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ON CONTINUOUS SIMPLE GAMES

DISSERTATION

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ATTESTATION DE CORRECTION

Nous soussignés, membres du jury lors de la soutenance de thèse de Doctorat Ph.D de Monsieur **TOUYEM Hilaire**, étudiant à l'Université de Yaoundé I sous le matricule **12V0911**, attestons que la thèse intitulée **« ON CONTINUOUS SIMPLE GAMES »**, présentée en soutenance publique le **jeudi 02 Décembre 2021 à 14 heures** dans la salle S01/S02 du bloc pédagogique par le candidat, a été corrigée conformément à nos recommandations.

En foi de quoi, la présente attestation lui est établie et délivrée pour servir et valoir ce que de droit.

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⋆ ⋆ Contents ⋆⋆

⋆⋆ Declaration ⋆⋆

I hereby declare that this submission is my own work and to the best of my knowledge, it contains no materials previously published or written by another person, no material which to a substantial extent has been accepted for the award of any other degree or diploma at The University of Yaoundé I or at any other educational institution except where due acknowledgment is made in this thesis. Any contribution made to the research by others, with whom I have worked at The University of Yaoundé I or elsewhere, is explicitly cited in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in style, presentation, and linguistic expression is acknowledged.

Hilaire TOUYEM.

⋆ ⋆ Dedication ⋆⋆

I dedicate this thesis to:
My mother **PEJINDA Pauline**;
In memory of my late father ***NGOUEUGCHE Barnabé***.

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⋆⋆ Résumé ⋆⋆

Dans une situation de prise de décision collective, les règles de décision sont généralement utilisées pour agréger les avis des membres de la collectivité en un résultat final. A cet effet, plusieurs modèles de décision ont été proposés et étudiés dans la littérature, notamment les jeux simples par Neumann et al. (1947), les jeux de vote avec abstention par Felsenthal and Machover (1997), les (j,k) jeux simples par Freixas and Zwicker (2003) et les jeux simples continus introduits par Kurz (2014). Notre travail porte sur l'étude des jeux simples continus. L'une des problématiques centrale commune aux classes de jeux précédents porte sur le problème de la mesure du pouvoir de décision, autrement dit, peut-on formaliser l'aptitude d'un membre de la collectivité à influencer le résultat final? L'indice de Shapley-Shubik défini par Shapley and Shubik (1954) ainsi que la relation d'influence de Isbell (1958) font partie des outils qui ont été conçus pour évaluer le pouvoir de décision dans un jeu simple. Ils ont été généralisés aux (j,k) jeux simples; puis aux jeux simples continus. Sur ces classes de jeux, beaucoup de défis mathématiques restent à relever, notamment ceux d'axiomatisation et de comparaison de ces mesures de pouvoir.

Dans la première partie de notre travail, nous montrons que les jeux simples continus comme fonctions à plusieurs variables sont Riemann intégrables; ce résultat permet de justifier que l'extension de l'indice de Shapley-Shubik aux jeux simples continus proposée par Kurz (2014) est bien défini. Nous montrons en plus que l'indice de Shapley-Shubik sur les jeux simples tout comme sur les (j,k) jeux simples est une discrétisation de celui des jeux simples continus. Dans la deuxième partie, nous proposons une formule simple et pratique de l'indice de Shapley-Shubik sur les (j,k) jeux simples et nous fournissons la toute première justification axiomatique de cet indice. Nous obtenons aussi deux caractérisations du même indice dans le cadre des jeux simples continus. Dans la dernière partie, nous étudions les propriétés de la relation d'influence introduite par Kurz (2014) sur les jeux simples continus. Principalement nous caractérisons la classe des jeux simples continus sur laquelle cette relation est complète et nous montrons qu'elle est un préordre dès qu'elle est complète. Pour comparer la relation d'influence et le préordre induit par l'indice de Shapley-Shubik, nous proposons une condition suffisante pour que ces deux relations coïncident.

Mots clés: Jeux simples; (j,k) jeux simples; jeux simples continus; indices de pouvoir; relation d'influence; indice de Shapley-Shubik.

⋆⋆ Abstract ⋆⋆

In the context of collective decision-making, decision rules are typically used to aggregate the opinions of community members into a final outcome. For this purpose, several decision models have been proposed and studied in the literature. This is the case for simple games by Neumann et al. (1947), voting games with abstention by Felsenthal and Machover (1997), (j, k) simple games by Freixas and Zwicker (2003) and continuous simple games introduced by Kurz (2014). Our work focuses on the study of the continuous simple games. One of the central problems common to the previous classes of games concerns the issue of power measurement. In other words, can we formalize the ability of a member of the community to influence the final outcome? The Shapley-Shubik index, see Shapley and Shubik (1954) and the influence relation introduced by Isbell (1958) are tools that were designed to evaluate power distribution in a simple game. They were generalised to (j, k) simple games and to continuous simple games, see Freixas (2005b), Pongou et al. (2011) and Kurz (2014). For these classes of games, many mathematical challenges are still to be tackled, principally those of axiomatizing and comparing of these power measurements.

In the first part of our contribution, we show that any continuous simple game viewed as a multivariate real-valued function is Riemann integrable; this result allows us to justify that the extension of the Shapley-Shubik index proposed by Kurz (2014) is well defined in the whole set of all continuous simple games. We also show that the Shapley-Shubik index for simple games as well as for (j, k) simple games appears as a special discretization of that one for continuous simple games. In the second part, we propose a rather simple and convenient formula of the Shapley-Shubik index for (j, k) simple games and provide the first axiomatic justification of this index. We also obtain two characterizations of the same index in the context of continuous simple games. The last part of our investigation leads us to the study of the properties of the influence relation introduced by Kurz (2014) on continuous simple games. We mainly characterize the class of continuous simple games on which this relation is complete, and show that it is a preordering whenever it is complete. In order to compare the influence relation and the preordering induced by the Shapley-Shubik index, we provide a sufficient condition for which these two relations coincide.

Keywords: Simple games; (j, k) simple games; continuous simple games; power indices; influence relation; Shapley-Shubik index.

⋆ ⋆ Introduction ⋆⋆

In many organizations around the world, the most important decisions are usually taken by a committee of experts or in a democratic context by voters (players) through a decision-making rule. For example, in a collective decision-making situation where the parliament must decide whether to adopt or reject a bill, each member is usually asked to vote "yes" if he is in favour of passing the bill and "no" if he is against it; the motion is adopted or not depending on the collective strength of members who vote "yes". Each parliamentarian therefore has two levels of approval: "yes" or "no" and the collective decision is either "adoption" or "rejection". Simple games (binary decision rules) are proposed in the literature to formalize such decision-making contexts. However, not all decisions are binary. For instance, in some real electoral systems such as the United Nations Security Council, the United States Federal System and the European Union Council of Ministers, abstention plays a key role in the decision-making process. It is seen as a third level of approval between "yes" and "no". Several models among which that of Fishburn (1973), the generalized binary constitutions, see Ferejohn and Fishburn (1977) or Andjiga and Moulen (1988) and the ternary voting games of Felsenthal and Machover (1997); take into account these contexts where players have three options of voting to express their opinions. In general, there might also be any number $j \geq 2$ of alternatives that can be chosen from. To this end, previous models were generalized to (j, k) simple games by Freixas and Zwicker (2003), where j is the number of ordered alternatives in the input, i.e., the voting possibilities, and k the number of ordered alternatives for the group decision. Simple games can be viewed as (2, 2) simple games; generalized binary constitutions and ternary voting games as (3, 2) simple games and Fishburn (1973) games as (3, 3) simple games. The models presented above can be considered as discrete, since the input and output alternatives are finite. Then those games can not covered some collection of economics problems such as, tax rates and spending, where we have a continuum of alternatives in the input. Mimicking the properties of a simple game, Kurz (2014) introduced the continuous simple games, where the set of individual alternatives as well as the output set is the real interval $[0, 1]$. More general models with a continuum of alternatives exist in the literature, see, for example Hsiao (1995), Calvo and Santos (2000), Grabisch and Lange (2007), Grabisch et al. (2009) and Grabisch and Rusinowska (2011), for some variants of games with a continuum of alternatives. Our thesis is focused on continuous simple games. In fact we aim to study the links between simple games, (j,k) simple games and continuous simple games. To this end, we show that simple games as well as (j,k) simple games are covered by continuous simple games using some natural embedding. This gives a coherent story condensing the different variants for committee decisions in one common framework.

One of the fundamental issues that is common to all models of collective decision making, we just mentioned, is the measurement of a decision maker's ability to affect the final outcome. In the literature, two approaches of solving this problem have been presented. The quantitative approach, consists in associating each n-players game v, with an n-tuples of real numbers, called *power distribution* in the game v, the *ith* component of which is interpreted as a numerical evaluation of the influence that player i can exert on the final outcome. In the context of simple games, several power indices were designed to evaluate the power distribution. For a short list of commonly used, voting power indices, the reader is referred to Shapley and Shubik (1954), Banzhaf (1965), Deegan and Packel (1978), Holler and Packel (1983) or Berg (1999). A larger list of power indices can be founded in Andjiga et al. (2003). A generalization of the Shapley-Shubik and Banzhaf-Coleman indices was proposed on voting games with abstention by Felsenthal and Machover (1997), and few years later on (j,k) simple games by Freixas (2005b) and Freixas (2005a). The Shapley-Shubik like index for continuous simple games was motivated and defined in Kurz (2014). The qualitative approach allows to classify players according to their *influence* in the game. For simple games, Isbell (1958) introduced the influence relation, and a weak version of this relation was introduced by Carreras and Freixas (2008). Tchantcho et al. (2008) extended the influence relation to voting games with abstention. For (j,k) simple games and continuous simple games, several variants of the influence relation were proposed respectively by Pongou et al. (2011) and Kurz (2014).

The existence of a multitude of power measurements on the same class of games has given rise to two major concerns: the axiomatization of power indices and the comparison of power theories resulting from the two approaches. The axiomatization of a power index allows to explore its features and its relevance using some intuitive properties called axioms. With regards to the issue of axiomatization, the Shapley-Shubik index is one of the most established power indices for committees drawing binary decisions. The first axiomatic justification of the Shapley-Shubik index for simple games was given by Dubey (1975) using axioms of *Efficiency, Symmetry, Null player and Transfer property.* Nowadays, several results of characterization of this index are available in the literature, for instance, Laruelle and Valenciano (2001) provided a new axiomatization by substituting the classical axioms by more transparent ones in terms of power in collective decision making procedures; Einy and Haimanko (2011) show that the characterization result of Dubey (1975) still holds if the efficiency is replaced by a weaker axiom called *gain-loss property.*

To the best of our knowledge, no axiomatization result of the generalized Shapley-Shubik index to the full class of (j, k) simple games, as well as to continuous simple games is available in the literature. These characterization problems constitute a major challenge in this thesis. Our investigations have successfully led us to alternative axiomatizations of the Shapley-Shubik index not only on (j, k) simple games; but also on continuous simple games. More precisely, we introduce on both classes of games the notion of *average game*, which is a TU-game associated to each game that is a (j, k) simple game or a continuous simple game. This notion is then used to formulate the new axiom of *average convexity*, which, combines to efficiency, symmetry and null player property leads to a characterization of the Shapley-Shubik index. The average game itself seems to be a very natural object on its own and has some nice properties. Indeed, the Shapley value of the average game of a given game v (a (j, k) simple game or a continuous simple game) coincides with the Shapley-Shubik index of the game v. Moreover, this latter property emerges to a functional formula for the Shapley-Shubik index for (j, k) simple games which is better suited for computation issues. A similar result was obtained in Pongou et al. (2012) for the Banzhaf index of a (j, k) simple game.

On continuous simple games, the first characterization result of the Shapley-Shubik index is obtained by transferring all the axioms used on (j, k) simple games. Within the framework of continuous simple games, we motivate and define two new axioms: our axiom of *Homogeneous Increments Sharing* can be seen as a correspondent to the axiom of Symmetry Gain-Loss introduced by Laruelle and Valenciano (2001) in the context of simple games; while the axiom of *Discreteness* bridges power distributions in continuous simple games with those from a specific class, the class of discrete continuous simple games that are regular. These axioms together with efficiency and null player property lead to a second axiomatization of the Shapley-Shubik index for continuous simple games. Furthermore, to justify the relevance of the Shapley-Shubik index for continuous simple games, we show that the Shapley-Shubik index for simple games, as well as for (j, k) simple games, occurs as a special discretization.

The problem in comparing power theories is whether or not any two theories of power induce the same ranking order on the set of players. This question was initially addressed within the class of simple games, Tomiyama (1987) proves that the preorderings induced by the Shapley-Shubik and Banzhaf-Coleman indices coincide on the subclass of weighted simple games. Diffo Lambo and Moulen (2002) generalize earlier result of Tomiyama (1987) by showing that the influence relation and the preorderings induced both by the Shapley-Shubik index and the Banzhaf-Colemann index coincide if and only if the game is swap-robust. On the class of voting games with abstention and that of (j, k) simple games, similar results were obtained, see Tchantcho et al. (2008), Parker (2012) and Pongou et al. (2014). To the best of our knowledge, neither a study of the influence relation of continuous simple games nor it comparison with the preordering induced by the Shapley-Shubik index has been made. One of our main goal in this thesis is to make an in-depth study of that relation and conduct an ordinal comparison with the Shapley-Shubik index. To this end, we firstly extend to continuous simple games the notion of swap-robustness introduced by Taylor and Zwicker (1993) on simple games and generalized to voting games with abstention by Tchantcho et al. (2008). After showing that the influence relation of continuous simple games is neither transitive nor complete in general, we show that this relation is complete if and only if the game is swap-robust; additionally we show that it is transitive whenever it is complete. This result thus generalizes that of Taylor and Zwicker (1999) on simple games and that of Tchantcho et al. (2008) on voting games with abstention. In order to compare the influence relation and the preordering induced by the Shapley-Shubik index, we provide a sufficient condition for which these two relations coincide.

A detailed presentation of the main ideas we just sketch above includes four chapters as follows. In Chapter 1, we present the models of simple games, (j, k) simple games and continuous simple games. We then show that the class of simple games and that of (j, k) simple games can be regarded as a subclass of continuous simple games. The relations between the Shapley-Shubik index defined on the different classes of games are also established. The axiomatization of the Shapley-Shubik index for (j, k) simple games is provided in Chapter 2. More precisely, after showing that the extension of classical axioms over (j, k) simple games are no longer sufficient to uniquely characterize the Shapley-Shubik index, we introduce the notion of *average game* that leads to the axiom of *average convexity*. The latter axiom with efficiency, symmetry and null player property characterize our index. The independence of axioms is also proven. Chapter 3 is devoted to the axiomatizations of the Shapley-Shubik index for continuous simple games and to the independence of the axioms used. In Chapter 4, we focus on the study of the influence relation of continuous simple games and its comparison with the preordering induced by the Shapley-Shubik index. The results obtained in this thesis suggest many other directions of research. In the conclusion, after a summary of the work carried out, we highlight some of these lines of future research. In order to illustrate the technical details and subtleties, detailed examples are given in the appendix.

Chapter One

From simple games to continuous simple games

In this chapter, we present some basic concepts related to simple games, (j,k) simple games and continuous simple games. We mainly establish some preliminary results that are useful in the sequel. More precisely, we provide a Shapley-type formula for (j,k) simple games which is an alternative to its cumbersome presentation using roll-call and pivotal voters of several levels. This result on the simplification of the Shapley-Shubik power index is similar with the one obtained by Pongou et al. (2012) on the analytical formula of the Banzhaf power index. Another achievement is the proof that, the Shapley-Shubik index for continuous simple games introduced by Kurz (2014), using integrals is well-defined. Moreover, we show that the Shapley-Shubik index for simple games as well as for (j,k) simple games, appears as a special discretization of the Shapley-Shubik index of continuous simple games.

The present chapter is organised as follows: Section 1.1 is devoted to the presentation of the simple games, (j,k) simple games and the Shapley-Shubik index of those classes of games. Section 1.2 emphases on preliminary results on the continuous simple games. We end with Section 1.3 which provides relationship between the Shapley-Shubik index for continuous simple games and its correspondents on simple games and (j,k) simple games.

It should be noted that throughout this thesis, we adopt the following notations and definitions:

$N=\{1,2,...,n\}$ is a finite set of players (or voters), with $n \geq 2$. A non-empty subset of N is called a coalition and the set of all coalitions of N is denoted by 2^N. Given two coalitions S and T, we write $S \subset T$ if $S \subseteq T$ and $S \neq T$. Consider a finite set A, $|A|$ denotes the cardinality of A. For easier reading, capital letters are reserved for coalitions (such as N, S, T, J, K, $\ldots$), while the corresponding small letters (n, s, t, j, k, $\ldots$) denote their respective cardinalities when there is no ambiguity. Given a coalition $S \subseteq N$ and a player $i \in N$, we will simple write $S+i$ instead of $S \cup \{i\}$ and S^c instead of $N \backslash S$.

If E is a nonempty subset of $\mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers and $\mathbb{R}_{\geq 0}$ the set of all non negative real numbers); E^n denotes the set of all n-tuples $x=(x_1,x_2,\cdots,x_n)$, where $x_i \in E$ for all $1 \leq i \leq n$. Given two n-tuples $x = (x_1, x_2, ..., x_n) \in E^n$ and $y = (y_1, y_2, ..., y_n) \in E^n$, we write $x \preceq y$ (resp. $x \prec y$) if $x_i \leq y_i$ for all $1 \leq i \leq n$ (resp. $x \preceq y$ and $x \neq y$). For instance, consider $x = (1, 2, 3)$, $y = (1, 3, 3)$ and $z = (3, 1, 0)$ elements of $\mathbb{R}^3$. We have $x \prec y$; x and z are not comparable.

Consider $x \in E^n$ and $S \in 2^N$, x_S denotes the s-tuple $(x_i)_{i\in S}$ and $x_{-S} = x_{S^c}$. Moreover, we simply write x_i (resp. x_{-i}) instead of $x_{\{i\}}$ (resp. $x_{-\{i\}}$). Consider $a \in E$, by slightly abusing notations we write $\mathbf{a} \in E^n$, for the n-tuple where each component is a, for example, $\mathbf{0} = (0, 0, \cdots, 0)$ and $\mathbf{1} = (1, 1, \cdots, 1)$.

Here are some distinctions on intervals of real numbers with bounds α and β:

- $[\alpha, \beta] = \{x \in \mathbb{R} : \alpha \leq x \leq \beta\}$;
- $]\alpha, \beta[= \{x \in \mathbb{R} : \alpha < x < \beta\}$;
- $[\alpha, \beta[= \{x \in \mathbb{R} : \alpha \leq x < \beta\}$;
- $]\alpha, \beta] = \{x \in \mathbb{R} : \alpha < x \leq \beta\}$.

A permutation of N is a bijection from N to N. We denote by $\mathcal{S}_n$ the set of all permutations on N. Given two players i and j, θ_{ij} denotes the permutation of N that only interchange i and j. In other words, θ_{ij} is the transposition of i and j. For any permutation $\pi \in \mathcal{S}_n$ and for any n-tuple $x \in \mathbb{R}^n$, $\pi(x)$ is the n-tuple defined by $\pi(x) = (x_{\pi(i)})_{i\in N}$.

1.1 Simple games and (j, k) simple games

This section comprises three subsections: the first one is devoted to the presentation of simple games while the second is dedicated to (j, k) simple games. The third subsection presents the Shapley-Shubik index of each of the two models of games. In the context of (j, k) simple games, we provide a simplified formula of the Shapley-Shubik index.

1.1.1 Simple games

In a parliament for example, amending a proposal generally opposes two issues (adoption and rejection) and each player is offered two opportunities (to vote for adoption or for rejection). A vote profile on a given proposal is then any collection that precises the opinion of each player. Given a decision rule, the outcome of each possible vote profile is either adoption; or rejection. A monotonicity condition guarantees that adding more support to an outcome is never harmful. For instance, if the outcome at a profile is adoption and some players turn their opinions from rejection to adoption, the collective decision remains unchanged.

More formally, for a given proposal let 1 stands for adoption and 0 stands for rejection.

Definition 1.1.1. *A vote profile (profile) is any collection $x \in \{0,\, 1\}^n$.*

Note that for a given player i, $x_i = 1$ means that i votes for adoption of the proposal, while $x_i = 0$ means that i votes for its rejection.

Definition 1.1.2. *A binary decision rule on N, is any map $R : \{0,\, 1\}^n \longrightarrow \{0,\, 1\}$.*

Given a binary decision rule R and a profile x, $R(x) = 1$ means that the proposal is adopted when profile x occurs and $R(x) = 0$ means the rejection of the proposal when profile x occurs. The following definition of simple games slightly reformulates the one given by Felsenthal and Machover (1997)

Definition 1.1.3. *A simple game on N is any binary decision rule v on N that satisfies the following conditions:*

(1) *if $x = \mathbf{1}$, then $v(x) = 1$;*

(2) *if $x = \mathbf{0}$, then $v(x) = 0$;*

(3) *if x and y are two profiles such that $x \preceq y$, then $v(x) \leq v(y)$.*

Condition (1) means that a proposal for which all players are in favor must be adopted. In the same way, condition (2) means that a proposal which is unanimously rejected by all the players must be rejected. Condition (3) describes the monotonicity of the rule, which is a desirable property of simple games. The set of all simple games on N will be denoted by $\mathcal{SG}_n$.

Denote by $\mathcal{P}(N)$ the set of all subsets of N. Given a profile $x \in \{0,\, 1\}^n$, pose $S_x = \{i \in N,\, x_i = 1\}$. The mapping that associates a profile x with the set S_x is one-to-one and onto. Therefore, $\{0,\, 1\}^n$ is in one-to-one correspondence to $\mathcal{P}(N)$. This allows us to give an alternative definition of a simple game using coalitions.

Definition 1.1.4. *A simple game on N is any map $v : \mathcal{P}(N) \longrightarrow \{0,\, 1\}$ such that $v(\emptyset) = 0$; $v(N) = 1$ and for all $S \subseteq T \subseteq N$, $v(S) \leq v(T)$.*

Given a coalition $S \subseteq N$, $v(S) = 1$ means that S is a winning and $v(S) = 0$ means that S is a losing. Given a simple game v on N, the set of all winning coalitions denoted $\mathcal{W}(v)$ is sufficient to uniquely define v. A winning coalition such that all proper subsets are losing, is called minimal winning coalition.

Definition 1.1.5. *A simple game v (on N) is a* unanimity game *if there exists a coalition $\emptyset \neq T \subseteq N$ such that $v(S) = 1$ iff $T \subseteq S$. As abbreviation, we use the notation γ_T for unanimity game with defining coalition T.*

Definition 1.1.6. *A simple game on N is :*

- ***proper*** *if the complement $N \backslash S$ of any winning coalition S is losing;*

- ***strong*** *if the complement* $N\backslash S$ *of any losing coalition* S *is winning;*
- ***constant-sum*** *(decisive) if it is proper and strong.*

The conditions of properness and strongness prevent instability in a simple game.

Definition 1.1.7. *A simple game* v *is weighted if there exist a quota* $q > 0$ *and positive real numbers* $w_1, w_2, \dots w_n$ *such that for all coalition* S, $v(S) = 1$ *if and only if* $w(S) = \sum_{i \in S} w_i \geq q$. *Such a game is denoted* $[q : w_1, w_2, ..., w_n]$.

Example 1.1.1. $v = [4 : 3, 2, 1, 1]$ *is a weighted simple game such that* $\mathcal{W}(v) = \{\{1,2\}; \{1,3\}; \{1,4\}; \{1,2,3\}; \{1,2,4\}; \{1,3,4\}; \{1,2,3,4\}; \{2,3,4\}\}$.

The minimal winning coalitions of this game are $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, *and* $\{2,3,4\}$.

Definition 1.1.8. *A TU-game on* N *is any mapping* $v : 2^N \longrightarrow \mathbb{R}$ *such that* $v(\emptyset) = 0$.

Note that any simple game on N is a $\{0,\ 1\}$-valued and monotonic TU-game on N. Shapley (1953) defined a value of a given TU-game v as follows:

$$\mathrm{Shap}_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S\backslash\{i\})] \quad \text{for any } i \in N\,. \tag{1.1}$$

1.1.2 (j,k) simple games

Several voting situations can be modeled by simple games that give each player two issues of voting: "adoption" and "rejection". These decision-making procedures do not take into account the fact that a player has the possibility to abstain, yet abstention plays a key role in some decision-making mechanisms such as the one of the United Nations Security Council in its current version. Felsenthal and Machover (1997) introduce ternary voting games that take into account abstention as an alternative for players. Freixas and Zwicker (2003) extend this class of games by introducing the (j,k) simple games where j is the number of ordered alternatives in the input, i.e. the voting possibilities, and k the number of ordered alternatives for the group decision. This section, presents the model of (j,k) simple games.

We assume that $j \geq 2$ and $k \geq 2$ and consider the alternative definition of a (j,k) simple game given by Pongou et al. (2014). For this, pose $J = \{0, \cdots, j-1\}$ the set of individual approval levels of the players ($j-1$ is the highest, follow by $j-2$ and so on), J^n the set of all *profiles of approval levels* (profile) and $K = \{v_1, v_2, \cdots, v_k\}$ the set of ordered voting outcomes ($v_1 < v_2 < \cdots < v_k$).

Definition 1.1.9. *A* $(j,\ k)$ *simple game on* N *is any mapping* $v : J^n \longrightarrow K$ *that satisfies:*

- $v(\mathbf{0}) = v_1$ *and* $v(\mathbf{j-1}) = v_k$;

- for all profiles x, y such that $x \preceq y$, $v(x) \leq v(y)$.

The set of all (j, k) simple games on N is denoted by $\mathcal{JK}_n$. Note that the values v_1, v_2, $\cdots$, v_k are not necessarily numerical. For example, an ordinary simple game may be identified to a $(2, 2)$ simple game for which the values of outputs set is {lose, win} with lose < win. We present below a way out (see Freixas (2005b)) to numerical evaluation of a (j, k) simple games with non real values as outputs.

Definition 1.1.10. *Let v be a (j, k) simple game on N. A real numeric evaluation of v is any mapping $\alpha : K \longrightarrow \mathbb{R}$ that assigns for each element v_l of K the real number $\alpha(v_l) = \alpha_l$ conserving the order i.e. $\alpha_l < \alpha_{l+1}$ for each $1 \leq l \leq k - 1$.*

It is usually assumed that $\alpha_1 = 0$. The *uniform numeric evaluation* is defined as the map μ such that $\mu_l = l - 1$ for all $1 \leq l \leq k$.

Notation 1.1.1. (Freixas, 2005b, Definition 2.3)
For any (j, k) simple game v and a numeric evaluation α, we will associate a new (j, k) simple game denote ${}^{\alpha}v$ which assigns a quantitative numeric output set instead of a qualitative ordered output set of v. Note that, in the case of uniform numeric evaluation, the associated game is simply denoted v instead of ${}^{\mu}v$.

In order to illustrate the definition of (j, k) simple game, consider the following example taken from Freixas and Zwicker (2003).

Example 1.1.2. [***The grading system at the Polytechnic School of Manresa***]
The academic committee is formed by three professors who evaluate students. Each professor evaluates a different aspect: one assesses the theoretical contents, another the laboratory training, and the last considers the exercises written by the student. Each student then is assigned a single, final mark compounded from the separate marks proposed by the three professors. The possible final marks that a student can get are: excellent (v_4), notable or creditable (v_3), pass (v_2), and fail (v_1) with $v_1 < v_2 < v_3 < v_4$. The possible marks that a professor may assign are: right, regular and wrong (right> regular > wrong), and the rule that determines final marks is given by the $(3, 4)$ simple game v defined as follows: let $N = \{1, 2, 3\}$ be the set of professors, where 1, 2 and 3 refers respectively to the theory professor, the practical laboratory professor and to the exercises professor; $J = \{\text{wrong}, \text{regular}, \text{right}\} := \{0, 1, 2\}$. For any $x = (x_1, x_2, x_3) \in J^3$ we set $N_l(x) = \{i \in N, x_i = l\}$, then the collective decision making rule v is defined by :

$$v(x) = \begin{cases} v_4 & \text{if } 2 \in N_2(x),\ |N_2(x)| \geq 2 \text{ and } N_0(x) = \emptyset \\ v_3 & \text{if } N_0(x) = \emptyset \text{ and } (N_2(x) = \{2\} \text{ or } N_1(x) = \{2\}) \\ v_2 & \text{if } (|N_2(x)| = 1,\ |N_1(x)| = 2 \text{ and } 2 \in N_1(x)) \text{ or} \\ & (|N_2(x)| = 2,\ N_1(x) = \emptyset \text{ and } 1 \in N_2(x)) \\ v_1 & \text{otherwise} \end{cases}$$

So, under the uniform numeric evaluation, the game v is defined as follows:

$$v(x) = \begin{cases} 3 & \text{if } 2 \in N_2(x),\ |N_2(x)| \geq 2 \text{ and } N_0(x) = \emptyset \\ 2 & \text{if } N_0(x) = \emptyset \text{ and } (N_2(x) = \{2\} \text{ or } N_1(x) = \{2\}) \\ 1 & \text{if } (|N_2(x)| = 1,\ |N_1(x)| = 2 \text{ and } 2 \in N_1(x)) \text{ or} \\ & (|N_2(x)| = 2,\ N_1(x) = \emptyset \text{ and } 1 \in N_2(x)) \\ 0 & \text{otherwise} \end{cases}$$

1.1.3 The Shapley-Shubik index for simple games and for (j,k) simple games

In a situation of collective decision-making modeled by a simple game or a (j,k) simple game, one of the major concerns consists in evaluating the importance of the vote of each player in the elaboration of the final decision. In other words, can we quantify the ability of a player to influence the collective decision? To answer the previous question, several mathematical tools have been introduced and studied in Social Choice Theory. We can mention power indices among which the well-known Shapley-Shubik index for simple games defined by Shapley and Shubik (1954) and its generalization to (j,k) simple games, see Freixas (2005b).

The Shapley-Shubik index for simple games

Definition 1.1.11. *Let v be a simple game on N. The Shapley-Shubik index of player i in v, denoted $\mathrm{SSI}_i(v)$ is defined by:*

$$\mathrm{SSI}_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S \backslash \{i\})] \tag{1.2}$$

For the game v of Example 1.1.1 we have,

$$\mathrm{SSI}(v) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)^{1}.$$

Definition 1.1.12. *Let v be a TU-game on N.*

- *a player $i \in N$ is said to be a null player in v if for all $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) = v(S)$;*
- *two players i and j are symmetric in v if for all $S \subseteq N \backslash \{i,j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$.*

Similar concepts are defined for (j,k) simple games as follows:

[1] From this example, we observe that the Shapley-Shubik index of players in a weighted simple game is not in general proportional to their weights.

Definition 1.1.13. *Let v be a (j,k) simple game on N.*

- *the player i is a null player in v if $v(x) = v(x_{-i}, a)$ for all $x \in J^n$ and for all $a \in J$;*
- *two players $i; h \in N$ are symmetric if $v(x) = v(y)$ for all $x; y \in J^n$, with $x_l = y_l$ for all $l \in N \backslash \{i,h\}$, $x_i = y_h$ and $x_h = y_i$.*

The Shapley-Shubik index for simple games was first axiomatized by Dubey (1975) using the axioms we now present:

Definition 1.1.14. *A power index $\phi : \mathcal{SG}_n \longrightarrow \mathbb{R}^n$:*

- *is* **anonymous** *if for all $v \in \mathcal{SG}_n$, for any permutation π of N and for all $i \in N$, $\phi_{\pi(i)}(\pi v) = \phi_i(v)$; where $\pi v(S) = v(\pi^{-1}(S))$ for $S \subseteq N$;*
- *is* **efficient** *if for all $v \in \mathcal{SG}_n$, $\sum_{i \in N} \phi_i(v) = 1$;*
- *satisfies the* **null player property** *if for all $v \in \mathcal{SG}_n$ and for all $i \in N$, $\phi_i(v) = 0$ whenever i is a null player in v;*
- *satisfies the* **transfer property** *if for all $u, v \in \mathcal{SG}_n$ and for all $i \in N$, $\phi_i(u) + \phi_i(v) = \phi_i(u \vee v) + \phi_i(u \wedge v)$; where $(u \vee v)(S) = \max\{u(S), v(S)\}$ and $(u \wedge v)(S) = \min\{u(S), v(S)\}$ for all $S \subseteq N$.*

Theorem 1.1.1.

The unique power index on $\mathcal{SG}_n$ that simultaneously satisfies anonymity, efficiency, null player property and transfer property is the Shapley-Shubik index.

The Shapley-Shubik index for (j,k) simple games

The Shapley-Shubik index for simple games have been extended to ternary voting games by Felsenthal and Machover (1997). Few years later, Freixas (2005b) extends this index to the class of (j,k) simple games. We recall his definition and simplify it to a more handy one.

Definition 1.1.15. *A roll call of N is an ordered pair (π, x) where π is a permutation of N and x is a vote profile. It follows that, the set of all roll calls is $\mathcal{S}_n \times J^n$.*

We suppose that,

- $\mathcal{S}_n$ is the probabilistic space consisting of the set of all permutations of N, with each permutation assigned probability $\dfrac{1}{n!}$;
- the set J^n of all vote profiles is the probabilistic space and the probability of given vote profile is assumed to be $\dfrac{1}{j^n}$;

- the roll call space $\mathcal{S}_n \times J^n$ is the probabilistic space with equal probability, i.e. each roll call assigned the same probability $\dfrac{1}{n! \times j^n}$.

For a given permutation $\pi \in \mathcal{S}_n$ and $i \in N$, we set:

- $\pi_{<i} = \{p \in N \,:\, \pi(p) < \pi(i)\}$;
- $\pi_{\leq i} = \{p \in N \,:\, \pi(p) \leq \pi(i)\}$;
- $\pi_{>i} = \{p \in N \,:\, \pi(p) > \pi(i)\}$;
- $\pi_{\geq i} = \{p \in N \,:\, \pi(p) \geq \pi(i)\}$.

Using above notations, we provide a useful and equivalent definition of h-pivotal player as defined in (Freixas, 2005b, Definition 3.5)

Definition 1.1.16. *Let v be a (j, k) simple game and (π, x) the roll call of N. The player i is said to be a v-h-pivot in (π, x) for $h = 1, 2, \cdots, k-1$ if one of the two following excluding conditions is satisfied:*

(C_1) $\quad v(a_{\pi_{\geq i}}, x_{\pi_{<i}}) > v_h$ *for some* $a \in J^n$, *and* $v(b_{\pi_{>i}}, x_{\pi_{\leq i}}) \leq v_h$ *for any* $b \in J^n$;

(C_2) $\quad v(a_{\pi_{\geq i}}, x_{\pi_{<i}}) < v_{h+1}$ *for some* $a \in J^n$, *and* $v(b_{\pi_{>i}}, x_{\pi_{\leq i}}) \geq v_{h+1}$ *for any* $b \in J^n$.

Literally the condition (C_1) means that, just before player i gets in, some output level greater than v_h is reachable; but, as player i gets in, the collective decision will be, independently of the approval levels of the subsequent players to i according to the ordering π, at most v_h. Similarly, condition (C_2) expresses the fact that, just before player i gets in, some output level lower than v_{h+1} is observable; but, as player i gets in, no matter how all subsequent players to i according to the ordering π are to change their votes, the final outcome will be at least v_{h+1}.

Note that if a player i is v-h-pivot in (π, x), then he is unique. We will therefore write "$i = h - piv(v, \pi, x)$". If this occurs for condition (C_1), we note $i = h^+ - piv(v, \pi, x)$ and $i = h^- - piv(v, \pi, x)$ if this occurs according to condition (C_2). In practice, the proposition below helps to identify all different levels for which a given player is pivotal in a given roll call.

Proposition 1.1.1. *Let v be a (j, k) simple game, (π, x) lect a roll call of N and $h = 1, 2, \cdots, k-1$. Given a player i:*

(P_1) $\quad i = h^+ - piv(v, \pi, x) \Longleftrightarrow v((\mathbf{j-1})_{\pi_{>i}}, x_{\pi_{\leq i}}) \leq v_h < v((\mathbf{j-1})_{\pi_{\geq i}}, x_{\pi_{<i}})$;

(P_2) $\quad i = h^- - piv(v, \pi, x) \Longleftrightarrow v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}}) < v_{h+1} \leq v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})$.

Proof.

Suppose that, $i = h^+ - piv(v,\, \pi,\, x)$, then from Definition 1.1.16, there exists $a \in J^n$ such that, $v_h < v(a_{\pi_{\geq i}}, x_{\pi_{<i}})$. Since $(a_{\pi_{\geq i}}, x_{\pi_{<i}}) \preceq ((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}})$, by monotonicity of v it follows that $v_h < v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}})$. Moreover, $v(b_{\pi_{>i}}, x_{\pi_{\leq i}}) \leq v_h$ holds for all $b \in J^n$. In particular, for $b = \mathbf{j} - \mathbf{1}$, we have $v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) \leq v_h$. Thus, $v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) \leq v_h < v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}})$.

Conversely, assume that $v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) \leq v_h < v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}})$, then for all $b \in J^n$, we can conclude using the monotonicity of v that: $v(b_{\pi_{>i}}, x_{\pi_{\leq i}}) \leq v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{<i}}) \leq v_h < v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{\leq i}})$. Thus, by Definition 1.1.16, $i = h^+ - piv(v,\, \pi,\, x)$. The proof of P_2) is obtained in the same way. ■

Example 1.1.3. Consider the game v in Example 1.1.2. Given the roll call (π, x) with $\pi = 213$ and $x = (1, 1, 2)$, we have:

Player	$v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}})$	$v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}})$	h^+-pivot for	$v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})$	$v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})$	h^--pivot for
1	v_2	v_3	$h = 2$	v_1	v_1	$---$
2	v_3	v_4	$h = 3$	v_1	v_1	$---$
3	v_2	v_2	$---$	v_1	v_2	$h = 1$

Remark 1.1.1. Note that if v is a (j, k) simple game with uniform numeric evaluation, the output value v_h of the game can be identified with $h - 1$. Thus the set K of outputs becomes $K = \{0, 1, ..., k-1\}$. In this case, we can observe that:

- $i = h^+ - piv(v,\, \pi,\, x) \iff h \in \{v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) + 1, \cdots, v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}})\}$;
- $i = h^- - piv(v,\, \pi,\, x) \iff h \in \{v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}}) + 1, \cdots, v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})\}$.

