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OPERATIONS AND METRICS ON INTUITIONISTIC FUZZY SETS AND APPLICATIONS TO DECISION MAKING

DISSERTATION

Presented in partial fulfillment of the requirement for the award of

Ph.D. in MATHEMATICS

Option

Applied Mathematics to Social Sciences

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Academic year : 2020 - 2021

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Nous soussignés, membres du jury lors de la soutenance de thèse de Doctorat Ph.D de Monsieur **DZATI KAMGA Romuald Thierry**, étudiant à l'Université de Yaoundé I sous le matricule **16Z5043**, attestons que la thèse intitulée « **OPERATIONS AND METRICS ON INTUITIONISTIC FUZZY SETS AND APPLICATIONS TO DECISION MAKING** », présentée en soutenance publique le Lundi 29 Novembre 2021 à 10 heures dans la salle Multimédia de la faculté des sciences par le candidat, a été corrigée conformément à nos recommandations.

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♣ Dedications ♣

I dedicate this work to

my wife Sylvanes DJOMBOU KOUCHOU

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♣ Declaration of Honor ♣

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, no material which to a substantial extent has been accepted for the award of any other degree or diploma at The University of Yaoundé I or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at The University of Yaoundé I or elsewhere, is explicitly cited in the thesis and listed in the bibliography. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in style, presentation and linguistic expression is acknowledge.

Romuald Thierry DZATI KAMGA

♣ Résumé ♣

La théorie des ensembles flous intuitionniste (IFS) introduit par Atanassov [1] généralise celle des ensembles flous proposée par Zadeh [46] et contribue à résoudre les problèmes de la vie réelle dans un environnement incertain et non probabiliste. Afin d'évaluer la proximité ou la similarité entre deux ensembles dans cet environnement, plusieurs auteurs ont proposé, étudié et utilisé des outils importants tels que la mesure de distance, la mesure de similarité et la métrique. Cependant les outils basés sur les différences symétrique n'ont pas encore été explorés.

Dans cette thèse, nous définissons, à l'aide des R-implication et co-implication flou, les opérations de différence et de différence symétrique de deux ensembles flous intuitionnistes (IFS). Nous étudions leurs propriétés. Nous proposons des classes de mesures de distance et de similarité sur les IFS. Nous déterminons des conditions sur les opérateurs d'implication et de co-implication pour lesquelles plusieurs de ces classes deviennent des métriques (distances). Nous appliquons ces mesures de distance et ces mesures de similarité dans le cas des t-représentables t-normes floues intuitionnistes de Lukasiewicz, Maximum et Produit pour contribuer à la prise de décision dans les problèmes de reconnaissance de formes et de diagnostic médical.

Mots clés: Sous ensemble flou intuitionniste; Différence symétrique; Mesure de distance; Reconnaissance de formes et de Diagnostic médical.

♣ Abstract ♣

The theory of intuitionistic fuzzy sets (IFSs) introduced by Atanassov [1] generalizes fuzzy set theory proposed by Zadeh [46] and solves some problems in real life dealing with imprecision and vagueness. In order to evaluate closeness or similarity between two sets in these problems, many authors have proposed, studied and used important tools such as distance measure, similarity measure and metric. But there are no tools based on symmetric difference.

In this thesis, based on R-implication and co-implication, we define difference and symmetric difference operations of two Intuitionistic fuzzy sets (IFSs). We study their properties. We propose some classes of distance and similarity measures on IFSs. We determine conditions on both fuzzy implication operators and fuzzy co-implication operators under which many of those classes become metrics (distances). We apply those distances measures and those similarity measures in the case of the t-representable IF t-norms of Lukasiewicz, Maximum and Product to contribute to the decision making in the problems of pattern recognition and medical diagnosis.

Keywords: Intuitionistic fuzzy set; Symmetric difference; Distance measure; Pattern recognition and medical diagnosis.

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♣ Introduction ♣

Our society harbors problems that are very often the cause of social conflicts, stress and even the high rate of mortality. Pattern recognition problem and medical diagnosis problem are among these. Some of those problems are solved in supply chain management ([23, 28, 35]) and social choice ([15]). The social sciences are part of the life sciences which identifies and offers solutions in order to resolve social conflicts. This then requires more in-depth tools. In the search for such tools, Atanassov [1] introduced the theory of intuitionistic fuzzy sets (IFSs) to generalize fuzzy set theory introduced by Zadeh [46] and to solve some problems in real life dealing with imprecision and vagueness. For example, it has been applied in logic programming in [5], pattern recognition in [31, 43] and medical diagnosis in [30, 45] to solve some decisions making problems.

Evaluate closeness or similarity between two intuitionistic fussy sets draw attention of authors these recent years. More precisely, authors have proposed, studied and used important tools such as similarity measures, distance measures and metrics. Szmidt and Kacprzyk [39] proposed four distance measures: Hamming distance, Euclidean distance, Normalized Hamming distance and Normalized Euclidean distance. Wang and Xin [43] showed that the normalized Euclidean distance may not classify patterns in some cases. They provided a more generalized definition of distance measures between IFSs inspired from the definition of geometric distance between crisp sets, and then they proposed many new distance measures. Grzegorzewski [22] proposed other Hamming and Euclidean distance measures for IFSs based on Hausdorff metric. Hung and Yang [26, 27] proposed some distance measures one based on L_p metric and other based on exponential, and then compared them to existing measures. Mitchell [32] proposed other distance measure.

It should be noted that in our society, many problems arise from the differences that resiles between the parties, which is what physicists call force action. It is therefore imperative to create a symmetrization of this difference to neutralize the causes of these differences, this is what physicists call reciprocal force. The symmetry difference thus appears as a tool for social balance. So, according to the literature review, distance measures and metrics for IFSs based on symmetric difference have not yet been studied as in the case of fuzzy sets and crisp sets. In addition, the existing literature on difference and symmetric difference operations on IFS, introduced by Huawen [25], Bustince et al. [4], Ejepwa [13] presents some intuitive difficulties. The two major concerns of this thesis are to introduce and study (i) Difference and symmetric difference for two IFSs (associated to a IF t-norm) preserving usual properties of those operations on fuzzy and crisp sets and (ii) distance measures, similarity measures and metrics generated by the obtained symmetric difference operations on IFSs(X).

The following are the contributions of our study. Inspired by the work of Fono et al. [17] on fuzzy sets, we introduce new definitions for difference and symmetric difference for intuitionistic fuzzy sets, by means of intuitionistic fuzzy R-implications and we study their properties. We establish that some common properties of the difference operations for fuzzy sets defined by Fono et al. [17] and for crisp sets are preserved by the new difference and symmetric difference operations for intuitionistic fuzzy sets. Then the new proposed difference and symmetric difference operations for intuitionistic fuzzy sets generalize the case for fuzzy sets introduced by Fono et al. [17]. This constitutes the first main stage of the thesis. The second main stage of this work is based on some classes of distance measure, similarity measures and metrics based on cardinality of new symmetric difference. For that, we firstly use the means of symmetric difference between two IFSs to give new definition for cardinality of symmetric difference between IFSs associated to a t-representable IF t-norm. We use this new definition to propose eight new classes of distance measures based on IF-cardinality symmetric difference of IFSs and by extension eight classes of similarity measures using dual construction. And we determine two conditions based on both fuzzy R-implications and fuzzy co-implication operators under which many of those distances measures become metrics. We apply the new distance measures (under the $t-$representable IF $t-$norms of Lukasiewicz, Maximum and Product to) solve the problem of pattern recognition introduced by Wang and Xin [43]. And we use the new similarity measures (under the $t-$representable IF $t-$norms of Lukasiewicz and Product) to solve the problem of medical diagnosis recalled in [40, 42, 47].

It is important to outline three notes from our contribution. The eight classes of distances measures are inspired from (i) the three distance measures proposed by Wang and Xin [43], (ii) the two Hamming and Euclidean distances for IFSs based on Hausdorff metric proposed by Grzegorzewski [22], (iii) the two distance measures one based on L_p metric proposed by Hung and Yang [26] and (iii) the distance measures proposed by Mitchell [32]. The obtained Euclidean and Hamming distance measure are the same as those proposed by Grzegorzewski [22] and the distance measure based on exponential function is the same as the one proposed by Hung and Yang [27] when we consider the $t-$representable IF $t-$norms of Lukasiewicz. Using those classes of distance measures, we determine two conditions (condition C^* and condition C^2) on both fuzzy implication operators and fuzzy co-implication operators under which many of those classes become metrics. Furthermore, we show that all the $t-$representable IF t-norms of the Frank and Mayor-torrens usual parametric families satisfy condition C^* and then generated metrics.

The continuation of this thesis is next organized as follows. Chapter 1 focuses on literature review where we recall some preliminaries on fuzzy sets and intuitionistic fuzzy sets and their operations. And we present the problems of pattern recognition and medical diagnosis. In Chapter 2, we introduce difference and symmetric difference between two intuitionistic fuzzy sets and based on fuzzy implications and fuzzy co-implications. We study their properties and we propose some classes of IF cardinality for difference and symmetric difference of IFSs. In Chapter 3, we propose and study new distance measures, similarity measures and metrics based on cardinality components of symmetric difference between IFSs. Chapter 4 is based on some applications of the obtained distance and similarity measures for decision making. More precisely, we use the IF t-norms T_L of Lukasiewicz, T_M Maximum and T_P Product in order to apply six classes of the proposed distance measures to pattern recognition problem and two classes of proposed similarity measures to medical diagnosis problem. Finally we give some concluding remarks.

Chapter 1

Literature review

Contents

Throughout this chapter, X is a non empty set.

1.1 On fuzzy sets

1.1.1 Description

Fuzzy Logic has been successfully applied in various areas pertaining to wireless communication systems. As fuzzy logic is used to model systems and situations, taking into consideration uncertainty and ambiguity, it can be an efficient tool to be used in problems for which knowledge of all factors is insufficient or impossible to obtain. The following definition of fuzzy set is as such.

Definition 1.1.1 ([46]). *A fuzzy set A in X is given by $A = \{(x, \mu_A(x)),\ x \in X\}$ where $\mu_A : X \to [0,1]$ is the membership function of A and $\mu_A(x) \in [0,1]$ is the membership degree of $x \in X$ in A.*

If for all $x \in X,\ \mu_A(x) \in \{0,1\}$, then A is a crisp set.
The following example display the membership function to define a fuzzy set A.

Example 1. Assume that X denote the set of age. The membership function to define a fuzzy set $A = \{$To have about twenty years old$\}$ is given in the following Figure 1.1:

We observe that the membership function can be fixed arbitrarily. So the explicite

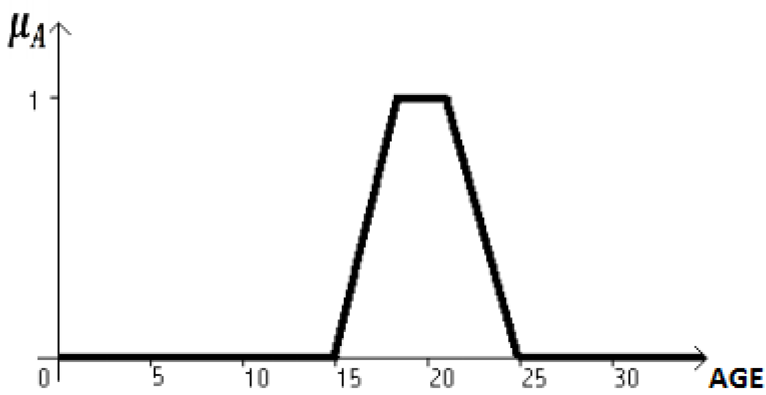

Figure 1.1: To have about twenty years hold

form is: for all $x \in X = \mathbb{R}_+$,

$$\mu_A(x) = \begin{cases} 0.4x - 6, & \text{if } 15 \leq x < 17.5; \\ 1, & \text{if } 17.5 \leq x < 22.5; \\ -0.4x + 10, & \text{if } 22.5 \leq x < 25; \\ 0, & \text{otherwise.} \end{cases}$$

The following Subsection presents the fuzzy operators and operations.

1.1.2 Fuzzy operators and operations

Let us give first the fuzzy t-norms and t-conorms.

Fuzzy t-norms and t-conorms

Definition 1.1.2. *1. A fuzzy triangular-norm (fuzzy t-norm) is a binary operation $\top : [0,1] \times [0,1] \longrightarrow [0,1]$ such that for any $x \in [0,1]$, $\top(x,1) = x$ and $\top$ satisfies commutativity ($\forall a, b \in [0,1]$, $\top(a,b) = \top(b,a)$), monotonicity (increasing) ($\forall a,b,c,d \in [0,1]$, if $a \leq b$ and $c \leq d$, then $\top(a,c) \leq \top(b,d)$) and associativity ($\forall a, b, c, \in [0,1]$, $\top(a, \top(b,c) = \top(\top(a,b),c)$).*

2. *A fuzzy t-conorm is a binary operation $S : [0,1] \times [0,1] \longrightarrow [0,1]$ such that for any $x \in [0,1]$, $S(x,0) = x$ and S satisfies commutativity, monotonicity (increasing) and associativity.*

3. *A fuzzy negation N is a non-increasing mapping $N : [0,1] \longrightarrow [0,1]$ with $N(0) = 1$ and $N(1) = 0$. If $N(N(x)) = x$, $\forall x \in [0,1]$ (i.e. N satisfies the involutive property), then N is called strong fuzzy negation.*

4. *The dual of a fuzzy t-norm $\top$ is a fuzzy t-conorm S, such that, for all $a, b \in [0,1]$, $\top(a,b) = 1 - S(1-a, 1-b)$.*

Klement et al. [29] displayed seven families of $t-$norms and their dual $t-$conorms. We give in the following examples the expressions of t-norms and their dual t-conorms of two families namely Frank and Mayor-Torrens. These two families will commonly be referred to in our study.

Example 2. ([17, 20, 29])

1. The Frank t-norms $\left(\top_F^l\right)_{l\in[0,+\infty]}$ are given by: for all $a, b \in [0,1]$,

$$\top_F^l(a,b) = \begin{cases} \top_M(a,b) = \min(a,b), & \text{if } l = 0, \\ \top_P(a,b) = ab, & \text{if } l = 1, \\ \top_L(a,b) = \max(a+b-1, 0), & \text{if } l = +\infty, \\ \log_l\left(1 + \frac{(l^a-1)(l^b-1)}{l-1}\right), & \text{otherwise,} \end{cases} \tag{1.1}$$

where $\top_M$, $\top_P$ and $\top_L$ are respectively the minimum, product and Lukasiewicz fuzzy t-norms. The Frank t-conorms $\left(S_F^l\right)_{l\in[0,+\infty]}$ are given by: for all $a, b \in [0,1]$,

$$S_F^l(a,b) = \begin{cases} S_M(a,b) = \max(a,b), & \text{if } l = 0, \\ S_P(a,b) = a+b-ab, & \text{if } l = 1, \\ S_L(a,b) = \min(a+b, 1), & \text{if } l = +\infty, \\ 1 - \log_l\left(1 + \frac{(l^{1-a}-1)(l^{1-b}-1)}{l-1}\right), & \text{otherwise,} \end{cases} \tag{1.2}$$

where S_M, S_P and S_L are respectively the maximum, probabilistic sum and Lukasiewicz fuzzy t-conorms.

2. The Mayor-Torrens $t-$norms $(\top^{\lambda}_{M\top})_{\lambda\in[0,1]}$ are given by: for all $\lambda \in [0,1]$ and $a, b \in [0,1]$,

$$\top^{\lambda}_{M\top}(a,b) = \begin{cases} \max(a+b-\lambda, 0), & \text{if } \lambda \in]0,1] \text{ and } (a,b) \in [0,\lambda]^2 \\ \min(a,b), & \text{otherwise} \end{cases} ; \quad (1.3)$$

The Mayor-Torrens t-conorms $(S^{\lambda}_{M\top})_{\lambda\in[0,1]}$ are given as: for all $\lambda \in [0,1]$ and $a, b \in [0,1]$,

$$S^{\lambda}_{M\top}(a,b) = \begin{cases} max(a+b+\lambda-1, 1), & \text{if } \lambda \in]0,1] \text{ and } (a,b) \in [1-\lambda, 1]^2 \\ max(a,b), & \text{otherwise.} \end{cases} . \quad (1.4)$$

We now give the definition of fuzzy implication and co-implication associated respectively with $t-$norm and $t-$conorm.

Fuzzy implications and co-implications

Definition 1.1.3. *Let $\top$ be a t-norm and S be a t-conorm.*

- *The fuzzy residual implication (for short R-implication) operator associated to $\top$ is the binary operator $I^1_\top$ on $[0,1]$ defined as follows: for all $a, b \in [0,1]$ by $I^1_\top(a,b) = \sup\{t \in [0,1] \mid \top(a,t) \leq b\}$.*

- *The fuzzy symmetric contraposition implication operator associated to $\top$ is the binary operator $I^2_\top$ on $[0,1]$ defined as follows: for all $a, b \in [0,1]$, $I^2_\top(a,b) = 1 - \inf\{t \in [0,1],\ S(b,t) \geq a\}$.*

- *The $QL-$implication operator associated to $\top$ is the binary operator $I^3_\top$ on $[0,1]$ defined as follows: for all $a, b \in [0,1]$, $I^3_\top(a,b) = S(n(a), \top(a,b))$.*

- *The $S-$implication operator associated to $\top$ is the binary operator $I^4_\top$ on $[0,1]$ defined as follows: for all $a, b \in [0,1]$, $I^4_\top(a,b) = S(n(a), b)$.*

- *The fuzzy residual co-implication (for short co-implication) operator associated to a t-conorm S is the binary operator J_S on $[0,1]$ defined as follow: for all $a, b \in [0,1]$ by $J_S(a,b) = \inf\{r \in [0,1] \mid b \leq S(a,r)\}$.*

It is important to notice that if $\top$ and S are left-continuous, the previous operators ($I^1_\top$, $I^2_\top$ and J_S) become: for all $a,\ b \in [0,1]$,

$$\begin{cases} I^1_\top(a,b) = \max\{t \in [0,1] \mid \top(a,t) \leq b\}; \\ I^2_\top(a,b) = 1 - \min\{t \in [0,1],\ S(b,t) \geq a\}; \\ J_S(a,b) = \min\{r \in [0,1] \mid b \leq S(a,r)\}. \end{cases} \tag{1.4}$$

We will go further in this thesis by assuming that, $\top$ and S are left-continuous t-norm and t-conorm respectively. $I_\top = I^1_\top$ is the fuzzy R-implication operator associated to $\top$ and J_S is the fuzzy co-implication operator associated to S.

Fono and Fotso. [16] displayed the fuzzy R-implication and the fuzzy co-implication associated with seven families of $t-$norms and $t-$co-norms of Klement et al. [29]. The following are examples of fuzzy R-implicators and fuzzy co-implicators associated with family of Frank t-norms and their dual $t-$conorm, and Mayor-Torrens t-norms, and their dual $t-$conorm.

Example 3. ([16], [17], [20]) For all $a, b \in [0,1]$:

1. Fuzzy R-implication associated with T_M and fuzzy co-implication associated with S_M are respectively given by:

$$I_{\top_M}(a,b) = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{if } a > b. \end{cases} \quad \textit{and } J_{S_M}(a,b) = \begin{cases} b, & \text{if } a < b, \\ 0, & \text{if } a \geq b. \end{cases} \tag{1.4}$$

2. Fuzzy R-implication associated with T_P and fuzzy co-implication associated with S_P are respectively given by:

$$I_{\top_P}(a,b) = \begin{cases} 1, & \text{if } a \leq b, \\ \frac{b}{a}, & \text{if } a > b. \end{cases} \quad \textit{and } J_{S_P}(a,b) = \begin{cases} \frac{b-a}{1-a}, & \text{if } a < b, \\ 0, & \text{if } a \geq b. \end{cases} \tag{1.4}$$

3. Fuzzy R-implication associated with T_L and fuzzy co-implication associated with S_L are respectively given by:

$$I_{\top_L}(a,b) = \begin{cases} 1, & \text{if } a \leq b, \\ 1-a+b, & \text{if } a > b. \end{cases} \quad \textit{and } J_{S_L}(a,b) = \begin{cases} b-a, & \text{if } a < b, \\ 0, & \text{if } a \geq b. \end{cases} \tag{1.4}$$

4. Fuzzy R-implication associated with $\top_F^l$ and fuzzy co-implication associated with S_F^l for all $l \in (0,1) \cup (1,+\infty)$ are respectively given by:

$$I_{\top_F^l}(a,b) = \begin{cases} 1, & \text{if } a \leq b, \\ \log_l\left(1 + \frac{(l-1)(l^b-1)}{l^a-1}\right), & \text{if } a > b. \end{cases} \quad and \ J_{S_F}^l(a,b) = \begin{cases} 1 - \log_l\left(1 + \frac{(l-1)(l^{1-b}-1)}{l^{1-a}-1}\right), & \text{if } a < b, \\ 0, & \text{if } a \geq b. \end{cases} \tag{1.4}$$

5. Fuzzy R-implications associated with $(\top_{M\top}^\lambda)_{\lambda\in[0,1]}$ and fuzzy co-implications associated with $(S_{M\top}^\lambda)_{\lambda\in[0,1]}$ are respectively given by: for all $\lambda \in [0,1]$ and $a,b \in [0,1]$,

$$I_{\top_{M\top}^\lambda}(a,b) = \begin{cases} \begin{cases} 1 \text{ if } a \leq b \\ b \text{ if } a > b \end{cases} & \text{if } \left(\begin{array}{c} \lambda = 0 \\ or \\ \lambda \neq 0,\ a \in [0,\lambda] \ and \ b \in (\lambda,1] \\ or \\ \lambda \neq 0,\ a \in (\lambda,1] \ and \ b \in [0,\lambda] \\ or \\ \lambda \neq 0 \ and \ (a,b) \in (\lambda,1]^2 \end{array} \right) \\ \begin{cases} 1 \text{ if } a \leq b \\ \lambda + b - a \text{ if } a > b \end{cases} & \text{if } \lambda \in (0,1] \ and \ (a,b) \in [0,\lambda]^2 \end{cases} \tag{1.4}$$

and

$$J_{S_{M\top}^\lambda}(a,b) = \begin{cases} 0 \text{ if } a \geq b \\ 1 + b - (a+\lambda) \text{ if } (a < b,\ a \in [1-\lambda,1] \text{ and } \lambda \in (0,1]) \\ b \text{ if } \left(\begin{array}{c} \lambda = 0,\ a \neq 1 \ and \ a < b \\ or \\ \lambda \neq 0,\ a \in [0,1-\lambda] \ and \ a < b \end{array} \right) \end{cases} \tag{1.4}$$

We will require the following useful results to establish the proofs of some basic findings in this thesis.

Proposition 1.1.1 ([17, 19]). *For all $a,b,c \in [0,1]$,*

1. $I_\top(a,a) = 1$; $J_S(a,a) = 0$; $J_S(a,b) \leq b \leq I_\top(a,b)$ *and* $I_\top(1,a) = a = J_S(0,a)$;

2. $b < a \Longleftrightarrow (I_\top(a,b) < 1$ *or* $J_S(b,a) > 0)$;

3. $a \leq b \Rightarrow \begin{cases} I_\top(b,c) \leq I_\top(a,c), \\ I_\top(c,a) \leq I_\top(c,b). \end{cases}$ *and* $a \leq b \Rightarrow \begin{cases} J_S(b,c) \leq J_S(a,c), \\ J_S(c,a) \leq J_S(c,b). \end{cases}$
   Thus $I_\top$ *and* J_S *are left decreasing and right increasing operators.*

4. *Let* S *and* $\top$ *be such that, for all* $a, b \in [0,1]$, $\top(a,b) \leq 1 - S(1-a, 1-b)$. *Then*

   i. *for all* $a, b \in [0,1]$, $I_\top(a,b) \geq 1 - J_S(1-a, 1-b)$;

   ii. *if* $\top$ *and* S *are dual, then for all* $a, b \in [0,1]$, $I_\top(a,b) = 1 - J_S(1-a, 1-b)$.

Fuzzy operations of fuzzy sets

Definition 1.1.4 ([17, 19])**.** *Let* A *and* B *be any two fuzzy sets defined on* X. *The following operations are defined by associated membership function as follows:*

i) *Inclusion:* $A \subseteq B$ *if and only,* $\mu_A(x) \leq \mu_B(x)$, $\forall\, x \in X$;

ii) *Intersection:* $A \cap B$ *is defined by:* $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$, $\forall\, x \in X$;

iii) *Union:* $A \cup B$ *is defined by:* $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$, $\forall\, x \in X$;

iv) *Complement:* A^c *is defined by:* $\mu_{A^c}(x) = 1 - \mu_A(x)$, $\forall\, x \in X$.
   Where $\vee$ *and* $\wedge$ *are* max *and* min *respectively.*

In the next Subsection, we recall the difference and symmetric difference operations for fuzzy sets, some examples and their properties as proposed by Fono et al. [17]

1.1.3 Difference and symmetric difference of fuzzy sets based on fuzzy implications

The following definition give difference and symmetric difference of fuzzy sets based on fuzzy implications.

Definition 1.1.5 (Difference and Symmetric Difference Operations for Fuzzy Sets [17])**.**

a. *Let* M, N *be any two fuzzy sets defined on* X *and* $i \in \{1,2,3,4\}$. *The fuzzy difference of type* i *associated to* $\top$ *of* M *and* N *is the fuzzy set of* X *denoted by* $M \overset{i}{\underset{\top}{-}} N$ *and defined by:*

$$\mu_{M \overset{i}{\underset{\top}{-}} N}(x) = \overline{I^i_\top}(\mu_M(x), \mu_N(x)) = 1 - I^i_\top(\mu_M(x), \mu_N(x)), \quad \textit{for all } x \in X.$$

b. *The fuzzy symmetric difference of type $i \in \{1,2\}$ associated to $\top$ of M and N is the fuzzy set of X denoted by $M \overset{i}{\underset{\top}{\triangle}} N$ and defined for all $x \in X$ by:*

$$\mu_{M \overset{i}{\underset{\top}{\triangle}} N}(x) = \mu_{M\cup N \overset{i}{\underset{\top}{-}} M\cap N}(x) = \begin{cases} 1 - I^1_\top\left(\mu_M(x) \vee \mu_N(x), \mu_M(x) \wedge \mu_N(x)\right), & \text{if } i = 1 \\ 1 - I^2_\top\left(\mu_M(x) \vee \mu_N(x), \mu_M(x) \wedge \mu_N(x)\right), & \text{if } i = 2. \end{cases}$$

We recall the examples of these operations for fuzzy sets of type 1 and 2 associated to the usual three fuzzy t-norms in what follows.

Example 4. For any fuzzy sets M and N defined on X,

1. Examples of fuzzy difference operations

   (a) The difference operation associated with $\top_M$ is given by, for all $x \in X$

$$\mu_{M \overset{1}{\underset{\top_M}{-}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) \leq \mu_N(x), \\ 1 - \mu_N(x), & \text{if } \mu_M(x) > \mu_N(x), \end{cases}$$

$$\mu_{M \overset{2}{\underset{\top_M}{-}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) \leq \mu_N(x), \\ \mu_M(x), & \text{if } \mu_M(x) > \mu_N(x) \end{cases}$$

   (b) The difference operation associated with $\top_P$ is given by, for all $x \in X$

$$\mu_{M \overset{1}{\underset{\top_P}{-}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) \leq \mu_N(x), \\ 1 - \frac{\mu_N(x)}{\mu_M(x)}, & \text{if } \mu_M(x) > \mu_N(x), \end{cases}$$

$$\mu_{M \overset{2}{\underset{\top_P}{-}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) \leq \mu_N(x), \\ \frac{\mu_M(x) - \mu_N(x)}{1 - \mu_N(x)}, & \text{if } \mu_M(x) > \mu_N(x). \end{cases}$$

   (c) The difference operation associated with $\top_L$ is given by, for all $x \in X$ and $i \in \{1,2\}$,

$$\mu_{M \overset{i}{\underset{\top_L}{-}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) \leq \mu_N(x), \\ \mu_M(x) - \mu_N(x), & \text{if } \mu_M(x) > \mu_N(x). \end{cases}$$

2. Examples of fuzzy symmetric difference operations

   (a) The symmetric difference operation associated with $\top_M$ is given by, for all $x \in X$

$$\mu_{M \overset{1}{\underset{\top_M}{\triangle}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) = \mu_N(x), \\ \max\left(1 - \mu_M(x), 1 - \mu_N(x)\right), & \text{if } \mu_M(x) \neq \mu_N(x), \end{cases}$$

$$\mu_{M \overset{2}{\underset{\top_M}{\Delta}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) = \mu_N(x), \\ \max\left(\mu_M(x), \mu_N(x)\right), & \text{if } \mu_M(x) \neq \mu_N(x). \end{cases}$$

(b) The symmetric difference operation associated with $\top_P$ is given by, for all $x \in X$

$$\mu_{M \overset{1}{\underset{\top_P}{\Delta}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) = \mu_N(x) = 0, \\ \frac{|\mu_M(x) - \mu_N(x)|}{\max\{\mu_M(x), \mu_N(x)\}}, & \text{if } \mu_M(x) \neq 0, \text{ or } \mu_N(x) \neq 0\ , \end{cases}$$

$$\mu_{M \overset{2}{\underset{\top_P}{\Delta}} N}(x) = \begin{cases} 0, & \text{if } \mu_M(x) = \mu_N(x) = 1, \\ \frac{|\mu_M(x) - \mu_N(x)|}{1 - \min\{\mu_M(x), \mu_N(x)\}}, & \text{if } \mu_M(x) \wedge \mu_N(x) < 1. \end{cases}$$

(c) The symmetric difference operation associated with $\top_L$ is given by, for all $x \in X$ and $i \in \{1, 2\}$,

$$\mu_{M \overset{i}{\underset{\top_L}{\Delta}} N}(x) = \mid \mu_M(x) - \mu_N(x) \mid .$$

Fono et al. [17] have also proved that the difference and symmetric difference operations for fuzzy sets of type 1 and 2 associated to any continuous t-norm $\top$ so defined preserve the properties of the classical difference and symmetric difference operation for crisp sets. We recall these results as follow:

Proposition 1.1.2. *Let $i \in \{1, 2\}$ and M, M', N be any arbitrary fuzzy sets on X. The following properties hold [17]:*

1. *Properties of fuzzy difference operation;*
   (a) if $M \subseteq N$, then $M \overset{i}{\underset{\top}{-}} N = \emptyset$, (b) if $M \subseteq M'$, then $M \overset{i}{\underset{\top}{-}} N \subseteq M' \overset{i}{\underset{\top}{-}} N$, (c) if $M \subseteq M'$, then $N \overset{i}{\underset{\top}{-}} M' \subseteq N \overset{i}{\underset{\top}{-}} M$, (d) $\left(M \overset{i}{\underset{\top}{-}} N\right) \cap \left(N \overset{i}{\underset{\top}{-}} M\right) = \emptyset$ and (e) $M \overset{i}{\underset{\top}{-}} N = M \overset{i}{\underset{\top}{-}} (M \cap N)$.

2. *Properties of fuzzy symmetric difference operation;*
   (a) $M \overset{i}{\underset{\top}{\Delta}} N = \left(M \overset{i}{\underset{\top}{-}} N\right) \cup \left(N \overset{i}{\underset{\top}{-}} M\right)$,
   (b) if $M \subseteq N$, then $M \overset{i}{\underset{\top}{\Delta}} N = N \overset{i}{\underset{\top}{-}} M$ and (c) $M \overset{i}{\underset{\top}{\Delta}} M = \emptyset$.

The following result shows that, the fuzzy complement of fuzzy sets associated with any continuous t-norm $\top$ so defined, preserve the property of the classical complement for crisp sets.

Corollary 1. Let $\top$ be any continuous t-norm, A a fuzzy set X, and A^c the fuzzy complement of A associated with $\top$.
Then $A^c = X -_\top A$.

Proof. Let $x \in X$. From Definition 1.1.5, it is sufficient to show that: $\mu_{X-_\top A}(x) = 1 - \mu_A(x)$.
Since $\mu_X(x) = 1$ and $I_\top(1, a) = a$ (see Proposition 1.1.1) for all $a \in [0; 1]$, from Definition 1.1.5,
$\mu_{X-_\top A}(x) = 1 - I_\top(\mu_X(x), \mu_A(x)) = 1 - \mu_A(x)$. □

In the following section, We describe some useful knowledge in of intuitionistic fuzzy sets.

1.2 On intuitionistic fuzzy sets (IFSs)

Fuzzy set is not appropriate to describe some situation in real life which shared ambiguous, vagueness and incomplete informations. It takes into account only a positive interpretation and looses a negative one. Note that there are situations where positive and negative interpretation are not dual. According to Example 1, in the context of FS "to have 25 years old" is the same situation as "to have 100 years old". But such situation is not right in real context. Intuitionistic fuzzy sets (IFSs) are very important tools to describe these situations.
In the following Subsection, we will describe IFSs on X (IFSs(X))

1.2.1 Description

To generalize Fuzzy set, IFS has been proposed by Atanassov [1] and has been successfully applied in various areas pertaining to decision making. As intuitionistic fuzzy logic is used to model systems and situations, taking into consideration uncertainty and ambiguity, it can be an efficient tool to be used in problems for which knowledge of all factors is vague. In the following, we will give the definition and geometrical interpretation of an example of IFSs.

Definition 1.2.1 (Atanassov [1]). *An intuitionistic fuzzy set D on X is defined by:*

$$D = \{(x, \mu_D(x), \nu_D(x)) \mid \mu_D(x), \nu_D(x) \in [0, 1],\ 0 \leq \mu_D(x) + \nu_D(x) \leq 1,\ \forall x \in X\},$$

where $\mu_D(x)$, $\nu_D(x)$ are the degrees of membership and non-membership of x in D respectively.

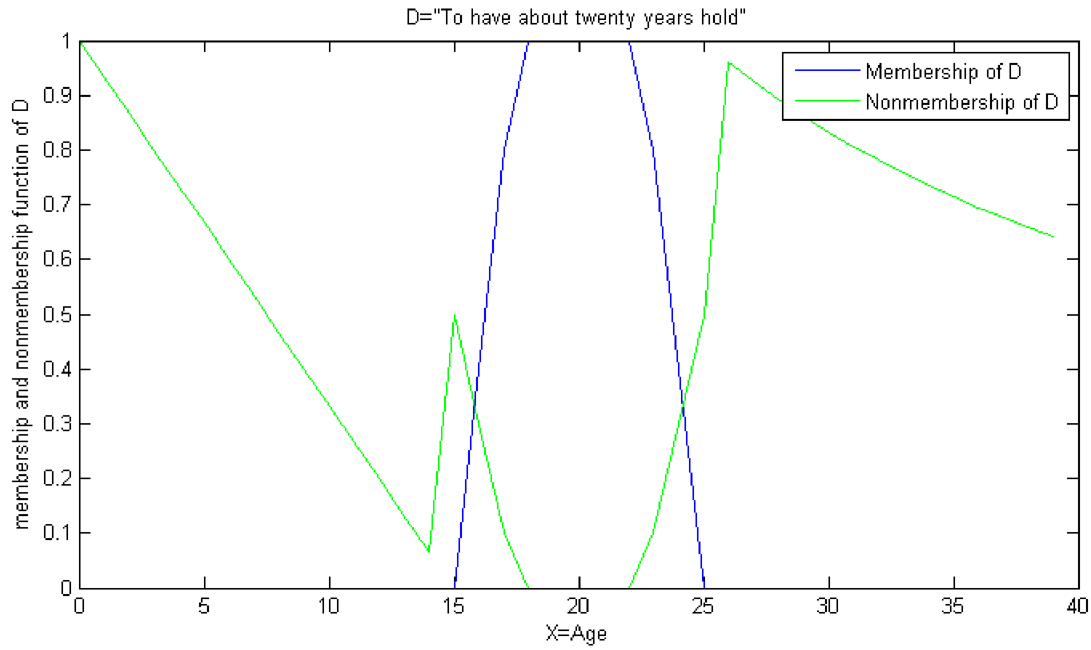

Figure 1.2: To have about twenty years hold

If $\mu_D(x) + \nu_D(x) = 1$, then D is a fuzzy set on X.
For each intuitionistic fuzzy set D on X, we call $\pi_D(x) = 1 - \mu_D(x) - \nu_D(x)$ the intuitionistic index of x in D. It is the hesitancy degree of x to D, and it is obvious that $0 \leq \pi_D(x) \leq 1$ for each $x \in X$.
The following example display the membership and nonmembership function to define a IFS D.

Example 5. Assume that X denote the set of age. The membership and nonmembership function to define a IFS $D = \{\text{To have about twenty years hold}\}$ is given in Figure 1.2:
We observe that the membership and nonmembership functions can be fixed arbitrarily. So the explicite form is: For all $x \in X = \mathbb{R}_+$,

$$\begin{cases} \mu_D(x) = 0, & \nu_D(x) = -\frac{1}{15}x + 1, & \text{if } 0 \leq x < 15; \\ \mu_D(x) = 0.4x - 6, & \nu_D(x) = -0.2x + 3.5, & \text{if } 15 \leq x < 17.5; \\ \mu_D(x) = 1, & \nu_D(x) = 0, & \text{if } 17.5 \leq x < 22.5; \\ \mu_D(x) = -0.4x + 10, & \nu_D(x) = 0.2x - 4.5, & \text{if } 22.5 \leq x < 25; \\ \mu_D(x) = 0, & \nu_D(x) = \frac{25}{x}, & \text{otherwise.} \end{cases}$$

The following Subsection introduce some basic definitions and provide some preliminary results along on Intuitionistic Fuzzy Operators and Operations, and needed in the rest of this thesis.

1.2.2 Intuitionistic fuzzy operators and operations

In this subsection, we recall the definitions of some intuitionistic fuzzy operators and operations ([6, 19, 36]).
We first recall in the following the definition of lattice and complete lattice.
Notice that, on L-fuzzy sets, Goguen [21] defined L-fuzzy set has a function $f : X \longrightarrow L$ where X is a universe. When $L = [0, 1]$, we have $[0, 1]$-fuzzy set. On Intuitionistic fuzzy sets, Atanassov [1] defined Atanassov's intuitionistic fuzzy set (AIFS) as defined in Definition 1.2.1. On the relationship between some extensions of fuzzy set theory, Deschrijver and Kerre [10] gave an alternative approach for AIFS by justifying that AIFS can also be seen as an L-fuzzy set in the sense of Goguen when the complete lattice $L = L^* = \{(x, y) \in [0, 1]^2,\ x + y \leq 1\}$ that we will use in this thesis.

Definition 1.2.2 (Lattice and complete lattice ([37])). *Let $(E, \mathcal{R})$ be an ordered set with an order relation $\mathcal{R}$.*

1. *$(E, \mathcal{R})$ is a lattice if for all $(x, y) \in E^2$, $\sup(\{x, y\})$ and $\inf(\{x, y\})$ exist.*
2. *$(E, \mathcal{R})$ is a complete lattice if for all $D \subset E$, $\sup(D)$ and $\inf(D)$ exist*

We will subsequently be referring to the complete lattice $(L^*, \leq_{L^*})$ (with $0_{L^*} = (0, 1)$ as bottom element and the unit $1_{L^*} = (1, 0)$ as top element; where $L^* = \{(x_1, x_2) \in [0, 1] \times [0, 1] \mid x_1 + x_2 \leq 1\}$ and $\leq_{L^*}$ is an order on L^* defined by: for all $(x_1, x_2), (y_1, y_2) \in L^*$, $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \geq y_2$.
The meet operator $\wedge$ and the join operator $\vee$ on this lattice are defined for all $(x_1, x_2), (y_1, y_2) \in L^*$ as:

$$(x_1, x_2) \wedge (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2))$$
$$(x_1, x_2) \vee (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)).$$

Intuitionistic Fuzzy Operators

Definition 1.2.3 (Intuitionistic Fuzzy t-norm and t-conorm, ([6, 9, 19])). *1. An intuitionistic fuzzy t-norm is a binary operation $\mathcal{T} : L^* \times L^* \longrightarrow L^*$ such that for any $\mathbf{x} \in L^*$, $\mathcal{T}(\mathbf{x}, 1_{L^*}) = \mathbf{x}$ (neutral element) and, $\mathcal{T}$ satisfies commutativity, monotonicity (increasing) and associativity.*

2. *An intuitionistic fuzzy t-conorm is a binary operation $\mathcal{J} : L^* \times L^* \longrightarrow L^*$ such that for any $\mathbf{x} \in L^*$, $\mathcal{J}(\mathbf{x}, 0_{L^*}) = \mathbf{x}$ and, $\mathcal{J}$ is commutative, monotone increasing and associative.*

We denote by IF-t-norm the intuitionistic fuzzy t-norm and, by IF-t-conorm the intuitionistic fuzzy t-conorm.

Definition 1.2.4 (Intuitionistic Fuzzy Negation, ([6, 9, 19]))**.** *An intuitionistic fuzzy negation is a non-increasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(\mathbf{x})) = \mathbf{x}$, $\forall \mathbf{x} \in L^*$, then $\mathcal{N}$ is said to be involutive. An involutive intuitionistic fuzzy negation is called strong intuitionistic fuzzy negation.*

The following definitions recall useful classes of intuitionistic fuzzy t-norm and t-conorm and, their implications and co-implications.

Definition 1.2.5 (t-Representable intuitionistic fuzzy t-norm and t-conorm ([6, 9, 36, 19]))**.** *An intuitionistic fuzzy t-norm $\mathcal{T}$ (respectively intuitionistic fuzzy t-conorm $\mathcal{J}$) is t-representable if there exists a fuzzy t-norm $\top$ and a fuzzy t-conorm S (respectively a fuzzy t-conorm S' and a fuzzy t-norm $\top'$) such that, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$, $\mathcal{T}(\mathbf{x}, \mathbf{y}) = (\top(x_1, y_1), S(x_2, y_2))$ and $\mathcal{J}(\mathbf{x}, \mathbf{y}) = (S'(x_1, y_1), \top'(x_2, y_2))$, respectively.*

Definition 1.2.6 (Intuitionistic fuzzy R-implication and co-implicator [6, 9, 19])**.**

1. *An intuitionistic fuzzy R-implication (for short, IF-R-implication) associated with an IF-t-norm $\mathcal{T} = (\top, S)$, is a mapping $I_{\mathcal{T}} : L^* \times L^* \longrightarrow L^*$ such that, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$,*

$$\begin{aligned} I_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) &= \sup\{\mathbf{z} \in L^* \mid \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq_{L^*} \mathbf{y}\} \\ &= \sup\{\mathbf{z} = (z_1, z_2) \in L^* \mid \top(x_1, z_1) \leq y_1 \text{ and } S(x_2, z_2) \geq y_2\}. \end{aligned}$$

2. *An intuitionistic fuzzy co-implication (for short, IF-co-implication) associated with an IF-t-conorm, $\mathcal{J} = (S, \top)$, is a mapping $J_{\mathcal{J}} : L^* \times L^* \longrightarrow L^*$ such that, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$,*

$$\begin{aligned} J_{\mathcal{J}}(\mathbf{x}, \mathbf{y}) &= \inf\{\mathbf{z} \in L^* \mid \mathbf{y} \leq_{L^*} \mathcal{J}(\mathbf{x}, \mathbf{z})\} \\ &= \inf\{\mathbf{z} = (z_1, z_2) \in L^* \mid y_1 \leq S(x_1, z_1) \text{ and } y_2 \geq \top(x_2, z_2)\}. \end{aligned}$$

The following known result is very helpful to construct t-representable intuitionistic fuzzy t-norms and t-conorms from fuzzy t-norms and t-conorms.

Proposition 1.2.1. *([6, 9, 19]) Given a fuzzy t-norm $\top$ and fuzzy t-conorm S satisfying $\top(a,b) \leq 1 - S(1-a, 1-b)$ for all $a, b \in [0,1]$. Then $\mathcal{T}(\mathbf{x},\mathbf{y}) = (\top(x_1,y_1), S(x_2,y_2))$ and $\mathcal{J}(\mathbf{x},\mathbf{y}) = (S(x_1,y_1), \top(x_2,y_2))$ for all $\mathbf{x} = (x_1,x_2)$, $\mathbf{y} = (y_1,y_2) \in L^*$, are t-representable intuitionistic fuzzy t-norm and t-representable intuitionistic fuzzy t-conorm respectively.*

The following useful result relates IF-co-implication and IF-R-implication associated with an IF-t-conorm, $\mathcal{J} = (S, \top)$ and IF-t-norm, $\mathcal{T} = (\top, S)$, respectively to corresponding fuzzy co-implication, J_S associated to S and fuzzy R-implication, $I_\top$ associated to $\top$.

