 1 we get $|\mathfrak{B}(\mathbb{K})| \leq |\mathfrak{B}(\mathbb{K}_a)|$. Moreover, the increase due to adding a, which is the difference $|\mathfrak{B}(\mathbb{K}_a)| - |\mathfrak{B}(\mathbb{K})|$, can be computed as the number of concepts of $\mathbb{K}_a$ that cannot be identified (via ϕ_a) with a concept in $\mathfrak{B}(\mathbb{K})$.

Let $(A, B) \in \mathfrak{B}(\mathbb{K})$ with $A \nsubseteq a'$. It holds: $\mathfrak{B}(\mathbb{K}_a) \ni (A, B) \equiv (A, B) \in \mathfrak{B}(\mathbb{K})$. Moreover, $A \cap a'$ is in $\mathrm{Ext}(\mathbb{K}_a)$. If $A \cap a'$ is also in $\mathrm{Ext}(\mathbb{K})$, then

$$\mathfrak{B}(\mathbb{K}_a) \ni \left(A \cap a', (A \cap a')^{\mathrm{I}_a}\right) \equiv \left(A \cap a', (A \cap a')^{\mathrm{I}}\right) \in \mathfrak{B}(\mathbb{K}).$$

Note that $(A \cap a')^{\mathrm{I}_a} = (A \cap a')^{\mathrm{I}} \cup \{a\}$ and $(A \cap a')^{\mathrm{I}} = (A \cap a')^{\mathrm{I}_a} \cap M$. Although (A, B) and $(A \cap a', (A \cap a')^{\mathrm{I}} \cup \{a\})$ are two different concepts of $\mathbb{K}_a$, they are equivalent to two concepts of $\mathbb{K}$ when $A \cap a'$ is an extent of $\mathbb{K}$. A concept (A, B) of $\mathbb{K}$ induces two concepts of $\mathbb{K}_a$ whenever $A \nsubseteq a'$. In the definition of ϕ_a in Proposition 1 we went for (A, B) instead of $(A \cap a', (A \cap a')^{\mathrm{I}} \cup \{a\})$. This choice is motivated by the injectivity of ϕ_a being straightforward. If $A \nsubseteq a'$ and $A \cap a'$ is an extent of $\mathbb{K}$ then the two concepts induced by (A, B) in $\mathbb{K}_a$ have their respective equivalents in $\mathfrak{B}(\mathbb{K})$. Then adding a to $\mathbb{K}$ will increase the size of the concept lattice only if there is A in $\mathrm{Ext}(\mathbb{K})$ such that $A \cap a'$ is not in $\mathrm{Ext}(\mathbb{K})$.

Each extent of $\mathbb{K}_a$ is an extent of $\mathbb{K}$ or an intersection of an extent of $\mathbb{K}$ with a'. The concepts of $\mathbb{K}_a$ that cannot be identified (via ϕ_a) with a concept of $\mathbb{K}$ are

$$\left\{\left(A \cap a', (A \cap a')^{\mathrm{I}} \cup \{a\}\right) \mid A \in \mathrm{Ext}(\mathbb{K}) \text{ and } A \cap a' \notin \mathrm{Ext}(\mathbb{K})\right\}.$$

Note that it is possible to have two different extents $A_1, A_2 \in \mathrm{Ext}(\mathbb{K})$ with $A_1 \cap a' = A_2 \cap a' \notin \mathrm{Ext}(\mathbb{K})$. In this case we say that the extents A_1 and A_2 coincide on a'. The increase is then less than or equal to $|\mathfrak{B}(\mathbb{K})|$. We can now sum up the finding of the above discussion in the next proposition.

Proposition 2. *Let $\mathbb{K}_a$ be a context obtained by adding an attribute a to a context $\mathbb{K}$. We set*

$$\mathcal{H}(a) := \left\{A \cap a' \mid A \in \mathit{Ext}(\mathbb{K}) \text{ and } A \cap a' \notin \mathit{Ext}(\mathbb{K})\right\} \quad \text{and} \quad h(a) := |\mathcal{H}(a)|.$$

1. *The increase in the number of concepts due to adding the attribute a to $\mathbb{K}$ is*

$$|\mathfrak{B}(\mathbb{K}_a)| - |\mathfrak{B}(\mathbb{K})| = h(a) \leq |\mathfrak{B}(\mathbb{K})|.$$

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$\mathbb{K}$	v	u	a	b	m
a		×	×		×
b	×			×	×
c	×	×			×
g	×	×	×	×	

$\mathbb{K}$	v	u	a	b	m
a		×	×		×
b	×			×	
c	×	×			×
g	×	×	×	×	

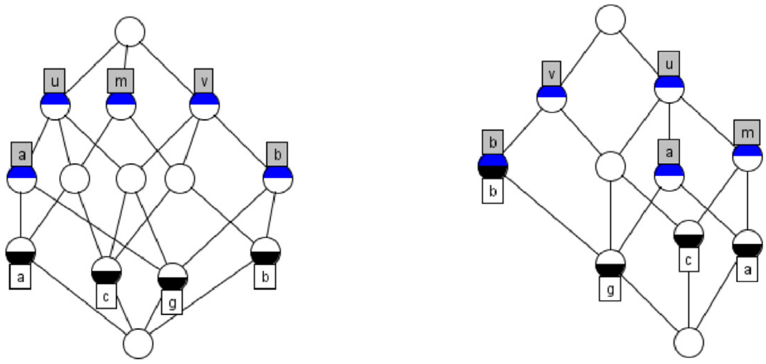

Fig. 2. Two tables obtained by adding an attribute m to the context in Table 1, and their corresponding concept lattices.