Definition 1.1.17. (Freixas, 2005b, Definition 3.7)
Let v be a (j, k) simple game with numeric evaluation α. The Shapley-Shubik power index for a player i is:

$$\Phi_i({}^{\alpha}v) = \frac{1}{n!j^n(\alpha_k)} \sum_{h=1}^{k-1} (\alpha_{h+1} - \alpha_h) |\{(\pi,\, x) \in \mathcal{S}_n \times J^n : \, i = h - \mathrm{piv}(v,\, \pi,\, x)\}| \qquad (1.3)$$

If we consider (j, k) simple games with uniform numeric evaluation, we provide in the next result a functional and more tractable formula of the Shapley-Shubik power index defined in Equation (1.3). Moreover, our new reformulation helps us in identifying some typos made by (Freixas, 2005b, pp.193) during the computation of the Shapley-Shubik index of the game of Example 1.1.2. Other extensions of the Shapley-Shubik power index are considered by some other authors; see for example (Mbama Engoulou, 2016, Chapter 3).

Theorem 1.1.2.

Let v be a (j, k) simple game with uniform numeric evaluation. The Shapley-Shubik power index for a player i is given by:

$$\Phi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot [\mathcal{C}_v(S) - \mathcal{C}_v(S \backslash \{i\})] \tag{1.4}$$

where $\mathcal{C}_v(T) = \dfrac{1}{j^n(k-1)} \cdot \displaystyle\sum_{x \in J^n} [v((\mathbf{j}-\mathbf{1})_T, x_{-T}) - v(\mathbf{0}_T, x_{-T})]$ for all $T \subseteq N$.

Proof. Let v be a (j, k) simple game with uniform numeric evaluation and i a given player. We first show that,

$$\Phi_i(v) = \frac{1}{n!j^n(k-1)} \sum_{(\pi,x) \in \mathcal{S}_n \times J^n} \left(\left[v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}}) \right] - \left[v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}}) \right] \right).$$

Since v is a (j, k) simple game with uniform numeric evaluation, then $\alpha_k = k - 1$. Hence according to Equation (1.3) we have :

$$\begin{aligned}
\Phi_i(v) &= \frac{1}{n!j^n(k-1)} \sum_{h=1}^{k-1} |\{(\pi, x) \in \mathcal{S}_n \times J^n : i = h - piv(v, \pi, x)\}| \\
&= \frac{1}{n!j^n(k-1)} \sum_{(\pi, x) \in \mathcal{S}_n \times J^n} |\{h \in \{1, \cdots, k-1\} : i = h - piv(v, \pi, x)\}| \\
&= \frac{1}{n!j^n(k-1)} \sum_{(\pi, x) \in \mathcal{S}_n \times J^n} |\{h \in \{1, \cdots, k-1\} : i = h^+ - piv(v, \pi, x)\}| \\
&\quad + \frac{1}{n!j^n(k-1)} \sum_{(\pi, x) \in \mathcal{S}_n \times J^n} |\{h \in \{1, \cdots, k-1\} : i = h^- - piv(v, \pi, x)\}| \quad \text{by Definition 1.1.16} \\
&= \frac{1}{n!j^n(k-1)} \sum_{(\pi, x) \in \mathcal{S}_n \times J^n} [v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})] \\
&\quad + \frac{1}{n!j^n(k-1)} \sum_{(\pi, x) \in \mathcal{S}_n \times J^n} \left[v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}}) - v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) \right] \quad \text{by Remark 1.1.1} \\
&= \frac{1}{n!j^n(k-1)} \sum_{(\pi,x) \in \mathcal{S}_n \times J^n} \left(\left[v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}}) \right] - \left[v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}}) \right] \right).
\end{aligned}$$

Now, we prove Equation (1.4). Consider $S \subseteq N$ such that $i \in S$. Pose $\Pi_i^S = \{\pi \in \mathcal{S}_n, S = \pi_{\geq i}\}$, then the set $\left\{\Pi_i^S,\, i \in S \subseteq N\right\}$ is a partition of $\mathcal{S}_n$ with $|\Pi_i^S| = (s-1)!(n-s)!$. Therefore,

$$\begin{aligned}
\Phi_i(v) &= \frac{1}{n!j^n(k-1)} \sum_{(\pi,x)\in\mathcal{S}_n\times J^n} \left(\left[v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})\right] - \left[v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})\right]\right) \\
&= \frac{1}{n!j^n(k-1)} \sum_{x\in J^n} \left(\sum_{\pi\in\mathcal{S}_n} \left(\left[v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})\right] - \left[v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})\right]\right)\right) \\
&= \frac{1}{n!j^n(k-1)} \sum_{x\in J^n} \left(\sum_{i\in S\subseteq N} \sum_{\pi\in\Pi_i^S} \left(\left[v((\mathbf{j}-\mathbf{1})_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})\right] - \left[v((\mathbf{j}-\mathbf{1})_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})\right]\right)\right) \\
&= \frac{1}{n!j^n(k-1)} \sum_{x\in J^n} \left(\sum_{i\in S\subseteq N} \sum_{\pi\in\Pi_i^S} \left([v((\mathbf{j}-\mathbf{1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] - \left[v((\mathbf{j}-\mathbf{1})_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}) - v(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})})\right]\right)\right) \\
&= \sum_{x\in J^n} \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!j^n(k-1)} \left([v((\mathbf{j}-\mathbf{1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] - \left[v((\mathbf{j}-\mathbf{1})_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}) - v(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})})\right]\right) \\
&= \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{j^n(k-1)} \left(\sum_{x\in J^n} [v((\mathbf{j}-\mathbf{1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})]\right. \\
&\qquad \left. - \sum_{x\in J^n} \left[v((\mathbf{j}-\mathbf{1})_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}) - v(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})})\right]\right) \\
&= \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!} \times [\mathcal{C}_v(S) - \mathcal{C}_v(S\setminus\{i\})]
\end{aligned}$$

■

Note that, for a given (j,k) simple game v with uniform numeric evaluation, $\mathcal{C}_v$ is a TU-game on N. Hence the Shapley-Shubik power index of a player in v is equal to its Shapley value in $\mathcal{C}_v$.

Example 1.1.4. *For the game of Example 1.1.2 with the uniform numeric evaluation, we compute the Shapley-Shubik power index of each professor using formula 1.4. We first define $\mathcal{C}_v(T)$, for all T. The detailed computations are stated in Appendix A*

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{23\}$	$\{13\}$	N
$\mathcal{C}_v(S)$	0	$\frac{11}{27}$	$\frac{11}{27}$	$\frac{9}{27}$	$\frac{7}{9}$	$\frac{6}{9}$	$\frac{6}{9}$	1

By Equation (1.4) we have:

$$\Phi(v) = \text{Shap}(\mathcal{C}_v) = \left(\frac{59}{162}, \frac{59}{162}, \frac{22}{81}\right).$$

This result is different from that given in (Freixas, 2005b, pp.193). Our investigations allowed us to identify the error made by the author in counting the number of times for which each player is pivot in the game; see Appendix B for more details.

1.2 Continuous simple games

To address some economic problems such as tax rates or judgment aggregations in which the opinion of a player is picked out from a continuum of alternatives, Kurz (2014) proposes the model of continuous simple games. This model, on which we focus in this section is related to contexts of collective decision-making where the opinion of each player as well as the collective decision, is a real number of the continuum interval $[0, 1]$.

1.2.1 The model

Hereafter and unless other indications, I will designate the interval $[0,\, 1]$.

Definition 1.2.1. *A (vote) profile is any n-tuples of elements of I. The set of all vote profiles is I^n.*

Given a vote profile x and player i, x_i represents the opinion (action) of player i in x.

Definition 1.2.2. *A continuous simple game (CSG) on N is any mapping $v : I^n \longrightarrow I$ with:*

- *$v(\mathbf{0}) = 0$ and $v(\mathbf{1}) = 1$;*
- *$v(x) \leq v(y)$ for all profiles x and y such that $x \preceq y$.*

The set of all CSGs on N is denoted by $\mathcal{CSG}_n$.

The monotonicity condition simply means that, if some of the input values increase, the collective decision should not decrease.

By relaxing the monotonicity property or by changing the domain and/or the co-domain in Definition 1.2.2, one can define alternative models of CSGs. For instance, Calvo and Santos (2000) define multilevel continuum game as they suppose that each player i has his own set of opinions $[0,\, a_i]$ $(a_i > 0)$ and the output can be any real number; Hsiao (1995) defines continuously-many-choice cooperative game with $[0,\, a]^n$ as domain and the whole set $\mathbb{R}$ of real numbers as co-domain. In Grabisch et al. (2009), CSGs are known under the name *aggregation function*.

Example 1.2.1. *[Inspired from Grabisch and Rusinowska (2011)]*

Suppose that a three-member committee evaluates a scientific project and decides about the amount of funding for the project. Each of the referees writes his report and proposes an amount of funding, i.e. chooses a percentage (from 0% till 100%) of the grant demanded by the project coordinator to be assigned to the project. The final amount of project funding is given by the arithmetic average of the referees' proposals.

This decision-making rule can be describe a CSG defined with $N = \{1,\, 2,\, 3\}$ (the set of the referees or players). Indeed, we can assume that each player has to choose an action from the interval $[0,\, 1]$, where each action means a proportion of the demanded grant. For example, the actions 0, 0.5 and 1 respectively means that the referee assigns no grant, one half of the grant and the whole amount of the grant to the project. Since the collective decision is given by the arithmetic average of the referees's actions, the output also belongs to interval $[0, 1]$. More formally, this decision rule is identified to the CSG v such that: for all $(x_1,\, x_2,\, x_3) \in [0,\, 1]^3$,

$$v(x_1,\, x_2,\, x_3) = \frac{x_1 + x_2 + x_3}{3}.$$

Example 1.2.2. Three shareholders 1, 2 and 3 of an agricultural firm hold respectively 3, 4 and 5 shares of the capital of the company. Knowing that this company can produce a maximum of 100 tonnes of maize, the three shareholders must decide about the their level of production of maize for the current year. To this end, they have to aggregate their individual levels of production into a collective one. The method of aggregation involved is the weighted geometric average where the weight of a shareholder is the proportion of his share in the company. This collective decision process can be modeled by the CSG u such that the set of players is $N = \{1, 2, 3\}$; and for all profiles $x = (x_1, x_2, , x_3) \in [0, 1]^3$,

$$u(x) = x_1^{1/4} x_2^{1/3} x_3^{5/12}.$$

Properties of continuous simple games and special subclasses

Some properties and specials subclasses of CSGs present here were proposed in Kurz (2014). In this section, we specially present CSGs that are weighted, proper, strong or constant-sum.

Definition 1.2.3. A CSG $v : I^n \longrightarrow I$ is said to be :

- ***proper*** if $v(x) + v(\mathbf{1} - x) \leq 1$ for all vote profile $x \in I^n$;
- ***strong*** if $v(x) + v(\mathbf{1} - x) \geq 1$ for all vote profile $x \in I^n$;
- ***constant-sum (decisive)*** if it is proper and strong.

Hereafter, we will consider the norm $\|\cdot\|_1$ defined on $\mathbb{R}^n$ as follows, for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\|x\|_1 = \sum_{i=1}^n |x_i|$.

Continuous linearly weighted simple games

Definition 1.2.4. A CSG $v : I^n \longrightarrow I$ is **linearly weighted** if there exists a positive weight vector $w = (w_i)_{i \in N}$ with $\|w\|_1 = 1$ such that $v(x) = \sum_{i=1}^n w_i x_i$, for all $x \in I^n$. The set of all continuous linearly weighted games on N is denoted by $\mathbb{L}_n$.

For instance, the game v from Example 1.2.1 is a continuous linearly weighted game with weight vector $w = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Proposition 1.2.1. (Kurz, 2014, Lemma 7)
All continuous linearly weighted games are proper, strong, and constant-sum.

Threshold continuous simple games

Definition 1.2.5. A CSG $v : I^n \longrightarrow I$ is called **threshold** if there exists a real number $q \in]0, 1]$ called quota and the positive weight vector $w = (w_i)_{i \in N}$ with $\|w\|_1 = 1$ such that for all $x \in I^n$, $v(x) = 1$ if $\sum_{i=1}^n w_i x_i \geq q$ and $v(x) = 0$ otherwise.

The set of all threshold CSGs (threshold games) on N is denoted by $\mathbb{T}_n$.

Proposition 1.2.2. (Kurz, 2014, Lemma 8 and 9)
A continuous threshold game with a weight vector w and a quota $q \in]0, 1]$ is :

- proper if and only if $q > \frac{1}{2}$;
- strong if and only if $q \leq \frac{1}{2}$.

Corollary 1.2.1. There is no decisive threshold CSG.

Weighted continuous simple games

Definition 1.2.6. A CSG $v : I^n \longrightarrow I$ is **weighted** if there exists a positive weight vector $w = (w_i)_{i \in N}$ with $\|w\|_1 = 1$ and a monotonously increasing quota function $q : [0, 1] \longrightarrow [0, 1]$ such that such that for all $x \in I^n$, $v(x) = q\left(\sum_{i=1}^n w_i x_i\right)$.

The set of all weighted continuous (simple) games on N is denoted by $\mathbb{W}_n$.

Note that, from the definition of CSG, the quota function q of a weighted continuous game satisfies $q(0) = 0$ and $q(1) = 1$.

Proposition 1.2.3. (Kurz, 2014, Lemma 10 and 11)
A weighted continuous game with weight vector w and a quota function q is:

- proper if and only if $q(y) + q(1 - y) \leq 1$ for all $y \in [0, 1]$;
- strong if and only if $q(y) + q(1 - y) \geq 1$ for all $y \in [0, 1]$.

Corollary 1.2.2. A continuous weighted game with weight vector w and a quota function q is decisive if and only if $q(y) + q(1 - y) = 1$ for all $y \in [0, 1]$.

Exponential product games

Definition 1.2.7. A CSG $v : I^n \longrightarrow I$ is **an exponential product game** if there exists a positive vector $\alpha = (\alpha_i)_{i \in N}$ such that, for all $x \in I^n$, $v(x) = \prod_{i \in N} x_i^{\alpha_i}$. The set of all exponential product games on N is denoted by $\mathbb{E}_n$.

The game from Example 1.2.2 is an exponential product game with vector $\alpha = \left(\frac{1}{4}, \frac{1}{3}, \frac{5}{12}\right)$.

1.2.2 Integrability of continuous simple games

We pay our attention on the (Riemann) integrability of CSGs. More generally, we provide a sufficient monotonicity condition for integrability of multivariate bounded real-valued functions defined on n-dimensional boxes. But before, we need some further notations and definitions. The principal results of this section are essentially taken from Touyem et al. (2021) working paper.

Definition 1.2.8. *A cell in $\mathbb{R}^n$ is a cartesian product $C = \bigtimes_{i=1}^{n} E_i := E_1 \times E_2 \times ... \times E_n$ such that for some real numbers α_i, β_i, $E_i \in \{[\alpha_i, \beta_i]\,,]\alpha_i, \beta_i[\,, [\alpha_i, \beta_i[\,,]\alpha_i, \beta_i]\}$.*

In this case, the interior of C is the cartesian product $int(C) = \bigtimes_{i=1}^{n}]\alpha_i, \beta_i[$ and its (n-dimensional) volume is the positive number $vol(C) = \prod_{i=1}^{n}(\beta_i - \alpha_i)$. Moreover, if $vol(C) > 0$ and $E_i = [\alpha_i, \beta_i]$ for all $1 \leq i \leq n$, C is called a non-degenerate closed cell.

Given a positive integer $n \geq 1$, two n-tuples a and b in $\mathbb{R}^n$ such that, $a_i < b_i$ for $1 \leq i \leq n$, let $C(a, b) = \bigtimes_{i=1}^{n} [a_i,\, b_i]$ be a non-degenerate closed cell.

Definition 1.2.9. *A real-valued function $f : C(a, b) \longrightarrow \mathbb{R}$ is monotone if one of the two following conditions is satisfied:*

- *($C1$): for all $x, y \in C(a, b) : x \preceq y \Longrightarrow f(x) \leq f(y)$;*
- *($C2$): for all $x, y \in C(a, b) : x \preceq y \Longrightarrow f(x) \geq f(y)$;*

Our aim is to prove that that all monotone real-valued function $f : C(a, b) \longrightarrow \mathbb{R}$ are integrable. Of course, this is an n-dimensional extension for $n \geq 2$, of the classic result stating that, for $n = 1$ all real-valued functions that are monotone on a closed interval are integrable, see (Protter and Morrey, 1977, Corollary 2, pp.106).

Definition 1.2.10. *A cell-partition of $C(a, b)$ is a collection $P = (C_j)_{1 \leq j \leq q}$ of disjoint cells in $C(a, b)$, which union is $C(a, b)$; that is $C_j \cap C_k = \emptyset$ if $j \neq k$ and $\cup_{j=1}^{q} C_j = C(a, b)$.*

Definition 1.2.11. *A step function on $C(a, b)$ is a real valued function f such that for some partition $P = (C_j)_{1 \leq j \leq q}$ of $C(a, b)$, f is constant on the interior of each C_j; that is, there exists a sequence $\lambda = (\lambda_j)_{1 \leq j \leq q}$ of real numbers such that $f(x) = \lambda_j$ for all $x \in \overset{o}{C}_j, j = 1, 2, ..., q$.*

In this case, f is said to be a step function associated to the partition P and the collection λ, or simply that f is associated to (P, λ). It is well-known that if f is associated to both (P, λ) and (P', λ') with $P' = \left(C'_j\right)_{1 \leq j \leq q'}$ and $\lambda' = \left(\lambda'_j\right)_{1 \leq j \leq q'}$, then it holds that

$$\sum_{j=1}^{q} \lambda_j vol\left(C_j\right) = \sum_{j=1}^{q'} \lambda'_j vol\left(C'_j\right).$$

The (Riemann) integral of f on $C(a,b)$ is, by definition, the real number :

$$\int_{C(a,b)} f = \sum_{j=1}^{q} \lambda_j vol\,(C_j)\,. \tag{1.5}$$

It is straightforward that for any other cell-partition $P' = \left(C'_j\right)_{1\leq j\leq q'}$ of $C(a,b)$,

$$\int_{C(a,b)} f = \sum_{j=1}^{q'} \int_{C(a,b)} f_{|_{C'_j}}$$

where $f_{|_{C'_j}}(x) = f(x)$ if $x \in C'_j$ and $f_{|_{C'_j}}(x) = 0$ if $x \notin C'_j$.

Note that, step functions on $C(a,b)$ constitute the simplest family of integrable real-valued functions on $C(a,b)$. Moreover, step functions are used to test whether a real-valued function on $C(a,b)$ is integrable or not; see Protter and Morrey (1977) for a more detailed theory of integration. To ease the presentation, we use the following characterization of an integrable multivariate real-valued functions, see (Jacob and Evans, 2016, Theorem 18. 13).

Definition 1.2.12. *A real-valued function f on $C(a,b)$ is integrable if there exists two sequences $(u_k)_k$ and $(v_k)_k$ of step functions such that:*

- *for all non negative integers k and for all $x \in C(a,b)$, $u_k(x) \leq f(x) \leq v_k(x)$;*
- *$\int_{C(a,b)} (v_k - u_k)$ tends to 0 as k tends to infinity.*

Theorem 1.2.1.

All monotone functions on $C(a,b)$ are Riemann integrable. ★

Proof. Suppose that $f : C(a,b) \to \mathbb{R}$ is monotone and assume that f satisfies $(C1)$) (otherwise consider $-f$). Let us prove that f is integrable on $C(a,b)$ by constructing two sequences of step functions (h_p) and (g_p) on $C(a,b)$ such that $h_p \leq f \leq g_p$ and $\lim\limits_{p\to+\infty} \int_{C(a,b)}(g_p - h_p) = 0$. To do this, let p be a positive integer. Then we split each $[a_i, b_i]$ into 2^p intervals of equal length and $C(a,b)$ into $(2^p)^n$ cells of equal volume by considering the sequence $a^p_{i,j} = a_i + \frac{j}{2^p}(b_i - a_i)$ with $0 \leq j \leq 2^p$, the collection $\left(C^p_{i,j}\right)_{1\leq j\leq 2^p}$ of intervals, the n-cartesian product $R_p = \{1, 2, ..., 2^p\}^n$ and the collection $\left(C^{p,k}\right)_{k\in R_p}$ of cells in $C(a,b)$ defined as follows:

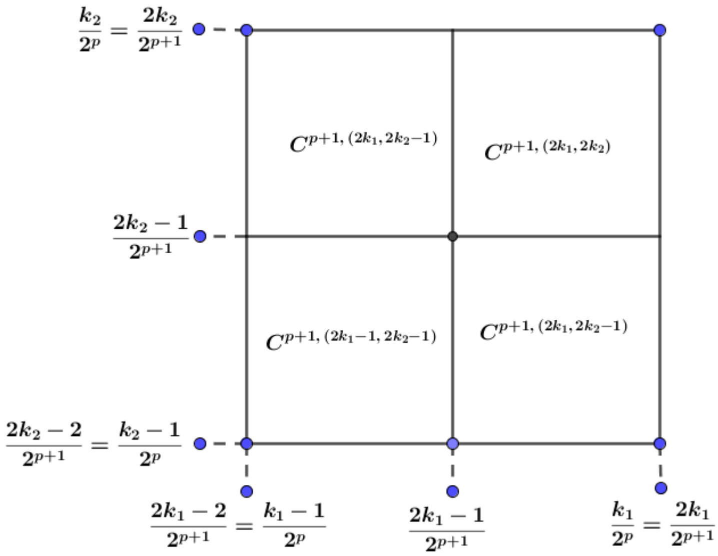
$$\begin{aligned} C^p_{i,j} &= \left[a^p_{i,j-1}, a^p_{i,j}\right[\text{ for } 1 \leq j < 2^p \text{ and } C^p_{i,j} = \left[a^p_{i,j-1}, a^p_{i,j}\right] \text{ for } j = 2^p \\ C^{p,k} &= C^p_{1,k_1} \times C^p_{2,k_2} \times ... \times C^p_{n,k_n}, \text{ for all } k = (k_1, k_2, ..., k_n) \in R_p. \end{aligned}$$

By construction, $\left\{C^p_{i,j}, j = 1, ..., 2^p\right\} = \left\{\left[a^p_{i,0}, a^p_{i,1}\right[, \cdots, \left[a^p_{i,2^p-2}, a^p_{i,2^p-1}\right[, \left[a^p_{i,2^p-1}, a^p_{i,2^p}\right]\right\}$ is a partition of $[a_i, b_i]$; $\left\{C^{p,k}, k \in R_p\right\}$ is a cell-partition of $C(a,b)$ and for all $1 \leq j \leq 2^p$,

$C^p_{i,j} = C^{p+1}_{i,2j-1} \cup C^{p+1}_{i,2j}$. Moreover for each $k \in R_p$, we split $C^{p,k}$ into 2^n cells of equal volume by considering

$$C^{p,k} = \bigtimes_{i=1}^{n} \left(C^{p+1}_{i,2k_i-1} \cup C^{p+1}_{i,2k_i}\right) = \bigcup_{l \in S_{p,k}} C^{p+1,l} \tag{1.6}$$

with $S_{p,k} = \{l \in R_p : l_i \in \{2k_i - 1, 2k_i\}, 1 \leq i \leq n\}$. Here follows an illustration of the decomposition of $C^{p,k}$ when $n = 2$.



Now define h_p and g_p as follows: for all $x \in C(a,b)$ there exists an unique $k \in R_p$ such that $x \in C^{p,k}$; pose

$$h_p(x) = f(\underline{C^{p,k}}) \text{ and } g_p(x) = f(\overline{C^{p,k}}) \text{ for all } x \in C^{p,k}$$

where $\underline{C^{p,k}} = \left(a^p_{1,k_1-1}, a^p_{2,k_2-1}, ..., a^p_{n,k_n-1}\right)$ and $\overline{C^{p,k}} = \left(a^p_{1,k_1}, a^p_{2,k_2}, ..., a^p_{n,k_n}\right)$. Note that h_p and g_p are both step functions on $C(a,b)$. Moreover, for all $k \in R_p$ and for all $x \in C^{p,k}$, we have $\underline{C^{p,k}} \preceq x \preceq \overline{C^{p,k}}$. Since f is monotone, then $f(\underline{C^{p,k}}) \leq f(x) \leq f(\overline{C^{p,k}})$. Hence for all $x \in C(a,b)$, $h_p(x) \leq f(x) \leq g_p(x)$.

To complete the proof, we show that $\lim\limits_{p\to+\infty} \int_{C(a,b)} (g_p - h_p) = 0$. For this purpose, let $\delta_p = \int_{C(a,b)} (g_p - h_p)$. By the definition of h_p and g_p we compute:

$$\delta_p = \sum_{k \in R_p} [f(\overline{C^{p,k}}) - f(\underline{C^{p,k}})] \times vol(C^{p,k}) = \frac{v_0}{2^{np}} \sum_{k \in R_p} [f(\overline{C^{p,k}}) - f(\underline{C^{p,k}})] \tag{1.7}$$

where $v_0 = vol(C(a,b))$. Since $\left\{C^{p,k} : k \in R_p\right\}$ is a cell-partition of $C(a,b)$, it follows that

$$\delta_{p+1} = \sum_{k \in R_p} \int_{C(a,b)} (g_{p+1} - h_{p+1})_{|_{C^{p,k}}}. \tag{1.8}$$

Furthermore, for each $k \in R_p$, $(g_{p+1} - h_{p+1})_{|_{C^{p,k}}}$ is null out of $C^{p,k}$ and $C^{p,k}$ is the union of disjoint cells $C^{p+1,l}$ for $l \in S_{p,k}$. Thus, for each $k = (k_1, k_2, \cdots, k_n) \in R_p$ we have:

$$\begin{aligned}
&\int_{C(a,b)} (g_{p+1} - h_{p+1})_{|C^{p,k}} \\
&= \sum_{l \in S_{p,k}} \left(g_{p+1}(\overline{C^{p+1,l}}) - h_{p+1}(\underline{C^{p+1,l}}) \right) \times vol(C^{p+1,l}) \\
&= \frac{v_0}{2^{n(p+1)}} \left[\sum_{l \in \{l^1, l^2\}} \left(f(\overline{C^{p+1,l}}) - f(\underline{C^{p+1,l}}) \right) + \sum_{l \in S^*_{p,k}} \left(f(\overline{C^{p+1,l}}) - f(\underline{C^{p+1,l}}) \right) \right]
\end{aligned}$$

where,

$$l^1 = (2k_i - 1)_{1 \leq i \leq n}, \quad l^2 = (2k_i)_{1 \leq i \leq n} \quad \text{and} \quad S^*_{p,k} = S_{p,k} \backslash \{l^1, l^2\}.$$

Note that,

$$\underline{C^{p+1,l^1}} = \underline{C^{p,k}}, \quad \underline{C^{p+1,l^2}} = \overline{C^{p+1,l^1}}, \quad \overline{C^{p+1,l^2}} = \overline{C^{p,k}} \tag{1.9}$$

and for all $l \in S^*_{p,k}$ we have:

$$\underline{C^{p,k}} \preceq \underline{C^{p+1,l}} \preceq \overline{C^{p+1,l}} \preceq \overline{C^{p,k}}. \tag{1.10}$$

So, thanks to the monotonicity of f and Equation (1.10) we get:

$$f(\underline{C^{p,k}}) \leq f(\underline{C^{p+1,l}}) \leq f(\overline{C^{p+1,l}}) \leq f(\overline{C^{p,k}}),$$

This implies

$$f(\overline{C^{p+1,l}}) - f(\underline{C^{p+1,l}}) \leq f(\overline{C^{p,k}}) - f(\underline{C^{p,k}}). \tag{1.11}$$

By combining equations (1.9) and (1.11), we obtain:

$$\begin{aligned}
&\int_{C(a,b)} (g_{p+1} - h_{p+1})|_{C^{p,k}} \\
&= \frac{v_0}{2^{n(p+1)}} \left[\sum_{l \in \{l^1, l^2\}} \left(f(\overline{C^{p+1,l}}) - f(\underline{C^{p+1,l}}) \right) + \sum_{l \in S^*_{p,k}} \left(f(\overline{C^{p+1,l}}) - f(\underline{C^{p+1,l}}) \right) \right] \\
&\leq \frac{v_0}{2^{n(p+1)}} [f(\overline{C^{p,k}}) - f(\underline{C^{p,k}})] + \frac{v_0}{2^{n(p+1)}} \sum_{l \in S^*_{p,k}} [f(\overline{C^{p,k}}) - f(\underline{C^{p,k}})] \\
&\leq \frac{v_0}{2^n} \frac{f(\overline{C^{p,k}}) - f(\underline{C^{p,k}})}{2^{np}} (1 + |S^*_{p,k}|) = \frac{2^n - 1}{2^n} \frac{v_0}{2^{np}} [f(\overline{C^{p,k}}) - f(\underline{C^{p,k}})]
\end{aligned}$$

Finally for all $k \in R_p$, we have

$$\int_{C(a,b)} (g_{p+1} - h_{p+1})|_{C^{p,k}} \leq \frac{2^n - 1}{2^n} \frac{v_0}{2^{np}} [f(\overline{C^{p,k}}) - f(\underline{C^{p,k}})] \tag{1.12}$$

By summing over $k \in R_p$ all left-hand-side terms and all right-hand-side terms from (1.12), equations (1.7) and (1.8) imply that

$$\delta_{p+1} \leq \frac{2^n - 1}{2^n} \delta_p.$$

for all positive integer p. Therefore $0 \leq \delta_p \leq \left(\frac{2^n - 1}{2^n} \right)^p \delta_0$, where $\delta_0 = v_0(f(b) - f(a))$.

Since $\lim_{p \to +\infty} \left(\frac{2^n - 1}{2^n} \right)^p = 0$, then $\lim_{p \to +\infty} \delta_p = 0$. ■

Corollary 1.2.3. *All continuous simple games are integrable on I^n.*

We generalize the result of Theorem 1.2.1 to monotonous bounded functions, defined on a nonempty and bounded *Jordan measurable subset* of $\mathbb{R}^n$. A nonempty subset D of $\mathbb{R}^n$ is said to be bounded if it is contained in some cell. The characteristic function of D is the function χ_D defined on $\mathbb{R}^n$ by

$$\chi_D(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.2.13. *Let $f : D \longrightarrow \mathbb{R}$ be bounded and C a non-degenerate closed cell such that $D \subseteq C$. Then f is Riemann integrable on D if $\widehat{f} : C \longrightarrow \mathbb{R}$ is Riemann integrable on C, with $\widehat{f}(x) = f(x)$ if $x \in D$ and $\widehat{f}(x) = 0$ otherwise.*

Definition 1.2.14. *(Jacob and Evans, 2016, Definition 19.23) A bounded set D contained in a non-degenerate closed cell C of $\mathbb{R}^n$ is said to be Jordan measurable if the restriction $\chi_{D_{|C}}$ of χ_D on C is Riemann integrable.*

Theorem 1.2.2.

Let D be a nonempty Jordan measurable subset of a non-degenerate closed cell C of $\mathbb{R}^n$. Then any monotone and bounded real-valued function $f : D \longrightarrow \mathbb{R}$ is integrable.

Proof. Let D be a nonempty Jordan measurable subset of a cell C of $\mathbb{R}^n$ and $f : D \longrightarrow \mathbb{R}$ a monotone and bounded function. Without loss of generality (w.l.o.g.), assume that f satisfies condition $(C1)$. We show that f is integrable on D by constructing a monotone function $g : C \longrightarrow \mathbb{R}$ such that $\widehat{f} = g \cdot \chi_{D_{|C}}$. The result comes from the fact that, $\widehat{f}$ is the product of two integrable functions.

Consider $x \in C$ we set $D^-(x) = \{t \in D\,, t \preceq x\}$. Since f is bounded on a nonempty set D, we define $g : C \longrightarrow \mathbb{R}$ as follows:

$$g(x) = \begin{cases} \sup\{f(t)\,, t \in D^-(x)\} & \text{if} \quad D^-(x) \neq \emptyset \\ \inf\{f(t)\,, t \in D\} & \text{otherwise} \end{cases} \tag{1.13}$$

Let us show that g is monotone and $\widehat{f} = g \cdot \chi_{D_{|C}}$. To do this, consider $x, y \in D$ such that $x \preceq y$, then $D^-(x) \subseteq D^-(y)$.

- If $D^-(y) = \emptyset$ then $D^-(x) = \emptyset$. So, from Equation (1.13), we get

$$g(x) = \inf\{f(t)\,, t \in D\} = g(y)\,;$$

- If $D^-(y) \neq \emptyset$, two cases are possible. First assume that $D^-(x) = \emptyset$, then $g(x) = \inf\{f(t)\,, t \in D\} \leq f(t_0)$ for all $t_0 \in D^-(y)$. Hence,

$$g(x) \leq \sup\{f(t)\,, t \in D^-(y)\} = g(y)\,.$$

Second, suppose that $D^-(x) \neq \emptyset$, then $g(x) = \sup\{f(t)\,, t \in D^-(x)\}$. Since $D^-(x) \subseteq D^-(y)$ then,

$$g(x) = \sup\{f(t)\,, t \in D^-(x)\} \leq \sup\{f(t)\,, t \in D^-(y)\} = g(y)\,.$$

Finally for all $x, y \in C$ such that $x \preceq y$ we have $g(x) \leq g(y)$. We then conclude that g is monotone on C and it follows from Theorem 1.2.1 that g is integrable on C.

To conclude this proof, it is sufficient to show that $g_{|D} = f$. Consider $x \in D$, then $x \in D^-(x)$ and by Equation (1.13) we can write:

$$f(x) \leq g(x) = \sup\{f(t),\, t \in D^-(x)\}\,. \tag{1.14}$$

Moreover for all $t \in D^-(x)$ we have $t \preceq x$. This implies that $f(t) \leq f(x)$, for all $t \in D^-(x)$. Hence

$$g(x) = \sup\{f(t),\, t \in D^-(x)\} \leq f(x)\,. \tag{1.15}$$

and from relations (1.14) and (1.15) we conclude that $g(x) = f(x)$, for all $x \in D$, i.e. $g_{|D} = f$. This implies that $\widehat{f} = g \cdot \chi_{D_{|C}}$. Since D is a bounded Jordan measurable set then $\chi_{D_{|C}}$ is integrable on C; so is the product $\widehat{f}$ of g and $\chi_{D_{|C}}$. ■

1.2.3 The Shapley-Shubik index for continuous simple games

Given a CSG, it is still of great importance to measure the capacity of each player to affect the final result of the game. On this class of games, Kurz (2014) extended the Shapley-Shubik index.

Definition 1.2.15. *Let v be a CSG on N and $i \in N$ an arbitrary player. The Shapley-Shubik index $\Psi_i(v)$ of a player i in the game v is given by:*

$$\Psi_i(v) = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \int_{I^n} \left(\left[v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}}) \right] - \left[v(\mathbf{1}_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}}) \right] \right) \mathrm{d}\, x\,. \tag{1.16}$$

where, $\mathrm{d}\, x := \mathrm{d}\, x_1 \ldots \mathrm{d}\, x_n$.

Note that, the existence of Ψ is guaranteed by the integrability of CSGs provided by Corollary 1.2.3.

Interpretation (see Kurz (2014) and Kurz (2018))

Let v be a CSG on N and $i \in N$.

- Assume that the players are ranked in a sequence π and called one by one to express their opinion according to the profile x;
- due to the monotonicity of v, the highest value that can be attained by $v(x)$ before the vote of player i is $v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}})$. Similarly, $v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})$, is the lowest value that can be attained by $v(x)$ before the vote of player i;
- since all real numbers between $v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})$ and $v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}})$, can be attained by some vote profile $y \in I^n$, the quantity $v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})$ is a suitable measure for the uncertainty of the game v before the player i announces his or her opinion, with respect to the ordering π and the profile x. Similarly, the uncertainty after the announcement of player i is given by $v(\mathbf{1}_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})$. Hence the difference $[v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})] - [v(\mathbf{1}_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}})]$ measures the uncertainty in the game v due to player i, according to the ordering π and the profile x;
- the Shapley-Shubik power index $\Psi_i(v)$ can thus be interpreted as the ability of player i to reduce the uncertainty in a game v.

Reformulation of the Shapley-Shubik index for continuous simple games

As shown in Theorem 1.1.2 in the context of (j, k) simple games, we provide a functional formula of the Shapley-Shubik power index for CSGs.

Theorem 1.2.3.

For every CSG v with player set N and every player $i \in N$ we have

$$\Psi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \times [\mathcal{C}(v, S) - \mathcal{C}(v, S \backslash \{i\})], \tag{1.17}$$

where $\mathcal{C}(v, T) = \int_{I^n} [v(\mathbf{1}_T, x_{-T}) - v(\mathbf{0}_T, x_{-T})] \, \mathrm{d}\, x$ for all $T \subseteq N$.

Proof. Let $S_\pi^i := \{j \in N : \pi(j) \geq \pi(i)\}$. Then $S_\pi^i = S$ for exactly $(s-1)!(n-s)!$ permutations $\pi \in \mathcal{S}_n$ and an arbitrary coalition S such that $\{i\} \subseteq S \subseteq N$.

$$\begin{aligned}
\Psi_i(v) &= \frac{1}{n!} \cdot \sum_{\pi \in \mathcal{S}_n} \int_{I^n} \left[v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{1}_{\pi_{>i}}, x_{\pi_{\leq i}}) + v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})\right] dx \\
&= \frac{1}{n!} \cdot \sum_{\pi \in \mathcal{S}_n} \int_{I^n} \left[(v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})) - (v(\mathbf{1}_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}}))\right] dx \\
&= \frac{1}{n!} \cdot \sum_{i \in S \subseteq N} \sum_{\pi \in \Pi_i^S} \int_{I^n} \left[(v(\mathbf{1}_{\pi_{\geq i}}, x_{\pi_{<i}}) - v(\mathbf{0}_{\pi_{\geq i}}, x_{\pi_{<i}})) - (v(\mathbf{1}_{\pi_{>i}}, x_{\pi_{\leq i}}) - v(\mathbf{0}_{\pi_{>i}}, x_{\pi_{\leq i}}))\right] dx \\
&= \frac{1}{n!} \cdot \sum_{i \in S \subseteq N} \sum_{\pi \in \Pi_i^S} \int_{I^n} \left[(v(\mathbf{1}_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})) - (v(\mathbf{1}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}) - v(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}))\right] dx \\
&= \frac{1}{n!} \cdot \sum_{i \in S \subseteq N} \left|\Pi_i^S\right| \int_{I^n} \left[(v(\mathbf{1}_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})) - (v(\mathbf{1}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}) - v(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}))\right] dx \\
&= \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \int_{I^n} \left[(v(\mathbf{1}_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})) - (v(\mathbf{1}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}) - v(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}))\right] dx \\
&= \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \left[\mathcal{C}(v, S) - \mathcal{C}(v, S\setminus\{i\})\right]
\end{aligned}$$

■

The simplified formula of Ψ given by the Equation (1.17) is more practical. This reformulation highlights the fact that, the Shapley-Shubik index for a given CSG is equal to the Shapley value of the TU-game $\mathcal{C}(v, .)$. To facilitate the computation of Ψ, an explicit formulas of $\mathcal{C}(., .)$ can be obtained for a special classes of CSGs.

Proposition 1.2.4.