Lemma 1 ([19])**.** For any $\mathbf{x} = (x_1,x_2)$, $\mathbf{y} = (y_1,y_2) \in L^*$, we have

1. $J_{\mathcal{J}}(\mathbf{x},\mathbf{y}) = \left(J_S(x_1,y_1), \min\left(I_\top(x_2,y_2), 1 - J_S(x_1,y_1)\right)\right).$

2. $I_{\mathcal{T}}(\mathbf{x},\mathbf{y}) = \left(\min\left(I_\top(x_1,y_1), 1 - J_S(x_2,y_2)\right), J_S(x_2,y_2)\right).$

The following are examples of t-representable IF-t-norms and IF-t-conorms [19].

Example 6. i. $\mathcal{T}_M = (\top_M, S_M)$ and $\mathcal{J}_M = (S_M, \top_M)$ are t-representable IF-t-norm and IF-t-conorm respectively associated to $\top_M$ and S_M.

ii. $\mathcal{T}_P = (\top_P, S_P)$ and $\mathcal{J}_P = (S_P, \top_P)$ are t-representable IF-t-norm and IF-t-conorm respectively associated to $\top_P$ and S_P.

iii. $\mathcal{T}_L = (\top_L, S_L)$ and $\mathcal{J}_L = (S_L, \top_L)$ are t-representable IF-t-norm and IF-t-conorm respectively associated to $\top_L$ and S_L.

iv. Also, by verifying that $\top_F^l(a,b) \leq 1 - S_F^l(1-a, 1-b)$ holds for all $a, b \in [0,1]$, $l \in (0,1) \cup (1,+\infty)$, $\mathcal{T}_F^l = (\top_F^l, S_F^l)$ and $\mathcal{J}_F^l = (S_F^l, \top_F^l)$ are t-representable IF-t-norm and IF-t-conorm respectively associated to $\top_F^l$ and S_F^l for all $l \in (0,1) \cup (1,+\infty)$.

Using Lemma 1 and Example 6, we construct the following examples of IF-R-implication and IF-co-implication associated with an IF-t-norm, $\mathcal{T} = (\top, S)$ and IF-t-conorm, $\mathcal{J} = (S, \top)$.

Example 7. For all $\mathbf{x} = (x_1,x_2), \mathbf{y} = (y_1,y_2) \in L^*$,

1. The IF-R-implication associated with $\mathcal{T}_M = (\top_M, S_M)$ and the IF-co-implication associated with $\mathcal{J}_M = (S_M, \top_M)$ are respectively given by:

$$I_{\mathcal{T}_M}(\mathbf{x},\mathbf{y}) = \begin{cases} (1,0), & \text{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ (\min(y_1, 1-y_2), y_2), & \text{if } \mathbf{x} >_{L^*} \mathbf{y}. \end{cases}$$

and

$$J_{\mathcal{J}_M(\mathbf{x},\mathbf{y})} = \begin{cases} (0,1), & \text{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ (y_1, \min(y_2, 1-y_1)), & \text{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

2. IF-R-implication associated with $\mathcal{T}_P = (\top_P, S_P)$ and IF-co-implication associated with $\mathcal{J}_P = (S_P, \top_P)$ are respectively given by:

$$I_{\mathcal{T}_P}(\mathbf{x},\mathbf{y}) = \begin{cases} (1,0), & \text{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ \left(\min\left(\frac{y_1}{x_1}, \frac{1-y_2}{1-x_2}\right), \frac{y_2-x_2}{1-x_2}\right), & \text{if } \mathbf{x} >_{L^*} \mathbf{y}. \end{cases}$$

and

$$J_{\mathcal{J}_P(\mathbf{x},\mathbf{y})} = \begin{cases} (0,1), & \text{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ \left(\frac{y_1-x_1}{1-x_1}, \min\left(\frac{y_2}{x_2}, \frac{1-y_1}{1-x_1}\right)\right), & \text{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

3. The IF-R-implication associated with $\mathcal{T}_L = (\top_L, S_L)$ and the IF-co-implication associated with $\mathcal{J}_L = (S_L, \top_L)$ are respectively given by:

$$I_{\mathcal{T}_L}(\mathbf{x},\mathbf{y}) = \begin{cases} (1,0), & \text{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ (\min(1-x_1+y_1, 1+x_2-y_2), y_2-x_2), & \text{if } \mathbf{x} >_{L^*} \mathbf{y}. \end{cases}$$

and

$$J_{\mathcal{J}_L(\mathbf{x},\mathbf{y})} = \begin{cases} (0,1), & \text{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ (y_1-x_1, \min(1-x_2+y_2, 1+x_1-y_1)), & \text{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

4. The IF-R-implication associated with $\mathcal{T}_F^l = (\top_F^l, S_F^l)$ and the IF-co-implication associated with $\mathcal{J}_F^l = (S_F^l, \top_F^l)$ for all $l \in (0,1) \cup (1,+\infty)$ are respectively given by:

$$I_{\mathcal{T}_F^l}(\mathbf{x},\mathbf{y}) = \begin{cases} (1,0), & \text{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ \left(\min\left(\log_l\left(1+\frac{(l-1)(l^{y_1}-1)}{l^{x_1}-1}\right), \log_l\left(1+\frac{(l-1)(l^{1-y_2}-1)}{l^{1-x_2}-1}\right)\right), 1-\log_l\left(1+\frac{(l-1)(l^{1-y_2}-1)}{l^{1-x_2}-1}\right)\right), & \text{if } \mathbf{x} >_{L^*} \mathbf{y}. \end{cases}$$

and $$J_{\mathcal{J}_F^l(\mathbf{x},\mathbf{y})} = \begin{cases} (0,1), & \text{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ \left(1-\log_l\left(1+\frac{(l-1)(l^{1-y_1}-1)}{l^{1-x_1}-1}\right), \min\left(\log_l\left(1+\frac{(l-1)(l^{y_2}-1)}{l^{x_2}-1}\right), \log_l\left(1+\frac{(l-1)(l^{1-y_2}-1)}{l^{1-x_2}-1}\right)\right)\right), & \text{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

We end this Subsection by recalling inclusion and some operations on intuitionistic fuzzy sets.

Intuitionistic fuzzy operations

Definition 1.2.7 (Intuitionistic Fuzzy Operations [14, 17, 18]). *Let A and B be any two IFSs defined on X. The following operations are defined by associated membership and non-membership functions as follows:*

i. Inclusion: $A \subseteq B$ *if* $\forall\, x \in X, \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$;

ii. Intersection: $A \cap B$ *is defined by:* $\forall\, x \in X, (\mu_{A\cap B}(x),\, \nu_{A\cap B}(x)) = (\mu_A(x) \wedge \mu_B(x),\, \nu_A(x) \vee \nu_B(x))$;

iii. Union: $A \cup B$ *is defined by:* $\forall\, x \in X, (\mu_{A\cup B}(x),\, \nu_{A\cup B}(x)) = (\mu_A(x) \vee \mu_B(x),\, \nu_A(x) \wedge \nu_B(x))$;

iv. Complement: A^c *is defined by:* $\forall\, x \in X, (\mu_{A^c}(x),\, \nu_{A^c}(x)) = (\nu_A(x),\, \mu_A(x))$;

v. Difference: $A - B$ *is defined by:* $\forall\, x \in X, (\mu_{A-B}(x),\, \nu_{A-B}(x)) = (\mu_A(x) \wedge \nu_B(x),\, \nu_A(x) \vee \mu_B(x))$;

vi. Symmetric Difference: $A \vartriangle B$ *is defined by:* $\forall\, x \in X, (\mu_{A\vartriangle B}(x),\, \nu_{A\vartriangle B}(x)) =$
$(\max\{\mu_A(x) \wedge \nu_B(x),\, \nu_A(x) \wedge \mu_B(x)\},\, \min\{\nu_A(x) \vee \mu_B(x),\, \mu_A(x) \vee \nu_B(x)\})$.

In the sequel, $\top$ is a $t-$norm, S is a $t-$conorm, $\mathcal{J} = (S, \top)$ is a $t-$representable IF$-t-$conorm, $\mathcal{T} = (\top, S)$ is a $t-$representable IF$-t-$norm, $I_\top$ and J_S are fuzzy implication and co-implication operators associated respectively with $\top$ and S.

1.2.3 Distance measures and similarity measures for IFSs

Distance measure is another important tool in IFS theory measuring the difference between IFSs. In the following, we recall definitions and useful properties of a distance measure, similarity measure and metric between two IFSs.

Definition 1.2.8. *([8, 10, 39])*

1. *A mapping* $s : IFSs(X) \times IFSs(X) \to [0, 1]$ *is a similarity measure if for all* $A,\ B,\ C \in IFSs(X)$, *the following properties hold: (i)* $0 \leq s(A, B) \leq 1$; *(ii)* $A = B$ *if and only if* $s(A, B) = 1$; *(iii)* $s(A, B) = s(B, A)$ *and (iv) If* $A \subseteq B \subseteq C$, *then* $s(A, C) \leq s(A, B)$ *and* $s(A, C) \leq s(B, C)$.

2. *Let* $d : IFSs(X) \times IFSs(X) \to [0, 1]$ *be a mapping.*

   a) d *is a distance measure if for all* $A,\ B,\ C \in IFSs(X)$, *the following properties hold:(i)* $0 \leq d(A, B) \leq 1$; *(ii)* $d(A, B) = 0$ *if and only if* $A = B$; *(iii)* $d(A, B) = d(B, A)$; *(iv) If* $A \subseteq B \subseteq C$, *then* $d(A, C) \geq d(A, B)$ *and* $d(A, C) \geq d(B, C)$.

   b) d *is a distance on* $IFSs(X)$ *if for all* $A, B, C \in IFSs(X)$, d *satisfies the following properties: (i):* $d(A, B) = 0$ *if and only if* $A = B$; *(ii):* $d(A, B) = d(B, A)$ *and (iii):* $d(A, C) \leq d(A, B) + d(B, C)$.

Note that we can rewrite the last property of a distance measure as follows: If $A, B, C, D \in IFSs(X)$ such that $A \subseteq B \subseteq C \subseteq D$, then $d(A, D) \geq d(B, C)$.

Corollary 2. [Theorems 3.4 and 3.5 of [43]] Let d be a distance measure, then $s = 1 - d$ is associated similarity measure and vis verso .

Recently Zhou and Wu [47] took into account the hesitancy degree to define some distance based-similarity measures and applied it in medical diagnostic.
In the following example, we recall some usual distances measures.

Example 8. (Some distance measures on IFSs) Let A and B be any two IFSs of $X = \{x_1, x_2, ..., x_n\}$.

(i) Wang and Xin [43] defined the tree following distance measures between A and B:

$$d_1^n(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|}{4} + \frac{\max(|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|)}{2}\right]. \quad (1.\text{-}3)$$

Usually, the weight of the element $x_i \in X$ should be taken into account (assume the weight of the element $x_i \in X$, $(i = 1...n)$ is w_i $(i = 1...n)$, where $0 \leq w_i \leq 1$), so they present the following weighted distance measures between IFSs A and B.

$$d_1^{nw}(A, B) = \frac{1}{nw}\sum_{i=1}^{n}w_i\left[\frac{|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|}{4} + \frac{\max(|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|)}{2}\right]. \quad (1.\text{-}3)$$

where $w = \sum_{i=1}^{n} w_i$.
Let p be a strictly positive integer, they proposed also the following distance measure:

$$d_1^{np}(A, B) = \frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\frac{|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|}{2}\right)^p}. \quad (1.\text{-}3)$$

(ii) Grzegorzewski [22] proposed the two following distance measures (derived from Hamming and Euclidean distance for IFSs based on Hausdorff metric).

$$d_1^{nH}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|\}. \quad (1.\text{-}3)$$

and

$$d_1^{nE}(A, B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\max\{(\mu_A(x_i) - \mu_B(x_i))^2, (\nu_A(x_i) - \nu_B(x_i))^2\}}. \quad (1.\text{-}3)$$

(iii) Hung and Yang [26] proposed some similarity measures based on L_p metric and compared them to existing measures. They deduced the following distance measure based on L_p metric:

$$d_{Lp}^{HY}(A,B) = \frac{1}{n\sqrt[p]{2}}\sum_{i=1}^{n}\left(|\mu_A(x_i) - \mu_B(x_i)|^p + |\nu_A(x_i) - \nu_B(x_i)|^p\right)^{\frac{1}{p}}. \quad (1.\text{-}3)$$

Hung and Yang [27] also proposed the following distance measure:

$$d_1^{HY}(A,B) = \frac{1 - e^{-\frac{1}{n}\sum\limits_{i=1}^{n}\max\{|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|\}}}{1 - e^{-1}}. \quad (1.\text{-}3)$$

(iv) Mitchell [32]

$$d_1^M(A,B) = \frac{1}{2}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)|^p} + \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}|\nu_A(x_i) - \nu_B(x_i)|^p}\right). \quad (1.\text{-}3)$$

As we had presented in the introduction of this thesis, symmetric difference can be consider as an important tool for social balance, nevertheless distance measures based on symmetric difference have not yet be defined. Subsequently, we will refer to distance measures defined by Eqs.(i)-(iv) to propose and study the new distance measures based on symmetric difference between IFSs in Chapter 3.

In the following section we present some societal problems.

1.3 On some society problems

In real life, there are many problems dealing with imprecision and vagueness in decision making. IFSs are a suitable tool to cope with imperfect information. In literature, some of those problems are solved using distance measures or similarity measures between IFSs. In this Section we present the basic concepts of pattern recognition and medical diagnostic.

1.3.1 On pattern recognition

The theory of IFSs has been used to perform pattern recognition in [43, 47]. The pattern recognition problem in the classification of objects using IFSs is defined as follows: We have m known objects $A_1, ..., A_m$ and one unknown object B (test sample) described as IFSs by n features of the universe $X = \{x_1, x_2, ..., x_n\}$ with the weights $w_1, ..., w_n$. The main objective is to determine which known objects is close to B with respect to a distance measure or similarity measure on $IFSs(X)$.

1.3.2 On medical diagnosis

Medical diagnosis comprises of uncertainties and increased volume of information available to physicians from new medical technologies. A diagnosis procedure usually starts off with an interview of patient and doctor [33]. Therefore, the screening method using questionaire is helpful in diagnosis of headache and interview chart is a leading part. The process of classifying different set of symptoms under a single name of a disease is a very difficult task. In some practical situations, there exists possibility of each element within a lower and an upper approximation of IFSs. It can deal with the medical diagnosis involving more indeterminacy. Actually this approach is more flexible and easy to use.
The theory of IFSs has been used to perform medical diagnosis in [40, 34, 44]. The medical diagnosis problem using IFSs is given as follow: We have a set of m diseases $D = \{d_1,\ d_2, ..., d_m\}$ described as IFSs by a set of n symptoms $S = \{s_1,\ s_2, ..., s_n\}$ and a set of k patients $P = \{p_1,\ p_2, ..., p_k\}$. One needs to find a proper diagnosis for each patient $p_1,\ p_2, ..., p_k$. So medical diagnostic problem sometimes use pattern recognition approached.

Chapter 2

New operations of difference and symmetric difference between IFSs

Contents

In this Chapter we define and study the properties of new operations of difference and symmetric difference between IFSs based on fuzzy implications and fuzzy co-implications Operators. In addition, we establish some cardinality properties of the proposed operations.

2.1 Difference operations between IFSs

Huawen [25] used standard definition of classical difference to define difference operation $-$ between two IFSs A and B as an IFS $A - B = A \cap B^c$ such that,

$$\forall x \in X,\ (\mu_{A-B}(x), \nu_{A-B}(x)) = (\mu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \mu_B(x))\,. \tag{2.0}$$

But Huawen's difference operation does not satisfy three of the five basic properties of standard difference operation (see Table 2.1).

	Satisfy	Counter-example
If $A \subseteq B$, then $A - B = \emptyset$	NO	$A = \{(x, 0.3, 0.2), x \in X\}$, $B = \{(x, 0.6, 0.1), x \in X\}$ $A - B = \{(x, 0.1, 0.6), x \in X\} \neq \emptyset = \{(x, 0, 1), x \in X\}$
If $A \subseteq B$, then $A - C \subseteq B - C$	YES	
If $A \subseteq B$, then $C - B \subseteq C - A$	YES	
$(A - B) \cap (B - A) = \emptyset$	NO	$A = \{(x, 0.2, 0.6), x \in X\}$, $B = \{(x, 0.7, 0.1), x \in X\}$ $A - B \cap B - A = \{(x, 0.1, 0.7), x \in X\} \neq \emptyset$
$A - B = A - (A \cap B)$	NO	$A = \{(x, 0.7, 0.2), x \in X\}$, $B = \{(x, 0.2, 0.1), x \in X\}$ $A - A \cap B = \{(x, 0.2, 0.2), x \in X\} \neq A - B = \{(x, 0.1, 0.2), x \in X\}$

Table 2.1: Properties of Huawen's difference operation

In the following Subsections, we will give the definition of difference operations between IFSs, we then study their properties and display some examples.

2.1.1 Definitions and examples

Let $I_{\mathcal{T}} = ({}_1I_{\mathcal{T}}, {}_2I_{\mathcal{T}})$ be an IF-R-implication operator. We define the negation of $I_{\mathcal{T}}$ as $\mathcal{N}(I_{\mathcal{T}}) = ({}_2I_{\mathcal{T}}, {}_1I_{\mathcal{T}})$. In particular, using Lemma 1 we define the negation of IF-R-implication as

$$\forall \mathbf{x} = (x_1, x_2),\ \mathbf{y} = (y_1, y_2) \in L^*, \mathcal{N}(I_{\mathcal{T}}(\mathbf{x}, \mathbf{y})) = (J_S(x_2, y_2), \min\{I_{\top}(x_1, y_1), 1 - J_S(x_2, y_2)\}).$$

Definition 2.1.1. *Let A, B be any two intuitionistic fuzzy sets defined on X. The intuitionistic fuzzy difference associated to $\mathcal{T}$ of A and B is the intuitionistic fuzzy set on X denoted by $A -_{\mathcal{T}} B$ and define by the membership and non-membership degrees as follows: for all $x \in X$,*

$$\begin{aligned}(\mu_{A -_{\mathcal{T}} B}(x), \nu_{A -_{\mathcal{T}} B}(x)) &= \mathcal{N}\left(I_{\mathcal{T}}\left((\mu_A(x), \nu_A(x)), (\mu_B(x), \nu_B(x))\right)\right) \\ &= (J_S(\nu_A(x), \nu_B(x)), \min\{I_{\top}(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}).\end{aligned}$$

The following are typical examples of difference operations associated with the three usual and well-known $\mathcal{T}$.

Example 9. For any intuitionistic fuzzy sets A and B defined on X,

1. The difference operation associated with $\mathcal{T}_M$ is given by:

   for all $x \in X$ $\left(\mu_{A -_{\mathcal{T}_M} B}(x), \nu_{A -_{\mathcal{T}_M} B}(x)\right) = \begin{cases} (0, 1), & \text{if } (\mu_A(x), \nu_A(x)) \leq_{L^*} (\mu_B(x), \nu_B(x)), \\ (\nu_B(x), \min\{\mu_B(x), 1 - \nu_B(x)\}), & \text{else.} \end{cases}$

2. The difference operation associated with $\mathcal{T}_P$ is given by:

for all $x \in X$ $$\left(\mu_{A-_{\mathcal{T}_P}B}(x), \nu_{A-_{\mathcal{T}_P}B}(x)\right) = \begin{cases} (0,1), & \text{if } (\mu_A(x), \nu_A(x)) \leq_{L^*} (\mu_B(x), \nu_B(x)) \\ \left(\frac{\nu_B(x)-\nu_A(x)}{1-\nu_A(x)}, \min\left\{\frac{\mu_B(x)}{\mu_A(x)}, \frac{1-\nu_B(x)}{1-\nu_A(x)}\right\}\right), & \text{else.} \end{cases}$$

3. The difference operation associated with $\mathcal{T}_L$ is given by:

for all $x \in X$ $$\left(\mu_{A-_{\mathcal{T}_L}B}(x), \nu_{A-_{\mathcal{T}_L}B}(x)\right) = \begin{cases} (0,1), & \text{if } (\mu_A(x), \nu_A(x)) \leq_{L^*} (\mu_B(x), \nu_B(x)) \\ \left(\nu_B(x) - \nu_A(x), \min\left\{1 - \mu_A(x) + \mu_B(x), 1 + \nu_A(x) - \nu_B(x)\right\}\right), & \text{else.} \end{cases}$$

We now establish the classical properties of a difference operation.

2.1.2 Some properties allowing to difference between IFSs

In the following results, we will establish four classical properties for difference operation which are satisfied by the new intuitionistic fuzzy difference operation.

Proposition 2.1.1 (Properties of Intuitionistic Fuzzy Difference Operation). *Let A, B, C be intuitionistic fuzzy sets on X. The following properties for intuitionistic fuzzy difference operations hold:*

1. *if $A \subseteq B$, then $A -_{\mathcal{T}} B = \emptyset$;*
2. *if $A \subseteq B$, then $A -_{\mathcal{T}} C \subseteq B -_{\mathcal{T}} C$;*
3. *if $A \subseteq B$, then $C -_{\mathcal{T}} B \subseteq C -_{\mathcal{T}} A$;*
4. *$A -_{\mathcal{T}} B = A -_{\mathcal{T}} (A \cap B)$.*

Proof. By Proposition 1.1.1 and Definition 2.1.1, we establish the results for all $x \in X$ as follows:

1. Assume that, $A \subseteq B$, then $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
   Since $J_S(\nu_A(x), \nu_B(x)) = 0$, whenever $\nu_A(x) \geq \nu_B(x)$, and $I_\top(\mu_A(x), \mu_B(x)) = 1$, whenever $\mu_A(x) \leq \mu_B(x)$ then by Definition 2.1.1, we have $(\mu_{A-_{\mathcal{T}}B}(x), \nu_{A-_{\mathcal{T}}B}(x)) = (0, 1)$ and the result follows.

2. Assume that $A \subseteq B$, then $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

   $$(\mu_{A-_{\mathcal{T}}C}(x), \nu_{A-_{\mathcal{T}}C}(x)) = (J_S(\nu_A(x), \nu_C(x)), \min\{I_\top(\mu_A(x), \mu_C(x)), 1 - J_S(\nu_A(x), \nu_C(x))\}),$$

   $$(\mu_{B-_{\mathcal{T}}C}(x), \nu_{B-_{\mathcal{T}}C}(x)) = (J_S(\nu_B(x), \nu_C(x)), \min\{I_\top(\mu_B(x), \mu_C(x)), 1 - J_S(\nu_B(x), \nu_C(x))\}).$$

   Since $\nu_B(x) \leq \nu_A(x)$, then from Proposition 1.1.1 $J_S(\nu_A(x), \nu_C(x)) \leq J_S(\nu_B(x), \nu_C(x))$.

   So, $\mu_{A-_{\mathcal{T}}C}(x) \leq \mu_{B-_{\mathcal{T}}C}(x)$.

   For the non-membership degree, there are four possibilities:

Case i: $\nu_{A -_{\mathcal{T}} C}(x) = I_{\top}(\mu_A(x), \mu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = I_{\top}(\mu_B(x), \mu_C(x))$.

Since $\mu_A(x) \leq \mu_B(x)$, then $I_{\top}(\mu_A(x), \mu_C(x)) \geq I_{\top}(\mu_B(x), \mu_C(x))$ and we have $\nu_{A -_{\mathcal{T}} C}(x) \geq \nu_{B -_{\mathcal{T}} C}(x)$.

Case ii: $\nu_{A -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_A(x), \nu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_B(x), \nu_C(x))$.

Since $\nu_B(x) \leq \nu_A(x) then, J_S(\nu_A(x), \nu_C(x)) \leq J_S(\nu_B(x), \nu_C(x))$, then we have $\nu_{A -_{\mathcal{T}} C}(x) \geq \nu_{B -_{\mathcal{T}} C}(x)$.

Case iii: $\nu_{A -_{\mathcal{T}} C}(x) = I_{\top}(\mu_A(x), \mu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_B(x), \nu_C(x))$.

Since $\mu_A(x) \leq \mu_B(x)$, then

$$I_{\top}(\mu_A(x), \mu_C(x)) \geq I_{\top}(\mu_B(x), \mu_C(x)) \geq 1 - J_S(\nu_B(x), \nu_C(x)).$$

So, $\nu_{A -_{\mathcal{T}} C}(x) \geq \nu_{B -_{\mathcal{T}} C}(x)$.

Case iv: $\nu_{A -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_A(x), \nu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = I_{\top}(\mu_B(x), \mu_C(x))$.

Since $\nu_A(x) \geq \nu_B(x)$, then

$$1 - J_S(\nu_A(x), \nu_C(x)) \geq 1 - J_S(\nu_B(x), \nu_C(x)) \geq I_{\top}(\mu_B(x), \mu_C(x)).$$

So, $\nu_{A -_{\mathcal{T}} C}(x) \geq \nu_{B -_{\mathcal{T}} C}(x)$.

Thus for all $x \in X$, $\mu_{A -_{\mathcal{T}} C}(x) \leq \mu_{B -_{\mathcal{T}} C}(x)$ and $\nu_{A -_{\mathcal{T}} C}(x) \geq \nu_{B -_{\mathcal{T}} C}(x)$.

So, $A -_{\mathcal{T}} C \subseteq B -_{\mathcal{T}} C$.

3. Assume that $A \subseteq B\ then, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

$$(\mu_{C -_{\mathcal{T}} B}(x), \nu_{C -_{\mathcal{T}} B}(x)) = (J_S(\nu_C(x), \nu_B(x)), \min\{I_{\top}(\mu_C(x), \mu_B(x)), 1 - J_S(\nu_C(x), \nu_B(x))\}),$$

$$(\mu_{C -_{\mathcal{T}} A}(x), \nu_{C -_{\mathcal{T}} A}(x)) = (J_S(\nu_C(x), \nu_A(x)), \min\{I_{\top}(\mu_C(x), \mu_A(x)), 1 - J_S(\nu_C(x), \nu_A(x))\}).$$

Since $\nu_B(x) \leq \nu_A(x)$, then $J_S(\nu_C(x), \nu_B(x)) \leq J_S(\nu_C(x), \nu_A(x))$. So, $\mu_{C -_{\mathcal{T}} B}(x) \leq \mu_{C -_{\mathcal{T}} A}(x)$.

For the non-membership degree, there are four possibilities:

Case i: $\nu_{C -_{\mathcal{T}} B}(x) = I_{\top}(\mu_C(x), \mu_B(x))$ and $\nu_{C -_{\mathcal{T}} A}(x) = I_{\top}(\mu_C(x), \mu_A(x))$.

Since $\mu_A(x) \leq \mu_B(x)$, then $I_{\top}(\mu_C(x), \mu_B(x)) \geq I_{\top}(\mu_C(x), \mu_A(x))$ and we have $\nu_{C -_{\mathcal{T}} B}(x) \geq \nu_{C -_{\mathcal{T}} A}(x)$.

Case ii: $\nu_{C -_{\mathcal{T}} B}(x) = 1 - J_S(\nu_C(x), \nu_B(x))$ and $\nu_{C -_{\mathcal{T}} A}(x) = 1 - J_S(\nu_C(x), \nu_A(x))$.

Since $\nu_B(x) \leq \nu_A(x)$ then $J_S(\nu_C(x), \nu_B(x)) \leq J_S(\nu_C(x), \nu_A(x))$, then we have $\nu_{C -_{\mathcal{T}} B}(x) \geq \nu_{C -_{\mathcal{T}} A}(x)$.

Case iii: $\nu_{C\,-_{\mathcal{T}}\,B}(x) = I_{\top}(\mu_C(x),\mu_B(x))$ and $\nu_{C\,-_{\mathcal{T}}\,A}(x) = 1 - J_S(\nu_C(x),\nu_A(x))$. Since $\mu_A(x) \leq \mu_B(x)$, then

$$I_{\top}(\mu_C(x),\mu_B(x)) \geq I_{\top}(\mu_C(x),\mu_A(x)) \geq 1 - J_S(\nu_C(x),\nu_A(x)).$$

So, $\nu_{C\,-_{\mathcal{T}}\,B}(x) \geq \nu_{C\,-_{\mathcal{T}}\,A}(x)$.

Case iv: $\nu_{C\,-_{\mathcal{T}}\,B}(x) = 1 - J_S(\nu_C(x),\nu_B(x))$ and $\nu_{C\,-_{\mathcal{T}}\,A}(x) = I_{\top}(\mu_C(x),\mu_A(x))$.
Since $\nu_A(x) \geq \nu_B(x)$, then

$$1 - J_S(\nu_C(x),\nu_B(x)) \geq 1 - J_S(\nu_C(x),\nu_A(x)) \geq I_{\top}(\mu_C(x),\mu_A(x)).$$

So, $\nu_{C\,-_{\mathcal{T}}\,B}(x) \geq \nu_{C\,-_{\mathcal{T}}\,A}(x)$.

Thus for all $x \in X$, $\mu_{C\,-_{\mathcal{T}}\,B}(x) \leq \mu_{C\,-_{\mathcal{T}}\,A}(x)$ and $\nu_{C\,-_{\mathcal{T}}\,B}(x) \geq \nu_{C\,-_{\mathcal{T}}\,A}(x)$.

So, $C\ -_{\mathcal{T}}\, B \subseteq C\ -_{\mathcal{T}}\, A$.

4. From Definition 2.1.1 we have,

$$\mu_{A\,-_{\mathcal{T}}\,(A\cap B)}(x) = J_S(\nu_A(x),\nu_{A\cap B}(x)) = J_S(\nu_A(x),\max\{\nu_A(x),\nu_B(x)\}), \qquad (2.-1)$$

$$\begin{aligned}\nu_{A\,-_{\mathcal{T}}\,(A\cap B)}(x) &= \min\Big\{I_{\top}(\mu_A(x),\mu_{A\cap B}(x)), 1 - J_S(\nu_A(x),\nu_{A\cap B}(x))\Big\}\\ &= \min\Big\{I_{\top}(\mu_A(x),\min\{\mu_A(x),\mu_B(x)\}), 1 - J_S(\nu_A(x),\max\{\nu_A(x),\nu_B(x)\})\Big\} \qquad (2.-1)\end{aligned}$$

$$\begin{aligned}\mu_{A\,-_{\mathcal{T}}\,B}(x) &= J_S(\nu_A(x),\nu_B(x)), &(2.0)\\ \nu_{A\,-_{\mathcal{T}}\,B}(x) &= \min\{I_{\top}(\mu_A(x),\mu_B(x)), 1 - J_S(\nu_A(x),\nu_B(x))\}. &(2.1)\end{aligned}$$

Claim:
We claim $\mu_{A\,-_{\mathcal{T}}\,(A\cap B)}(x) = \mu_{A\,-_{\mathcal{T}}\,B}(x)$ and $\nu_{A\,-_{\mathcal{T}}\,(A\cap B)}(x) = \nu_{A\,-_{\mathcal{T}}\,B}(x)$ for all $x \in X$.
We note the following properties:
$J_S(\nu_A(x),\nu_B(x)) = 0$, whenever $\nu_A(x) \geq \nu_B(x)$; $I_{\top}(\mu_A(x),\mu_B(x)) = 1$, whenever $\mu_A(x) \leq \mu_B(x)$; $J_S(\nu_A(x),\nu_A(x)) = 0$ and $I_{\top}(\mu_A(x),\mu_A(x)) = 1$.
Then consider the following cases:

Case i: If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, then by Equations (2.-1)-(2.1), we have

$$\begin{aligned}\mu_{A\,-_{\mathcal{T}}\,(A\cap B)}(x) &= J_S(\nu_A(x),\nu_B(x)) = \mu_{A\,-_{\mathcal{T}}\,B}(x), \quad \text{and}\\ \nu_{A\,-_{\mathcal{T}}\,(A\cap B)}(x) &= \min\Big\{I_{\top}(\mu_A(x),\mu_A(x)), 1 - J_S(\nu_A(x),\nu_B(x))\Big\}\\ &= \min\Big\{1, 1 - J_S(\nu_A(x),\nu_B(x))\Big\}\\ &= \min\Big\{I_{\top}(\mu_A(x),\mu_B(x)), 1 - J_S(\nu_A(x),\nu_B(x))\Big\} = \nu_{A\,-_{\mathcal{T}}\,B}(x).\end{aligned}$$

Case ii: If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then by Equations (2.-1)-(2.1), we have

$$\begin{aligned}
\mu_{A -_{\mathcal{T}} (A\cap B)}(x) &= J_S(\nu_A(x), \nu_A(x)) = 0 \\
&= J_S(\nu_A(x), \nu_B(x)) = \mu_{A -_{\mathcal{T}} B}(x), \quad \text{and} \\
\nu_{A -_{\mathcal{T}} (A\cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_A(x)), 1 - J_S(\nu_A(x), \nu_A(x))\Big\} = \min\Big\{1, 1-0\Big\} \\
&= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} = \nu_{A -_{\mathcal{T}} B}(x).
\end{aligned}$$

Case iii: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, then by Equations (2.-1)-(2.1), we have

$$\begin{aligned}
\mu_{A -_{\mathcal{T}} (A\cap B)}(x) &= J_S(\nu_A(x), \nu_B(x)) = \mu_{A -_{\mathcal{T}} B}(x), \quad \text{and} \\
\nu_{A -_{\mathcal{T}} (A\cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} = \nu_{A -_{\mathcal{T}} B}(x).
\end{aligned}$$

Case iv: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then by Equations (2.-1)-(2.1), we have

$$\begin{aligned}
\mu_{A -_{\mathcal{T}} (A\cap B)}(x) &= J_S(\nu_A(x), \nu_A(x)) = 0 \\
&= J_S(\nu_A(x), \nu_B(x)) = \mu_{A -_{\mathcal{T}} B}(x), \quad \text{and} \\
\nu_{A -_{\mathcal{T}} (A\cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_A(x))\Big\} \\
&= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - 0\Big\} \\
&= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} = \nu_{A -_{\mathcal{T}} B}(x).
\end{aligned}$$

Hence, $\big(\mu_{A -_{\mathcal{T}} (A\cap B)}(x), \nu_{A -_{\mathcal{T}} (A\cap B)}(x)\big) = (\mu_{A -_{\mathcal{T}} B}(x), \nu_{A -_{\mathcal{T}} B}(x))$ for all $x \in X$, and the result follows.

□

The following result shows that, the intuitionistic fuzzy complement of fuzzy sets associated with a t-representable IF-t-norm $\mathcal{T} = (\top, S)$ so defined, preserves the properties of the classical complement for crisp sets.

Corollary 3. Let A be any intuitionistic fuzzy set of X. $A^c_{\mathcal{T}}$ be the intuitionistic fuzzy complement of A. Then $A^c_{\mathcal{T}} = X -_{\mathcal{T}} A$.

Proof. Let $x \in X$. Since $(\mu_X(x), \nu_X(x)) = (1, 0)$, then from Definition 2.1.1,

$$\begin{aligned}
(\mu_{X -_{\mathcal{T}} A}(x), \nu_{X -_{\mathcal{T}} A}(x)) &= \left(J_S(0, \nu_A(x)), \min\{I_\top(1, \mu_A(x)), 1 - J_S(0, \nu_A(x))\}\right), \\
&= \left(\nu_A(x),\ \min\{\mu_A(x), 1 - \nu_A(x)\}\right), \ \text{(recalling Prop. 1.1.1(1))}, \\
&= \left(\nu_A(x), \mu_A(x)\right), \quad \text{since } \mu_A(x) \leq 1 - \nu_A(x). \qquad (2.\text{-}15)
\end{aligned}$$

From Definition 1.2.7, the result follows. □

The following is the definition of intersection of two IFSs.

Definition 2.1.2. *Let A and B be two intuitionistic fuzzy sets and $\mathcal{T} = (\top, S)$ be a t-representable IF t-norm. The intuitionistic fuzzy intersection of A and B associated with $\mathcal{T}$ is defined as follows:*

$$A \cap_{\mathcal{T}} B = \{\langle x, \top(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}. \tag{2.-15}$$

The following result also establish a property of the new difference operation.

Proposition 2.1.2. *Let A and B be any intuitionistic fuzzy sets on X.*

1. *Then $(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)$ is an intuitionistic fuzzy set with membership function,*
   $\mu_{(A -_{\mathcal{T}} B)\cap_{\mathcal{T}}(B -_{\mathcal{T}} A)}(x) = 0$, $\forall x \in X$ and non-membership function defined by: for all $x \in X$,

$$\nu_{(A -_{\mathcal{T}} B)\cap_{\mathcal{T}}(B -_{\mathcal{T}} A)}(x) = \tag{2.-14}$$

$$\begin{cases} S\left(1 - J_S(\nu_A(x), \nu_B(x)), I_\top(\mu_B(x), \mu_A(x))\right), & \text{if } \mu_A(x) \le \mu_B(x) \text{ and } \nu_A(x) \le \nu_B(x)\ , \\ S\left(I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_B(x), \nu_A(x))\right), & \text{if } \mu_A(x) \ge \mu_B(x) \text{ and } \nu_A(x) \ge \nu_B(x)\ , \\ 1, \quad \text{otherwise.} \end{cases} \tag{2.-13}$$

2. *If $\mathcal{T}$ is a Lukasiewicz IF-t-norm, then*

$$(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A) = \emptyset.$$

Proof. 1. From Definition 2.1.2,

$$\mu_{(A -_{\mathcal{T}} B)\cap_{\mathcal{T}}(B -_{\mathcal{T}} A)}(x) = \top\left(J_S(\nu_A(x), \nu_B(x)), J_S(\nu_B(x), \nu_A(x))\right), \tag{2.-12}$$

$$\begin{aligned}\nu_{(A -_{\mathcal{T}} B)\cap_{\mathcal{T}}(B -_{\mathcal{T}} A)}(x) = S\,(&\min\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}, \\ &\min\{I_\top(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}). \end{aligned} \tag{2.-12}$$

We note that $J_S(\nu_A(x), \nu_B(x)) = 0$, whenever $\nu_A(x) \ge \nu_B(x)$ and $I_\top(\mu_A(x), \mu_B(x)) = 1$, whenever $\mu_A(x) \le \mu_B(x)$, then consider the following cases: for all $x \in X$,

Case i: If $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \le \nu_B(x)$, then from Equation (2.-12) $\mu_{(A -_{\mathcal{T}} B)\cap_{\mathcal{T}}(B -_{\mathcal{T}} A)}(x) = \top\left(J_S(\nu_A(x), \nu_B(x)), 0\right) = 0$, and from Equation (2.-12) we have

$$\begin{aligned}\nu_{(A -_{\mathcal{T}} B)\cap_{\mathcal{T}}(B -_{\mathcal{T}} A)}(x) &= S\left(\min\{1, 1 - J_S(\nu_A(x), \nu_B(x))\}, \min\{I_\top(\mu_B(x), \mu_A(x)), 1 - 0\}\right), \\ &= S\left(1 - J_S(\nu_A(x), \nu_B(x)), I_\top(\mu_B(x), \mu_A(x))\right).\end{aligned}$$

Case ii: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then from Equation (2.-12) $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \top(0, J_S(\nu_B(x), \nu_A(x))) = 0$, and from Equation (2.-12)

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = S\left(\min\{I_\top(\mu_A(x), \mu_B(x)), 1 - 0\}, \min\{1, 1 - J_S(\nu_B(x), \nu_A(x))\}\right),$$

$$= S\left(I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_B(x), \nu_A(x))\right).$$

Other possible cases are:

Case iii: If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then from Equation (2.-12) $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \top(0, J_S(\nu_B(x), \nu_A(x))) = 0$, and from Equation (2.-12)

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = S\left(\min\{1, 1 - 0\}, \min\{I_\top(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}\right),$$

$$= S\left(1, \min\{I_\top(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}\right) = 1.$$

Case iv: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, then from Equation (2.-12) $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \top(J_S(\nu_A(x), \nu_B(x)), 0) = 0$, and from Equation (2.-12)

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = S\left(\min\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}, \min\{1, 1 - 0\}\right),$$

$$= S\left(\min\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}, 1\right) = 1.$$

So, we have established the result 1.

2. If $\mathcal{T}$ is Lukasiewicz IF-t-norm, then $\mathcal{T} = \mathcal{T}_L = (\top_L, S_L)$. Since from the result in 1 above, we have the membership function $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = 0$, $\forall x \in X$, then from Equation (2.-13) it suffices to prove that the non-membership function, $\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = 1$, $\forall x \in X$, for the first two cases in (2.-13). From Equation (2.-12),

   i If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, we obtain by applying Proposition 1.1.1 and Example 6,

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \min\left(1 - J_{S_L}(\nu_A(x), \nu_B(x)) + I_{\top_L}(\mu_B(x), \mu_A(x)), 1\right),$$

$$= 1, \quad \text{if } \mu_A(x) = \mu_B(x) \text{ or } \nu_A(x) = \nu_B(x).$$

   If $\mu_A(x) < \mu_B(x)$ and $\nu_A(x) < \nu_B(x)$, then we have

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \min\left(1 - \nu_B(x) + \nu_A(x) + 1 - \mu_B(x) + \mu_A(x), 1\right),$$

$$= \min\left(2 - (\mu_B(x) + \nu_B(x)) + \mu_A(x) + \nu_A(x), 1\right) = 1, \quad \text{since } \mu_B(x) + \nu_B(x) \leq 1.$$

ii If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, we obtain by applying Proposition 1.1.1 and Example 6,

$$\nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) = \min\left(I_{\top_L}(\mu_A(x), \mu_B(x)) + 1 - J_{S_L}(\nu_B(x), \nu_A(x)), 1\right),$$

$$= 1, \quad \text{if } \mu_A(x) = \mu_B(x) \text{ or } \nu_A(x) = \nu_B(x).$$

If $\mu_A(x) > \mu_B(x)$ and $\nu_A(x) > \nu_B(x)$, then we have

$$\nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) = \min\left(1 - \mu_A(x) + \mu_B(x) + 1 - \nu_A(x) + \nu_B(x), 1\right),$$

$$= \min\left(2 - (\mu_A(x) + \nu_A(x)) + \mu_B(x) + \nu_B(x), 1\right) = 1, \quad \text{since } \mu_A(x) + \nu_A(x) \leq 1.$$

So $\left(\mu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x), \nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x)\right) = (0,1)$, for all $x \in X$. Hence result 2 is established.

□

Remark 1. 1. Note that, $(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A) = \emptyset$ whenever either $A \subseteq B$ or $B \subseteq A$. This follows immediately from the third case in Equation (2.-13).

2. Proposition 2.1.1 specifies conditions which are preserved by the intuitionistic fuzzy difference operation. These four conditions shall be referred to as the minimal conditions to require of difference operation on (even in crisp, fuzzy and intuitionistic) sets in general.

Note that for all two crisp sets A and B, the following equation holds.

$$(A - B) \cap B = \emptyset \tag{2.-12}$$

The following result shows that Eq.(2.1.2) is not satisfies for IFSs in general.

Proposition 2.1.3. *There exists two IFSs(X) A and B, such that:*

$$(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} B \neq \emptyset \tag{2.-12}$$

Proof. Let A and B be two IFSs(X) such that for all $x \in X$, $\mu_A(x) > \mu_B(x)$ and $\nu_A(x) > \nu_B(x)$. Then from Definition 2.1.1,

$$\mu_{A-_{\mathcal{T}}B}(x) = 0 \quad \textit{and} \quad \nu_{A-_{\mathcal{T}}B}(x) = 0. \tag{2.-12}$$

We most show that: $(\mu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}B}(x), \nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}B}(x)) \neq (0,1)$.
Since $\mu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}B}(x) = \min\{\mu_{A-_{\mathcal{T}}B}(x), \mu_B(x)\}$ and $\nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}B}(x) = \max\{\nu_{A-_{\mathcal{T}}B}(x), \nu_B(x)\}$ then from Eq. (2.1.2) the result follows. □

The following result gives a necessary and sufficient condition for difference of intuitionistic fuzzy sets to be a fuzzy set.