2. *The maximal increase is $h(a) = |\mathfrak{B}(\mathbb{K})|$ and is reached if each $A \in Ext(\mathbb{K})$ satisfies $A \cap a' \notin Ext(\mathbb{K})$ and no pairs $A_1, A_2 \in Ext(\mathbb{K})$ coincide on a'.*

Before we continue with the discussion on the maximal increase, let us look at two examples, where an attribute m is added to the context of Table 1. In the first case (left of Fig. 2) the concept lattice of Fig. 1 has been doubled and the maximal increase is reached. In the second case (right of Fig. 2) only the concepts in the interval $[\emptyset^{II}; \{a, c\}^{II}]$ of the concept lattice of Fig. 1 has been doubled. Note that in both cases, $g \in \emptyset^{II} \neq \emptyset$.

Based on the examples in Fig. 2 and Proposition 2, we can now discuss the maximal increase. Note that if $A \in \mathrm{Ext}(\mathbb{K})$ and $A \cap a' \notin \mathrm{Ext}(\mathbb{K})$, then $A \nsubseteq a'$. Moreover, if $A \nsubseteq a'$ for every extent A of $\mathbb{K}$, then in particular $\emptyset^{II} \nsubseteq a'$. Thus there is $g \in \emptyset^{II}$ such that $g \notin a'$. This element g is in every extent of $\mathbb{K}$, but is not in a'. Conversely, if an element g is in $\emptyset^{II} \setminus a'$, then g is in every extent A of $\mathbb{K}$, and g is not in $A \cap a'$. Thus $g \in \left(A \cap a'\right)^{II}$ and $g \notin A \cap a'$, i.e. $A \cap a'$ is not closed in $\mathbb{K}$. Thus $A \cap a' \notin \mathrm{Ext}(\mathbb{K})$ for each $A \in \mathrm{Ext}(\mathbb{K})$.

Proposition 3. *Let $\mathbb{K}$ be a formal context and a an attribute added to $\mathbb{K}$. The following are equivalent:*

(i) $\forall A \in Ext(\mathbb{K}),\ A \cap a' \notin Ext(\mathbb{K})$.
(ii) $\emptyset^{II} \setminus a' \neq \emptyset$.

Both contexts of Fig. 2 satisfy the above conditions (the added attribute a is m). Each extent of $\mathbb{K}$ generates two extents of $\mathbb{K}_m$ and one of these cannot be identified (via ϕ_m) with an extent of $\mathbb{K}$. However, some of these new concepts can be equal. In fact if two extents coincide on m', then they generate the same new concept. To avoid coincidences on m', it is enough to have $m' = G \setminus \{g\}$.

Corollary 1. *Let $\mathbb{K}$ be a formal context such that $\emptyset^{II} \neq \emptyset$ and $\mathbb{K}_a$ a context obtained by adding an attribute a to $\mathbb{K}$ such that $a' = G \setminus \emptyset^{II}$. Then we have*

$$|\mathfrak{B}(\mathbb{K}_a)| = 2 \cdot |\mathfrak{B}(\mathbb{K})|\,.$$

Using these results we can now present some huge increases after generalizing only two attributes.

3. Number of concepts generated by an ∃-generalization

Let $\mathbb{K} := (G, M, \mathrm{I})$ be a formal context. A generalized attribute of $\mathbb{K}$ is a subset $s \subseteq M$. We denote by S the set of generalized attributes of $\mathbb{K}$. Since the final goal is to reduce the size of the lattice, we assume that S forms a partition of M.[1] Then at least the number of attributes is reduced. For a ∃-generalization, an object g has the generalized attribute s iff g has at least one of the attributes in s; i.e. $s' = \bigcup\{m' \mid m \in s\}$. We get a relation J on $G \times S$ defined by:

$$g\mathrm{J}s \iff \exists m \in s \text{ such that } g\mathrm{I}m.$$

[1] It is possible to allow some attributes to appear in different groups. In that case the number of generalized attributes can be larger than in the initial context.

$\mathbb{K}_2^1$	1	2	m_1	m_2
1		×	×	
2	×			×
g_1	×	×		

$\mathbb{K}_{2ge}^1$	1	2	m_{12}
1		×	×
2	×		×
g_1	×	×	

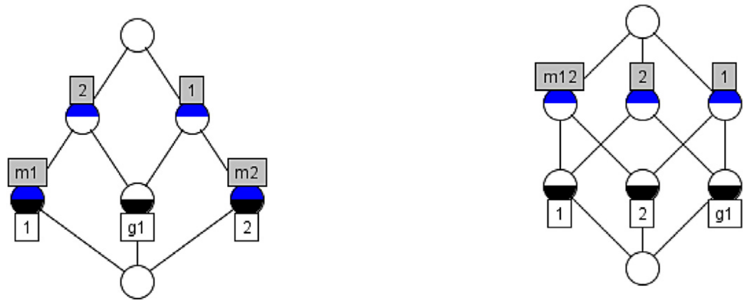

Fig. 3. $\mathfrak{B}(\mathbb{K}_2^1)$ (left) and $\mathfrak{B}(\mathbb{K}_{2ge}^1)$ (right), as defined by (1) with $n = 2$.

Table 2
Examples of increases after a ∃-generalization.

n	2	3	4	5	10	20
$\lvert\mathfrak{B}(\mathbb{K}_n^1)\rvert$	7	13	25	49	1537	1 572 865
$\lvert\mathfrak{B}(\mathbb{K}_{nge}^1)\rvert$	8	16	32	64	2048	2 097 152
$\lvert\mathfrak{B}(\mathbb{K}_{nge}^1)\rvert - \lvert\mathfrak{B}(\mathbb{K}_n^1)\rvert$	1	3	7	15	511	524 287

In this section we look at a very simple case, where two attributes $a, b \in M$ are generalized to get a new one, s. This means that from a context (G, M, I), we remove the attributes a and b from M and add an attribute $s \notin M$ to M, with $s' = a' \cup b'$. In particular we show that the number of concepts of $(G, (M \setminus \{a, b\}) \cup \{s\}, \mathrm{I}_s)$ can be very much larger than that of (G, M, I).