1. *Consider $w \in \mathbb{R}^n_{\geq 0}$ with $\|w\|_1 = 1$ and $f_i : I \longrightarrow I$ weakly monotonic increasing functions with $f_i(0) = 0$ and $f_i(1) = 1$ for all $i \in N$. Then $v : I^n \longrightarrow I$ defined by $v(x) = \sum_{i=1}^n w_i \cdot f_i(x_i)$ is a CSG such that*

$$\mathcal{C}(v, S) = \sum_{i \in S} w_i \quad \text{for all } S \subseteq N\,.$$

2. *For every exponential product game v with vector α, we have:*

$$\mathcal{C}(v, S) = \frac{1}{\Lambda_{N\setminus S}} \quad \text{for all } S \subseteq N\,.$$

where, $\Lambda_T = \prod_{i \in T} (\alpha_i + 1)$ for all $T \subseteq N$ with $\Lambda_\emptyset = 1$.

Proof.

1. Let v be a CSG such that for all $x \in I^n$, $v(x) = \sum_{i \in N} w_i \cdot f_i(x_i)$. Consider $S \subseteq N$, then for all $x \in I^n$, $v(\mathbf{1}_S, x_{-S}) = \sum_{i \in S} w_i + \sum_{i \notin S} w_i \cdot f_i(x_i)$ and $v(\mathbf{0}_S, x_{-S}) = \sum_{i \notin S} w_i \cdot f_i(x_i)$. Therefore,

$$\mathcal{C}(v, S) = \int_{I^n} [v(\mathbf{1}_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] dx = \sum_{i \in S} w_i\,.$$

2. Let v be an exponential product game with vector α, $S \subseteq N$ and $x \in I^n$. We have, $v(\mathbf{1}_S,\, x_{-S}) = \prod_{i \notin S} x_i^{\alpha_i}$ and $v(\mathbf{0}_S,\, x_{-S}) = 0$. Thus,

$$\begin{aligned}\mathcal{C}(v,\, S) &= \int_0^1 \cdots \int_0^1 \prod_{i \notin S} x_i^{\alpha_i} dx_1 \cdots dx_n \\ &= \prod_{i \notin S} \frac{1}{\alpha_i + 1} = \frac{1}{\Lambda_{N \setminus S}}\end{aligned}$$

∎

Example 1.2.3. *Consider the games v of Example 1.2.1 and u of Example 1.2.2. Applying Proposition 1.2.4, one obtains:*

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	$\{123\}$
$\mathcal{C}(v, S)$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1
$\mathcal{C}(u, S)$	0	$\frac{9}{17}$	$\frac{48}{85}$	$\frac{3}{5}$	$\frac{12}{17}$	$\frac{3}{4}$	$\frac{4}{5}$	1

Therefore, formula 1.17 gives :

$$\Psi(v) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \quad \text{and} \quad \Psi(u) = \left(\frac{7}{24}, \frac{341}{1020}, \frac{763}{2040}\right)$$

Corollary 1.2.4. *The Shapley-Shubik index for any continuous linearly weighted game coincides with its weight vector.*

1.3 The Shapley-Shubik index for CSGs as a generalisation

In this section, we show that the Shapley-Shubik index for simple games as well as for (j, k) simple games appears as special discretizations of the one of CSGs. To establish this result, we first show that the set $\mathcal{SG}_n$ (resp. $\mathcal{JK}_n$) of all simple games (resp. (j, k) simple games) can be viewed as some subsets of $\mathcal{CSG}_n$.

1.3.1 Simple games and (j, k) simple games viewed as CSGs

We show that simple games as well as (j, k) simple games can be embedded in a CSGs via an injective map. To do this, consider $\tau \in]0,\, 1]$ and $j \geq 2$ an integer. For each $x \in I^n$, pose $\theta_\tau(x) = \{i \in N : x_i \geq \tau\}$ and $X^x \in J^n$ the profile defined as follows:

$$X_i^x = \begin{cases} l_i & \text{if } \dfrac{l_i}{j} \leq x_i < \dfrac{l_i + 1}{j} \\ & \\ j - 1 & \text{if } \dfrac{j-1}{j} \leq x_i \leq 1 \end{cases} \qquad \text{for all } i \in N\,.$$

Proposition 1.3.1. *Let v be a simple game on N and $\tau \in]0, 1]$.*

1. *The mapping $C_v^\tau : I^n \longrightarrow I$ defined by $C_v^\tau(x) = v(\theta_\tau(x))$, for all $x \in I^n$ is a CSG.*

2. *The mapping $T^\tau : \mathcal{SG}_n \longrightarrow \mathcal{CSG}_n$ that associates each simple game v with the CSG C_v^τ is one-to-one.*

Proof.

1. Consider $v \in \mathcal{SG}_n$, then $C_v^\tau(\mathbf{0}) = v(\theta_\tau(\mathbf{0})) = v(\emptyset) = 0$ and $C_v^\tau(\mathbf{1}) = v(\theta_\tau(\mathbf{1})) = v(N) = 1$. Consider $x, y \in I^n$ such that $x \preceq y$, then $\theta_\tau(x) \subseteq \theta_\tau(y)$. Since v is monotone, we have $v(\theta_\tau(x)) \leq v(\theta_\tau(y))$, that is $C_v^\tau(x) \leq C_v^\tau(y)$. So, $C_v^\tau \in \mathcal{CSG}_n$.

2. From (1), T^τ is well defined. Consider $u, v \in \mathcal{SG}_n$ such that $T^\tau(u) = T^\tau(v)$ and $S \in 2^N$. We prove that $u(S) = v(S)$. First note that, $S = \theta_\tau(x)$ where $x \in I^n$ with $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise. Since $T^\tau(u) = T^\tau(v)$, then $T^\tau(u)(x) = T^\tau(v)(x)$, therefore $u(\theta_\tau(x)) = v(\theta_\tau(x))$ i.e. $u(S) = v(S)$. Thus $u = v$ and we conclude that T^τ is one-to-one.

■

Proposition 1.3.1 highlights the fact that the set $\mathcal{SG}_n$ of simple games can be identified to a subset of $\mathcal{CSG}_n$. However, the embedding transformation is not unique. Nevertheless, we show below that the embedding T^τ preserves some properties of simple games.

Definition 1.3.1. *Let v be a CSG.*

- *the player i is a null player in v if $v(x) = v(x_{-i}, a)$ for all $x \in I^n$ and for all $a \in I$;*

- *two players $i; h \in N$ are symmetric if $v(x) = v(y)$ for all $x; y \in I^n$, with $x_l = y_l$ for all $l \in N \backslash \{i; h\}$, $x_i = y_h$ and $x_h = y_i$.*

In words, i is a null player in v if for any vote profile x, the collective decision $v(x)$ does not depend on the opinion x_i of i. If interchanging the input x_i and x_h of two players never alters the output $v(x)$, then players i and h are symmetric. In other words, i and h are symmetric players if, and only if, $v(x) = v(\theta_{ih} x)$ for all $x \in I^n$.

Proposition 1.3.2. *Consider $v \in \mathcal{SG}_n$ and $T^\tau(v)$ an associated CSG, with $\tau \in]0, 1]$.*

1. *any null player in v is a null player in $T^\tau(v)$;*

2. *any two symmetric players in v are symmetric in $T^\tau(v)$;*

3. *if v is proper (resp. strong) and $\tau > \dfrac{1}{2}$ then $T^\tau(v)$ is proper (resp. strong).*

Proof. Let v be a simple game on N and $T^\tau(v)$ the associated CSG, with $\tau \in]0,\,1]$.

1. Suppose that i be a null player in v. We prove that i is a null player in $T^\tau(v)$. Consider $x \in I^n$ and $a \in [0,\,1]$.

   - If $x_i \geq \tau$ and $a \geq \tau$ (or $x_i < \tau$ and $a < \tau$), then $\theta_\tau(x) = \theta_\tau(x_{-i}, a)$. By the definition of $T^\tau(v)$, it follows that $T^\tau(v)(x) = T^\tau(v)(x_{-i}, a)$;
   - If $x_i \geq \tau$ and $a < \tau$, then $\theta_\tau(x) = \theta_\tau(x_{-i}, a) \cup \{i\}$. Therefore, $v\left(\theta_\tau(x)\right) = v\left(\theta_\tau(x_{-i}, a) \cup \{i\}\right) = v\left(\theta_\tau(x_{-i}, a)\right)$, since player i is a null player in v. Therefore, by definition, $T^\tau(v)(x) = T^\tau(v)(x_{-i}, a)$;
   - If $x_i < \tau$ and $a \geq \tau$, then $\theta_\tau(x_{-i}, a) = \theta_\tau(x) \cup \{i\}$. Similar to the previous case.

   Finally $T^\tau(v)(x) = T^\tau(v)(x_{-i}, a)$ for all $x \in I^n$ and for all $a \in [0, 1]$. We then conclude that i is a null player in $T^\tau(v)$.

2. Let i and h be two symmetric players in v and $x \in I^n$. We have $\theta_\tau(\theta_{ih}(x)) = \theta_{ih}(\theta_\tau(x))$.

   - If $i, h \in \theta_\tau(x)$ or $i, h \notin \theta_\tau(x)$, then $\theta_{ih}(\theta_\tau(x)) = \theta_\tau(x)$. So, $T^\tau(v)(\theta_{ih}(x)) = v(\theta_{ih}(\theta_\tau(x))) = v(\theta_\tau(x)) = T^\tau(v)(x)$.
   - If $i \in \theta_\tau(x)$ and $h \notin \theta_\tau(x)$, then $\theta_{ih}(\theta_\tau(x)) = [\theta_\tau(x)\backslash\{i\}] \cup \{h\}$. Since i and h are symmetric in v, then $v\left(\theta_\tau(x)\backslash\{i\} \cup \{h\}\right) = v\left(\theta_\tau(x)\backslash\{i\} \cup \{i\}\right)$ i.e. $v\left(\theta_\tau(\theta_{ih}(x))\right) = v\left(\theta_\tau(x)\right)$. By the definition of $T^\tau(v)$, one obtains $T^\tau(v)(\theta_{ih}(x)) = v(\theta_{ih}(\theta_\tau(x))) = v(\theta_\tau(x)) = T^\tau(v)(x)$.
   - If $i \notin \theta_\tau(x)$ and $h \in \theta_\tau(x)$, then $\theta_{ih}(\theta_\tau(x)) = [\theta_\tau(x)\backslash\{h\}] \cup \{i\}$. By interchanging the role of i and h in the previous step, we get $T^\tau(v)(\theta_{ih}(x)) = T^\tau(v)(x)$.

   It follows that, for all $x \in I^n$, $T^\tau(v)(\theta_{ih}(x)) = T^\tau(v)(x)$. This means that players i and h are symmetric in $T^\tau(v)$.

3. Let v be a simple game and $\tau > \frac{1}{2}$ a given parameter. Since $\tau > \frac{1}{2}$, we obviously have $\theta_\tau(\mathbf{1} - x) = N\backslash\theta_\tau(x)$, for all $x \in I^n$. Therefore, $T^\tau(v)(\mathbf{1} - x) + T^\tau(v)(x) = v\left(N\backslash\theta_\tau(x)\right) + v\left(\theta_\tau(x)\right)$. Hence,

   - If v is proper, then $T^\tau(v)(\mathbf{1} - x) + T^\tau(v)(x) \leq 1$, i.e. $T^\tau(v)$ is proper;
   - If v is strong, then $T^\tau(v)(\mathbf{1} - x) + T^\tau(v)(x) \geq 1$, i.e. $T^\tau(v)$ is strong.

■

Proposition 1.3.3. *Let v be a (j, k) simple game with numerical evaluation α.*

1. *The mapping $C'_{\alpha_v} : I^n \longrightarrow I$ defined by $C'_{\alpha_v}(x) = \alpha(v(X^x))/\alpha_k$, for all $x \in I^n$ is a*

CSG.

2. The mapping $T' : \mathcal{JK}_n \longrightarrow \mathcal{CSG}_n$ that associates a (j,k) simple game v with numerical evaluation α with the game C'_{α_v} is one-to-one.

Proof.

1. Consider $v \in \mathcal{JK}_n$ with numerical evaluation α. Since $\mathbf{0} \in \left[0, \frac{1}{j}\right[^n$ and $\mathbf{1} \in \left[\frac{j-1}{j}, 1\right]^n$, then $C'_{\alpha_v}(\mathbf{0}) = \frac{\alpha(v_1)}{\alpha_k} = 0$ and $C'_{\alpha_v}(\mathbf{1}) = \frac{\alpha(v_k)}{\alpha_k} = 1$. Consider $x, y \in I^n$ such that, $x \preceq y$, let us show that $X^x \preceq Y^y$. Suppose the contrary, this means that $X_i^x > Y_i^y$ for some player i. We pose $l_i = Y_i^y$ and $k_i = X_i^x$, then $0 \leq l_i \leq j-2$ and $l_i + 1 \leq \ k_i$. Since $Y_i^y = l_i \leq j-2$, it follows that $y_i < \dfrac{l_i+1}{j} \leq \dfrac{k_i}{j} \leq x_i$. Hence $y_i < x_i$. This contradicts the fact that $x \preceq y$. Therefore $X^x \preceq Y^y$. By monotonicity of v, one obtains $v(X^x) \leq v(Y^y)$. This implies $\dfrac{\alpha(v(X^x))}{\alpha_k} \leq \dfrac{\alpha(v(Y^y))}{\alpha_k}$ i.e. $C'_{\alpha_v}(x) \leq C'_{\alpha_v}(y)$. Finally, $C'_{\alpha_v} \in \mathcal{CSG}_n$.

2. We now prove that T' is a one-to-one mapping. From item 1, T' is well defined. Let u and v be two (j,k) simple games with numeric evaluation α such that, $T'(u) = T'(v)$. Consider $y \in J^n$, then $y = X^x$ where $x \in I^n$ is defined by $x_i = \dfrac{y_i}{j}$ for all $i \in N$. Since $T'(u) = T'(v)$, then $C'_{\alpha_u}(x) = C'_{\alpha_v}(x)$ that is $\alpha(u(X^x)) = \alpha(v(X^x))$ i.e. $\alpha(u(y)) = \alpha(v(y))$, this implies $u(y) = v(y)$; thus $u = v$. We conclude that T' is a one-to-one mapping.

∎

Using very similar arguments to those of Proposition 1.3.2, one can easily check that, the embedding T' transforms a null player (resp. two symmetric players) into a null player (resp. two symmetric players).

1.3.2 The Shapley-Shubik index from $\mathcal{SG}_n$ to $\mathcal{CSG}_n$

In Proposition 1.3.1, we have shown that simple games can be embedded in to CSGs via T^τ. But one may now be dubious whether the Shapley-Shubik index Ψ for CSGs is a natural extension of the Shapley-Shubik index SSI for simple games. In this section, we provide positive arguments to discard such a worrying concern by showing that, each transformation T^τ for a given $\tau \in \left]0,1\right]$ preserves the SSI power index for simple games. More precisely, we show that, $\Psi\left(T^\tau\left(v\right)\right) = \mathrm{SSI}\left(v\right)$ for all $v \in \mathcal{SG}_n$. This equality allows us to conclude that the Shapley-Shubik index Ψ on $\mathcal{CSG}_n$ is an extension of that defined on $\mathcal{SG}_n$. In other words, the SSI index can be seen as a discretization of Ψ. To establish the main result of this section, we need the following propositions:

Proposition 1.3.4. Let $n \geq 2$ be an integer. Consider the mapping f defined on $]0,\, 1[$ by :

$$f(x) = 1 + x + x^2 + \cdots + x^{n-1} \tag{1.18}$$

and denote by $f^{(k)}$ the k^{th} derivative of f. Then, for all $x \in]0,\, 1[$ and for all $1 \leq k \leq n-1$,

$$(1-x)f^{(k)}(x) - kf^{(k-1)}(x) = -\frac{n!}{(n-k)!}x^{n-k} \tag{1.19}$$

Moreover for all k, with $1 \leq k \leq n-1$,

$$f^{(k)}(x) = \sum_{s=0}^{n-k-1} \frac{(k+s)!}{s!}x^s \tag{1.20}$$

Proof.

We first note that, $f^{(0)} = f$. Consider $x \in]0, 1[$, following Equation (1.18), $f(x)$ can be rewritten as $f(x) = \dfrac{1-x^n}{1-x}$, thus $(1-x)f(x) = 1 - x^n$. Since $(x^n)^{(k)} = \dfrac{n!}{(n-k)!}x^{n-k}$ for $1 \leq k \leq n$, then by Leibniz derivative formula, we have

$$(1-x)f^{(k)}(x) - kf^{(k-1)}(x) = -\frac{n!}{(n-k)!}x^{n-k}$$

for all $1 \leq k \leq n-1$.

Moreover, for all $x \in]0, 1[$, $f(x) = \sum\limits_{s=0}^{n-1} x^s$ thus,

$$f^{(k)}(x) = \sum_{s=k}^{n-1} \frac{s!}{(s-k)!}x^{s-k} = \sum_{s=0}^{n-k-1} \frac{(k+s)!}{s!}x^s$$

∎

Notation 1.3.1. Consider the mapping f defined by Equation (1.18) and $\tau \in]0,\, 1[$. For all $1 \leq k \leq n$, we pose:

- $K_1\,(n,\, k,\, \tau) = (n-k)!\,(1-\tau)^k\, f^{(k-1)}(\tau)$;
- $K_2\,(n,\, k,\, \tau) = \tau^{n-k}\,(k-1)!f^{(n-k)}(1-\tau)$.

Proposition 1.3.5. Considering the previous notations, we obtain:

$$K_1\,(n,\, k,\, \tau) + \tau K_2\,(n,\, k,\, \tau) = (k-1)!(n-k)! \quad \text{for all} \quad 1 \leq k \leq n \tag{1.21}$$

Proof. Given $k \in \{1, \cdots, n\}$, consider the following assertion :

$$\mathcal{A}(k) : K_1(n,\, k,\, \tau) + \tau K_2(n,\, k,\, \tau) = (k-1)!(n-k)!$$

We prove by induction on $k \in \{1, \cdots, n\}$ that $\mathcal{A}(k)$ holds.

Suppose that $k = 1$, then $K_1(n,\, 1,\, \tau) + \tau K_2(n,\, 1,\, \tau) = (n-1)!(1-\tau^n) + (n-1)!\tau^n = (n-1)!\,(1-\tau^n+\tau^n) = (1-1)!(n-1)!$. It follows that $\mathcal{A}(1)$ holds.

Now assume that $\mathcal{A}(k)$ holds for some $k \in \{1, \cdots, n-1\}$, we prove that, $\mathcal{A}(k+1)$ holds.

$$\begin{aligned}
K_1(n,\, k+1,\, \tau) &= (n-k-1)!\,(1-\tau)^{k+1} f^{(k)}(\tau) \\
&= (n-k-1)!\,(1-\tau)^{k+1} \left(\frac{k}{1-\tau} f^{(k-1)}(\tau) - \frac{n!\tau^{n-k}}{(n-k)!(1-\tau)}\right) \quad \text{by Equation (1.19)} \\
&= \frac{k}{n-k}\left((n-k)!(1-\tau)^k f^{(k-1)}(\tau)\right) - \frac{n!}{n-k}\tau^{n-k}(1-\tau)^k \\
&= \frac{k}{n-k} K_1(n,\, k,\, \tau) - \frac{n!}{n-k}\tau^{n-k}(1-\tau)^k
\end{aligned}$$

and

$$\begin{aligned}
K_2(n,\, k+1,\, \tau) &= k!\tau^{n-k-1} f^{(n-k-1)}(1-\tau) \\
&= k!\tau^{n-k-1}\left(\frac{\tau}{n-k} f^{(n-k)}(1-\tau) + \frac{n!(1-\tau)^k}{(n-k)k!}\right) \quad \text{by Equation (1.19)} \\
&= \frac{k}{n-k}\left((k-1)!\tau^{n-k} f^{(k-1)}(\tau)\right) + \frac{n!}{n-k}\tau^{n-k-1}(1-\tau)^k \\
&= \frac{k}{n-k} K_2(n,\, k,\, \tau) + \frac{n!}{n-k}(1-\tau)^k\tau^{n-k-1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& K_1(n,\, k+1,\, \tau) + \tau K_2(n,\, k+1,\, \tau) \\
=\ & \frac{k}{n-k} K_1(n,\, k,\, \tau) - \frac{n!}{n-k}(1-\tau)^k\tau^{n-k} + \frac{k}{n-k}\cdot \tau K_2(n,\, k,\, \tau) + \frac{n!}{n-k}(1-\tau)^k\tau^{n-k} \\
=\ & \frac{k}{n-k}\left(K_1(n,\, k,\, \tau) + \tau K_2(n,\, k,\, \tau)\right) \\
=\ & \frac{k}{n-k}(k-1)!(n-k)! \quad \text{since } \mathcal{A}(k) \text{ holds} \\
=\ & k!(n-k-1)!
\end{aligned}$$

In conclusion, for all $1 \le k \le n$ we have $K_1(n,\, k,\, \tau) + \tau K_2(n,\, k,\, \tau) = (k-1)!(n-k)!$. ■

Theorem 1.3.1.

Given $\tau \in]0,\, 1]$; $\Psi(T^\tau(v)) = \mathrm{SSI}(v)$ for all $v \in \mathcal{SG}_n$. ★

Proof. Consider $v \in \mathcal{SG}_n$, $i \in N$ and $\tau \in\,]0,\, 1]$. First suppose that $\tau = 1$. Then for all

$S \subseteq N$, we have :

$$\begin{aligned}\mathcal{C}\left(C_v^\tau,\, S\right) =& \int_{I^n} C_v^\tau\left(\mathbf{1}_S,\, x_{-S}\right) - C_v^\tau\left(\mathbf{0}_S,\, x_{-S}\right)\, dx\\ =& \int_{L^n} v\left(\theta_\tau(\mathbf{1}_S,\, x_{-S})\right) - v\left(\theta_\tau(\mathbf{0}_S,\, x_{-S})\right)\, dx \quad \text{where} \quad L =]0,\, 1[\\ =& v(S) - v(\emptyset) = v(S) \quad \text{since} \quad \tau = 1\,.\end{aligned}$$

By substituting $v\left(S\right)$ to $\mathcal{C}\left(T^\tau\left(v\right),\, S\right)$ in Equation (1.17), one obtains $\Psi_i\left(T^\tau(v)\right) = \mathrm{SSI}_i(v)$.

Now, suppose that $0 < \tau < 1$. Then for all $S \subseteq N$,

$$\begin{aligned}\mathcal{C}\left(C_v^\tau,\, S\right) =& \int_{I^n} C_v^\tau\left(\mathbf{1}_S,\, x_{-S}\right) - C_v^\tau\left(\mathbf{0}_S,\, x_{-S}\right)\, dx\\ =& \int_{L^n} v\left(\theta_\tau(\mathbf{1}_S,\, x_{-S})\right) - v\left(\theta_\tau(\mathbf{0}_S,\, x_{-S})\right)\, dx\\ =& \sum_{T \subseteq N\setminus S} \tau^{n-s-t}(1-\tau)^t\left[v\left(S \cup T\right) - v\left(T\right)\right] \quad \text{by definition of } \theta_\tau\end{aligned}$$

Hence, for all $i \in N$,

$$\begin{aligned}\Psi_i\left(C_v^\tau\right) =& \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \sum_{T \subseteq S^c} \tau^{n-t-s}(1-\tau)^t\left[v\left(S \cup T\right) - v\left(T\right)\right]\\ & - \sum_{S \subseteq N\setminus\{i\}} \frac{s!(n-s-1)!}{n!} \sum_{T \subseteq S^c} \tau^{n-t-s}(1-\tau)^t\left[v\left(S \cup T\right) - v\left(T\right)\right]\\ =& \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4\end{aligned}$$

where

$$\begin{aligned}\Sigma_1 &= -\sum_{S: i \in S} \frac{(s-1)!\,(n-s)!}{n!} \sum_{\emptyset \neq T \subseteq S^c} (1-\tau)^t\, \tau^{n-s-t} v\left(T\right)\\ &= -\sum_{\emptyset \neq K: i \notin K} v\left(K\right) \sum_{S: i \in S \subseteq K^c} \frac{(s-1)!\,(n-s)!}{n!} (1-\tau)^k\, \tau^{n-s-k}\\ &= -\sum_{\emptyset \neq K: i \notin K} v\left(K\right)(1-\tau)^k \sum_{s=1}^{n-k} \begin{pmatrix} n-k-1\\ s-1\end{pmatrix} \frac{(s-1)!\,(n-s)!}{n!}\tau^{n-s-k} \text{ since } i \in S \subseteq K^c\\ &= -\sum_{K: i \notin K} v\left(K\right) \frac{(n-k-1)!}{n!} (1-\tau)^k \sum_{s=0}^{n-k-1} \frac{(k+s)!}{s!}\tau^s\\ &= -\frac{1}{1-\tau}\left(\sum_{K: i \notin K} \frac{v\left(K\right)}{n!} \cdot K_1\left(n,\, k+1,\, ,\tau\right)\right) \text{(by Equation (1.20) and Notation 1.3.1.)}\end{aligned}$$

$$\begin{aligned}\Sigma_2 &= \sum_{S \subseteq N\setminus\{i\}} \frac{s!\,(n-s-1)!}{n!} \sum_{T \subseteq S^c} (1-\tau)^t\, \tau^{n-s-t} v\left(T\right)\\ &= \sum_{K \subseteq N} v\left(K\right)(1-\tau)^k \sum_{S: i \notin S \subseteq K^c} \frac{s!\,(n-s-1)!}{n!}\tau^{n-k-s}\\ &= \sum_{K: i \in K} v\left(K\right)(1-\tau)^k \sum_{s=0}^{n-k} \begin{pmatrix} n-k\\ s\end{pmatrix} \frac{s!\,(n-s-1)!}{n!}\tau^{n-k-s}\\ &\quad + \sum_{K: i \notin K} v\left(K\right)(1-\tau)^k \sum_{s=0}^{n-k-1} \begin{pmatrix} n-k-1\\ s\end{pmatrix} \frac{s!\,(n-s-1)!}{n!}\tau^{n-k-s}\\ &= \sum_{K: i \in K} v\left(K\right) \frac{(n-k)!}{n!} (1-\tau)^k \sum_{s=0}^{n-k} \frac{(k+s-1)!}{s!}\tau^s + \sum_{K: i \notin K} v\left(K\right) \frac{(n-k-1)!}{n!} (1-\tau)^k \sum_{s=0}^{n-k-1} \frac{(k+s)!}{s!}\tau^{s+1}\\ &= \sum_{K: i \in K} \frac{v\left(K\right)}{n!} \cdot K_1\left(n,\, k,\, \tau\right) + \frac{\tau}{1-\tau}\left(\sum_{K: i \notin K} \frac{v\left(K\right)}{n!} \cdot K_1\left(n,\, k+1,\, \tau\right)\right)\end{aligned}$$

$$\begin{aligned}
\Sigma_3 &= \sum_{S:i\in S} \frac{(s-1)!\,(n-s)!}{n!} \sum_{T\subseteq S^c} (1-\tau)^t \tau^{n-s-t} v(S\cup T)\\
&= \sum_{K:i\in K} v(K) \sum_{S:i\in S\subseteq K} \frac{(s-1)!\,(n-s)!}{n!} (1-\tau)^{k-s}\tau^{n-k}\\
&= \sum_{K:i\in K} v(K) \sum_{s=1}^{k} \begin{pmatrix} s-1\\ k-1 \end{pmatrix} \frac{(s-1)!\,(n-s)!}{n!} (1-\tau)^{k-s}\tau^{n-k} \text{ since } i\in S\subseteq K\\
&= \sum_{K:i\in K} v(K)\tau^{n-k}\frac{(k-1)!}{n!}\sum_{s=0}^{k-1}\frac{(n-k+s)!}{s!}(1-\tau)^s = \sum_{K:i\in K}\frac{v(K)}{n!}\cdot K_2(n,k,\tau)
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_4 &= -\sum_{S\subseteq N\setminus\{i\}} \frac{s!\,(n-s-1)!}{n!} \sum_{T\subseteq S^c} (1-\tau)^t\tau^{n-s-t} v(S\cup T)\\
&= -\sum_{K:i\in K} v(K) \sum_{S:S\subseteq K\setminus\{i\}} \frac{s!\,(n-s-1)!}{n!}(1-\tau)^{k-s}\tau^{n-k} - \sum_{K:i\notin K} v(K)\sum_{S:S\subseteq K}\frac{s!\,(n-s-1)!}{n!}(1-\tau)^{k-s}\tau^{n-k}\\
&= -\sum_{K:i\in K} v(K)\sum_{s=0}^{k-1}\begin{pmatrix} s\\ k-1\end{pmatrix}\frac{s!\,(n-s-1)!}{n!}(1-\tau)^{k-s}\tau^{n-k}\\
&\quad -\sum_{K:i\notin K} v(K)\sum_{s=0}^{k}\begin{pmatrix} s\\ k\end{pmatrix}\frac{s!\,(n-s-1)!}{n!}(1-\tau)^{k-s}\tau^{n-k}\\
&= -\sum_{K:i\in K} v(K)\frac{(k-1)!}{n!}\tau^{n-k}\sum_{s=0}^{k-1}\frac{(n-k+s)!}{s!}(1-\tau)^{s+1} - \sum_{K:i\notin K} v(K)\tau^{n-k}\frac{k!}{n!}\sum_{s=0}^{k}\frac{(n-k+s-1)!}{s!}(1-\tau)^s\\
&= -(1-\tau)\sum_{K:i\in K}\frac{v(K)}{n!}\cdot K_2(n,k,\tau) - \tau\cdot\sum_{K:i\notin K}\frac{v(K)}{n!}\cdot K_2(n,k+1,\tau)
\end{aligned}$$

Therefore, the share $\Psi_i(C_v^\tau)$ can be rewriting as:

$$\Psi_i(C_v^\tau) = \sum_{K:i\in K}\frac{v(K)}{n!}\left(K_1(n,k,\tau)+\tau K_2(n,k,\tau)\right) - \sum_{K:i\notin K}\frac{v(K)}{n!}\left(K_1(n,k+1,\tau)+\tau K_2(n,k+1,\tau)\right) \tag{1.22}$$

Hence by Proposition 1.3.5, Equation (1.22) becomes:

$$\Psi_i(C_v^\tau) = \sum_{K:i\in K}\frac{v(K)}{n!}(k-1)!(n-k)! - \sum_{K:i\notin K}\frac{v(K)}{n!}k!(n-k-1)! = \mathrm{SSI}_i(v).$$

Finally, we obtain $\Psi(T^\tau(v)) = \mathrm{SSI}(v)$. ■

1.3.3 The Shapley-Shubik index from $\mathcal{JK}_n$ to $\mathcal{CSG}_n$

We prove that, the Shapley-Shubik index for CSGs is an extension of that for (j,k) simple games with uniform numerical evaluation. To achieve this, we show that the embedding T' (see, Proposition (1.3.3)) preserves the Shapley-Shubik index Φ. For $j=2$, a more general result is obtained by using parameterized embedding as in the case of simple games.

Consider $e=(e_1,e_2,\cdots,e_n)\in J^n$. We denote by $\mathcal{B}_e$ the cartesian product $I_{e_1}\times I_{e_2}\times\cdots\times I_{e_n}$ of intervals, where each interval I_{e_i} is given by:

$$I_{e_i} = \begin{cases} \left[\frac{e_i}{j}, \frac{e_i+1}{j}\right[& \text{if } e_i < j-1 \\ \left[\frac{j-1}{j}, 1\right] & \text{otherwise} \end{cases}$$

PROPOSITION 1.3.6. *The collection $\{\mathcal{B}_e, e \in J^n\}$ is a partition of I^n and the n-dimensional volume of each $\mathcal{B}_e$ is equal to $\frac{1}{j^n}$.*

Proof. We first prove that $I^n = \bigcup_{e\in J^n} \mathcal{B}_e$. Note that, for each $e \in J^n$, $\mathcal{B}_e \subseteq I^n$, thus $\bigcup_{e\in J^n} \mathcal{B}_e \subseteq I^n$. Consider $x \in I^n$, for all $i \in N$. We pose $e_i = \lfloor jx_i \rfloor$ if $x_i < 1$ and $e_i = j-1$ otherwise. We easily check that, $x \in \mathcal{B}_e$, then $I^n \subseteq \bigcup_{e\in J^n} \mathcal{B}_e$. Finally we have $I^n = \bigcup_{e\in J^n} \mathcal{B}_e$.

Now consider $e, e' \in J^n$ such that $e \neq e'$. Then $e_i \neq e'_i$ for some $i \in N$. Consider $i \in N$, with $e_i \neq e'_i$ and assume that $I_{e_i} \cap I_{e'_i} \neq \emptyset$. Then there is some $t \in [0;1]$ such that $t \in I_{e_i} \cap I_{e'_i}$. W.l.o.g., suppose that $e_i < e'_i$, then $e_i < j-1$. Since $t \in I_{e_i} \cap I_{e'_i}$ and $e_i < j-1$, then $t \neq 1$, therefore we have:

$$\frac{e_i}{j} \leq t < \frac{e_i+1}{j} \quad \text{and} \quad \frac{e'_i}{j} \leq t < \frac{e'_i+1}{j} \tag{1.23}$$

It follows from the relation (1.23) that $e_i = e'_i$. A contradiction arises since $e_i \neq e'_i$. So $e \neq e'$ implies $\mathcal{B}_e \cap \mathcal{B}_{e'} = \emptyset$. ■

DEFINITION 1.3.2. *For integers $j, k \geq 2$, let v be a (j,k) simple game with uniform numerical evaluation. A natural embedding of v is a CSG $\hat{v}$ defined by $\hat{v}(x) = \frac{v(e)}{k-1}$ for all $x \in I^n$ such that $x \in \mathcal{B}_e$, for some $e \in J^n$.*

By Proposition 1.3.3, we remark that $T'(v) = \hat{v}$.

Theorem 1.3.2.

For integers $j, k \geq 2$, let v be a (j,k) simple game and $\hat{v}$ its natural embedding, then $\Psi(\hat{v}) = \Phi(v)$.

Proof. Due to Equations (1.1.3) and (1.17), it is sufficient to verify the coincidence of the two different expressions for $\mathcal{C}_v(T)$ and $\mathcal{C}(\hat{v},T)$ for all $T \subseteq N$.

$$\begin{aligned}\mathcal{C}(\hat{v},T) &= \int_{I^n} (\hat{v}(\mathbf{1}_T, x_{-T}) - \hat{v}(\mathbf{0}_T, x_{-T}))\,\mathrm{d}\,x \\ &= \sum_{e\in J^n} \int_{\mathcal{B}_e} (\hat{v}(\mathbf{1}_T, x_{-T}) - \hat{v}(\mathbf{0}_T, x_{-T}))\,\mathrm{d}\,x \quad \text{(by Proposition 1.3.6)} \\ &= \frac{1}{k-1}\cdot \sum_{e\in J^n} [\mathrm{vol}(\mathcal{B}_e)\cdot v((\mathbf{j}-\mathbf{1})_T, e_{-T}) - v(\mathbf{0}_T, e_{-T})] \\ &= \frac{1}{j^n(k-1)}\cdot \sum_{x\in J^n} [v((\mathbf{j}-\mathbf{1})_T, x_{-T}) - v(\mathbf{0}_T, x_{-T})] = \mathcal{C}_v(T).\end{aligned}$$

Since $\mathcal{C}_v(T) = \mathcal{C}(\hat{v},T)$ for all $T \subseteq N$, we then conclude that $\Psi(\hat{v}) = \Phi(v)$. ■

For $j = 2$, there is an even more general statement. For any parameter $\tau \in]0,1]$, we can replace the natural embedding $\hat{v}$ by the embedding $\widehat{v_\tau}$ defined by:

$$\widehat{v_\tau}(x) = \frac{v(l^\tau(x))}{k-1} \tag{1.24}$$

where for all $x \in I^n$, $l^\tau(x) \in J^n$ is a profile such that, for all $i \in N$, $l_i^\tau(x) = 0$ if $x_i \in [0,\tau[$ and $l_i^\tau(x) = 1$ otherwise. We prove below that $\Psi(\widehat{v_\tau}) = \Phi(v)$, for all $\tau \in]0,\,1]$.

But before, note that $\mathcal{C}_v(T)$ and $\mathcal{C}(\widehat{v_\tau},T)$ do not necessarily coincide for any coalition T as in the proof of Theorem 1.3.2.

Example 1.3.1. Consider $\tau \in]0,\,1]$. To see that $\mathcal{C}_v(T)$ and $\mathcal{C}(\widehat{v_\tau},T)$ may not coincide, consider the n-player $(2,\,k)$ simple game defined as follows:

$$v\,(x) = 1 \text{ if } x = \mathbf{1} \text{ and } v\,(x) = 0 \text{ otherwise}$$

It follows that, $\widehat{v_\tau}\,(x) = 1$ if $x \in [\tau,1]^n$ and $v\,(x) = 0$ otherwise. Thus, for $S \subset N$, we have

$$\mathcal{C}_v\,(S) = \frac{1}{2^{n-s}} \quad \text{and} \quad \mathcal{C}\,(\widehat{v_\tau},\,S) = (1-\tau)^{n-s}.$$

Therefore, $\mathcal{C}_v\,(S) \neq \mathcal{C}\,(\widehat{v_\tau},\,S)$ for any $\tau \neq \frac{1}{2}$.

Theorem 1.3.3.

Given an integer $k \geq 2$ and $\tau \in]0,1]$, let v be a $(2,k)$ simple game with uniform numerical evaluation and $\widehat{v_\tau}$ its parametric embedding. Then, $\Psi(\widehat{v_\tau}) = \Phi(v)$.

Proof. Due to Example 1.3.1, we adopt a different approach from that used in the proof of Theorem 1.3.2.