Proposition 2.1.4. *Let A and B be any intuitionistic fuzzy sets defined on X. Then the intuitionistic fuzzy difference $A -_{\mathcal{T}} B$ is a fuzzy set if and only if for all $x \in X$,*

$$I_{\top}(\mu_A(x), \mu_B(x)) \geq 1 - J_S(\nu_A(x), \nu_B(x)).$$

Proof. Let $x \in X$. Then from the Definition 2.1.1,

$$(\mu_{A -_{\mathcal{T}} B}(x), \nu_{A -_{\mathcal{T}} B}(x)) = (J_S(\nu_A(x), \nu_B(x)), \min\{I_{\top}(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}).$$

$A -_{\mathcal{T}} B$ is a fuzzy set if and only if $\nu_{A -_{\mathcal{T}} B}(x) = 1 - \mu_{A -_{\mathcal{T}} B}(x)$,

if and only if $\min\{I_{\top}(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\} = 1 - J_S(\nu_A(x), \nu_B(x))$,

if and only if $I_{\top}(\mu_A(x), \mu_B(x)) \geq 1 - J_S(\nu_A(x), \nu_B(x))$.

□

Note that $A -_{\mathcal{T}} B$ also becomes a fuzzy set if $A \subset B$, because in this case $A -_{\mathcal{T}} B = \emptyset$ (Proposition 2.1.1), $I_{\top}(\mu_A(x), \mu_B(x) = 1$ and $J_S(\nu_A(x), \nu_B(x)) = 0$. Furthermore, in the case where $A -_{\mathcal{T}} B$ becomes a fuzzy set, we deduce from Proposition 2.1.4 that for $x \in X$: $(\mu_{A -_{\mathcal{T}} B}(x), \nu_{A -_{\mathcal{T}} B}(x)) = (J_S(\nu_A(x), \nu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x)))$. This can be considered as fuzzy part of $A -_{\mathcal{T}} B$.

The following are typical applications of Proposition 2.1.4 to difference operators associated with the three usual and well-known $\mathcal{T}$.

Notation 2.1.1. Let A and B be any fuzzy sets, $A \equiv B$ if and only if for all $x \in X$, $\mu_A(x) = \mu_B(x)$.

The following result shows that, the intuitionistic fuzzy difference operator defined in Definition 2.1.1 associated with t-representable IF t-norm $\mathcal{T} = (\top, S)$ is a generalization of fuzzy difference operator proposed by Fono et al. [17] associated with a t-norm $\top$ if and only if the fuzzy t-norm $\top$ and fuzzy t-conorm S are dual.

Proposition 2.1.5 (Generalization of Difference Operation for Fuzzy Sets)**.** *Let $\top$ and S be any fuzzy t-norm and t-conorm respectively, and $\mathcal{T} = (\top, S)$ a t-representable IF t-norm associated with any intuitionistic fuzzy set. $\top$ and S are dual if and only if for any fuzzy sets A and B, $A -_{\mathcal{T}} B$ is a fuzzy set and $A -_{\top} B \equiv A -_{\mathcal{T}} B$.*

Proof. Let $x \in X$, and A and B be any fuzzy sets.

a. Assume that $\top$ and S are dual.

   i. Let us show that $A -_{\mathcal{T}} B$ is a fuzzy set.
   Since $\top$ and S are dual, then From Proposition 1.1.1, $I_{\top}(\mu_A(x), \mu_B(x)) = 1 - J_S(1-\mu_A(x), 1-\mu_B(x))$ and from Proposition 2.1.4, the result follows.

   ii. Now we shall show that, $A -_{\top} B \equiv A -_{\mathcal{T}} B$. It is sufficient to prove that $\mu_{A-_{\top}B}(x) = \mu_{A-_{\mathcal{T}}B}(x)$.
   According to Fono and al. [17], $\mu_{A-_{\top}B}(x) = 1 - I_{\top}(\mu_A(x), \mu_B(x))$ and from Definition 2.1.1 $\mu_{A-_{\mathcal{T}}B}(x) = J_S(1-\mu_A(x), 1-\mu_B(x))$.
   Since $\top$ and S are dual, the Proposition 1.1.1 shows that, $I_{\top}(\mu_A(x), \mu_B(x)) = 1 - J_S(1-\mu_A(x), 1-\mu_B(x))$ and the result follows.

b. Assume now that $A -_{\mathcal{T}} B$ is a fuzzy set and $A -_{\top} B \equiv A -_{\mathcal{T}} B$. Let us show that $\top$ and S are dual.
   We have,

$$\begin{aligned}
\mu_{A-_{\top}B}(x) &= 1 - I_{\top}(\mu_A(x), \mu_B(x)) \\
&= 1 - \max\{t \in [0;1],\ \top(\mu_A(x), t) \leq \mu_B(x)\} \\
&= \min\{1 - t \in [0;1],\ \top(\mu_A(x), t) \leq \mu_B(x)\} \\
&= \min\{1 - t \in [0;1],\ \top(\mu_A(x), t) \leq \mu_B(x)\}, \qquad (2.\text{-}14)
\end{aligned}$$

   and

$$\begin{aligned}
\mu_{A-_{\mathcal{T}}B}(x) &= J_S(1-\mu_A(x), 1-\mu_B(x)) \\
&= \min\{r \in [0;1],\ S(1-\mu_A(x), r) \geq 1 - \mu_B(x)\} \\
&= \min\{1 - t \in [0;1],\ S(1-\mu_A(x), 1-t) \geq 1 - \mu_B(x)\} \\
&= \min\{1 - t \in [0;1],\ 1 - S(1-\mu_A(x), 1-t) \leq \mu_B(x)\} \qquad (2.\text{-}16)
\end{aligned}$$

   Since $A -_{\top} B \equiv A -_{\mathcal{T}} B$ then, $\mu_{A-_{\top}B}(x) = \mu_{A-_{\mathcal{T}}B}(x)$. From Equations (2.-11) and (2.-13) $\top(\mu_A(x), t) = 1 - S(1-\mu_A(x), 1-t),\ \forall t \in [0;1]$ and the result follows.

$\square$

In the following Subsection, we will define a new symmetric difference operation for intuitionistic fuzzy sets based on the IF-R-implication and IF-co-implication and we will study its properties.

2.2 Symmetric difference operation between IFSs

Ejegwa [13] defines IF symmetric difference of IFs A and B as the IFS $A \vartriangle B$ of X such that,

$$\forall x \in X, \quad (\mu_{A\vartriangle B}(x), \nu_{A\vartriangle B}(x)) = (\max\{\mu_A(x) \wedge \nu_B(x), \nu_A(x) \wedge \mu_B(x)\}, \min\{\nu_A(x) \vee \mu_B(x), \mu_A(x) \vee \nu_B(x)\}) . \tag{2.-16}$$

But Ejegwa's symmetric difference operation does not satisfy one of the four basic properties of standard symmetric difference operation (see Table 2.2).

	Satisfy	Counter-example
$A \vartriangle B = (A - B) \cup (B - A)$	YES	
$A \vartriangle B = B \vartriangle A$	YES	
If $A \subseteq B$, then $A \vartriangle B = B - A$	YES	
$A \vartriangle A = \emptyset$.	NO	$A = \{(x, 0.6, 0.2), \; x \in X\}$, $A \vartriangle A = \{(x, 0.2, 0.6), x \in X\} \neq \emptyset$

Table 2.2: Properties of Ejegwa's symmetric difference operation

In the following Subsections, we will first give the definition of the new symmetric difference operations between IFSs and display appropriate examples.

2.2.1 Definition and examples

The idea for the new definition is derive from the classical formula for symmetric difference and the operations of union and intersection alongside with the proposed difference for intuitionistic fuzzy sets.

Definition 2.2.1. *Let A, B be any two IFSs defined on X. The intuitionistic fuzzy symmetric difference associated to $\mathcal{T}$ of A and B is the IFS on X denoted by $A \vartriangle_{\mathcal{T}} B$ and defined by the membership and non-membership degrees as follows:*
For all $x \in X$,

$$\begin{aligned} \mu_{A \vartriangle_{\mathcal{T}} B}(x) &= J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)), \qquad (2.\text{-}15) \\ \nu_{A \vartriangle_{\mathcal{T}} B}(x) &= \min\{I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)), 1 - J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x))\}. \end{aligned}$$

Example 10. For any two intuitionistic fuzzy sets A and B defined on X,

1. The symmetric difference operator associated with $\mathcal{T}_M$ is given by, for all $x \in X$

$$\mu_{A \triangle_{\mathcal{T}_M} B}(x) = \begin{cases} 0, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \nu_B(x) \vee \nu_A(x), & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$\nu_{A \triangle_{\mathcal{T}_M} B}(x) = \begin{cases} 1, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\{\min\{\mu_B(x), 1-\nu_B(x)\}, \min\{\mu_A(x), 1-\nu_A(x)\}\}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$= \begin{cases} 1, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\{\mu_A(x) \wedge \mu_B(x), 1 - \nu_A(x) \vee \nu_B(x)\}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)). \end{cases}$$

2. The symmetric difference operator associated with $\mathcal{T}_P$ is given by, for all $x \in X$

$$\mu_{A \triangle_{\mathcal{T}_P} B}(x) = \begin{cases} 0, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \max\left\{\frac{\nu_B(x)-\nu_A(x)}{1-\nu_A(x)}, \frac{\nu_A(x)-\nu_B(x)}{1-\nu_B(x)}\right\}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$= \begin{cases} 0, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \frac{(\nu_A(x)-\nu_B(x)) \vee (\nu_B(x)-\nu_A(x))}{(1-\nu_A(x)) \wedge (1-\nu_B(x))}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$\nu_{A \triangle_{\mathcal{T}_P} B}(x) = \begin{cases} 1, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\left\{\min\left\{\frac{\mu_B(x)}{\mu_A(x)}, \frac{\mu_A(x)}{\mu_B(x)}\right\}, 1 - \max\left\{\frac{\nu_B(x)-\nu_A(x)}{1-\nu_A(x)}, \frac{\nu_A(x)-\nu_B(x)}{1-\nu_B(x)}\right\}\right\}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$= \begin{cases} 1, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\left\{\frac{\mu_A(x) \wedge \mu_B(x)}{\mu_A(x) \vee \mu_B(x)}, 1 - \frac{(\nu_A(x)-\nu_B(x)) \vee (\nu_B(x)-\nu_A(x))}{(1-\nu_A(x)) \wedge (1-\nu_B(x))}\right\}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)). \end{cases}$$

3. The symmetric difference operator associated with $\mathcal{T}_L$ is given by, for all $x \in X$

$$\mu_{A \triangle_{\mathcal{T}_L} B}(x) = \begin{cases} 0, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ (\nu_A(x) - \nu_B(x)) \vee (\nu_B(x) - \nu_A(x)), & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

and

$$\nu_{A \triangle_{\mathcal{T}_L} B}(x) = \begin{cases} 1, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)), \\ \wedge\Big\{ \wedge\{1 - \mu_A(x) + \mu_B(x), 1 + \nu_A(x) - \nu_B(x)\}, \wedge\{1 - \mu_B(x) + \mu_A(x), 1 + \nu_B(x) - \nu_A(x)\}\Big\}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)). \end{cases}$$

$$= \begin{cases} 1, & \text{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)), \\ \min\Big\{1 - (\mu_A(x) - \mu_B(x)) \vee (\mu_B(x) - \mu_A(x)), 1 - (\nu_A(x) - \nu_B(x)) \vee (\nu_B(x) - \nu_A(x))\Big\}, & \text{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)). \end{cases}$$

In what follows, we are going to establish some results showing that some properties of the classical set symmetric difference are preserved by this new proposed intuitionistic fuzzy symmetric difference operation.

2.2.2 Some properties allowing to symmetry difference between IFSs

In the following results, we establish four classical properties for intuitionistic fuzzy symmetric difference operations.

Proposition 2.2.1 (Properties of intuitionistic fuzzy symmetric difference operation)**.** *Let A, B be any two IFSs on X. The following properties for intuitionistic fuzzy symmetric difference operation hold:*

1. $A \vartriangle_{\mathcal{T}} B = (A -_{\mathcal{T}} B) \cup (B -_{\mathcal{T}} A)$;
2. $A \vartriangle_{\mathcal{T}} B = B \vartriangle_{\mathcal{T}} A$;
3. *If* $A \subseteq B$, *then* $A \vartriangle_{\mathcal{T}} B = B -_{\mathcal{T}} A$;
4. $A \vartriangle_{\mathcal{T}} A = \emptyset$.

Proof. 1. The following are properties for fuzzy-R-implication, $I_\top$ and fuzzy co-implication, J_S which we require here:

$$\begin{aligned}
I_\top(a \vee b, c) &= I_\top(a, c) \wedge I_\top(b, c), \text{ and } J_S(a \vee b, c) = J_S(a, c) \wedge J_S(b, c);\\
I_\top(a \wedge b, c) &= I_\top(a, c) \vee I_\top(b, c), \text{ and } J_S(a \wedge b, c) = J_S(a, c) \vee J_S(b, c);\\
I_\top(a, b \vee c) &= I_\top(a, b) \vee I_\top(a, c), \text{ and } J_S(a, b \vee c) = J_S(a, b) \vee J_S(a, c);\\
I_\top(a, b \wedge c) &= I_\top(a, b) \wedge I_\top(a, c), \text{ and } J_S(a, b \wedge c) = J_S(a, b) \wedge J_S(a, c).
\end{aligned}$$

These can easily be verified.

Now, we proceed to prove 1 and 2 consequently as follows: From Equation (2.-15) and applying above properties of $I_\top$ and J_S we have, for all $x \in X$

$$(\mu_{A \vartriangle_{\mathcal{T}} B}(x), \nu_{A \vartriangle_{\mathcal{T}} B}(x)) = (J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x) \wedge \nu_B(x), \nu_B(x)) \,,$$

$$\min\{I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x)) \wedge I_\top(\mu_A(x) \vee \mu_B(x), \mu_B(x)) \,,$$

$$1 - J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x) \wedge \nu_B(x), \nu_B(x))\}) \,.$$

So we have

$$\mu_{A \vartriangle_{\mathcal{T}} B}(x) = (J_S(\nu_A(x), \nu_A(x)) \vee J_S(\nu_B(x), \nu_A(x))) \vee (J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_B(x))) \,.$$

$$\nu_{A \vartriangle_{\mathcal{T}} B}(x) = \min\{(I_\top(\mu_A(x), \mu_A(x)) \wedge I_\top(\mu_B(x), \mu_A(x))\} \wedge \{I_\top(\mu_A(x), \mu_B(x)) \wedge I_\top(\mu_B(x), \mu_B(x))) \,.$$

$$1 - (J_S(\nu_A(x), \nu_A(x)) \vee J_S(\nu_B(x), \nu_A(x))\} \vee \{J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_B(x)))\},$$

and applying Proposition 1.1.1 we have the following:

$$\begin{aligned} \mu_{A \,\triangle_{\mathcal{T}}\, B}(x) &= J_S(\nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x), \nu_B(x)), \\ &= J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_A(x)), \\ &= \mu_{A \,-_{\mathcal{T}}\, B}(x) \vee \mu_{B \,-_{\mathcal{T}}\, A}(x), \\ &= \mu_{(A \,-_{\mathcal{T}}\, B)\cup(B \,-_{\mathcal{T}}\, A)}(x). \end{aligned} \tag{2.-20}$$

$$\begin{aligned} \nu_{A \,\triangle_{\mathcal{T}}\, B}(x) &= \min\{I_{\top}(\mu_B(x), \mu_A(x)) \wedge I_{\top}(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x), \nu_B(x))\}, \\ &= \min\{I_{\top}(\mu_A(x), \mu_B(x)) \wedge I_{\top}(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_A(x))\}, \end{aligned}$$

$$= \min\{I_{\top}(\mu_A(x), \mu_B(x)) \wedge (1 - J_S(\nu_A(x), \nu_B(x))), I_{\top}(\mu_B(x), \mu_A(x)) \wedge (1 - J_S(\nu_B(x), \nu_A(x)))\},$$

$$= \nu_{A \,-_{\mathcal{T}}\, B}(x) \wedge \nu_{B \,-_{\mathcal{T}}\, A}(x),$$

$$= \nu_{(A \,-_{\mathcal{T}}\, B)\cup(B \,-_{\mathcal{T}}\, A)}(x).$$

So, result 1 is established.

2. By commutativity of Equations (2.-20) and (1), result 2 follows, since $A \,\triangle_{\mathcal{T}}\, B = (A \,-_{\mathcal{T}}\, B) \cup (B \,-_{\mathcal{T}}\, A) = (B \,-_{\mathcal{T}}\, A) \cup (A \,-_{\mathcal{T}}\, B) = B \,\triangle_{\mathcal{T}}\, A$.

3. If $A \subseteq B$, then for all $x \in X$, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
   Applying the above inequalities to the Equation (2.-15), we get

$$\begin{aligned} (\mu_{A \,\triangle_{\mathcal{T}}\, B}(x), \nu_{A \,\triangle_{\mathcal{T}}\, B}(x)) &= (J_S(\nu_B(x), \nu_A(x)), \min\{I_{\top}(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}), \\ &= (\mu_{B \,-_{\mathcal{T}}\, A}(x), \nu_{B \,-_{\mathcal{T}}\, A}(x)), \end{aligned}$$

   and the result follows.

4. By Equation (2.-15) we have, for all $x \in X$

$$\begin{aligned} \mu_{A \,\triangle_{\mathcal{T}}\, A}(x) &= J_S(\nu_A(x) \wedge \nu_A(x), \nu_A(x) \vee \nu_A(x)), \\ &= J_S(\nu_A(x), \nu_A(x)) = 0. \end{aligned}$$

$$\begin{aligned} \nu_{A \,\triangle_{\mathcal{T}}\, A}(x) &= \min\{I_{\top}(\mu_A(x) \vee \mu_A(x), \mu_A(x) \wedge \mu_A(x)), 1 - J_S(\nu_A(x) \wedge \nu_A(x), \nu_A(x) \vee \nu_A(x))\}, \\ &= \min\{I_{\top}(\mu_A(x), \mu_A(x)), 1 - J_S(\nu_A(x), \nu_A(x))\}, \\ &= \min\{1, 1\} = 1. \end{aligned}$$

   So the result is established.

□

We are going to establish important results which will be very useful in the study of distance measure between IFSs.

Lemma 2. Let $A, B, C \in IFSs(X)$.

1. $A\Delta_{\mathcal{T}}B = \emptyset$ if and only if $A = B$.

2. If $A \subseteq B \subseteq C$, then $A\Delta_{\mathcal{T}}B \subseteq A\Delta_{\mathcal{T}}C$ and $B\Delta_{\mathcal{T}}C \subseteq A\Delta_{\mathcal{T}}C$.

Proof. 1. To prove the first result of Lemma 2, let us recall the well-know properties of J_S and $I_{\top}$: for all $a, b \in [0, 1]$.

$$J_S(a, b) = 0 \;\text{ if and only if } \quad a \geq b \tag{2.-31}$$

and

$$I_{\top}(a, b) = 1 \;\text{ if and only if } \quad a \leq b \tag{2.-31}$$

Let us prove now that $A\Delta_{\mathcal{T}}B = \emptyset_{\mathcal{T}}$ if and only if $A = B$.

If $A = B$, then from Proposition 2.2.1, $A\Delta_{\mathcal{T}}B = A\Delta_{\mathcal{T}}A = \emptyset_{\mathcal{T}}$.

On the other hand, suppose that $A\Delta_{\mathcal{T}}B = \emptyset_{\mathcal{T}}$. Let us show that $A = B$, that is, for all $x \in X$, $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$.
Since $A\Delta_{\mathcal{T}}B = \emptyset_{\mathcal{T}}$ then

$$\mu_{A\Delta_{\mathcal{T}}B}(x) = 0 \quad \text{and} \;\; \nu_{A\Delta_{\mathcal{T}}B}(x) = 1, \;\text{for all } x \in X. \tag{2.-31}$$

From Definition 2.2.1, (1) implies that, for all $x \in X$,

$$J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)) = 0 \quad \text{and} \;\; I_{\top}(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) = 1. \tag{2.-31}$$

Since $\mu_A(x) \vee \mu_B(x) \geq \mu_A(x) \wedge \mu_B(x)$ and $\nu_A(x) \wedge \nu_B(x) \leq \nu_A(x) \vee \nu_B(x)$, then from (1), (1) and (1) we have $\mu_A(x) \vee \mu_B(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_A(x) \wedge \nu_B(x) = \nu_A(x) \vee \nu_B(x)$. And the result follows.

2. Let us prove the second result of Lemma 2.

   i) Let us show first that: $A\Delta_{\mathcal{T}}B \subseteq A\Delta_{\mathcal{T}}C$.
   It is sufficient to show that: for all $x \in X$, $\mu_{A\Delta_{\mathcal{T}}C}(x) \geq \mu_{A\Delta_{\mathcal{T}}B}(x)$ and $\nu_{A\Delta_{\mathcal{T}}C}(x) \leq \nu_{A\Delta_{\mathcal{T}}B}(x)$.
   Since $A \subseteq B \subseteq C$, then: for all $x \in X$,

$$\begin{cases} \mu_A(x) \leq \mu_B(x) \leq \mu_C(x) \\ \nu_A(x) \geq \nu_B(x) \geq \nu_C(x). \end{cases} \tag{2.-31}$$

By using (2i) and Definition 2.2.1, we have the following system for all $x \in X$,:

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}C}(x) = J_S(\nu_C(x), \nu_A(x)), \\ \mu_{A\Delta_{\mathcal{T}}B}(x) = J_S(\nu_B(x), \nu_A(x)), \\ \nu_{A\Delta_{\mathcal{T}}C}(x) = \min\{I_\top(\mu_C(x), \mu_A(x)), 1 - J_S(\nu_C(x), \nu_A(x))\}, \\ \nu_{A\Delta_{\mathcal{T}}B}(x) = \min\{I_\top(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}. \end{cases} \tag{2.-31}$$

Since J_S and $I_\top$ are left decreasing, (2i) and (2i) show that:
for all $x \in X$, $\mu_{A\Delta_{\mathcal{T}}C}(x) \geq \mu_{A\Delta_{\mathcal{T}}B}(x)$ and $\nu_{A\Delta_{\mathcal{T}}C}(x) \leq \nu_{A\Delta_{\mathcal{T}}B}(x)$ and the result follows.

ii) Let us show now that $B\Delta_{\mathcal{T}}C \subseteq A\Delta_{\mathcal{T}}C$.
It is sufficient to show that: for all $x \in X$, $\mu_{A\Delta_{\mathcal{T}}C}(x) \geq \mu_{B\Delta_{\mathcal{T}}C}(x)$ and $\nu_{A\Delta_{\mathcal{T}}C}(x) \leq \nu_{B\Delta_{\mathcal{T}}C}(x)$.
Since $A \subseteq B \subseteq C$, then, by using (2i) and Definition 2.2.1, we have the following system for all $x \in X$,:

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}C}(x) = J_S(\nu_C(x), \nu_A(x)), \\ \mu_{B\Delta_{\mathcal{T}}C}(x) = J_S(\nu_C(x), \nu_B(x)), \\ \nu_{A\Delta_{\mathcal{T}}C}(x) = \min\{I_\top(\mu_C(x), \mu_A(x)), 1 - J_S(\nu_C(x), \nu_A(x))\}, \\ \nu_{B\Delta_{\mathcal{T}}C}(x) = \min\{I_\top(\mu_C(x), \mu_B(x)), 1 - J_S(\nu_C(x), \nu_B(x))\}. \end{cases} \tag{2.-31}$$

Since J_S and $I_\top$ are right increasing, (2i) and (2ii) show that:
for all $x \in X$, $\mu_{A\Delta_{\mathcal{T}}C}(x) \geq \mu_{B\Delta_{\mathcal{T}}C}(x)$ and $\nu_{A\Delta_{\mathcal{T}}C}(x) \leq \nu_{B\Delta_{\mathcal{T}}C}(x)$ and the result follows.

□

The following result shows that, the intuitionistic fuzzy symmetric difference operator in Definition 2.2.1 associated with t-representable IF t-norm $\mathcal{T} = (\top, S)$ is a generalization of fuzzy symmetric difference operation proposed by Fono et al. [17] associated with the t-norm $\top$ if and only if $\top$ and S are dual.

Proposition 2.2.2. *[Generalization of symmetric difference operation for fuzzy sets] Let $\top$ and S be any fuzzy $t-$norm and $t-$conorm respectively, $\mathcal{T} = (\top, S)$ a t-representable IF t-norm associated with any intuitionistic fuzzy set. $\top$ and S are dual if and only if for any fuzzy sets C and D, $C \ \triangle_{\mathcal{T}} D$ is a fuzzy set and for all $x \in X$, $\mu_{C\Delta_\top D}(x) = \mu_{C \,\triangle_{\mathcal{T}}\, D}(x)$.*

Proof. Let $x \in X$, and C and D be any fuzzy sets.

a. Assume that $\top$ and S are dual.

i. Let us show that $C \, \triangle_{\mathcal{T}} D$ is a fuzzy set.
Since C and D are fuzzy sets $(1-\mu_C(x)=\nu_C(x)$ and $1-\mu_D(x)=\nu_D(x))$, and $\top$ and S are dual, then from Proposition 1.1.1,

$$\begin{aligned}
& I_\top(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x)) \\
= \ & 1-J_S(1-\mu_C(x)\vee\mu_D(x),1-\mu_C(x)\wedge\mu_D(x)) \\
= \ & 1-J_S((1-\mu_C(x))\wedge(1-\mu_D(x)),(1-\mu_C(x))\vee(1-\mu_D(x))) \\
= \ & 1-J_S(\nu_C(x)\wedge\nu_D(x),\nu_C(x)\vee\nu_D(x)). \qquad (2.\text{-}33)
\end{aligned}$$

From Definition 2.2.1, the result follows.

ii. Now we shall show that, $C \, \triangle_{\top} D \equiv C \, \triangle_{\mathcal{T}} D$. It is sufficient to prove that $\mu_{C\triangle_\top D}(x)=\mu_{C\,\triangle_{\mathcal{T}}\,D}(x)$.
Definition 1.1.5 and Definition 2.2.1 shows that,

$$\mu_{C\overset{i}{\underset{\top}{\triangle}}D}(x)=\mu_{C\cup D\overset{i}{\underset{\top}{-}}C\cap D}(x)=\begin{cases}1-I^1_\top\left(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x)\right), & \text{if } i=1\\ 1-I^2_\top\left(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x)\right), & \text{if } i=2,\end{cases}$$

and $\mu_{C\,\triangle_{\mathcal{T}}\,D}(x)=J_S(\nu_C(x)\wedge\nu_D(x),\nu_C(x)\vee\nu_D(x))$.
From Equation (2.-30), the result follows.

b. Assume now that $C \, \triangle_{\mathcal{T}} D$ is a fuzzy set and $C \triangle_{\top} D \equiv C \, \triangle_{\mathcal{T}} D$.
Let us show that $\top$ and S are dual.
We have,

$$\begin{aligned}
\mu_{C\triangle_\top D}(x) &= 1-I_\top(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x)) \\
&= 1-\max\{t\in[0;1],\ \top(\mu_C(x)\vee\mu_D(x),t)\le\mu_C(x)\wedge\mu_D(x)\} \\
&= \min\{1-t\in[0;1],\ \top(\mu_C(x)\vee\mu_D(x),t)\le\mu_C(x)\wedge\mu_D(x)\} \\
&= \min\{1-t\in[0;1],\ \top(\mu_C(x)\vee\mu_D(x),t)\le\mu_C(x)\wedge\mu_D(x)\}, \qquad (2.\text{-}35)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{C\,\triangle_{\mathcal{T}}\,D}(x) &= J_S(\nu_C(x)\wedge\nu_D(x),\nu_C(x)\vee\nu_D(x)) \\
&= \min\{r\in[0;1],\ S(\nu_C(x)\wedge\nu_D(x),r)\ge\nu_C(x)\vee\nu_D(x)\} \\
&= \min\{1-t\in[0;1],\ S(1-\mu_C(x)\vee\mu_D(x),1-t)\ge1-\mu_C(x)\wedge\mu_D(x)\} \\
&= \min\{1-t\in[0;1],\ 1-S(1-\mu_C(x)\vee\mu_D(x),1-t)\le\mu_C(x)\wedge\mu_D(x)\} \qquad (2.\text{-}37)
\end{aligned}$$

Since $C \bigtriangleup {}_{\top}D \equiv C \ \bigtriangleup {}_{\mathcal{T}} D$ then, $\mu_{C\bigtriangleup_{\top}D}(x) = \mu_{C\,\bigtriangleup_{\mathcal{T}}\,D}(x)$. From Equation (2.-32) and (2.-34),

$\top(\mu_C(x) \vee \mu_D(x), t) = 1 - S(1 - \mu_C(x) \vee \mu_D(x), 1 - t)$, $\forall t \in [0; 1]$, and the result follows.

$\square$

In the following Section, we will first give a generalized definition of intuitionistic fuzzy cardinality for difference and symmetric difference and we will then investigate their properties.

2.3 Some classes of IF cardinality for difference and symmetric difference of IFSs

Throughout this Section, the universal set X is finite. On fuzzy sets, Deschrijver and Král [11] introduce the cardinalities of interval-valued fuzzy sets and study their properties.

In this subsection, we will discuss the cardinality of IFSs. First, we will recall the definition of intuitionistic fuzzy cardinality introduced by Tripathy et al. [41].

Definition 2.3.1. *(Cardinality of Intuitionistic Fuzzy Set [41]) Let $A = \{(x_i, \mu_A(x_i), \nu_A(x_i)),\ x_i \in X\}$ be an IFS on X. The intuitionistic fuzzy cardinality (IF-cardinality) of A denoted by $Count(A)$ is given by*

$$Count(A) = \left(\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n} (1 - \nu_A(x_i)) \right). \tag{2.-36}$$

One of the properties of this cardinality operation is given here.

Theorem 1 (Property of *Count* [41])**.** Let A and B be any two intuitionistic fuzzy sets on X. Then

$$Count(A \cup B) + Count(A \cap B) = Count(A) + Count(B). \tag{2.-35}$$

Using Definition 2.3.1, we obtain the IF-cardinality of symmetric difference between IFSs A and B associated with $\mathcal{T}$ as follows:

$$Count(A\Delta_{\mathcal{T}}B) = \left(\sum_{i=1}^{n} \mu_{A\Delta_{\mathcal{T}}B}(x_i), \sum_{i=1}^{n} (1 - \nu_{A\Delta_{\mathcal{T}}B}(x_i)) \right). \tag{2.-34}$$

Given IFSs $A = \{(x_i, \mu_A(x_i), \nu_A(x_i)),\ x_i \in X\}$ and $B = \{(x_i, \mu_B(x_i), \nu_B(x_i)),\ x_i \in X\}$, we let

$$f_S^{AB}(i) = J_S(\nu_A(x_i) \wedge \nu_B(x_i), \nu_A(x_i) \vee \nu_B(x_i))\ for\ i = 1, \ldots, n. \tag{2.-34}$$

and

$$g_{\top}^{AB}(i) = I_{\top}(\mu_A(x_i) \vee \mu_B(x_i), \mu_A(x_i) \wedge \mu_B(x_i)) \ for \ i = 1, \dots, n. \quad (2.\text{-}34)$$

Using Equation (2.-34), we can rewrite the IF-cardinality for symmetric difference between any two IFSs A and B as follows.

$$Count(A\Delta_{\mathcal{T}}B) = \left(\sum_{i=1}^{n} f_S^{AB}(i), \sum_{i=1}^{n} \max\{1 - g_{\top}^{AB}(i), f_S^{AB}(i)\}\right). \quad (2.\text{-}34)$$

If A is a fuzzy set we denote by $Card(A)$ the cardinality of A.
Let β be the positive real number and B be IFS. We introduce the following notation: $Count(B) \equiv \beta$ if and only if $Count(B) = (\beta, \beta)$.

The following remark gives fuzzy version of the IF-cardinality for the intuitionistic fuzzy symmetric difference.

Remark 2. Let A and B be two fuzzy sets. If $\top$ and S are dual, then the IF-cardinality of $A\Delta_{\mathcal{T}}B$ is given by:

$$Count(A\Delta_{\mathcal{T}}B) \equiv Card(A\overset{i}{\underset{\top}{\Delta}}B) = \sum_{i=1}^{n} f_S^{AB}(i) = \sum_{i=1}^{n} \left(1 - g_{\top}^{AB}(i)\right) \quad (2.\text{-}34)$$

according to $i \in \{1, 2\}$. Where $A\overset{i}{\underset{\top}{\Delta}}B$ is define in Definition 1.1.5.

Remark 2 shows that the cardinality of the new symmetric difference between IFSs(X) generalizes definition of the cardinality of symmetric difference between FSs(X) defined by Fono et al. [17].

In what follows, we establish a cardinality property that is satisfied by the intuitionistic fuzzy difference and symmetric difference proposed.

Proposition 2.3.1. *Let A, B, C be any intuitionistic fuzzy sets on X. The following property holds:*

$$Count(A \vartriangle_{\mathcal{T}} B) \leq_{L^*} Count(A -_{\mathcal{T}} B) + Count(B -_{\mathcal{T}} A).$$

Proof. Recall from Proposition 2.2.1, we have $A \vartriangle_{\mathcal{T}} B = (A -_{\mathcal{T}} B) \cup (B -_{\mathcal{T}} A)$ and by Theorem 1 we obtain

$$\text{Count}(A \vartriangle_{\mathcal{T}} B) = \text{Count}(A -_{\mathcal{T}} B) + \text{Count}(B -_{\mathcal{T}} A) - \text{Count}\left((A -_{\mathcal{T}} B) \cap (B -_{\mathcal{T}} A)\right) \quad (2.\text{-}33)$$

Since by Proposition 2.1.2 we have $(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A) \neq \emptyset$ in general, then we have

$$\text{Count}\left((A -_{\mathcal{T}} B) \cap (B -_{\mathcal{T}} A)\right) \geq_{L^*} 0_{L^*}. \quad (2.\text{-}32)$$

Putting Equation (2.-32) into (2.-33) we obtain the required result. $\square$

Chapter 3

New distance measures, similarity measures and metrics based on symmetric difference between IFSs

Contents

In this Chapter we first propose eight classes of distance measures and eight classes of similarity measures between IFSs using cardinality components of the new symmetric difference between IFSs. We further determine conditions on both fuzzy $t-$norm $\top$ and fuzzy $t-$conorm S under which many of those classes become metrics.
Throughout this Chapter $A = \{(x_i, \mu_A(x_i), \nu_A(x_i)),\ x_i \in X\}$ and $B = \{(x_i, \mu_B(x_i), \nu_B(x_i)),\ x_i \in X\}$ are two IFS on X.
The following Section display the new distance measures and similarity measures.

3.1 Some classes of distances measures and similarity measures for IFSs based on symmetric difference for IFSs

Distance measures or similarity measures have been proposed to solve some problems of decision making. The existing distance or similarity measures are not based on symmetric difference between IFSs.

In this Section we proposed such tools which are based on symmetric difference.

Let us introduce our eight proposed mappings on pairs of IFSs A and B.

$$d^Z_{2n,\mathcal{T}}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\frac{3+\mu_{A\Delta_{\mathcal{T}}B}(x_i) - 3\nu_{A\Delta_{\mathcal{T}}B}(x_i)}{4}. \tag{3.0}$$

Assume the weight of the element $x_i \in X,\ (i=1,...,n)$ is $w_i\ (i=1,...,n)$, were $0 \le w_i \le 1$.

$$d^Z_{2w,\mathcal{T}}(A,B) = \frac{1}{nw}\sum_{i=1}^{n}w_i\left(\frac{3+\mu_{A\Delta_{\mathcal{T}}B}(x_i) - 3\nu_{A\Delta_{\mathcal{T}}B}(x_i)}{4}\right) \tag{3.0}$$

with $w = \sum\limits_{i=1}^{n} w_i$.

$$d^Z_{2H,\mathcal{T}}(A,B) = \frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)), \tag{3.0}$$

$$d^Z_{2E,\mathcal{T}}(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^2}. \tag{3.0}$$

Let p be a strictly positive integer.

$$d^Z_{2p,\mathcal{T}}(A,B) = \frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\frac{1+\mu_{A\Delta_{\mathcal{T}}B}(x_i) - \nu_{A\Delta_{\mathcal{T}}B}(x_i)}{2}\right)^p}; \tag{3.0}$$

$$d^Z_{2L_p,\mathcal{T}}(A,B) = \frac{1}{n\sqrt[p]{2}}\sum_{i=1}^{n}\sqrt[p]{(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^p + (1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^p}; \tag{3.0}$$

$$d^Z_{HY,\mathcal{T}}(A,B) = \frac{1-e^{-\frac{1}{n}\sum\limits_{i=1}^{n}\left(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)\right)}}{1-e^{-1}}. \tag{3.0}$$

and

$$d^Z_{M,\mathcal{T}}(A,B) = \frac{1}{2}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^p} + \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^p}\right). \tag{3.0}$$

Notice that we replace $|\mu_A(x_i) - \mu_B(x_i)|$ by $\mu_{A\Delta_{\mathcal{T}}B}(x_i)$ and $|\nu_A(x_i) - \nu_B(x_i)|$ by $1 - \nu_{A\Delta_{\mathcal{T}}B}(x_i)$ in each of the eight distance measures proposed in the literature and recalled in Eq. (i), Eq. (i), Eq. (i), Eq. (ii), Eq. (ii), Eq. (iii), Eq. (iii) and Eq. (iv). Thereby, our first main result of this chapter establishes that the eight mappings in Eqs. (3.1)-(3.1) are distance measures. In order words the replacement preserves the structure of distance measures.

Theorem 2. The mappings $d^Z_{2n,\mathcal{T}}$, $d^Z_{2w,\mathcal{T}}$, $d^Z_{2H,\mathcal{T}}$, $d^Z_{2E,\mathcal{T}}$, $d^Z_{2p,\mathcal{T}}$, $d^Z_{2L_p,\mathcal{T}}$, $d^Z_{HY,\mathcal{T}}$ and $d^Z_{M,\mathcal{T}}$ are distance measures associated with $\mathcal{T}$.

Note that, Theorem 2 can be proved using Definition 1.2.8 and Lemma 2.

Proof. 1. Let us prove that $d^Z_{2n,\mathcal{T}}$ is a distance measure.

i. Let A and B be $IFSs(X)$, we show that $0 \leq d^Z_{2n,\mathcal{T}}(A,B) \leq 1$.
Since $A\Delta_{\mathcal{T}}B \in IFSs(X)$, then for all $x_i \in X$,
$0 \leq \mu_{A\Delta_{\mathcal{T}}B}(x_i) + 3(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)) \leq 4$ implies $0 \leq d^Z_{2n,\mathcal{T}}(A,B) \leq 1$.

ii. Let A and B be $IFSs(X)$, we show that $d^Z_{2n,\mathcal{T}}(A,B) = 0$ if and only if $A = B$.
Since for all $x_i \in X$,

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}B}(x_i) \geq 0; \\ 1-\nu_{A\Delta_{\mathcal{T}}B}(x_i) \geq 0, \end{cases} \tag{3.0}$$

then from Eq. (3.1) $d^Z_{2n,\mathcal{T}}(A,B) = 0$ is equivalent to: for all $x_i \in X$,

$$\mu_{A\Delta_{\mathcal{T}}B}(x_i) = 0 \text{ and } 1-\nu_{A\Delta_{\mathcal{T}}B}(x_i) = 0, \tag{3.0}$$

Eq. (1ii) is equivalent to $A\Delta_{\mathcal{T}}B = \emptyset_{\mathcal{T}}$. The preceding and Lemma 2 give the result.

iii. Let A and B be $IFSs(X)$, we show that $d^Z_{2n,\mathcal{T}}(A,B) = d^Z_{2n,\mathcal{T}}(B,A)$.
Since $A\Delta_{\mathcal{T}}B = B\Delta_{\mathcal{T}}A$ (see Proposition 2.2.1), then from (3.1), the result follows.

iv. Let A, B and C be $IFSs(X)$ such that $A \subseteq B \subseteq C$. We show that
$d^Z_{2n,\mathcal{T}}(A,C) \geq d^Z_{2n,\mathcal{T}}(A,B)$ and $d^Z_{2n,\mathcal{T}}(A,C) \geq d^Z_{2n,\mathcal{T}}(B,C)$.
Since $A \subseteq B \subseteq C$, then from Lemma 2, we have: for all $x_i \in X$.

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}C}(x_i) \geq \mu_{A\Delta_{\mathcal{T}}B}(x_i) \\ \nu_{A\Delta_{\mathcal{T}}C}(x_i) \leq \nu_{A\Delta_{\mathcal{T}}B}(x_i), \end{cases} \tag{3.0}$$

and

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}C}(x_i) \geq \mu_{B\Delta_{\mathcal{T}}C}(x_i) \\ \nu_{A\Delta_{\mathcal{T}}C}(x_i) \leq \nu_{B\Delta_{\mathcal{T}}C}(x_i), \end{cases} \tag{3.0}$$

Therefore: for all $x_i \in X$

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}C}(x_i) \geq \mu_{A\Delta_{\mathcal{T}}B}(x_i) \\ 1-\nu_{A\Delta_{\mathcal{T}}C}(x_i) \geq 1-\nu_{A\Delta_{\mathcal{T}}B}(x_i), \end{cases} \tag{3.0}$$

and

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}C}(x_i) \geq \mu_{B\Delta_{\mathcal{T}}C}(x_i) \\ 1-\nu_{A\Delta_{\mathcal{T}}C}(x_i) \geq 1-\nu_{B\Delta_{\mathcal{T}}C}(x_i), \end{cases} \tag{3.0}$$

In consequent Eqs. (3.1) and (1iv) show that $d^Z_{2n,\mathcal{T}}(A,C) \geq d^Z_{2n,\mathcal{T}}(A,B)$. Also, Eqs. (3.1) and (1iv) show that $d^Z_{2n,\mathcal{T}}(A,C) \geq d^Z_{2n,\mathcal{T}}(B,C)$.

2. The proofs of $d^Z_{2H,\mathcal{T}}$, $d^Z_{2E,\mathcal{T}}$, $d^Z_{2w,\mathcal{T}}$, $d^Z_{2p,\mathcal{T}}$, $d^Z_{2L_p,\mathcal{T}}$, $d^Z_{HY,\mathcal{T}}$, and $d^Z_{M,\mathcal{T}}$. are analogous to the proof of $d^Z_{2n,\mathcal{T}}$.

□

We observe that many of the new previous distance measures use cardinality components of symmetric difference between two IFSs.
The following remark establishes that $d^Z_{2n,\mathcal{T}}$ and $d^Z_{2H,\mathcal{T}}$ are equivalents; and $d^Z_{2H,\mathcal{T}}$ is lower than $d^Z_{2E,\mathcal{T}}$.

Remark 3. Let A and B be IFSs(X), Since $\mu_{A\Delta_{\mathcal{T}}B}(x_i)) \leq 1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))$ we have, $\frac{3}{4}d^Z_{2n,\mathcal{T}}(A,B) \leq d^Z_{2H,\mathcal{T}}(A,B) \leq d^Z_{2n,\mathcal{T}}(A,B)$ and $d^Z_{2H,\mathcal{T}}(A,B) \leq \sqrt{n}d^Z_{2E,\mathcal{T}}(A,B)$.

In the following remark, we give the fuzzy version of our proposed eight distance measures. Three distance measures coincide on fuzzy sets and becomes the distance measure introduced by Fono et al. [17], and the five other ones become new distance measures on fuzzy sets.

Remark 4. The restrictions of $d^Z_{2n,\mathcal{T}}$, $d^Z_{2w,\mathcal{T}}$, $d^Z_{2H,\mathcal{T}}$, $d^Z_{2E,\mathcal{T}}$, $d^Z_{2p,\mathcal{T}}$, $d^Z_{2L_p,\mathcal{T}}$, $d^Z_{HY,\mathcal{T}}$, and $d^Z_{M,\mathcal{T}}$ to FSs(X) become: for all $A, B \in FSs(X)$:

$$d^Z_{2n,\mathcal{T}}(A,B) = d^Z_{2H,\mathcal{T}}(A,B) = d^Z_{2L_p,\mathcal{T}}(A,B) = \frac{1}{n}\sum_{i=1}^{n}\mu_{A\Delta_{\mathcal{T}}B}(x_i), \tag{3.0}$$

$$d^Z_{2w,\mathcal{T}}(A,B) = \frac{1}{nw}\sum_{i=1}^{n}w_i\mu_{A\Delta_{\mathcal{T}}B}(x_i), \tag{3.0}$$

$$d^Z_{2p,\mathcal{T}}(A,B) = d^Z_{M,\mathcal{T}}(A,B) = \frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^p}, \tag{3.0}$$

$$d^Z_{2E,\mathcal{T}}(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^2}, \tag{3.0}$$

and

$$d^Z_{HY,\mathcal{T}}(A,B) = \frac{1-e^{-\frac{1}{n}\sum_{i=1}^{n}\mu_{A\Delta_\top B}(x_i)}}{1-e^{-1}}. \tag{3.0}$$

The above equality hold because $\mu_{A\Delta_\mathcal{T} B}(x_i) = 1-\nu_{A\Delta_\mathcal{T} B}(x_i) = \mu_{A\Delta_\top B}(x_i)$ holds due to results of Proposition 2.2.2.