By S_n we denote a set with n elements where $n \geq 2$, and write for simplicity $S_n := \{1, 2, \ldots, n\}$. We define a context $\mathbb{K}_n^1$ by:

$$\mathbb{K}_n^1 := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I}) \quad \text{with}$$

$$g\mathrm{I}m :\iff \begin{cases} g, m \in S_n & \text{and } g \neq m, \text{ or} \\ g = g_1 & \text{and } m \in S_n, \text{ or} \\ g = 1 & \text{and } m = m_1, \text{ or} \\ g \in S_n \setminus \{1\} & \text{and } m = m_2. \end{cases} \tag{1}$$

We generalize the attributes m_1 and m_2 to get m_{12} and denote the resulting context by $\mathbb{K}_{nge}^1 := (S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I})$ with $m_{12}' = m_1' \cup m_2'$. For the case $n = 2$, the contexts and their concept lattices are displayed in Fig. 3.

We want to compare the numbers of concepts of $\mathbb{K}_{nge}^1$ and that of $\mathbb{K}_n^1$. Table 2 shows the difference in the size of these concept lattices.

We denote by I the restriction of the incidence relation of $\mathbb{K}_n^1$ on any subcontext of $\mathbb{K}_n^1$, and also by I the incidence relation in the generalized context $\mathbb{K}_{nge}^1$. We set

$$\begin{aligned} \mathbb{K}_{00} &:= (S_n \cup \{g_1\}, S_n, \mathrm{I}), \\ \mathbb{K}_{02} &:= (S_n \cup \{g_1\}, S_n \cup \{m_2\}, \mathrm{I}), \\ \mathbb{K}_{01} &:= (S_n \cup \{g_1\}, S_n \cup \{m_1\}, \mathrm{I}), \\ \mathbb{K}_{0s} &:= (S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I}) = \mathbb{K}_{nge}^1, \\ \mathbb{K}_{12} &:= (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I}) = \mathbb{K}_n^1. \end{aligned}$$

The context $\mathbb{K}_{00}$ has 2^n concepts since g_1 is a reducible object in $\mathbb{K}_{00}$ and the remaining context after removing g_1 is $(S_n, S_n, \neq)$. The context $\mathbb{K}_n^1$ is obtained by adding successively m_2 to $\mathbb{K}_{00}$ to get $\mathbb{K}_{02}$, and then m_1 to $\mathbb{K}_{02}$. The generalized context is obtained by adding $s = m_{12}$ to $\mathbb{K}_{00}$ (see Fig. 4).

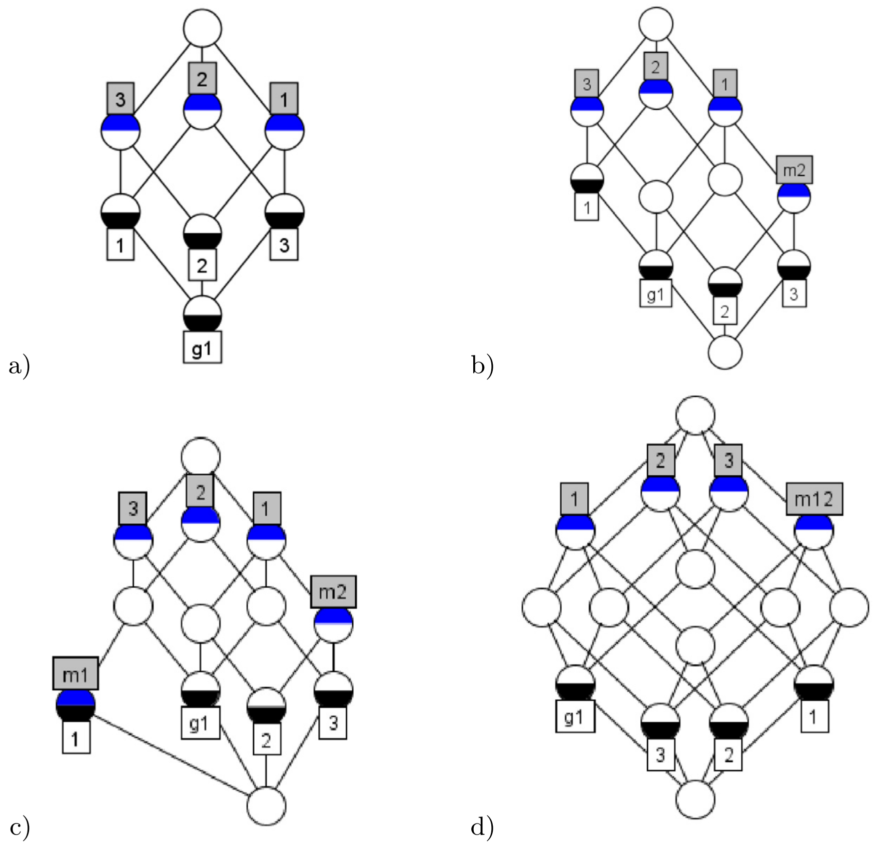

Fig. 4. (a) $\mathfrak{B}(\mathbb{K}_{00})$, (b) $\mathfrak{B}(\mathbb{K}_{02})$, (c) $\mathfrak{B}(\mathbb{K}_{12})$ and (d) $\mathfrak{B}(\mathbb{K}_{0s})$, for $n = 3$.