Consider $\tau \in]0,\,1]$ and $i \in N$ a given player. Since $j = 2$, we have:

$$\Phi_i\left(v\right) = \sum_{i\in S\subseteq N}\theta(s,k)\times\sum_{x\in J^n}\left(\left[v(\mathbf{1}_S,x_{-S})-v(\mathbf{0}_S,x_{-S})\right]-\left[v(\mathbf{1}_{S\setminus\{i\}},x_{-(S\setminus\{i\})})-v(\mathbf{0}_{S\setminus\{i\}},x_{-(S\setminus\{i\})})\right]\right)$$

where $\theta(s,k)=\dfrac{(s-1)!(n-s)!}{n!2^n(k-1)}$. Furthermore,

$$\begin{aligned}\Psi_i(\widehat{v_\tau}) &= \sum_{i\in S\subseteq N}\frac{(s-1)!(n-s)!}{n!(k-1)}\cdot\int_{I^n}\left(\left[v(l^\tau(\mathbf{1}_S,x_{-S}))-v(l^\tau(\mathbf{0}_S,x_{-S}))\right]\right.\\ &\quad\left.-\left[v(l^\tau(\mathbf{1}_{S\setminus\{i\}},x_{-(S\setminus\{i\})}))-v(l^\tau(\mathbf{0}_{S\setminus\{i\}},x_{-(S\setminus\{i\})}))\right]\right)dx\end{aligned}$$

Following Equation (1.24), one concludes that the ranges of $\widehat{v_\tau}$ and v coincide. Thus, by collecting all terms that contain each $v\left(x\right), x\in J^n$, one gets

$$\Phi_i\left(v\right)=\sum_{x\in J^n}a\left(i,x\right)v\left(x\right). \tag{1.25}$$

and

$$\Psi_i(\widehat{v_\tau})=\sum_{x\in J^n}b\left(\tau,\,i,\,x\right)v\left(x\right). \tag{1.26}$$

To complete the proof, we show that for all $x\in J^n$, $a(i,\,x)=b(\tau,\,i,\,x)$. For this purpose, consider the following notations in the sequel,

$$\theta\left(s,k\right)=\frac{(s-1)!(n-s)!}{n!2^n(k-1)},\quad \beta(s,k)=2^n\cdot\theta(s,\,k),\quad\text{and}\quad \varepsilon=1-\tau\,.$$

For each $y\in J^n$, we pose

$$A_y=\left\{p\in N: y_p=1\right\}, B_y=\left\{p\in N: y_p=0\right\}\ \text{and}\ D_y=\left\{x\in[0,1]^n:\widehat{v_\tau}\left(x\right)=\frac{v\left(y\right)}{k-1}\right\}.$$

Therefore, each term of $\Psi_i(\widehat{v_\tau})$ that depends on $v\left(y\right)$ comes from an integration over a subset of D_y. By definition of $\widehat{v_\tau}$, $D_y=[0,\tau[^{B_y}\times[\tau,1]^{A_y}$. We also note that, A_y and B_y form a partition of N.

Suppose that $i\in A_y$. We compute $a(i,\,y)$ and $b(\tau,\,i,\,y)$.

$$\begin{aligned} a\left(i,y\right) &= \sum_{i\in S\subseteq A_y}\theta\left(s,k\right)\sum_{\substack{x\in J^n\\ x_{-S}=y_{-S}}}1-\sum_{i\in S\subseteq A_y:S\setminus\{i\}\neq\emptyset}\theta\left(s,k\right)\sum_{\substack{x\in J^n\\ x_{S^c+i}=y_{S^c+i}}}1+\sum_{i\in S:\emptyset\neq S\setminus\{i\}\subseteq B_y}\theta\left(s,k\right)\sum_{\substack{x\in J^n\\ x_{S^c+i}=y_{S^c+i}}}1\\ &= \sum_{i\in S\subseteq A_y}\theta\left(s,k\right)\cdot 2^s-\sum_{i\in S\subseteq A_y:S\setminus\{i\}\neq\emptyset}\theta\left(s,k\right)2^{s-1}+\sum_{i\in S:\emptyset\neq S\setminus\{i\}\subseteq B_y}\theta\left(s,k\right)\cdot 2^{s-1}\\ &= \frac{(n-1)!}{n!2^{n-1}(k-1)}+\sum_{s=2}^{a_y}\binom{a_y-1}{s-1}\frac{(s-1)!(n-s)!}{n!2^n(k-1)}2^{s-1}+\sum_{s=2}^{b_y+1}\binom{b_y}{s-1}\frac{(s-1)!(n-s)!}{n!2^n(k-1)}2^{s-1}\\ &= \frac{(n-1)!}{n!2^{n-1}(k-1)}+\frac{(a_y-1)!}{n!2^{b_y+1}(k-1)}\sum_{p=0}^{a_y-2}\frac{(b_y+p)!}{p!}2^{-p}+\frac{(b_y)!2^{b_y}}{n!2^n(k-1)}\sum_{p=0}^{b_y-1}\frac{(a_y-1+p)!}{p!}2^{-p}\end{aligned}$$

Thus by Equation (1.20), we obtain:

$$a(i,y)=2\cdot\theta(1,\,k)+\frac{(a_y-1)!}{n!2^{b_y+1}(k-1)}\cdot g^{(b_y)}\left(\frac{1}{2}\right)+\frac{(b_y)!}{n!2^{a_y}(k-1)}\cdot g^{(a_y-1)}\left(\frac{1}{2}\right)=F_1(a_y,\,b_y) \tag{1.27}$$

where $g(x) = f(x) - x^{n-1}$ and f is the function defined by Equation (1.18).

In order to compute $b(\tau, i, y)$, we pose $D_{y,T} = [0,\tau]^{B_y\cap T} \times [\tau,1]^{A_y\cap T}$ for all $T \subseteq N$. It follows that:

$$\begin{aligned}
b(\tau,\, i,\, y) &= \sum_{i\in S\subseteq A_y} \beta(s,k) \int_{[0,1]^S\times D_{y,\,S^c}} dx - \sum_{\{i\}\subsetneqq S\subseteq A_y} \beta(s,k) \int_{[0,1]^{S\setminus\{i\}}\times D_{y,\,S^c+i}} dx \\
&\quad + \sum_{\{i\}\subsetneqq S:S\setminus\{i\}\subseteq B_y} \beta(s,k) \int_{[0,1]^{S\setminus\{i\}}\times D_{y,\,S^c+i}} dx \\
&= \sum_{i\in S\subseteq A_y} \beta(s,k)\varepsilon^{a_y-s}\tau^{b_y} - \sum_{i\in S\subseteq A_y:S\setminus\{i\}\neq\emptyset} \beta(s,k)\varepsilon^{a_y-s+1}\tau^{b_y} + \sum_{i\in S:\emptyset\neq S\setminus\{i\}\subseteq B_y} \beta(s,k)\varepsilon^{a_y}\tau^{b_y-s+1} \\
&= \beta(1,k)\tau^{b_y}\varepsilon^{a_y-1} + \tau^{b_y+1}\sum_{s=2}^{a_y}\beta(s,k)\begin{pmatrix} a_y-1 \\ s-1\end{pmatrix}\varepsilon^{a_y-s+1} + \varepsilon^{a_y}\sum_{s=2}^{b_y+1}\beta(s,k)\begin{pmatrix} b_y \\ s-1\end{pmatrix}\tau^{b_y-s+1} \\
&= \beta(1,k)\tau^{b_y}\varepsilon^{a_y-1} + \frac{\tau^{b_y+1}(a_y-1)!}{n!\,(k-1)}\sum_{p=0}^{a_y-2}\frac{(b_y+p)!}{p!}\varepsilon^p + \frac{(b_y)!\varepsilon^{a_y}}{n!\,(k-1)}\sum_{p=0}^{b_y-1}\frac{(a_y-1+p)!}{p!}\tau^p
\end{aligned}$$

Hence,

$$b(\tau,\, i,\, y) = \beta(1)\,\tau^{b_y}\varepsilon^{a_y-1} + \frac{\tau^{b_y+1}(a_y-1)!}{n!\,(k-1)}g^{(b_y)}(\varepsilon) + \frac{(b_y)!\varepsilon^{a_y}}{n!\,(k-1)}g^{(a_y-1)}(\tau) = F_2(\tau,\, a_y,\, b_y)\,. \tag{1.28}$$

We conclude from Equations (1.27) and (1.28) that:

$$a(i,y) = b\left(\frac{1}{2},\, i,\, y\right). \tag{1.29}$$

Furthermore, by differentiating $F_2(\tau,\, a_y,\, b_y)$ with respect to τ, we obtain:

$$\begin{aligned}
&(k-1)F_2'(\tau,\, a_y,\, b_y) \\
&= \frac{(1+b_y)\tau^{b_y}(a_y-1)!}{n!}g^{(b_y)}(\varepsilon) - \frac{\tau^{1+b_y}(a_y-1)!}{n!}g^{(1+b_y)}(\varepsilon) + \frac{b_y!\varepsilon^{a_y}}{n!}g^{(a_y)}(\tau) \\
&\quad - \frac{b_y!\varepsilon^{a_y-1}}{n!}g^{(a_y-1)}(\tau) + \frac{b_y\tau^{b_y-1}\varepsilon^{a_y-1}}{n} - \frac{(a_y-1)\tau^{b_y}\varepsilon^{a_y-2}}{n} \\
&= \tfrac{\tau^{b_y}(a_y-1)!}{n!}\left((1+b_y)g^{(b_y)}(\varepsilon) - (1-\varepsilon)g^{(1+b_y)}(\varepsilon)\right) + \tfrac{b_y!\varepsilon^{a_y-1}}{n!}\left((1-\tau)g^{(a_y)}(\tau) - a_y g^{(a_y-1)}(\tau)\right) \\
&\quad + \frac{b_y\tau^{b_y-1}\varepsilon^{a_y-1}}{n} - \frac{(a_y-1)\tau^{b_y}\varepsilon^{a_y-2}}{n} \\
&= \frac{(a_y-1)\tau^{b_y}\varepsilon^{a_y-2}}{n} - \frac{b_y\tau^{b_y-1}\varepsilon^{a_y-1}}{n} + \frac{b_y\tau^{b_y-1}\varepsilon^{a_y-1}}{n} - \frac{(a_y-1)\tau^{b_y}\varepsilon^{a_y-2}}{n} = 0 \quad \text{(by Equation (1.19))}
\end{aligned}$$

It follows that, $F_2(\tau,\, a_y,\, b_y)$ is independent of any arbitrary value of τ. So, for all $\tau \in]0;\, 1]$, for all $y \in J^n$ and for all $i \in A_y$,

$$b(\tau,\, i,\, y) = F_2(\tau,\, a_y,\, b_y) = b\left(\frac{1}{2},\, i,\, y\right) = a(i,\, y) = F_1(a_y,\, a_y)\,. \tag{1.30}$$

Now assume that, $i \in B_y$. We compute $a(i,\, y)$ and $b(\tau, i,\, y)$.

$$\begin{aligned}
a(i,y) &= -\sum_{i\in S\subseteq B_y}\theta(s,k)\sum_{\substack{x\in J^n \\ x_{-S}=y_{-S}}} 1 - \sum_{i\in S:\emptyset\neq S\setminus\{i\}\subseteq A_y}\theta(s,k)\sum_{\substack{x\in J^n \\ x_{S^c+i}=y_{S^c+i}}} 1 + \sum_{i\in S\subseteq B_y:S\setminus\{i\}\neq\emptyset}\theta(s,k)\sum_{\substack{x\in J^n \\ x_{S^c+i}=y_{S^c+i}}} 1 \\
&= -2\cdot\theta(1,k) - \frac{(b_y-1)!}{n!2^{a_y+1}(k-1)}\sum_{p=0}^{b_y-2}\frac{(a_y+p)!}{p!}2^{-p} - \frac{(a_y)!}{n!2^{b_y}(k-1)}\sum_{p=0}^{a_y-1}\frac{(b_y-1+p)!}{p!}2^{-p} \\
&= = -F_1(b_y,\, a_y)\,.
\end{aligned}$$

and

$$\begin{aligned}
b\,(\tau, i, y) &= -\sum_{i\in S\subseteq B_y}\int_{[0,1]^S\times D_{y,S^c}}\beta\,(s,k)dx - \sum_{\{i\}\subsetneq S:S\setminus\{i\}\subseteq A_y}\int_{[0,1]^{S\setminus\{i\}}\times D_{y,S^c+i}}\beta\,(s,k)dx \\
&\quad + \sum_{\{i\}\subsetneq S\subseteq B_y}\int_{[0,1]^{S\setminus\{i\}}\times D_{y,S^c+i}}\beta\,(s,k)dx \\
&= -\beta\,(1,k)\varepsilon^{a_y}\tau^{b_y-1} - \frac{\varepsilon^{a_y+1}\,(b_y-1)!}{n!(k-1)}\sum_{p=0}^{b_{y-2}}\frac{(a_y+p)!}{p!}\varepsilon^p - \frac{a_y!\tau^{b_y}}{n!(k-1)}\sum_{p=0}^{a_y-1}\frac{(b_y-1+p)!}{p!}\varepsilon^p \\
&= = -F_2(\varepsilon,\, b_y,\, a_y)\,.
\end{aligned}$$

As shown above, $F_2(\varepsilon,\, b_y,\, a_y)$ is independent of ε and $F_2(\frac{1}{2},\, b_y,\, a_y) = F_1(b_y,\, a_y)$, thus,

$$b\,(\tau,\, i,\, y) = -F_2(\tau,\, a_y,\, b_y) = b\left(\frac{1}{2},\, i,\, y\right) = a(i,\, y) = -F_1(a_y,\, a_y)\,. \tag{1.31}$$

Finally, for all $i\in N$, for all $y\in J^n$, $b\,(\tau,\, i,\, y) = a(i,y)$. Hence, by Equations (1.25) and (1.26), we conclude that, $\Psi(\widehat{v_\tau}) = \Phi\,(v)$ for all $(2,k)$ simple games v and all $\tau\in]0,\, 1]$. ■

The presentation of simple games, (j,k) simple games and CSGs as well as the Shapley-Shubik index of these classes of games raises several research problems among which the question of axiomatizing power indices with appropriate sets of axioms that highlight the intuition and ideas behind each power index. In the next chapter, we focus on the axiomatization of the Shapley-Shubik power index for (j,k) simple games, a problem that remains open for the past decade.

⋆ ⋆ Chapter Two ⋆ ⋆

Axiomatization of the Shapley-Shubik index for (j, k) simple games

From an axiomatic approach, Shapley (1953) introduced a function, the so-called Shapley value, that maps each TU-game with individual shares of the grand coalition in such a way that, the share of each player corresponds to his/her expected utility in the game. An axiomatization of the Shapley-Shubik index, the restriction of the Shapley value on the set of simple games, was given quite a few years later by Dubey (1975). A Shapley-Shubik index for (j, k) simple games was introduced in Freixas (2005b) with an axiomatic justification only for $k = 2$ and $j \geq 2$, see (Freixas, 2005b, Theorem 5.1). Here, we fill this gap by providing a characterization of the Shapley-Shubik index (Φ) for arbitrary (j, k) with the uniform numerical evaluation; see Kurz et al. (2020).

This chapter, partially published in Kurz et al. (2020) comprises three sections presented as follows. Section 2.1 is devoted to a presentation of the notions of (j, k) *simple game with point-veto* and *average game*, immediately followed by some preliminary results. An axiomatization of Φ is presented in Section 2.2. We end this chapter in Section 2.3, with an introduction of a new axiom of power indices for (j, k) simple games called *Symmetry Gain Loss** which is as an extension of the symmetry gain-loss axiom introduced by Laruelle and Valenciano (2001) on the class of simple games.

To emphasis on our assumption of uniform numerical evaluation, a (j, k) simple game will be called "*uniform* (j, k) *simple game*" and the set of such games denoted by $\mathcal{U}_n^{j,k}$ (or $\mathcal{U}_n$, whenever j and k are clear from the context).

2.1 Preliminaries

In this section, we mostly present a special subclass of uniform (j, k) simple games and introduce an operator that associates each uniform (j, k) simple game v with a TU-game $\widetilde{v}$ called *average game*. Additionally, we present some intuitive axioms of power indices for uniform (j, k) simple games.

2.1.1 Uniform (j,k) simple games with point-veto

We introduce a subclass of uniform (j,k) simple games with the property that for each profile x, the collective decision $v(x)$ is either 0 *(the lowest level of approval)* or it is $k-1$ *(the highest level of approval)* depending on whether some given players report some minimum approval levels. For example, when any full support of the proposal needs a full support of each player in a given coalition S, players in S are each empowered with a veto. One may require from each player in S only a certain level of approval for a full support of the proposal. All such games will be called uniform (j,k) simple games with point-veto.

Definition 2.1.1. *A uniform (j,k) simple game with a point-veto is a (j,k) simple game v such that there exists some $a \in J^n \backslash \{\mathbf{0}\}$ satisfying*

$$v(x) = \begin{cases} k-1 & \text{if} \quad a \preceq x \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } x \in J^n. \tag{2.1}$$

In this case, a is the veto and the game v is denoted by u^a. For each coalition $S \in 2^N$, we abbreviate $w^S = u^a$, where $a_i = j-1$ for all $i \in S$ and $a_i = 0$ otherwise.

We remark that uniform $(2,2)$ simple games with a point veto are in one-to-one correspondence to the subclass of unanimity games within simple games presented in Definition 1.1.5, page 7. Hereafter, given a veto a, the set of all players who report a non-null approval level is denoted by N^a, i.e. $N^a = \{i \in N : 0 < a_i \leq j-1\}$. Every player in N^a will be called a *vetoer* of the game u^a. Note that for the profile a defined via $w^S = u^a$, we have $N^a = S$.

Null players as well as symmetric players can be identified easily in a given uniform (j,k) simple game with point-veto:

Proposition 2.1.1. *Consider $a \in J^n \backslash \{\mathbf{0}\}$.*

- *a player $i \in N$ is a null player of u^a iff $i \in N \backslash N^a$;*
- *two players $i, h \in N$ are symmetric in u^a iff $a_i = a_h$.*

Proof.

- Consider $a \in J^n \backslash \{\mathbf{0}\}$ and $i \in N \backslash N^a$, then $a_i = 0$ by the definition of N^a. Therefore, for any $x \in J^n$ and any $y_i \in J$, $a \preceq x$ iff $a \preceq (x_{-i}, y_i)$. Thus, $u^a(x) = u^a(x_{-i}, y_i)$, i.e. i is a null player in u^a. Now consider $i \in N^a$, then $a_i > 0$. Since $u^a(a) = k-1 \neq 0 = u^a(a_{-i}, 0)$, player i is not a null player in u^a.
- Consider $i, h \in N$ and $a \in J^n \backslash \{\mathbf{0}\}$. If $a_i = a_h$, then for any $x \in J^n$, $a \preceq x$ iff $a \preceq \pi_{ih} x$. Thus, the definition of u^a leads to $u^a(x) = u^a(\pi_{ih} x)$, so i and h are symmetric in u^a.

Now assume that the players i and h are symmetric in u^a. Since $a \preceq a$, we obtain $u^a(a) = u^a(\pi_{ih}a) = k-1$. This implies $a \preceq \pi_{ih}a$. Hence, $a_i \le a_h$ and $a_h \le a_i$, that is $a_i = a_h$.

■

Note that uniform (j,k) simple games can be combined using the disjunction $(\vee)$ or the conjunction $(\wedge)$ operations to obtain new games.

Definition 2.1.2. *Let v and v' be two uniform (j,k) simple games with player set N. By $v \vee v'$ and $v \wedge v'$ we denote the uniform (j,k) simple game defined as follows:*

$$\text{for } x \in J^n,\ (v \vee v')(x) = \max\{v(x),\, v'(x)\} \text{ and } (v \wedge v')(x) = \min\{v(x),\, v'(x)\}\,.$$

One can easily check that the above mentioned games are uniform (j,k) simple games. This can be specialized to the subclass of uniform (j,k) simple games with point veto, i.e. uniform (j,k) simple games with point-veto can be combined using the disjunction $(\vee)$ or the conjunction $(\wedge)$ operations to obtain new games. To see this, consider a non-empty subset E of $J^n \backslash \{\mathbf{0}\}$ and define the uniform (j,k) simple game denoted by u^E as follows:

$$\text{for all } x \in J^n, \quad u^E(x) = \begin{cases} k-1 & \text{if } a \preceq x \text{ for some } a \in E \\ 0 & \text{otherwise} \end{cases}$$

Note that for any $a \in J^n \backslash \{\mathbf{0}\}$, $u^{\{a\}} = u^a$ (see Definition 2.1.1).

Given $a, b \in J^n$ $c = \max\{a,b\}$ is the element of J^n defined by $c_i = \max\{a_i, b_i\}$ for all $i \in N$.

Proposition 2.1.2. *Let E and E' be two non-empty subsets of $J^n \backslash \{\mathbf{0}\}$. Then, we have $u^E \vee u^{E'} = u^{E \cup E'}$ and $u^E \wedge u^{E'} = u^{E''}$, where $E'' = \{max\{a,b\},\, a \in E \text{ and } b \in E'\}$.*

Proof.

In order to prove that $u^E \vee u^{E'} = u^{E\cup E'}$ we consider an arbitrary $x \in J^n$. If $u^{E\cup E'}(x) = k-1$, then there exists $a \in E \cup E'$ such that, $a \preceq x$. Therefore $u^E(x) = k-1$ or $u^{E'}(x) = k-1$ and $(u^E \vee u^{E'})(x) = k-1$. Now suppose that $u^{E\cup E'}(x) = 0$. Then, for all $a \in E \cup E'$, $a \npreceq x$. Since $E \subseteq E \cup E'$ and $E' \subseteq E \cup E'$ then $b \npreceq x$ and $c \npreceq x$ for all $b \in E$ and all $c \in E'$. This implies $u^E(x) = u^{E'}(x) = 0$ and $(u^E \vee u^{E'})(x) = 0$. Thus, $u^E \vee u^{E'} = u^{E\cup E'}$.

Similarly to prove that $u^E \wedge u^{E'} = u^{E''}$, consider an arbitrary $x \in J^n$. If $u^{E''}(x) = k-1$, then there exists $c \in E''$ such that $c \preceq x$. But, by the definition of E'', $c = \max(a,b)$ for some $a \in E$ and $b \in E'$, that is $a \preceq c \preceq x$ and $b \preceq c \preceq x$. Hence, $u^E(x) = u^{E'}(x) = k-1$ and $(u^E \wedge u^{E'})(x) = k-1$. Now assume that $u^{E''}(x) = 0$ and $(u^E \wedge u^{E'})(x) \neq 0$. By the definition of u^E and $u^{E'}$ we have, $(u^E \wedge u^{E'})(x) = k-1$. Thus, there exists $a \in E$ and $b \in E'$ such that $a \preceq x$ and $b \preceq x$. It follows that $c = \max(a,b) \preceq x$, which is a contradiction to $u^{E''}(x) = 0$. This proves that $u^E \wedge u^{E'} = u^{E''}$. ■

Example 2.1.1. *For $(j,k) = (5,3)$ and $n = 3$, pose $E = \{(1,2,3),(2,1,2)\}$, $E' = \{(4,1,1),(1,1,3)\}$. With this, $E'' = \{(4,2,3),(1,2,3),(2,1,3),(4,1,2)\}$. Note that one may have $E'' = \{(1,2,3),(2,1,3),(4,1,2)\}$ since $(1,2,3) \preceq (4,2,3)$ or $(4,1,2) \preceq (4,2,3)$.*

Especially, Proposition 2.1.2 yields that every uniform (j,k) simple game of the form u^E is a disjunction of some uniform (j,k) simple games with point-veto. So, each uniform (j,k) simple game of the form u^E will be called a uniform (j,k) simple game with *veto*.

The previous proposition leads to the following technical result:

Corollary 2.1.1. *Let E be a non-empty subset of $J^n \backslash \{\mathbf{0}\}$. Then,*

$$u^E = \bigvee_{a\in E} u^a = \sum_{l=1}^{|E|} (-1)^{l-1} \left(\sum_{L\subseteq E,\, |L|=l} u^{\max\{a, a\in L\}} \right) \quad \text{and} \quad \bigwedge_{a\in E} u^a = u^{\max\{a, a\in E\}}.$$

In the game u^E, E can be viewed as some minimum requirements (or thresholds) on the approval levels of players' inputs for the full support of the proposal. It is worth noticing that u^E is $\{0, k-1\}$-valued; the final decision at all profiles is either a no-support or a full-support. In the context of uniform (2, 2) simple games, u^E corresponds to the simple game whose minimal winning coalitions are the subsets N^a, $a \in E$. The set of all veto uniform (j,k) simple games on N is denoted $\mathcal{V}_n$. Note that Proposition 2.1.2 shows that $\mathcal{V}_n$ is a lattice.

The sum of two uniform (j,k) simple games cannot be a uniform (j,k) simple game. However, we will show that each uniform (j,k) simple game is a convex combination of uniform (j,k) simple games with veto.

Definition 2.1.3. *A convex combination of the games $v_1, v_2, \ldots, v_p \in \mathcal{U}_n$ is given by $v = \sum_{t=1}^{p} \alpha_t \cdot v_t$ for some non-negative numbers α_t, $t = 1,2,\ldots,p$, such that $\sum_{t=1}^{p} \alpha_t = 1$.*

Note that there exists convex combinations of uniform (j,k) simple games that are not uniform (j,k) simple games. As an example, consider the uniform $(3,3)$ simple games u and v with two players such that, $u(2,1) = 2$ and $v(2,1) = 1$. Then $w = \frac{1}{2}u + \frac{1}{2}v \notin \mathcal{U}_2$, since $w(2,1) = \frac{3}{2} \notin \{0,1,2\}$.

Proposition 2.1.3. *For each uniform (j,k) simple game v there exists a collection of positive numbers $(\alpha_t)_{1\leq t\leq p}$ such that $\sum_{t=1}^{p} \alpha_t = 1$ and a collection $(F_t(v))_{1\leq t\leq p}$ of non-empty subsets of J^n such that $v = \sum_{t=1}^{p} \alpha_t \cdot u^{F_t(v)}$.*

Proof.

Consider $v \in \mathcal{U}_n$. If $v = u^E$ for some $E \subseteq J^n \backslash \{\mathbf{0}\}$, the result is obvious. Now assume that v is not a veto game and pose $\mathcal{F}(v) = \{x \in J^n, v(x) > 0\}$. Since J^n is finite and v is monotone, the elements of $\mathcal{F}(v)$ can be labeled in such a way that $\mathcal{F}(v) = \{x^1, x^2, \ldots, x^p\}$, where $x^p = \mathbf{j-1}$, $v(x^t) \leq v(x^{t+1})$ for all $1 \leq t < p$, and $t \leq s$ whenever $x^t \preceq x^s$. Now, set

$x^0 = \mathbf{0}$ and $F_t(v) = \{x^s,\ t \leq s \leq p\}$, $\alpha_t = \frac{v(x^t)-v(x^{t-1})}{k-1}$ for all $1 \leq t \leq p$. By assumption on x^t, $\alpha_t \geq 0$ for all $1 \leq t \leq p$. Moreover,

$$\sum_{t=1}^{p} \alpha_t = \frac{1}{k-1} \cdot \sum_{t=1}^{p} v(x^t) - v(x^{t-1}) = \frac{v(x^p) - v(x^0)}{k-1} = 1\,.$$

Pose $u = \sum_{t=1}^{p} \alpha_t \cdot u^{F_t(v)}$. In order to prove that $v = u$, we consider $x \in J^n$. First suppose that $x \notin \mathcal{F}(v)$. Since v is monotone, there is no $a \in \mathcal{F}(v)$ such that $a \preceq x$. By definition, it follows that $v^{F_t(v)}(x) = 0$ for all $t = 1, 2, \ldots, p$. Therefore $v(x) = u(x) = 0$. Now assume that $x \in \mathcal{F}(v)$, then $x = x^s$ for some $s = 1, 2, \ldots, p$. It follows that for all $t = 1, 2, \ldots, p$, $v^{F_t(v)}(x) = k - 1$ if $1 \leq t \leq s$ and $v^{F_t(v)}(x) = 0$ otherwise. For this reason,

$$u(x) = \sum_{t=1}^{s} \alpha_t \cdot (k-1) = \sum_{t=1}^{s} \left[\frac{v(x^t) - v(x^{t-1})}{k-1} \cdot (k-1)\right] = v(x^s) = v(x).$$

Clearly, the game v is a convex combination of the games $u^{F_t(v)}$, where $t = 1, 2, \ldots, p$. ∎

Proposition 2.1.3 underlines the importance of uniform (j, k) simple games with veto, i.e. any uniform (j, k) simple game can be obtained from uniform (j, k) simple games with veto via convex combination. To illustrate the Proposition 2.1.3, we consider the following example.

Example 2.1.2. *Let v be the $(3, 4)$ simple game v for 2-players defined by*

x	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$
$v(x)$	0	1	2	0	1	2	1	3	3

We have $\mathcal{F}(v) = \{(1,0); (1,1); (0,2); (2,0); (2,1); (1,2); (2,2)\}$ and the decomposition of v as convex combination of uniform $(3, 4)$ simple games with veto is given by:

$$v = \frac{1}{3}u^{F_1} + \frac{1}{3}u^{F_2} + \frac{1}{3}u^{F_3}$$

where $F_1 = \mathcal{F}(v)$; $F_2 = \{(2,0); (2,1); (1,2); (2,2)\}$ and $F_3 = \{(1,2); (2,2)\}$.

We end this section with the following useful result on TU-games.

Proposition 2.1.4. *Let v be a TU-game on N. If the player $i \in N$ is a null player in v then, there exists a collection $(x_S^v)_{S \in 2^N}$ of real numbers such that,*

$$v = \sum_{i \notin S \in 2^N} x_S^v \cdot \gamma_S{}^{1}\,. \tag{2.2}$$

Proof.

[1]The definition of unanimity games has already been given in Definition 1.1.5, page 7.

It is well known that the set Γ^N of all TU-games on N is a vector space which a basis is $(\gamma_S)_{S\in 2^N}$, see (Shapley, 1953, Lemma 3). So, for a given TU-game v on N, there exists a unique collection $(x_S^v)_{S\in 2^N}$ of real numbers such that

$$v = \sum_{S\in 2^N} x_S^v \cdot \gamma_S \tag{2.3}$$

where the coefficients

$$x_S^v = \sum_{T\subseteq S} (-1)^{s-t} \cdot v(T)$$

are the well known Harsanyi (1963) dividends.

Now let $i \in N$ be a null player in v. Consider $S \in 2^N$ such that $i \in S$, then:

$$\begin{aligned} x_S^v &= \sum_{i\notin T\subseteq S} (-1)^{s-t} \cdot v(T) + \sum_{i\in T\subseteq S} (-1)^{s-t} \cdot v(T) \\ &= \sum_{i\notin T\subseteq S} (-1)^{s-t} \cdot v(T) + \sum_{i\notin L\subseteq S} (-1)^{s-l-1} \cdot v(L\cup\{i\}) \quad \text{with } L = T\setminus\{i\} \\ &= \sum_{i\notin T\subseteq S} (-1)^{s-t-1} \left[v(T\cup\{i\}) - v\left(T\right)\right] = 0 \quad \text{since } i \text{ is null player in } v. \end{aligned}$$

Finally, Equation (2.3) implies,

$$v = \sum_{i\in S\in 2^N} x_S^v \cdot \gamma_S + \sum_{i\notin S\in 2^N} x_S^v \cdot \gamma_S = \sum_{i\notin S\in 2^N} x_S^v \cdot \gamma_S$$

■

2.1.2 The average game of a uniform (j, k) simple game

In Chapter 1, given a (j, k) simple game v, we derive a TU-game $\mathcal{C}_v$ such that the Shapley-Shubik index of v coincides with the Shapley value of $\mathcal{C}_v$. In this section, we pay a particular attention to this new defined TU-game called *average game*. The average game itself seems to be a very natural object on its own and have some nice properties. Indeed, they are used to obtain another formula of the Shapley-Shubik index for (j, k) simple games which is better suited for computation issues.

Definition 2.1.4. *Let $v \in \mathcal{U}_n$ be an arbitrary uniform (j, k) simple game. The* average game*, denoted by $\widetilde{v}$, associated with v is defined by*

$$\widetilde{v}(S) = \frac{1}{j^n(k-1)} \sum_{x\in J^n} \left[v((\mathbf{j-1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})\right] \quad \text{for all } S \subseteq N. \tag{2.4}$$

Remark 2.1.1. *The Equation (2.4) can be simplified to:*

$$\widetilde{v}(S) = \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S}\in J^{n-s}} \left[v((\mathbf{j-1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})\right] \quad \text{for all } S \subseteq N. \tag{2.5}$$

Given a uniform (j,k) simple game v, the average game $\widetilde{v}$ can be seen as a coalitional representation of v. With that notation the Theorem 1.1.2, page 14 can be restated as follows:

Theorem 2.1.1.

For any uniform (j,k) simple game v the Shapley-Shubik power index $\Phi(v)$ is equals to the Shapley value of $\widetilde{v}$. More precisely,

$$\Phi(v) = \text{Shap}(\widetilde{v}) \tag{2.6}$$

Example 2.1.3. *Consider the uniform $(3,4)$ simple game v from Example 2.1.2. The average simple game is given by*

$$\tilde{v}(\emptyset) = 0, \tilde{v}(\{1\}) = \frac{2}{3}, \tilde{v}(\{2\}) = \frac{4}{9}, \text{ and } \tilde{v}(N) = 1,$$

Consequently,

$$\Phi(v) = \text{Shap}(\widetilde{v}) = \left(\frac{11}{18}, \frac{7}{18}\right).$$

As illustrated in the following example, two distinct uniform (j,k) simple games may have the same average game. This highlights the fact that the average game operator is not injective.

Example 2.1.4. *Consider $u, v \in \mathcal{U}_n$ defined as follows. For all $x \in J^n$,*

- *$u(x) = k-1$ if $x = \mathbf{j-1}$ and $u(x) = 0$ otherwise;*
- *$v(x) = k-1$ if $x \neq \mathbf{0}$ and $v(x) = 0$ otherwise*

Consider $S \in 2^N$, then following Equation (2.5) we have:

$$\widetilde{u}(S) = \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S}=(\mathbf{j-1})_{-S}} [u((\mathbf{j-1})_S, x_{-S}) - u(\mathbf{0}_S, x_{-S})] = \frac{1}{j^{n-s}},$$

and

$$\widetilde{v}(S) = \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S}=\mathbf{0}_{-S}} [v((\mathbf{j-1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] = \frac{1}{j^{n-s}}.$$

It is obvious that $u \neq v$, however $\widetilde{u} = \widetilde{v}$.

The average game operator has some nice properties among which are the following:

Proposition 2.1.5. *Given a uniform (j,k) simple game $v \in \mathcal{U}_n$,*

(a) $\widetilde{v}$ is a TU-game on N that is $[0,1]$-valued and monotone;

(b) any null player in v is a null player in $\widetilde{v}$;

(c) any two symmetric players in v are symmetric in $\widetilde{v}$;

(d) if $v = \sum_{t=1}^{p} \alpha_t v_t$ is a linear combination for some $v_1, \ldots, v_p \in \mathcal{U}_n$, then $\widetilde{v} = \sum_{t=1}^{p} \alpha_t \widetilde{v}_t$.

Proof. Let $v \in \mathcal{U}_n$ be an uniform (j,k) simple game with n players.

(a) By definition,

$$\widetilde{v}(\emptyset) = \frac{1}{j^n(k-1)} \sum_{x \in J^n} [v(x) - v(x)] = 0 \quad \text{and} \quad \widetilde{v}(N) = \frac{1}{j^n(k-1)} \sum_{x \in J^n} [v((\mathbf{j-1})) - v(\mathbf{0})] = 1\,.$$

Now consider $x \in J^n$ and $S, T \in 2^N$ such that $S \subseteq T$. Since $\mathbf{0} \preceq x \preceq \mathbf{j-1}$ and v is monotone, then $v((\mathbf{j-1})_S, x_{-S}) \leq v((\mathbf{j-1})_T, x_{-T})$ and $v(\mathbf{0}_S, x_{-S}) \geq v(\mathbf{0}_T, x_{-T})$. Thus, by Equation (2.4) one concludes that $0 \leq \widetilde{v}(S) \leq \widetilde{v}(T) \leq 1$.

(b) Let $i \in N$ be a null player in v. Consider $S \subseteq N\backslash\{i\}$. Since i is null in v, then $v(x_{S\cup\{i\}}, y_{-(S\cup\{i\})}) = v(x_S, y_{-S})$ for all $x, y \in J^n$. Therefore,

$$\begin{aligned}\widetilde{v}(S \cup \{i\}) =& \frac{1}{j^n(k-1)} \sum_{x \in J^n} \left[v((\mathbf{j-1})_{S\cup\{i\}}, x_{-(S\cup\{i\})}) - v(\mathbf{0}_{S\cup\{i\}}, x_{-(S\cup\{i\})}) \right] \\ =& \frac{1}{j^n(k-1)} \sum_{x \in J^n} [v((\mathbf{j-1})_S,\, x_{-S}) - v(\mathbf{0}_S,\, x_{-S})] \text{ since } i \text{ is a null player in } v \\ =& \widetilde{v}(S)\,.\end{aligned}$$

Thus, $\widetilde{v}(S \cup \{i\}) = \widetilde{v}(S)$. It follows that i is a null player in $\widetilde{v}$.

(c) Let $i, h \in N$ be two symmetric players in v, $S \subseteq N\backslash\{i,h\}$. Then, for any $x \in J^n$ and any $a \in J$, $v(\mathbf{a}_{S\cup\{i\}}, x_{-(S\cup\{i\})}) = v(\mathbf{a}_{S\cup\{h\}}, (\theta_{ih}x)_{-(S\cup\{h\})})$. It follows that,

$$\begin{aligned}\widetilde{v}(S \cup \{i\}) &= \frac{1}{j^n(k-1)} \sum_{x \in J^n} \left[v((\mathbf{j-1})_{S\cup\{i\}}, x_{-(S\cup\{i\})}) - v(\mathbf{0}_{S\cup\{i\}}, x_{-(S\cup\{i\})}) \right] \\ &= \frac{1}{j^n(k-1)} \sum_{x \in J^n} \left[v((\mathbf{j-1})_{S\cup\{h\}}, (\theta_{ih}x)_{-(S\cup\{h\})}) - v(\mathbf{0}_{S\cup\{h\}}, (\theta_{ih}x)_{-(S\cup\{h\})}) \right] \\ &= \frac{1}{j^n(k-1)} \sum_{y \in J^n} \left[v((\mathbf{j-1})_{S\cup\{h\}}, y_{-(S\cup\{h\})}) - v(\mathbf{0}_{S\cup\{h\}}, y_{-(S\cup\{h\})}) \right] \text{ with } y = \theta_{ih}(x) \\ &= \widetilde{v}(S \cup \{h\})\,.\end{aligned}$$

We conclude that, players i and h are symmetric in $\widetilde{v}$.

(d) Now suppose that $v = \sum_{t=1}^{p} \alpha_t v_t$ is a linear combination for some $v_1, v_2, \cdots, v_p \in \mathcal{U}_n$. Since for any $x \in J^n$ and for any $a \in J$, $v(\mathbf{a}_S,\, x_{-S}) = \sum_{t=1}^{p} \alpha_t v_t(\mathbf{a}_S,\, x_{-S})$, Equation (2.4) gives $\widetilde{v}(S) = \sum_{t=1}^{p} \alpha_t \widetilde{v}_t(S)$ for all $\emptyset \subseteq S \subseteq N$.