The following result justifies that, distance measure $d^Z_{HY,\mathcal{T}}$ and, the new Hamming and Euclidean distance measures generalize the respective distance measure d^{HY}_1 defined by (iii) and, the Hamming and Euclidean distance measures proposed by Grzegorzewski [22] and defined by (ii) and (ii) when $\mathcal{T} = \mathcal{T}_L = (\top_L, S_L)$ is t-representable IF t-norm of Lukasiewicz. And the five other one are greater than their corresponding in the literature of distance measures.

Corollary 4. Let $\mathcal{T} = \mathcal{T}_L = (\top_L, S_L)$ be t-representable IF t-norm of Lukasiewicz. Then for all $A,\ B \in IFSs(X)$,

- $d^Z_{HY,\mathcal{T}}(A,B) = d^{HY}_1(A,B)$; $d^Z_{2H,\mathcal{T}}(A,B) = d^{nH}_1(A,B)$ and $d^Z_{2E,\mathcal{T}}(A,B) = d^{nE}_1(A,B)$.
- $d^Z_{2n,\mathcal{T}}(A,B) \geq d^n_1(A,B)$; $d^Z_{2w,\mathcal{T}}(A,B) \geq d^{nw}_1(A,B)$; $d^Z_{2p,\mathcal{T}}(A,B) \geq d^{np}_1(A,B)$; $d^Z_{2L_p,\mathcal{T}}(A,B) \geq d^{HY}_{L_p}(A,B)$ and $d^Z_{M,\mathcal{T}}(A,B) \geq d^M_1(A,B)$.

Proof. Let A and B be two intuitionistic fuzzy sets defined on X. From Example 10, the membership and non-membership of IF symmetric difference between A and B associated with $\mathcal{T}_L$ are defined for all $x \in X$ as:

$$\mu_{A\,\triangle_{\mathcal{T}_L}\,B}(x) = \begin{cases} 0, & \text{if } (\mu_A(x),\nu_A(x)) = (\mu_B(x),\nu_B(x)) \\ \mid \nu_A(x)-\nu_B(x) \mid, & \text{if } (\mu_A(x),\nu_A(x)) \neq (\mu_B(x),\nu_B(x)), \end{cases} \tag{3.0}$$

$$\nu_{A\,\triangle_{\mathcal{T}_L}\,B}(x) = \begin{cases} 1, & \text{if } (\mu_A(x),\nu_A(x)) = (\mu_B(x),\nu_B(x)) \\ \min\Big\{1- \mid \mu_A(x)-\mu_B(x) \mid, 1- \mid \nu_A(x)-\nu_B(x) \mid \Big\}, & \text{otherwise.} \end{cases} \tag{3.0}$$

a) We show that $d^Z_{HY,\mathcal{T}}(A,B) = d^{HY}_1(A,B)$; $d^Z_{2H,\mathcal{T}}(A,B) = d^{nH}_1(A,B)$ and $d^Z_{2E,\mathcal{T}}(A,B) = d^{nE}_1(A,B)$.

It follows from Eq. (3.1) that

$$1-\nu_{A\,\triangle_{\mathcal{T}_L}\,B}(x) = \begin{cases} 0, & \text{if } (\mu_A(x),\nu_A(x)) = (\mu_B(x),\nu_B(x)) \\ \max\Big\{ \mid \mu_A(x)-\mu_B(x) \mid, \mid \nu_A(x)-\nu_B(x) \mid \Big\}, & \text{otherwise.} \end{cases} \tag{3.0}$$

Therefore from Eq. (a),

$$d^Z_{HY,\mathcal{T}}(A,B) = \frac{1-e^{-\frac{1}{n}\sum\limits_{i=1}^{n}\left(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)\right)}}{1-e^{-1}} = \frac{1-e^{-\frac{1}{n}\sum\limits_{i=1}^{n}\max\{|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|\}}}{1-e^{-1}}, \tag{3.0}$$

$$d^Z_{2H,\mathcal{T}}(A,B) = \frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)) = \frac{1}{n}\sum_{i=1}^{n}\max\left\{ \mid \mu_A(x_i)-\mu_B(x_i) \mid , \mid \nu_A(x_i)-\nu_B(x_i) \mid \right\} \tag{3.0}$$

and

$$d^Z_{2E,\mathcal{T}}(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\max\left\{ (\mu_A(x_i)-\mu_B(x_i))^2, (\nu_A(x_i)-\nu_B(x_i))^2 \right\}}. \tag{3.0}$$

From Eqs. (iii) and (a), Eqs. (ii) and (a) and, Eqs. (ii) and (a), the first result of Corollary 4 holds.

b) The proof of the second result of Corollary 4 is obviously since for all $x \in X$, $\mu_{A\,\triangle_{\mathcal{T}_L}\,B}(x) \leq 1-\nu_{A\,\triangle_{\mathcal{T}_L}\,B}(x)$.

□

Similarity measure is also important to measure degree of similarity between two objets. We deduce the following result for similarity measures from the proposed corresponding distance measures. The proof is obviously followed from Corollary 2 and Theorem 2.

Corollary 5. a) The following mappings $s^Z_{2n,\mathcal{T}} = 1-d^Z_{2n,\mathcal{T}}$, $s^Z_{2w,\mathcal{T}} = 1-d^Z_{2w,\mathcal{T}}$, $s^Z_{2p,\mathcal{T}} = 1-d^Z_{2p,\mathcal{T}}$, $s^Z_{HY,\mathcal{T}} = 1-d^Z_{HY,\mathcal{T}}$ and $s^Z_{M,\mathcal{T}} = 1-d^Z_{M,\mathcal{T}}$ are classes of similarity measures on IFSs associated with $\mathcal{T}$.

b) The following mappings $s^Z_{2H,\mathcal{T}} = 1-d^Z_{2H,\mathcal{T}}$, $s^Z_{2E,\mathcal{T}} = 1-d^Z_{2E,\mathcal{T}}$ and $s^Z_{2L_p,\mathcal{T}} = 1-d^Z_{2L_p,\mathcal{T}}$ are classes of similarity measures on IFSs associated with $\mathcal{T}$, provided to Hamming metric, Euclidean metric and L_p metric respectively.

Although distance measure is important for us to study difference between two objets, metric (distance) between IFSs gives good geometric properties on the set $IFSs(X)$. Note that a normalized metric is a distance measure but the converse is not necessarily true.

In the following Section, we define and study condition on both fuzzy implication operators and fuzzy co-implication operators under which some of our classes of distance measures become metrics.

3.2 Some classes of metric for IFSs based on cardinality components of symmetric difference for IFSs

De Baets and Mesiar [7] introduced the condition C_1 for fuzzy R-implication operators $I_\top$ as follows. For all $a, b, c \in [0, 1]$,

$$a > b > c \;\text{ implies }\; I_\top(a, c) \geq \top(I_\top(a, b), I_\top(b, c)). \tag{3.0}$$

Fono et al. [17] introduced and studied the following condition C for fuzzy R-implication operators (see [17] Definition 2 page 317).
$I_\top$ satisfies condition C if for all $a, b, c \in [0, 1]$,

$$a > b > c \;\text{ implies }\; 1 + I_\top(a, c) \geq I_\top(a, b) + I_\top(b, c). \tag{3.0}$$

They showed that conditions C and C_1 are equivalent when $\top = \top_L$ is the Lukasiewicz t-norm.
We first recall that De Baets and Mesiar [7] have showed that the R-implicators and contraposition symmetrical operators of the Lukasievicz t-norm satisfy condition C_1 (see [7] Theorem 5 p. 5). Fono et al. [17] have completed their results by showing that the R-implicators and contraposition symmetrical operators of the family of Frank t-norms $(\top_F^\lambda)_{\lambda\in[0;+\infty]}$ satisfy condition C (see [17] Proposition 2 p. 318). In the same view, Fotso et al. [20] displayed in each seven usual parameterized families of t-norms recalled in [29], those which satisfy condition C and thereby generate metrics from dissimilarity measures. Their results have showed that, all the t-norms of the family of Mayor-Torrens $\left(\top_{MT}^\lambda\right)_{\lambda\in[0,1]}$ generate metrics from the cardinality of symmetric difference for fuzzy sets (see [20], Proposition 4 p. 7).

We now introduce and study new conditions for both fuzzy implication operators and fuzzy co-implication operators which will be useful in the next of this section.

3.2.1 New Conditions for both fuzzy implication and fuzzy co-implication operators

Condition C^*

The following are the definition of condition C^* which will be helpful in this study.

Definition 3.2.1 (Condition C^*)**.** *$(I_\top, J_S)$ satisfies condition C^* if for all $(a,a'),(b,b'),(c,c') \in L^*$,*

$$\begin{cases} a > b > c \text{ implies } 1 + I_\top(a,c) \geq I_\top(a,b) + I_\top(b,c), \\ a' < b' < c' \text{ implies } J_S(a',c') \leq J_S(a',b') + J_S(b',c'). \end{cases} \tag{3.0}$$

In the next, we will say $\mathcal{T}$ satisfies condition C^* if $(I_\top, J_S)$ satisfies condition C^*.

The two lines of the Eq. (3.2.1) are linked due to the fact that $(a,a'),(b,b'),(c,c') \in L^*$. On the other hand condition C^* implies condition C because the existence of the triplet $(a,b,c) \in [0,1]^3$ such that $a > b > c$ induces the existence of the triplet $(a',b',c') \in [0,1]^3$ such that a'<b'<c' and $(a,a'),(b,b'),(c,c') \in L^*$.
The following result establishes that condition C^* becomes condition C when t-norm $\top$ and t-conorm S are dual.

Corollary 6. Suppose that $\top$ and S are dual, then conditions C^* and C are equivalent.

Proof. From (3.2) and (3.2.1) to show that condition C^* is the same as condition C, it is sufficient to show that

$$\begin{aligned} &\{(a,b,c) \in [0,1]^3,\ a > b > c \Rightarrow 1 + I_\top(a,c) \geq I_\top(a,b) + I_\top(b,c)\} \\ &= \{(a',b',c') \in [0,1]^3,\ a' < b' < c' \Rightarrow J_S(a',c') \leq J_S(a',b') + J_S(b',c')\}. \end{aligned}$$

From Proposition 1.1.1, $\top$ and S are dual, implies for all $a,b \in [0,1]$,
$I_\top(a,b) = 1 - J_S(1-a, 1-b)$.

$$\begin{aligned} Thus\ \ &\{(a,b,c) \in [0,1]^3,\ a > b > c \Rightarrow 1 + I_\top(a,c) \geq I_\top(a,b) + I_\top(b,c)\} \\ &= \{(a,b,c) \in [0,1]^3,\ 1-a < 1-b < 1-c \Rightarrow J_S(1-a,1-c) \leq J_S(1-a,1-b) + J_S(1-b,1-c)\} \\ &= \{(1-a,1-b,1-c) \in [0,1]^3,\ 1-a < 1-b < 1-c \Rightarrow J_S(1-a,1-c) \leq J_S(1-a,1-b) + J_S(1-b,1-c)\} \\ &= \{(a',b',c') \in [0,1]^3,\ a' < b' < c' \Rightarrow J_S(a',c') \leq J_S(a',b') + J_S(b',c')\}. \end{aligned}$$

□

The following result gives necessary and sufficient conditions for C^*.

Lemma 3. $\mathcal{T}$ satisfies condition C^* if and only if for all IFSs $A,\ B$ and C, and for all $x \in X$,

$$\begin{cases} 1 + I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) \geq I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) + I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)); \\ J_S(\nu_A(x) \wedge \nu_C(x), \nu_A(x) \vee \nu_C(x)) \leq J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)) + J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x)). \end{cases} \tag{3.0}$$

Proof. 1. Assume that $\mathcal{T}$ satisfies condition C^*. Let $A,\ B$ and C be IFSs and $x \in X$.

(a) Let us show first that

$$J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))\leq J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x))+J_S(\nu_B(x)\wedge\nu_C(x),\nu_B(x)\vee\nu_C(x)). \tag{3.0}$$

We distinguish nine cases.

Case 1: if $\nu_A(x)=\nu_C(x)$, then $J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_A(x),\nu_A(x))=0$ and (1a) holds.

Case 2: if $\nu_A(x)=\nu_B(x)$, then $J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_B(x)\wedge\nu_C(x),\nu_B(x)\vee\nu_C(x))$ and (1a) holds.

Case 3: if $\nu_B(x)=\nu_C(x)$, then $J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x))$ and (1a) holds.

Case 4: if $\nu_A(x)<\nu_C(x)<\nu_B(x)$, then $J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_A(x),\nu_C(x))\leq J_S(\nu_A(x),\nu_B(x))=J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x))$ since J_S is right increasing operator (see Proposition 1.1.1), and (1a) holds.

Case 5: if $\nu_B(x)<\nu_A(x)<\nu_C(x)$, then $J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_A(x),\nu_C(x))\leq J_S(\nu_B(x),\nu_C(x))=J_S(\nu_B(x)\wedge\nu_C(x),\nu_B(x)\vee\nu_C(x))$ since J_S is left decreasing operator (see Proposition 1.1.1), and (1a) holds.

Case 6: if $\nu_B(x)<\nu_C(x)<\nu_A(x)$, then $J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_C(x),\nu_A(x))\leq J_S(\nu_B(x),\nu_A(x))=J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x))$ since J_S is left decreasing operator and (1a) holds.

Case 7: if $\nu_C(x)<\nu_A(x)<\nu_B(x)$, then $J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_C(x),\nu_A(x))\leq J_S(\nu_C(x),\nu_B(x))=J_S(\nu_B(x)\wedge\nu_C(x),\nu_B(x)\vee\nu_C(x))$ since J_S is right increasing operator and (1a) holds.

Case 8: if $\nu_A(x)<\nu_B(x)<\nu_C(x)$, then

$$\begin{cases} J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x))=J_S(\nu_A(x),\nu_C(x));\\ J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x))=J_S(\nu_A(x),\nu_B(x));\\ J_S(\nu_B(x)\wedge\nu_C(x),\nu_B(x)\vee\nu_C(x))=J_S(\nu_B(x),\nu_C(x)).\end{cases}$$

and (1a) follows from Definition 3.2.1 because $\mathcal{T}$ satisfies condition C^*.

Case 9: if $\nu_C(x) < \nu_B(x) < \nu_A(x)$, then

$$\begin{cases} J_S(\nu_A(x) \wedge \nu_C(x), \nu_A(x) \vee \nu_C(x)) = J_S(\nu_C(x), \nu_A(x)); \\ J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)) = J_S(\nu_B(x), \nu_A(x)); \\ J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x)) = J_S(\nu_C(x), \nu_B(x)). \end{cases}$$

and (1a) follows from Definition 3.2.1 because $\mathcal{T}$ satisfies condition C^*.

(b) Let us show now that

$$1 + I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) \geq I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) + I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)). \tag{3.0}$$

As before, we distinguish nine cases.

Case 1: If $\mu_A(x) = \mu_C(x)$, then $I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_A(x), \mu_A(x)) = 1$ and (1b) holds.

Case 2: If $\mu_A(x) = \mu_B(x)$, then $I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x))$ and (1b) holds.

Case 3: If $\mu_B(x) = \mu_C(x)$, then $I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x))$ and (1b) holds.

Case 4: If $\mu_A(x) < \mu_C(x) < \mu_B(x)$, then $I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_C(x), \mu_A(x)) \geq I_\top(\mu_B(x), \mu_A(x)) = I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x))$ since $I_\top$ is left decreasing operator and (1b) holds.

Case 5: If $\mu_B(x) < \mu_C(x) < \mu_A(x)$, then $I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_A(x), \mu_C(x)) \geq I_\top(\mu_A(x), \mu_B(x)) = I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x))$ since $I_\top$ is right increasing operator and (1b) holds.

Case 6: If $\mu_B(x) < \mu_A(x) < \mu_C(x)$, then $I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_C(x), \mu_A(x)) \geq I_\top(\mu_C(x), \mu_B(x)) = I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x))$ since $I_\top$ is right increasing operator and (1b) holds.

Case 7: If $\mu_C(x) < \mu_A(x) < \mu_B(x)$, then $I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_A(x), \mu_C(x)) \geq I_\top(\mu_B(x), \mu_C(x)) = I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x))$ since $I_\top$ is left decreasing operator and (1b) holds.

Case 8: If $\mu_A(x) < \mu_B(x) < \mu_C(x)$ then

$$\begin{cases} I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_C(x), \mu_A(x)); \\ I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) = I_\top(\mu_B(x), \mu_A(x)); \\ I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)) = I_\top(\mu_C(x), \mu_B(x)). \end{cases}$$

and (1b) follows from Definition 3.2.1 because $\mathcal{T}$ satisfies condition C^*.

Case 9: If $\mu_C(x) < \mu_B(x) < \mu_A(x)$ then

$$\begin{cases} I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) = I_\top(\mu_A(x), \mu_C(x)); \\ I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) = I_\top(\mu_A(x), \mu_B(x)); \\ I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)) = I_\top(\mu_B(x), \mu_C(x)). \end{cases}$$

and (1b) follows from Definition 3.2.1 because $\mathcal{T}$ satisfies condition C^*.

2. Assume that for all IFSs A, B and C, and for all $x \in X$, (3) hold. We show that $\mathcal{T}$ satisfies condition C^*.

   Let $(a, a'), (b, b'), (c, c') \in L^*$, such that $a > b > c$ and $a' < b' < c'$, we need to show that Equation (3.2.1) holds.

$$\begin{cases} 1 + I_\top(a, c) \geq I_\top(a, b) + I_\top(b, c); \\ J_S(a', c') \leq J_S(a', b') + J_S(b', c'). \end{cases} \tag{3.0}$$

   Define A, B and C such that for all $x \in X$, $(\mu_A(x), \nu_A(x)) = (a, a')$, $(\mu_B(x), \nu_B(x)) = (b, b')$ and $(\mu_C(x), \nu_C(x)) = (c, c')$.

   Since $a > b > c$ and $a' < b' < c'$, then (3) reduces to (3.2.1) and the Definition 3.2.1 is established.

□

The following result shows that condition C^* is sufficient for triangular inequalities of the membership and non-membership of the cardinality of symmetric difference between IFSs.

Proposition 3.2.1. *If $\mathcal{T}$ satisfies condition C^* then, for all IFSs A, B and C and for all $x \in X$,*

$$\begin{cases} \mu_{A\Delta_\mathcal{T}C}(x) \leq \mu_{A\Delta_\mathcal{T}B}(x) + \mu_{B\Delta_\mathcal{T}C}(x); \\ 1 + \nu_{A\Delta_\mathcal{T}C}(x) \geq \nu_{A\Delta_\mathcal{T}B}(x) + \nu_{B\Delta_\mathcal{T}C}(x). \end{cases} \tag{3.0}$$

Proof. Assume that $\mathcal{T}$ satisfies condition C^*. Let A, B and C be IFSs and $x \in X$. We must show that:

$$\begin{cases} \mu_{A\Delta_\mathcal{T}C}(x) \leq \mu_{A\Delta_\mathcal{T}B}(x) + \mu_{B\Delta_\mathcal{T}C}(x); \\ 1 + \nu_{A\Delta_\mathcal{T}C}(x) \geq \nu_{A\Delta_\mathcal{T}B}(x) + \nu_{B\Delta_\mathcal{T}C}(x). \end{cases}$$

1. Let us show first that

$$\mu_{A\Delta_\mathcal{T}C}(x) \leq \mu_{A\Delta_\mathcal{T}B}(x) + \mu_{B\Delta_\mathcal{T}C}(x). \tag{3.0}$$

   From Definition 2.2.1, (1) is equivalent to

   $J_S(\nu_A(x) \wedge \nu_C(x), \nu_A(x) \vee \nu_C(x)) \leq J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)) + J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x))$.

   Since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3 the result follows.

2. Let us show now that

$$1 + \nu_{A\Delta_{\mathcal{T}}C}(x) \geq \nu_{A\Delta_{\mathcal{T}}B}(x) + \nu_{B\Delta_{\mathcal{T}}C}(x). \tag{3.0}$$

From Definition 2.2.1, we have:

$$\begin{cases} \nu_{A\Delta_{\mathcal{T}}C}(x) = \min\{I_{\top}(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)), 1 - J_S(\nu_A(x) \wedge \nu_C(x), \nu_A(x) \vee \nu_C(x))\}; \\ \nu_{A\Delta_{\mathcal{T}}B}(x) = \min\{I_{\top}(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)), 1 - J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x))\}; \\ \nu_{B\Delta_{\mathcal{T}}C}(x) = \min\{I_{\top}(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)), 1 - J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x))\}. \end{cases}$$

We distinguish eight cases:

i) If

$$\begin{cases} \nu_{A\Delta_{\mathcal{T}}C}(x) = 1 - J_S(\nu_A(x) \wedge \nu_C(x), \nu_A(x) \vee \nu_C(x)) \\ \nu_{A\Delta_{\mathcal{T}}B}(x) = 1 - J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)) \\ \nu_{B\Delta_{\mathcal{T}}C}(x) = 1 - J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x)), \end{cases}$$

then (2) is equivalent to (1a). Since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3, the result follows.

ii) If

$$\begin{cases} \nu_{A\Delta_{\mathcal{T}}C}(x) = I_{\top}(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) \\ \nu_{A\Delta_{\mathcal{T}}B}(x) = I_{\top}(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) \\ \nu_{B\Delta_{\mathcal{T}}C}(x) = I_{\top}(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)), \end{cases}$$

then (2) is equivalent to (1b). Since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3, the result follows.

iii) If

$$\begin{cases} \nu_{A\Delta_{\mathcal{T}}C}(x) = I_{\top}(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) \\ \nu_{A\Delta_{\mathcal{T}}B}(x) = I_{\top}(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) \\ \nu_{B\Delta_{\mathcal{T}}C}(x) = 1 - J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x)), \end{cases}$$

then $I_{\top}(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)) \geq 1 - J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x))$; and since $\mathcal{T}$ satisfies condition C^*, from Lemma 3, $1 + I_{\top}(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) \geq I_{\top}(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) + I_{\top}(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x))$ implies $1 + I_{\top}(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) \geq I_{\top}(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)) + 1 - J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x))$, and (2) holds.

iv) If

$$\begin{cases} \nu_{A\Delta_{\mathcal{T}}C}(x) = I_{\top}(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)) \\ \nu_{A\Delta_{\mathcal{T}}B}(x) = 1 - J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)) \\ \nu_{B\Delta_{\mathcal{T}}C}(x) = I_{\top}(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)), \end{cases}$$

then $I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x)) \geq 1-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x))$ and since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3, $1+I_\top(\mu_A(x)\vee\mu_C(x), \mu_A(x)\wedge\mu_C(x)) \geq I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x))+I_\top(\mu_B(x)\vee\mu_C(x), \mu_B(x)\wedge\mu_C(x))$ implies $1+I_\top(\mu_A(x)\vee\mu_C(x), \mu_A(x)\wedge\mu_C(x)) \geq 1-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x))+I_\top(\mu_B(x)\vee\mu_C(x), \mu_B(x)\wedge\mu_C(x))$, and (2) holds.

v) If

$$\begin{cases} \nu_{A\Delta_\mathcal{T}C}(x) = I_\top(\mu_A(x)\vee\mu_C(x), \mu_A(x)\wedge\mu_C(x)) \\ \nu_{A\Delta_\mathcal{T}B}(x) = 1-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x)) \\ \nu_{B\Delta_\mathcal{T}C}(x) = 1-J_S(\nu_B(x)\wedge\nu_C(x), \nu_B(x)\vee\nu_C(x)), \end{cases}$$

then

$$\begin{cases} I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x)) \geq 1-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x)); \\ I_\top(\mu_B(x)\vee\mu_C(x), \mu_B(x)\wedge\mu_C(x)) \geq 1-J_S(\nu_B(x)\wedge\nu_C(x), \nu_B(x)\vee\nu_C(x)). \end{cases}$$

and since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3, $1+I_\top(\mu_A(x)\vee\mu_C(x), \mu_A(x)\wedge\mu_C(x)) \geq I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x))+I_\top(\mu_B(x)\vee\mu_C(x), \mu_B(x)\wedge\mu_C(x))$ implies $1+I_\top(\mu_A(x)\vee\mu_C(x), \mu_A(x)\wedge\mu_C(x)) \geq 1-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x))+1-J_S(\nu_B(x)\wedge\nu_C(x), \nu_B(x)\vee\nu_C(x))$, and (2) holds.

vi) If

$$\begin{cases} \nu_{A\Delta_\mathcal{T}C}(x) = 1-J_S(\nu_A(x)\wedge\nu_C(x), \nu_A(x)\vee\nu_C(x)) \\ \nu_{A\Delta_\mathcal{T}B}(x) = I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x)) \\ \nu_{B\Delta_\mathcal{T}C}(x) = 1-J_S(\nu_B(x)\wedge\nu_C(x), \nu_B(x)\vee\nu_C(x)), \end{cases}$$

then $1-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x)) \geq I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x))$ and since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3, $1-J_S(\nu_A(x)\wedge\nu_C(x), \nu_A(x)\vee\nu_C(x)) \geq 2-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x))-J_S(\nu_B(x)\wedge\nu_C(x), \nu_B(x)\vee\nu_C(x))$ implies $1-J_S(\nu_A(x)\wedge\nu_C(x), \nu_A(x)\vee\nu_C(x)) \geq I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x))+1-J_S(\nu_B(x)\wedge\nu_C(x), \nu_B(x)\vee\nu_C(x))$, and (2) holds.

vii) If

$$\begin{cases} \nu_{A\Delta_\mathcal{T}C}(x) = 1-J_S(\nu_A(x)\wedge\nu_C(x), \nu_A(x)\vee\nu_C(x)) \\ \nu_{A\Delta_\mathcal{T}B}(x) = I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x)) \\ \nu_{B\Delta_\mathcal{T}C}(x) = I_\top(\mu_B(x)\vee\mu_C(x), \mu_B(x)\wedge\mu_C(x)), \end{cases}$$

then

$$\begin{cases} 1-J_S(\nu_A(x)\wedge\nu_B(x), \nu_A(x)\vee\nu_B(x)) \geq I_\top(\mu_A(x)\vee\mu_B(x), \mu_A(x)\wedge\mu_B(x)); \\ 1-J_S(\nu_B(x)\wedge\nu_C(x), \nu_B(x)\vee\nu_C(x)) \geq I_\top(\mu_B(x)\vee\mu_C(x), \mu_B(x)\wedge\mu_C(x)). \end{cases},$$

and since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3, $2-J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x)) \geq 1+I_\top(\mu_A(x)\vee\mu_C(x),\mu_A(x)\wedge\mu_C(x)) \geq I_\top(\mu_A(x)\vee\mu_B(x),\mu_A(x)\wedge\mu_B(x))+I_\top(\mu_B(x)\vee\mu_C(x),\mu_B(x)\wedge\mu_C(x))$, and (2) holds.

viii) If

$$\begin{cases} \nu_{A\Delta_\mathcal{T}C}(x) = 1-J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x)) \\ \nu_{A\Delta_\mathcal{T}B}(x) = 1-J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x)) \\ \nu_{B\Delta_\mathcal{T}C}(x) = I_\top(\mu_B(x)\vee\mu_C(x),\mu_B(x)\wedge\mu_C(x)), \end{cases}$$

then $1-J_S(\nu_B(x)\wedge\nu_C(x),\nu_B(x)\vee\nu_C(x)) \geq I_\top(\mu_B(x)\vee\mu_C(x),\mu_B(x)\wedge\mu_C(x))$, and since $\mathcal{T}$ satisfies condition C^*, then from Lemma 3, $2-J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x)) \geq 2-J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x))-J_S(\nu_B(x)\wedge\nu_C(x),\nu_B(x)\vee\nu_C(x))$ implies $2-J_S(\nu_A(x)\wedge\nu_C(x),\nu_A(x)\vee\nu_C(x)) \geq 1-J_S(\nu_A(x)\wedge\nu_B(x),\nu_A(x)\vee\nu_B(x))+I_\top(\mu_B(x)\vee\mu_C(x),\mu_B(x)\wedge\mu_C(x))$, and (2) holds.

□

Condition C^2

Condition C^2 is defined in this thesis as follows. We shall refer to this definition in establishing related results subsequently.

Definition 3.2.2 (Condition C^2)**.** *$(I_\top, J_S)$ satisfies condition C^2 if for all $(a,a'),(b,b'),(c,c') \in L^*$,*

$$\begin{cases} a>b>c \text{ implies } (1-I_\top(a,c))^2 \leq (1-I_\top(a,b))^2+(1-I_\top(b,c))^2, \\ a'<b'<c' \text{ implies } (J_S(a',c'))^2 \leq (J_S(a',b'))^2+(J_S(b',c'))^2. \end{cases} \tag{3.0}$$

Subsequently, we say $\mathcal{T}$ satisfies condition C^2 if $(I_\top, J_S)$ satisfies condition C^2. We deduce the following result which stipulates that condition C^2 is stronger than condition C^*.

Corollary 7. If $\mathcal{T}$ satisfies condition C^2, then $\mathcal{T}$ satisfies condition C^*.

Proof. The proof is immediate since

$(1-I_\top(a,c))^2 \leq (1-I_\top(a,b))^2+(1-I_\top(b,c))^2$ implies $(1-I_\top(a,c))^2 \leq (1-I_\top(a,b))^2+(1-I_\top(b,c))^2+2(1-I_\top(a,b))(1-I_\top(b,c))$

and $(J_S(a',c'))^2 \leq (J_S(a',b'))^2+(J_S(b',c'))^2$ implies $(J_S(a',c'))^2 \leq (J_S(a',b'))^2+(J_S(b',c'))^2+2(J_S(a',b'))(J_S(b',c'))$.

□

The following result gives necessary and sufficient conditions for the existence of condition C^2.

Lemma 4. $\mathcal{T}$ satisfies condition C^2 if and only if for all IFSs $A,\ B$ and C, and for all $x \in X$,

$$\begin{cases} (1 - I_\top(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x)))^2 \leq (1 - I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)))^2 + (1 - I_\top(\mu_B(x) \vee \mu_C(x), \mu_B(x) \wedge \mu_C(x)))^2, \\ (J_S(\nu_A(x) \wedge \nu_C(x), \nu_A(x) \vee \nu_C(x)))^2 \leq (J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)))^2 + (J_S(\nu_B(x) \wedge \nu_C(x), \nu_B(x) \vee \nu_C(x)))^2. \end{cases} \tag{3.0}$$

Proof. The proof is analogous to the Proof of Lemma 3. □

The following result shows that condition C^2 is sufficient for triangular inequalities of the square of membership and non-membership of the cardinality of symmetric difference between IFSs.

Proposition 3.2.2. *If $\mathcal{T}$ satisfies condition C^2 then, for all IFSs $A,\ B$ and C, and for all $x \in X$,*

$$\begin{cases} (\mu_{A\Delta_{\mathcal{T}}C}(x))^2 \leq (\mu_{A\Delta_{\mathcal{T}}B}(x))^2 + (\mu_{B\Delta_{\mathcal{T}}C}(x))^2; \\ (1 - \nu_{A\Delta_{\mathcal{T}}C}(x))^2 \leq (1 - \nu_{A\Delta_{\mathcal{T}}B}(x))^2 + (1 - \nu_{B\Delta_{\mathcal{T}}C}(x))^2. \end{cases} \tag{3.0}$$

Proof. Using Lemma 4, the proof is analogous to the Proof of Proposition 3.2.1. □

In the next Subsection, we establish sufficient condition under which seven of our eight proposed classes of distance measures become metrics.

3.2.2 Some classes of metrics for IFSs

Our second main result of this chapter gives sufficient conditions on $\mathcal{T}$ under which seven of our eight proposed distance measures become metrics on IFSs(X) generated by $\mathcal{T}$.

Theorem 3.
1. If $\mathcal{T}$ satisfies condition C^*, then (a) $d^Z_{2n,\mathcal{T}}$, (b) $d^Z_{2w,\mathcal{T}}$, (c) $d^Z_{2H,\mathcal{T}}$, (d) $d^Z_{2E,\mathcal{T}}$, (e) $d^Z_{2p,\mathcal{T}}$ with $p = 2$ and (f) $d^Z_{2p,\mathcal{T}} = d^Z_{2L_p,\mathcal{T}} = d^Z_{M,\mathcal{T}}$ with $p = 1$ are metrics on IFSs(X) generated by $\mathcal{T}$.

2. If $\mathcal{T}$ satisfies condition C^2 and $p = 2$ then $d^Z_{2L_p,\mathcal{T}}$ and $d^Z_{M,\mathcal{T}}$ are metric generated by $\mathcal{T}$.

Proof.
1. From Definition 1.2.8 and Theorem 2, all of (a) – (f) satisfy properties (i) and (ii) of metric. Thus, it remains to show that (a) – (f) satisfy the triangular inequality property.

   Assume that $\mathcal{T}$ satisfies condition C^*.

   (a) Since $d^Z_{2n,\mathcal{T}} = n d^Z_{2w,\mathcal{T}}$ when $w_i = \frac{1}{n}$ for all $i = 1, ..., n$, it is sufficient to show that $d^Z_{2w,\mathcal{T}}$ is a metric.

(b) To prove that $d^Z_{2w,\mathcal{T}}$ satisfies the triangle inequality, we let $A, B, C \in IFSs(X)$ and show that $d^Z_{2w,\mathcal{T}}(A,C) \leq d^Z_{2w,\mathcal{T}}(A,B) + d^Z_{2w,\mathcal{T}}(B,C)$ holds.
From (3.1) we have

$$\begin{cases} d^Z_{2w,\mathcal{T}}(A,C) = \frac{1}{nw}\sum_{i=1}^{n} w_i \frac{3+\mu_{A\Delta_{\mathcal{T}}C}(x_i)-3\nu_{A\Delta_{\mathcal{T}}C}(x_i)}{4}; \\ d^Z_{2w,\mathcal{T}}(A,B) = \frac{1}{nw}\sum_{i=1}^{n} w_i \frac{3+\mu_{A\Delta_{\mathcal{T}}B}(x_i)-3\nu_{A\Delta_{\mathcal{T}}B}(x_i)}{4}; \\ d^Z_{2w,\mathcal{T}}(B,C) = \frac{1}{nw}\sum_{i=1}^{n} w_i \frac{3+\mu_{B\Delta_{\mathcal{T}}C}(x_i)-3\nu_{B\Delta_{\mathcal{T}}C}(x_i)}{4}. \end{cases} \tag{3.0}$$

Since $\mathcal{T}$ satisfies condition C^*, then from Proposition 3.2.1 we have that (3.2.1) is satisfied.
For all $x \in X$,

$$\begin{cases} \mu_{A\Delta_{\mathcal{T}}C}(x) \leq \mu_{A\Delta_{\mathcal{T}}B}(x) + \mu_{B\Delta_{\mathcal{T}}C}(x); \\ 1 + \nu_{A\Delta_{\mathcal{T}}C}(x) \geq \nu_{A\Delta_{\mathcal{T}}B}(x) + \nu_{B\Delta_{\mathcal{T}}C}(x). \end{cases} \tag{3.0}$$

Hence, the result follows from (1b).

(c) For $A, B, C \in IFSs(X)$ arbitrary, we show that $d^Z_{2H,\mathcal{T}}(A,C) \leq d^Z_{2H,\mathcal{T}}(A,B) + d^Z_{2H,\mathcal{T}}(B,C)$ holds.
From (3.1) we have

$$\begin{cases} d^Z_{2H,\mathcal{T}}(A,C) = \frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}C}(x_i)); \\ d^Z_{2H,\mathcal{T}}(A,B) = \frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)); \\ d^Z_{2H,\mathcal{T}}(B,C) = \frac{1}{n}\sum_{i=1}^{n}(1-\nu_{B\Delta_{\mathcal{T}}C}(x_i)). \end{cases} \tag{3.0}$$

Since $\mathcal{T}$ satisfies condition C^*, then from Proposition 3.2.1 we have $1+\nu_{A\Delta_{\mathcal{T}}C}(x) \geq \nu_{A\Delta_{\mathcal{T}}B}(x)+\nu_{B\Delta_{\mathcal{T}}C}(x)$ holds for all $x \in X$. Thus, the required result follows from (1c).

(d) For $A, B, C \in IFSs(X)$ arbitrary, we show that $d^Z_{2E,\mathcal{T}}(A,C) \leq d^Z_{2E,\mathcal{T}}(A,B) + d^Z_{2E,\mathcal{T}}(B,C)$ holds.
From (3.1) we have

$$\begin{cases} d^Z_{2E,\mathcal{T}}(A,C) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}C}(x_i))^2}; \\ d^Z_{2E,\mathcal{T}}(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^2}; \\ d^Z_{2E,\mathcal{T}}(B,C) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{B\Delta_{\mathcal{T}}C}(x_i))^2}. \end{cases} \tag{3.0}$$

Since $\mathcal{T}$ satisfies condition C^*, then from Proposition 3.2.1 $1+\nu_{A\Delta_{\mathcal{T}}C}(x) \geq \nu_{A\Delta_{\mathcal{T}}B}(x)+\nu_{B\Delta_{\mathcal{T}}C}(x)$ holds for all $x \in X$.

This implies that

$$\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}C}(x_i))^2 \leq \frac{1}{n}\sum_{i=1}^{n}[(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))+(1-\nu_{B\Delta_{\mathcal{T}}C}(x_i))]^2. \quad (3.0)$$

Recall that by Minkowski inequality in $\mathbb{R}^n$, we have

$$\sqrt{\sum_{i=1}^{n}(x_k+y_k)^2} \leq \sqrt{\sum_{i=1}^{n}x_k^2}+\sqrt{\sum_{i=1}^{n}y_k^2} \quad (3.0)$$

holds for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)$.

By combining (1d), (1d) and (1d), we obtain the required result.

(e) For $A,B,C \in IFSs(X)$ arbitrary, we show that $d^Z_{22,\mathcal{T}}(A,C) \leq d^Z_{22,\mathcal{T}}(A,B) + d^Z_{22,\mathcal{T}}(B,C)$ holds.

From (3.1) we have

$$\begin{cases} d^Z_{22,\mathcal{T}}(A,C) = \frac{1}{\sqrt{n}}\sqrt{\sum\limits_{i=1}^{n}\left(\frac{1+\mu_{A\Delta_{\mathcal{T}}C}(x_i)-\nu_{A\Delta_{\mathcal{T}}C}(x_i)}{2}\right)^2}; \\ d^Z_{22,\mathcal{T}}(A,B) = \frac{1}{\sqrt{n}}\sqrt{\sum\limits_{i=1}^{n}\left(\frac{1+\mu_{A\Delta_{\mathcal{T}}B}(x_i)-\nu_{A\Delta_{\mathcal{T}}B}(x_i)}{2}\right)^2}; \\ d^Z_{22,\mathcal{T}}(B,C) = \frac{1}{\sqrt{n}}\sqrt{\sum\limits_{i=1}^{n}\left(\frac{1+\mu_{B\Delta_{\mathcal{T}}C}(x_i)-\nu_{B\Delta_{\mathcal{T}}C}(x_i)}{2}\right)^2}. \end{cases} \quad (3.0)$$

Since $\mathcal{T}$ satisfies condition C^*, then from Proposition 3.2.1, (3.2.1) holds for all $x \in X$, and this implies that

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1+\mu_{A\Delta_{\mathcal{T}}C}(x_i)-\nu_{A\Delta_{\mathcal{T}}C}(x_i)}{2}\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1+\mu_{A\Delta_{\mathcal{T}}B}(x_i)-\nu_{A\Delta_{\mathcal{T}}B}(x_i)}{2}\right)+\left(\frac{1+\mu_{B\Delta_{\mathcal{T}}C}(x_i)-\nu_{B\Delta_{\mathcal{T}}C}(x_i)}{2}\right)\right]^2. \quad (3.0)$$

By combining (1d),(1e) and (1e), the required result is obtained.

(f) For $A,B,C \in IFSs(X)$ arbitrary, we show that $d^Z_{21,\mathcal{T}}(A,C) \leq d^Z_{21,\mathcal{T}}(A,B) + d^Z_{21,\mathcal{T}}(B,C)$.

From (3.1) and (3.1) we have

$$\begin{cases} d^Z_{21,\mathcal{T}}(A,C) = \sum\limits_{i=1}^{n}\frac{1+\mu_{A\Delta_{\mathcal{T}}C}(x_i)-\nu_{A\Delta_{\mathcal{T}}C}(x_i)}{2n}; \\ d^Z_{21,\mathcal{T}}(A,B) = \sum\limits_{i=1}^{n}\frac{1+\mu_{A\Delta_{\mathcal{T}}B}(x_i)-\nu_{A\Delta_{\mathcal{T}}B}(x_i)}{2n}; \\ d^Z_{21,\mathcal{T}}(B,C) = \sum\limits_{i=1}^{n}\frac{1+\mu_{B\Delta_{\mathcal{T}}C}(x_i)-\nu_{B\Delta_{\mathcal{T}}C}(x_i)}{2n}. \end{cases} \quad (3.0)$$

Since $\mathcal{T}$ satisfies condition C^*, then from Proposition 3.2.1, (3.2.1) holds for all $x \in X$, and by consequence, the result follows from (1f).

2. From Definition 1.2.8 and Theorem 2, $d^Z_{2L_p,\mathcal{T}}$ and $d^Z_{HY,\mathcal{T}}$ satisfies properties (i) and (ii) of metrics.

We assume that $\mathcal{T}$ satisfy condition C^2 and $p=2$; and show that, $d^Z_{2L_p,\mathcal{T}}$ and $d^Z_{M,\mathcal{T}}$ satisfies the triangle inequality property.

a) For $A,B,C \in IFSs(X)$ arbitrary, we prove that $d^Z_{2L_2,\mathcal{T}}(A,C) \leq d^Z_{2L_2,\mathcal{T}}(A,B) + d^Z_{2L_2,\mathcal{T}}(B,C)$.

From (3.1) we have

$$\begin{cases} d^Z_{2L_2,\mathcal{T}}(A,C) = \frac{1}{n\sqrt{2}}\sum\limits_{i=1}^{n}\sqrt{(\mu_{A\Delta_{\mathcal{T}}C}(x_i))^2 + (1-\nu_{A\Delta_{\mathcal{T}}C}(x_i))^2}; \\ d^Z_{2L_2,\mathcal{T}}(A,B) = \frac{1}{n\sqrt{2}}\sum\limits_{i=1}^{n}\sqrt{(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^2 + (1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^2}; \\ d^Z_{2L_2,\mathcal{T}}(B,C) = \frac{1}{n\sqrt{2}}\sum\limits_{i=1}^{n}\sqrt{(\mu_{B\Delta_{\mathcal{T}}C}(x_i))^2 + (1-\nu_{B\Delta_{\mathcal{T}}C}(x_i))^2}. \end{cases} \tag{3.0}$$

Since $\mathcal{T}$ satisfies condition C^2, then from Proposition 3.2.2, (3.2.2) holds for all $x \in X$, and this implies that

$$(\mu_{A\Delta_{\mathcal{T}}C}(x))^2 + (1-\nu_{A\Delta_{\mathcal{T}}C}(x))^2 \leq \left(\sqrt{(\mu_{A\Delta_{\mathcal{T}}B}(x))^2 + (1-\nu_{A\Delta_{\mathcal{T}}B}(x))^2} + \sqrt{(\mu_{B\Delta_{\mathcal{T}}C}(x))^2 + (1-\nu_{B\Delta_{\mathcal{T}}C}(x))^2}\right)^2 \tag{3.0}$$

holds for all $x \in X$. Hence, we combine (2) and (2) to obtain the required result.

b) For $A,B,C \in IFSs(X)$ arbitrary, we prove that $d^Z_{M,\mathcal{T}}(A,C) \leq d^Z_{M,\mathcal{T}}(A,B) + d^Z_{M,\mathcal{T}}(B,C)$.