What happens when m_2 is added to $\mathbb{K}_{00}$? Every extent A of $\mathbb{K}_{00}$ is of the form $A = A_1 \cup \{g_1\}$ with $A_1 \subseteq S_n$ and satisfies $A \cap m_2' \notin \mathrm{Ext}(\mathbb{K}_{00})$. It therefore generates two concepts in $\mathbb{K}_{02}$. The extents A with $A_1 \subseteq m_2' = \{2, \ldots, n\}$ do not coincide on m_2', and therefore generate 2^{n-1} concepts in $\mathbb{K}_{02}$ that cannot be identified (via ϕ_{m_2}) with a concept of $\mathbb{K}_{00}$. If A is an extent of $\mathbb{K}_{00}$ containing 1 then $A \setminus \{1\}$ is also an extent of $\mathbb{K}_{00}$, and both extents coincide on m_2'. Thus by Proposition 2 we get $|\mathfrak{B}(\mathbb{K}_{02})| = 2^n + 2^{n-1}$. Now adding m_1 to $\mathbb{K}_{02}$ generates at most two concepts, since $m_1' = \{1\}$ and $\mathcal{H}(m_1) \subseteq \{A \cap m_1' \mid A \in \mathrm{Ext}(\mathbb{K}_{02})\} = \{\emptyset, m_1'\}$. The extents generated by $\{1\}$ in $\mathbb{K}_{00}$ and in $\mathbb{K}_{02}$ are equal to $\{1, g_1\}$. Thus $\{1, g_1\} \cap m_1' = m_1' \notin \mathrm{Ext}(\mathbb{K}_{02})$. However, $\emptyset \in \mathrm{Ext}(\mathbb{K}_{02})$ and $\emptyset \cap m_1' = \emptyset$. Therefore $\mathcal{H}(m_1) = \{m_1'\}$ and $|\mathfrak{B}(\mathbb{K}_{12})| = 2^n + 2^{n-1} + 1$. The context $(S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I})$ is isomorphic to $(S_{n+1}, S_{n+1}, \neq)$. The object g_1 is identified with $n+1$ and the attribute m_{12} with $n+1$. Thus generalizing m_1 and m_2 to m_{12} leads to a lattice with 2^{n+1} concepts. The increase is then $2^{n+1} - \left(2^n + 2^{n-1} + 1\right) = 2^{n-1} - 1$ which is exponential in the number of attributes of the initial context. We can summarize the discussion above in the following proposition:

Proposition 4. *Let $n \geq 2$ and $(S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, I)$ defined by (1). Putting the attributes m_1 and m_2 with a $\exists$-generalization increases the size of the concept lattice by $2^{n-1} - 1$.*

In the above construction of $\mathbb{K}^1_{nge}$ the idea is to construct a context $(E, E, \neq)$ from the initial context, via a $\exists$-generalization. The objects in S_n are split between m_1 and m_2 with no overlap. We can choose a split that assigns k objects of S_n to m_1 and the other $n-k$ to m_2. Let $\mathbb{K}^k_n := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$ be such a context, where I is defined by

$$g\mathrm{I}m :\iff \begin{cases} g, m \in S_n & \text{and } g \neq m, \text{ or} \\ g = g_1 & \text{and } m \in S_n \text{ or} \\ g \in \{1, 2, \ldots, k\} & \text{and } m = m_1 \text{ or} \\ g \in S_n \setminus \{1, 2, \ldots, k\} & \text{and } m = m_2 \end{cases} \tag{2}$$

Then the existential generalization of the attributes m_1 and m_2 to m_{12} leads to the generalized context $\mathbb{K}^k_{nge} := (S_n \cup \{g_1\}, S_n \cup \{m_{12}\}, \mathrm{I}) \cong (S_{n+1}, S_{n+1}, \neq)$. To get the cardinality of $\mathfrak{B}(\mathbb{K}^k_n)$, we observe that

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(i) $\mathbb{K}_{00} := (S_n \cup \{g_1\}, S_n, \mathrm{I})$ has 2^n concepts. The extents of $\mathbb{K}_{00}$ are of the form $A \uplus \{g_1\}, A \subseteq S_n$.
(ii) $\mathbb{K}_{02} := (S_n \cup \{g_1\}, S_n \cup \{m_2\}, \mathrm{I})$ has $2^n + 2^{n-k}$ concepts. They are of the form $(A \uplus \{g_1\}, S_n \setminus A)$ with $A \subseteq S_n$ or $(A, (S_n \setminus A) \uplus \{m_2\})$ with $A \subseteq m_2'$.
(iii) $\mathbb{K}_{01} := (S_n \cup \{g_1\}, S_n \cup \{m_1\}, \mathrm{I})$ has $2^n + 2^k$ concepts, which are of the form $(A \uplus \{g_1\}, S_n \setminus A)$ with $A \subseteq S_n$ or the form $(A, (S_n \setminus A) \uplus \{m_1\})$ with $A \subseteq m_1'$.

We get $\mathbb{K}_{12} := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$ from $\mathbb{K}_{02}$ by adding m_1. To compute $\mathcal{H}(m_1)$ with respect to $\mathbb{K}_{02}$ we take $A \in \mathrm{Ext}(\mathbb{K}_{02})$ and distinguish two cases:

(i) If $g_1 \notin A$, then $A \subseteq m_2'$, and $A \cap m_1' = \emptyset$ is an extent of $\mathbb{K}_{02}$. No new concept is generated.
(ii) If $g_1 \in A$, then the extent A is of the form $A = A_1 \uplus \{g_1\}$ with $A_1 \subseteq S_n$. Since $m_1' \cap m_2' = \emptyset$, we get $A_1 \cap m_1' \notin \mathrm{Ext}(\mathbb{K}_{02}) \iff A_1 \cap m_1' \nsubseteq m_2'$. Thus the number of additional concept generated is

$$\left|\left\{A \cap m_1' \mid A \in \mathrm{Ext}(\mathbb{K}_{02}) \text{ and } A \cap m_1' \nsubseteq m_2'\right\}\right|$$

Among the extents of $\mathbb{K}_{02}$ with $A \cap m_1' \nsubseteq m_2'$, there are $2^k - 1$ that do not coincide on m_1', for example those with $\emptyset \neq A_1 \subseteq m_1'$. This means that adding m_1 to $\mathbb{K}_{02}$ will generate $2^k - 1$ new concepts that cannot be identified with concepts in $\mathbb{K}_{02}$. Therefore $\mathbb{K}_n^k$ has $2^n + 2^{n-k} + 2^k - 1$ concepts.