■

Proposition 2.1.5 highlights the fact that the average game of a given game preserves some properties of that game. The average game of a uniform (j,k) simple game with a point-veto is provided by:

PROPOSITION 2.1.6. *Given $a \in J^n \backslash \{\mathbf{0}\}$, the average game $\widetilde{u^a}$ satisfies for every coalition $S \neq N$*

$$\widetilde{u^a}(S) = \begin{cases} \prod_{i \in N \backslash S} \left(\frac{j - a_i}{j} \right) & \text{if} \quad S \cap N^a \neq \emptyset \\ 0 & \text{if} \quad S \cap N^a = \emptyset \end{cases} \tag{2.7}$$

Proof. Consider $a \in J^n \backslash \{\mathbf{0}\}$ and $\emptyset \subseteq S \subset N$. First assume that $S \cap N^a = \emptyset$. Then, for all $x \in J^n$, $a \preceq ((\mathbf{j-1})_S, x_{-S})$ iff $a \preceq (\mathbf{0}_S, x_{-S})$. Thus, $u^a(((\mathbf{j-1})_S, x_{-S})) = u^a((\mathbf{0}_S, x_{-S}))$. It follows from Equation (2.4) that $\widetilde{u^a}(S) = 0$.

Now suppose that $S \cap N^a \neq \emptyset$. Then, for all $x \in J^n$, $a \npreceq (\mathbf{0}_S, x_{-S})$. Thus, $u^a((\mathbf{0}_S, x_{-S})) = 0$. Note that $a \preceq ((\mathbf{j-1})_S, x_{-S})$ iff $a_{-S} \preceq x_{-S}$. Hence, by Equation (2.5) we obtain:

$$\begin{aligned} \widetilde{u^a}(S) &= \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S} \in J^{n-s}} u^a((\mathbf{j-1})_S, x_{-S}) \\ &= \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S} \in J^{n-s} \wedge a_{-S} \preceq x_{-S}} u^a((\mathbf{j-1})_S, x_{-S}) \\ &= \frac{1}{k-1} \cdot (k-1) \frac{|\{x_{-S} \in J^{n-s},\, a_{-S} \preceq x_{-S}\}|}{j^{n-s}} \\ &= \frac{\left| \times_{i \in N \backslash S} \{a_i, \cdots, j-1\} \right|}{j^{n-s}} \\ &= \prod_{i \in N \backslash S} \left(\frac{j - a_i}{j} \right). \end{aligned}$$

In both cases, $\widetilde{u^a}(S)$ is completely determined. ■

It may be interesting to check whether or not each uniform (j,k) simple game may be decomposed as a combination of uniform (j,k) simple game with a point-veto of the form $a \in \{0, j-1\}^n$. The response is affirmative when one considers combinations between average games. Before this, recall that the average game associated with each uniform (j,k) simple game is a TU-game on N. The set of all TU-games on N is vector space and a famous basis consists in all unanimity games $(\gamma_S)_{S \in 2^N}$, where $\gamma_S(T) = 1$ if $S \subseteq T$ and $\gamma_S(T) = 0$ otherwise.

Let us recall the notation provided in Definition 2.1.1. For $S \in 2^N$, $w^S = u^a$ where $a \in J^n$ is specified by $a_i = j-1$ if $i \in S$ and $a_i = 0$ otherwise.

Proposition 2.1.7. *For every coalition $C \in 2^N$, there exists a collection of real numbers $(y_S)_{S\in 2^C}$ such that*

$$\widetilde{w^C} = \sum_{S\in 2^C} y_S \cdot \gamma_S.$$

Proof. Note that $\widetilde{w^C}$ is a TU-game on N. Therefore, for some real numbers $(y_S)_{S\in 2^N}$ we have

$$\widetilde{w^C} = \sum_{S\in 2^N} y_S \cdot \gamma_S. \tag{2.8}$$

This proves the result for $C = N$.

Now, suppose that $C \neq N$ and pose $\mathcal{E}_k = \{T \in 2^N, T\backslash C \neq \emptyset \quad \text{and} \quad |T| = k\}$ for $1 \leq k \leq n$. We prove by induction on k that $y_T = 0$ for all coalitions $T \in \mathcal{E}_k$. Consider the assertion $\mathcal{P}(k)$: for all $T \in \mathcal{E}_k$, $y_T = 0$.

First assume that, $k = 1$. Consider $T \in \mathcal{E}_k$, then there exists $i \in N\backslash C$ such that $T = \{i\}$. Since player i is not contained in C, Propositions 2.1.1 and 2.1.5 yield that i is a null player in $\widetilde{w^C}$, so that $\widetilde{w^C}(T) = 0$. Since Equation (2.8) implies $\widetilde{w^C}(T) = \sum_{S\in 2^T} y_S = y_T$, then $y_T = 0$. Therefore $\mathcal{P}(1)$ holds.

Now consider $2 \leq k \leq n$ and suppose that $\mathcal{P}(l)$ holds for all $1 \leq l < k$. Consider $T \in \mathcal{E}_k$, then there exists $i \in N\backslash C$ such that $T = K \cup \{i\}$, $i \notin K \neq \emptyset$. Since i is a null player in $\widetilde{w^C}$, then $\widetilde{w^C}(T) - \widetilde{w^C}(K) = 0$. By Equation (2.8) we have:

$$0 = \widetilde{w^C}(T) - \widetilde{w^C}(K) = \sum_{S\in 2^T} y_S - \sum_{S\in 2^K} y_S = y_T + \sum_{i\in S\subsetneq T} y_S = y_T$$

using $S\backslash C \neq \emptyset$ and $1 \leq |S| < |T| = k$. Thus $y_T = 0$, which proves that $\mathcal{P}(k)$ holds.

One concludes that, for any $C \neq N$ and all $T \in 2^N$ such that $T\backslash C \neq \emptyset$, $y_T = 0$. So, Equation (2.8) is reduced to

$$\widetilde{w^C} = \sum_{S\in 2^C} y_S \cdot \gamma_S\,.$$

∎

Lemma 2.1.1. *For every uniform (j, k) simple game $u \in \mathcal{U}_n$, there exists a collection of real numbers $(x_S)_{S\in 2^N}$ such that*

$$\widetilde{u} = \sum_{S\in 2^N} x_S \cdot \widetilde{w^S}. \tag{2.9}$$

Proof. Let u be a uniform (j, k) simple. First assume that $j = 2$, from Propositions 2.1.3 and 2.1.5 we can write

$$\widetilde{u} = \sum_{t=1}^{p} \alpha_t \cdot \widetilde{u^{F_t}}\,. \tag{2.10}$$

Besides, for any $F_t \subseteq J^n \backslash \{\mathbf{0}\}$, Corollary 2.1.1 and Proposition 2.1.5 give:

$$\widetilde{u^{F_t}} = \sum_{l=1}^{|F_t|} (-1)^{l-1} \sum_{L \subseteq F_t, |L|=l} \widetilde{u^{M^L}}, \text{ where } M^L = \max\{a,\, a \in L\}\,. \tag{2.11}$$

Since $j = 2$ then $u^{M^L} = w^S$ with $S = \{i \in N,\, M_i^L = 1\}$. Thus by substituting each $\widetilde{u^{F_t}}$ in Equation (2.10) by its expression found in Equation (2.11), we collect all the terms that lead to the same $\widetilde{w^S}$ and then write $\widetilde{u}$ as a linear combination of $\widetilde{w^S}$, $S \in 2^N$.

Now suppose that $j \geq 3$. Note that all TU-games on N can be written as a linear combination of unanimity games $(\gamma_S)_{S \in 2^N}$. It is then sufficient to only prove that each TU-game γ_C for $C \in 2^N$ is a linear combination of the TU-games $(\widetilde{w^S})_{S \in 2^C}$. The proof is done by induction on $1 \leq k = |C| \leq n$. More precisely, we prove the assertion $\mathcal{A}(k)$ that for all $C \in 2^N$ such that $|C| \leq k$, there exists a collection $(z_S)_{S \in 2^C}$ such that

$$\gamma_C = \sum_{S \in 2^C} z_S \widetilde{w^S}. \tag{2.12}$$

First assume that $k = 1$. Using Proposition 2.1.6, it can be easily checked that we have $\gamma_{\{i\}} = \widetilde{w^{\{i\}}}$ for all $i \in N$. Therefore $\mathcal{A}(1)$ holds. Now, consider a coalition C such that $|C| = k \in \{2, \ldots, n\}$ and assume that $\mathcal{A}(l)$ holds for all l such that $1 \leq l < k$. By Proposition 2.1.7, there exists some real numbers $(\alpha_S)_{S \in 2^C}$ and $(\beta_S)_{S \in 2^C \backslash \{C\}}$ such that

$$\widetilde{w^C} = \sum_{S \in 2^C} \alpha_S \cdot \gamma_S = \alpha_C \cdot \gamma_C + \sum_{S \in 2^C \backslash \{C\}} \alpha_S \cdot \gamma_S = \alpha_C \cdot \gamma_C + \sum_{S \in 2^C \backslash \{C\}} \beta_S \cdot \widetilde{w^S}\,.$$

where the last equality holds by the induction hypothesis. Moreover, α_C can be determined using Proposition 2.1.6 for $c = |C|$ by:

$$\alpha_C = \sum_{S \in 2^C} (-1)^{|C \backslash S|} \widetilde{w^C}(S) = \sum_{s=1}^{c} (-1)^{c-s} \binom{c}{s} \left(\frac{1}{j}\right)^{c-s} = \frac{(j-1)^c - (-1)^c}{j^c} \neq 0 \text{ since } j-1 \geq 2.$$

Therefore we get

$$\gamma_C = \sum_{S \in 2^C} z_S \cdot \widetilde{w^S}$$

where for all $S \in 2^C$, $z_S = -\frac{1}{\alpha_C}$ if $S = C$ and $z_S = -\frac{\beta_S}{\alpha_C}$ otherwise. This gives $\mathcal{A}(k)$. In summary, each $\gamma_S, S \in 2^N$ is a linear combination of $\widetilde{w^C}, C \in 2^N$. Thus, the proof is completed since $\widetilde{u}$ is a linear combination of $\gamma_S, S \in 2^N$ ■

Before we continue, note that by Equation (2.12), for $C \in 2^N$ each TU-game γ_C is a linear combination of the TU-games $\left(\widetilde{w^S}\right)_{S \in 2^N}$. Since $(\gamma_S)_{S \in 2^N}$ is a basis of the vector space Γ^N, it follows that $\left(\widetilde{w^S}\right)_{S \in 2^N}$ is also a basis of Γ^N *(using incomplete basis theorem)*. Form Lemma 2.1.1 we extract the following technical result.

Corollary 2.1.2. *If a player i is null in the game u, then Equation (2.9) can be* reduced to:

$$\widetilde{u} = \sum_{S \in 2^{N \setminus \{i\}}} x_S \cdot \widetilde{w^S}. \tag{2.13}$$

Proof.

Consider a null player i in the uniform (j, k) simple game u and let show that, in Equation (2.9), $x_S = 0$ for all $S \in 2^N$ such that $i \in S$. Since $\widetilde{u}$ is a TU-game on N and i is null in $\widetilde{u}$, then Propositions 2.1.4 and 2.1.5 yield,

$$\begin{aligned} \widetilde{u} &= \sum_{i \notin S \subseteq N} x_S^{\widetilde{u}} . \gamma_S \\ &= \sum_{i \notin S \subseteq N} x_S^{\widetilde{u}} \left(\sum_{T \in 2^S} z_T \cdot \widetilde{w^T} \right) \quad \text{by Equation (2.12)} \end{aligned}$$

The last equality is a linear combination of games $\widetilde{w^T}$, $T \in 2^N$ such that $i \notin T$. Since $(\widetilde{w^S})_{S \in 2^N}$ is a basis of Γ^N, then by Equation (2.9), we conclude that $x_S = 0$ for all $S \in 2^N$ such that $i \in S$. So, Equation (2.9) is reduced to Equation (2.13). ■

2.1.3 Axioms of power indices on $\mathcal{U}_n$

We present some desirable axioms of the power indices for uniform (j, k) simple games.

Definition 2.1.5. *A power index F on $\mathcal{U}_n$ is a mapping that associates each game v with a n-tuple $(F_i(v))_{i \in N}$ of real numbers. F satisfies:*

- ***positivity*** *(P) if $F(v) \neq \mathbf{0}$ and $F_i(v) \geq 0$ for all $i \in N$ and all $v \in \mathcal{U}_n$;*
- ***anonymity*** *(A) if $F_{\pi(i)}(\pi v) = F_i(v)$ for all permutations π of N, $i \in N$, and $v \in \mathcal{U}_n$, where $\pi v(x) = v(\pi(x))$ and $\pi(x) = \left(x_{\pi(i)}\right)_{i \in N}$;*
- ***symmetry*** *(S) if $F_i(v) = F_j(v)$ for all $v \in \mathcal{U}_n$ and all players $i, j \in N$ that are symmetric in v;*
- ***efficiency*** *(E) if $\sum_{i \in N} F_i(v) = 1$ for all $v \in \mathcal{U}_n$;*
- *the null player property (NP) if $F_i(v) = 0$ for every game $v \in \mathcal{U}_n$ such that i is a null player in v;*
- *the **transfer property** (T) if for any $u, v \in \mathcal{U}_n$ and any $i \in N$, $F_i(u) + F_i(v) = F_i(u \vee v) + F_i(u \wedge v)$;*
- ***convexity*** *(C) if $F(w) = \sum_{t=1}^{p} \alpha_t \cdot F(u_t)$ for any $u_1, u_2, \cdots u_p \in \mathcal{U}_n$ and any $(\alpha_t)_{1 \leq t \leq p}$, non-negative numbers such that $\sum_{t=1}^{p} \alpha_t = 1$; with $w = \sum_{t=1}^{p} \alpha_t \cdot u_t \in \mathcal{U}_n$;*

- ***linearity*** (L) *if* $F(w) = \sum_{t=1}^{p} \alpha_t \cdot F(u_t)$ *for all* $u_1, u_2, \cdots u_p \in \mathcal{U}_n$ *and all* $(\alpha_t)_{1\leq t\leq p}$, *real numbers such that,* $w = \sum_{t=1}^{p} \alpha_t \cdot u_t \in \mathcal{U}_n$.

Obviously, (L) implies (C) and (A) implies (S). Since $x+y = \max\{x,y\} + \min\{x,y\}$ for all $x, y \in \mathbb{R}$ so, (T) is also implied by (L).

Lemma 2.1.2. *The Shapley-Shubik index* Φ *satisfies the axioms (P), (A), (S), (E), (NP), (T), (C) and (L).*

Proof. Combine Proposition 2.1.5 and Equation (2.6) we easily check that Φ satisfies (E), (NP), (S), (NP) and (L). Moreover (L) implies (T) and (C) thus Φ also satisfies (T) as well as (C). Since $\widetilde{v}$ is monotone and Φ is efficient, then $\Phi(v) \neq \mathbf{0}$, so that Φ is positive.

In order to show that Φ is anonymous, consider a permutation $\pi \in \mathcal{S}_n$, $S \subseteq N$ and $\mathbf{a} \in J^n$. Then for all $x \in J^n$, we obviously obtain $\pi(\mathbf{a}_S, x_{-S}) = \big(\mathbf{a}_{\pi^{-1}(S)}, (\pi x)_{-(\pi^{-1}(S))}\big)$. So, by Equation (2.4) we have:

$$\begin{aligned}
\widetilde{\pi v}(S) =& \frac{1}{j^n(k-1)} \sum_{x\in J^n} \left[\pi v((\mathbf{j}-\mathbf{1})_S,\, x_{-S}) - \pi v(\mathbf{0}_S,\, x_{-S})\right]\\
=& \frac{1}{j^n(k-1)} \sum_{x\in J^n} \left[v(\pi((\mathbf{j}-\mathbf{1})_S,\, x_{-S})) - v(\pi(\mathbf{0}_S,\, x_{-S}))\right]\\
=& \frac{1}{j^n(k-1)} \sum_{x\in J^n} \Big[v((\mathbf{j}-\mathbf{1})_{\pi^{-1}(S)},\, (\pi x)_{-(\pi^{-1}(S))}) - v(\mathbf{0}_{\pi^{-1}(S)},\, (\pi x)_{-(\pi^{-1}(S))})\Big]\\
=& \frac{1}{j^n(k-1)} \sum_{y\in J^n} \Big[v((\mathbf{j}-\mathbf{1})_{\pi^{-1}(S)},\, y_{-(\pi^{-1}(S))}) - v(\mathbf{0}_{\pi^{-1}(S)},\, y_{-(\pi^{-1}(S))})\Big] \quad \text{with } y = \pi x\\
=& \widetilde{v}(\pi^{-1}(S))
\end{aligned}$$

Therefore, for all $i \in N$,

$$\begin{aligned}
\Phi_{\pi(i)}(\pi v) &= \sum_{\pi(i)\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!} \left[\widetilde{\pi v}(S) - \widetilde{\pi v}(S\backslash\{\pi(i)\})\right]\\
&= \sum_{i\in \pi^{-1}(S)\subseteq N} \frac{(s-1)!(n-s)!}{n!} \left[\widetilde{v}(\pi^{-1}(S)) - \widetilde{v}(\pi^{-1}(S\backslash\{\pi(i)\}))\right]\\
&= \sum_{i\in T\subseteq N} \frac{(t-1)!(n-t)!}{n!} \left[\widetilde{v}(T) - \widetilde{v}(T\backslash\{i\})\right] \quad \text{with } T = \pi^{-1}(S)\\
&= \text{Shap}(\widetilde{v}) = \Phi_i(v) \quad \text{by Equation (2.6)}.
\end{aligned}$$

So, Φ satisfies anonymity. Therefore, Φ satisfies (S) since (A) implies (S). ■

2.2 Axiomatization of Φ on $\mathcal{U}_n$

We provide an axiomatization of the Shapley-Shubik index (Φ) for uniform (j,k) simple games. For this fact, after showing that the extension of Dubey (1975) axioms (efficiency (E), symmetry (S), null player property (NP) and transfer (T)) to uniform (j,k) simple games are no longer sufficient to uniquely determinate Φ, we introduce a new axiom called *average convexity* (AC). This latter axiom together with (E), (S) and (NP) lead to an axiomatic justification of Φ. We end this section by the proof of the independence of the characterization axioms.

2.2.1 Average convexity axiom

Insufficiency of Dubey (1975) axioms over $\mathcal{U}_n$

Actually the proof of Lemma 2.1.2 is valid for a larger class of power indices on $\mathcal{U}_n$. To prove this, we construct a parametric class of power indices on $\mathcal{U}_n$ as follows. For a given uniform (j,k) simple game v and a profile $a \in J^n$, we associate a TU-game v_a defined as follows:

$$\text{for any } S \subseteq N\,,\ v_a(S) = \frac{1}{k-1}\cdot\left[v((\mathbf{j-1})_S, a_{-S}) - v(\mathbf{0}_S, a_{-S})\right].$$

With this, we define the mapping Φ^a on $\mathcal{U}_n$ by

$$\Phi_i^a(v) = \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!}\left[v_a(S) - v_a\left(S\backslash\{i\}\right)\right], \tag{2.14}$$

for any $v \in \mathcal{U}_n$ and any $i \in N$.

Proposition 2.2.1. *For every $a \in J^n$ such that $a_i = a_h$ for all $i, h \in N$, the mapping Φ^a is a power index on $\mathcal{U}_n$ that satisfies the axioms (P), (A), (S), (E), (NP), (T), (C), and (L).*

Proof. Similar as in the proof of the properties of Φ in Lemma 2.1.2 ∎

While the Shapley-Shubik index for simple games is the unique power index that is symmetric, efficient, satisfies both the null player property and the transfer property, see Dubey (1975), this result is not transferred to uniform (j,k) simple games.

Proposition 2.2.2. *When $j \geq 3$, there exists $\mathbf{a} = (a, a, \cdots, a) \in J^n$ such that $\Phi^{\mathbf{a}} \neq \Phi$.*

Proof. Consider the uniform (j,k) simple game u^b with point-veto $b = (1, j-1, 0, \cdots, 0)$ and $a = j-2$. From Equation (2.1) we have

$$u^b(x) = \begin{cases} k-1 & \text{if } x_1 \geq 1 \text{ and } x_2 = j-1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, Equation (2.14) yields $\Phi^{\mathbf{a}}(u^b) = (0,\, 1,\, 0,\, \cdots,\, 0)$. Moreover Equation (2.7) implies that

$$\text{for all } T \subseteq N, \quad \widetilde{u^b}(T) = \begin{cases} 1 & \text{if} \quad 1, 2 \in T \\ \frac{j-1}{j} & \text{if} \quad 2 \in T \subseteq N\backslash\{1\} \\ \frac{1}{j} & \text{if} \quad 1 \in T \subseteq N\backslash\{2\} \\ 0 & \text{if} \quad T \subseteq N\backslash\{1, 2\} \end{cases} \tag{2.15}$$

So, by Equation (2.6) and Proposition 2.1.5 we obtain $\Phi(u^b) = \left(\frac{1}{j},\, \frac{j-1}{j},\, 0,\, \cdots,\, 0\right) \neq \Phi^{\mathbf{a}}(u^b)$. ∎

We remark that the condition $j \geq 3$ is necessary in Proposition 2.2.2, since for $(2, 2)$ simple games the roll-call interpretation of Mann and Shapley (1964), for the Shapley-Shubik index for simple games yields $\Phi^{\mathbf{0}} = \Phi^{\mathbf{1}} = \Phi$.

Average Convexity axiom

As state in Propositions 2.2.1 and 2.2.2 the axioms of Definition 2.1.5 are no longer sufficient to uniquely identify the Shapley-Shubik power index Φ on $\mathcal{U}_n$ [2]. Therefore we introduce a newer axiom on $\mathcal{U}_n$ called *average convexity (AC)* which, combined with (E), (NP) and (S) provide a characterization of Φ.

Definition 2.2.1. *A power index F on $\mathcal{U}_n$ is averagely convex (AC) if we always have*

$$\sum_{t=1}^{p} \alpha_t \cdot F(u_t) = \sum_{t=1}^{q} \beta_t \cdot F(v_t) \tag{2.16}$$

whenever

$$\sum_{t=1}^{p} \alpha_t \cdot \widetilde{u}_t = \sum_{t=1}^{q} \beta_t \cdot \widetilde{v}_t, \tag{2.17}$$

where $u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q \in \mathcal{U}_n$ and $(\alpha_t)_{1 \leq t \leq p}$, $(\beta_t)_{1 \leq t \leq q}$ are non-negative numbers such that $\sum_{t=1}^{p} \alpha_t = 1$ each.

One may motivate the axiom (AC) as follows. In a game, the *a priori strength* of a coalition, given the profile of the other individuals, is the difference between the outputs observed when all of her members respectively give each her *maximum support* and her *minimum support*. The *average strength* game associates each coalition with her expected strength when the profile of other individuals uniformly varies. Average convexity for power indices is the requirement that whenever two convex combinations of average games coincide, the corresponding convex combinations of the power distributions also coincide.

Proposition 2.2.3. *The Shapley-Shubik index Φ satisfies (AC).*

[2] Except on $\mathcal{U}^{2,k}$ where we show that Φ is the unique index satisfying (E), (S), (NP), (T) and (C).

Proof. Let show that Φ satisfies (AC). Consider $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_q \in \mathcal{U}_n$ $(\alpha_t)_{1\leq t\leq p}$, $(\beta_t)_{1\leq t\leq q}$ non-negative numbers such that $\sum_{t=1}^{p} \alpha_t = 1$ and $\sum_{t=1}^{p} \alpha_t \cdot \widetilde{u_t} = \sum_{t=1}^{q} \beta_t \cdot \widetilde{v_t}$, then

$$\begin{aligned}
\sum_{t=1}^{p} \alpha_t \cdot \Phi(u_t) &= \sum_{t=1}^{p} \alpha_t \cdot \mathrm{Shap}(\widetilde{u_t}) \quad \text{by Equation (2.6)} \\
&= \mathrm{Shap}\left(\sum_{t=1}^{p} \alpha_t \cdot \widetilde{u_t}\right) \quad \text{since the Shapley value is linear} \\
&= \sum_{t=1}^{q} \beta_t \cdot \mathrm{Shap}(\widetilde{v_t}) \quad \text{by hypothesis and linearity of the Shapley value} \\
&= \sum_{t=1}^{q} \beta_t \cdot \Phi(v_t)
\end{aligned}$$

So, the power index Φ satisfies (AC). ∎

We remark that the axiom of Average Convexity is much stronger than the axiom of Convexity. A minor technical point is that $\sum_{t=1}^{p} \alpha_t u_t$ as well as $\sum_{t=1}^{q} \beta_t v_t$ do not need to be a uniform (j,k) simple game. However, the more important issue is that, when $\sum_{t=1}^{p} \alpha_t \cdot u_t = \sum_{t=1}^{q} \beta_t \cdot v_t \in \mathcal{U}_n$ then,

$$\widetilde{\sum_{t=1}^{p} \alpha_t \cdot u_t} \overset{\text{Proposition 2.1.5.(d)}}{=} \sum_{t=1}^{p} \alpha_t \cdot \widetilde{u_t} = \sum_{t=1}^{q} \beta_t \cdot \widetilde{v_t} \overset{\text{Proposition 2.1.5.(d)}}{=} \widetilde{\sum_{t=1}^{q} \beta_t \cdot v_t}\,,$$

i.e. Equation (2.17) is far less restrictive than

$$\sum_{t=1}^{p} \alpha_t \cdot u_t = \sum_{t=1}^{q} \beta_t \cdot v_t$$

since two different uniform (j,k) simple games may have the same average game, see Example 2.1.4.

Further evidence is given by the fact that the parametric power indices $\Phi^{\mathbf{a}}$ in Proposition 2.2.1 do not all satisfy (AC) as showing the following proposition.

Proposition 2.2.4. *When $j \geq 3$, there exists $a \in J$ such that $\Phi^{\mathbf{a}}$ does not satisfy (AC).*

Proof. To see this, considerer the uniform (j,k) simple game with point-veto $b = (1, j-1, 0, \cdots, 0) \in J^n$ and $a = j-2$. From the proof of Proposition 2.2.2, we have $\Phi^{\mathbf{a}}(u^b) = (0,\ 1,\ 0,\ \cdots,\ 0)$. Furthermore, due to Equation (2.15) we can easily check that,

$$\widetilde{u^b} = \frac{1}{j} \cdot \widetilde{w^{\{1\}}} + \frac{j-1}{j} \cdot \widetilde{w^{\{2\}}}\,. \tag{2.18}$$

Since $\Phi^{\mathbf{a}}$ satisfies (NP), (E), (S) we directly obtain $\Phi^{\mathbf{a}}\left(w^{\{1\}}\right) = (1, 0, \cdots, 0)$ and $\Phi^{\mathbf{a}}\left(w^{\{2\}}\right) = (0, 1, 0, \cdots, 0)$. Therefore,

$$\frac{1}{j} \cdot \Phi^{\mathbf{a}}\left(w^{\{1\}}\right) + \frac{j-1}{j} \cdot \Phi^{\mathbf{a}}\left(w^{\{2\}}\right) = \left(\frac{1}{j}, \frac{j-1}{j}, 0, \cdots, 0\right) \neq \Phi(u^b). \tag{2.19}$$

It follows from Equations (2.18) and (2.19) that $\Phi^{\mathbf{a}}$ does not satisfy (AC). ■

2.2.2 Results of axiomatisation

In this section, a characterization of the Shapley-Subik power index Φ for uniform (j, k) simple games is given. More precisely, the axiom (AC) together with (E), (NP) and (S) provide an axiomatization of the Φ. When $j = 2$, i.e. uniform (j, k) simple games with two alternatives in the input, an alternative axiomatization is provided using the axioms (E), (S), (NP), (T) and (C).

As a preliminary step to our characterization result, we establish the following lemma:

Lemma 2.2.1. *If a power index F on $\mathcal{U}_n$ satisfies (E), (S) and (NP), then we have $F(w^C) = \Phi(w^C)$ for all $C \in 2^N$.*

Proof. Let F be a power index on $\mathcal{U}_n$ that satisfies (E), (S), (NP) and let $C \in 2^N$.

According to Proposition 2.1.1, all players $i, h \in C$ are symmetric in w^C and those outside of C are null players in w^C. Since both F and Φ satisfy (E), (S), and (NP) then, $F_i(w^C) = \Phi_i(w^C) = \dfrac{1}{|C|}$ if $i \in C$ and $F_i(w^C) = \Phi_i(w^C) = 0$ otherwise. It clear that $F(w^C) = \Phi(w^C)$. ■

Theorem 2.2.1. A power index F on $\mathcal{U}_n$ satisfies (E), (S), (NP), and (AC) if and only if $F = \Phi$. ★

Proof. *Necessity*: As shown in Lemma 2.1.2 , Φ satisfies (E), (S), and (NP). For (AC) the proof follows from Proposition 2.2.3.

Sufficiency: Consider a power index F on $\mathcal{U}_n$ that satisfies (E), (S), (NP), and (AC). Consider an arbitrary uniform (j, k) simple game $u \in \mathcal{U}_n$. By Lemma 2.1.1, there exists a collection of real numbers $(x_S)_{S \in 2^N}$ such that

$$\widetilde{u} = \sum_{S \in 2^N} x_S \cdot \widetilde{w^S} = \sum_{S \in E_1} x_S \cdot \widetilde{w^S} + \sum_{S \in E_2} x_S \cdot \widetilde{w^S}, \tag{2.20}$$

where $E_1 = \{S \in 2^N : x_S > 0\}$ and $E_2 = \{S \in 2^N : x_S < 0\}$. Note that $E_1 \neq \emptyset$ since $\widetilde{u}(N) = 1$. We pose

$$\varpi = \sum_{S \in E_1} x_S \cdot \widetilde{w^S}(N) = \sum_{S \in E_1} x_S > 0. \tag{2.21}$$

It follows that

$$\frac{1}{\varpi}\widetilde{u} + \sum_{S\in E_2} \frac{-x_S}{\varpi}\ \widetilde{w^S} = \sum_{S\in E_1} \frac{x_S}{\varpi}\ \widetilde{w^S}. \tag{2.22}$$

Since Equation (2.22) is an equality among two convex combinations, axiom (AC) yields

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\ F(w^S) = \sum_{S\in E_1} \frac{x_S}{\varpi}\ F(w^S).$$

Hence by Lemma 2.2.1,

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\ \Phi(w^S) = \sum_{S\in E_1} \frac{x_S}{\varpi}\ \Phi(w^S). \tag{2.23}$$

Since Φ also satisfies (AC), we have

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\ \Phi(w^S) = \frac{1}{\varpi}\Phi(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\ \Phi(w^S)\,. \tag{2.24}$$

Therefore $F(u) = \Phi(u)$, for all $u \in \mathcal{U}_n$. That is $F = \Phi$. ■

Alternative characterisation of Φ on $\mathcal{U}^{2,k}$

We provide here an alternative characterization of Φ on $\mathcal{U}_n^{2,k}$ by proving that, axioms (E), (NP), (S), (T) and (C) uniquely determine Φ on this class of games. A preliminary result is given by the following lemma.

Lemma 2.2.2. *If F is a power index on $\mathcal{U}_n^{2,k}$ that satisfies (E), (NP) and (S) then for all $a \in J^n\backslash\{\mathbf{0}\}$, $F(u^a) = \Phi(u^a)$.*

Additionally, if F satisfies (T), then for all nonempty subset E of $J^n\backslash\{\mathbf{0}\}$, $F(u^E) = \Phi(u^E)$.

Proof. Let F be a power index on $\mathcal{U}_n^{2,k}$ that satisfies (E), (S) and (NP), and $a \in J^n\backslash\{\mathbf{0}\}$.

Since $j = 2$, it follows from Proposition 2.1.1 that all players in N^a are symmetric in u^a and those outside N^a are null players. But, since both F and Φ satisfy (E), (S) and (NP) then, $F_i(u^a) = \dfrac{1}{|N^a|} = \Phi_i(u^a)$ for any $i \in N^a$ and $F_i(u^a) = \Phi(u^a) = 0$ otherwise. So, $F(u^a) = \Phi(u^a)$.

Moreover, assume that F also satisfies (T). Let E be a nonempty subset of $J^n\backslash\{\mathbf{0}\}$. Then $u^E = \bigvee\limits_{a\in E} u^a$ by Corollary 2.1.1. As prove in (Einy, 1987, Lemma 2.3) we can write:

$$\begin{aligned}
F(u^E) &= \sum_{\emptyset \neq L \subseteq E} (-1)^{|L|+1} F\left(\bigwedge_{a \in L} u^a\right) \quad \text{since } F \text{ satisfies (T)} \\
&= \sum_{\emptyset \neq L \subseteq E} (-1)^{|L|+1} F\left(u^{\max\{a,\, a \in L\}}\right) \quad \text{by Corollary 2.1.1} \\
&= \sum_{\emptyset \neq L \subseteq E} (-1)^{|L|+1} \Phi\left(u^{\max\{a,\, a \in L\}}\right) \quad \text{since } \max\{a,\, a \in L\} \in J^n \backslash \{\mathbf{0}\}, \text{ for all } L \subseteq E \\
&= \Phi(u^E) \quad \text{applying Corollary 2.1.1 and (T)}
\end{aligned}$$

∎

Note that the previous lemma characterizes Φ on $(2,2)$ simple games, since all these games are in the form u^E. For $k \geq 3$, a similar result of characterization is obtained by adding convexity among the axioms.

Theorem 2.2.2.

The Shapley-Shubik index Φ is the unique power index on $\mathcal{U}_n^{2,k}$ that simultaneously satisfies de axioms (E), (NP), (S), (T) and (C). ⋆

Proof.

Necessity: We have already proven that Φ satisfies (E), (NP), (S), (T) and (C), (see Lemma 2.1.2).

Sufficiency: Consider $F : \mathcal{U}_n^{2,k} \longrightarrow \mathbb{R}^n$ a power index that simultaneous meets (E), (NP), (S), (T) and (C). Consider a uniform $(2,k)$ simple game v. Then, from Proposition 2.1.3 there exists a collection $(\alpha_t)_{1 \leq t \leq p}$ of non-negative real numbers sum to 1 and a collection $(E_t)_{1 \leq t \leq p}$ of nonempty subsets of $J^n \backslash \{\mathbf{0}\}$ such that $v = \sum_{t=1}^{p} \alpha_t \cdot u^{E_t}$. Thus,

$$\begin{aligned}
F(v) &= \sum_{t=1}^{p} \alpha_t \cdot F(u^{E_t}) \quad \text{since } F \text{ satisfies (C)} \\
&= \sum_{t=1}^{p} \alpha_t \cdot \Phi(u^{E_t}) \quad \text{by Lemma 2.2.2} \\
&= \Phi(v) \quad \text{since } \Phi \text{ satisfies (C)}
\end{aligned}$$

Therefore $F(v) = \Phi(v)$, for all $v \in \mathcal{U}^{2,k}$. This means that $F = \Phi$. ∎

2.2.3 Independence of axioms

We now prove that, the four axioms in Theorem 2.2.1 are independent. To this end, we provide a power index on $\mathcal{U}_n$ that meets the three other axioms but not the chosen one.

Efficiency can not be dropped

Proposition 2.2.5. *The power index $F^1 = 2 \cdot \Phi$ satisfies (NP), (S), and (AC) but not (E).*

Proof. It is straightforward from its definition that F^1 satisfies (S), (NP); and (AC) but not (E). ■

Null player property can not be dropped

Denote by ED the *equal division* power index which assigns $\frac{1}{n}$ to each player in every uniform (j, k) simple game.

Proposition 2.2.6. *The power index $F^2 = \frac{1}{2} \cdot \Phi + \frac{1}{2} \cdot \mathrm{ED}$ satisfies (E), (S) and (AC), but not (NP).*

Proof. One can easily check that F^2 satisfies (E) and (S) but not (NP). In order to prove that F^2 meets (AC), consider $u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q \in \mathcal{U}_n$ and $(\alpha_t)_{1\leq t\leq p}$, $(\beta_t)_{1\leq t\leq q}$ non-negative numbers with $\sum_{t=1}^{p} \alpha_t = \sum_{t=1}^{q} \beta_t = 1$, and $\sum_{t=1}^{p} \alpha_t \cdot \widetilde{u_t} = \sum_{t=1}^{q} \beta_t \cdot \widetilde{v_t}$. We have:

$$
\begin{aligned}
\sum_{t=1}^{p} \alpha_t \cdot F^2(u_t) &= \frac{1}{2}\left(\sum_{t=1}^{p} \alpha_t \cdot \Phi(u_t) + \frac{1}{n} \cdot \sum_{t=1}^{p} \alpha_t\right) \\
&= \frac{1}{2}\left(\sum_{t=1}^{q} \beta_t \cdot \Phi(v_t) + \frac{1}{n} \cdot \sum_{t=1}^{q} \beta_t\right) \quad \text{since } \Phi \text{ satisfies (AC) and } \sum_{t=1}^{p} \alpha_t = \sum_{t=1}^{q} \beta_t = 1 \\
&= \sum_{t=1}^{q} \beta_t \cdot F^2(v_t)
\end{aligned}
$$

So, F^2 satisfies (AC). ■

Average convexity can not be dropped

Proposition 2.2.7. *The power index $F^3 = \Phi^{\mathbf{a}}$ for $a = j - 2$ defined in the proof of Proposition 2.2.2 satisfies (E), (S), and (NP); but not (AC).*

Proof. The proof that F^3 satisfies (E), (NP) and (S); but not (AC) is given in Propositions 2.2.2 and 2.2.4. ■

Symmetry can not be dropped

To construct a power index that satisfies (E), (NP) and (AC); but not (S), we recall that $\left(\widetilde{w^S}\right)_{S\in 2^N}$ is a basis of the vector space of all TU-games on N. Thus given a (j,k) simple game u, there exists a unique collection of real numbers $(x_S^u)_{S\in 2^N}$ such that

$$\widetilde{u} = \sum_{S\in 2^N} x_S^u \cdot \widetilde{w^S}. \tag{2.25}$$

Let $i_0 \in N$ be a given player and Φ^{i_0} be a power index on $\mathcal{U}_n$ such that

$$\Phi_p^{i_0}(w^N) = \begin{cases} \frac{2}{n+1} & \text{if } p = i_0 \\ \frac{1}{n+1} & \text{if } p \neq i_0 \end{cases}$$

Proposition 2.2.8. *The power index F^4 defined on $\mathcal{U}_n$ such that for all $S \in 2^N\setminus\{N\}$, $F^4\left(w^S\right) = \Phi\left(w^S\right)$ and $F^4(w^N) = \Phi^1(w^N)$ and*

$$F^4(u) = \sum_{S\in 2^N} x_S^u \cdot F^4\left(w^S\right) \qquad \text{for any } u \in \mathcal{U}_n, \tag{2.26}$$

satisfies (E), (NP), (AC); but not (S).