From (3.1), we have:

$$\begin{cases} d^Z_{M,\mathcal{T}}(A,C) = \frac{1}{2}\left(\sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}\left(\mu_{A\Delta_{\mathcal{T}}C}(x_i)\right)^2} + \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}\left(1-\nu_{A\Delta_{\mathcal{T}}C}(x_i)\right)^2}\right); \\ d^Z_{M,\mathcal{T}}(A,B) = \frac{1}{2}\left(\sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}\left(\mu_{A\Delta_{\mathcal{T}}B}(x_i)\right)^2} + \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}\left(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)\right)^2}\right); \\ d^Z_{M,\mathcal{T}}(B,C) = \frac{1}{2}\left(\sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}\left(\mu_{B\Delta_{\mathcal{T}}C}(x_i)\right)^2} + \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}\left(1-\nu_{B\Delta_{\mathcal{T}}C}(x_i)\right)^2}\right). \end{cases} \tag{3.0}$$

Since $\mathcal{T}$ satisfies condition C^2, then from Proposition 3.2.2, (3.2.2) implies

$$\begin{cases} \frac{1}{n}\sum\limits_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}C}(x_i))^2 \leq \frac{1}{n}\sum\limits_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^2 + \frac{1}{n}\sum\limits_{i=1}^{n}(\mu_{B\Delta_{\mathcal{T}}C}(x_i))^2; \\ \frac{1}{n}\sum\limits_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}C}(x_i))^2 \leq \frac{1}{n}\sum\limits_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^2 + \frac{1}{n}\sum\limits_{i=1}^{n}(1-\nu_{B\Delta_{\mathcal{T}}C}(x_i))^2. \end{cases} \tag{3.0}$$

And Eq. (2) implies

$$\begin{cases} \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}C}(x_i))^2} \leq \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^2} + \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}(\mu_{B\Delta_{\mathcal{T}}C}(x_i))^2}; \\ \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}C}(x_i))^2} \leq \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^2} + \sqrt{\frac{1}{n}\sum\limits_{i=1}^{n}(1-\nu_{B\Delta_{\mathcal{T}}C}(x_i))^2}. \end{cases} \tag{3.0}$$

Hence we combine Eq. (2) and Eq. (2) to obtain the required result.

□

The following remark extends our proposed metrics.

Remark 5. By multiplying each of our seven metrics by a strictly positive real number, we obtain a metric.

Let us end this Chapter by establishing two t-representable family of IF-t-norms (IF-t-norm of Frank and IF-t-norm of Mayor-Torrens defined in Example 2 respectively by Eq. (1.1), Eq. (1.2), Eq. (1.3) and Eq. (1.4)) generating previous metrics (since those metrics are based on IF-t-norms).

Corollary 8. i. If $\mathcal{T} = \mathcal{T}_M = (\top_M, S_M)$ and $p = 2$ then $d^Z_{2L_p,\mathcal{T}}$ and $d^Z_{M,\mathcal{T}}$ are metrics.

ii. If $\mathcal{T} = (\mathcal{T}^l_F)_{l\in[0;+\infty]}$ or $\mathcal{T} = \left(\mathcal{T}^\lambda_{M\top}\right)_{\lambda\in[0,1]}$, then $d^Z_{2n,\mathcal{T}}$, $d^Z_{2w,\mathcal{T}}$, $d^Z_{22,\mathcal{T}}$, $d^Z_{2H,\mathcal{T}}$, $d^Z_{2E,\mathcal{T}}$, and $d^Z_{2L_1,\mathcal{T}}$ are metrics.

Proof. i. Assume that $\mathcal{T} = \mathcal{T}_M = (\top_M, S_M)$ and $p = 2$. Let us recall the following fuzzy R-implication and fuzzy co-implication associated with $\top_M$ and S_M respectively given in Example 3 by:

$$I_{\top_M}(a,b) = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{if } a > b. \end{cases} \tag{3.0}$$

and

$$J_{S_M}(a,b) = \begin{cases} b, & \text{if } a < b, \\ 0, & \text{if } a \geq b. \end{cases} \tag{3.0}$$

From the second result of Theorem 3, to prove that $d^Z_{2L_p,\mathcal{T}}$ and $d^Z_{M,\mathcal{T}}$ are metrics, it's sufficient to prove that $\mathcal{T}$ satisfies condition C^2.
Let $(a,a'),\ (b,b'),\ (c,c') \in L^*$ such that:

$$\begin{cases} a > b > c, \\ a' < b' < c'. \end{cases}$$

Let us show that

$$\begin{cases} (1 - I_{\top_M}(a,c))^2 \leq (1 - I_{\top_M}(a,b))^2 + (1 - I_{\top_M}(b,c))^2, \\ (J_{S_M}(a',c'))^2 \leq (J_{S_M}(a',b'))^2 + (J_{S_M}(b',c'))^2. \end{cases} \tag{3.0}$$

Since

$$\begin{cases} a > b > c, \\ a' < b' < c', \end{cases}$$

then from Eq. (i) we have:

$$\begin{cases} I_{\top_M}(a,c) = c; \\ I_{\top_M}(a,b) = b; \\ I_{\top_M}(b,c) = c. \end{cases} \tag{3.0}$$

and from Eq. (i) we have:

$$\begin{cases} J_{S_M}(a',c') = c'; \\ J_{S_M}(a',b') = b'; \\ J_{S_M}(b',c') = c'. \end{cases} \tag{3.0}$$

Since $(1-c)^2 \leq (1-b)^2 + (1-c)^2$ and $c'^2 \leq b'^2 + c'^2$, then from Eqs. (i) and (i) , Eq. (i) hold.

ii. Assume that $\mathcal{T} = (\mathcal{T}_F^l)_{l\in[0;+\infty]}$. From Theorem 3, to prove that $d_{2n,\mathcal{T}}$, $d^Z_{2w,\mathcal{T}}$, $d^Z_{2p,\mathcal{T}}$ with $p = 2$, $d^Z_{2H,\mathcal{T}}$, $d^Z_{2E,\mathcal{T}}$, and $d^Z_{2L_p,\mathcal{T}}$ with $p = 1$ are metrics, it's sufficient to prove that $\mathcal{T}$ satisfies condition C^*.

From Klement et al. [29], $(\top_F^l)_{l\in[0;+\infty]}$ and $(S_F^l)_{l\in[0;+\infty]}$ are dual; and $\left(\top^\lambda_{M\top}\right)_{\lambda\in[0,1]}$ and $\left(S^\lambda_{M\top}\right)_{\lambda\in[0,1]}$ are dual. Since Fono et al. [17] had prove that $(\top_F^l)_{l\in[0;+\infty]}$ satisfies condition C; and Fotso et al. [20] had prove that $\left(\top^\lambda_{M\top})\right)_{\lambda\in[0,1]}$ satisfies condition C, then from Corollary 6, the result follows.

□

Chapter 4

Some applications of the new distance and similarity measures for decision making

Contents

Some societal problems are been solved using distance measures or similarity measures. In this chapter we will apply our proposed distance measures and similarity measures to solve some societal problems related to pattern recognition and medical diagnosis.

4.1 Application of distances measures to solving the problems of pattern recognition using IFSs

In this section, we will apply six classes of our proposed distance measures to solve the following example of problem of pattern recognition.

Example 11. Given four classes of building material $A_1 = Brick$, $A_2 = Concrete$, $A_3 = wood$ and $A_4 = Stone$, each one is represented by the intuitionistic fuzzy sets (rows 2 to 5 of Table 4.1) in the feature space $X = \{x_1, x_2, ..., x_{12}\}$ = {solidity, stability, resistance, expensive, economical, performance,...} (columns 3 to 14 of Table 4.1). For example if x_1 is solidity then, $\mu_{A_1}(x_1)$ is the degree with which we can accept the hypothesis that "the brick is solid" and, $\nu_{A_1}(x_1)$ is the degree with which we can also reject that hypothesis.

The weight of each feature is given in row 6 of Table 4.1.

Another kind of unknown building material B has been given as test sample in row 7 of Table 4.1. The second colum of the Table gives memberships and non-memberships of IFSs describing the objects.

The objective is to justify that, using our proposed distance measures, which classes the unknown pattern B belongs to and which of the distances is highly confident.

Table 4.1: **4 classes, 12 attributes problem, patterns, test sample [43], and weights**

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
Pattern	$\mu_{A_1}(x)$	0.173	0.102	0.530	0.965	0.420	0.008	0.331	1.000	0.215	0.432	0.750	0.432
∓1	$\nu_{A_1}(x)$	0.524	0.818	0.326	0.008	0.351	0.956	0.512	0.000	0.625	0.534	0.126	0.432
Pattern	$\mu_{A_2}(x)$	0.510	0.627	1.000	0.125	0.026	0.732	0.556	0.650	1.000	0.145	0.047	0.760
∓2	$\nu_{A_2}(x)$	0.365	0.125	0.000	0.648	0.823	0.153	0.303	0.267	0.000	0.762	0.923	0.231
Pattern	$\mu_{A_3}(x)$	0.495	0.603	0.987	0.073	0.037	0.690	0.147	0.213	0.501	1.000	0.324	0.045
∓3	$\nu_{A_3}(x)$	0.387	0.298	0.006	0.849	0.923	0.268	0.812	0.653	0.284	0.000	0.483	0.912
Pattern	$\mu_{A_4}(x)$	1.000	1.000	0.857	0.734	0.021	0.076	0.152	0.113	0.489	1.000	0.386	0.028
∓4	$\nu_{A_4}(x)$	0.000	0.000	0.123	0.158	0.896	0.912	0.712	0.756	0.389	0.000	0.485	0.912
weight	w_i	0.2	0.2	0.2	0.4	0.4	0.4	0.7	0.7	0.7	0.8	0.8	0.8
Test	$\mu_B(x)$	0.978	0.980	0.798	0.693	0.051	0.123	0.152	0.113	0.494	0.987	0.376	0.012
Sample	$\nu_B(x)$	0.003	0.012	0.132	0.213	0.876	0.756	0.721	0.732	0.368	0.000	0.423	0.897

To solve the main objective, scholars used the principle of minimum degree of difference between IFSs stipulating that the lesser the difference between IFSs, the more likely these IFSs approach. More formally, they used the minimum-distance classifier in defining the class from which its distance to the test sample is minimum and defined as follows:

$$j^* = arg \min_j \{dist(A_j, B)\}, \tag{4.0}$$

where A_j is the intuitionistic fuzzy set representing pattern j and B is the unknown pattern considered as test sample.
In order to compare the different distance measures, a performance index called Degree of Confidence (DoC), introduced by Hatzimichailidis et al. [24], measures the confidence

of each distance measures in recognizing a specific sample that belongs to the pattern (j) and it is defined by:

$$DoC^{(j)} = \sum_{i=1, i\neq j}^{n} \mid dist(A_j, B) - dist(A_i, B) \mid . \tag{4.0}$$

where $dist$ is a given distance. It follows from Equation (6), that the greater the $DoC^{(j)}$ the more confident the result of the specific distance measure. This index can be used in experiment in order to give a more accurate measurement of the distance behavior along with the absolute recognition rate.
The following remark justifies that for a similarity measure s, This index can be used in experiment in order to give a more accurate measurement of the similarity behavior along with the absolute recognition rate.

Remark 6. Since $s(A_j, B) = 1 - dist(A_j, B)$ and $s(A_i, B) = 1 - dist(A_i, B)$ for all i and all j then,

$$DoC^{(j)} = \sum_{i=1, i\neq j}^{n} \mid s(A_j, B) - s(A_i, B) \mid . \tag{4.0}$$

In the following paragraphs of this Section, we recall some findings on Example 11, we tackle the problem in pattern recognition using six of our distance measures based on IF t-norms of Lukasiewicz ($\mathcal{T}_L$), Maximum ($\mathcal{T}_M$) and Product ($\mathcal{T}_P$) and, we display comparisons.
Liang and Shi [31] used the principle of the maximum degree of similarity between IFSs and Wang and Xin [43] used the principle of minimum degree of difference between IFSs to solve the problem. Notice that the two proposed methods coincide.
Table 4.2 recalls results for existing distance measure and degree of confidence.

Table 4.2: **Literature results of distance measures and degree of confidence**

Distance	*Results*							
	$dist(A_1, B)$	$dist(A_2, B)$	$dist(A_3, B)$	$dist(A_4, B)$	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$
d_1^n	0.45375	0.45992	0.21075	**0.03382**	0.66910	0.68144	0.66910	**1.02298**
d_1^{nw}	0.03637	0.04296	0.01219	**0.00249**	0.06465	0.07784	0.06465	**0.08405**
d_1^{np}	0.43108	0.43617	0.19817	**0.02704**	0.64204	0.65221	0.64204	**0.98429**
d_1^{nH}	0.47642	0.48367	0.22333	**0.04058**	0.69617	0.71067	0.69617	**1.06167**
d_1^{nE}	0.53640	0.52586	0.31219	**0.05627**	0.71489	0.69381	0.69381	**1.20565**$\checkmark$
$d_{L_p}^{HY}$	0.43108	0.43617	0.19817	**0.02704**	0.64204	0.65221	0.64204	**0.98429**

We now use our proposed distance measures to solve the problem. This problem can be approached with the following algorithm (steps) denoted by **ALG1-steps**.

step 1 For each feature x_i, i=1,...,12, use the data of the Table 4.1 to obtain the membership $\mu_{A_j \Delta_{\mathcal{T}} B}(x_i)$ and non-membership $\nu_{A_j \Delta_{\mathcal{T}} B}(x_i)$ degrees of symmetric difference between building material A_j and unknown pattern B.

step 2 Determine the degree of distance measure $d(A_j, B)$ between building material A_j and unknown pattern B where d is the given distance.

step 3 Classify the degrees of distance measure between each building material A_j and unknown pattern B.

step 4 Use the principe of minimum degree of distance measure to choose the best building material.

Table 4.3, 4.4 and 4.5 gives the results of our proposed distance measures and corresponding degree of confidence obtained for IF t-norms $\mathcal{T}_L$, $\mathcal{T}_M$ and $\mathcal{T}_P$ respectively based on the data in Table 4.1. Those results are obtained using **ALG1-steps**. Here, we assume that parameter $p = 1$.

Table 4.3: **Results of distance measures and degree of confidence with $\mathcal{T}_L$**

Distance	*Results*							
	$dist(A_1, B)$	$dist(A_2, B)$	$dist(A_3, B)$	$dist(A_4, B)$	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$
$d^Z_{2n,\mathcal{T}_L}$	0.46033	0.46435	0.21533	**0.03848**	0.67088	0.67892	0.67088	**1.02458**
$d^Z_{2w,\mathcal{T}_L}$	0.03688	0.04348	0.01265	**0.00290**	0.06480	0.07802	0.06480	**0.08430**
$d^Z_{2p,\mathcal{T}_L}$	0.44425	0.44504	0.207333	**0.03638**	0.64558	0.64717	0.64558	**0.9875**
$d^Z_{2H,\mathcal{T}_L}$	0.47642	0.48367	0.22333	**0.04058**	0.69617	0.71067	0.69617	**1.06167**
$d^Z_{2E,\mathcal{T}_L}$	0.53640	0.52586	0.31219	**0.05627**	0.71489	0.69381	0.69381	**1.20565**$\checkmark$
$d^Z_{2L_p,\mathcal{T}_L}$	0.44425	0.44504	0.20733	**0.03638**	0.64558	0.64717	0.64558	**0.9875**

Table 4.4: **Results of distance measures and degree of confidence with $\mathcal{T}_M$**

Distance	Results							
	$dist(A_1, B)$	$dist(A_2, B)$	$dist(A_3, B)$	$dist(A_4, B)$	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$
$d^Z_{2n,\mathcal{T}_M}$	0.73071	0.70394	0.62571	**0.49233**	0.37015	0.31661	0.31661	**0.58337**
$d^Z_{2w,\mathcal{T}_M}$	0.0616	0.06516	0.05408	**0.04674**	0.02594	0.03306	0.02594	**0.04062**
$d^Z_{2p,\mathcal{T}_M}$	0.6995	0.67221	0.60192	**0.47908**	0.34529	0.29071	0.29071	**0.53638**
$d^Z_{2H,\mathcal{T}_M}$	0.76192	0.73567	0.6495	**0.50558**	0.39501	0.34251	0.34251	**0.63035**
$d^Z_{2E,\mathcal{T}_M}$	0.79059	0.77893	0.71955	**0.62359**	0.2497	0.22638	0.22638	**0.4183**
$d^Z_{2L_p,\mathcal{T}_M}$	0.6995	0.67221	0.60192	**0.47908**	0.34529	0.29071	0.29071	**0.53638**

Table 4.5: **Results of distance measures and degree of confidence with $\mathcal{T}_P$**

Distance	Results							
	$dist(A_1, B)$	$dist(A_2, B)$	$dist(A_3, B)$	$dist(A_4, B)$	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$
$d^Z_{2n,\mathcal{T}_P}$	0.64968	0.62773	0.38469	**0.16765**	0.76897	0.72507	0.72507	**1.15915**$\checkmark$
$d^Z_{2w,\mathcal{T}_P}$	0.05393	0.05917	0.029	**0.0141**	0.07	0.08048	0.07	**0.0998**
$d^Z_{2p,\mathcal{T}_P}$	0.61802	0.59266	0.35579	**0.14783**	0.75778	0.70706	0.70706	**1.12298**$\checkmark$
$d^Z_{2H,\mathcal{T}_P}$	0.68134	0.6628	0.41359	**0.18747**	0.78016	0.74308	0.74308	**1.19532** $\checkmark$
$d^Z_{2E,\mathcal{T}_P}$	0.72034	0.70311	0.49544	**0.30455**	0.65792	0.62346	0.62346	**1.00524**
$d^Z_{2L_p,\mathcal{T}_P}$	0.61802	0.59266	0.35579	**0.14783**	0.75778	0.70706	0.70706	**1.12298**$\checkmark$

Let us discuss our obtained results.

Remark and interpretation 4.1.1. *i. When $\mathcal{T} \in \{\mathcal{T}_L, \mathcal{T}_M, \mathcal{T}_P\}$ and the parameter $p = 1$, results of Tables 4.3, 4.4 and 4.5 show that: for each of our six distance measures and for $j \in \{1, 2, 3, 4\}$, $dist(A_j, B) \geq dist(A_4, B)$ and $DoC^{(j)} \leq DoC^{(4)}$. Consequently, we can say that A_4 should approach B this means that **stone** is the best building material which approache unknown pattern B. This result coincides with the result of Liang and Shi [31]. In addition, $d^Z_{2n,\mathcal{T}}$, $d^Z_{2w,\mathcal{T}}$, $d^Z_{2p,\mathcal{T}}$, $d^Z_{2H,\mathcal{T}}$ and $d^Z_{2L_p,\mathcal{T}}$ are more highly confident than the respective corresponding measures d^n_1, d^{nw}_1, d^{np}_1, d^{nH}_1 and $d^{HY}_{L_p}$.*

ii. Assume that $DoC^{(4)}_{nE}$, $DoC^{(4)}_{2E,\mathcal{T}_L}$, $DoC^{(4)}_{2H,\mathcal{T}_P}$, $DoC^{(4)}_{2n,\mathcal{T}_P}$, $DoC^{(4)}_{2p,\mathcal{T}_P}$, $DoC^{(4)}_{2L_p,\mathcal{T}_P}$ are degrees of confidence of distance measures d^{nE}_1, $d^Z_{2E,\mathcal{T}_P}$, $d^Z_{2H,\mathcal{T}_P}$, $d^Z_{2n,\mathcal{T}_P}$, $d^Z_{2p,\mathcal{T}_P}$, and $d^Z_{2L_p,\mathcal{T}_P}$

respectively, then from the results above, Tables 4.3, 4.4 and 4.5 show that:

$$DoC^4_{nE} = DoC^4_{2E,\mathcal{T}_L} > DoC^{(4)}_{2H,\mathcal{T}_P} > DoC^{(4)}_{2n,\mathcal{T}_P} > DoC^{(4)}_{2p,\mathcal{T}_P} = DoC^{(4)}_{2L_p,\mathcal{T}_P}. \qquad (4.0)$$

This equation gives in the same order the confidence order of the distance measures corresponding to the associated degrees of confidence.

Remark 7. The comparison of $d^Z_{2p,\mathcal{T}}$ and d^{np}_1 depend on the parameter p. For example when $p = 2$, d^{np}_1 is more highly confident than $d^Z_{2p,\mathcal{T}}$ (see Table 4.6 below).

Table 4.6: **Results of distance measures and degree of confidence with $\mathcal{T}_P$ when $p = 2$**

	$dist(A_1,B)$	$dist(A_2,B)$	$dist(A_3,B)$	$dist(A_4,B)$	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$
d^{np}_1	0.4876	0.48308	0.29045	0.03732	0.65194	0.64291	0.64291	**1.149171**
$d^Z_{2p,\mathcal{T}_P}$	0.49753	0.49038	0.29370	0.05384	0.65468	0.64038	0.64038	**1.12011**

4.2 Similary measures solving some problems of medical diagnostics using IFSs

A diagnosis, in the sense of diagnostic procedure, can be regarded as an attempt at classifying an individual's condition into separate and distinct categories that allow medical decisions about treatment and prognosis to be made. Subsequently, a diagnostic opinion is often described in terms of a disease or other condition.

Many methods are distinguish for diagnostic procedure (see [30, 45]).

Differential diagnosis:

The method of differential diagnosis is based on finding as many of the candidate's diseases or conditions as possible that can possibly cause the signs or symptoms, followed by a process of elimination or at least of rendering the entries more or less probable by further medical tests and other processing, aiming to reach the point where only one candidate disease or condition remains as probable. The final result may also remain a list of possible conditions, ranked in order of probability or severity.

Pattern recognition:

In a pattern recognition method the provider uses experience to recognize a pattern of clinical characteristics. It is mainly based on certain symptoms or signs being associated with certain diseases or conditions, not necessarily implying the more cognitive processing involved in a differential diagnosis.

This may be the primary method used in cases where diseases are "obvious", or the provider's experience may enable him or her to recognize the condition quickly. Theoretically, a certain pattern of signs or symptoms can be directly associated with a certain therapy, even without a definite decision regarding what is the actual disease, but such a compromise carries a substantial risk of missing a diagnosis which actually has a different therapy so it may be limited to cases where no diagnosis can be made.

It is important to note that there exist diseases that have the same symptoms. Pattern recognition method can be used in this case to recognize the appropriate disease.
The theory of IFSs has been used to perform medical diagnosis. Using pattern recognition method, the following example shows how to solve medical diagnostic problem with intuitionistic fuzzy information by two of our proposed similarity measures defined in Corollary 5 and recalled in the followings Eqs. (4.2) and (4.2). For all $A,\ B \in IFS(X)$,

$$s^Z_{M,\mathcal{T}}(A,B) = 1 - \frac{1}{2}\left(\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}(\mu_{A\Delta_{\mathcal{T}}B}(x_i))^p} + \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i))^p}\right). \tag{4.0}$$

and

$$s^Z_{HY,\mathcal{T}}(A,B) = 1 - \frac{1 - e^{-\frac{1}{n}\sum\limits_{i=1}^{n}\left(1-\nu_{A\Delta_{\mathcal{T}}B}(x_i)\right)}}{1-e^{-1}}. \tag{4.0}$$

Here we use the IF t-norms of Lukasiewicz ($\mathcal{T}_L$) and Product ($\mathcal{T}_P$) and we let $p = 2$.

Example 12. We consider the same data recalled in [40, 42, 47]. Assuming that the set of diseases is $D = \{$Viral fever, Malaria, Typhoid, Stomach problem, Chest problem$\}$, the set of symptoms is $S = \{$Temperature, Headache, Stomach pain, Cough, Chest pain$\}$ and the set of patients is $P = \{$Adeline, Albert, Ronald, Tom$\}$.
Table 4.7 presents the characteristic symptoms for the considered diagnosis, and Table 4.8 gives the symptoms for each patient. Each element of the tables is given in the form IFS represented by the membership μ and non-membership ν. The aims are to justify for each patient what disease approches.

Table 4.7: **Symptoms characteristic for the considered diagnosis.**

		Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
		D_1	D_2	D_3	D_4	D_5
Temperature (S_1)	μ_{S_1}	0.4	0.7	0.3	0.1	0.1
$\mp_1$	ν_{S_1}	0.0	0.0	0.3	0.7	0.8
Headache (S_2)	μ_{S_2}	0.3	0.2	0.6	0.2	0.0
$\mp_2$	ν_{S_2}	0.5	0.6	0.1	0.4	0.8
Stomach pain (S_3)	μ_{S_3}	0.1	0.0	0.2	0.8	0.2
$\mp_3$	ν_{S_3}	0.7	0.9	0.7	0.0	0.8
Cough (S_4)	μ_{S_4}	0.4	0.7	0.2	0.2	0.2
$\mp_4$	ν_{S_4}	0.3	0.0	0.6	0.7	0.8
Chest pain (S_5)	μ_{S_5}	0.1	0.1	0.1	0.2	0.8
$\mp_5$	ν_{S_5}	0.7	0.8	0.9	0.7	0.1

Table 4.8: **Symptoms characteristic for the considered patients.**

		Temperature	Headache	Stomach pain	cough	Chest pain
		S_1	S_2	S_3	S_4	S_5
Adeline (P_1)	μ_{P_1}	0.8	0.6	0.2	0.6	0.1
$\mp_1$	ν_{P_1}	0.1	0.1	0.8	0.1	0.6
Albert (P_2)	μ_{P_2}	0.0	0.4	0.6	0.1	0.1
$\mp_2$	ν_{P_2}	0.8	0.4	0.1	0.7	0.8
Ronald (P_3)	μ_{P_3}	0.8	0.8	0.0	0.2	0.0
$\mp_3$	ν_{P_3}	0.1	0.1	0.6	0.7	0.5
Tom (P_4)	μ_{P_4}	0.6	0.5	0.3	0.7	0.3
$\mp_4$	ν_{P_4}	0.1	0.4	0.4	0.2	0.4

Now we will show how to use the proposed similarity measures defined by Eqs. (4.2) and (4.2) to derive a proper diagnosis for each patient $P_j,\ j = 1, ..., 4$.

To solve the main objective, Liang and Shi [31] used the principle of maximum degree of similarity between IFSs stipulating that the greatest the similarity between IFSs, the more likely these IFSs approach.

In order to compare similarity measures before making a better choice in a case of necessity in this example, we can used Degree of Confidence (DoC), introduced by Hatzimichailidis et al. [24] to measure the confidence of each similarity measures in recognizing a specific disease that belongs to the patient P_j and that is defined here by:

$$DoC^{(k)} = \sum_{i=1, i\neq k}^{5} \mid s(D_k, P_j) - s(D_i, P_j) \mid . \tag{4.0}$$

where s is a given similarity measure and $D_k,\ k = 1, ..., 5$ are diseases. It follows from Equation (4.2), that the greater the $DoC^{(k)}$ the more confident the result of the specific similarity measure. This index is used in this experiment in order to give a more accurate measurement of the similarity behavior along with the absolute recognition rate.

Like Liang and Shi [31] we use the principle of the maximum degree of similarity between IFSs to solve the problem. This problem can be approached with the following algorithm (steps) denoted by **ALG2-steps**.

step 1 For each symptoms S_i, i=1,...,5, use the previous data to obtain the membership $\mu_{P_j \Delta_{\mathcal{T}} D_k}(S_i)$ and non-membership $\nu_{P_j \Delta_{\mathcal{T}} D_k}(S_i)$ degrees of symmetry difference between patient P_j and disease D_k.

step 2 Use data of the Tables 4.7 and 4.8 to determine the degree of similarity measures $s(P_j, D_k)$ between patient P_j and disease D_k where s is a given similarity measure.

step 3 Classify the degrees of similarity measure between each patient and diseases.

step 4 Use the principle of the maximum degree of similarity to show the proper diagnostic for each patient P_j $(j = 1, ..., 4)$.

In the following Table 4.9, we recall some findings on Example 12. Table 4.9 gives results of the following defined similarity measures s_1^M and s_1^{HY} dual of existing distance measures d_1^M (with $p = 2$) and d_1^{HY} respectively, and degrees of confidence of each disease. For all $A, B \in IFSs(X)$,

$$s_1^M(A,B) = 1 - \frac{1}{2}\left(\sqrt{\frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)|^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}|\nu_A(x_i) - \nu_B(x_i)|^2}\right). \tag{4.0}$$

$$S_1^{HY}(A,B) = 1 - \frac{1 - e^{-\frac{1}{n}\sum_{i=1}^{n}\max\{|\mu_A(x_i)-\mu_B(x_i)|,|\nu_A(x_i)-\nu_B(x_i)|\}}}{1 - e^{-1}}. \tag{4.0}$$

Table 4.9: **Result for literature similarity measures and degrees of confidence**

Patients	Similarity measures	D_1	D_2	D_3	D_4	D_5	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$	$DoC^{(5)}$
		Results									
P_1	$s_1^M(P_1, D_k)$	**0.77028**	0.76862	0.71717	0.48691	0.43288	**0.67554**	0.67056	0.61911	0.84938	1.01148
	$s_1^{HY}(P_1, D_k)$	0.66245	**0.68758**	0.61365	0.35854	0.32166	0.71863	**0.79404**	0.66983	0.92495	1.03558
P_2	$s_1^M(P_2, D_k)$	0.59741	0.44916	0.68778	**0.88710**	0.55649	0.56923	0.93213	0.65961	**1.25756**	0.61015
	$s_1^{HY}(P_2, D_k)$	0.47845	0.37754	0.56677	**0.79332**	0.47845	0.50410	0.80684	0.59242	**1.27208**	0.50410
P_3	$s_1^M(P_3, D_k)$	0.70886	0.60687	**0.76473**	0.51725	0.4552	0.6030	0.5010	**0.77069**	0.59071	0.7767
	$s_1^{HY}(P_3, D_k)$	0.56677	0.47845	**0.61365**	0.41669	0.35854	0.49350	0.40518	**0.63415**	0.46694	0.64141
P_4	$s_1^M(P_4, D_k)$	**0.78572**	0.73464	0.68109	0.58648	0.52427	**0.6164**	0.46316	0.4096	0.50423	0.69083
	$s_1^{HY}(P_4, D_k)$	**0.63780**	0.58998	0.49987	0.43687	0.33991	**0.68458**	0.54110	0.45099	0.51400	0.80486

We now use **ALG2-steps** to solve the problem.

Computation with IF t-norm of Lukasiewicz

The following Table 4.10, gives results of the proposed similarity measures s_M^Z and s_{HY}^Z and degrees of confidence of each disease for each patient obtained for the IF t-norms $\mathcal{T}_L$ based on data in Tables 4.7 and 4.8.

Table 4.10: **Results of the similarity measures and degree of confidence with** T_L

Patients	Similarity measures	D_1	D_2	D_3	D_4	D_5	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$	$DoC^{(5)}$
		results									
P_1	$s_M^Z(P_1, D_k)$	**0.75492**	0.74122	0.68715	0.44171	0.39369	**0.7559**	0.71480	0.6607	0.90616	1.0502
	$s_{HY}^Z(P_1, D_k)$	0.66245	**0.68758**	0.61365	0.35854	0.32166	0.71863	**0.79404**	0.66983	0.92495	1.03558
P_2	$s_M^Z(P_2, D_k)$	0.51420	0.39833	0.62052	**0.88710**	0.51937	0.60027	0.94787	0.69624	**1.49599**	0.59509
	$s_{HY}^Z(P_2, D_k)$	0.47845	0.37754	0.56677	**0.79332**	0.47845	0.50410	0.80684	0.59242	**1.27208**	0.50410
P_3	$s_M^Z(P_3, D_k)$	0.68609	0.55632	**0.73700**	0.51725	0.45441	0.58119	0.45142	**0.73393**	0.4904	0.67903
	$s_{HY}^Z(P_3, D_k)$	0.56677	0.47845	**0.61365**	0.41669	0.35854	0.49350	0.40518	**0.63415**	0.46694	0.64141
P_4	$s_M^Z(P_4, D_k)$	**0.76524**	0.67605	0.62630	0.5646	0.47422	**0.71978**	0.45221	0.40246	0.46417	0.73528
	$s_{HY}^Z(P_4, D_k)$	**0.63780**	0.58998	0.49987	0.43687	0.33991	**0.68458**	0.54110	0.45099	0.51400	0.80486

Let us discuss our obtained results.

Remark and interpretation **4.2.1.** *1. For all* $k \in \{1, 2, 3, 4, 5\}$,

$$s_1^{HY}(P_4, D_k) = s_{HY}^Z(P_4, D_k).$$

This is justified by the first result of Corollary 4.

2. *Assume that* $DOC^{(j)}_{s^Z_{HY}}$ *and* $DOC^{(j)}_{s^Z_M}$ *are degrees of confidence of disease* D_j *for similarity measures* s^Z_{HY} *and* s^Z_M *respectively under the t-norm* $\mathcal{T}$*.*
According to Table 4.10, the following table classifies similarity measures follower from diseases and gives a proper diagnosis for each patient.

Table 4.11: **Classification of similarity measures and proper diagnosis for each patient**

Patients	Classification of similary measure	Diagnosis	Degree of confidence
P_1	$s^Z_M(P_1,D_1) > s^Z_M(P_1,D_2) > s^Z_M(P_1,D_3) > s^Z_M(P_1,D_4) > s^Z_M(P_1,D_5)$	**Viral fever**	$DoC^{(2)}_{s^Z_{HY},\mathcal{T}_L} > DoC^{(1)}_{s^Z_M,\mathcal{T}_L}$
	$s^Z_{HY}(P_1,D_2) > s^Z_{HY}(P_1,D_1) > s^Z_{HY}(P_1,D_3) > s^Z_{HY}(P_1,D_4) > s^Z_{HY}(P_1,D_5)$	**Malaria**	
P_2	$s^Z_M(P_2,D_4) > s^Z_M(P_2,D_3) > s^Z_M(P_2,D_5) > s^Z_M(P_2,D_1) > s^Z_M(P_2,D_2)$	**Stomach problem**	$DoC^{(4)}_{s^Z_M,\mathcal{T}_L} > DoC^{(4)}_{s^Z_{HY},\mathcal{T}_L}$
	$s^Z_{HY}(P_2,D_4) > s^Z_{HY}(P_2,D_3) > s^Z_{HY}(P_2,D_1) = s^Z_{HY}(P_2,D_5) > s^Z_{HY}(P_2,D_2)$	**Stomach problem**	
P_3	$s^Z_M(P_3,D_3) > s^Z_M(P_3,D_1) > s^Z_M(P_3,D_2) > s^Z_M(P_3,D_4) > s^Z_M(P_3,D_5)$	**Typhoid**	$DoC^{(3)}_{s^Z_M,\mathcal{T}_L} > DoC^{(3)}_{s^Z_{HY},\mathcal{T}_L}$
	$s^Z_{HY}(P_3,D_3) > s^Z_{HY}(P_3,D_1) > s^Z_{HY}(P_3,D_2) > s^Z_{HY}(P_3,D_4) > s^Z_{HY}(P_3,D_5)$	**Typhoid**	
P_4	$s^Z_M(P_4,D_1) > s^Z_M(P_4,D_2) > s^Z_M(P_4,D_3) > s^Z_M(P_4,D_4) > s^Z_M(P_4,D_5)$	**Viral fever**	$DoC^{(4)}_{s^Z_M,\mathcal{T}_L} > DoC^{(4)}_{s^Z_{HY},\mathcal{T}_L}$
	$s^Z_{HY}(P_4,D_1) > s^Z_{HY}(P_4,D_2) > s^Z_{HY}(P_4,D_3) > s^Z_{HY}(P_4,D_4) > s^Z_{HY}(P_4,D_5)$	**Viral fever**	

From Table 4.11 the proper diagnosis for each patient using IF t-norm $\mathcal{T}_L$ *are:*

- ***Adeline*** *suffer of* ***Malaria***
- ***Albert*** *suffer of* ***Stomach problem***
- ***Ronald*** *suffer of* ***Typhoid***
- ***Tom*** *suffer of* ***Viral fever***

3. *In oder to compare our similarity measure for existent one, we look after degree of confidence. For patients* P_1*,* P_2*, and* P_4 *for example similarity measure* s^Z_M *is higher confident than corresponding literature one* s^M_1 *since* $DoC^{(1)}_{s^Z_M,\mathcal{T}_L} > DoC^{(1)}_{s^M_1,\mathcal{T}_L}$, $DoC^{(2)}_{s^Z_M,\mathcal{T}_L} > DoC^{(2)}_{s^M_1,\mathcal{T}_L}$ *and* $DoC^{(4)}_{s^Z_M,\mathcal{T}_L} > DoC^{(4)}_{s^M_1,\mathcal{T}_L}$*. But for patient* P_3 s^Z_M *is not higher confident than* s^M_1 *since* $DoC^{(3)}_{s^Z_M,\mathcal{T}_L} < DoC^{(3)}_{s^M_1,\mathcal{T}_L}$*.*

Computation with IF t-norm Product

The following Table 4.12 gives results of the proposed similarity measures s_M^Z and s_{HY}^Z and degrees of confidence of each disease for each patient obtained for the IF t-norms $\mathcal{T}_P$ based on data in Tables 4.7 and 4.8.

Table 4.12: **Results of the similarity measures and degree of confidence with T_P**

Patients	Similarity measures	results									
		D_1	D_2	D_3	D_4	D_5	$DoC^{(1)}$	$DoC^{(2)}$	$DoC^{(3)}$	$DoC^{(4)}$	$DoC^{(5)}$
P_1	$s_M^Z(P_1, D_k)$	**0.63838**	0.50230	0.49823	0.35155	0.26897	**0.93248**	0.52423	0.52016	0.66684	0.91456
	$s_{HY}^Z(P_1, D_k)$	0.39018	**0.40183**	0.20230	0.19925	0.19925	0.65012	**0.46119**	0.47284	0.64908	0.65821
P_2	$s_M^Z(P_2, D_k)$	0.37301	0.29500	0.41689	**0.59120**	0.27742	0.4356	0.51369	0.47956	**1.00248**	0.56643
	$s_{HY}^Z(P_2, D_k)$	0.2577	0.22637	0.28623	**0.33074**	0.10745	0.28313	0.31452	0.31159	**0.44514**	0.67128
P_3	$s_M^Z(P_3, D_k)$	0.42033	0.26236	**0.44355**	0.36017	0.26236	0.39931	0.43695	**0.46897**	0.33915	0.43695
	$s_{HY}^Z(P_3, D_k)$	0.17334	0.09984	**0.26479**	0.18421	0.09984	0.24932	0.32282	**0.50193**	0.26019	0.32282
P_4	$s_M^Z(P_4, D_k)$	**0.57445**	0.43969	0.43840	0.41843	0.27760	**0.72369**	0.31939	0.31810	0.33808	0.76055
	$s_{HY}^Z(P_4, D_k)$	**0.37845**	0.35675	0.30715	0.24015	0.14684	**0.46291**	0.39780	0.34820	0.41519	0.69513

Let us discuss our obtained results.

Remark and interpretation 4.2.2. *Assume that $DOC^{(j)}_{s^Z_{HY},\mathcal{T}}$ and $DOC^{(j)}_{s^Z_M,\mathcal{T}}$ are degrees of confidence of disease j for similarity measures s^Z_{HY} and s^Z_M respectively under the t-norm $\mathcal{T}$.*
According to Table 4.12, the following Table 4.13 classify similarity measures follower from diseases and gives a proper diagnosis for each patient.

Table 4.13: **Classification of similarity measures and proper diagnosis for each patient**

Patients	Classification of similary measure	Diagnosis	Degree of confidence
P_1	$s_M^Z(P_1,D_1) > s_M^Z(P_1,D_2) > s_M^Z(P_1,D_3) > s_M^Z(P_1,D_4) > s_M^Z(P_1,D_5)$	**Viral fever**	$DoC^{(1)}_{s_M^Z,\mathcal{T}_P} > DoC^{(2)}_{s_{HY}^Z,\mathcal{T}_P}$
	$s_{HY}^Z(P_1,D_2) > s_{HY}^Z(P_1,D_1) > s_{HY}^Z(P_1,D_3) > s_{HY}^Z(P_1,D_4) = s_{HY}^Z(P_1,D_5)$	**Malaria**	
P_2	$s_M^Z(P_2,D_4) > s_M^Z(P_2,D_3) > s_M^Z(P_2,D_1) > s_M^Z(P_2,D_2) > s_M^Z(P_2,D_5)$	**Stomach problem**	$DoC^{(4)}_{s_M^Z,\mathcal{T}_P} > DoC^{(4)}_{s_{HY}^Z,\mathcal{T}_P}$
	$s_{HY}^Z(P_2,D_4) > s_{HY}^Z(P_2,D_3) > s_{HY}^Z(P_2,D_1) > s_{HY}^Z(P_2,D_2) > s_{HY}^Z(P_2,D_5)$	**Stomach problem**	
P_3	$s_M^Z(P_3,D_3) > s_M^Z(P_3,D_1) > s_M^Z(P_3,D_4) > s_M^Z(P_3,D_2) = s_M^Z(P_3,D_5)$	**Typhoid**	$DoC^{(3)}_{s_{HY}^Z,\mathcal{T}_P} > DoC^{(3)}_{s_M^Z,\mathcal{T}_P}$
	$s_{HY}^Z(P_3,D_3) > s_{HY}^Z(P_3,D_4) > s_{HY}^Z(P_3,D_1) > s_{HY}^Z(P_3,D_2) = s_{HY}^Z(P_3,D_5)$	**Typhoid**	
P_4	$s_M^Z(P_4,D_1) > s_M^Z(P_4,D_2) > s_M^Z(P_4,D_3) > s_M^Z(P_4,D_4) > s_M^Z(P_4,D_5)$	**Viral fever**	$DoC^{(1)}_{s_M^Z,\mathcal{T}_P} > DoC^{(1)}_{s_{HY}^Z,\mathcal{T}_P}$
	$s_{HY}^Z(P_4,D_1) > s_{HY}^Z(P_4,D_2) > s_{HY}^Z(P_4,D_3) > s_{HY}^Z(P_4,D_4) > s_{HY}^Z(P_4,D_5)$	**Viral fever**	

From Table 4.13 the proper diagnosis for each patient using IF t-norm $\mathcal{T}_P$ are:

- ***Adeline*** *suffer of* ***Viral fever***
- ***Albert*** *suffer of* ***Stomach problem***
- ***Ronald*** *suffer of* ***Typhoid***
- ***Tom*** *suffer of* ***Viral fever***

Conclusion 4.2.1. : General diagnosis under IF t-norms $\mathcal{T}_L$ and $\mathcal{T}_P$ simultaneously

On the question of which diagnosis is appropriate for each patient, many false diagnosis are often made. It would therefore be important to combine many similarity measures in order to make the right diagnosis. In the context of this example, the degrees of confidence of the similarity measures proposed by Hatzimichailidis et al. [24] allow us to take the following most general diagnosis:

- **Adeline** suffer of **Viral fever** since $DoC^{(1)}_{s_M^Z,\mathcal{T}_L} > DoC^{(1)}_{s_{HY}^Z,\mathcal{T}_P}$
- **Albert** suffer of **Stomach problem**
- **Ronald** suffer of **Typhoid**
- **Tom** suffer of **Viral fever**.

♣ Concluding remarks ♣

In this thesis, we propose some classes of distance measures, similarity measures and metrics based on symmetric difference of intuitionistic fuzzy sets. Some of these classes are based on cardinality components of symmetric difference. Our study was carried out in two stages and applications.

In the first stage, we proposed new difference and symmetric difference operations for intuitionistic fuzzy sets by means of intuitionistic fuzzy R-implications associated to IF $t-$norm. We constructed some examples of difference and symmetric difference operations associated to the well-known IF $t-$norms (minimum $\mathcal{T}_M$, product $\mathcal{T}_P$ and Lukasiewicz $\mathcal{T}_L$). We established that the intuitionistic fuzzy difference operation preserves four properties out of five, which we referred to as the four minimal conditions to require of a difference operation on sets in general (even in crisp, fuzzy and intuitionistic fuzzy cases). Whereas Huawen's IF difference operation preserves only two of the five. We investigated and established some sufficient conditions under which the fifth property was satisfied. We also established that the intuitionistic fuzzy symmetric difference operation preserves all the four basic properties of sets. Whereas Ejegwa's IF symmetric difference operation does not preserves one of the four. In addition, we established one cardinality property that was satisfied by these operations.