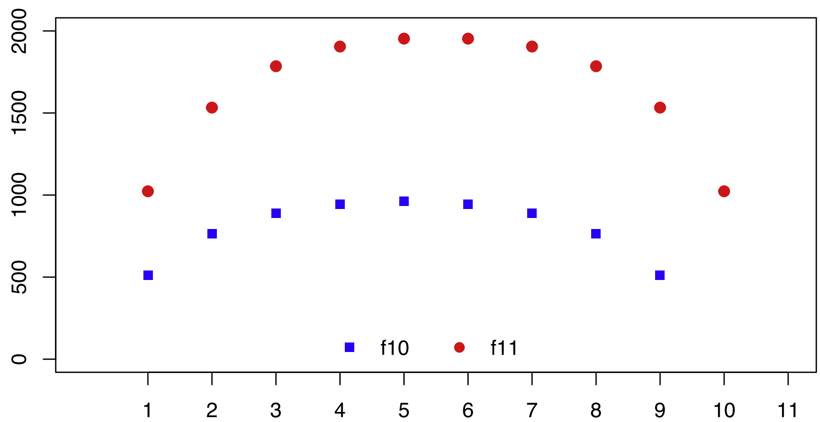
Proposition 5. *Let $n \geq 2$, $1 \leq k < n$ and $\mathbb{K}_n^k$ defined by (2).*

(a) The context $\mathbb{K}_n^k$ has $2^n + 2^{n-k} + 2^k - 1$ concepts.
(b) Generalizing m_1 and m_2 increases the number of concepts by

$$f_n(k) := 2^n - 2^k - 2^{n-k} + 1.$$

A natural question is to find out which $\mathbb{K}_n^k$ has a maximal increase. The function $f_n(k)$ is convex and its slope vanishes at $k = \frac{n}{2}$. In fact,

$$\frac{d}{dk} f_n(k) = \ln 2\left(-2^k + 2^{n-k}\right) = 0 \iff n = 2k.$$
$$\frac{d^2}{dk^2} f_n(k) = -(\ln 2)^2\left(2^k + 2^{n-k}\right) < 0.$$



The maximum is reached when the objects are evenly split; i.e. $k \in \left\{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\right\}$. The above diagram shows f_n for $n = 10$ and $n = 11$. If $n = 2q$, then the maximal increase is $f_{2q}(q) = 2^{2q} - 2 \cdot 2^q + 1 = (2^q - 1)^2$. If $n = 2q + 1$, then the maximal increase is $f_{2q+1}(q) = 2^{2q+1} - 2^q - 2^{q+1} + 1 = (2^q - 1)\left(2^{q+1} - 1\right)$.

We could allow overlap in constructing $\mathbb{K}_n^k$ by using any covering of S_n with two proper subsets m_1' and m_2'; i.e. $m_1' \cup m_2' = S_n$ with $\emptyset \subsetneq m_1', m_2' \subsetneq S_n$. An $\exists$-generalization that puts m_1 and m_2 together to get m_{12}, will also have 2^{n+1} concepts. However, the concept lattice of $\mathbb{K}_{12}$ will have more concepts when $m_1' \cap m_2' \neq \emptyset$ compared to when $m_1' \cap m_2' = \emptyset$. The minimal increase in that case is achieved with $|m_1'| = n - 2 = |m_2'|$ and $|m_1' \cap m_2'| = n - 3$.

Let $\mathbb{K}_{12} := (S_n \cup \{g_1\}, S_n \cup \{m_1, m_2\}, \mathrm{I})$ with $m_1' \cap m_2' \neq \emptyset$. If $m_1' \subseteq m_2'$ or $m_2' \subseteq m_1'$ then putting m_1 and m_2 together will not increase the size of the concept lattice. Therefore, we assume that $m_1' \parallel m_2'$.

Proposition 6. *Let $n > 2$ and $\mathbb{K}_s$ the $\exists$-generalized context obtained from $\mathbb{K}_{12}$ by putting m_1 and m_2 together. Then:*

1. *The context $\mathbb{K}_{12}$ has $2^n + 2^{|m_2'|} + 2^{|m_1'|} - 2^{|m_2' \cap m_1'|}$ concepts.*
2. *After the generalization, the size of the initial lattice increases by*

$$2^n - 2^{|m_1'|} - 2^{|m_2'|} + 2^{|m_1' \cap m_2'|}.$$

Before we start with the proof we look at a concrete case with $n = 3$. Its context is isomorphic to

	1	2	3	m_1	m_2
1		$\times$	$\times$	$\times$	
2	$\times$		$\times$	$\times$	$\times$
3	$\times$	$\times$			$\times$
g_1	$\times$	$\times$	$\times$		

and has $2^3 + 2^2 + 2^2 - 2^1 = 14$ concepts.

Proof. $\mathbb{K}_{12} := (S_n \uplus \{g_1\}, S_n \uplus \{m_1, m_2\}, \mathrm{I})$ has $2^n + 2^{|m_1'|} + \mathcal{H}(m_2)$ concepts, where $\mathcal{H}(m_2)$ is to be determined with respect to $\mathbb{K}_{01} := (S_n \cup \{g_1\}, S_n \cup \{m_1\}, \mathrm{I})$. The concepts of $\mathbb{K}_{01}$ are of the form $(A_1 \uplus \{g_1\}, S_n \setminus A_1)$ with $A_1 \subseteq S_n$ or of the form $(A_1, (S_n \setminus A_1) \cup \{m_1\})$ with $A_1 \subseteq m_1'$. Let $A \in \mathrm{Ext}\,(\mathbb{K}_{01})$.