Proof. We first remark that, from Equation (2.25) , the power index F^4 is well defined. Since $F^4(w^N) = \Phi^1(w^N)$, then F^4 does not satisfy (S). Let u be a uniform (j,k) simple game.

$$\begin{aligned}
\sum_{p\in N} F_p^4(u) &= \sum_{S\in 2^N\setminus\{N\}} x_S^u \left(\sum_{p\in N} \Phi_p(w^S)\right) + x_N^u \cdot \sum_{p\in N} \Phi_p^1(w^N) \quad \text{by Equation (2.26)}\\
&= \left(\sum_{S\in 2^N\setminus\{N\}} x_S^u\right) + x_N^u \quad \text{since } \Phi \text{ and } \Phi^1 \text{ are efficient}\\
&=\widetilde{u}(N) = 1\,.
\end{aligned}$$

This proves that F^4 is efficient.

Now let i be a null player in u. By Corollary 2.1.2, Equation (2.25) becomes

$$\widetilde{u} = \sum_{i\notin S\in 2^N} x_S^u \cdot \widetilde{w^S}. \tag{2.27}$$

Since i is null player in all w^S such that $i \notin S$, (see Proposition 2.1.1) and Φ satisfies (NP), then Equations (2.26) and (2.27) give

$$F_i^4(u) = \sum_{i\notin S\in 2^N} x_S^u \cdot \Phi_i\left(w^S\right) = 0\,.$$

So, F^4 satisfies (NP).

To prove that F^4 satisfies (AC), consider $u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q \in \mathcal{U}_n$ and $(\alpha_t)_{1\leq t\leq p}$, $(\beta_t)_{1\leq t\leq q}$ non-negative numbers such that $\sum_{t=1}^{p}\alpha_t = \sum_{t=1}^{q}\beta_t = 1$ and

$$\sum_{t=1}^{p}\alpha_t\cdot\widetilde{u}_t = \sum_{t=1}^{q}\beta_t\cdot\widetilde{v}_t\,. \tag{2.28}$$

Note that, $\widetilde{u}_t$ and $\widetilde{v}_t$ are TU-games on N. Hence, Equation (2.25) implies

$$\sum_{t=1}^{p}\alpha_t\cdot\widetilde{u}_t = \sum_{t=1}^{p}\alpha_t\left(\sum_{S\in 2^N}x_S^{u_t}\cdot\widetilde{w^S}\right) = \sum_{S\in 2^N}\left(\sum_{t=1}^{p}\alpha_t\cdot x_S^{u_t}\right)\widetilde{w^S}, \tag{2.29}$$

and

$$\sum_{t=1}^{q}\beta_t\cdot\widetilde{v}_t = \sum_{t=1}^{q}\beta_t\left(\sum_{S\in 2^N}x_S^{v_t}\cdot\widetilde{w^S}\right) = \sum_{S\in 2^N}\left(\sum_{t=1}^{q}\alpha_t\cdot x_S^{v_t}\right)\widetilde{w^S}\,. \tag{2.30}$$

Since $(\widetilde{w^S})_{S\in 2^N}$ is a basis of the vector space Γ^N, then Equations (2.28)–(2.30) impose,

$$\sum_{t=1}^{p}\alpha_t\cdot x_S^{u_t} = \sum_{t=1}^{q}\beta_t\cdot x_S^{v_t} \quad \text{for all } S\in 2^N. \tag{2.31}$$

Finally, we have,

$$\begin{aligned}
\sum_{t=1}^{p}\alpha_t\cdot F^4(u_t) &= \sum_{t=1}^{p}\alpha_t\left(\sum_{S\in 2^N}x_S^{u_t}\cdot F^4(w^S)\right) \quad \text{by Equations (2.26) and (2.29)}\\
&= \sum_{S\in 2^N}\left(\sum_{t=1}^{p}\alpha_t\cdot x_S^{u_t}\right)F^4(w^S)\\
&= \sum_{S\in 2^N}\left(\sum_{t=1}^{q}\beta_t\cdot x_S^{v_t}\right)F^4(w^S) \quad \text{by Equation (2.31)}\\
&= \sum_{t=1}^{q}\beta_t\left(\sum_{S\in 2^N}x_S^{v_t}\cdot F^4(w^S)\right)\\
&= \sum_{t=1}^{q}\beta_t\cdot F^4(v_t) \quad \text{by Equations (2.26) and (2.30)}
\end{aligned}$$

We conclude that F^4 satisfies (AC). ■

2.3 An analog of Symmetry Gain-Loss axiom on $\mathcal{U}_n$

The Symmetry Gain-Loss (SymGL) axiom for power indices was introduced in the context of simple games by Laruelle and Valenciano (2001). We introduced an analog of this axiom on the class of uniform (j,k) simple games. In terms of simple games, i.e. (2, 2) simple games, our newer axiom called *Symmetry Gain-Loss* (SymGL*)* is a strong version of

(SymGL).

2.3.1 Symmetry Gain-Loss axiom on $\mathcal{SG}_n$

We briefly present the axiom of Symmetry Gain-Loss within the class of simple games.

Let u and v be two simple games such that the winning coalitions of v are given by the winning coalitions of u and a coalition $S \in 2^N \backslash \{N\}$ that is losing in u, i.e. $\mathcal{W}(v) = \mathcal{W}(u) \cup \{S\}$. As notation we write $v = u \oplus S$. Note that $v = u \oplus S$ implies that the coalition S is a minimal winning coalition in v. So, u is the *modified game* of v as defined in (Laruelle and Valenciano, 2001, Definition 1).

Definition 2.3.1. *A power index F for simple games satisfies (SymGL) if for any games $u, v \in \mathcal{SG}_n$ such that $v = u \oplus S$ for some $S \in 2^N \backslash \{N\}$,*

$$F_i(v) - F_i(u) = F_j(v) - F_j(u) \quad \text{for all } i, j \in S \text{ (resp. for all } i, j \in N \backslash S). \tag{2.32}$$

This axiom states that the effect of adding (or dropping) a single minimal winning coalition is the same for any two players belonging to it and for any two players outside it.

Proposition 2.3.1. *(Laruelle and Valenciano, 2001, Lemma 1)*

The Shapley-Shubik index SSI satisfies (SymGL) and for any games $u, v \in \mathcal{SG}_n$ such that $v = u \oplus S$ for some $S \in 2^N \backslash \{N\}$,

$$\mathrm{SSI}_i(v) - \mathrm{SSI}_i(u) = \begin{cases} \dfrac{(s-1)!(n-s)!}{n!} & \text{if } i \in S \\[2ex] -\dfrac{s!(n-s-1)!}{n!} & \text{if } i \in N \backslash S \end{cases} \tag{2.33}$$

2.3.2 Symmetry Gain-Loss* axiom on $\mathcal{U}_n$

Note that a profile in a simple game is any element $x \in \{0,1\}^n$ which can be viewed as a 2-partition that consists in $\mathbf{1}_x = \{i \in N / x_i = 1\}$ and $\mathbf{0}_x = \{i \in N / x_i = 0\}$, i.e. $x \equiv (\mathbf{1}_x, \mathbf{0}_x)$. Similarly, a simple game v partitions the set of all profiles into two components $\mathcal{C}_0(v)$ and $\mathcal{C}_1(v)$ that respectively correspond to the set of all profiles at which the collective decision is 0 and the set of all profiles at which the collective decision is 1. The Symmetry Gain-Loss axiom captures how a reasonable power index changes when a coalition or a profile in a given simple game v changes from $\mathcal{C}_0(v)$ to $\mathcal{C}_1(v)$; or conversely.

The same analysis can be carried on uniform (j,k) simple games. Before this, recall that, the set of all possible profiles J^n and each element $x \in J^n$ is a j-partition that consists in the collection $(\mathbf{m}_x)_{m \in J}$ where $\mathbf{m}_x = \{i \in N / x_i = m\}$.

Definition 2.3.2. *Let v be a uniform (j,k) simple game and $t = 0, 1, \cdots, k-1$. A t-profile in v is any profile $x \in J^n$ such that $v(x) = t$. The set all t-profiles of v is denoted by $\mathcal{C}_t(v)$.*

For this reason, a uniform (j,k) simple game v partitions the set of all profiles into k components $\mathcal{C}_0(v)$, $\mathcal{C}_1(v)$, $\cdots$, $\mathcal{C}_{k-1}(v)$. These components uniquely defined the game v. As an example the uniform $(3,4)$ simple game v in Example 2.1.2 is clearly defined by $\mathcal{C}_0(v) = \{(0,0);(0,1)\}$, $\mathcal{C}_1(v) = \{(1,0);(1,1);(0,2)\}$, $\mathcal{C}_2(v) = \{(2,0);(2,1)\}$ and $\mathcal{C}_3(v) = \{(1,2);(2,2)\}$.

For simple games, (SymGL) requires that when a profile x shifts from $\mathcal{C}_1(v)$ to $\mathcal{C}_0(v)$ (or conversely) in a given game v, the powers of all players with the same level of approbation in profile x in the new obtained game are affected in the same way. Since this changes occurs in the expense of players in $\mathbf{1}_x$ who each gives a full support, these players lost (gain) the same amount of their respective powers while other players gain (loss) each the same amount of power, see Definition 2.3.1. In the case of uniform (j,k) simple games, a generalization of (SymGL) may be phrased for example as follows: in a given uniform (j,k) simple game v, when a profile x shifts from $\mathcal{C}_{t+1}(v)$ to $\mathcal{C}_t(v)$ (or conversely) for some $t \in \{0,\ldots,k-2\}$, there should exists a vector $\delta(x)$ that specifies what each player in each subset $\mathbf{m}_x$, $m \in J$ gains or loses, in a reasonable way to be specified.

Definition 2.3.3. *Let u be a uniform (j,k) simple game on N and $a \in J^n$ a profile such that $u(a) \neq k-1$. The uniform (j,k) simple game v is an* elementary improvement *of u at a and we write $u \xrightarrow{a} v$ if v is defined as follows:*

$$\text{for all } x \in J^n, \quad v(x) = \begin{cases} u(a)+1 & \text{if } x = a \\ u(x) & \text{otherwise} \end{cases} \tag{2.34}$$

In words, $u \xrightarrow{a} v$ means that, the game v differs from u only on a single profile a and the output values $u(a)$ and $v(a)$ are consecutive in K. By the definition of a uniform (j,k) simple game, it is clear that each (j,k) simple game admits no improvement neither at $\mathbf{0}$; nor at $\mathbf{j-1}$. Hereafter, we pose $J^* = J^n \backslash \{\mathbf{0}, \mathbf{j-1}\}$.

Note that, in the context of simple games, i.e. uniform $(2,2)$ simple games, the elementary improvement from u to v at a means that, the set of winning coalitions of v consists of the set of winning coalitions of u and an additional winning coalition (generated by a) that was losing in u, see (Weber, 1988, pp.17).

Example 2.3.1. *Let v, v_1, v_2 and v_3 be uniform $(3,4)$ simple games with two players defined as follows:*

x	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$
$v(x)$	0	1	2	0	1	2	1	3	3
$v_1(x)$	0	1	2	0	2	2	1	3	3
$v_2(x)$	0	1	2	0	1	2	2	3	3
$v_3(x)$	0	1	2	1	1	2	1	3	3

It follows that $v \xrightarrow{(1,1)} v_1$, $v \xrightarrow{(0,1)} v_2$ and $v \xrightarrow{(0,2)} v_3$.

Notation 2.3.1.

- For any $a \in J^*$, we set $J_a = \{m \in J, \mathbf{m}_a \neq \emptyset\}$;
- Given $S \subseteq N$, we denote by $\mathbb{1}_S$ the mapping defined on N, by $\mathbb{1}_S(i) = 1$ if $i \in S$ and $\mathbb{1}_S(i) = 0$ otherwise.

Definition 2.3.4. *A power index F for uniform (j,k) simple games on N satisfies the Symmetry Gain-Loss* (SymGL*) if for any games $u, v \in \mathcal{U}_n$ such that $u \xrightarrow{a} v$ for some $a \in J^*$,*

$$F(v) - F(u) = \sum_{m \in J_a} \Delta_F(\mathbf{m}_a) \cdot \mathbb{1}_{\mathbf{m}_a} \tag{2.35}$$

where $(\Delta_F(\mathbf{m}_a))_{m \in J_a}$ is a collection of real numbers that does not depend on u and v but only on a.

The (SymGL*) requires that, in case of an elementary improvement at the profile $a \in J^*$, the variation of the powers of players should depend only on the profile a, but not on the game in consideration. Moreover, this change is the same for any two players who have the same level of approval in a.

Hereafter, in order to ease the presentation we pose $\Delta_F(a) = \sum\limits_{m \in J_a} \Delta_F(\mathbf{m}_a) \cdot \mathbb{1}_{\mathbf{m}_a}$, for $a \in J^*$. $\Delta_F(a)$ is the improvement vector associated with F at a.

Remark 2.3.1. *On uniform $(2,k)$ simple games, the Equation (2.35) becomes*

$$F_i(v) - F_i(u) = \begin{cases} \Delta_F(\mathbf{1}_a) & \text{if } i \in \mathbf{1}_a \\ \Delta_F(\mathbf{0}_a) & \text{if } i \in \mathbf{0}_a \end{cases} \tag{2.36}$$

So, the restriction of (SymGL*) axiom on $(2,2)$ simple games can be considered as a strong version of (SymGL) axiom presented in Definition 2.3.1.

Lemma 2.3.1. *Consider a power index F on $\mathcal{U}_n$ that satisfies (SymGL*) and let $(\Delta_F(a))_{a \in J^*}$ be its collection of improvement vectors. Then for all $v \in \mathcal{U}_n$,*

$$F(v) = F(w^N) + \sum_{a \in J^*} v(a) \cdot \Delta_F(a). \tag{2.37}$$

where for all $x \in J^n$, $w^N(x) = k-1$ if $x = \mathbf{j-1}$ and $w^N(x) = 0$ otherwise.

Proof.
Suppose that F is a power index on $\mathcal{U}_n$ that satisfies (SymGL*) and let $(\Delta_F(a))_{a \in J^*}$ be its collection of improvement vectors. Since J^* is finite, it can be labeled in such a way that $J^* = \{a^1, a^2, \ldots, a^p\}$ and for all $s, t \in \{1, 2, \ldots, p\}$, $a^t \preceq a^s$ implies $s \leq t$. Pose $a^0 = \mathbf{j-1}$ and define the sequence $(v^t)_{0 \leq t \leq p}$ of uniform (j,k) simple games as follows:

$$\text{for all } x \in J^n, \quad v^t(x) = \begin{cases} v(x) & \text{if } a^t \preceq x \\ 0 & \text{otherwise} \end{cases} \tag{2.38}$$

Note that for $t \in \{0, 1, \ldots, p-1\}$ and for $x \in J^n$,

$$v^{t+1}(x) = \begin{cases} v^t(a^{t+1}) + v(a^{t+1}) & \text{if } x = a^{t+1} \\ v^t(x) & \text{otherwise} \end{cases} \tag{2.39}$$

Consequently, v^{t+1} can be obtained from v^t after $v(a^{t+1})$ elementary improvements at a^{t+1} So, applying (SymGL*) $v(a^{t+1})$-times yields

$$F(v^{t+1}) = F(v^t) + v(a^{t+1})\Delta_F(a^{t+1}) \tag{2.40}$$

Since $v^p = v$ and $v^0 = w^N$, then by summing over $t \in \{0, \cdots p-1\}$ all left-hand-side terms and all right-hand-side terms in Equation (2.40), we have:

$$F(v) = F(w^N) + \sum_{t=1}^{p} v(a^t)\Delta_F(a^t)\,.$$

Since $J^* = \{a^1, a^2, \ldots, a^p\}$, the proof is completed. ∎

It appears that a power index for uniform (j,k) simple games that satisfies (SymGL*) is completely defined by its collection of improvement vectors and its distribution of power for the game w^N.

2.3.3 Symmetry Gain-Loss* and the Shapley-Shubik index

We prove that the Shapley-Shubik index Φ for uniform (j,k) simple games satisfies (SymGL*) and we determinate its collections of improvement vectors.

Definition 2.3.5. *Given a profile $x \in J^*$, we say that a player $i \in N$ is fully committed if $x_i = j-1$ (i.e. he reports the highest level of approval to the proposal) and fully opposed if $x_i = 0$ (i.e. he gives the lowest level of approval to the proposal).*

The set of all fully committed players in x is denoted by $H(x)$ and the set of all fully opposed players is $L(x)$. We denote $I(x) = N \backslash (H(x) \cup L(x))$.

For each $S \subseteq N$, we define a TU-game τ_S as follows:

$$\text{for } T \subseteq N, \tau_S(T) = \begin{cases} \dfrac{1}{j^{n-t}(k-1)} & \text{if } \emptyset \neq T \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.3.2. *For any games $u, v \in \mathcal{U}_n$ such that $u \xrightarrow{a} v$, for some $a \in J^*$, we have,*

$$\widetilde{v} = \widetilde{u} + \tau_{H(a)} - \tau_{L(a)}\,. \tag{2.41}$$

Proof.

Consider u and v two uniform (j,k) simple games such that $u \xrightarrow{a} v$. To prove Equation (2.41), we distinguish four cases. Consider $S \in 2^N \backslash \{N\}$ and pose $J^n(a_{-S}) = \{x \in J^n,\ x_{-S} = a_{-S}\}$.

Case 1: $H(a) = L(a) = \emptyset$. Then, for any $x \in J^n$, $((\mathbf{j-1})_S,\ x_{-S}) \neq a$ and $(\mathbf{0}_S,\ x_{-S}) \neq a$. Thus,

$$\begin{aligned}
\widetilde{v}(S) =& \frac{1}{j^n(k-1)} \cdot \sum_{x \in J^n} [v((\mathbf{j-1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] \\
=& \frac{1}{j^n(k-1)} \cdot \sum_{x \in J^n} [u((\mathbf{j-1})_S, x_{-S}) - u(\mathbf{0}_S, x_{-S})] \quad \text{by Definition 2.3.3} \\
=& \widetilde{u}(S) + \tau_{H(a)}(S) - \tau_{L(a)}(S) \quad \text{since } \tau_{H(a)}(S) = \tau_{L(a)}(S) = 0\,.
\end{aligned}$$

Case 2: $L(a) \neq \emptyset$ and $H(a) = \emptyset$. For this reason, for every $x \in J^n$, $((\mathbf{j-1})_S,\ x_{-S}) \neq a$ and $(\mathbf{0}_S,\ x_{-S}) = a \Leftrightarrow (S \subseteq L(a) \quad \text{and} \quad x_{-S} = a_{-S})$. So,

$$\begin{aligned}
\widetilde{v}(S) &= \frac{1}{j^n(k-1)} \left(\sum_{x \in J^n(a_{-S})} [v((\mathbf{j-1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] + \sum_{x \notin J^n(a_{-S})} [v((\mathbf{j-1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] \right) \\
&= \frac{1}{j^n(k-1)} \left(\sum_{x \in J^n(a_{-S})} [u((\mathbf{j-1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] + \sum_{x \notin J^n(a_{-S})} [u((\mathbf{j-1})_S, x_{-S}) - u(\mathbf{0}_S, x_{-S})] \right) \\
&= \begin{cases} \widetilde{u}(S) - \frac{1}{j^{n-s}(k-1)} & \text{if } \emptyset \neq S \subseteq L(a) \\ \widetilde{u}(S) & \text{otherwise} \end{cases} \quad \text{by Definition 2.3.3} \\
&= \widetilde{u}(S) + \tau_{H(a)}(S) - \tau_{L(a)}(S) \quad \text{since } \tau_{H(a)}(S) = 0\,.
\end{aligned}$$

Case 3: $L(a) = \emptyset$ and $H(a) \neq \emptyset$. Therefore, for any $x \in J^n$, $(\mathbf{0}_S,\ x_{-S}) \neq a$ and $((\mathbf{j-1})_S,\ x_{-S}) = a \Leftrightarrow (S \subseteq H(a) \quad \text{and} \quad x_{-S} = a_{-S})$. Thus, similar to the case 2, we obtain

$$\begin{aligned}
\widetilde{v}(S) =& \begin{cases} \widetilde{u}(S) + \frac{1}{j^{n-s}(k-1)} & \text{if } \emptyset \neq S \subseteq H(a) \\ \widetilde{u}(S) & \text{otherwise} \end{cases} \\
=& \widetilde{u}(S) + \tau_{H(a)}(S) - \tau_{L(a)}(S) \quad \text{since } \tau_{L(a)}(S) = 0\,.
\end{aligned}$$

Case 4: $L(a) \neq \emptyset$ and $H(a) \neq \emptyset$. Since $L(a) \cap H(a) = \emptyset$, then by taking in consideration the previous cases we have

$$\begin{aligned}
\widetilde{v}(S) =& \begin{cases} \widetilde{u}(S) - \frac{1}{j^{n-s}(k-1)} & \text{if } \emptyset \neq S \subseteq L(a) \\ \widetilde{u}(S) + \frac{1}{j^{n-s}(k-1)} & \text{if } \emptyset \neq S \subseteq H(a) \\ \widetilde{u}(S) & \text{otherwise} \end{cases} \\
=& \widetilde{u}(S) + \tau_{H(a)}(S) - \tau_{L(a)}(S)\,.
\end{aligned}$$

So, the four cases yield that $\widetilde{v}(S) = \widetilde{u}(S) + \tau_{H(a)}(S) - \tau_{L(a)}(S)$ for all $S \in 2^N$. That is $\widetilde{v} = \widetilde{u} + \tau_{H(a)} - \tau_{L(a)}$. ■

To illustrate the former result, we consider the uniform $(3,4)$ simple games v, v_1, v_2 and v_3 in Example 2.3.1. The average game of v is defined by $\widetilde{v}(\emptyset) = 0$, $\widetilde{v}(\{1\}) = \frac{2}{3}$, $\widetilde{v}(\{1\}) = \frac{4}{9}$ and $\widetilde{v}(\{12\}) = 1$.

Since

$v \xrightarrow{a} v_i$	$L(a)$	$H(a)$
$v \xrightarrow{(1,1)} v_1$	$\emptyset$	$\emptyset$
$v \xrightarrow{(0,2)} v_2$	$\{1\}$	$\{2\}$
$v \xrightarrow{(0,1)} v_3$	$\{1\}$	$\emptyset$

then

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\widetilde{v_1}(S)$	0	$\frac{2}{3}$	$\frac{4}{9}$	1
$\widetilde{v_2}(S)$	0	$\frac{5}{9}$	$\frac{5}{9}$	1
$\widetilde{v_3}(S)$	0	$\frac{5}{9}$	$\frac{4}{9}$	1

Remark 2.3.2. *Consider $S \subseteq N$, then any two players $i, j \in S$ (resp., $i, j \in N \backslash S$) are symmetric in τ_S. Therefore, we pose*

$$\mathrm{Shap}(\tau_S) = \beta_S \cdot \mathbb{1}_S + \lambda_{N\backslash S} \cdot \mathbb{1}_{N\backslash S} \tag{2.42}$$

where β_S and $\lambda_{N\backslash S}$ are real numbers.

Lemma 2.3.2. *The Shapley-Shubik index Φ satisfies (SymGL*) and for all $a \in J^*$,*

$$\Delta_\Phi(a) = \left(\beta_{H(a)} - \lambda_{N\backslash L(a)}\right) \cdot \mathbb{1}_{H(a)} + \left(\lambda_{N\backslash H(a)} - \lambda_{N\backslash L(a)}\right) \cdot \mathbb{1}_{I(a)} + \left(\lambda_{N\backslash H(a)} - \beta_{L(a)}\right) \cdot \mathbb{1}_{L(a)} \,. \tag{2.43}$$

Proof.

Consider $u, v \in \mathcal{U}_n$ such that $u \xrightarrow{a} v$ for some $a \in J^*$. By Proposition 2.3.2, we have, $\widetilde{v} = \widetilde{u} + \tau_{H(a)} - \tau_{L(a)}$. It follows from Theorem 2.1.1 that, $\Phi(v) - \Phi(u) = \mathrm{Shap}(\tau_{H(a)} - \tau_{L(a)})$. Since $\mathrm{Shap}(\tau_{H(a)} - \tau_{L(a)})$ is independent of u and v, we pose $\Delta_\Phi(a) = \mathrm{Shap}(\tau_{H(a)} - \tau_{L(a)})$. Then by Equation (2.42), we have

$$\Delta_\Phi(a) = \left(\beta_{H(a)} - \lambda_{N\backslash L(a)}\right) \cdot \mathbb{1}_{H(a)} + \left(\lambda_{N\backslash H(a)} - \lambda_{N\backslash L(a)}\right) \cdot \mathbb{1}_{I(a)} + \left(\lambda_{N\backslash H(a)} - \beta_{L(a)}\right) \cdot \mathbb{1}_{L(a)} \,.$$

It follows that the Shapley-Shubik index satisfies (SymGL*). Besides, for any $a \in J^*$ and any $m \in J_a$,

$$\Delta_\Phi(\mathbf{m}_a) = \begin{cases} \beta_{H(a)} - \lambda_{N\backslash L(a)} & \text{if } m = j-1 \\ \lambda_{N\backslash H(a)} - \lambda_{N\backslash L(a)} & \text{if } 0 < m < j-1 \\ \lambda_{N\backslash H(a)} - \beta_{L(a)} & \text{if } m = 0 \end{cases}$$

∎

So, in case of an elementary improvement of a uniform (j,k) simple game, variations of the Shapley-Shubik index depend only on the set of fully committed players and the set of fully opposed players.

Hereafter, following Freixas (2019), we use the numerical coefficients $\gamma_j^n(t)$ given by :

$$\gamma_j^n(t) = t!j^t \sum_{l=0}^{t} \frac{(n-t-1+l)!}{l!j^l}\,, \quad \text{for } t = 0, 1, \cdots, n-1. \tag{2.44}$$

Proposition 2.3.3. In case of an elementary improvement of the game at a profile a, the total change $\Delta_{\Phi_i}(a)$ of a player i is given by:

$$\Delta_{\Phi_i}(a) = \begin{cases} \dfrac{\gamma_j^n(l_a) + (j-1)\gamma_j^n(h_a - 1)}{(k-1)n!j^n} & \text{if } i \in H(a) \\[2ex] \dfrac{\gamma_j^n(l_a) - \gamma_j^n(h_a)}{(k-1)n!j^n} & \text{if } i \in I(a) \\[2ex] -\dfrac{\gamma_j^n(h_a) + (j-1)\gamma_j^n(l_a - 1)}{(k-1)n!j^n} & \text{if } i \in L(a) \end{cases} \tag{2.45}$$

where $h_a = |H(a)|$ and $l_a = |L(a)|$.

Proof. From Equation (2.43), the total change $\Delta_{\Phi_i}(a)$ of any player $i \in N$ in Φ due to an elementary improvement of the game at a profile a can be rewritten as:

$$\Delta_{\Phi_i}(a) = \sum_{\{i\}\subseteq S\subseteq H(a)} \frac{\zeta(s)}{j^{n-s}} - \sum_{\emptyset\neq S\subseteq H(a)\setminus\{i\}} \frac{\zeta(s+1)}{j^{n-s}} - \sum_{\{i\}\subseteq S\subseteq L(a)} \frac{\zeta(s)}{j^{n-s}} + \sum_{\emptyset\neq S\subseteq L(a)\setminus\{i\}} \frac{\zeta(s+1)}{j^{n-s}} \tag{2.46}$$

where, for all $\emptyset \neq S \subseteq N$, $\zeta(s) = \dfrac{(s-1)!(n-s)!}{n!(k-1)}$ and $\displaystyle\sum_{t\in\{\}} x_t = 0$.

Case 1: if $i \in H(a)$, then we have

$$\begin{aligned} \Delta_{\Phi_i}(a) &= \sum_{\{i\}\subseteq S\subseteq H(a)} \frac{\zeta(s)}{j^{n-s}} - \sum_{\emptyset\neq S\subseteq H(a)\setminus\{i\}} \frac{\zeta(s+1)}{j^{n-s}} + \sum_{\emptyset\neq S\subseteq L(a)} \frac{\zeta(s+1)}{j^{n-s}} \\ &= \sum_{s=1}^{h_a} \binom{h_a-1}{s-1} \cdot \frac{\zeta(s)}{j^{n-s}} - \sum_{s=1}^{h_a-1} \binom{h_a-1}{s} \cdot \frac{\zeta(s+1)}{j^{n-s}} + \sum_{s=1}^{l_a} \binom{s}{l_a} \cdot \frac{\zeta(s+1)}{j^{n-s}} \\ &= \frac{1}{n!(k-1)} \left[(h_a-1)! \left(\sum_{s=1}^{h_a} \frac{(n-s)!}{(h_a-s)!} \cdot \frac{1}{j^{n-s}} - \sum_{s=1}^{h_a-1} \frac{(n-s-1)!}{(h_a-s-1)!} \cdot \frac{1}{j^{n-s}} \right) + l_a! \cdot \sum_{s=1}^{l_a} \frac{(n-s-1)!}{(l_a-s)!} \cdot \frac{1}{j^{n-s}} \right] \\ &= \frac{1}{n!(k-1)j^n} \left[(j-1)(h_a-1)! j^{h_a-1} \cdot \sum_{s=0}^{h_a-1} \frac{(n-h_a+s)!}{s! j^s} + l_a! j^{l_a} \cdot \sum_{s=0}^{l_a} \frac{(n-l_a-1+s)!}{s! j^s} \right] \\ &= \frac{\gamma_j^n(l_a) + (j-1)\gamma_j^n(h_a-1)}{(k-1)n!j^n} \quad \text{by Equation (2.44).} \end{aligned}$$

Case 2: if $i \in I(a)$, then from Equation (2.46), we have:

$$\begin{aligned} \Delta_{\Phi_i}(a) &= -\sum_{\emptyset\neq S\subseteq H(a)} \frac{\zeta(s+1)}{j^{n-s}} + \sum_{\emptyset\neq S\subseteq L(a)} \frac{\zeta(s+1)}{j^{n-s}} \\ &= \sum_{s=1}^{l_a} \binom{s}{l_a} \cdot \frac{\zeta(s+1)}{j^{n-s}} - \sum_{s=1}^{h_a} \binom{s}{h_a} \cdot \frac{\zeta(s+1)}{j^{n-s}} \\ &= \frac{\gamma_j^n(l_a) - \gamma_j^n(h_a)}{(k-1)n!j^n} \quad \text{using case 1 and Equation (2.44).} \end{aligned}$$

Case 3: if $i \in L(a)$, one can use the two previous cases to obtain,

$$\Delta_{\Phi_i}(a) = -\frac{\gamma_j^n(h_a) + (j-1)\gamma_j^n(l_a-1)}{(k-1)n!j^n}.$$

■

From Proposition 2.3.3, we deduce Corollary 2.3.1 below. This result allows us to confirm that, Equation (2.45) is a generalization of Equation (2.33) (*who gives the total change of Shapley-Shubik index after adding one winning coalition in a simple game*) to uniform (j,k) simple games.

COROLLARY 2.3.1. *For any games $u, v \in \mathcal{U}_n^{2,k}$ such that $u \xrightarrow{a} v$, for some $a \in J^*$,*

$$\Phi_i(v) - \Phi_i(u) = \begin{cases} \dfrac{(h_a-1)! \cdot (n-h_a)!}{n!(k-1)} & \text{if } i \in H(a) \\ \\ -\dfrac{h_a! \cdot (n-h_a-1)!}{n!(k-1)} & \text{if } i \in L(a) \end{cases} \tag{2.47}$$

Proof.

Consider $u, v \in \mathcal{U}_n^{2,k}$ such that $u \xrightarrow{a} v$ for some $a \in J^*$. Since $j = 2$, then $H(a) \neq \emptyset$, $L(a) \neq \emptyset$ and $I(a) = \emptyset$. It appears that, $h_a + l_a = n$. [1] Consider $i \in H(a)$, then by Equation (2.45),

$$\begin{aligned} \Delta_{\Phi_i}(a) &= \frac{1}{n!(k-1)} \left[\frac{(h_a-1)!}{2^{n-h_a+1}} \cdot \sum_{s=0}^{h_a-1} \frac{(n-h_a+s)!}{s!} \cdot \frac{1}{2^s} + \frac{l_a!}{2^{n-l_a}} \cdot \sum_{s=0}^{l_a} \frac{(n-l_a-1+s)!}{s!} \cdot \frac{1}{2^s} \right] \\ &= \frac{1}{n!(k-1)} \left[\frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-h_a} \cdot (h_a-1)! \cdot f^{(n-h_a)}\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{h_a} \cdot (n-h_a)! \cdot f^{(h_a-1)}\left(\frac{1}{2}\right) \right] \text{ since } h_a + l_a = n.^{2} \\ &= \frac{1}{n!(k-1)} \left[\frac{1}{2} \cdot K_2\left(n, h_a, \frac{1}{2}\right) + K_1\left(n, h_a, \frac{1}{2}\right) \right] \quad \text{using Notation 1.3.1, page 31.}^{3} \\ &= \frac{(h_a-1)!(n-h_a)!}{n!(k-1)} \quad \text{by Proposition 1.3.5, page 31.} \end{aligned}$$

Similarly, for $i \in L(a)$ we easily compute

$$\Delta_{\Phi_i}(a) = -\frac{(l_a-1)! \cdot (n-l_a)!}{n!(k-1)} = -\frac{(n-h_a-1)! \cdot h_a!}{n!(k-1)}$$

In particular for uniform $(2,2)$ simple games, Equation (2.47) leads to Equation (2.33). ■

The characterization of the Shapley-Shubik index for (j,k) simple games suggests a characterization of that for CSGs. In the next chapter, we extend some axioms introduced in this chapter to provide two axiomatizations of the Shapley-Shubik index for CSGs.

[1] This equality plays a key role in this proof.
[2] f is defined by Equation (1.18), page 31.
[3] $K_1(n,k,\tau) = (1-\tau)^k (n-k)! f^{(k-1)}(\tau)$ and $K_2(n,k,\tau) = \tau^{n-k}(k-1)! f^{(n-k)}(1-\tau)$.

Chapter Three

Axiomatizations of the Shapley-Shubik index for continuous simple games

The Shapley-Shubik index was designed to evaluate the power distribution in committee systems drawing binary decisions and is one of the most established power indices. It has been extended to cover more classes of games such as ternary voting games, see Felsenthal and Machover (1997); or (j, k) voting games, see Freixas (2005b). In each case, a minimal set of axioms is brought out to identify the Shapley-Shubik index as the unique power index that succeeds in enjoying that set of axioms; except for continuous simple games on which we show that in this chapter that resizing, as usual, the classical axioms, namely efficiency, symmetry, null player property and transfer condition, is no longer sufficient to completely described the newly introduced power index. Instead, we provide two axiomatizations of the Shapley-Shubik index (Ψ) for CSGs thanks to our newly introduced axioms of average game convexity, homogeneous increments sharing and discreteness.

This chapter is organized into three sections. Section 3.1 is devoted to the presentation of the axioms. More precisely, after showing that the extension of Dubey (1975) axioms to CSGs is no longer sufficient to characterize the Shapley-Shubik index (Ψ) over CSGs, we motivate and introduce two new axioms. In Section 3.2, we provide the first axiomatization of Ψ and show that axioms used to achieve it independence. We end with Section 3.3 in which we give another axiomatization for Ψ similar to the one we did for the Shapley-Shubik index for uniform (j, k) simple games in Chapter 2. Note that most of the tools and results in Sections 3.1 and 3.2 are mainly taken from Kurz et al. (2019) working paper.

3.1 Axioms of characterization

The Shapley-Shubik index for simple games was identified by Dubey (1975) as the unique power index that simultaneously meets efficiency (E), symmetry (S), null player (NP) and transfer (T) property, see Theorem 1.1.1, page 11. Proposition 1.3.1, page 28 shows that the class of simple games can be identified with a subclass of CSGs via T^{τ}, for each $\tau \in]0, 1]$. It is also established that these embedding mapping preserve some properties of simple games (see Proposition 1.3.2, page 28 and Theorem 1.3.1, page 32). It is therefore natural to know whether the extension of Dubey's axioms to CSGs allows us to obtain a characterization of the Shapley-Shubik index (Ψ) as conjectured in Kurz (2014). Unfortunately, we prove here that Dubey's axioms over CSGs are no longer sufficient to uniquely characterize Ψ.

We first extend to CSGs some classical axioms of power indices defined on simple games.

Definition 3.1.1. *A power index $P : \mathcal{CSG}_n \longrightarrow \mathbb{R}^n$ satisfies :*

- ***positivity* (P)** *if $P(v) \neq \mathbf{0}$ and $P_i(v) \geq 0$ for all $i \in N$ and all $v \in \mathcal{CSG}_n$;*
- ***anonymity* (A)** *if for all $v \in \mathcal{CSG}_n$, for all $\pi \in \mathcal{S}_n$ and for all $i \in N$, $P_{\pi(i)}(\pi v) = P_i(v)$; where for all $x \in I^n$, $(\pi v)(x) = v(\pi(x))$;*
- ***symmetry* (S)** *if for all $v \in \mathcal{CSG}_n$ and for all symmetric players i and j of v, $P_i(v) = P_j(v)$;*
- ***efficiency* (E)** *if for all $v \in \mathcal{CSG}_n$, $\sum_{i\in N} P_i(v) = 1$;*
- *the* **null player (NP)** *axiom if for all $v \in \mathcal{CSG}_n$ and for all null player i in v, $P_i(v) = 0$;*
- ***transfer* (T)** *if for all $u,\, v \in \mathcal{CSG}_n$, for all $i \in N$, $P_i(u)+P_i(v) = P_i(u\vee v)+P_i(u\wedge v)$; where $(u\vee v)(x) = \max\{u(x),\, v(x)\}$ and $(u\wedge v)(x) = \min\{u(x),\, v(x)\}$ for all $x \in I^n$.*

3.1.1 Insufficiency of Dubey (1975) axioms over CSGs

We concretely set up a power index on CSGs that is efficient, symmetric, satisfies the null player property and transfer property but is different to the Shapley-Shubik index. To achieve this, we first define two families of power indices as follows:

Definition 3.1.2. *Given $x \in I^n$ and v be a CSG on N. The punctual Shapley-Shubik index in x associates the game v with the n-tuple $\Psi^x(v)$ defined for all $i \in N$ by :*

$$\Psi_i^x(v) = \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!}\left[\mathcal{C}^x(v,S) - \mathcal{C}^x(v,S\backslash\{i\})\right] \tag{3.1}$$

where, $\mathcal{C}^x(v,T) = v\left(\mathbf{1}_T,\, x_{-T}\right) - v\left(\mathbf{0}_T,\, x_{-T}\right)$, for all $\quad T \subseteq N$.

In other words, Ψ^x does not consider all possible vote vectors with equal probability but just a specific vote vector x. For simple games, the analogy of Definition 3.1.2 is the roll call model where either all players say "yes" (corresponding to $x = \mathbf{1}$) or all players say "no" (corresponding to $x = \mathbf{0}$). The punctual Shapley-Shubik index can easily be generalized by introducing a notion of density function.

Definition 3.1.3. Let v be a CSG on N, $f : I^n \to \mathbb{R}_{\geq 0}$ be a density function on I^n i.e. $\displaystyle\int_{I^n} f(x)\,\mathrm{d}\,x = 1$.