In the second stage, on the basis of properties of symmetric difference operations, we proposed eight classes of distance measures and their associated classes of similarity measures. We justified that in the case of a $t-$representable IF $t-$norm of Lukasiewicz, the new obtained Hamming and Euclidean distance measures becomes the well-known Hamming and Euclidean distance measures proposed by Grzegorzewski [22], also the new obtained distance measure based on exponential function is the same as that of Hung and Yang's distance measure based on exponential function. We introduced conditions C^* and C^2 on $t-$representable IF $t-$norm $\mathcal{T}$ (and thus conditions on fuzzy R-implication and co-implication operators) under which the obtained classes of distance measures become metrics, namely, t-norm-based metrics. Specifically, we established that $d^Z_{2n,\mathcal{T}}$, $d^Z_{2w,\mathcal{T}}$, $d^Z_{2L_1,\mathcal{T}}$, $d^Z_{22,\mathcal{T}}$, $d^Z_{2H,\mathcal{T}}$ and $d^Z_{2E,\mathcal{T}}$ are metrics if $\mathcal{T}$ satisfies condition C^*, and, $d^Z_{2L_2,\mathcal{T}}$ and $d^Z_{M,\mathcal{T}}$ with $p = 2$ are metrics if $\mathcal{T}$ satisfies condition C^2. The following Table summarizes those results.

Distance measures	$d^Z_{2n,\mathcal{T}}$	$d^Z_{2H,\mathcal{T}}$	$d^Z_{2E,\mathcal{T}}$	$d^Z_{2w,\mathcal{T}}$	$d^Z_{2p,\mathcal{T}}$	$d^Z_{2L_p,\mathcal{T}}$	$d^Z_{M,\mathcal{T}}$
Metric if $\mathcal{T}$ satisfy C^*	$\surd$	$\surd$	$\surd$	$\surd$	$\surd\ if\ p = 1\ or\ p = 2$	$\surd\ if\ p = 1\ or\ p = 2$	$\surd\ if\ p = 1\ or\ p = 2$
Metric if $\mathcal{T}$ satisfy C^2						$\surd\ if\ p = 2$	$\surd\ if\ p = 2$

To display some versions of these metrics for practical use, we proved that all the $t-$representable IF t-norms of Frank and Mayor-Torrens families generate metrics.

All these theoretical results generalize those obtained by Fono et al. [17] and Fotso et al. [20] on metrics on fuzzy sets.

We derived two applications from some of the previewed theoretical results. Firstly we applied particular case of the obtained distance measures to an Example of the problem of pattern recognition introduced by Wang and Xin [43]. The obtained classification of our numerical results corroborate with the results of Liang and Shi [31] and Wang and Xin [43]. However, we noticed that by using Degree of Confidence (DoC), introduced by Hatzimichailidis et al. [24], our proposed metrics, defined with $t-$representables IF $t-$norms of Lukasiewicz and Product, are more highly confident than other measures described in the review.

Secondly we applied, particular case of the obtained similarity measures to an Example of the problem of medical diagnosis recalled in [40, 42, 47]. The diagnosis depends on the used similarity measure. But by using degrees of confidence of the similarity measures, we deduced the most general diagnosis. The obtained result shows that, for the four patients: **Adeline** suffers of **Viral fever**, **Albert** suffers of **Stomach problem**, **Ronald** suffers of **Typhoid**, and **Tom** suffers of **Viral fever**.

An open problem will be to investigate other families of t-norms generating metric from our distance measures (thereby satisfying our two conditions on IF t-norm). This will display more expressions of metrics usefulness to analyze some decision making problems. Another open question, is to generalize all our obtained results for intuitionistic L-fuzzy set, where L is a distributive complete lattice.

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♣ Publications ♣

Metrics of Symmetric Difference on Fuzzy Sets Based on R-implicators of the Usual Families of t-norms

The first publication of this thesis characterizes by means of R-implications operators of the usual families of t-norms defined by Klement et al. [29], which of that families generate metrics from dissimilarity measures on fuzzy sets.

Metrics of Symmetric Difference on Fuzzy Sets Based on R-implicators of the Usual Families of t-norms

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Abstract. Symmetric differences of two fuzzy sets by means of R-implications operators of a norm have been studied and their cardinalities provided classes of dissimilarity measures on fuzzy sets.

In this paper, we characterize t-norms which generate metrics from these measures. The obtained metrics are fuzzy versions of the well-known distance of cardinality of the symmetric difference of crisp sets. For an application point of view, we consider the seven usual parameterized families of t-norms and we use their R-implications to examine conditions on parameters under which these t-norms generate metrics on fuzzy sets from those measures.

Keywords: t-norm · R-implication operator · Fuzzy set · Symmetric difference · Cardinality · Metric

1 Introduction

In real life, there are many situations where we have to compare two objects and the question of the determination of tools for comparisons of objects are of interest (topicality). More specifically, Gottwald [6] introduced metrics based on a t-norm to measure a kind of distinctness of fuzzy sets. Georgescu [5] used four metrics (distances) to investigate the way the changes in fuzzy preferences are reflected in the changes of fuzzy choices associated with them. Valverde [9] introduced S-metric and studied its relationship with T-indistinguishability (T-similarity relation, that is, reflexive, symmetric and T-transitive fuzzy binary

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relation) where S and T are t-conorm and t-norm respectively. Others scholars (De Baets and Mesiar [1], Gwet [7] and Fono et al. [4]) proposed such tools (metrics and similarity measures) to evaluate similarity and dissimilarity between objects described by fuzzy characteristics.

In this paper, we focus on two families of dissimilarity measures based on R-implicator of a t-norm (for shortly based on a t-norm) introduced by Fono et al. [4]. They established a sufficient condition on a t-norm in order to obtain a metric. They displayed an example by showing that t-norms of the Frank family satisfy that condition and thereby generate such metric. Our modest contribution is to complement previous results by showing that the sufficient condition is necessary. More precisely, we characterize t-norms under which the proposed measures become metrics and we display in each seven usual parameterized families of t-norms recalled in [8], those which satisfy the obtained condition and thereby generate metrics from these dissimilarity measures.

It is important to notice that our approach (idea) is similar to the one of Gottwald [6] but results are different. In fact, he introduced the negative of fuzzy equation measure and determine condition on t-norm under which such mapping are metrics. We justify that, for a given t-norm, his measure and metric are different to ours.

The paper is organized as follows. Section 2 gives some basic notions on operations (difference and symmetric difference) for fuzzy sets and on fuzzy implication operators associated with a t-norm. We recall the cardinality of symmetric differences of two fuzzy sets introduced by Fono et al. [4] and the sufficient condition C under which these mappings are metrics. We establish links between that family of metrics and those introduced by Gottwald [6]. Section 3 contains 2 Subsections. In Sect. 3.1, we prove the converse of the result established by Fono et al. [4], that is, if a t-norm generates the distance of cardinality of symmetric difference on fuzzy sets, then it satisfies condition C. In Sect. 3.2, for each of seven parameterized families of t-norms (Aczèl-Alsima, Dombi, Hamacher, Mayor Torrens, Schweizer-Sklar, Sugeno-Weber and Yager), we check those satisfying condition C by using expressions of R-implication operators of these t-norms determined by Fono and Fotso [3] and recalled in Appendix. Since some obtained inequalities are often so far insolvable analytically, that leads us to write, for four families, programmes in MATLAB to verify the results. Section 4 contains some concluding remarks. At the end, we have an Appendix on Figures illustrating some proofs.

2 Preliminaries

2.1 Fuzzy Implications, Difference and Symmetric Difference

Throughout this paper E is a nonempty set. A fuzzy set A of E is defined by its membership function μ_A. $\mathcal{F}(E)$ is the set of all fuzzy sets of E. n is the classical negation, i.e., $\forall a \in [0,1],\ n(a) = 1 - a = \overline{a}$. We assume that T is a continuous t-norm and S its dual t-conorm.

We recall the definitions of two types of fuzzy implication operators useful in this paper [1,2,4,7,10]: The residual implicator (R-implicator) associated to T defined by $\forall a, b \in [0,1],\ I_T^1(a,b) = \max\{t \in [0,1], T(a,t) \leq b\}$ and the contrapositive symmetrical operator of the R-implicator associated to T defined by $\forall a, b \in [0,1],\ I_T^2(a,b) = 1 - \min\{t \in [0,1], S(b,t) \geq a\}$.

We end this Subsection by recalling definitions of differences and symmetric differences of two fuzzy sets introduced by Fono et al. [4]. Let $A, B \in \mathcal{F}(E)$ and $i \in \{1,2\}$.

- The fuzzy difference of type i associated to T of A and B is the fuzzy set of E denoted $A \overset{i}{\underset{T}{-}} B$ and defined by: $\forall x \in E$,

$$\mu_{A \overset{i}{\underset{T}{-}} B}(x) = \overline{I_T^i}(\mu_A(x), \mu_B(x)) = 1 - I_T^i(\mu_A(x), \mu_B(x)).$$

- The fuzzy symmetric difference of type i associated to T of A and B is the fuzzy set of E denoted $A \overset{i}{\underset{T}{\triangle}} B$ and defined by: $\forall x \in E$,

$$\mu_{A \overset{i}{\underset{T}{\triangle}} B}(x) = 1 - I_T^i(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)).$$

Notice that difference and symmetric difference operations for fuzzy sets of types 1 and 2 associated with T preserve properties of these operations for crisp sets (see Propositions 3, 4 and 5 of Fono et al. [4]). In addition, the authors established that property $Card(A \triangle B) \leq Card(A \triangle F) + Card(F \triangle B)$ for three crisp sets A, B, F is preserved by the symmetric differences between fuzzy sets if I_T^i satisfies condition C defined by:

$$\forall a, b, c \in [0,1],\ b < c < a \Rightarrow I_T^i(a,c) + I_T^i(c,b) \leq 1 + I_T^i(a,b). \tag{1}$$

Let us remark that since $\forall a, b \in [0,1]$ such that $b < a$, we have $I_T^2(a,b) = I_T^1(1-b, 1-a)$, then I_T^2 satisfies C iff I_T^1 satisfies C. With that equivalence, in the sequel we will consider only the residual implication I_T^1 which will henceforth be denoted as I_T. We will simply say T satisfies C instead of I_T^i satisfies C.

Let us recall expressions of usual family of t-norms (see [8]) and those of R-implication operators of t-norms of the seven families usual t-norms obtained by Fono and Fotso [3]. We first recall the expressions of t-norm of Lukasiewicz, the drastic's t-norm and the family of t-norms of Frank (see [8]).

Let $\lambda \in [0, +\infty]$ and $a, b \in [0,1]$. The t-norm of Lukasiewicz is defined by: $\top_L(a,b) = max(a+b-1, 0) = (a+b-1) \vee 0$. The drastic's t-norm is defined by:

$$\top_D(a,b) = \begin{cases} 0 \text{ if } (a,b) \in [0,1[^2, \\ min(a,b) = a \wedge b \text{ elsewhere.} \end{cases} \tag{2}$$

The t-norms of the family of Frank denoted by

$$\top_F^\lambda(a,b) = \begin{cases} a \wedge b \text{ if } \lambda = 0 \\ ab \text{ if } \lambda = 1 \\ \top_L(a,b) \text{ si } \lambda = +\infty \\ log_\lambda(1 + \frac{(\lambda^a-1)(\lambda^b-1)}{\lambda-1}) \text{ elsewhere.} \end{cases} \tag{3}$$

The following result recalls expressions of R-implication operators of t-norms of the seven families usual t-norms obtained by Fono and Fotso [3].

Proposition 1 *(Fono and Fotso [3]). (1) Let* $(T_{AA}^\lambda)_{\lambda\in]0,+\infty[}$ *the family of t-norms of Aczèl-Alsima defined by*

$$\top_{AA}^\lambda(a,b) = \begin{cases} \top_D(a,b) \ \textit{if } \lambda = 0 \\ a \wedge b \ \textit{if } \lambda = +\infty \\ e^{-((-loga)^\lambda+(-logb)^\lambda)^{\frac{1}{\lambda}}} \ \textit{elsewhere} \end{cases}, \tag{4}$$

we have, $\forall\lambda \in]0,+\infty[, \forall a,b \in [0,1]$,

$$I_{T_{AA}^\lambda}(a,b) = \begin{cases} 1 \ \textit{if } a \leq b \\ e^{-((-logb)^\lambda-(-loga)^\lambda)^{\frac{1}{\lambda}}} \ \textit{if } a > b. \end{cases} \tag{5}$$

(2) Let $(T_{D_o}^\lambda)_{\lambda\in]0,+\infty[}$ *the family of t-norms of Dombi defined by:*

$$\top_{D_o}^\lambda(a,b) = \begin{cases} \top_D(a,b) \ \textit{if } \lambda = 0 \\ a \wedge b \ \textit{if } \lambda = +\infty \\ \frac{1}{1+((\frac{1-a}{a})^\lambda+(\frac{1-b}{b})^\lambda)^{\frac{1}{\lambda}}} \ \textit{elsewhere} \end{cases}, \tag{6}$$

we have, $\forall\lambda \in]0,+\infty[, \forall a,b \in [0,1]$,

$$I_{T_{D_o}^\lambda}(a,b) = \begin{cases} 1 \ \textit{if } a \leq b \\ \frac{1}{1+((\frac{1-b}{b})^\lambda-(\frac{1-a)}{a})^\lambda)^{\frac{1}{\lambda}}} \ \textit{if } a > b. \end{cases} \tag{7}$$

(3) Let $(T_H^\lambda)_{\lambda\in[0,+\infty[}$ *the family of t-norms of Hamacher defined by*

$$\top_H^\lambda(a,b) = \begin{cases} 0 \ \textit{if } \lambda = a = b = 0 \\ \top_D(a,b) \ \textit{if } \lambda = +\infty \\ \frac{ab}{\lambda+(1-\lambda)(a+b-ab)} \ \textit{elsewhere} \end{cases}, \tag{8}$$

we have $\forall\lambda \in [0,+\infty[, \forall a,b \in [0,1]$,

$$I_{T_H^\lambda}(a,b) = \begin{cases} 1 \ \textit{if } a \leq b \\ \frac{b\lambda+ab(1-\lambda)}{a-b(1-a)(1-\lambda)} \ \textit{if } a > b. \end{cases} \tag{9}$$

(4) Let $(T_{MT}^{\lambda})_{\lambda\in[0,1]}$ *the family of t-norms of Mayor-Torrens defined by:*

$$\top_{M\top}^{\lambda}(a,b)=\begin{cases}(a+b-\lambda)\vee 0 \text{ if } \lambda\in]0,1] \text{ and } (a,b)\in[0,\lambda]^2\\ a\wedge b \text{ elsewhere}\end{cases}, \quad (10)$$

then $\forall\lambda\in[0,1]$ *and* $a,b\in[0,1]$.

(i) If $\left(\begin{array}{c}\lambda=0\\ or\\ \lambda\neq 0, a\in[0,\lambda] \text{ and } b\in]\lambda,1]\\ or\\ \lambda\neq 0, a\in]\lambda,1] \text{ and } b\in[0,\lambda]\\ or\\ \lambda\neq 0 \text{ and } (a,b)\in]\lambda,1]^2\end{array}\right)$, *then* $I_{T_{MT}^{\lambda}}(a,b)=\begin{cases}1 \text{ if } a\leq b\\ b \text{ if } a>b.\end{cases}$

(ii) If $\lambda\in\ ]0,1]$ *and* $(a,b)\in[0,\lambda]^2$, *then* $I_{T_{MT}^{\lambda}}(a,b)=\begin{cases}1 \text{ if } a\leq b\\ \lambda+b-a \text{ if } a>b.\end{cases}$

(5) Let $(T_{SS}^{\lambda})_{\lambda\in\mathbb{R}^*}$ *the family of t-norms of Schweizer-Sklar defined by:*

$$\top_{SS}^{\lambda}(a,b)=\begin{cases}\top_M(a,b) \text{ if } \lambda=-\infty\\ \top_D(a,b) \text{ if } \lambda=+\infty\\ \top_P(a,b) \text{ if } \lambda=0\\ (max(a^{\lambda}+b^{\lambda}-1,0))^{\frac{1}{\lambda}} \text{ elsewhere}\end{cases}, \quad (11)$$

then $\forall\lambda\in\ ]-\infty,0[\cup]0,+\infty[$ *and* $\forall a,b\in[0,1]$, *we have:*

$$I_{T_{SS}^{\lambda}}(a,b)=\begin{cases}1 \text{ if } a\leq b\\ (1+b^{\lambda}-a^{\lambda})^{\frac{1}{\lambda}} \text{ if } a>b.\end{cases} \quad (12)$$

(6) Let $(T_{SW}^{\lambda})_{\lambda\in]-1,+\infty[}$ *the family of t-norms of Sugeno-Weber defined by:*

$$\top_{SW}^{\lambda}(a,b)=\begin{cases}\top_D(a,b) \text{ if } \lambda=-1\\ \top_P(a,b) \text{ if } \lambda=+\infty\\ max(\frac{a+b-1+\lambda ab}{1+\lambda},0) \text{ elsewhere}\end{cases}, \quad (13)$$

then $\forall\lambda\in]-1,+\infty[$ *and* $\forall a,b\in[0,1]$,

$$I_{T_{SW}^{\lambda}}(a,b)=\begin{cases}1 \text{ if } a\leq b\\ \frac{1-a+b(1+\lambda)}{1+a\lambda} \text{ if } a>b.\end{cases} \quad (14)$$

(7) Let $(T_{Y}^{\lambda})_{\lambda\in]0,+\infty[}$ *the family of t-norms of Yager defined by:*

$$\top_{Y}^{\lambda}(a,b)=\begin{cases}\top_D(a,b) \text{ if } \lambda=0\\ \top_M(a,b) \text{ if } \lambda=+\infty\\ max(1-((1-a)^{\lambda}+(1-b)^{\lambda})^{\frac{1}{\lambda}},0) \text{ elsewhere}\end{cases}, \quad (15)$$

then $\forall\lambda\in]0,+\infty[,\forall a,b\in[0,1]$, *we have:*

$$I_{T_{Y}^{\lambda}}(a,b)=\begin{cases}1 \text{ if } a\leq b\\ 1-((1-b)^{\lambda}-(1-a)^{\lambda})^{\frac{1}{\lambda}} \text{ if } a>b.\end{cases} \quad (16)$$

In the next Subsection, we recall metrics introduced by Gottwald [6] and the ones introduced by Fono et al. [4] and we display links between those metrics.

2.2 From Measures to Metrics: Definition and relationship with Gotwald Metrics

Gottwald [6] introduced the following mapping on $\mathcal{F}(E)$: $\forall A, B \in \mathcal{F}(E), Q_T(A,B) = 1 - T(\inf_{x \in E} I_T(\mu_A(x), \mu_B(x)), \inf_{x \in E} I_T(\mu_B(x), \mu_A(x)))$. He characterized t-norms under which such measures are metrics by establishing that for a t-norm T, Q_T is a metric iff $T \geq T_L$.

Fono et al. [4] also introduced the following mapping on $\mathcal{F}(E)$: For $i \in \{1, 2\}$, a t-norm T and $A, B \in \mathcal{F}(E)$,

$$d_\alpha^{i,T}(A,B) = \alpha Card(A \overset{i}{\underset{T}{\triangle}} B) \tag{17}$$

where α is a positive real. They established that $d_\alpha^{i,T}$ is a metric on $\mathcal{F}(E)$ if I_T^i satisfies condition C defined by Eq. (1).

In the following, we outline similarities and differences between the two mappings or measures. The two measures (metrics) are based on t-norm and the second one is the cardinality of the symmetric difference between fuzzy sets.

If $T = T_M$ the minimum t-norm, then Q_T and $d^{i,T}$ simply become respectively $Q_M(A,B) = \sup\limits_{\substack{x \in E \\ \mu_A(x) \neq \mu_B(x)}} \{1 - \mu_{A \cap B}(x)\}$, $d^{1,T_M}(A,B) = \sup\limits_{\substack{x \in E \\ \mu_A(x) \neq \mu_B(x)}} \{1 - \mu_A(x), 1 - \mu_B(x)\}$ and $d^{2,T_M}(A,B) = \sup\limits_{\substack{x \in E \\ \mu_A(x) \neq \mu_B(x)}} \{\mu_A(x), \mu_B(x)\}$.

When $T = T_L$ the Lukasiewicz t-norm, Q_T becomes

$$Q_L(A,B) = min\{1, max\{0, \sup_{x \in E}(\mu_A(x) - \mu_B(x))\} + max\{0, \sup_{x \in E}(\mu_B(x) - \mu_A(x))\}\}. \tag{18}$$

And if $A \subset B$, $Q_L(A,B)$ yields the same value as *Čbyšev* metric on A and B defined by $Q_L(A,B) = \sup_{x \in E} |\mu_B(x) - \mu_A(x)|$. Furthermore, $d_\alpha^{i,T}$ becomes the Hamming metric defined by $d_\alpha^{i,T_L}(A,B) = \sum_{x \in E} |\mu_B(x) - \mu_A(x)|$.

It is obvious to justify that these metrics are topologically equivalent for $T = T_L$, that is, they satisfy $\forall A, B \in \mathcal{F}(E), Q_L(A,B) \leq d^{i,T_L}(A,B) \leq nQ_L(A,B)$.

The following example shows that there exists three fuzzy sets A, B and F of E such that A is closed to F than B w.r.t. the metric Q_L whereas A is closed to B than F w.r.t d^{i,T_L}.

Example 1. *Consider the universe* $E = \{x_1, x_2, x_3\}$ *and three fuzzy sets* $A = \{(x_1, 0.3), (x_2, 0.4), (x_3, 0.5)\}, B = \{(x_1, 0.3), (x_2, 0.5), (x_3, 0.1)\}$ *and* $F = \{(x_1, 0.5), (x_2, 0.6), (x_3, 0.7)\}$. *We have:* $Q_L(A,B) = \min\{1, \max\{0, \sup\{0, -0.1, 0.4\}\} + \max\{0, \sup\{0, 0.1, -0.4\}\}\} = 0.5$; $Q_L(A,C) = 0.2$; $d^{i,T_L}(A,B) = 0 + 0.1 + 0.4 = 0.5$ *and* $d^{i,T_L}(A,C) = 0.2 + 0.2 + 0.2 = 0.6$. *Thus* $Q_L(A,B) > Q_L(A,F)$ *and* $d^{i,T_L}(A,B) < d^{i,T_L}(A,F)$.

In the remainder of this paper, we establish our results on the measures of the cardinality of the difference of two fuzzy sets defined by (17).

3 t-norms generating Metrics from Symmetric Difference Operations

In the next Subsection, we first prove that if a t-norm generates a metric, then it satisfies condition C.

3.1 Condition on t-norms for generating Metrics

Proposition 2. *Let α a positive real. If $d_\alpha^T(A,B) = \alpha Card(A \overset{1}{\underset{T}{\triangle}} B)$ is a metric on $\mathcal{F}(E)$, then T satisfies C.*

Proof. Suppose d_α^T is a metric on $\mathcal{F}(E)$ and show that T satisfies condition C. Let $a, b, c \in [0,1]$ such that $b < c < a$ and let us show (1).Sinced_α^T is a metric on $\mathcal{F}(E)$, we have $\forall A, B, C \in \mathcal{F}(E)$,

$$\alpha Card(A \overset{1}{\underset{T}{\triangle}} B) \leq \alpha Card(A \overset{1}{\underset{T}{\triangle}} C) + \alpha Card(C \overset{1}{\underset{T}{\triangle}} B). \tag{19}$$

With the definition of cardinality, inequality (19) is equivalent to

$$\sum_{x \in E} (1 - I_T(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)))$$
$$\leq$$
$$\sum_{x \in E} (1 - I_T(\mu_A(x) \vee \mu_C(x), \mu_A(x) \wedge \mu_C(x))) + \sum_{x \in E} (1 - I_T(\mu_C(x) \vee \mu_B(x), \mu_C(x) \wedge \mu_B(x))).$$

Let $x_0 \in E$, define A, B and C as follows

$$\begin{cases} \mu_A(t) = \mu_B(t) = \mu_C(t) = 0 \text{ if } t \in E \text{ and } t \neq x_0 \\ \mu_A(x_0) = a, \ \mu_B(x_0) = b, \text{ and } \mu_C(x_0) = c. \end{cases}$$

Previous inequality becomes $1 - I_T(a,b) \leq 1 - I_T(a,c) + 1 - I_T(c,b)$, that is, $I_T(a,c) + I_T(c,b) \leq 1 + I_T(a,b)$. □

Consequently, we deduce the general result which stipulates that condition C is necessary and sufficient under which a t-norm generates metric.

Corollary 1. *Let α a positive real.*
$d_\alpha^T(A,B) = \alpha Card(A \overset{1}{\underset{T}{\triangle}} B)$ is a metric on $\mathcal{F}(E)$ iff T satisfies C.

We now determine for the seven usual parameterized families of t-norms, and by means of expressions of their R-implicators, those satisfying condition C, that is, generating a metric with the cardinality of symmetric difference operation on fuzzy sets. For that, we will use expressions of R-implications of usual and well-known seven families of t-norms (see Klement et al. [8]) determined by Fono and Fotso [3].

In the sequel, (a,b,c) is a triplet such that $0 \leq b < c < a \leq 1$; and if T^λ is a t-norm depending on the parameter λ, we define the function

$$f_{abc}(\lambda) = I_{T^\lambda}(a,c) + I_{T^\lambda}(c,b) - 1 - I_{T^\lambda}(a,b).$$

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3.2 Metrics of Cardinality of Symmetric Difference of Fuzzy Sets generated by Parameterized Families of t-norms

3.2.1 Metrics associated with t-norms of Aczèl-Alsima and t-norms of Dombi

The following result determines the t-norms of Aczèl-Alsina family and the t-norms of Dombi satisfying condition C.

Proposition 3. *(1) Let $(T_{AA}^{\lambda})_{\lambda\in]0,+\infty[}$ the family of t-norms of Aczèl-Alsima.*
(a) If $\lambda \in [1,+\infty[$, then T_{AA}^{λ} satisfies condition C.
(b) $\forall\lambda \in]0,1[$, T_{AA}^{λ} violates condition C.
(2) Let $(T_{D_o}^{\lambda})_{\lambda\in]0,+\infty[}$ the family of t-norms of Dombi.
(a) $\forall\lambda \in [1,+\infty[$, T_{Do}^{λ} satisfies condition C.
(b) $\forall\lambda \in]0,1[$, T_{Do}^{λ} violates condition C.

Proof. (1-a) Let us prove that $\forall\lambda \in [1,+\infty[$, T_{AA}^{λ} satisfy condition C. Let (a,b,c) a triplet; let us show that

$$\forall\lambda \in [1,+\infty[,\ f_{abc}(\lambda) \leq 0. \tag{20}$$

We distinguish two cases.

(i) If $\lambda = 1$, from (5) we have $I_{T_{AA}^1}(a,c) = \frac{c^{\frac{1}{\ln 10}}}{a^{\frac{1}{\ln 10}}}$, $I_{T_{AA}^1}(c,b) = \frac{b^{\frac{1}{\ln 10}}}{c^{\frac{1}{\ln 10}}}$ and $I_{T_{AA}^1}(a,b) = \frac{b^{\frac{1}{\ln 10}}}{a^{\frac{1}{\ln 10}}}$. Set $x = a^{\frac{1}{\ln 10}}$, $y = b^{\frac{1}{\ln 10}}$ and $z = c^{\frac{1}{\ln 10}}$, inequality (20) becomes

$$\frac{z}{x} + \frac{y}{z} \leq 1 + \frac{y}{x} \text{ with } 0 \leq y < z < x \leq 1 \tag{21}$$

since $0 \leq b < c < a \leq 1$. Inequalities (21) can be written $(x-z)(y-z) < 0$. Hence the result.

(ii) For $\lambda \in]1,+\infty[$, in order to get the sign of $f_{(a,b,c)}(\lambda)$, we write a programm in MATLAB which enables us to sketch the curve of $f_{(a,b,c)}$ for different values of a, b and c. Figure AA1 obtained by simulating several triplets (a,b,c) shows that $f_{(a,b,c)}$ is always below the x axis. Thus we can conclude that T_{AA}^{λ} satisfies condition C for $\lambda \in]1,+\infty[$. This graphical result remains true if $\lambda = 1$.

(1-b) Figure AA2 shows that for all $\lambda \in]0,1[$, there exists at least a triplet (a,b,c) such that the curve of $f_{(a,b,c)}$ is above the x axis. It follows that T_{AA}^{λ} violates condition C in this case.

(2-a) Let $\lambda \in [1,+\infty[$, prove that T_{Do}^{λ} satisfies condition C. We distinguish two cases.

(i) If $\lambda = 1$. As in the case of Aczèl-Alsima t-norms, we can use (7) to prove that $T_{D_o}^{1}$ satisfies C.

(ii) If $\lambda \in]1,+\infty[$, figure Do1 obtained by simulating several triplets (a,b,c) shows that $f_{(a,b,c)}$ is always below the x axis. Thus T_{Do}^{λ} satisfies condition C for $\lambda \in]1,+\infty[$. The graphical result remains true if $\lambda = 1$.

(2-b) Figure Do2 shows that for all $\lambda \in]0,1[$, there exists at least a triplet (a,b,c) such that the curve of $f_{(a,b,c)}$ is above the x axis. It follows that T_{Do}^{λ} violates condition C in this case. □

The following result determines the t-norms of Aczèl-Alsima and those of Dombi for which the cardinal of symmetric difference operation on fuzzy sets becomes a metric.

Proposition 4. *Let α a positive real, $d_\alpha^{T_{AA}^\lambda}$ and $d_\alpha^{T_{Do}^\lambda}$ are metrics on $\mathcal{F}(E)$ iff $\lambda \in [1, +\infty]$.*

3.2.2 Metrics associated with t-norms of Hamacher and t-norms of Mayor-Torrens

The following result determines t-norms of Hamacher satisfying condition C. It also justifies that all t-norms of Mayor-Torrens family satisfy condition C.

Proposition 5.*(1) Let $(T_H^\lambda)_{\lambda\in[0,+\infty[}$ the family of t-norms of Hamacher. T_H^λ satisfies condition C iff $\lambda \in [0, 2]$.*

(2) Let $(T_{MT}^\lambda)_{\lambda\in[0,1]}$ the family of t-norms of Mayor-Torrens. $\forall\lambda \in [0, 1]$, T_{MT}^λ satisfies condition C.

Proof. (1) We determine the expression of $f_{abc}(\lambda)$, then the sign of $f_{abc}(\lambda)$ and we deduce the result.

(1-a) From (9) we have $f_{abc}(\lambda) = \frac{(a-c)(c-b)[(1-\lambda)^2(1-a)b-(1-\lambda)ab-a]}{[c-(1-\lambda)(1-c)b][a-(1-\lambda)(1-a)b][a-(1-\lambda)(1-a)c]}$. Since the denominator of $I_{T_H^\lambda}$ is strictly positive and given the constraints on a, b and c, $f_{abc}(\lambda)$ has the same sign as the polynomial of degree 2 in $1 - \lambda$

$$P_{abc}(\lambda) = (1-\lambda)^2(1-a)b - (1-\lambda)ab - a.$$

Let us study the sign of $P_{abc}(\lambda)$.

If $b = 0$, then $P_{abc}(\lambda) \leq 0 \ \ \forall\lambda \in [0, +\infty[$. If $a = 1$ and $b \neq 0$, then $P_{abc}(\lambda) \leq 0$ iff $0 \leq \lambda \leq \frac{b+1}{b}$. If $0 < b < c < a < 1$, then $P_{abc}(\lambda) \leq 0$ iff $\lambda \in [0, \lambda_{abc}]$ where λ_{abc} is the unique root of P_{abc}.

It is easy to prove that $\forall(a, b, c), P_{abc}(\lambda) \leq 0$ iff $\lambda \in [0, 2]$.

(2) Let $a, b, c \in [0, 1]$ such that $b < c < a$ and show that: $\forall\lambda \in [0, 1]$,

$$I_{T_{MT}^\lambda}(a, c) + I_{T_{MT}^\lambda}(c, b) \leq 1 + I_{T_{MT}^\lambda}(a, b). \tag{22}$$

For $(x, y) \in [0, 1]^2$, we say (x, y) satisfies condition C1 iff $(\lambda = 0)$ or $(\lambda \neq 0,\ x \in [0, \lambda]$ and $y \in]\lambda, 1])$, or $(\lambda \neq 0,\ x \in]\lambda, 1]$ and $y \in [0, \lambda])$ or $(\lambda \neq 0$ and $(x, y) \in]\lambda, 1]^2)$. Given conditions on (x, y) in the expression of $I_{T_{MT}^\lambda}(x, y)$ (see Proposition 1), we have the following situations on $(a, b), (a, c)$ and (b, c):

$$\begin{cases} \left((a,b) \text{ satisfies } C1 \text{ or } \begin{cases} \lambda \in]0,1] \\ (a,b) \in [0,\lambda]^2 \end{cases} \right); \\ \left((a,c) \text{ satisfies } C1 \text{ or } \begin{cases} \lambda \in]0,1] \\ (a,c) \in [0,\lambda]^2 \end{cases} \right); \\ \left((c,b) \text{ satisfies } C1 \text{ or } \begin{cases} \lambda \in]0,1] \\ (c,b) \in [0,\lambda]^2 \end{cases} \right). \end{cases} \tag{23}$$

(23) and condition $b < c < a$ lead to the following four cases:

(i) If $(a,b), (a,c)$ and (c,b) satisfy condition $C1$, with the expression of $I_{T^{\lambda}_{MT}}$ given in Proposition 1, we have $I_{\top^{\lambda}_{MT}}(a,c) = c, I_{\top^{\lambda}_{MT}}(c,b) = b$ and $I_{\top^{\lambda}_{MT}}(a,b) = b$. Thus (22) holds.

Proofs of the two last cases are analogous to the previous cases. □

The following result determines t-norms of Hamacher family for which the cardinal of symmetric difference operation on fuzzy sets is a metric.

Proposition 6. *Let α a positive real.*

(1) $d_{\alpha}^{T^{\lambda}_{H}}$ *is a metric on* $\mathcal{F}(E)$ *iff* $\lambda \in [0,2]$.
(2) $\forall\lambda \in [0,1], \quad d_{\alpha}^{T^{\lambda}_{MT}}$ *is a metric on* $\mathcal{F}(E)$.

3.2.3 Metrics associated with t-norms of Schweizer-Sklar, t-norms of Sugeno-Weber and t-norms of Yager

The following result determines the t-norms of Schweizer-Sklar, the t-norms of Sugeno-Weber and the t-norms of Yager satisfying condition C.

Proposition 7. *(1) Let* $(T^{\lambda}_{SS})_{\lambda\in\mathbb{R}^*}$ *the family of t-norms of Schweizer-Sklar.*
(a) $\forall\lambda \in]-\infty, 0[\cup]0,1]$, T^{λ}_{SS} *satisfies condition* C.
(b) $\forall\lambda \in]1,+\infty[$, T^{λ}_{SS} *violates condition* C.
(2) Let $(T^{\lambda}_{SW})_{\lambda\in]-1,+\infty[}$ *the family of t-norms of Sugeno-Weber.* T^{λ}_{SW} *satisfies condition* C *iff* $\lambda \in [0,+\infty[$.
(3) Let $(T^{\lambda}_{Y})_{\lambda\in]0,+\infty[}$ *the family of t-norms of Yager.*
(a) $\forall\lambda \in [1,+\infty[$, T^{λ}_{Y} *satisfies condition* C.
(b) $\forall\lambda \in]0,1[$, T^{λ}_{Y} *violates condition* C.

Proof. (1-a) Let $\lambda \in]-\infty, 0[\cup]0,1]$, prove that T^{λ}_{SS} satisfies condition C. For a triplet (a,b,c), let us show that

$$f_{abc}(\lambda) \leq 0. \tag{24}$$

If $\lambda = 1$, (12) implies that (24) is an equality. For $\lambda \neq 1$, Figure SS1 obtained by simulation of different values of the triplet (a,b,c) shows that the curve of $f_{(a,b,c)}$ is always below the x axis.

(1-b) Figure SS2 shows that for all $\lambda \in]1,+\infty[$, there exists many triplets (a,b,c) such that the curve of $f_{(a,b,c)}$ is above the x axis. It follows that T^{λ}_{SS} violates condition C.

(2) With (14), we have $I_{\top^{\lambda}_{SW}}(c,b) - I_{\top^{\lambda}_{SW}}(a,b) = \frac{(1+\lambda)(1+\lambda b)(a-c)}{(1+\lambda a)(1+\lambda c)}$. It follows from the previous equalities $f_{abc}(\lambda) = \frac{(a-c)(1+\lambda)}{1+\lambda a}\left[\frac{1+\lambda b}{1+\lambda c} - 1\right]$. Given that the denominator of $I_{\top^{\lambda}_{SW}}$ is stricly positive and $b < c < a$, we have $f_{abc}(\lambda) \leq 0 \Longleftrightarrow \lambda(b-c) \leq 0 \Longleftrightarrow \lambda \geq 0$.

(3) With (16), Figures Y1 and Y2, the proof is analogous to one of the first result. □

The following result gives the t-norms of Schweizer-Sklar family, the t-norms of Sugeno-Weber and the t-norms of Yager for which the cardinal of symmetric difference operation on fuzzy sets becomes a metric.

Proposition 8. *Let α a positive real.*

$d_\alpha^{T_{SS}^\lambda}$ *(resp.* $d_\alpha^{T_{SW}^\lambda}$, *resp.* $d_\alpha^{T_Y^\lambda}$*) is a metric on* $\mathcal{F}(E)$ *iff* $\lambda \in [-\infty, 1]$ *(resp.* $\lambda \in [0, +\infty]$, *resp.* $\lambda \in [1, +\infty]$*).*

4 Concluding Remarks

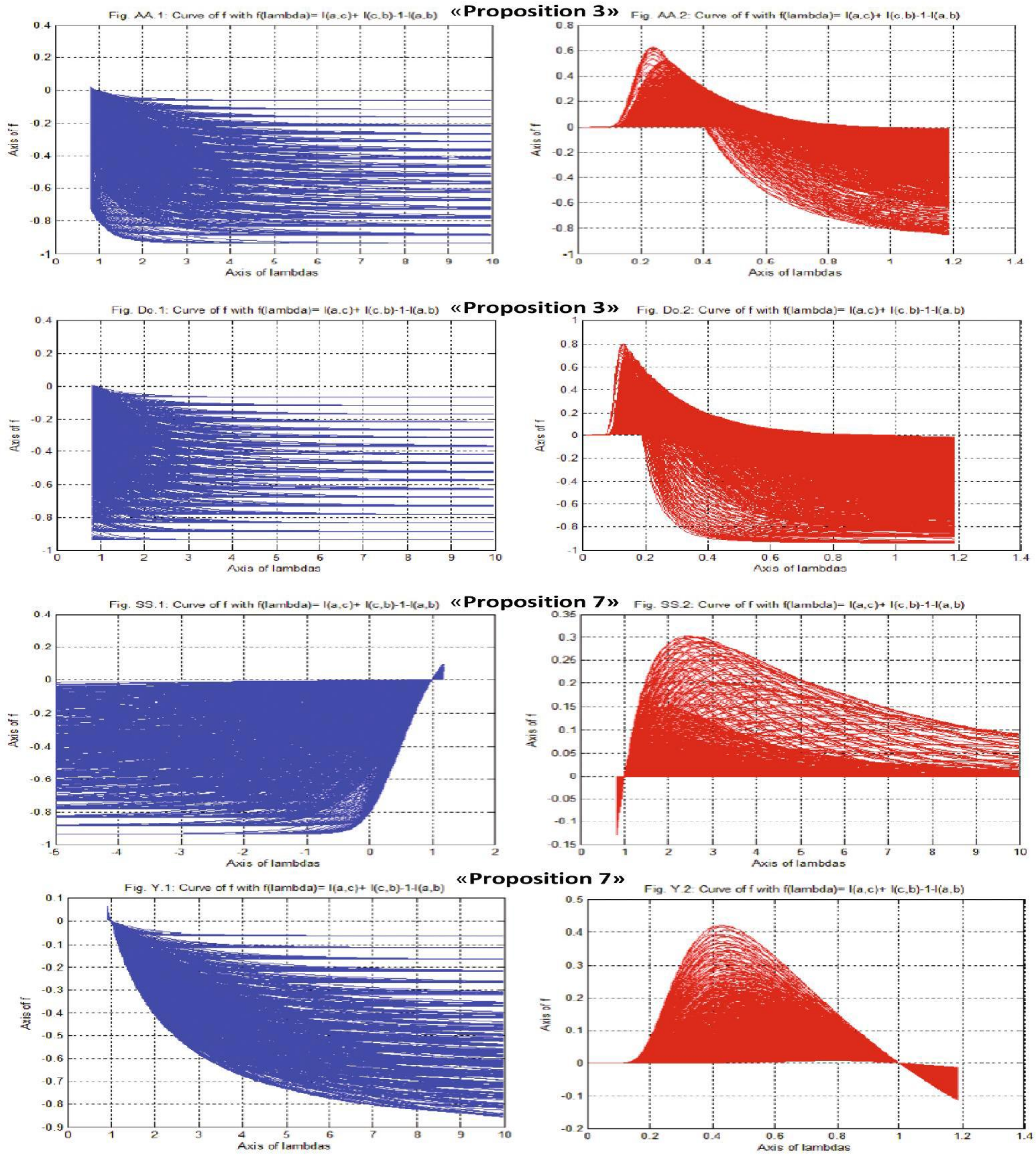
In this paper, we prove that a t-norm generates a metric of the cardinality of symmetric difference on fuzzy sets iff it satisfies condition C. Regarding the eight families of t-norms, only all the t-norms of Frank and all the t-norms of Mayor-Torrens generate metrics from the cardinality of symmetric difference, the other families contain t-norms which generate metrics and those which did not. Our results are summarized in the following table.

	Generate
Aczèl-Alsina, Dombi and Yager	iff $\lambda \in [1, +\infty]$
Frank and Mayor-Torrens	all
Hamacher	iff $\lambda \in [0, 2]$
Schweizer-Sklar	iff $\lambda \in [-\infty, 1]$
Sugeno-Weber	iff $\lambda \in [0, +\infty]$

To sum up, we complement the literature on metrics for fuzzy sets by displaying a set of distances of cardinality of symmetric difference between fuzzy sets. It is important to notice that for the Lukasiewicz t-norm, our metrics is the Hamming metric whereas those of Gotwald are closely related to *Čbyšev* metric. An open question is to reformulate and analyze our results for generalized fuzzy operators such as copulas.

Appendix

Figures illustrating some proofs.



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Difference and symmetric difference for intuitionistic fuzzy sets

The second publication of this thesis proposes new difference and symmetric difference operations for intuitionistic fuzzy sets based on intuitionistic fuzzy R-implication operators and standard intuitionistic fuzzy negation operator. It also studies theirs properties.

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Difference and symmetric difference for intuitionistic fuzzy sets

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Abstract: Fono et al. [10] determined some classes of difference and symmetric difference operations for fuzzy sets using fuzzy implication operators. Intuitionistic fuzzy sets are known to be generalizations of fuzzy sets. So, in this paper, we propose new difference and symmetric difference operations for intuitionistic fuzzy sets based on intuitionistic fuzzy R-implication operators and standard intuitionistic fuzzy negation operator. We establish that some common properties of the difference operations for fuzzy sets established earlier by Fono et al. in [10] and for crisp sets are preserved by the new obtained operations for intuitionistic fuzzy sets. We display a specific property satisfied by difference operation in crisp and fuzzy cases and violated in intuitionistic fuzzy case. The proposed difference and symmetric difference operations for intuitionistic fuzzy sets generalize the case for fuzzy sets. This strength provides a more dynamic perspective into the studies and applications of these operations.

Keywords: Intuitionistic fuzzy set, Difference operation, Symmetric difference operation, Intuitionistic fuzzy R-implication, Intuitionistic fuzzy negation.
2010 Mathematics Subject Classification: 03F55.

1 Introduction

The framework of fuzzy set provides us with tools to handle problems in which the source of vagueness is the ambiguity in criteria of class membership rather than randomness [20]. In this framework, any element of a universal crisp set is allowed to belong to a subset partially with a membership grade usually between 0 and 1 assigned to it. Furthermore, the sum of the membership grade and non-membership grade of an element is always 1. But in reality, this case is not always true because there may be some hesitation degree [9] and this led to the introduction of intuitionistic fuzzy sets as generalization of fuzzy sets by Atanassov [1] in which the degree of hesitation is accounted for, so that the sum of the membership grade, non-membership grade of an element and its degree of hesitation is always 1. Throughout this paper, we consider that definition of an intuitionistic fuzzy set.