– If $g_1 \notin A$ then $A \subseteq m_1'$ and $A \cap m_2'$ is a subset of m_1', and by then an extent of $\mathbb{K}_{01}$. No new concept is generated.
– If $g_1 \in A$ then $A = A_1 \uplus \{g_1\}$ with $A_1 \subseteq S_n$. Since all subsets of m_1' are extents of $\mathbb{K}_{01}$ and the object g_1 does not have the attribute m_2, we have

$$A \cap m_2' \notin \mathrm{Ext}(\mathbb{K}_{01}) \iff A \cap m_2' \in \mathcal{P}(m_2') \setminus \mathcal{P}(m_1').$$

Thus, adding m_2 to $\mathbb{K}_{01}$ will generate $2^{|m_2'|} - 2^{|m_1' \cap m_2'|}$ new concepts that cannot be identified (via ϕ_{m_2}) with concepts in $\mathbb{K}_{01}$. Then $\mathbb{K}_{12}$ has

$$2^n + 2^{|m_1'|} + 2^{|m_2'|} - 2^{|m_1' \cap m_2'|}$$

concepts. The increase of the size of the lattice is then

$$\begin{aligned} |\mathfrak{B}(\mathbb{K}_{12})| - |\mathfrak{B}(\mathbb{K}_{01})| &= 2^{n+1} - \left(2^n + 2^{|m_1'|} + 2^{|m_2'|} - 2^{|m_1' \cap m_2'|}\right) \\ &= 2^n - 2^{|m_1'|} - 2^{|m_2'|} + 2^{|m_1' \cap m_2'|} \quad \square \end{aligned}$$

Remark 1. Note that $n = |m_1' \cup m_2'|$ and the increase is

$$2^{|m_1' \cup m_2'|} - 2^{|m_1'|} - 2^{|m_2'|} + 2^{|m_1' \cap m_2'|}$$

which is a general formula that holds, even if $m_1' \cup m_2' \neq S_n$. The starting context is $\mathbb{K}_{00} := (S_n \uplus \{g_1\}, S_n, \mathrm{I})$ and has 2^n extents. After adding an attribute m_1 to $\mathbb{K}_{00}$ we increase the number of extents by $2^{|m_1'|}$. After adding m_2 to $\mathbb{K}_{00}$ we increase the number of extents by $2^{|m_2'|}$. After adding an attribute s with $s' = m_1' \cup m_2'$ to $\mathbb{K}_{00}$ we increase the extents by $2^{|m_1' \cup m_2'|}$. If we add an attribute t with $t' = m_1' \cap m_2'$ to $\mathbb{K}_{00}$ we will increase the extents by $2^{|m_1' \cap m_2'|}$. However, these extents "appear" already when m_1 or m_2 is added to $\mathbb{K}_{00}$, and are therefore counted twice when both m_1 and m_2 are added to $\mathbb{K}_{00}$.

Remark 2. The counting with $\mathbb{K}_{12}$ has been made easy by the fact that each "subset" of S_n identifies an extent of $\mathbb{K}_{00}$. If $m_1' \cap m_2'$ is not empty, then $\mathbb{K}_{12}$ has more concepts while the number of generalized concept remains the same. Then the condition $m_1' \cap m_2' = \emptyset$ is necessary (but not sufficient) to get the maximal increase. For the contexts $\mathbb{K}_{12}$, putting m_1 and m_2 together can increase the size of the concept lattice by up to $\left(2^{\lfloor \frac{n}{2} \rfloor} - 1\right)\left(2^{\lceil \frac{n}{2} \rceil} - 1\right)$ concepts. Is this the maximal increase for contexts of similar size?

Remark 3. Note that all contexts $\mathbb{K}_{12}$ we have constructed are reduced. Requiring the contexts to be reduced is a fair assumption. If not then we should first remove reducible attributes before processing with a generalization. This removal does not affect the size of the concept lattice. However, putting together two reducible attributes will for sure not decrease the size, but probably increases it.

Remark 4. B_4 is the smallest lattice for which there are two attributes whose $\exists$-generalization increases the size of the concept lattice. All lattices presented in this section contain a labeled copy of B_4 (as subposet!). Is there any characterization of contexts for which generalizing increases the size, for example in terms of forbidden subcontexts or subposets?

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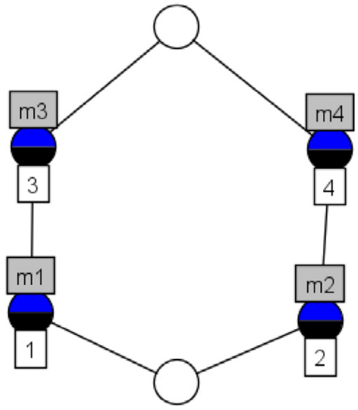

Fig. 5. A concept lattice for B_4.

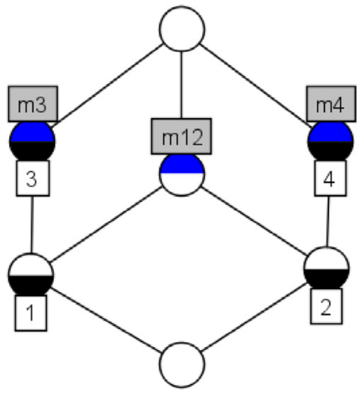

Fig. 6. Generalizing m_1 and m_2.

In this section we have found out that the size of the generalized concept lattice can be exponentially larger than that of the initial concept lattice after an existential generalization. In the next section we will discuss the maximum of the increase after a ∃-generalization (see Figs. 5 and 6).

4. Maximum increase after an existential generalization

In this section we investigate the maximal increase in the general case. This means that from a context $\mathbb{K} := (G, M, \mathrm{I})$ which is attribute reduced, two attributes a, b are removed and replaced with an attribute s defined by $s' = a' \cup b'$. We set $M_0 = M \setminus \{a, b\}$ and adopt the following notation:

$$\begin{aligned}
\mathbb{K}_{00} &:= (G, M_0, \mathrm{I}), && \text{(removing } a, b \text{ from } \mathbb{K}) \\
\mathbb{K}_{01} &:= (G, M_0 \mathbin{\dot\cup} \{a\}, \mathrm{I}), && \text{(adding } a \text{ to } \mathbb{K}_{00}) \\
\mathbb{K}_{02} &:= (G, M_0 \mathbin{\dot\cup} \{b\}, \mathrm{I}), && \text{(adding } b \text{ to } \mathbb{K}_{00}) \\
\mathbb{K}_{0s} &:= (G, M_0 \mathbin{\dot\cup} \{s\}, \mathrm{I}), && \text{(generalized context)} \\
\mathbb{K}_{12} &:= (G, M_0 \mathbin{\dot\cup} \{a, b\}, \mathrm{I}) = \mathbb{K}. && \text{(initial context)}
\end{aligned}$$