The *density Shapley-Shubik index* $\Psi_i^f(v)$ is defined by:

$$\Psi_i^f(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \left[\mathcal{C}^f(v,S) - \mathcal{C}^f(v, S\backslash\{i\})\right] \tag{3.2}$$

where, $\mathcal{C}^f(v,T) = \displaystyle\int_{I^n} \left(f(x) \cdot \left[v\left(\mathbf{1}_T, x_{-T}\right) - v\left(\mathbf{0}_T, x_{-T}\right)\right]\right) dx$, for all $\quad T \subseteq N$.

The analogy of Definition 3.1.3 for simple games is the roll call model with exchangeable probabilities for vote vectors $x \in \{0,1\}^n$ as shown in Hu (2006).

Lemma 3.1.1. Let $\mathscr{C}$ be a mapping that associates a pair (v, S) of a CSG v with player set N and a subset S of N to a real number $\mathscr{C}(v,S)$ such that, $\mathscr{C}(v,\emptyset) = 0$. With this, we set:

$$\Psi_i^{\mathscr{C}}(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot \left[\mathscr{C}(v,S) - \mathscr{C}(v, S\backslash\{i\})\right] \tag{3.3}$$

for all $i \in N$.

1. If $\mathscr{C}(v,N) = 1$ and $\mathscr{C}(v,S) \geq \mathscr{C}(v,S\backslash\{i\})$ for all $i \in N$ and all coalitions S such that $i \in S$, then $\Psi^{\mathscr{C}}$ is positive and efficient.

2. If $\mathscr{C}(u,S)+\mathscr{C}(v,S) = \mathscr{C}(u \vee v, S)+\mathscr{C}(u \wedge v, S)$ for all coalitions S, then $\Psi^{\mathscr{C}}$ satisfies the transfer property.

3. If $\mathscr{C}(v,S) = \mathscr{C}(\pi v, \pi S)$ for all $\pi \in \mathcal{S}_n$, then $\Psi^{\mathscr{C}}$ is anonymous.

4. If $\mathscr{C}(v,S\backslash\{i\}) = \mathscr{C}(v,S\backslash\{j\})$ for any pair of symmetric players i,j in v and any coalition S such that $\{i,j\} \subseteq S$, then $\Psi^{\mathscr{C}}$ is symmetric.

5. If $\mathscr{C}(v,S) = \mathscr{C}(v,S\backslash\{i\})$ for all null player i in v and any coalition S such that $i \in S$, then $\Psi^{\mathscr{C}}$ satisfies the null player property.

Proof. We first remark that given a CSG v, the function $\mathscr{C}(v,\cdot)$ that maps each coalition S to a real number $\mathscr{C}(v,\,S)$ is a TU-game on N. Thus, $\Psi^{\mathscr{C}}(v)$ is just the Shapley value of the TU-game $\mathscr{C}(v,\cdot)$. Now consider $u, v \in \mathcal{CSG}_n$.

1. If $\mathscr{C}(v,S) \geq \mathscr{C}(v,S\backslash\{i\})$ for all coalition $\{i\} \subseteq S \subseteq N$, then $\mathscr{C}(v,S) - \mathscr{C}(v,S\backslash\{i\}) \geq 0$. It follows from Equation (3.3) that $\Psi_i^{\mathscr{C}}(v) \geq 0$. Moreover, by the efficiency of the Shapley value, $\sum_{p \in N} \Psi_p^{\mathscr{C}}(v) = \mathscr{C}(v,N) = 1$, since $\mathscr{C}(v,N) = 1$. One concludes that $\Psi^{\mathscr{C}}$ is efficient and positive.

2. Assume that for all $S \subseteq N$, $\mathscr{C}(u,S) + \mathscr{C}(v,S) = \mathscr{C}(u \vee v, S) + \mathscr{C}(u \wedge v, S)$. Since the Shapley value is additive, then $\Psi^{\mathscr{C}}(u) + \Psi^{\mathscr{C}}(v) = \Psi^{\mathscr{C}}(u \vee v) + \Psi^{\mathscr{C}}(u \wedge v)$. So, $\Psi^{\mathscr{C}}$ satisfies the transfer property.

3. Consider $\pi \in \mathcal{S}_n$ such that $\mathscr{C}(\pi v, \pi S) = \mathscr{C}(v, S)$, for all $S \subseteq N$. Let $i \in N$ be a player, by Equation (3.3), we have:

$$\begin{aligned}
\Psi^{\mathscr{C}}_{\pi(i)}(\pi v) &= \sum_{\pi(i)\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!}\cdot[\mathscr{C}(\pi v, S)-\mathscr{C}(\pi v, S\backslash\{i\})] \\
&= \sum_{i\in\pi^{-1}(S)\subseteq N} \frac{(s-1)!(n-s)!}{n!}\cdot[\mathscr{C}(\pi v, S)-\mathscr{C}(\pi v, S\backslash\{i\})] \\
&= \sum_{i\in T\subseteq N} \frac{(t-1)!(n-t)!}{n!}\cdot[\mathscr{C}(\pi v, \pi(T))-\mathscr{C}(\pi v, \pi(T\backslash\{i\}))] \quad \text{with } T=\pi^{-1}(S) \\
&= \sum_{i\in T\subseteq N} \frac{(s-1)!(n-s)!}{n!}\cdot[\mathscr{C}(v, T)-\mathscr{C}(v, T\backslash\{i\})] \quad \text{since } \mathscr{C}(\pi v,\, \pi T)=\mathscr{C}(v,\, T) \text{ for all } T\subseteq N \\
&= \Psi^{\mathscr{C}}_i(v)\,.
\end{aligned}$$

Therefore, $\Psi^{\mathscr{C}}$ is anonyme.

4. Consider two symmetric players i and j in v such that $\mathscr{C}(v, S\backslash\{i\}) = \mathscr{C}(v, S\backslash\{j\})$ for all $S \subseteq N$, then, the players i and j are symmetric in $\mathscr{C}(v, \cdot)$. So, by symmetry of the Shapley value, $\Psi^{\mathscr{C}}_i(v) = \Psi^{\mathscr{C}}_j(v)$.

5. Directly from Equation (3.3) we obtain $\Psi^{\mathscr{C}}_i(v) = 0$.

■

Corollary 3.1.1.

1. *For every $\alpha \in [0,1]$, the punctual Shapley-Shubik index Ψ^a, where $a = (\alpha, \dots, \alpha) \in I^n$ is positive, efficient, anonymous, symmetric, and satisfies both the null player and the transfer property.*

2. *For all symmetric density function f (i.e. $f(x) = f(\pi(x))$ for any $x \in I^n$ and any $\pi \in \mathcal{S}_n$), the density Shapley-Shubik index Ψ^f is positive, efficient, anonymous, symmetric, and satisfies both the null player and the transfer property.*

Proof. Consider $a = (\alpha, \dots, \alpha) \in I^n$ and f a symmetric density function on I^n. We remark that $\Psi^a = \Psi^{\mathcal{C}^a}$ and $\Psi^f = \Psi^{\mathcal{C}^f}$. Therefore, by Lemma 3.1.1 one directly concludes that Ψ^a and Ψ^f are positive, efficient, symmetric and satisfy both the null player and the transfer property.

To prove the anonymity of Ψ^a and Ψ^f, consider $\pi \in \mathcal{S}_n$, $S \subseteq N$, $b \in [0,\, 1]$ and $c \in I^n$. Pose $x = \big(\mathbf{b}_{\pi(S)},\, c_{-\pi(S)}\big)$ and $y = \pi x$. Then for $i \in N$, $x_i = b$ if $\pi^{-1}(i) \in S$ and $x_i = c_i$ otherwise. Therefore,

$$y_i = x_{\pi(i)} = \begin{cases} b & \text{if} \quad i \in S \\ c_{\pi(i)} & \text{if} \quad i \notin S \end{cases} = (\mathbf{b}_S,\, (\pi c)_{-S})_i\;. \tag{3.4}$$

So, $y = \pi x = (\mathbf{b}_S, (\pi c)_{-S})$. Now consider a CSG v:

$$\begin{aligned}
\mathcal{C}^a(\pi v, \pi(S)) =& \pi v\left(\mathbf{1}_{\pi(S)}, a_{-\pi(S)}\right) - \pi v\left(\mathbf{0}_{\pi(S)}, a_{-\pi(S)}\right) \\
=& v\left(\pi(\mathbf{1}_{\pi(S)}, a_{-\pi(S)})\right) - v\left(\pi(\mathbf{0}_{\pi(S)}, a_{-\pi(S)})\right) \\
=& v\left(\mathbf{1}_S, (\pi a)_{-S}\right) - v\left(\mathbf{0}_S, (\pi a)_{-S}\right) \quad \text{(by Equation (3.4))} \\
&= v\left(\mathbf{1}_S, a_{-S}\right) - v\left(\mathbf{0}_S, a_{-S}\right) \quad \text{Since } \pi a = a \\
&= \mathcal{C}^a(v, S)\,.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}^f(\pi v, \pi S) &= \int_{I^n} \left(f(x)\cdot\left[\pi v\left(\mathbf{1}_{\pi(S)}, x_{-\pi(S)}\right) - \pi v\left(\mathbf{0}_{\pi(S)}, x_{-\pi(S)}\right)\right]\right) dx \\
&= \int_{I^n} \left(f(x)\cdot\left[v\left(\pi(\mathbf{1}_{\pi(S)}, x_{-\pi(S)})\right) - v\left(\pi(\mathbf{0}_{\pi(S)}, x_{-\pi(S)})\right)\right]\right) dx \\
&= \int_{I^n} \left(f(x)\cdot\left[v\left(\mathbf{1}_S, (\pi x)_{-S}\right) - v\left(\mathbf{0}_S, (\pi x)_{-\pi(S)}\right)\right]\right) dx, \text{ by Equation (3.4)} \\
&= \int_{I^n} \left(f(\pi^{-1}y)\cdot\left[v\left(\mathbf{1}_S, y_{-S}\right) - v\left(\mathbf{0}_S, y_{-S}\right)\right]\right) dy, \quad \text{with } y = \pi x \\
&= \mathcal{C}^f(v, S), \text{ since } f \text{ is a symmetric density function.}
\end{aligned}$$

By Lemma 3.1.1 (item 3), one concludes that Ψ^a and Ψ^f satisfy anonymity. ■

Kurz (2014) conjectured that every power index for CSGs that satisfies symmetry, efficiency, the null player property, and the transfer property coincides with Ψ. However, the construction in the Corollary 3.1.1 combined with the following proposition prove that this conjecture fails.

Proposition 3.1.1. *$\Psi^a \neq \Psi$ for some $a = (\alpha, \ldots, \alpha) \in I^n$ and $\Psi^f \neq \Psi$ for at least one symmetric density function f.*

Proof. Let v be a CSG on N defined as follows:

$$\text{for all } x = (x_1, \cdots, x_n) \in I^n, \quad v(x_1, \cdots, x_n) = x_1 x_2^2\,.$$

Consider $x \in I^n$ and $T \subseteq N$ then

$$\mathcal{C}^x(v, T) = v(\mathbf{1}_T, x_{-T}) - v(\mathbf{0}_T, x_{-T}) = \begin{cases} 1 & \text{if } 1 \in T \text{ and } 2 \in T \\ x_2^2 & \text{if } 1 \in T \text{ and } 2 \notin T \\ x_1 & \text{if } 1 \notin T \text{ and } 2 \in T \\ 0 & \text{if } 1 \notin T \text{ and } 2 \notin T \end{cases} \tag{3.5}$$

Consider $a = (\alpha, \ldots, \alpha) \in I^n$. By Equation (3.1)

$$\Psi^a(v) = \left(\frac{1}{2} - \frac{\alpha - \alpha^2}{2}, \frac{1}{2} + \frac{\alpha - \alpha^2}{2}, 0, \ldots, 0\right)$$

Thus for any $\alpha \neq \frac{1}{2} \pm \frac{1}{2\sqrt{3}}$, $\Psi^a(v) \neq \Psi(v)$.

Now let f_β be a symmetric density function on I^n defined as follows: for all $x \in I^n$, $f_\beta(x) = (\beta+1)^n \cdot \left(\prod_{i\in N} x_i^\beta\right)$, $\beta \in \mathbb{R}_{\geq 0}$. Note that for $\beta = 0$, $\Psi^{f_0} = \Psi$. From Equation (3.5), one gets:

$$\mathcal{C}^{f_\beta}(v,T) = \begin{cases} 1 & \text{if} \quad 1 \in T \text{ and } 2 \in T \\ \frac{\beta+1}{\beta+3} & \text{if} \quad 1 \in T \text{ and } 2 \notin T \\ \frac{\beta+1}{\beta+2} & \text{if} \quad 1 \notin T \text{ and } 2 \in T \\ 0 & \text{if} \quad 1 \notin T \text{ and } 2 \notin T \end{cases}$$

Thus, by Equation (3.2),

$$\Psi^{f_\beta}(v) = \left(\frac{\beta^2+4\beta+5}{2\beta^2+10\beta+12}, \frac{\beta^2+6\beta+7}{2\beta^2+10\beta+12}, 0, \ldots, 0\right)$$

Therefore, for $\beta \in \mathbb{R}_{\geq 0}$ such that $\beta \notin \{0,1\}$, $\Psi^{f_\beta}(v) \neq \Psi(v)$. ∎

3.1.2 Axiom of Homogeneous Increments Sharing

Corollary 3.1.1 together with Proposition 3.1.1 point out the fact that, the axioms of efficiency (E), symmetry (S), null player property (NP) and the transfer property (T) are not sufficient to characterize the Shapley-Shubik power index (Ψ). Also anonymity (A) and positivity (P) are satisfied by our parametric examples of power indices. So, for an axiomatization we need some further axioms. Inspired by the axiom of (SymGL*) introduced in Page 62, we set up a new axiom of power index for CSGs called *Homogeneous Increments Sharing*.

Definition 3.1.4. *Let v be a CSG and $S \in 2^N \backslash \{N\}$. The potential influence (potential) of the coalition S in v is the mapping $\Delta v(S, \cdot)$, that associates each profile $x \in I^n$ to the real number $\Delta v(S, x_{-S})$, given by $\Delta v(S, x_{-S}) = v(\mathbf{1}_S,\, x_{-S}) - v(\mathbf{0}_S,\, x_{-S})$.*

Given a profile $x \in I^n$, the potential influence $\Delta v(S, x_{-S})$ of S measures the greatest change in the social decision that may be observed when players in S change their respective opinions from 0 to 1 while the opinions of players not in S remain constant accordingly to profile x.

Example 3.1.1. *Let v be an exponential game with vector $\alpha = (\alpha_1, \cdots, \alpha_n)$. Given a profile $x \in I^n$ and a coalition S, it can be easily checked that the potential of S in v is given by*

$$\Delta v(S,\, x_{-S}) = \prod_{i\in N\backslash S} x_i^{\alpha_i}.$$

Definition 3.1.5. *Consider $T \in 2^N$. A T-domain is a cartesian product $D = \bigtimes_{i\in T} [a_i, b_i]$ with $a_i, b_i \in [0,1]$ for all $i \in T$.*

Considering the T-domain $D = \bigtimes_{i\in T} [a_i, b_i]$ amounts to assuming that each player $i \in T$, freely and independently, from others chooses his level of approbation from $[a_i, b_i] \subseteq [0,\, 1]$.

Definition 3.1.6. Let v be a CSG on N, $S \in 2^N\backslash\{N\}$, $\varepsilon \in \mathbb{R}_{\geq 0}$, and $D = \bigtimes_{i\in N\backslash S} [a_i, b_i]$ be an $(N\backslash S)$-domain, where $0 \leq a_i < b_i \leq 1$ for all $i \in N\backslash S$. If a CSG u satisfies

- $\forall x_{-S} \in]0,1[^{N\backslash S}$ $\Delta v\,(S, x_{-S}) = \begin{cases} \Delta u\,(S,\, x_{-S}) & \text{if} \quad x_{-S} \notin \bigtimes_{i\in N\backslash S} [a_i, b_i] \\ \\ \Delta u\,(S, x_{-S}) + \varepsilon & \text{if} \quad x_{-S} \in \bigtimes_{i\in N\backslash S}]a_i, b_i[\end{cases}$

- $\forall T \in 2^N \backslash \{S\}, \forall x_{-T} \in]0,1[^{N\backslash T}$, $\Delta v\,(T, x_{-T}) = \Delta u\,(T, x_{-T})$,

then v is a *local increment (improvement)* of u and we write $u \xrightarrow{S,\,\varepsilon,\,D} v$.

In words, $u \xrightarrow{S,\,\varepsilon,\,D} v$ can be interpreted as follows: On the one hand, the potential of coalition S increases by a constant increment ε whenever each player $i \in N\backslash S$ picks his opinion from $]a_i, b_i[$, but remains unchanged if the opinion of at least one player $i \in N\backslash S$ is outside of $[a_i, b_i]$. It is then reasonable that the corresponding increment in the collective decision mainly comes from players in S and is uniform, local and elsewhere valid on $\bigtimes_{i\in S}]a_i, b_i[$. On the other hand, the potential of any other coalition T remains unchanged unless some players in S show a full support ($x_i = 1$), or no support ($x_i = 0$). In such situations, the shares by a conceivable power index from u to v are expected to change accordingly by only uniformly rewarding players in S in the expense of players outside of S. The following example is an illustration of local improvement with $n = 2$.

Example 3.1.2. Let u and v be 2-players CSGs defined as follows: for all $x \in [0,1]^2\backslash\{\mathbf{0}, \mathbf{1}\}$,

$$u(x) = \begin{cases} 0.1 & \text{if} \quad x \in \left[0, \frac{1}{2}\right[\times \left[0, \frac{1}{2}\right[\\ 0.6 & \text{if} \quad x \in \left[0, \frac{1}{2}\right[\times \left[\frac{1}{2}, 1\right] \\ 0.3 & \text{if} \quad x \in \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right[\\ 0.9 & \text{if} \quad x \in \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 0.1 & \text{if} \quad x \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right[\\ 0.6 & \text{if} \quad x \in \left[0, \frac{1}{2}\right[\times \left[\frac{1}{2}, 1\right[\\ 0.8 & \text{if} \quad x \in \left[0, \frac{1}{2}\right[\times \{1\} \\ 0.3 & \text{if} \quad x \in \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right[\\ 0.9 & \text{if} \quad x \in \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right] \end{cases}$$

Consider $t \in]0,1[$. Then we compute,

$$\Delta u(\{1\},\, t) = \begin{cases} 0.2 & \text{if} \quad 0 \leq t < \frac{1}{2} \\ 0.3 & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases} \quad \text{and} \quad \Delta v(\{1\},\, t) = \begin{cases} 0.2 & \text{if} \quad 0 \leq t < \frac{1}{2} \\ 0.3 & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$\Delta u(\{2\},\, t) = \begin{cases} 0.5 & \text{if} \quad 0 \leq t < \frac{1}{2} \\ 0.6 & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases} \quad \text{and} \quad \Delta v(\{2\},\, t) = \begin{cases} 0.7 & \text{if} \quad 0 \leq t < \frac{1}{2} \\ 0.6 & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

Thus, $\Delta u(\{1\}, t) = \Delta v(\{1\}, t)$ and $\Delta v(\{2\}, t) = \begin{cases} \Delta u(\{2\}, t) + 0.2 & \text{if} \quad 0 \leq t < \frac{1}{2} \\ \Delta u(\{2\}, t) & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$

One concludes that $u \xrightarrow{\{2\},\, 0.2,\, \left[0,\, \frac{1}{2}\right]} v$.

Remark 3.1.1. *Let u, v and w be three CSGs.*

1. *We extend the notion of local improvement to negative parameters ε by the equivalence: $u \xrightarrow{S,\, \varepsilon,\, D} v \Longleftrightarrow v \xrightarrow{S,\, -\varepsilon,\, D} u$.*

2. *If $u \xrightarrow{S,\, \varepsilon,\, D} v$ and $\Delta v = \Delta w$ then $u \xrightarrow{S,\, \varepsilon,\, D} w$.*

Definition 3.1.7. *A power index F for CSGs satisfies the* homogeneous increments sharing *(HIS) axiom, if for all $S \in 2^N \backslash \{N\}$ and for all CSGs u and v such that $u \xrightarrow{S,\, \varepsilon,\, D} v$ for some $\varepsilon > 0$ and some $(N \backslash S)$-domain D, then for any player $i \in N$,*

$$F_i(v) - F_i(u) = \begin{cases} \lambda_F(S) \cdot \varepsilon \cdot \text{vol}(D) & \text{if} \quad i \in S, \\ \\ -\gamma_F(S) \cdot \varepsilon \cdot \text{vol}(D) & \text{if} \quad i \notin S, \end{cases} \tag{3.6}$$

where $\lambda_F(S)$ and $\gamma_F(S)$ are two real constants that do only depend on S, i.e. they do neither depend on u and v nor on ε and D, and $\text{vol}(D)$ denotes the volume of D.

The quantity $\varepsilon \cdot \text{vol}(D)$ captures the fact that the change in the share of a player is both proportional to the magnitude ε of the homogeneous increment and to the (local) volume $\text{vol}(D)$ of the domain on which this change occurs.

For simple games, an analog of (HIS) is the axiom of *Symmetric Gain-Loss* (SymGL) given in Page 62, while an analog for (j, k) simple games is the axiom of (SymGL*) defined in Page 64.

Lemma 3.1.2. *The power index Ψ for CSGs satisfies (HIS) and for any $S \in 2^N \backslash \{N\}$,*

$$(\lambda_\Psi(S), \gamma_\Psi(S)) = \left(\frac{(s-1)!(n-s)!}{n!}, \frac{s!(n-s-1)!}{n!} \right) \tag{3.7}$$

Proof.

Let u, v, S, ε, and D be given such that $u \xrightarrow{S,\, \varepsilon,\, D} v$ and $S \notin \{\emptyset, N\}$. Consider $T \neq N$, due to the formula for $\Delta v(\cdot, \cdot)$ in Definition 3.1.6, we have

$$\mathcal{C}(v, T) = \int_{I^n} \Delta(T, x_{-T})\, dx = \begin{cases} \mathcal{C}(u, T) + \varepsilon \cdot \text{vol}(D) & \text{if} \quad T = S \\ \mathcal{C}(u, T) & \text{if} \quad T \neq S \end{cases} \tag{3.8}$$

Thus for any $i \in N$, Equation (1.17), page 25 implies:

$$\begin{aligned} \Psi_i(v) - \Psi_i(u) &= \sum_{i \in T \subseteq N} \frac{(t-1)!(n-t)!}{n!} [\mathcal{C}(v, T) - \mathcal{C}(u, T)] - \sum_{i \notin T \subseteq N} \frac{t!(n-t-1)!}{n!} [\mathcal{C}(v, T) - \mathcal{C}(u, T)] \\ &= \begin{cases} \frac{(s-1)!(n-s)!}{n!} \cdot \varepsilon \cdot \text{vol(D)} & \text{if} \quad i \in S \\ -\frac{s!(n-s-1)!}{n!} \cdot \varepsilon \cdot \text{vol(D)} & \text{if} \quad i \notin S \end{cases} \quad \text{by Equation (3.8)} \end{aligned}$$

Then $\lambda_\Psi(S) = \frac{(s-1)!(n-s)!}{n!}$ for any $i \in S$ and $\gamma_\Psi(S) = \frac{s!(n-s-1)!}{n!}$ for any $i \notin S$. ■

Hereafter, given $x \in I^n$, we denote by $\mathbf{1}_x = \{i \in N, x_i = 1\}$ and $\mathbf{0}_x = \{i \in N, x_i = 0\}$. For $S \in 2^N \setminus \{N\}$, we define the CSGs u_S and v_S as follows:

$$u_S(x) = \begin{cases} 1 & \text{if } |\mathbf{1}_x| > |S| \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad v_S(x) = \begin{cases} 1 & \text{if}\, u_S(x) = 1 \,\text{or}\, x_S = \mathbf{1}_S \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

Proposition 3.1.2. *If F is a power index on $\mathcal{CSG}_n$ that satisfies anonymity and (HIS), then for $S, T \in 2^N \setminus \{N\}$ such that $|S| = |T|$,*

$$\lambda_F(S) = \lambda_F(T) \text{ and } \gamma_F(S) = \gamma_F(T).$$

Proof.

Let F be a power index on $\mathcal{CSG}_n$ and $S, T \in 2^N \setminus \{N\}$ such that $|S| = |T|$. Then, there exists a permutation π on N such that $\pi(S) = T$ and $\pi(S^c) = T^c$. Consider $X \in \{S, T\}$, we easily obtain $\pi u_X = u_X$, $u_S = u_T$, $\pi v_S = v_T$ and $u_X \xrightarrow{X,\,1,\,D_X} v_X$, where $D_X = [0,\,1]^{N \setminus X}$. Given $i \in S$, we have $\pi(i) \in T$ and by (HIS) one gets:

$$\begin{aligned}
\lambda_F(T) =& F_{\pi(i)}(v_T) - F_{\pi(i)}(u_T) \quad \text{since}\, u_T \xrightarrow{T,\,1,\,D_T} v_T \\
=& F_{\pi(i)}(\pi v_S) - F_{\pi(i)}(\pi u_S) \quad \text{since } \pi v_S = v_T \text{ and } \pi u_S = u_S = u_T \\
=& F_i\,(v_S) - F_i\,(u_S) \quad \text{since } F \text{ satisfies (A)} \\
=& \lambda_F(S) \quad \text{by}(HIS)
\end{aligned}$$

Similarly, consider $j \notin S$. Then $\pi(j) \notin T$ and by (HIS) we have:

$$\begin{aligned}
-\gamma_F(T) =& F_{\pi(j)}(v_T) - F_{\pi(j)}(u_T) \quad \text{since } u_T \xrightarrow{T,\,1,\,D_T} v_T \\
=& F_{\pi(j)}(\pi v_S) - F_{\pi(j)}(\pi u_S) \quad \text{since } \pi v_S = v_T \text{ and } \pi u_S = u_S = u_T \\
=& F_j\,(v_S) - F_j\,(u_S) \quad \text{since } F \text{ satisfies (A)} \\
=& -\gamma_F(S)
\end{aligned}$$

Finally, $\lambda_F(S) = \lambda_F(T)$ and $\gamma_F(S) = \gamma_F(T)$. ■

In the following lemma, it is shown that all power indices on $\mathcal{CSG}_n$ that are (A), (E), (NP) and (HIS) all share homogeneous increments with respect to same pair of proportionality constants; those of the Shapley-Shubik index Ψ.

Lemma 3.1.3. *If F is a power index on $\mathcal{CSG}_n$ that simultaneously satisfies (E), (A), (NP) and (HIS), then for all $S \in 2^N \setminus \{N\}$:*

$$\lambda_F(S) = \lambda_\Psi(S) \text{ and } \gamma_F(S) = \gamma_\Psi(S). \tag{3.10}$$

Proof.

Consider $S \in 2^N \backslash \{N\}$, then there exists $i \in N$ such that $i \notin S$. We set $\mathcal{E} = \{T \in 2^N,\, i \notin T \text{ and } |T| = |S|\}$. The set $\mathcal{E}$ contains $q = \begin{pmatrix} n-1 \\ s \end{pmatrix}$ coalitions, so can be labelling by $\mathcal{E} = \{S_1, S_2, \cdots, S_q\}$. Consider the sequence of CSGs $(w_k)_{0 \leq k \leq q}$ defined by $w_0 = u_S$ and for all $1 \leq k \leq q$,

$$w_k(x) = \begin{cases} 1 & \text{if } u_S(x) = 1 \text{ or } \exists l \leq k,\, x_{S_l} = \mathbf{1}_{S_l} \\ 0 & \text{otherwise} \end{cases}$$

We can notice that, for all $1 \leq k \leq q$,

$$w_k(x) = \begin{cases} 1 & \text{if } x_{S_k} = \mathbf{1}_{S_k} \\ w_{k-1}(x) & \text{otherwise} \end{cases}$$

Therefore, we easily have $w_{k-1} \xrightarrow{S_k,\, 1,\, D_k} w_k$, where $D_k = [0,\, 1]^{N \backslash S_k}$. Consequently, Proposition 3.1.2 and (HIS) axiom lead to $F_i(w_k) = F_i(w_{k-1}) - \gamma_F(S)$, then $F_i(w_q) = F_i(w_0) - q \cdot \gamma_F(S) = \dfrac{1}{n} - q \cdot \gamma_F(S)$. But player i is null in the game w_q and all players in w_0 are symmetric. Thus, by (NP), (A) and (E) we then obtain,

$$\frac{1}{n} - q \cdot \gamma_F(S) = 0 \Longrightarrow \gamma_F(S) = \frac{1}{nq} = \frac{s!(n-s-1)!}{n!} = \gamma_\Psi(S)\,.$$

Moreover, $w_0 \xrightarrow{S_1,\, 1,\, D_1} w_1$, thus by (HIS) axiom we can write,

$$F_a(w_1) = F_a(w_0) + \lambda_F(S_1) \text{ and } F_b(w_1) = F_b(w_0) - \gamma_F(S_1)\,.$$

for all $(a, b) \in S_1 \times (N \backslash S_1)$. It follows that,

$$\sum_{p \in N} F_p(w_1) = \sum_{p \in N} F_p(w_0) + |S_1| \cdot \lambda_F(S_1) - (n - |S_1|) \cdot \gamma_F(S)$$

Finally, by (E) and Proposition 3.1.2 one gets,

$$\lambda_F(S) = \frac{n-s}{s}\gamma_F(S) = \frac{(s-1)!(n-s)!}{n!} = \lambda_\Psi(S)\,.$$

■

For CSGs with at least three players, we show that anonymity is redundant in the previous lemma. To do this, recall that if u and v are two simple games such that the winning coalitions of v are given by the winning coalitions of u and a coalition $S \in 2^N \backslash \{N\}$ that is losing in u (noted $v = u \oplus S$ in Page 62), then for all $i \in S$ and $j \in N \backslash S$ we have

$$\mathrm{SSI}_i(v) = \mathrm{SSI}_i(u) + \frac{(s-1)!(n-s)!}{n!} \quad \text{and} \quad \mathrm{SSI}_j(v) = \mathrm{SSI}_j(u) - \frac{s!(n-s-1)!}{n!}\,. \tag{3.11}$$

To each simple game v we associate a CSG $\widetilde{v}$ defined as follows: $\widetilde{v}\,(x) = 1$ if $v(\mathbf{1}_x) = 1$ and $\widetilde{v}\,(x) = 0$ otherwise. It can be easily checked that this embedding transfers the null player property, i.e. a null player in v is also a null player in $\widetilde{v}$.

PROPOSITION 3.1.3. *Let u and v be two simple games on N and $S \in 2^N \backslash \{N\}$.*

1. *If $v = u \oplus S$, then $\widetilde{u} \xrightarrow{S,\,1,\,D} \widetilde{v}$, with $D = \bigtimes_{i \in N \backslash S} [0,\,1]$.*

2. *If $\mathcal{W}(u) = \{T \subseteq N,\ S \subset T\}$ and $v = u \oplus S$, then all players in $N \backslash S$ are null in v.*

3. *If $u = [n : 1, \ldots, 1]$ with $n \geq 3$ and F is a power index on $\mathcal{CSG}_n$ that satisfies (E), (NP), and (HIS), then $F(\widetilde{u}) = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$.*

Proof.

Consider u and v two simple games on N and $S \in 2^N \backslash \{N\}$.

1. Assume that $v = u \oplus S$. Consider $x_{-S} \in]0,1[^{N \backslash S}$. Since S is loosing in u and winning in v, then $\Delta\widetilde{u}(S, x_{-S}) = 0$ and $\Delta\widetilde{v}(S, x_{-S}) = 1$. That is $\Delta\widetilde{v}(S, x_{-S}) = \Delta\widetilde{u}(S, x_{-S}) + 1$. Furthermore, for any other coalition $T \neq S$, $T \in \mathcal{W}(u)$ iff $T \in \mathcal{W}(v)$. Thus, for all $x_{-T} \in]0,1[^{N \backslash T}$, $\Delta\widetilde{v}(T, x_{-T}) = \Delta\widetilde{u}(T, x_{-T})$. It follows from Definition 3.1.7 that $\widetilde{u} \xrightarrow{S,\,1,\,D} \widetilde{v}$.

2. If $\mathcal{W}(u) = \{T \subseteq N,\ S \subset T\}$ and $v = u \oplus S$ then, $\mathcal{W}(v) = \{T \subseteq N,\ S \subseteq T\}$. Consider $i \in N \backslash S$ then, for all $T \subseteq N \backslash \{i\}$, $S \subseteq T$ iff $S \subseteq T \cup \{i\}$, thus $v(T \cup \{i\}) = v(T)$. Therefore, i is a null player in v.

3. Now, let $u = [n : 1, \ldots, 1]$, $n \geq 3$ and F be a power index on $\mathcal{CSG}_n$ that satisfies (E), (NP), and (HIS). Let us prove that $F_i(\widetilde{u}) = \frac{1}{n}$ for all $i \in N$. Let i and j be two arbitrary players and $X = N \backslash \{i, j\}$. Since $n \geq 3$, $X \neq \emptyset$. Pose $F(\widetilde{u}) = (a_1, \ldots, a_n)$ and $v_k = u \oplus (X \cup \{k\})$ for all $k \in \{i, j\}$. As above, player j is a null player in v_i and $\widetilde{u} \xrightarrow{X \cup \{i\},\,1,\,D} \widetilde{v_i}$, so $\gamma_F(X \cup \{i\}) = a_j$ by (HIS) and (NP). By (E) we conclude that $\lambda_F(X \cup \{i\}) = \dfrac{1}{n-1} \lambda_F(X \cup \{i\}) = \dfrac{a_j}{n-1}$. Similarly, we conclude $\gamma_F(X \cup \{j\}) = a_i$ and $\lambda_F(X \cup \{j\}) = \dfrac{a_i}{n-1}$.
   Now pose $v_{ij} = u \oplus (X \cup \{i\}) \oplus (X \cup \{j\}) = v_i \oplus (X \cup \{j\}) = v_j \oplus (X \cup \{i\})$. From the above constants and (HIS), $F_i(\widetilde{v_{ij}}) = \dfrac{a_j}{n-1}$ and $F_j(\widetilde{v_{ij}}) = \dfrac{a_i}{n-1}$. In $v = v_{ij} \oplus X$ the players i and j are null players and $\widetilde{v_{ij}} \xrightarrow{X,\,1,\,D} \widetilde{v}$, so $a_i = a_j$ by (NP) and (HIS). Since i and j were arbitrary, we have $F_i(\widetilde{u}) = \frac{1}{n}$ for all $i \in N$ by efficiency.

■

LEMMA 3.1.4. *If F is a power index for continuous simple games that simultaneously satisfies (E), (NP), and (HIS) and $n \geq 3$, then:*

$$(\lambda_F(S), \gamma_F(S)) = \left(\frac{(s-1)!(n-s)!}{n!}, \frac{s!(n-s-1)!}{n!}\right) \tag{3.12}$$

for all $S \in 2^N \backslash \{N\}$.

Proof. Consider $S \in 2^N \backslash \{N\}$. We prove Equation (3.12) by induction from $s = n-1$ to $s = 1$. Moreover, we show at each induction stage s that, $\mathrm{SSI}(v) = F(\widetilde{v})$ whenever all winning coalitions in v are of cardinality greater or equal to s.

Consider $u = [n : 1, \ldots, 1]$ and $v = u \oplus S$ for some coalition $S \subsetneq N$ of cardinality $s = n-1$. It follows from Proposition 3.1.3 that, $\widetilde{u} \xrightarrow{S,\,1,\,D} \widetilde{v}$, $F_i(\widetilde{u}) = \frac{1}{n}$ for all $i \in N$ and the unique player j in $N \backslash S$ is a null player in $\widetilde{v}$. Then, $F_j(\widetilde{v}) = 0$ and $\gamma_F(S) = \frac{s!(n-s-1)!}{n!}$ by (HIS). By efficiency we then conclude $\lambda_F(S) = \frac{(s-1)!(n-s)!}{n!}$. Note that $\mathrm{SSI}(u) = F(\widetilde{u})$ and by (HIS), $\mathrm{SSI}(v) = F(\widetilde{v})$. Moreover, $\mathrm{SSI}(v) = F(\widetilde{v})$ whenever $v = u \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_p$ for some coalitions S_k each of cardinality $n-1$ by applying (HIS) p-times together with Equation (3.11). So, the induction start is made.

Now, consider $S \subseteq N$ with $0 < s < n$. To determine $\gamma_F(S)$, let u be the simple game with $\mathcal{W}(u) = \{T \subseteq N,\, S \subset T\}$. For the corresponding CSG $\widetilde{u}$ we have $\mathrm{SSI}(u) = F(\widetilde{u})$ by the induction hypothesis. For $v = u \oplus S$, Proposition 3.1.3 permits to conclude that all players $j \in N \backslash S$ are null players in v, therefore $F_j(\widetilde{v}) = \mathrm{SSI}_j(v) = 0$. With this, we easily compute $\gamma_F(S) = F_j(\widetilde{u}) = \mathrm{SSI}_j(u) = \frac{s!(n-s-1)!}{n!}$ by (HIS) and Equation (3.11), which gives $\lambda_F(S) = \frac{(s-1)!(n-s)!}{n!}$ by efficiency. Moreover, suppose that u is any other simple game whose winning coalitions are of cardinality greater than s and that $v = u \oplus S_1 \oplus S_2 \oplus \ldots \oplus S_q$. Then $\mathrm{SSI}(u) = F(\widetilde{u})$ by induction hypothesis, and applying (HIS) q-times together with Equation (3.11) yields $\mathrm{SSI}(v) = F(\widetilde{v})$. ∎

Note that for $n = 2$, axioms (E), (NP) and (HIS) are not sufficient to uniquely determine the HIS's constants as in the previous lemma.

Lemma 3.1.5. *Consider $N = \{1, 2\}$, $a_1, a_2 \in \mathbb{R}_{\geq 0}$ with $a_1 + a_2 = 1$. The power index F^a defined for all $v \in \mathcal{CSG}_2$ and for $i, j \in N$ as follows:*

$$F_i^a(v) = a_i + a_j \int_0^1 \Delta v(\{i\},\, t)\, dt - a_i \int_0^1 \Delta v(\{j\},\, t)\, dt \tag{3.13}$$

satisfies (E), (NP) and (HIS). Moreover, the pair of HIS's constants are given by $(\lambda_{F^a}(\{i\}),\, \gamma_{F^a}(\{i\})) = (a_j,\, a_j)$.

Proof. Consider $u, v \in \mathcal{CSG}_2$ and $i, j \in N$ two distinct players. Note that $F_i^a(v) \geq 0$ by definition. By Equation (3.13), we easily compute $F_1^a(v) + F_2^a(v) = a_1 + a_2 = 1$, thus F^a is efficient. If i is a null player in v, then for all $t \in [0, 1]$, $\Delta v(\{i\}, t) = 0$ and $\Delta v(\{j\}, t) = 1$. Thus, Equation (3.13) gives $F_i^a(v) = 0$, hence F^a satisfies (NP).