Since intuitionistic fuzzy set theory is a generalization of the fuzzy set theory, a rigorous study was needful to be able to establish workable results when concepts under crisp sets and fuzzy sets are transferred. A new set of definitions for set operations needed to be proposed for this field. Many standard operations (such as inclusion, intersection, union, complement, etc) [2,7,9,11,12] have been unanimously agreed upon to serve as usual operations on intuitionistic fuzzy sets. Meanwhile, the need to study these operations in a more mathematical framework which allows for generalization has motivated many scholars [3–6, 11–13, 17–19] among others to undertake studies in intuitionistic fuzzy operators and generators. Of these operators which are germane to establishing results in our current research include intuitionistic fuzzy t-norms, t-conorms, R-implications, co-implications and negations. Cornelis et al. [5, 6] and Atanassov [3] have established many results in the study of intuitionistic fuzzy implications, co-implications, negations and their properties. Some of these results have provided in great measure some required mathematical background for our current study.

Fono et al. [10] have proposed two classes of difference operations for fuzzy sets and two classes of symmetric difference for fuzzy sets using the fuzzy implication operators. They established that these difference and symmetric difference operations for fuzzy sets of type 1 and 2 preserve the classical properties of difference and symmetric difference operations for crisp sets. Inspired by their work on fuzzy sets, we introduce new definitions for difference and symmetric difference for intuitionistic fuzzy sets by means of intuitionistic fuzzy R-implications and we study their properties.

Huawen [15] defined three difference operations for intuitionistic fuzzy sets, one based on the intuitionistic fuzzy t-norm $\mathcal{T}_M = (\min, \max)$ and the remaining based on any decreasing intuitionistic fuzzy generators as follows: For any two intuitionistic fuzzy sets A and B of X,

$$
\begin{aligned}
A -_1 B &= \{\langle x,\, \mu_A(x) \wedge \nu_B(x),\, \nu_A(x) \vee \mu_B(x)\rangle \mid x \in X\}, && (1)\\
A -_2 B &= \{\langle x,\, \mu_A(x) \wedge \varphi(1-\nu_B(x)),\, \nu_A(x) \vee (1-\varphi(\mu_B(x)))\rangle \mid x \in X\}, && (2)\\
A -_3 B &= \{\langle x,\, \mu_A(x) \wedge \varphi(1-\nu_B(x)),\, \nu_A(x) \vee \varphi(1-\mu_B(x))\rangle \mid x \in X\}, && (3)
\end{aligned}
$$

where φ is any decreasing intuitionistic fuzzy generator such that $\varphi(0) = 1$.

However, these definitions do not provide a sufficient endowment to explore the mathematical extensions of these operations to the more general cases that apply to any intuitionistic fuzzy t-norm. Huawen's difference operations preserve only two out of the four properties which are the minimal conditions (as we have established in the results of Proposition 2) to require for a difference operation on sets, fuzzy sets and intuitionistic fuzzy sets in general. It is worthy to note here that the Huawen's difference operation $-_1$ is trivially the generalization of the difference operation in the sense of crisp set given by $A - B = A \cap B^c$. As he noted, if we choose the generator φ to be the standard negator defined by $\varphi(x) = 1 - x$, then the difference operations $-_2$ and $-_3$ reduce to $-_1$. The complement functions (which are special examples of any difference operation) constructed from the difference operations $-_2$ and $-_3$ in Equations (2) and (3) are the same as the intuitionistic fuzzy complementation and intuitionistic fuzzy pseudo-complementation respectively, obtained by Bustince, et al. [4]. Thus, we can refer to the difference operation $-_3$ as intuitionistic fuzzy pseudo-difference operation, which in general does not inherit the general properties of the difference $-_2$.

It is also notable to remark here that, the intuitionistic fuzzy complementation associated to difference $-_2$ defined by Huawen [15] and Bustince, et al. [4] depends largely on the choice of the intuitionistic fuzzy generator. Thus, with different choice of intuitionistic fuzzy generator, the intuitionistic fuzzy complementation so defined may yield different result. Meanwhile, the intuitionistic fuzzy complementation associated to the difference operation we proposed, though by means of intuitionistic fuzzy R-implications, yet yields the same result for any choice of associated t-representable intuitionistic fuzzy t-norm.

By these new difference and symmetric difference operations for intuitionistic fuzzy sets we have proposed, we are able to construct typical examples of intuitionistic fuzzy difference and symmetric difference associated to the three usual known of intuitionistic fuzzy t-norms (the minimum, product and Lukasiewicz). More explicit examples of these new operations can be constructed for other t-representable intuitionistic fuzzy t-norms. This possibility provides a more robust knowledge and insight into the study of these operations in general cases and their applications would be more enriched.

The rest of this paper is organized as follows. Section 2 recalls some preliminaries on fuzzy sets and intuitionistic fuzzy sets. It also recalls known and useful results on difference and symmetric difference of fuzzy sets established by Fono et al. [10]. Section 3 introduces difference and symmetric difference of intuitionistic fuzzy sets and establishes their properties. Section 4 gives some concluding remarks. An Appendix recalls some known results of the fuzzy case that we use.

2 Preliminaries

Throughout this paper, X shall denote a nonempty universal set, $\top$ a t-norm and S a t-conorm.

In this Section, we introduce some basic definitions and provide some preliminary results needed in the rest of the paper. Some other useful notions and concepts on fuzzy sets are recalled in Appendix.

2.1 Intuitionistic fuzzy sets, intuitionistic fuzzy operators and operations

Here, we introduce the basic concepts of intuitionistic fuzzy sets, recall the definitions and examples of some intuitionistic fuzzy operators and operations ([5, 13, 17]).

Definition 1 (Intuitionistic Fuzzy Set [7, 9, 17]). *An intuitionistic fuzzy set D on X is defined by:*

$$D = \{(x, \mu_D(x), \nu_D(x)) \mid \mu_D(x), \nu_D(x) \in [0, 1],\ 0 \leq \mu_D(x) + \nu_D(x) \leq 1,\ \forall x \in X\},$$

where $\mu_D(x)$, $\nu_D(x)$ are the degrees of membership and non-membership of x in D, respectively.

If $\mu_D(x) + \nu_D(x) = 1$, then D is a fuzzy set of X where $\mu_D(x)$ is the degree of membership of x in D.

We will subsequently be referring to the complete lattice $(L^*, \leq_{L^*})$ with $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$ as the units where $L^* = \{(x_1, x_2) | (x_1, x_2) \in [0, 1] \times [0, 1],\ x_1 + x_2 \leq 1\}$ and $\leq_{L^*}$ is an order on L^* defined by: for all $(x_1, x_2), (y_1, y_2) \in L^*$, $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \geq y_2$. The meet operator $\wedge$ and the join operator $\vee$ on this lattice, $(L^*, \leq_{L^*})$ are defined for all $(x_1, x_2), (y_1, y_2) \in L^*$ as:

$$(x_1, x_2) \wedge (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2))$$
$$(x_1, x_2) \vee (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)).$$

Definition 2 (Intuitionistic Fuzzy t-norm and t-conorm, (see [5, 6, 13])).

1. *An intuitionistic fuzzy t-norm is a binary operation $\mathcal{T} : L^* \times L^* \longrightarrow L^*$ such that for any $\mathbf{x} \in L^*$, $\mathcal{T}(\mathbf{x}, 1_{L^*}) = \mathbf{x}$ (neutral element) and, $\mathcal{T}$ satisfies commutativity, monotonicity (increasing) and associativity.*

2. *An intuitionistic fuzzy t-conorm is a binary operation $\mathcal{J} : L^* \times L^* \longrightarrow L^*$ such that for any $\mathbf{x} \in L^*$, $\mathcal{J}(\mathbf{x}, 0_{L^*}) = \mathbf{x}$ and, $\mathcal{J}$ is commutative, monotone increasing and associative.*

Definition 3 (Intuitionistic Fuzzy Negation, (see [5–7, 18])). *An intuitionistic fuzzy negation is a non-increasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(\mathbf{x})) = \mathbf{x}$, $\forall \mathbf{x} \in L^*$, then $\mathcal{N}$ is said to be involutive. An involutive intuitionistic fuzzy negation is called strong intuitionistic fuzzy negation.*

Deschrijver et al. and, Reseir and Bedregal [6, 18] have shown that an involutive intuitionistic fuzzy negation, $\mathcal{N}$, can be characterized by an involutive fuzzy negation by proving that, if the fuzzy negation N is involutive, then $\mathcal{N}(\mathbf{x}) = (N(1-x_2), 1-N(x_1))$. An example of a strong (involutive) intuitionistic fuzzy negation is the standard negation $\mathcal{N}_s$ on L^* defined by $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$.

We now recall useful classes of intuitionistic fuzzy t-norm and t-conorm and, their implications and co-implications.

Definition 4 (t-Representable intuitionistic fuzzy t-norm and t-conorm (see [5, 6, 13, 17])). *An intuitionistic fuzzy t-norm $\mathcal{T}$ (respectively intuitionistic fuzzy t-conorm $\mathcal{J}$) is t-representable if there exists a fuzzy t-norm $\top$ and a fuzzy t-conorm S (respectively a fuzzy t-conorm S' and a fuzzy t-norm $\top'$) such that, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$, $\mathcal{T}(\mathbf{x}, \mathbf{y}) = (\top(x_1, y_1), S(x_2, y_2))$ and $\mathcal{J}(\mathbf{x}, \mathbf{y}) = (S'(x_1, y_1), \top'(x_2, y_2))$, respectively.*

The following result allows us to construct t-representable intuitionistic fuzzy t-norms and t-conorms from fuzzy t-norms and t-conorms.

Theorem 1. *[5, 6, 17] Given a fuzzy t-norm $\top$ and fuzzy t-conorm S satisfying $\top(a, b) \leq 1 - S(1-a, 1-b)$ for all $a, b \in [0, 1]$, then $\mathcal{T}(\mathbf{x}, \mathbf{y}) = (\top(x_1, y_1), S(x_2, y_2))$ and $\mathcal{J}(\mathbf{x}, \mathbf{y}) = (S(x_1, y_1), \top(x_2, y_2))$ for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$, are t-representable intuitionistic fuzzy t-norm and t-representable intuitionistic fuzzy t-conorm respectively.*

We denote by IF-t-norm the intuitionistic fuzzy t-norm and, by IF-t-conorm the intuitionistic fuzzy t-conorm.

Definition 5 (Intuitionistic fuzzy R-implication and co-implicator [5, 6, 13]).

1. *An intuitionistic fuzzy R-implication (for short, IF-R-implication) associated with an IF-t-norm, $\mathcal{T} = (\top, S)$, is a mapping $I_{\mathcal{T}} : L^* \times L^* \longrightarrow L^*$ such that, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$,*

$$\begin{aligned} I_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) &= \sup\{\mathbf{z} | \mathbf{z} \in L^*,\ \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq_{L^*} \mathbf{y}\} \\ &= \sup\{(z_1, z_2) | (z_1, z_2) \in L^*,\ \top(x_1, z_1) \leq y_1 \text{ and } S(x_2, z_2) \geq y_2\}. \end{aligned}$$

2. *An intuitionistic fuzzy co-implication (for short, IF-co-implication) associated with an IF-t-conorm, $\mathcal{J} = (S, \top)$, is a mapping $J_{\mathcal{J}} : L^* \times L^* \longrightarrow L^*$ such that, for all $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$,*

$$\begin{aligned} J_{\mathcal{J}}(\mathbf{x}, \mathbf{y}) &= \inf\{\mathbf{z} | \mathbf{z} \in L^*,\ \mathbf{y} \leq_{L^*} \mathcal{J}(\mathbf{x}, \mathbf{z})\} \\ &= \inf\{(z_1, z_2) | (z_1, z_2) \in L^*,\ y_1 \leq \top(x_1, z_1) \text{ and } y_2 \geq S(x_2, z_2)\}. \end{aligned}$$

The following useful result relates IF-co-implication and IF-R-implication associated with an IF-t-conorm, $\mathcal{J} = (S, \top)$ and IF-t-norm, $\mathcal{T} = (\top, S)$, respectively to corresponding fuzzy co-implication, J_S associated to S and fuzzy R-implication, $I_\top$ associated to $\top$.

Lemma 1 (see [13]). *For any $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*$, we have*

1. $J_{\mathcal{J}}(\mathbf{x}, \mathbf{y}) = (J_S(x_1, y_1), \min(I_\top(x_2, y_2), 1 - J_S(x_1, y_1)))$.
2. $I_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = (\min(I_\top(x_1, y_1), 1 - J_S(x_2, y_2)), J_S(x_2, y_2))$.

The following are examples of t-representable IF-t-norms and IF-t-conorms [13].

Example 1. *i.* $\mathcal{T}_M = (\top_M, S_M)$ *and* $\mathcal{J}_M = (S_M, \top_M)$ *are t-representable IF-t-norm and IF-t-conorm respectively associated to* $\top_M$ *and* S_M.

ii. $\mathcal{T}_P = (\top_P, S_P)$ *and* $\mathcal{J}_P = (S_P, \top_P)$ *are t-representable IF-t-norm and IF-t-conorm respectively associated to* $\top_P$ *and* S_P.

iii. $\mathcal{T}_L = (\top_L, S_L)$ *and* $\mathcal{J}_L = (S_L, \top_L)$ *are t-representable IF-t-norm and IF-t-conorm respectively associated to* $\top_L$ *and* S_L.

iv. Also, by verifying that $\top_F^l(a,b) \leq 1 - S_F^l(1-a, 1-b)$ *holds for all* $a, b \in [0,1]$, $l \in (0,1) \cup (1, +\infty)$, $\mathcal{T}_F^l = (\top_F^l, S_F^l)$ *and* $\mathcal{J}_F^l = (S_F^l, \top_F^l)$ *are t-representable IF-t-norm and IF-t-conorm respectively associated to* $\top_F^l$ *and* S_F^l *for all* $l \in (0,1) \cup (1, +\infty)$.

Using Lemma 1 and Example 8 (see the Appendix), we construct the following examples of IF-R-implication and IF-co-implication associated with an IF-t-norm, $\mathcal{T} = (\top, S)$ and IF-t-conorm, $\mathcal{J} = (S, \top)$.

Example 2. *For all* $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in L^*$,

1. *The IF-R-implication associated with* $\mathcal{T}_M = (\top_M, S_M)$ *and the IF-co-implication associated with* $\mathcal{J}_M = (S_M, \top_M)$ *are respectively given by:*

$$I_{\mathcal{T}_M}(\mathbf{x}, \mathbf{y}) = \begin{cases} (1,0), & \textit{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ (\min(y_1, 1-y_2), y_2), & \textit{if } \mathbf{x} >_{L^*} \mathbf{y}. \end{cases}$$

and

$$J_{\mathcal{J}_M(\mathbf{x}, \mathbf{y})} = \begin{cases} (0,1), & \textit{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ (y_1, \min(y_2, 1-y_1)), & \textit{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

2. *IF-R-implication associated with* $\mathcal{T}_P = (\top_P, S_P)$ *and IF-co-implication associated with* $\mathcal{J}_P = (S_P, \top_P)$ *are respectively given by:*

$$I_{\mathcal{T}_P}(\mathbf{x}, \mathbf{y}) = \begin{cases} (1,0), & \textit{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ \left(\min\left(\frac{y_1}{x_1}, \frac{1-y_2}{1-x_2}\right), \frac{y_2-x_2}{1-x_2}\right), & \textit{if } \mathbf{x} >_{L^*} \mathbf{y}. \end{cases}$$

and

$$J_{\mathcal{J}_P(\mathbf{x}, \mathbf{y})} = \begin{cases} (0,1), & \textit{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ \left(\frac{y_1-x_1}{1-x_1}, \min\left(\frac{y_2}{x_2}, \frac{1-y_1}{1-x_1}\right)\right), & \textit{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

3. *The IF-R-implication associated with* $\mathcal{T}_L = (\top_L, S_L)$ *and the IF-co-implication associated with* $\mathcal{J}_L = (S_L, \top_L)$ *are respectively given by:*

$$I_{\mathcal{T}_L}(\mathbf{x}, \mathbf{y}) = \begin{cases} (1,0), & \textit{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ (\min(1-x_1+y_1, 1+x_2-y_2), y_2-x_2), & \textit{if } \mathbf{x} >_{L^*} \mathbf{y}. \end{cases}$$

and

$$J_{\mathcal{J}_L(\mathbf{x},\mathbf{y})} = \begin{cases} (0,1), & \text{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ (y_1 - x_1, \min(1 - x_2 + y_2, 1 + x_1 - y_1)), & \text{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

4. *The IF-R-implication associated with $\mathcal{T}_F^l = (\top_F^l, S_F^l)$ and the IF-co-implication associated with $\mathcal{J}_F^l = (S_F^l, \top_F^l)$ for all $l \in (0,1) \cup (1, +\infty)$ are respectively given by:*

$$I_{\mathcal{T}_F^l}(\mathbf{x},\mathbf{y}) = \begin{cases} (1,0), & \text{if } \mathbf{x} \leq_{L^*} \mathbf{y}, \\ \Big(\min\Big(\log_l\Big(1 + \frac{(l-1)(l^{y_1}-1)}{l^{x_1}-1}\Big), \log_l\Big(1 + \frac{(l-1)(l^{1-y_2}-1)}{l^{1-x_2}-1}\Big)\Big), \\ 1 - \log_l\Big(1 + \frac{(l-1)(l^{1-y_2}-1)}{l^{1-x_2}-1}\Big)\Big), & \text{if } \mathbf{x} >_{L^*} \mathbf{y}, \end{cases}$$

and

$$J_{\mathcal{J}_F^l(\mathbf{x},\mathbf{y})} = \begin{cases} (0,1), & \text{if } \mathbf{x} \geq_{L^*} \mathbf{y}, \\ \Big(1 - \log_l\Big(1 + \frac{(l-1)(l^{1-y_1}-1)}{l^{1-x_1}-1}\Big), \\ \min\Big(\log_l\Big(1 + \frac{(l-1)(l^{y_2}-1)}{l^{x_2}-1}\Big), \log_l\Big(1 + \frac{(l-1)(l^{1-y_2}-1)}{l^{1-x_2}-1}\Big)\Big)\Big), & \text{if } \mathbf{x} <_{L^*} \mathbf{y}. \end{cases}$$

We end this Subsection by recalling inclusion and some operations on intuitionistic fuzzy sets.

Definition 6 (Intuitionistic Fuzzy Operations [2,7,9,11,12])**.** *Let A and B be any two intuitionistic fuzzy sets defined on X. Inclusion and the following operations are defined by associated membership and non-membership functions as follows:*

i. *Inclusion: $A \subseteq B$ if $\forall\, x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;*

ii. *Intersection: $A \cap B$ is defined by: $\forall\, x \in X, (\mu_{A\cap B}(x), \nu_{A\cap B}(x)) = (\mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x))$;*

iii. *Union: $A \cup B$ is defined by: $\forall\, x \in X, (\mu_{A\cup B}(x), \nu_{A\cup B}(x)) = (\mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x))$;*

iv. *Complement: A^c is defined by: $\forall\, x \in X, (\mu_{A^c}(x), \nu_{A^c}(x)) = (\nu_A(x), \mu_A(x))$;*

v. *Difference: $A - B$ is defined by: $\forall\, x \in X, (\mu_{A-B}(x), \nu_{A-B}(x)) = (\mu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \mu_B(x))$;*

vi. *Symmetric Difference: $A \vartriangle B$ is defined by: $\forall\, x \in X, (\mu_{A\vartriangle B}(x), \nu_{A\vartriangle B}(x)) =$ $(\min\{\mu_A(x) \vee \mu_B(x), \nu_A(x) \vee \nu_B(x)\}, \max\{\nu_A(x) \wedge \nu_B(x), \mu_A(x) \wedge \mu_B(x)\})$.*

In the next Subsection, we recall the difference and symmetric difference operations for fuzzy sets, some examples and their properties as proposed by Fono et al. [10].

2.2 Difference and symmetric difference of fuzzy sets based on fuzzy implications

Definition 7 (Difference and Symmetric Difference Operations for Fuzzy Sets [10])**.** *a. Let M, N be any two fuzzy sets defined on X and $i \in \{1,2,3,4\}$. The fuzzy difference of type i associated to $\top$ of M and N is the fuzzy set of X denoted by $M \overset{i}{\underset{\top}{-}} N$ and defined by:*

$$\mu_{M \overset{i}{\underset{\top}{-}} N}(x) = \overline{I_\top^i}(\mu_M(x), \mu_N(x)) = 1 - I_\top^i(\mu_M(x), \mu_N(x)), \quad \textit{for all } x \in X.$$

b. The fuzzy symmetric difference of type $i \in \{1,2\}$ associated to $\top$ of M and N is the fuzzy set of X denoted by $M \overset{i}{\underset{\top}{\triangle}} N$ and defined for all $x \in X$ by:

$$\mu_{M \overset{i}{\underset{\top}{\triangle}} N}(x) = \mu_{M\cup N \overset{i}{\underset{\top}{-}} M\cap N}(x) = \begin{cases} 1 - I_{\top}^{1}\left(\mu_M(x) \vee \mu_N(x), \mu_M(x) \wedge \mu_N(x)\right), & \textit{if } i = 1 \\ 1 - I_{\top}^{2}\left(\mu_M(x) \vee \mu_N(x), \mu_M(x) \wedge \mu_N(x)\right), & \textit{if } i = 2. \end{cases}$$

We recall the examples of these operations for fuzzy sets of type 1 and 2 associated to the usual three fuzzy t-norms in what follows.

Example 3. *For any fuzzy sets M and N defined on X,*

1. *Examples of fuzzy difference operations*

   (a) *The difference operation associated with $\top_M$ is given by, for all $x \in X$*

$$\mu_{M \overset{1}{\underset{\top_M}{-}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) \leq \mu_N(x), \\ 1 - \mu_N(x), & \textit{if } \mu_M(x) > \mu_N(x), \end{cases}$$

$$\mu_{M \overset{2}{\underset{\top_M}{-}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) \leq \mu_N(x), \\ \mu_M(x), & \textit{if } \mu_M(x) > \mu_N(x) \end{cases}$$

   (b) *The difference operation associated with $\top_P$ is given by, for all $x \in X$*

$$\mu_{M \overset{1}{\underset{\top_P}{-}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) \leq \mu_N(x), \\ 1 - \frac{\mu_N(x)}{\mu_M(x)}, & \textit{if } \mu_M(x) > \mu_N(x), \end{cases}$$

$$\mu_{M \overset{2}{\underset{\top_P}{-}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) \leq \mu_N(x), \\ \frac{\mu_M(x) - \mu_N(x)}{1 - \mu_N(x)}, & \textit{if } \mu_M(x) > \mu_N(x). \end{cases}$$

   (c) *The difference operation associated with $\top_L$ is given by, for all $x \in X$ and $i \in \{1,2\}$,*

$$\mu_{M \overset{i}{\underset{\top_L}{-}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) \leq \mu_N(x), \\ \mu_M(x) - \mu_N(x), & \textit{if } \mu_M(x) > \mu_N(x). \end{cases}$$

2. *Examples of fuzzy symmetric difference operations*

   (a) *The symmetric difference operation associated with $\top_M$ is given by, for all $x \in X$*

$$\mu_{M \overset{1}{\underset{\top_M}{\triangle}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) = \mu_N(x), \\ \max\left(1 - \mu_M(x), 1 - \mu_N(x)\right), & \textit{if } \mu_M(x) \neq \mu_N(x), \end{cases}$$

$$\mu_{M \overset{2}{\underset{\top_M}{\triangle}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) = \mu_N(x), \\ \max\left(\mu_M(x), \mu_N(x)\right), & \textit{if } \mu_M(x) \neq \mu_N(x). \end{cases}$$

(b) The symmetric difference operation associated with $\top_P$ is given by, for all $x \in X$

$$\mu_{M \underset{\top_P}{\overset{1}{\triangle}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) = \mu_N(x) = 0, \\ \frac{|\mu_M(x)-\mu_N(x)|}{\max\{\mu_M(x),\mu_N(x)\}}, & \textit{if } \mu_M(x) \neq 0, \textit{ or } \mu_N(x) \neq 0 \text{ ,} \end{cases}$$

$$\mu_{M \underset{\top_P}{\overset{2}{\triangle}} N}(x) = \begin{cases} 0, & \textit{if } \mu_M(x) = \mu_N(x) = 1, \\ \frac{|\mu_M(x)-\mu_N(x)|}{1-\min\{\mu_M(x),\mu_N(x)\}}, & \textit{if } \mu_M(x) \wedge \mu_N(x) < 1. \end{cases}$$

(c) The symmetric difference operation associated with $\top_L$ is given by, for all $x \in X$ and $i \in \{1, 2\}$,

$$\mu_{M \underset{\top_L}{\overset{i}{\triangle}} N}(x) = \mid \mu_M(x) - \mu_N(x) \mid .$$

Fono et al. [10] have also proved that the difference and symmetric difference operations for fuzzy sets of type 1 and 2 associated to any continuous t-norm $\top$ so defined preserve the properties of the classical difference and symmetric difference operation for crisp sets. We recall these results as follows:

Proposition 1. *Let $i \in \{1, 2\}$ and M, M', N be any arbitrary fuzzy sets on X. The following properties hold [10]:*

1. *Properties of fuzzy difference operation;*
   (a) if $M \subseteq N$, then $M \underset{\top}{\overset{i}{-}} N = \emptyset$, (b) if $M \subseteq M'$, then $M \underset{\top}{\overset{i}{-}} N \subseteq M' \underset{\top}{\overset{i}{-}} N$, (c) if $M \subseteq M'$, then $N \underset{\top}{\overset{i}{-}} M' \subseteq N \underset{\top}{\overset{i}{-}} M$, (d) $\left(M \underset{\top}{\overset{i}{-}} N\right) \cap \left(N \underset{\top}{\overset{i}{-}} M\right) = \emptyset$ and (e) $M \underset{\top}{\overset{i}{-}} N = M \underset{\top}{\overset{i}{-}} (M \cap N)$.
2. *Properties of fuzzy symmetric difference operation;*
   (a) $M \underset{\top}{\overset{i}{\triangle}} N = \left(M \underset{\top}{\overset{i}{-}} N\right) \cup \left(N \underset{\top}{\overset{i}{-}} M\right)$,
   (b) if $M \subseteq N$, then $M \underset{\top}{\overset{i}{\triangle}} N = N \underset{\top}{\overset{i}{-}} M$ and (c) $M \underset{\top}{\overset{i}{\triangle}} M = \emptyset$.

The following result shows that, the fuzzy complement of fuzzy sets associated with any continuous t-norm $\top$ so defined, preserve the property of the classical complement for crisp sets.

Corollary 1. *Let $\top$ be any continuous t-norm, A be a fuzzy set on X, and A^c be the fuzzy complement of A associated with $\top$.*
Then $A^c = X -_\top A$.

Proof. Let $x \in X$. From Definition 12 (see the Appendix), it is sufficient to show that: $\mu_{X-_\top A}(x) = 1 - \mu_A(x)$.
Since $\mu_X(x) = 1$ and $I_\top(1, a) = a$ (see Proposition 9 in the Appendix) for all $a \in [0; 1]$, from Definition 7,
$\mu_{X-_\top A}(x) = 1 - I_\top(\mu_X(x), \mu_A(x)) = 1 - \mu_A(x)$. □

In the following Section, we introduce new operations for intuitionistic fuzzy sets and establish some of their properties.

3 New operations for intuitionistic fuzzy sets: Difference and symmetric difference

3.1 Definitions and properties of difference operations

Let $I_{\mathcal{T}} = ({}_1I_{\mathcal{T}}, {}_2I_{\mathcal{T}})$ be an IF-R-implication operator. We define the negation of $I_{\mathcal{T}}$ as $\mathcal{N}(I_{\mathcal{T}}) = ({}_2I_{\mathcal{T}}, {}_1I_{\mathcal{T}})$. In particular, using Lemma 1 we define the negation of IF-R-implication as $\forall \mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in L^*, \mathcal{N}(I_{\mathcal{T}}(\mathbf{x}, \mathbf{y})) = (J_S(x_2, y_2), \min\{I_{\top}(x_1, y_1), 1 - J_S(x_2, y_2)\})$.

Definition 8. *Let A, B be any two intuitionistic fuzzy sets defined on X. The intuitionistic fuzzy difference associated to $\mathcal{T}$ of A and B is the intuitionistic fuzzy set on X denoted by $A -_{\mathcal{T}} B$ and defined by the membership and non-membership degrees as follows:*
For all $x \in X$,

$$\begin{aligned}(\mu_{A -_{\mathcal{T}} B}(x), \nu_{A -_{\mathcal{T}} B}(x)) &= \mathcal{N}\left(I_{\mathcal{T}}\left((\mu_A(x), \nu_A(x)), (\mu_B(x), \nu_B(x))\right)\right) \\ &= (J_S(\nu_A(x), \nu_B(x)), \min\{I_{\top}(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}).\end{aligned}$$

The following are typical examples of difference operations associated with the three usual and well-known $\mathcal{T}$.

Example 4. *For any intuitionistic fuzzy sets A and B defined on X,*

1. *The difference operation associated with $\mathcal{T}_M$ is given by, for all $x \in X$*

$$\left(\mu_{A -_{\mathcal{T}_M} B}(x), \nu_{A -_{\mathcal{T}_M} B}(x)\right) = \begin{cases} (0, 1), \qquad \text{if } (\mu_A(x), \nu_A(x)) \leq_{L^*} (\mu_B(x), \nu_B(x)), \\ (\nu_B(x), \min\{\mu_B(x), 1 - \nu_B(x)\}), \\ \text{if } (\mu_A(x), \nu_A(x)) >_{L^*} (\mu_B(x), \nu_B(x)). \end{cases}$$

2. *The difference operation associated with $\mathcal{T}_P$ is given by, for all $x \in X$*

$$\left(\mu_{A -_{\mathcal{T}_P} B}(x), \nu_{A -_{\mathcal{T}_P} B}(x)\right) = \begin{cases} (0, 1), \qquad \text{if } (\mu_A(x), \nu_A(x)) \leq_{L^*} (\mu_B(x), \nu_B(x)) \\ \left(\frac{\nu_B(x) - \nu_A(x)}{1 - \nu_A(x)}, \min\left\{\frac{\mu_B(x)}{\mu_A(x)}, \frac{1 - \nu_B(x)}{1 - \nu_A(x)}\right\}\right), \\ \text{if } (\mu_A(x), \nu_A(x)) >_{L^*} (\mu_B(x), \nu_B(x)). \end{cases}$$

3. *The difference operation associated with $\mathcal{T}_L$ is given by, for all $x \in X$*

$$\left(\mu_{A -_{\mathcal{T}_L} B}(x), \nu_{A -_{\mathcal{T}_L} B}(x)\right) = \begin{cases} (0, 1), \qquad \text{if } (\mu_A(x), \nu_A(x)) \leq_{L^*} (\mu_B(x), \nu_B(x)) \\ \Big(\nu_B(x) - \nu_A(x), \min\Big\{1 - \mu_A(x) + \mu_B(x), 1 + \nu_A(x) \\ -\nu_B(x)\Big\}\Big), \text{if } (\mu_A(x), \nu_A(x)) >_{L^*} (\mu_B(x), \nu_B(x)). \end{cases}$$

In the following results, we establish four classical properties for difference operation which are satisfied by the new intuitionistic fuzzy difference operation.

Proposition 2 (Properties of Intuitionistic Fuzzy Difference Operation). *Let A, B, C be intuitionistic fuzzy sets on X. The following properties for intuitionistic fuzzy difference operations hold:*

1. if $A \subseteq B$*, then* $A \; -_{\mathcal{T}} B = \emptyset$*;*

2. if $A \subseteq B$*, then* $A \; -_{\mathcal{T}} C \subseteq B \; -_{\mathcal{T}} C$*;*

3. if $A \subseteq B$*, then* $C \; -_{\mathcal{T}} B \subseteq C \; -_{\mathcal{T}} A$*;*

4. $A \; -_{\mathcal{T}} B = A \; -_{\mathcal{T}} (A \cap B)$*.*

Proof. By Proposition 9 and Definition 8, we establish the results for all $x \in X$ as follows:

1. Assume that, $A \subseteq B,$ then $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$.
   Since $J_S(\nu_A(x), \nu_B(x)) = 0$, whenever $\nu_A(x) \ge \nu_B(x)$, and $I_\top(\mu_A(x), \mu_B(x)) = 1$, whenever $\mu_A(x) \le \mu_B(x)$ then by Definition 8, we have $(\mu_{A -_{\mathcal{T}} B}(x), \nu_{A -_{\mathcal{T}} B}(x)) = (0, 1)$ and the result follows.

2. Assume that $A \subseteq B,$ then $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$.

   $$(\mu_{A -_{\mathcal{T}} C}(x), \nu_{A -_{\mathcal{T}} C}(x)) = (J_S(\nu_A(x), \nu_C(x)), \min\{I_\top(\mu_A(x), \mu_C(x)), 1 - J_S(\nu_A(x), \nu_C(x))\}),$$

   $$(\mu_{B -_{\mathcal{T}} C}(x), \nu_{B -_{\mathcal{T}} C}(x)) = (J_S(\nu_B(x), \nu_C(x)), \min\{I_\top(\mu_B(x), \mu_C(x)), 1 - J_S(\nu_B(x), \nu_C(x))\}).$$

   Since $\nu_B(x) \le \nu_A(x)$, then from Proposition 9 $J_S(\nu_A(x), \nu_C(x)) \le J_S(\nu_B(x), \nu_C(x))$.

   So, $\mu_{A -_{\mathcal{T}} C}(x) \le \mu_{B -_{\mathcal{T}} C}(x)$.

   For the non-membership degree, there are four possibilities:

   Case i: $\nu_{A -_{\mathcal{T}} C}(x) = I_\top(\mu_A(x), \mu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = I_\top(\mu_B(x), \mu_C(x))$.
   Since $\mu_A(x) \le \mu_B(x)$, then $I_\top(\mu_A(x), \mu_C(x)) \ge I_\top(\mu_B(x), \mu_C(x))$ and we have $\nu_{A -_{\mathcal{T}} C}(x) \ge \nu_{B -_{\mathcal{T}} C}(x)$.

   Case ii: $\nu_{A -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_A(x), \nu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_B(x), \nu_C(x))$.
   Since $\nu_B(x) \le \nu_A(x) then, J_S(\nu_A(x), \nu_C(x)) \le J_S(\nu_B(x), \nu_C(x))$, then we have $\nu_{A -_{\mathcal{T}} C}(x) \ge \nu_{B -_{\mathcal{T}} C}(x)$.

   Case iii: $\nu_{A -_{\mathcal{T}} C}(x) = I_\top(\mu_A(x), \mu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_B(x), \nu_C(x))$.
   Since $\mu_A(x) \le \mu_B(x)$, then

   $$I_\top(\mu_A(x), \mu_C(x)) \ge I_\top(\mu_B(x), \mu_C(x)) \ge 1 - J_S(\nu_B(x), \nu_C(x)).$$

   So, $\nu_{A -_{\mathcal{T}} C}(x) \ge \nu_{B -_{\mathcal{T}} C}(x)$.

   Case iv: $\nu_{A -_{\mathcal{T}} C}(x) = 1 - J_S(\nu_A(x), \nu_C(x))$ and $\nu_{B -_{\mathcal{T}} C}(x) = I_\top(\mu_B(x), \mu_C(x))$.
   Since $\nu_A(x) \ge \nu_B(x)$, then

   $$1 - J_S(\nu_A(x), \nu_C(x)) \ge 1 - J_S(\nu_B(x), \nu_C(x)) \ge I_\top(\mu_B(x), \mu_C(x)).$$

   So, $\nu_{A -_{\mathcal{T}} C}(x) \ge \nu_{B -_{\mathcal{T}} C}(x)$.

   Thus for all $x \in X,$ $\mu_{A -_{\mathcal{T}} C}(x) \le \mu_{B -_{\mathcal{T}} C}(x)$ and $\nu_{A -_{\mathcal{T}} C}(x) \ge \nu_{B -_{\mathcal{T}} C}(x)$.

   So, $A \; -_{\mathcal{T}} C \subseteq B \; -_{\mathcal{T}} C$.

3. Assume that $A \subseteq B$ *then*, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

$(\mu_{C -_{\mathcal{T}} B}(x), \nu_{C -_{\mathcal{T}} B}(x)) = (J_S(\nu_C(x), \nu_B(x)), \min\{I_\top(\mu_C(x), \mu_B(x)), 1 - J_S(\nu_C(x), \nu_B(x))\})$,

$(\mu_{C -_{\mathcal{T}} A}(x), \nu_{C -_{\mathcal{T}} A}(x)) = (J_S(\nu_C(x), \nu_A(x)), \min\{I_\top(\mu_C(x), \mu_A(x)), 1 - J_S(\nu_C(x), \nu_A(x))\})$.

Since $\nu_B(x) \leq \nu_A(x)$, then $J_S(\nu_C(x), \nu_B(x)) \leq J_S(\nu_C(x), \nu_A(x))$. So, $\mu_{C -_{\mathcal{T}} B}(x) \leq \mu_{C -_{\mathcal{T}} A}(x)$.

For the non-membership degree, there are four possibilities:

Case i: $\nu_{C -_{\mathcal{T}} B}(x) = I_\top(\mu_C(x), \mu_B(x))$ and $\nu_{C -_{\mathcal{T}} A}(x) = I_\top(\mu_C(x), \mu_A(x))$.
Since $\mu_A(x) \leq \mu_B(x)$, then $I_\top(\mu_C(x), \mu_B(x)) \geq I_\top(\mu_C(x), \mu_A(x))$ and we have $\nu_{C -_{\mathcal{T}} B}(x) \geq \nu_{C -_{\mathcal{T}} A}(x)$.

Case ii: $\nu_{C -_{\mathcal{T}} B}(x) = 1 - J_S(\nu_C(x), \nu_B(x))$ and $\nu_{C -_{\mathcal{T}} A}(x) = 1 - J_S(\nu_C(x), \nu_A(x))$.
Since $\nu_B(x) \leq \nu_A(x)$ then $J_S(\nu_C(x), \nu_B(x)) \leq J_S(\nu_C(x), \nu_A(x))$, then we have $\nu_{C -_{\mathcal{T}} B}(x) \geq \nu_{C -_{\mathcal{T}} A}(x)$.

Case iii: $\nu_{C -_{\mathcal{T}} B}(x) = I_\top(\mu_C(x), \mu_B(x))$ and $\nu_{C -_{\mathcal{T}} A}(x) = 1 - J_S(\nu_C(x), \nu_A(x))$. Since $\mu_A(x) \leq \mu_B(x)$, then

$$I_\top(\mu_C(x), \mu_B(x)) \geq I_\top(\mu_C(x), \mu_A(x)) \geq 1 - J_S(\nu_C(x), \nu_A(x)).$$

So, $\nu_{C -_{\mathcal{T}} B}(x) \geq \nu_{C -_{\mathcal{T}} A}(x)$.

Case iv: $\nu_{C -_{\mathcal{T}} B}(x) = 1 - J_S(\nu_C(x), \nu_B(x))$ and $\nu_{C -_{\mathcal{T}} A}(x) = I_\top(\mu_C(x), \mu_A(x))$.
Since $\nu_A(x) \geq \nu_B(x)$, then

$$1 - J_S(\nu_C(x), \nu_B(x)) \geq 1 - J_S(\nu_C(x), \nu_A(x)) \geq I_\top(\mu_C(x), \mu_A(x)).$$

So, $\nu_{C -_{\mathcal{T}} B}(x) \geq \nu_{C -_{\mathcal{T}} A}(x)$.

Thus for all $x \in X$, $\mu_{C -_{\mathcal{T}} B}(x) \leq \mu_{C -_{\mathcal{T}} A}(x)$ and $\nu_{C -_{\mathcal{T}} B}(x) \geq \nu_{C -_{\mathcal{T}} A}(x)$.
So, $C -_{\mathcal{T}} B \subseteq C -_{\mathcal{T}} A$.

4. From Definition 8 we have,

$$\mu_{A -_{\mathcal{T}} (A \cap B)}(x) = J_S(\nu_A(x), \nu_{A \cap B}(x)) = J_S(\nu_A(x), \max\{\nu_A(x), \nu_B(x)\}), \tag{4}$$

$$\begin{aligned} \nu_{A -_{\mathcal{T}} (A \cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_{A \cap B}(x)), 1 - J_S(\nu_A(x), \nu_{A \cap B}(x))\Big\} \\ &= \min\Big\{I_\top(\mu_A(x), \min\{\mu_A(x), \mu_B(x)\}), 1 - J_S(\nu_A(x), \max\{\nu_A(x), \nu_B(x)\})\Big\}, \end{aligned} \tag{5}$$

$$\mu_{A -_{\mathcal{T}} B}(x) = J_S(\nu_A(x), \nu_B(x)), \tag{6}$$
$$\nu_{A -_{\mathcal{T}} B}(x) = \min\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}. \tag{7}$$

<u>Claim:</u>
We claim $\mu_{A -_{\mathcal{T}} (A \cap B)}(x) = \mu_{A -_{\mathcal{T}} B}(x)$ and $\nu_{A -_{\mathcal{T}} (A \cap B)}(x) = \nu_{A -_{\mathcal{T}} B}(x)$ for all $x \in X$.

We note the following properties:
$J_S(\nu_A(x), \nu_B(x)) = 0$, whenever $\nu_A(x) \geq \nu_B(x)$; $I_\top(\mu_A(x), \mu_B(x)) = 1$, whenever $\mu_A(x) \leq \mu_B(x)$; $J_S(\nu_A(x), \nu_A(x)) = 0$ and $I_\top(\mu_A(x), \mu_A(x)) = 1$.
Then consider the following cases:

Case i: If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, then by Equations (4)-(7), we have

$$\begin{aligned}
\mu_{A -_\mathcal{T} (A\cap B)}(x) &= J_S(\nu_A(x), \nu_B(x)) = \mu_{A -_\mathcal{T} B}(x), \quad \text{and} \\
\nu_{A -_\mathcal{T} (A\cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_A(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} \\
&= \min\Big\{1, 1 - J_S(\nu_A(x), \nu_B(x))\Big\} \\
&= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} = \nu_{A -_\mathcal{T} B}(x).
\end{aligned}$$

Case ii: If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then by Equations (4)-(7), we have

$$\begin{aligned}
\mu_{A -_\mathcal{T} (A\cap B)}(x) &= J_S(\nu_A(x), \nu_A(x)) = 0 \\
&= J_S(\nu_A(x), \nu_B(x)) = \mu_{A -_\mathcal{T} B}(x), \quad \text{and} \\
\nu_{A -_\mathcal{T} (A\cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_A(x)), 1 - J_S(\nu_A(x), \nu_A(x))\Big\} = \min\Big\{1, 1 - 0\Big\} \\
&= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} = \nu_{A -_\mathcal{T} B}(x).
\end{aligned}$$

Case iii: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, then by Equations (4)-(7), we have

$$\begin{aligned}
\mu_{A -_\mathcal{T} (A\cap B)}(x) &= J_S(\nu_A(x), \nu_B(x)) = \mu_{A -_\mathcal{T} B}(x), \quad \text{and} \\
\nu_{A -_\mathcal{T} (A\cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} = \nu_{A -_\mathcal{T} B}(x).
\end{aligned}$$

Case iv: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then by Equations (4)-(7), we have

$$\begin{aligned}
\mu_{A -_\mathcal{T} (A\cap B)}(x) &= J_S(\nu_A(x), \nu_A(x)) = 0 \\
&= J_S(\nu_A(x), \nu_B(x)) = \mu_{A -_\mathcal{T} B}(x), \quad \text{and} \\
\nu_{A -_\mathcal{T} (A\cap B)}(x) &= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_A(x))\Big\} \\
&= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - 0\Big\} \\
&= \min\Big\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\Big\} = \nu_{A -_\mathcal{T} B}(x).
\end{aligned}$$

Hence, $\big(\mu_{A -_\mathcal{T} (A\cap B)}(x), \nu_{A -_\mathcal{T} (A\cap B)}(x)\big) = (\mu_{A -_\mathcal{T} B}(x), \nu_{A -_\mathcal{T} B}(x))$ for all $x \in X$, and the result follows.

□

The following result shows that, the intuitionistic fuzzy complement of fuzzy sets associated with a t-representable of an IF-t-norm $\mathcal{T} = (\top, S)$ so defined, preserve the property of the classical complement for crisp sets.