In general we get context $\mathbb{K}$ by adding a to $\mathbb{K}_{00}$ and get $\mathbb{K}_{01}$, and add b to $\mathbb{K}_{01}$. Recall that if an attribute m is added to any context $\mathbb{K}$, then the number of concepts increase by

$$h(m) = |\{A \cap m' \mid A \in \mathrm{Ext}(\mathbb{K}) \text{ and } A \cap m' \notin \mathrm{Ext}(\mathbb{K})\}|$$

We denote by $a \cap b$ the attribute defined by $(a \cap b)' := a' \cap b'$, and by $a \cup b$ the attribute defined by $(a \cup b)' := a' \cup b' = s'$. We start from $\mathbb{K}_{00}$. Adding the attribute a to $\mathbb{K}_{00}$ increases its number of concepts by

$$h(a) = |\{A \cap a' \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap a' \notin \mathrm{Ext}(\mathbb{K}_{00})\}| \leq 2^{|a'|}.$$

Adding the attribute b to $\mathbb{K}_{00}$ increases its number of concepts by

$$h(b) = |\{A \cap b' \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap b' \notin \mathrm{Ext}(\mathbb{K}_{00})\}| \leq 2^{|b'|}.$$

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Adding the attribute $a \cap b$ to $\mathbb{K}_{00}$ increases its number of concepts by

$$h(a \cap b) = |\{A \cap a' \cap b' \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap a' \cap b' \notin \mathrm{Ext}(\mathbb{K}_{00})\}| \leq 2^{|a' \cap b'|}.$$

However, these concepts appear in $\mathcal{H}(a)$ and $\mathcal{H}(b)$ and will be counted twice. If $a' \cap b'$ is empty then $h(a \cap b) \leq 1$.

Adding the attribute $a \cup b$ to $\mathbb{K}_{00}$ increases its number of concepts by

$$\begin{aligned} h(a \cup b) &= |\{A \cap (a' \cup b') \mid A \in \mathrm{Ext}(\mathbb{K}_{00}) \text{ and } A \cap (a' \cup b') \notin \mathrm{Ext}(\mathbb{K}_{00})\}| \\ &\leq 2^{|a' \cup b'|} \leq 2^{|a'|+|b'|}. \end{aligned}$$

If $h(a \cup b) = 2^{|a'|+|b'|}$ then $a' \cap b' = \emptyset$ and no subset of $a' \cup b'$ is an extent of $\mathbb{K}_{00}$, but each subset of $a' \cup b'$ is the restriction of an extent of $\mathbb{K}_{00}$ on $a' \cup b'$. In this case $h(a) = 2^{|a'|}$, $h(b) = 2^{|b'|}$ and $h(a \cap b) = 1$.

We denote by $h(a, b)$ the increase when two attributes a and b are both added to $\mathbb{K}_{00}$. Then we have

$$\begin{aligned} |\mathfrak{B}(\mathbb{K}_{12})| &= |\mathfrak{B}(\mathbb{K}_{01})| + h(b) - h(a \cap b) \\ &= |\mathfrak{B}(\mathbb{K}_{00})| + h(a) + h(b) - h(a \cap b). \end{aligned}$$

and $h(a, b) = h(a) + h(b) - h(a \cap b)$. The increase is then

$$|\mathfrak{B}(\mathbb{K}_{0s})| - |\mathfrak{B}(\mathbb{K})| = h(a \cup b) - h(a, b) = h(a \cup b) - h(a) - h(b) + h(a \cap b)$$

If $h(a \cup b) = 2^{|a'|+|b'|}$ then the increase is

$$h(a \cup b) - h(a) - h(b) + h(a \cap b) = 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1.$$

Now we are going to show that this increase is the least upper bound. Since we are interested in the maximal increase, we assume that $a' \cap b' = \emptyset$. In fact $\mathbb{K}$ has more concepts when $a' \cap b' \neq \emptyset$, than when $a' \cap b' = \emptyset$; But the number of concepts of $\mathbb{K}_{0s}$ will remain the same in both cases. The increase $|\mathfrak{B}(\mathbb{K}_{0s})| - |\mathfrak{B}(\mathbb{K})|$ is then larger if $|\mathfrak{B}(\mathbb{K})|$ is smaller.

There exists $d_1 \leq 2^{|a'|}$ such that $h(a) = 2^{|a'|} - d_1$. In fact

$$d_1 = |\{A \subseteq a' \mid A \in \mathrm{Ext}(\mathbb{K}_{00})\}|.$$

Similarly, there exists $d_2 \leq 2^{|b'|}$ such that $h(b) = 2^{|b'|} - d_2$. As above we have

$$d_2 = |\{A \subseteq b' \mid A \in \mathrm{Ext}(\mathbb{K}_{00})\}|.$$

Similarly, there exists $d_0 \leq 2^{|a' \cup b'|}$ such that $|\mathcal{H}(s)| = 2^{|a' \cup b'|} - d_0 = 2^{|a'|+|b'|} - d_0$, with

$$d_0 = |\{A \subseteq a' \cup b' \mid A \in \mathrm{Ext}(\mathbb{K}_{00})\}|.$$

Since we assume $a' \cap b' = \emptyset$, the following holds for any extent $A \neq \emptyset$ of $\mathbb{K}_{00}$:

$$A \subseteq a' \cup b' \iff A \subseteq a' \quad \text{xor} \quad A \subseteq b' \quad \text{xor} \quad A \subseteq a' \cup b', A \nsubseteq a' \text{ and } A \nsubseteq b'.$$

where xor denotes the exclusive or. Therefore $d_1 + d_2 \leq d_0$ and the increase is then

$$\begin{aligned} |\mathfrak{B}(\mathbb{K}_{0s})| - |\mathfrak{B}(\mathbb{K})| &= h(a \cup b) - h(a, b) \\ &= h(a \cup b) - h(a) - h(b) + h(a \cap b) \\ &= \left(2^{|a'|+|b'|} - d_0\right) - \left(2^{|a'|} - d_1\right) - \left(2^{|b'|} - d_2\right) + h(a \cap b) \\ &= 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + h(a \cap b) + \underbrace{d_1 + d_2 - d_0}_{\leq 0} \\ &\leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + h(a \cap b) \\ &\leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1, \end{aligned}$$

since $h(a \cap b) \in \{0, 1\}$ when $a' \cap b' = \emptyset$.