Corollary 2. *Let A be any intuitionistic fuzzy set of X. $A^c_\mathcal{T}$ be the intuitionistic fuzzy complement of A. Then $A^c_\mathcal{T} = X -_\mathcal{T} A$.*

Proof. Let $x \in X$. Since $(\mu_X(x), \nu_X(x)) = (1, 0)$, then from Definition 8,

$$
\begin{aligned}
(\mu_{X-_{\mathcal{T}}A}(x), \nu_{X-_{\mathcal{T}}A}(x)) &= (J_S(0, \nu_A(x)), \min\{I_\top(1, \mu_A(x)), 1 - J_S(0, \nu_A(x))\}), \\
&= (\nu_A(x),\ \min\{\mu_A(x), 1 - \nu_A(x)\}), \ \text{(recalling Prop. 9(1))}, \\
&= (\nu_A(x), \mu_A(x)), \quad \text{since } \mu_A(x) \le 1 - \nu_A(x). \qquad (8)
\end{aligned}
$$

From Definition 6, the result follows. □

The following result establishes a property of the new difference operation.

Proposition 3. *Let A and B be any intuitionistic fuzzy sets on X.*

1. *Then $(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)$ is an intuitionistic fuzzy set with membership function, $\mu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) = 0, \forall x \in X$ and non-membership function defined by: for all $x \in X$,*

$$
\nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) = \begin{cases} S\left(1 - J_S(\nu_A(x), \nu_B(x)), I_\top(\mu_B(x), \mu_A(x))\right), \\ \qquad \textit{if } \mu_A(x) \le \mu_B(x) \textit{ and } \nu_A(x) \le \nu_B(x), \\ S\left(I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_B(x), \nu_A(x))\right), \\ \qquad \textit{if } \mu_A(x) \ge \mu_B(x) \textit{ and } \nu_A(x) \ge \nu_B(x), \\ 1, \qquad \textit{otherwise.} \end{cases} \qquad (9)
$$

2. *If $\mathcal{T}$ is a Lukasiewicz IF-t-norm, then*

$$
(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A) = \emptyset.
$$

Proof. 1. Recall that for any two intuitionistic fuzzy sets A and B, we define the intersection by means of any t-representable IF-t-norm $\mathcal{T} = (\top, S)$ as follows:

$$
A \cap_{\mathcal{T}} B = \{\langle x, \top(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}.
$$

So,

$$
\mu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) = \top\left(J_S(\nu_A(x), \nu_B(x)), J_S(\nu_B(x), \nu_A(x))\right), \qquad (10)
$$

$$
\begin{aligned}
\nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) = S\,(&\min\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}, \\
&\min\{I_\top(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}). \qquad (11)
\end{aligned}
$$

We note that $J_S(\nu_A(x), \nu_B(x)) = 0,$ whenever $\nu_A(x) \ge \nu_B(x)$ and $I_\top(\mu_A(x), \mu_B(x)) = 1,$ whenever $\mu_A(x) \le \mu_B(x)$, then consider the following cases: for all $x \in X$,

Case i: If $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \le \nu_B(x)$, then from Equation (10) $\mu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) = \top\left(J_S(\nu_A(x), \nu_B(x)), 0\right) = 0$, and from Equation (11) we have

$$
\begin{aligned}
\nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x) &= S\left(\min\{1, 1 - J_S(\nu_A(x), \nu_B(x))\}, \min\{I_\top(\mu_B(x), \mu_A(x)), 1 - 0\}\right), \\
&= S\left(1 - J_S(\nu_A(x), \nu_B(x)), I_\top(\mu_B(x), \mu_A(x))\right).
\end{aligned}
$$

Case ii: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then from Equation (10) $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \top(0, J_S(\nu_B(x), \nu_A(x))) = 0$, and from Equation (11)

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = S\left(\min\{I_\top(\mu_A(x), \mu_B(x)), 1-0\}, \min\{1, 1 - J_S(\nu_B(x), \nu_A(x))\}\right),$$
$$= S\left(I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_B(x), \nu_A(x))\right).$$

Other possible cases are:

Case iii: If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, then from Equation (10) $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \top(0, J_S(\nu_B(x), \nu_A(x))) = 0$, and from Equation (11)

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = S\left(\min\{1, 1-0\}, \min\{I_\top(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}\right),$$
$$= S\left(1, \min\{I_\top(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}\right) = 1.$$

Case iv: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, then from Equation (10) $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \top(J_S(\nu_A(x), \nu_B(x)), 0) = 0$, and from Equation (11)

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = S\left(\min\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}, \min\{1, 1-0\}\right),$$
$$= S\left(\min\{I_\top(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}, 1\right) = 1.$$

So, we have established the result 1.

2. If $\mathcal{T}$ is Lukasiewicz IF-t-norm, then $\mathcal{T} = \mathcal{T}_L = (\top_L, S_L)$. Since from the result in 1 above, we have the membership function $\mu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = 0,\ \forall x \in X$, then from Equation (9) it suffices to prove that the non-membership function, $\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = 1,\ \forall x \in X$, for the first two cases in (9). From Equation (11),

   i If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, we obtain by applying Proposition 9 and Example 8,

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \min\left(1 - J_{S_L}(\nu_A(x), \nu_B(x)) + I_{\top_L}(\mu_B(x), \mu_A(x)), 1\right),$$
$$= 1, \quad \text{if } \mu_A(x) = \mu_B(x) \text{ or } \nu_A(x) = \nu_B(x).$$

   If $\mu_A(x) < \mu_B(x)$ and $\nu_A(x) < \nu_B(x)$, then we have

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \min\left(1 - \nu_B(x) + \nu_A(x) + 1 - \mu_B(x) + \mu_A(x), 1\right),$$
$$= \min\left(2 - (\mu_B(x) + \nu_B(x)) + \mu_A(x) + \nu_A(x), 1\right) = 1, \quad \text{since } \mu_B(x) + \nu_B(x) \leq 1.$$

   ii If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, we obtain by applying Proposition 9 and Example 8,

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \min\left(I_{\top_L}(\mu_A(x), \mu_B(x)) + 1 - J_{S_L}(\nu_B(x), \nu_A(x)), 1\right),$$
$$= 1, \quad \text{if } \mu_A(x) = \mu_B(x) \text{ or } \nu_A(x) = \nu_B(x).$$

   If $\mu_A(x) > \mu_B(x)$ and $\nu_A(x) > \nu_B(x)$, then we have

$$\nu_{(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A)}(x) = \min\left(1 - \mu_A(x) + \mu_B(x) + 1 - \nu_A(x) + \nu_B(x), 1\right),$$
$$= \min\left(2 - (\mu_A(x) + \nu_A(x)) + \mu_B(x) + \nu_B(x), 1\right) = 1, \quad \text{since } \mu_A(x) + \nu_A(x) \leq 1.$$

So $\left(\mu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x), \nu_{(A-_{\mathcal{T}}B)\cap_{\mathcal{T}}(B-_{\mathcal{T}}A)}(x)\right) = (0,1)$, for all $x \in X$. Hence result 2 is established. □

Remark 1. *1. Note that, $(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A) = \emptyset$ whenever either $A \subseteq B$ or $B \subseteq A$. This follows immediately from the third case in Equation* (9).

2. Proposition 2 specifies conditions which are preserved by the intuitionistic fuzzy difference operation. These four conditions shall be referred to as the minimal conditions to require of difference operation on (even in crisp, fuzzy and intuitionistic) sets in general.

The following result gives a necessary and sufficient condition for difference of intuitionistic fuzzy sets to be a fuzzy set.

Proposition 4. *Let A and B be any intuitionistic fuzzy sets defined on X. Then the intuitionistic fuzzy difference $A -_{\mathcal{T}} B$ is a fuzzy set if and only if for all $x \in X$,*

$$I_{\top}\left(\mu_A(x), \mu_B(x)\right) \geq 1 - J_S\left(\nu_A(x), \nu_B(x)\right).$$

Proof. Let $x \in X$. Then from the Definition 8,

$$(\mu_{A-_{\mathcal{T}}B}(x), \nu_{A-_{\mathcal{T}}B}(x)) = \left(J_S(\nu_A(x), \nu_B(x)), \min\{I_{\top}(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\}\right).$$

$A -_{\mathcal{T}} B$ is a fuzzy set if and only if $\nu_{A-_{\mathcal{T}}B}(x) = 1 - \mu_{A-_{\mathcal{T}}B}(x)$,
if and only if
$\min\{I_{\top}(\mu_A(x), \mu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\} = 1 - J_S(\nu_A(x), \nu_B(x))$,
if and only if $I_{\top}(\mu_A(x), \mu_B(x)) \geq 1 - J_S(\nu_A(x), \nu_B(x))$. □

Note that $A -_{\mathcal{T}} B$ also becomes a fuzzy set if $A \subset B$, because in this case $A -_{\mathcal{T}} B = \emptyset$ (Proposition 2), $I_{\top}(\mu_A(x), \mu_B(x) = 1$ and $J_S(\nu_A(x), \nu_B(x)) = 0$. Furthermore, in the case where $A -_{\mathcal{T}} B$ becomes a fuzzy set, we deduce from Proposition 4 that for

$$x \in X : (\mu_{A-_{\mathcal{T}}B}(x), \nu_{A-_{\mathcal{T}}B}(x)) = \left(J_S(\nu_A(x), \nu_B(x)), 1 - J_S(\nu_A(x), \nu_B(x))\right).$$

This can be considered as fuzzy part of $A -_{\mathcal{T}} B$.

The following are typical applications of Proposition 4 to difference operators associated with the three usual and well-known $\mathcal{T}$.

Example 5. *For any intuitionistic fuzzy sets A and B defined on X,*

1. The difference operation associated with $\mathcal{T}_M$ is given by, for all $x \in X$

$$\left(\mu_{A-_{\mathcal{T}_M}B}(x), \nu_{A-_{\mathcal{T}_M}B}(x)\right) = \begin{cases} (\nu_B(x), \mu_B(x)), & \textit{Intuitionistic Fuzzy Part} \\ (\nu_B(x), 1 - \nu_B(x)), & \textit{Fuzzy Part}. \end{cases}$$

2. *The difference operation associated with $\mathcal{T}_P$ is given by, for all $x \in X$*

$$\left(\mu_{A-_{\mathcal{T}_P}B}(x), \nu_{A-_{\mathcal{T}_P}B}(x)\right) = \begin{cases} \left(\frac{\nu_B(x)-\nu_A(x)}{1-\nu_A(x)}, \frac{\mu_B(x)}{\mu_A(x)}\right), & \textit{Intuitionistic Fuzzy Part} \\ \left(\frac{\nu_B(x)-\nu_A(x)}{1-\nu_A(x)}, \frac{1-\nu_B(x)}{1-\nu_A(x)}\right), & \textit{Fuzzy Part}. \end{cases}$$

3. *The difference operation associated with $\mathcal{T}_L$ is given by, for all $x \in X$*

$$\left(\mu_{A-_{\mathcal{T}_L}B}(x), \nu_{A-_{\mathcal{T}_L}B}(x)\right) = \begin{cases} \left(\nu_B(x)-\nu_A(x), 1-\mu_A(x)+\mu_B(x)\right), \textit{Intuitionistic Fuzzy} \\ \left(\nu_B(x)-\nu_A(x), 1+\nu_A(x)-\nu_B(x)\right), \textit{Fuzzy Part}. \end{cases}$$

Notation 1. *Let A and B be any fuzzy sets, $A \equiv B$ if and only if for all $x \in X$, $\mu_A(x) = \mu_B(x)$.*

The following result shows that the intuitionistic fuzzy difference operator defined in Definition 8 associated with t-representable IF t-norm $\mathcal{T} = (\top, S)$ is a generalization of fuzzy difference operator proposed by Fono et al. [10] associated with a t-norm $\top$ if and only if the fuzzy t-norm $\top$ and fuzzy t-conorm S are dual.

Proposition 5 (Generalization of Difference Operation for Fuzzy Sets)**.** *Let $\top$ and S be any t-norm and t-conorm respectively, and $\mathcal{T} = (\top, S)$ be a t-representable IF t-norm associated with any intuitionistic fuzzy set. $\top$ and S are dual if and only if for any fuzzy sets A and B, $A -_{\mathcal{T}} B$ is a fuzzy set and $A -_{\top} B \equiv A -_{\mathcal{T}} B$.*

Proof. Let $x \in X$, and A and B be any fuzzy sets.

a. Assume that $\top$ and S are dual.

   i. Let us show that $A -_{\mathcal{T}} B$ is a fuzzy set.
   Since $\top$ and S are dual, then From Proposition 10, $I_\top(\mu_A(x), \mu_B(x)) = 1 - J_S(1 - \mu_A(x), 1 - \mu_B(x))$ and from Proposition 4, the result follows.

   ii. Now we shall show that, $A -_{\top} B \equiv A -_{\mathcal{T}} B$. It is sufficient to prove that $\mu_{A-_{\top}B}(x) = \mu_{A-_{\mathcal{T}}B}(x)$.
   According to Fono and al. [10], $\mu_{A-_{\top}B}(x) = 1 - I_\top(\mu_A(x), \mu_B(x))$ and from Definition 8 $\mu_{A-_{\mathcal{T}}B}(x) = J_S(1 - \mu_A(x), 1 - \mu_B(x))$.
   Since $\top$ and S are dual, the Proposition 10 shows that, $I_\top(\mu_A(x), \mu_B(x)) = 1 - J_S(1 - \mu_A(x), 1 - \mu_B(x))$ and the result follows.

b. Assume now that $A -_{\mathcal{T}} B$ is a fuzzy set and $A -_{\top} B \equiv A -_{\mathcal{T}} B$. Let us show that $\top$ and S are dual.
   We have,

$$\begin{aligned} \mu_{A-_{\top}B}(x) &= 1 - I_\top(\mu_A(x), \mu_B(x)) \\ &= 1 - \max\{t \in [0;1],\ \top(\mu_A(x), t) \leq \mu_B(x)\} \\ &= \min\{1 - t \in [0;1],\ \top(\mu_A(x), t) \leq \mu_B(x)\} \\ &= \min\{1 - t \in [0;1],\ \top(\mu_A(x), t) \leq \mu_B(x)\}, \end{aligned} \tag{12}$$

and

$$\begin{aligned}
\mu_{A-_{\mathcal{T}}B}(x) &= J_S(1-\mu_A(x), 1-\mu_B(x)) \\
&= \min\{r \in [0;1],\ S(1-\mu_A(x), r) \geq 1-\mu_B(x)\} \\
&= \min\{1-t \in [0;1],\ S(1-\mu_A(x), 1-t) \geq 1-\mu_B(x)\} \\
&= \min\{1-t \in [0;1],\ 1-S(1-\mu_A(x), 1-t) \leq \mu_B(x)\} \qquad (13)
\end{aligned}$$

Since $A -_{\top} B \equiv A -_{\mathcal{T}} B$ then, $\mu_{A-_{\top}B}(x) = \mu_{A-_{\mathcal{T}}B}(x)$. From Equations (12) and (13) $\top(\mu_A(x), t) = 1 - S(1-\mu_A(x), 1-t),\ \forall t \in [0;1]$ and the result follows. □

In the following, we define a new symmetric difference operation for intuitionistic fuzzy sets based on the IF-R-implication and IF-co-implication and we study its properties.

3.2 Definitions and properties of symmetric difference operations

The idea for the new definition is derived from the classical formula for symmetric difference and the operations of union and intersection alongside with the proposed difference for intuitionistic fuzzy sets in Section 3.

Definition 9. *Let A, B be any two intuitionistic fuzzy sets defined on X. The intuitionistic fuzzy symmetric difference associated to $\mathcal{T}$ of A and B is the intuitionistic fuzzy set on X denoted by $A \bigtriangleup_{\mathcal{T}} B$ and defined by the membership and non-membership degrees as follows:*

For all $x \in X$,

$$\begin{aligned}
\mu_{A \bigtriangleup_{\mathcal{T}} B}(x) &= J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x)), \qquad (14) \\
\nu_{A \bigtriangleup_{\mathcal{T}} B}(x) &= \min\{I_{\top}(\mu_A(x) \vee \mu_B(x), \mu_A(x) \wedge \mu_B(x)), \\
&\qquad 1 - J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x) \vee \nu_B(x))\}.
\end{aligned}$$

In what follows, we establish some results showing that some properties of the classical set symmetric difference are preserved by this new proposed intuitionistic fuzzy symmetric difference operation.

Proposition 6 (Properties of Intuitionistic Fuzzy Symmetric Difference Operation)**.** *Let A, B be any intuitionistic fuzzy sets on X. The following properties for intuitionistic fuzzy symmetric difference operation hold:*

1. *$A \bigtriangleup_{\mathcal{T}} B = (A -_{\mathcal{T}} B) \cup (B -_{\mathcal{T}} A)$;*
2. *$A \bigtriangleup_{\mathcal{T}} B = B \bigtriangleup_{\mathcal{T}} A$;*
3. *If $A \subseteq B$, then $A \bigtriangleup_{\mathcal{T}} B = B -_{\mathcal{T}} A$;*
4. *$A \bigtriangleup_{\mathcal{T}} A = \emptyset$.*

Proof. 1. The following are properties for fuzzy-R-implication, $I_\top$ and fuzzy co-implication, J_S which we require here:

$$\begin{aligned}
I_\top(a \vee b, c) &= I_\top(a,c) \wedge I_\top(b,c), \text{ and } J_S(a \vee b, c) = J_S(a,c) \wedge J_S(b,c);\\
I_\top(a \wedge b, c) &= I_\top(a,c) \vee I_\top(b,c), \text{ and } J_S(a \wedge b, c) = J_S(a,c) \vee J_S(b,c);\\
I_\top(a, b \vee c) &= I_\top(a,b) \vee I_\top(a,c), \text{ and } J_S(a, b \vee c) = J_S(a,b) \vee J_S(a,c);\\
I_\top(a, b \wedge c) &= I_\top(a,b) \wedge I_\top(a,c), \text{ and } J_S(a, b \wedge c) = J_S(a,b) \wedge J_S(a,c).
\end{aligned}$$

These can easily be verified.
Now, we proceed to prove 1 and 2 consequently as follows: From Equation (14) and applying above properties of $I_\top$ and J_S we have, for all $x \in X$

$$\begin{aligned}
(\mu_{A \triangle_{\mathcal{T}} B}(x), \nu_{A \triangle_{\mathcal{T}} B}(x)) = (J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x) \wedge \nu_B(x), \nu_B(x)) \,,\\
\min\{I_\top(\mu_A(x) \vee \mu_B(x), \mu_A(x)) \wedge I_\top(\mu_A(x) \vee \mu_B(x), \mu_B(x)) \,,\\
1 - J_S(\nu_A(x) \wedge \nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x) \wedge \nu_B(x), \nu_B(x))\}) \,.
\end{aligned}$$

So we have

$$\mu_{A \triangle_{\mathcal{T}} B}(x) = (J_S(\nu_A(x), \nu_A(x)) \vee J_S(\nu_B(x), \nu_A(x))) \vee (J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_B(x))) \,,$$

$$\begin{aligned}
\nu_{A \triangle_{\mathcal{T}} B}(x) = \min\{(I_\top(\mu_A(x), \mu_A(x)) \wedge I_\top(\mu_B(x), \mu_A(x))\} \wedge \{I_\top(\mu_A(x), \mu_B(x))\\
\wedge I_\top(\mu_B(x), \mu_B(x))) \,, 1 - (J_S(\nu_A(x), \nu_A(x)) \vee J_S(\nu_B(x), \nu_A(x))\}\\
\vee\{J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_B(x)))\},
\end{aligned}$$

and applying Proposition 9 we have the following:

$$\begin{aligned}
\mu_{A \triangle_{\mathcal{T}} B}(x) &= J_S(\nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x), \nu_B(x)),\\
&= J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_A(x)),\\
&= \mu_{A -_{\mathcal{T}} B}(x) \vee \mu_{B -_{\mathcal{T}} A}(x),\\
&= \mu_{(A -_{\mathcal{T}} B) \cup (B -_{\mathcal{T}} A)}(x).
\end{aligned} \tag{15}$$

$$\begin{aligned}
\nu_{A \triangle_{\mathcal{T}} B}(x) &= \begin{array}{c}\min\{I_\top(\mu_B(x), \mu_A(x)) \wedge I_\top(\mu_A(x), \mu_B(x)),\\ 1 - J_S(\nu_B(x), \nu_A(x)) \vee J_S(\nu_A(x), \nu_B(x))\}\end{array}\\
&= \begin{array}{c}\min\{I_\top(\mu_A(x), \mu_B(x)) \wedge I_\top(\mu_B(x), \mu_A(x)),\\ 1 - J_S(\nu_A(x), \nu_B(x)) \vee J_S(\nu_B(x), \nu_A(x))\}\end{array}\\
&= \begin{array}{c}\min\{I_\top(\mu_A(x), \mu_B(x)) \wedge (1 - J_S(\nu_A(x), \nu_B(x))),\\ I_\top(\mu_B(x), \mu_A(x)) \wedge (1 - J_S(\nu_B(x), \nu_A(x)))\}\end{array}\\
&= \nu_{A -_{\mathcal{T}} B}(x) \wedge \nu_{B -_{\mathcal{T}} A}(x)\\
&= \nu_{(A -_{\mathcal{T}} B) \cup (B -_{\mathcal{T}} A)}(x).
\end{aligned} \tag{16}$$

So, result 1 is established.

2. By commutativity of Equations (15) and (16), result 2 follows, since $A \bigtriangleup_{\mathcal{T}} B = (A -_{\mathcal{T}} B) \cup (B -_{\mathcal{T}} A) = (B -_{\mathcal{T}} A) \cup (A -_{\mathcal{T}} B) = B \bigtriangleup_{\mathcal{T}} A$.

3. If $A \subseteq B$, then for all $x \in X$, $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.
   Applying the above inequalities to the Equation (14), we get

$$\begin{aligned}(\mu_{A \bigtriangleup_{\mathcal{T}} B}(x), \nu_{A \bigtriangleup_{\mathcal{T}} B}(x)) &= (J_S(\nu_B(x), \nu_A(x)), \min\{I_{\top}(\mu_B(x), \mu_A(x)), 1 - J_S(\nu_B(x), \nu_A(x))\}), \\ &= (\mu_{B -_{\mathcal{T}} A}(x), \nu_{B -_{\mathcal{T}} A}(x)),\end{aligned}$$

   and the result follows.

4. By Equation (14) we have, for all $x \in X$

$$\begin{aligned}\mu_{A \bigtriangleup_{\mathcal{T}} A}(x) &= J_S(\nu_A(x) \wedge \nu_A(x), \nu_A(x) \vee \nu_A(x)) \\ &= J_S(\nu_A(x), \nu_A(x)) = 0.\end{aligned}$$

$$\begin{aligned}\nu_{A \bigtriangleup_{\mathcal{T}} A}(x) &= \min\{I_{\top}(\mu_A(x) \vee \mu_A(x), \mu_A(x) \wedge \mu_A(x)), 1 - J_S(\nu_A(x) \wedge \nu_A(x), \nu_A(x) \vee \nu_A(x))\} \\ &= \min\{I_{\top}(\mu_A(x), \mu_A(x)), 1 - J_S(\nu_A(x), \nu_A(x))\} \\ &= \min\{1, 1\} = 1.\end{aligned}$$

So the result is established. □

The following are typical examples of symmetric difference operators associated with the three usual and well-known $\mathcal{T}$.

Example 6. *For any two intuitionistic fuzzy sets A and B defined on X,*

1. *The symmetric difference operator associated with $\mathcal{T}_M$ is given by, for all $x \in X$*

$$\mu_{A \bigtriangleup_{\mathcal{T}_M} B}(x) = \begin{cases} 0, & \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \nu_B(x) \vee \nu_A(x), & \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$\nu_{A \bigtriangleup_{\mathcal{T}_M} B}(x) = \begin{cases} 1, & \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\{\min\{\mu_B(x), 1 - \nu_B(x)\}, \min\{\mu_A(x), 1 - \nu_A(x)\}\}, & \\ \qquad \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$= \begin{cases} 1, & \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\{\mu_A(x) \wedge \mu_B(x), 1 - \nu_A(x) \vee \nu_B(x)\}, & \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)). \end{cases}$$

2. *The symmetric difference operator associated with $\mathcal{T}_P$ is given by, for all $x \in X$*

$$\mu_{A \bigtriangleup_{\mathcal{T}_P} B}(x) = \begin{cases} 0, & \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \max\left\{\frac{\nu_B(x) - \nu_A(x)}{1 - \nu_A(x)}, \frac{\nu_A(x) - \nu_B(x)}{1 - \nu_B(x)}\right\}, & \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$= \begin{cases} 0, & \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \frac{(\nu_A(x)-\nu_B(x))\vee(\nu_B(x)-\nu_A(x))}{(1-\nu_A(x))\wedge(1-\nu_B(x))}, & \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$\nu_{A \triangle_{\mathcal{T}_P} B}(x) = \begin{cases} 1, \quad \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\Big\{ \min\left\{\frac{\mu_B(x)}{\mu_A(x)}, \frac{\mu_A(x)}{\mu_B(x)}\right\}, 1 - \max\left\{\frac{\nu_B(x)-\nu_A(x)}{1-\nu_A(x)}, \frac{\nu_A(x)-\nu_B(x)}{1-\nu_B(x)}\right\}\Big\}, \\ \qquad \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$= \begin{cases} 1, \quad \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\left\{\frac{\mu_A(x)\wedge\mu_B(x)}{\mu_A(x)\vee\mu_B(x)}, 1 - \frac{(\nu_A(x)-\nu_B(x))\vee(\nu_B(x)-\nu_A(x))}{(1-\nu_A(x))\wedge(1-\nu_B(x))}\right\}, \\ \qquad \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)). \end{cases}$$

3. *The symmetric difference operator associated with $\mathcal{T}_L$ is given by, for all $x \in X$*

$$\mu_{A \triangle_{\mathcal{T}_L} B}(x) = \begin{cases} 0, & \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ (\nu_A(x) - \nu_B(x)) \vee (\nu_B(x) - \nu_A(x)), & \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$\nu_{A \triangle_{\mathcal{T}_L} B}(x) = \begin{cases} 1, \quad \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\Big\{ \min\{1 - \mu_A(x) + \mu_B(x), 1 + \nu_A(x) - \nu_B(x)\}, \\ \min\{1 - \mu_B(x) + \mu_A(x), 1 + \nu_B(x) - \nu_A(x)\}\Big\}, \\ \qquad \textit{if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)), \end{cases}$$

$$= \begin{cases} 1, \quad \textit{if } (\mu_A(x), \nu_A(x)) = (\mu_B(x), \nu_B(x)) \\ \min\Big\{1 - (\mu_A(x) - \mu_B(x)) \vee (\mu_B(x) - \mu_A(x)), \\ 1 - (\nu_A(x) - \nu_B(x)) \vee (\nu_B(x) - \nu_A(x))\Big\}, \textit{ if } (\mu_A(x), \nu_A(x)) \neq (\mu_B(x), \nu_B(x)). \end{cases}$$

The following result shows that, the intuitionistic fuzzy symmetric difference operator in Definition 9 associated with t-representable IF t-norm $\mathcal{T} = (\top, S)$ is a generalization of fuzzy symmetric difference operator proposed by Fono et al. [10] associated with a t-norm $\top$ if and only if the fuzzy t-norm $\top$ and fuzzy t-conorm S are dual.

Proposition 7 (Generalization of Symmetric Difference Operation for Fuzzy Sets)**.** *Let $\top$ and S be any t-norm and t-conorm, respectively, and $\mathcal{T} = (\top, S)$ be a t-representable IF t-norm associated with any intuitionistic fuzzy set. $\top$ and S are dual if and only if for any fuzzy sets* **C** *and* **D**, *$C \triangle_{\mathcal{T}} D$ is a fuzzy set and $C \triangle_{\top} D \equiv C \triangle_{\mathcal{T}} D$.*

Proof. Let $x \in X$, and C and D be any fuzzy sets.

a. Assume that $\top$ and S are dual.

   i. Let us show that $C \triangle_{\mathcal{T}} D$ is a fuzzy set.
   Since C and D are fuzzy sets ($1 - \mu_C(x) = \nu_C(x)$ and $1 - \mu_D(x) = \nu_D(x)$) , and $\top$ and

S are dual, then from Proposition 10,

$$\begin{aligned}I_\top(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x)) &= 1-J_S(1-\mu_C(x)\vee\mu_D(x),1-\mu_C(x)\wedge\mu_D(x))\\ &= 1-J_S((1-\mu_C(x))\wedge(1-\mu_D(x)),(1-\mu_C(x))\vee(1-\mu_D(x)))\\ &= 1-J_S(\nu_C(x)\wedge\nu_D(x),\nu_C(x)\vee\nu_D(x)).\end{aligned}\quad(17)$$

From Definition 9, the result follows.

ii. Now we shall show that, $C\ \triangle_{\top} D \equiv C\ \ \triangle_{\mathcal{T}} D$. It is sufficient to prove that $\mu_{C\triangle_\top D}(x) = \mu_{C\,\triangle_{\mathcal{T}}\,D}(x)$.
Definition 7 and Definition 9 show that,

$$\mu_{C\overset{i}{\underset{\top}{\triangle}}D}(x)=\mu_{C\cup D\overset{i}{\underset{\top}{-}}C\cap D}(x)=\begin{cases}1-I_\top^1\left(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x)\right), & \text{if } i=1\\ 1-I_\top^2\left(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x)\right), & \text{if } i=2,\end{cases}$$

and $\mu_{C\,\triangle_{\mathcal{T}}\,D}(x)=J_S(\nu_C(x)\wedge\nu_D(x),\nu_C(x)\vee\nu_D(x))$.
From Equation (17), the result follows.

b. Assume now that $C\ \ \triangle_{\mathcal{T}} D$ is a fuzzy set and $C\ \triangle_{\top} D\equiv C\ \ \triangle_{\mathcal{T}} D$.
Let us show that $\top$ and S are dual.
We have,

$$\begin{aligned}\mu_{C\triangle_\top D}(x) &= 1-I_\top(\mu_C(x)\vee\mu_D(x),\mu_C(x)\wedge\mu_D(x))\\ &= 1-\max\{t\in[0;1],\ \top(\mu_C(x)\vee\mu_D(x),t)\le\mu_C(x)\wedge\mu_D(x)\}\\ &= \min\{1-t\in[0;1],\ \top(\mu_C(x)\vee\mu_D(x),t)\le\mu_C(x)\wedge\mu_D(x)\}\\ &= \min\{1-t\in[0;1],\ \top(\mu_C(x)\vee\mu_D(x),t)\le\mu_C(x)\wedge\mu_D(x)\},\end{aligned}\quad(18)$$

and

$$\begin{aligned}\mu_{C\,\triangle_{\mathcal{T}}\,D}(x) &= J_S(\nu_C(x)\wedge\nu_D(x),\nu_C(x)\vee\nu_D(x))\\ &= \min\{r\in[0;1],\ S(\nu_C(x)\wedge\nu_D(x),r)\ge\nu_C(x)\vee\nu_D(x)\}\\ &= \min\{1-t\in[0;1],\ S(1-\mu_C(x)\vee\mu_D(x),1-t)\ge1-\mu_C(x)\wedge\mu_D(x)\}\end{aligned}$$

$$=\min\{1-t\in[0;1],\ 1-S(1-\mu_C(x)\vee\mu_D(x),1-t)\le\mu_C(x)\wedge\mu_D(x)\}.\quad(19)$$

Since $C\ \triangle_{\top} D\equiv C\ \ \triangle_{\mathcal{T}} D$ then, $\mu_{C\triangle_\top D}(x)=\mu_{C\,\triangle_{\mathcal{T}}\,D}(x)$. From Equation (18) and (19), $\top(\mu_C(x)\vee\mu_D(x),t)=1-S(1-\mu_C(x)\vee\mu_D(x),1-t),\ \forall t\in[0;1]$, and the result follows.

□

In the following Subsection, we investigate some properties of cardinality for intuitionistic fuzzy set difference and symmetric difference. For that, throughout this Subsection, the universal set X is finite.

3.3 Some cardinality properties of difference and symmetric difference for IFSs

We recall the definition and some results on intuitionistic fuzzy cardinality in what follows.

Definition 10 (Cardinality of Intuitionistic Fuzzy Set [19]). *Let A be an intuitionistic fuzzy set on X. The cardinality of A denoted by $\Sigma\, count(A)$ is given by*

$$\Sigma\, count(A) = \mathit{Card}(A) = \left(\sum_{i=1}^{n} \mu_A(x_i), \sum_{i=1}^{n} 1 - \nu_A(x_i) \right). \tag{20}$$

One of the properties of this cardinality operation is given here [See Property of $\Sigma\, count$ [19]]: Let A and B be any two intuitionistic fuzzy sets on X. Then

$$\Sigma\, count(A \cup B) + \Sigma\, count(A \cap B) = \Sigma\, count(A) + \Sigma\, count(B). \tag{21}$$

In what follows, we establish a cardinality property that is satisfied by the intuitionistic fuzzy difference and symmetric difference proposed.

Proposition 8. *Let A, B, C be any intuitionistic fuzzy sets on X. The following property holds:*

$$\mathit{Card}(A \bigtriangleup_{\mathcal{T}} B) \leq_{L^*} \mathit{Card}(A -_{\mathcal{T}} B) + \mathit{Card}(B -_{\mathcal{T}} A).$$

Proof. Recall from Proposition 6, we have $A \bigtriangleup_{\mathcal{T}} B = (A -_{\mathcal{T}} B) \cup (B -_{\mathcal{T}} A)$ and by Equation (21) we obtain

$$\text{Card}(A \bigtriangleup_{\mathcal{T}} B) = \text{Card}(A -_{\mathcal{T}} B) + \text{Card}(B -_{\mathcal{T}} A) - \text{Card}\left((A -_{\mathcal{T}} B) \cap (B -_{\mathcal{T}} A)\right). \tag{22}$$

Since by Proposition 3 we have $(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A) \neq \emptyset$ in general, then we have

$$\text{Card}\left((A -_{\mathcal{T}} B) \cap (B -_{\mathcal{T}} A)\right) \geq_{L^*} 0_{L^*}. \tag{23}$$

Putting Equation (23) into (22) we obtain the required result. □

4 Conclusion

In this study, we have proposed new difference and symmetric difference operations for intuitionistic fuzzy sets by means of intuitionistic fuzzy R-implications. We have also constructed some examples of difference and symmetric operations associated to the well-known intuitionistic fuzzy t-norms (minimum $\mathcal{T}_M$, product $\mathcal{T}_P$ and Łukasiewicz $\mathcal{T}_L$) and established conditions under which these operations yield the same results for fuzzy cases as obtained by Fono et al. [10].

We further established that the intuitionistic fuzzy difference operation preserves four properties out of five, which we referred to as the four minimal conditions to require of a difference operation on sets in general (even in crisp, fuzzy and intuitionistic fuzzy cases). We investigated and established some sufficient conditions under which the fifth property is satisfied. Meanwhile, we established that the intuitionistic fuzzy symmetric difference operation proposed preserves the properties of symmetric difference operations for crisp sets and fuzzy sets. We established out of many, one cardinality property that is satisfied by these operations.

The results of Proposition 3 have shown that the property, $(A -_{\mathcal{T}} B) \cap_{\mathcal{T}} (B -_{\mathcal{T}} A) = \emptyset$ do not hold true in general case for the difference operation for intuitionistic fuzzy sets proposed. The open problem will be to determine all intuitionistic fuzzy-t-norms under which the difference operation, so defined, preserves this property. We have not studied here, other cardinality properties and the cardinality-based measures of comparison for intuitionistic fuzzy sets by means of these new difference and symmetric difference operations for intuitionistic fuzzy sets proposed. This area is opened for further research studies.

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Appendix on Fuzzy Operators and Fuzzy Operations

Fuzzy Sets and Fuzzy Operators

Definition 11. *1. A fuzzy set B on X is defined by:*
$B = \{(x, \mu_B(x)) \mid \mu_B(x) \in [0,1], \forall x \in X\}$ where $\mu_B(x)$ is the degree of membership of x in B.

2. *A fuzzy triangular-norm (fuzzy t-norm) is a binary operation $\top : [0,1] \times [0,1] \longrightarrow [0,1]$ such that for any $x \in [0,1]$, $\top(x,1) = x$ and $\top$ satisfies commutativity ($\forall a, b \in [0,1],\ \top(a,b) = \top(b,a)$), monotonicity (increasing) ($\forall a, b, c, d \in [0,1]$, if $a \leq b$ and $c \leq d$, then $\top(a,c) \leq \top(b,d)$) and associativity ($\forall a, b, c, \in [0,1],\ \top(a, \top(b,c)) = \top(\top(a,b),c)$).*

3. *A fuzzy t-conorm is a binary operation $S : [0,1] \times [0,1] \longrightarrow [0,1]$ such that for any $x \in [0,1]$, $S(x,0) = x$ and S satisfies commutativity, monotonicity (increasing) and associativity.*

4. *A fuzzy negation N is a non-increasing mapping $N : [0,1] \longrightarrow [0,1]$ with $N(0) = 1$ and $N(1) = 0$. If $N(N(x)) = x, \forall x \in [0,1]$ (i.e. N satisfies the involutive property), then N is called strong fuzzy negation.*

5. *The dual of a fuzzy t-norm $\top$ is a fuzzy t-conorm S, such that, for all $a, b \in [0,1],\ \top(a,b) = 1 - S(1-a, 1-b)$.*

6. *A fuzzy R-implicator, $I_\top$ associated to a t-norm $\top$ is an operator $I_\top : [0,1] \times [0,1] \longrightarrow [0,1]$ defined for all $a, b \in [0,1]$ by $I_\top(a,b) = \max\{t \in [0,1] \mid \top(a,t) \leq b\}$.*
When $\top$ is left continued, we defined the residual implicator $I^1_\top$, the symetric contraposition implicator $I^2_\top$, the $QL-$implicator $I^3_\top$ and the $S-$implicator $I^4_\top$ as follows: for all $x, y \in [o,1]$, $I^1_\top(x,y) = max\{t \in [0,1],\ \top(x,t) \leq y\}$; $I^2_\top(x,y) = 1 - min\{t \in [0,1],\ S(y,t) \geq x\}$; $I^3_\top(x,y) = S(n(x), \top(x,y))$ and $I^4_\top(x,y) = S(n(x), y)$.

7. *A fuzzy co-implicator, J_S associated to S is an operator $J_S : [0,1] \times [0,1] \longrightarrow [0,1]$ defined for all $a, b \in [0,1]$ by $J_S(a,b) = \min\{r \in [0,1] \mid b \leq S(a,r)\}$.*

We will require the following useful results to establish the proofs of some basic findings in this research work.

Proposition 9 (See [10–13])**.** *For all $a, b, c \in [0,1]$,*

1. *$I_\top(a,a) = 1$; $J_S(a,a) = 0$; $J_S(a,b) \leq b \leq I_\top(a,b)$ and $I_\top(1,a) = a = J_S(0,a)$;*

2. *$b < a \Longleftrightarrow (I_\top(a,b) < 1$ or $J_S(b,a) > 0)$;*

3. $a \leq b \Rightarrow \begin{cases} I_\top(b,c) \leq I_\top(a,c), \\ I_\top(c,a) \leq I_\top(c,b). \end{cases}$ *and* $a \leq b \Rightarrow \begin{cases} J_S(b,c) \leq J_S(a,c), \\ J_S(c,a) \leq J_S(c,b). \end{cases}$
Thus $I_\top$ and J_S are left decreasing and right increasing operators.

Proposition 10 (see [13]). *Let S and $\top$ be such that, for all $a, b \in [0,1]$, $\top(a,b) \leq 1 - S(1-a, 1-b)$. Then*

i. for all $a, b \in [0,1]$, $I_{\top}(a,b) \geq 1 - J_S(1-a, 1-b)$;

ii. if $\top$ and S are dual, then for all $a, b \in [0,1]$, $I_{\top}(a,b) = 1 - J_S(1-a, 1-b)$.

The following examples of fuzzy t-norms and fuzzy t-conorms belonging to a family called Frank t-norms and Frank t-conorms will commonly be referred to in this study (see [10, 13]):

Example 7. *The Frank t-norms $\left(\top_F^l\right)_{l\in[0,+\infty]}$ such that, for all $a, b \in [0,1]$,*

$$\top_F^l(a,b) = \begin{cases} \top_M(a,b) = \min(a,b), & \text{if } l = 0, \\ \top_P(a,b) = ab, & \text{if } l = 1, \\ \top_L(a,b) = \max(a+b-1, 0), & \text{if } l = +\infty, \\ \log_l\left(1 + \frac{(l^a-1)(l^b-1)}{l-1}\right), & \text{otherwise}, \end{cases} \tag{24}$$

where $\top_M, \top_P, \top_L$ are the minimum, product and Lukasiewicz fuzzy t-norms, respectively. The Frank t-conorms $\left(S_F^l\right)_{l\in[0,+\infty]}$ such that, for all $a, b \in [0,1]$,

$$S_F^l(a,b) = \begin{cases} S_M(a,b) = \max(a,b), & \text{if } l = 0, \\ S_P(a,b) = a+b-ab, & \text{if } l = 1, \\ S_L(a,b) = \min(a+b, 1), & \text{if } l = +\infty, \\ 1 - \log_l\left(1 + \frac{(l^{1-a}-1)(l^{1-b}-1)}{l-1}\right), & \text{otherwise}, \end{cases} \tag{25}$$

where S_M, S_P, S_L are the maximum, probabilistic sum and Lukasiewicz fuzzy t-conorms, respectively (see [9, 10, 13, 16, 17]). .

The following are examples of fuzzy R-implications and fuzzy co-implications associated with Frank t-norms and Frank t-conorms respectively.

Example 8. *[10, 13, 17]: for all $a, b \in [0,1]$:*

1. *Fuzzy R-implication and fuzzy co-implication associated with $\top_M$ and S_M are respectively given by*

$$I_{\top_M}(a,b) = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{if } a > b. \end{cases}$$

and

$$J_{S_M}(a,b) = \begin{cases} b, & \text{if } a < b, \\ 0, & \text{if } a \geq b. \end{cases}$$

2. *Fuzzy R-implication and fuzzy co-implication associated with $\top_P$ and S_P are respectively given by*

$$I_{\top_P}(a,b) = \begin{cases} 1, & \text{if } a \leq b, \\ \frac{b}{a}, & \text{if } a > b. \end{cases}$$

and

$$J_{S_P}(a,b) = \begin{cases} \frac{b-a}{1-a}, & \textit{if } a < b, \\ 0, & \textit{if } a \geq b. \end{cases}$$

3. *Fuzzy R-implication and fuzzy co-implication associated with $\top_L$ and S_L are respectively given by*

$$I_{\top_L}(a,b) = \begin{cases} 1, & \textit{if } a \leq b, \\ 1-a+b, & \textit{if } a > b. \end{cases}$$

and

$$J_{S_L}(a,b) = \begin{cases} b-a, & \textit{if } a < b, \\ 0, & \textit{if } a \geq b. \end{cases}$$

4. *Fuzzy R-implication and fuzzy co-implication associated with $\top_F^l$ and S_F^l for all $l \in (0,1) \cup (1,+\infty)$ are respectively given by*

$$I_{\top_F^l}(a,b) = \begin{cases} 1, & \textit{if } a \leq b, \\ \log_l \left(1 + \frac{(l-1)(l^b-1)}{l^a-1}\right), & \textit{if } a > b. \end{cases}$$

and

$$J_{S_L}(a,b) = \begin{cases} 1 - \log_l \left(1 + \frac{(l-1)(l^{1-b}-1)}{l^{1-a}-1}\right), & \textit{if } a < b, \\ 0, & \textit{if } a \geq b. \end{cases}$$

Fuzzy Operations of Fuzzy Sets

Definition 12. *Let A and B be any two fuzzy sets defined on X. The following operations are defined by associated membership function as follows:*

i) Inclusion: $A \subseteq B$ if and only, $\mu_A(x) \leq \mu_B(x)$, $\forall\, x \in X$;

ii) Intersection: $A \cap B$ is defined by: $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$, $\forall\, x \in X$;

iii) Union: $A \cup B$ is defined by: $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$, $\forall\, x \in X$;

iv) Complement: A^c is defined by: $\mu_{A^c}(x) = 1 - \mu_A(x)$, $\forall\, x \in X$.
Where $\vee$ and $\wedge$ are max *and* min *respectively.*