Theorem 1. *Let (G, M, I) be an attribute reduced context and a, b be two attributes such that their existential generalization $s = a \cup b$ increases the size of the concept lattice. Then*

(i) $|\mathfrak{B}(G, M, I)| = |\mathfrak{B}(G, M \setminus \{a, b\}, I)| + h(a, b)$, *with*

$$h(a, b) = h(a) + h(b) - h(a \cap b).$$

(ii) The increase after generalizing is

$$h(a \cup b) - h(a) - h(b) + h(a \cap b) \leq 2^{|a'|+|b'|} - 2^{|a'|} - 2^{|b'|} + 1.$$

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Remark 5. If generalizing a and b does not increase the size of the lattice, then the difference

$$h(a \cup b) - h(a) - h(b) + h(a \cap b)$$

is at most zero, and will describe the reduction in the number of concepts.

5. Related works and conclusion

To the best of our knowledge generalized attributes were first considered by Srikant and Agrawal [14], as they discussed the problem of mining **generalized association rules**. In fact they consider an *is-a* taxonomy on items to extract relevant information in the form of association rules in transactions data. They called such rules generalized association rules. Note that using an is-a taxonomy is equivalent to the ∃-generalization of some attributes. The following example from [14] shows the added-value of generalizing attributes. If an is-a taxonomy is available and says for example that: Jacket is-a Outwear, Ski Pants is-a Outwear, Outwear is-a Clothes, etc., then generating rules that span different levels of the taxonomy could let to discovering interesting information, that were not possible without generalizing the attributes. A rule like "people who buy Outwear tend to buy Hiking Boots" may be inferred from the fact that people bought Jackets with Hiking Boots and Ski Pants with Hiking Boots. However, the rules "Jackets $\Longrightarrow$ Hiking Boots" and "Ski Pants $\Longrightarrow$ Hiking Boots" might be non-relevant if their support is smaller than the fixed threshold. Although the papers [3,14] present some applications of generalizing attributes, they do not discuss the size of the information discovered.

In the present work, we have shown a family of concepts lattices in which an existential generalization on a specific pair of attributes increases the size of the lattice. We have also found the maximal increase when two attributes are put together. We are planing to look at the numbers of rules that can be discovered after generalizing the attributes. If there are some cases where the number of concepts as well as the number of rules increase after a generalization, then not all is-a taxonomies are useful in terms of the size of extracted information. Therefore characterizing contexts where such a generalization does not increase the number of extracted concepts/rules is a crucial pretreatment to make sure that extracted information will not explode.

There are many other works dealing with the size of the concept lattice. In [1] the authors gave an upper bound for the size of concept lattices. In fact they proved for finite contexts that $|\mathfrak{B}(G, M, I)| \leq \sum_{i=0}^{k-1} \binom{n}{i}$, with $n = |G|$ and $k = \min\{|S| : (S, S, \neq) \text{ is not isomorphic to a subcontext of } (G, M, I)\}$. This upper bound is reached by the so-called (n, k)-*extremal* concept lattices. We will investigate in future works what happens if we generalize two attributes of an (n, k)-extremal concept lattice. Is it possible to get an $(n, k+1)$-extremal lattice from an (n, k)-extremal lattice after an existential generalization? The size of the concept lattice can be exponential in the size of the context. Moreover, it was proved in [11] that the problem of determining the size of the lattice is #P-complete. This explains why it is difficult to compute or estimate the size without computing the lattice. Deciding whether the number of concept increases after generalizing would have been easy if counting the number of concept of a given lattice were straightforward. Since counting the number of concept is #P-complete, it is worth to investigate whether the change in the number of concepts after generalization can be accessed without computing the concept lattices.

To control the size of the lattice some authors have proposed to constrain the concept lattice by attribute dependencies in the form of attribute implications [16] or in the form of attribute-dependency formulas [5]: i.e. to only select from the initial lattice the concepts that satisfy these attribute dependencies. In these two situations, the attribute dependencies are set in advance by an expert. The size of the constrained lattice is usually smaller than that of the initial lattice. In the absence of the expert, putting some attributes together is an option. Moreover, an expert can also be part of this pre-treatment by suggesting which attributes should be put together. Other authors working on pattern structures have applied extensional projection on pattern structures [7] to obtained projected pattern structures having a smaller number of pattern concepts than the initial pattern structures. However, projected patterns concepts are subsets of the set of pattern concepts of the initial pattern structure, which is not always the case after generalizing attributes. In [10], the discussion is on how to construct a concept lattice of a context resulting from another context with a known concept lattice by removing exactly one incidence. Removing that incidence leads to the construction of a closure operator and an interior operator such that if a concept of the initial context is a fix-point of both operators, then it is duplicated into two different concepts in the new concept lattice, else it vanishes from the new concept lattice. Hence, removing an incidence from a formal context may lead to a reduction or to an increase of the size of the initial concept lattice. This is similar to the situation observed when generalizing attributes. Moreover, generalizing attributes can be seen as adding many incidences to the initial context.

Generalizing attributes also has a close connection with boolean factorization [4], that we are studying now. Another direction of interest is to look at efficiently computable similarity measures that discriminate attributes if putting these together increases the number of concepts.

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