Now assume that, $u \xrightarrow{\{i\},\, \varepsilon,\, [a,b]} v$ for a given $\varepsilon > 0$ and $[a, b] \subseteq [0, 1]$. According to Definition 3.1.7, $\Delta v(\{i\}, t) = \Delta u(\{i\}, t) + \varepsilon$ if $t \in]a, b[$ and $\Delta v(\{i\}, t) = \Delta u(\{i\}, t)$ otherwise and $\Delta v(\{j\}, t) = \Delta u(\{j\}, t)$ for all $t \in]0, 1[$. Therefore, by Equation (3.13)

$F_i^a(v) - F_i^a(u) = a_j \times \varepsilon \times (b-a)$ and $F_j^a(v) - F_j^a(u) = -a_j \times \varepsilon \times (b-a)$. We conclude that, F^a satisfies (HIS) and $(\lambda_{F^a}(\{i\}), \gamma_{F^a}(\{i\})) = (a_j, a_j)$, by Equation (3.6). ■

So, for $n = 2$ including symmetry (S) as in Lemma 3.1.3 permits to uniquely determine the (HIS) constants, but this symmetry axiom may also be replaced by some technically weaker axiom.

The following proposition is useful in the axiomatization of Ψ.

Proposition 3.1.4. *Consider two power indices F and P which are (HIS) and such that $\lambda_F(S) = \lambda_P(S)$ and $\gamma_F(S) = \gamma_P(S)$ for all coalitions S. Then for all CSGs u and v such that v is a local improvement of potentials in u,*

$$F(u) - P(u) = F(v) - P(v).$$

Proof.
Suppose that F and P are two power indices that satisfy (HIS) and $\lambda_F(S) = \lambda_P(S)$ and $\gamma_F(S) = \gamma_P(S)$ for all coalitions S. Consider $u, v \in \mathcal{CSG}_n$ such that $u \xrightarrow{S,\,\varepsilon,\,D} v$ holds for some coalition S, some $\varepsilon > 0$ and some $N\backslash S$-domain D. Since both F and P satisfy (HIS),

$$F(v) = F(u) + q(S) \times \varepsilon \times vol(D) \text{ and } P(u) = P(v) + q(S) \times \varepsilon \times vol(D) \qquad (3.14)$$

where $q_i(S) = \lambda_F(S)$ for all $i \in S$ and $q_i(S) = -\gamma_F(S)$ otherwise. Therefore, $F(u) - P(u) = F(v) - P(v)$. ■

3.1.3 Discreteness axiom

We have shown that one can embed the set of simple games in the set of CSGs through out any transformation T^τ, $0 < \tau \leq 1$; thus simple games can be viewed as *discrete* CSGs. Moreover, each transformation T^τ preserves the distribution of power among players while moving from the Shapley-Shubik index SSI on simple games to the Shapley-Shubik index Ψ on CSGs. In this section, we formally define a special class of CSGs called discrete CSGs. Those games are used to formalize the new axiom of *discretization*, that captures the possibility for a power index to be completely determined by its restriction on the set of all discrete CSGs. Further notations and definitions are needed before the formalisation of this axiom.

For each integer $p \geq 2$, we pose:

$$\mathcal{D}_p = \left\{\alpha = (\alpha_0, \alpha_1, ..., \alpha_p) \in I^{p+1} : \alpha_0 = 0, \alpha_p = 1 \text{ and } \alpha_i < \alpha_{i+1} \text{ for } i = 0, 1, ..., p-1\right\}.$$

and $\mathcal{A}_{p,n} = \{1, 2, ..., p\}^n$, where n denotes the number of players. Given $\alpha \in \mathcal{D}_p$, we denote by $\omega(\alpha) = \max\limits_{1 \leq k \leq p} (\alpha_k - \alpha_{k-1})$ the maximal difference between two consecutive α_h and

$$[\alpha]_k = \begin{cases} [\alpha_{k-1}, \alpha_k[& \text{if} \quad k < p \\ [\alpha_{p-1}, \alpha_p] & \text{if} \quad k = p \end{cases} \quad \text{for any} k \in \{1, 2, ..., p\}$$

For each $e = (e_1, e_2, ..., e_n) \in \mathcal{A}_{p,n}$ and each $\alpha \in \mathcal{D}_p$, we abbreviate by $[\alpha]_e$ the box

$$[\alpha]_{e_1} \times [\alpha]_{e_2} \times ... \times [\alpha]_{e_n}$$

and c_e the center of $[\alpha]_e$, i.e.

$$c_e = \left(\frac{\alpha_{e_1-1} + \alpha_{e_1}}{2}, \frac{\alpha_{e_2-1} + \alpha_{e_2}}{2}, \ldots, \frac{\alpha_{e_n-1} + \alpha_{e_n}}{2} \right) \tag{3.15}$$

Remark 3.1.2. *For each integer $p \geq 2$ and each $\alpha \in \mathcal{D}_p$, the collection $\{[\alpha]_k : 1 \leq k \leq p\}$ form a partition of $I = [0, 1]$ and $\mathcal{P}_\alpha = \{[\alpha]_e : e \in \mathcal{A}_{p,n}\}$ is the paving of I^n.*

It will be useful in the sequel to order all cuboids provided by a paving $\mathcal{P}_\alpha$ of I^n. For this, we use the lexicographic order denoted $\preceq_{lex}$ and defined as follows. Consider $x, y \in \mathbb{R}^n$ then, $x \preceq_{lex} y$ if $x = y$ or there exists some $k \in \{1, 2, ..., n\}$ such that $x_k < y_k$ and $x_j = y_j$ for all $j > k$. Note that, we compare each component of x with the corresponding one of y starting at the n^{th} component and moving downward.

For example, in $\mathbb{R}^2$, we have $(1, 1) \prec_{lex} (2, 1) \prec_{lex} (1, 2) \prec_{lex} (2, 2)$.

Proposition 3.1.5.

1. *The lexicographic order $\preceq_{lex}$ is a linear order on $\mathbb{R}^n$.*

2. *Let $p \geq 2$ be an integer, $\alpha \in \mathcal{D}_p$ and $x, y \in I^n$ such that $(x, y) \in [\alpha]_e \times [\alpha]_{e'}$ for some $e, e' \in \mathcal{A}_{p,n}$. If $x \preceq y$, then $e \preceq_{lex} e'$.*

Proof.
By definition, $\preceq_{lex}$ is reflexive, complete and antisymmetric. To see that $\preceq_{lex}$ is transitive, consider three distinct n-tuples a, b and c such that $a \preceq_{lex} b$ and $b \preceq_{lex} c$. Assume that $a = b$ or $b = c$, then we obtain $a \preceq_{lex} c$ without difficulties. Now suppose that $a \prec_{lex} b$ and $b \prec_{lex} c$, then there exists k and k' such that $a_k < b_k$ and $a_j = b_j$ for any $j > k$; and $b_{k'} < c_{k'}$ and $b_j = c_j$ for any $j > k'$. If $k \leq k'$, then $a_{k'} \leq b_{k'} < c_k$ and for all $j > k'$, $a_j = b_j = c_j$. Thus $a \preceq_{lex} c$. If $k' < k$, then $a_k < b_{k'} \leq c_k$ and for all $j > k$, $a_j = b_j = c_j$. Thus $a \preceq_{lex} c$. Therefore $\preceq_{lex}$ is transitive.

Now Consider $x, y \in I^n$ such that $(x, y) \in [\alpha]_e \times [\alpha]_{e'}$ for some $e, e' \in \mathcal{A}_{p,n}$. Assume that $x \preceq y$ and $e' \prec_{lex} e$; then $e'_i < e_i$ for some $i \in N$, i.e, $e'_i \leq e_i - 1 < p$. Since $y_i \in [\alpha]_{e'_i}$ and $e'_i < p$, then $y_i < \alpha_{e'_i} \leq \alpha_{e_i-1} \leq x_i$. This implies $y_i < x_i$ which is a contradiction since $x \preceq y$. Consequently, $x \preceq y$, $x \in [\alpha]_e$ and $y \in [\alpha]_{e'}$ implies $e \preceq_{lex} e'$. ■

Simple games as well as (j, k) simple games are discrete in the sense that each such games take a finite set of values. With CSGs, an analogous class of games is obtained when one considers for example CSGs that are constant over each cuboid provided by some paving $\mathcal{P}_\alpha$ of I^n. More formally,

Definition 3.1.8. *A CSG u is discrete if there exists a partition $P = \{P_1, P_2, \cdots, P_m\}$ of I^n such that for all $P_j \in P$ $(1 \leq j \leq m)$ there exists $a_j \in [0, 1]$ such that $u(x) = a_j$ for all $x \in P_j$.*

In other words, u is *discrete* if there exists a partition $P = \{P_1, P_2, \cdots, P_m\}$ such that the restriction of u over each P_j is constant. In this case, we say that u is a discrete CSG associated with P.

Notation 3.1.1. *Let $p \geq 2$ be an integer and $\alpha \in \mathcal{D}_p$.*

- *We denote by Γ_p^α the set of all discrete CSGs associated with $\mathcal{P}_\alpha$,*
- *for any $c \in \Gamma_p^\alpha$, we will write $c = c_{a,\alpha}$, where $a = (a_e)_{e \in \mathcal{A}_{p,\,n}}$ is the collection of real numbers such that, $c(x) = a_e$; for all $x \in [\alpha]_e \setminus \{\mathbf{0}, \mathbf{1}\}$ and for all $e \in \mathcal{A}_{p,\,n}$.*

Remark 3.1.3. *Any 2-players discrete game $c_{a,\alpha} \in \Gamma_p^\alpha$ can be represented by the following matrix:*

$a_{1,p}$	$a_{2,p}$	$\cdots$	$a_{p,p}$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
$a_{1,2}$	$a_{2,2}$	$\cdots$	$a_{p,2}$
$a_{1,1}$	$a_{2,1}$	$\cdots$	$a_{p,1}$

Definition 3.1.9. *A discrete CSG u associated with the partition $\mathcal{P} = \{P_1, P_2, \cdots, P_m\}$ is a discretization of a CSG v if for all $P_j \in \mathcal{P}$, there exists $t_j \in P_j$ such that $u(x) = v(t_j)$ for all $x \in P_j$.*

A discretization u of a CSG v is called regular if $u \in \Gamma_p^\alpha$ for some integer $p \geq 2$ and some $\alpha \in \mathcal{D}_p$.

Example 3.1.3. *Consider $v \in \mathcal{CSG}_n$, $p \geq 2$ an integer and $\alpha \in \mathcal{D}_p$. Then, the discrete game u defined by $u(x) = v(c_e)$ for all $x \in [\alpha]_e$, with $e \in \mathcal{A}_{p,n}$ is a regular discretization of v.*

While discretizing a CSG, one may expect that, the potentials of coalitions to be preserved in some sense.

Definition 3.1.10. *A regular discretization u of a CSG v is compatible with the potentials in v if for all $S \in 2^N \setminus \{N\}$, for all $e \in \mathcal{A}_{p,n}$, there exists $t_e \in [\alpha]_e$ such that,*

$$\forall x \in [\alpha]_e\,,\ \Delta u\,(S, x_{-S}) = \Delta v\,(S, (t_e)_{-S})\ .$$

In this case, we write $u \sim_\Delta v$.

In words u, is compatible with potential influence in v if the potential of each coalition in u is a discretization of the potential in v. Moreover, a regular discretization u of a CSG v, being potentials compatible requires that, for each $S \in 2^N \setminus \{N\}$, the graphs of the potential of S in u and in v should intercept over each $[\alpha]_e$, $e \in \mathcal{A}_{p,n}$. Thus, when $\omega\,(\alpha)$ is sufficiently small, potentials of coalitions in u mimic those in v. The question is whether such a discretization always exists. The answer is "yes", an example is provided below:

Example 3.1.4. *Let v be a CSG with three players. Given $\alpha = (0, \alpha_1, 1) \in \mathcal{D}_2$, the discrete game $u \in \Gamma_2^\alpha$ defined by the constants $(a_e)_{e \in \mathcal{A}_{2,3}}$*

e	$(1,1,1)$	$(2,1,1)$	$(1,2,1)$	$(2,2,1)$	$(1,1,2)$	$(2,1,2)$	$(1,2,2)$	$(2,2,2)$
a_e	$v(0,0,0)$	$v(1,0,0)$	$v(0,1,0)$	$v(1,1,0)$	$v(0,0,1)$	$v(1,0,1)$	$v(0,1,1)$	$v(1,1,1)$

is compatible with potentials in v.

More generally, we have:

Proposition 3.1.6. *For all $c \in \mathcal{CSG}_n$, there exists a sequence $(c_{a^p,\alpha^p})_{p\geq 2}$ of regular discetizations of c compatible with potentials such that for all $p \geq 2$, $\alpha^p \in \mathcal{D}_p$ and $\lim\limits_{p\to+\infty} \omega\,(\alpha^p) = 0$.*

Proof.

Consider $c \in \mathcal{CSG}_n$. For all integer $p \geq 2$, we pose $\alpha^p = \left(0, \frac{1}{p}, \cdots, \frac{p-1}{p}, 1\right) \in \mathcal{D}_p$, for each $e \in \mathcal{A}_{p,n}$, we define the profile t_e^p by $(t_e^p)_i = \frac{e_i - 1}{p}$ if $e_i < p$ and $(t_e^p)_i = 1$ otherwise and pose $a_e^p = c(t_e^p)$. Consider the sequence $(u_p)_{p\geq 2}$ of discrete games such that for any $p \geq 2$, $u_p = c_{a^p,\alpha^p}$, where $a^p = (a_e^p)_{e\in\mathcal{A}_{p,n}}$. By construction, u_p is a regular discretization of c such that $u_p \in \Gamma_p^{\alpha^p}$ and $\lim\limits_{p\to+\infty} \omega\,(\alpha^p) = 0$.

Now let prove that, for all $p \geq 2$, $u_p \sim_\Delta c$. To this end, consider $S \in 2^N \setminus \{N\}$, $e \in \mathcal{A}_{p,n}$ and $x \in [\alpha^p]_e$, then, the potential of coalition S in u_p satisfies:

$$\begin{aligned} \Delta u_p\,(S, x_{-S}) &= u_p\,(\mathbf{1}_S, x_{-S}) - u_p\,(\mathbf{0}_S, x_{-S}) \\ &= u_p\,(y) - u_p\,(z) \end{aligned}$$

where

$$y_i = \begin{cases} 1 & \text{if} \quad i \in S \\ x_i & \text{otherwise} \end{cases} \quad \text{and } z_i = \begin{cases} 0 & \text{if} \quad i \in S \\ x_i & \text{otherwise} \end{cases}$$

Since $x \in [\alpha^p]_e$, it follows that

$$y \in [\alpha^p]_{e'} \text{ and } z \in [\alpha^p]_{e''}$$

with

$$e'_i = \begin{cases} p & \text{if} \quad i \in S \\ e_i & \text{otherwise} \end{cases} \quad \text{and } e''_i = \begin{cases} 1 & \text{if} \quad i \in S \\ e_i & \text{otherwise} \end{cases}$$

So, according to the definition of u_p, we have,

$$u_p\,(y) = a^p_{e'} = c\,(t^p_{e'}) = c\,(\mathbf{1}_S,\,(t^p_e)_{-S}) \text{ and } u_p\,(z) = a^p_{e''} = c\,(t^p_{e''}) = c\,(\mathbf{0}_S,\,(t^p_e)_{-S})\,.$$

Thus, by rewriting $\Delta u_p\,(S, x_{-S})$ we get,

$$\begin{aligned} \Delta u_p\,(S, x_{-S}) &= u_p\,(y) - u_p\,(z) \\ &= c\,(\mathbf{1}_S,\,(t^p_e)_{-S}) - c\,(\mathbf{0}_S,\,(t^p_e)_{-S}) \\ &= \Delta c\,(S,\,(t^p_e)_{-S}) \end{aligned}$$

Since $t^p_e \in [\alpha^p]_e$ and does not depend on x, we conclude that $u_p \sim_\Delta c$. ∎

Definition 3.1.11. *A power index F for CSGs satisfies the discreteness axiom (D) or is discretizable if for any CSG c and for any regular discretizations sequence $(u_p)_{p\geq 2}$ of c compatible with potentials such that*

$$u_p \in \Gamma_p^{\alpha^p} \qquad \text{and} \qquad \lim_{p\to+\infty} \omega\,(\alpha^p) = 0,$$

it follows that,

$$F\,(c) = \lim_{p\to+\infty} F\,(u_p)\,.$$

Lemma 3.1.6. *The Shapley-Shubik power index Ψ is discretizable.*

Proof. Let c be a CSG on N and $(u_p)_{p\geq 2}$ a sequence of regular discretizations of c such that, for all $p \geq 2$, $u_p \in \Gamma_p^{\alpha^p}$, $u_p \sim_\Delta c$ and $\lim\limits_{p\to+\infty} \omega\,(\alpha^p) = 0$. Consider $i \in N$, by Theorem 1.2.3, page 25 we can write,

$$\Psi_i(u_p) = \frac{1}{n} + \sum_{i\in T\subset N} \frac{(t-1)!(n-t)!}{n!}\mathcal{C}\,(u_p, T) - \sum_{\emptyset\neq T\subseteq N\setminus\{i\}} \frac{t!(n-t-1)!}{n!}\mathcal{C}\,(u_p, T) \quad (3.16)$$

Since u_p is a discrete CSG associated with $\mathcal{P}_{\alpha^p}$, then

$$\forall\,\emptyset \subset T \subset N,\, \mathcal{C}\,(u_p, T) = \int_{I^n} \Delta u_p(T, x_{-T})\,dx = \sum_{e\in\mathcal{A}_{p,\,n}} \mathrm{vol}\,([\alpha^p]_e)\,\Delta u_p(T,\,(c^p_e)_{-T}) \quad (3.17)$$

where for any $e \in \mathcal{A}_{p,n}$, c^p_e is the center of $[\alpha^p]_e$. Since $u_p \sim_\Delta c$ then, Definition 3.1.10 and Equations (3.16)–(3.17) lead to:

$$\Psi_i(u_p) = \frac{1}{n} + \sum_{i\in T\subset N} \frac{(t-1)!(n-t)!}{n!} \left(\sum_{e\in\mathcal{A}_{p,\,n}} \mathrm{vol}\left([\alpha^p]_e\right) \Delta c(T,\,(x_e^p)_{-T}) \right)$$
$$- \sum_{\emptyset\neq T\subseteq N\setminus\{i\}} \frac{t!(n-t-1)!}{n!} \left(\sum_{e\in\mathcal{A}_{p,\,n}} \mathrm{vol}\left([\alpha^p]_e\right) \Delta c(T,\,(x_e^p)_{-T}) \right)$$

Since c is integrable (see Corollary 1.2.3, page 23), then for any $T \in 2^N\setminus\{N\}$, $\Delta c(T,\,.)$ is integrable on I^n. Additionally, since $\lim\limits_{p\to+\infty} \omega(\alpha^p) = 0$, then,

$$\lim_{p\to+\infty} \left(\sum_{e\in\mathcal{A}_{p,\,n}} \mathrm{vol}\left([\alpha^p]_e\right) \Delta c(T,\,x_e^p) \right) = \int_{I^n} \Delta c(T, y_{-T})\, dy = \mathcal{C}\left(c,\,T\right) \qquad (3.18)$$

Finally,

$$\begin{aligned}
\lim_{p\to+\infty} \Psi_i(u_p) &= \frac{1}{n} + \sum_{i\in T\subset N} \frac{(t-1)!(n-t)!}{n!} \lim_{p\to+\infty} \left(\sum_{e\in\mathcal{A}_{p,\,n}} \mathrm{vol}\left([\alpha^p]_e\right) \Delta c(T,\,(x_e^p)_{-T}) \right) \\
&\quad - \sum_{\emptyset\neq T\subseteq N\setminus\{i\}} \frac{t!(n-t-1)!}{n!} \lim_{p\to+\infty} \left(\sum_{e\in\mathcal{A}_{p,\,n}} \mathrm{vol}\left([\alpha^p]_e\right) \Delta c(T,\,(x_e^p)_{-T}) \right) \\
&= \frac{1}{n} + \sum_{i\in T\subset N} \frac{(t-1)!(n-t)!}{n!} \mathcal{C}(c,\,T) - \sum_{\emptyset\neq T\subseteq N\setminus\{i\}} \frac{t!(n-t-1)!}{n!} \mathcal{C}(c,\,T), \text{ by Equation (3.18)} \\
&= \Psi_i(c)
\end{aligned}$$

Therefore, $\Psi(c) = \lim\limits_{p\to+\infty} \Psi(u_p)$. ∎

3.2 Results of axiomatization

We prove that, axiom of symmetry (S) together with efficiency (E), null player property (NP), homogeneous increments sharing (HIS) and discreteness (D) are sufficient to uniquely identify the Shapley-Shubik index (Ψ) on the set of 2-players CSGs. In the context of CSGs with at least three players, we prove that this result of characterization still holds even if axiom of symmetry is dropped. We end this section with proving the independence of those axioms of characterization.

3.2.1 An axiomatization of Ψ on 2-players CSGs

It is proved that when regular CSGs are involved, axioms (E), (NP) and (HIS) characterize the parameterized power indices defined in Lemma 3.1.5. Moreover, we show that adding axiom (S) to the previous axioms allows to uniquely identify the Shapley-Shubik power index (Ψ). Secondly, we use the axioms (E), (NP), (HIS), (S) and (D) to provide an axiomatization of the power index Ψ on $\mathcal{CSG}_2$.

To ease the presentation, we denote by $\mathcal{F} = \{F^a,\ a = (a_1, a_2) \in \mathbb{R}^2_{\geq 0} \quad \text{with} \quad a_1 + a_2 = 1\}$, where F^a is the power index defined by Equation (3.13), 81. Note that, $\Psi \in \mathcal{F}$; just taking $a_1 = a_2 = \frac{1}{2}$.

Theorem 3.2.1.

Let P be a power index on $\mathcal{CSG}_2$ that satisfies (E), (NP) and (HIS). Then, there exists $F \in \mathcal{F}$ such that, $P(c) = F(c)$ for every regular CSG c. ⋆

Proof.

Consider $N = \{1,\, 2\}$ the set of players and P a power index on $\mathcal{CSG}_2$ that satisfies (E), (NP) and (HIS). In order to show that the restriction of P on the subset of regular games coincide with some element of $\mathcal{F}$, for a each player $i \in N$ consider the CSGs u_0^i and u_0 respectively defined as follows:

$$\text{for all } x \in I^2,\ u_0^i(x) = \begin{cases} 1 & \text{if } x_i = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } u_0(x) = 0 \text{ if } x \neq \mathbf{1}\,.$$

We can easily check that, for all $i \in N$, $u_0 \xrightarrow{\{i\},\,1,\,[0,1]} u_0^i$ and the unique player $j \in N\backslash\{i\}$ is a null player in u_0^i. Thus, by (HIS), (NP) and (E) one get,

$$\lambda_P(\{i\}) = \gamma_P(\{i\}) = P_j(u_0)\,.$$

Hereafter, we pose $P(u_0) = (a_1, a_2)$, then $a_1, a_2 \in \mathbb{R}_{\geq 0}$ and $a_1 + a_2 = 1$ by (E). Consider a regular CSG v, then there exists an integer $p \geq 2$ and $\alpha \in \mathcal{D}_p$ such that, for all $e = (e_1, e_2) \in \mathcal{A}_{p,n}$ and for all $x \in [\alpha]_e$, $v(x) = a_{e_1,e_2}$. Throughout this proof, we set $[p] =: \{1, \cdots, p\}$ for a given integer $p \geq 2$ and $I_k = [\alpha_{p-k},\, \alpha_{p-k+1}]$, for any $k \in [p]$. In order to prove that $P(v) = F^a(v)$, four steps are needed.

Step 1 Consider a sequence of CSGs $(v_k)_{0 \leq k \leq p}$ defined by $v_0 = u_0$ and for all $k \in [p]$,

$$v_k(x) = \begin{cases} v(x) & \text{if} \quad x \in \{1\}\times]\alpha_{p-k}, \alpha_{p-k+1}] \\ v_{k-1}(x) & \text{otherwise} \end{cases}$$

For all $k \in [p]$, $v_{k-1} \xrightarrow{\{1\},\,\varepsilon_k,\,I_k} v_k$ with $\varepsilon_k = a_{p,\,p-k+1}$. It follows by (HIS) that,

$$P_1(v_k) = P_1(v_{k-1}) + a_2 \cdot \varepsilon_k \cdot (\alpha_{p-k+1} - \alpha_{p-k}) \quad \text{and} \quad P_2(v_k) = P_2(v_{k-1}) - a_2 \cdot \varepsilon_k \cdot (\alpha_{p-k+1} - \alpha_{p-k})\,.$$

By summing over $k \in [p]$ all left-hand-side terms and all right-hand-side terms of each of the above equations, we finally obtain:

$$P_1(v_p) = P_1(v_0) + a_2 \cdot \sum_{k=1}^{p} a_{p,\,k} \cdot (\alpha_k - \alpha_{k-1}) = a_1 + a_2 \int_0^1 v(1,\, t)\, dt, \tag{3.19}$$

and

$$P_2(v_p) = P_2(v_0) - a_2 \cdot \sum_{k=1}^{p} a_{p,k} \cdot (\alpha_k - \alpha_{k-1}) = a_2 - a_2 \int_0^1 v(1,\, t)\, dt\,. \tag{3.20}$$

Step 2 Consider a sequence of CGSs $(w_k)_{0\leq k\leq p}$ defined by $w_0 = v_p$ and for all $k \in [p]$,

$$w_k(x) = \begin{cases} v(x) & \text{if} \quad x \in (]\alpha_{p-k}, \alpha_{p-k+1}]\backslash\{1\}) \times]0,1] \\ w_{k-1}(x) & \text{otherwise} \end{cases}$$

For all $k \in [p]$,$w_{k-1} \xrightarrow{\{2\},\, \theta_k,\, I_k} w_k$ with $\theta_k = a_{p-k+1,\,p}$. Thus, by (HIS) we one obtain,

$$P_1(w_k) = P_1(w_{k-1}) - a_1 \cdot \theta_k \cdot (\alpha_{p-k+1} - \alpha_{p-k}) \quad and \quad P_2(w_k) = P_2(v_{k-1}) + a_1 \cdot \theta_k \cdot (\alpha_{p-k+1} - \alpha_{p-k})$$

By summing over $k \in [p]$ all left-hand-side terms and all right-hand-side terms of each of the above equations, it follows from Equations (3.19) and (3.20) that:

$$P_1(w_p) = a_1 + a_2 \int_0^1 v(1,\, t)\, dt - a_1 \int_0^1 v(t,\, 1)\, dt \quad \text{and} \quad P_2(w_p) = a_2 - a_2 \int_0^1 v(1,\, t)\, dt + a_1 \int_0^1 v(t,\, 1)\, dt\,. \tag{3.21}$$

Step 3 Consider a sequence of CSGs $(f_k)_{0\leq k\leq p}$ defined by $f_0 = w_p$ and for all $k \in [p]$,

$$f_k(x) = \begin{cases} v(x) & \text{if} \quad x \in \{0\} \times]\alpha_{p-k}, \alpha_{p-k+1}] \\ f_{k-1}(x) & \text{otherwise} \end{cases}$$

For all $k \in [p]$, $f_{k-1} \xrightarrow{\{1\},\, \theta'_k,\, I_k} f_k$ where $\theta'_k = -a_{1,\,p-k+1}$. Hence by (HIS) we one get,

$$P_1(f_k) = P_1(f_{k-1}) + a_2 \cdot \theta'_k \cdot (\alpha_{p-k+1} - \alpha_{p-k}) \quad and \quad P_2(f_k) = P_2(f_{k-1}) - a_2 \cdot \theta'_k \cdot (\alpha_{p-k+1} - \alpha_{p-k})\,.$$

By summing over $k \in [p]$ all left-hand-side terms and all right-hand-side terms of each of the above equations, it follows from Equation (3.21) that:

$$P_1(f_p) = a_1 + a_2 \int_0^1 v(1,\, t)\, dt - a_1 \int_0^1 v(t,\, 1)\, dt - a_2 \int_0^1 v(0,\, t)\, dt \tag{3.22}$$

and

$$P_2(f_p) = a_2 - a_2 \int_0^1 v(1,\, t)\, dt + a_1 \int_0^1 v(t,\, 1)\, dt + a_2 \int_0^1 v(0,\, t)\, dt \tag{3.23}$$

Step 4 Consider a sequence of CSGs $(g_k)_{0\leq k\leq p}$ defined by $g_0 = f_p$ and for all $k \in [p]$,

$$g_k(x) = \begin{cases} v(x) & if \quad x_2 = 0 \quad and \quad \alpha_{p-k} < x_1 \leq \alpha_{p-k+1} \\ g_{k-1}(x) & otherwise \end{cases}$$

We note that, $g_p = v$ and for all $k \in [p]$, $g_{k-1} \xrightarrow{\{2\},\, \theta"_k,\, I_k} g_k$ where $\theta"_k = -a_{p-k+1,\,1}$. It follows from (HIS) that,

$$P_1(g_k) = P_1(g_{k-1}) - a_1 \cdot \theta"_k \cdot (\alpha_{p-k+1} - \alpha_{p-k}) \quad and \quad P_2(g_k) = P_2(g_{k-1}) + a_1 \cdot \theta"_k \cdot (\alpha_{p-k+1} - \alpha_{p-k})$$

By summing over $k \in [p]$ all left-hand-side terms and all right-hand-side terms of each of the above equations, it follows from Equations (3.22) and (3.23) that:

$$P_1(g_p) = a_1 + a_2 \int_0^1 \Delta(\{1\}, t)\, dt - a_1 \int_0^1 \Delta(\{2\}, t)\, dt \quad \text{and} \quad P_2(g_p) = a_2 - a_2 \int_0^1 \Delta(\{1\}, t)\, dt + a_1 \int_0^1 \Delta(\{2\}, t)\, dt. \tag{3.24}$$

Since $g_p = v$ Equations (3.13) and (3.24) imply $P(v) = F^a(v)$, with $a = (a_1, a_2) = P(u_0)$. ■

Note that if P also satisfies (S), then $P(u_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Therefore $F^a(v) = \Psi(v)$. We derive from the proof of Theorem 3.2.1 the following result:

Corollary 3.2.1. *If P is a power index on $\mathcal{CSG}_2$ satisfying* (E), (NP), (HIS) *and* (S) *then,* $P(c) = \Psi(c)$ *for every regular CSG* c.

Lemmas 3.1.1 and 3.1.2 combined with Corollary 3.2.1 provide a full characterization of Ψ using four axioms on the subclass of all regular CSGs . We extend this result to the whole set of all 2-players CSGs using (D) as follows:

Theorem 3.2.2.

The unique power index on $\mathcal{CSG}_2$ that simultaneously meets (E), (NP), (S), (HIS) and (D) is the Shapley-Shubik power index. ★

Proof. Let F be a power index on $\mathcal{CSG}_n$.

Necessity: If $F = \Psi$, it follows from Lemmas 3.1.2 and 3.1.6 that F satisfies (HIS) and (D). The others properties (E), (NP) and (S) come directly from Lemma 3.1.1.

Sufficiency: Now assume that F satisfies (E), (NP), (S), (HIS) and (D). Consider $c \in \mathcal{CSG}_2$, by Proposition 3.1.6, there exists a sequence of regular discretizations $(u_p)_{p\geq 2}$ of c compatible with potentials such that, $u_p \in \Gamma_p^{\alpha^p}$ and $\lim\limits_{p\longrightarrow+\infty} \omega\left(\alpha^p\right) = 0$. Hence :

$$\begin{aligned} F(c) &= \lim_{p\longrightarrow+\infty} F(u_p) \quad \text{since } F \text{ is discretizable} \\ &= \lim_{p\longrightarrow+\infty} \Psi(u_p) \quad \text{by Corollary 3.2.1} \\ &= \Psi(c) \quad \text{since } \Psi \text{ is discretizable.} \end{aligned}$$

Therefore $F = \Psi$. ■

3.2.2 An axiomatization of Ψ on CSGs with at least three players

Corollary 3.2.1 provides a characterization of Ψ using the axioms (E), (NP), (S) and (HIS) on the subclass of all 2-players CSGs that are regular. We now show that axioms (E), (NP) and (HIS) are sufficient to characterize Ψ on the subclass of regular CSGs with at least three players. We extend this result to the whole set of all CSGs using the discreteness axiom (D). Some notations and preliminary results are needed.

Let $p \geq 2$ be an integer and $\alpha \in \mathcal{D}_p$. We pose,

$$\mathcal{A}^*_{p,n} = \{e \in \mathcal{A}_{p,n},\, U(e) \cup L(e) \neq \emptyset\}$$

where $U(e) = \{i \in N,\, e_i = p\}$ and $L(e) = \{i \in N,\, e_i = 1\}$. Since $\preceq_{lex}$ is a linear order, the set $\mathcal{A}^*_{p,n}$ can be labelling in such way that $\mathcal{A}^*_{p,n} = \{e^1, e^2, \cdots, e^{p^*}\}$, with $p^* = \left|\mathcal{A}^*_{p,n}\right|$ and $e^k \prec_{lex} e^{k+1}$ for all $k < p^*$.

For each $e \in \mathcal{A}_{p,n}$ we denote by

$$\mathcal{B}(e) = \bigcup_{e' \preceq_{lex} e} [\alpha]_{e'} \quad \text{and} \quad \mathcal{B}^-(e) = \bigcup_{e' \prec_{lex} e} [\alpha]_{e'} = \mathcal{B}(e) \backslash [\alpha]_e\,.$$

$\mathcal{B}^-(e)$ is the union of all boxes of $\mathcal{P}_\alpha$ which precede $[\alpha]_e$ according to the lexicographic order $\preceq_{lex}$.

Consider $c = c_{a,\alpha} \in \Gamma^\alpha_p$. We define a sequence of CSGs $(c_k)_{0\leq k\leq p^*}$ as follows: $c_0(x) = 1$ if $x \neq \mathbf{0}$ and

$$c_k(x) = \begin{cases} c(x) & \text{if} \quad x \in \mathcal{B}(e^k) \\ 1 & \text{otherwise} \end{cases} \qquad (\text{for all } 1 \leq k \leq p^*)\,. \tag{3.25}$$

Note that, the sequence $(c_k)_{0\leq k\leq p^*}$ can help to build the game c in a finite number of steps. To illustrate this, consider the game u in Example 3.1.2. u is a 2-players discrete game, i.e. $u = c_{a,\alpha}$ with $\alpha = \left(0, \frac{1}{2}, 1\right)$, $a_{1,1} = 0.1$, $a_{2,1} = 0.3$, $a_{1,2} = 0.6$ and $a_{2,2} = 0.9$. Note that $\mathcal{A}_{2,2} = \mathcal{A}^*_{2,2} = \{(1,1); (2,1); (1,2); (2,2)\}$. By Equation (3.25) and Remark 3.1.3, the games c_0, c_1, c_2, c_3 and c_4 are represented below:

$$c_0 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array} \quad c_1 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0.1 & 1 \\ \hline\end{array} \quad c_2 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0.1 & 0.3 \\ \hline\end{array} \quad c_3 := \begin{array}{|c|c|}\hline 0.6 & 1 \\ \hline 0.1 & 0.3 \\ \hline\end{array} \quad c_4 := \begin{array}{|c|c|}\hline 0.6 & 0.9 \\ \hline 0.1 & 0.3 \\ \hline\end{array} = u\,.$$

The following property of the sequence $(c_k)_{0\leq k\leq p^*}$ is useful to prove our main result. It simply states that from c_k to c_{k+1}, one can use a finite moves each consisting in a (backward or forward) local improvement of potentials of two CSGs.

Lemma 3.2.1. *Let $p \geq 2$ be an integer, $\alpha \in \mathcal{D}_p$, $c = c_{a,\alpha} \in \Gamma^\alpha_N$ such that $c \neq c_0$ and $(c_k)_{0\leq k\leq p^*}$ be defined by Equation (3.25). For each $k \in \{0, 1, 2, ..., p^* - 1\}$ such that $a_{e^{k+1}} \neq 1$, there exists a sequence $(f_l)_{0\leq l\leq m}$ of CSGs such that $f_0 = c_k$, $f_m = c_{k+1}$ and for all $l < m$, either f_{l+1} is an improvement of potentials in f_l; or f_l is an improvement of potentials in f_{l+1}.*

Proof.

Consider an integer $p \geq 2$, $\alpha \in \mathcal{D}_p$, $c = (a_e)_{e\in\mathcal{A}_{p,\,n}} \in \Gamma_p^\alpha$ such that $c \neq c_0$ and $k \in \{0, 1, 2, ..., p^* - 1\}$ such that $a_{e^{k+1}} \neq 1$. We pose $\varepsilon_k = 1 - a_{e^{k+1}} > 0$. In order to construct the sequence $(f_l)_{0\leq l\leq m}$, three distinct cases are considered:

Case 1: If $L(e^{k+1}) \neq \emptyset$ **and** $U\left(e^{k+1}\right) = \emptyset$**.** Pose $q = \left|2^{L(e^{k+1})}\backslash\{N\}\right|$ and consider a labelling $\{S_t,\, 1 \leq t \leq q\}$ of $2^{L(e^{k+1})}\backslash\{N\}$ such that $|S_t| \geq |S_{t+1}|$ for all $1 \leq t < q$. Let $(u_l)_{0\leq l\leq q}$ be the sequence of mapping defined on I^n as follows:

$$u_0(x) = \begin{cases} c(x) & \text{if } x \in \mathcal{B}^-(e^{k+1}) \\ c_k(x) & \text{otherwise} \end{cases} \tag{3.26}$$

and for all $1 \leq l \leq q$,

$$u_l(x) = \begin{cases} c(x) & \text{if } x \in \mathcal{B}^-(e^{k+1}) \text{ or } x \in [\alpha]_{e^{k+1}} \text{ and } \mathbf{0}_x = S_t \text{ for some } t \leq l \\ c_k(x) & \text{otherwise} \end{cases} \tag{3.27}$$

Using Proposition 3.1.5, we easily check that, u_l is a CSG.

Now, we show that for all $l = 1, 2, \cdots, q$; $u_{l-1} \xrightarrow{S_l,\,\varepsilon_k,\,D_l} u_l$ where $D_l = \bigtimes_{i\in N\backslash S_l} [\alpha]_{e_i^{k+1}}$.

We first remark that, for all $l = 1, 2, \cdots, q$,

$$u_l(x) = \begin{cases} c(x) & \text{if } x \in [\alpha]_{e^{k+1}} \text{ and } \mathbf{0}_x = S_l \\ u_{l-1}(x) & \text{otherwise} \end{cases} \tag{3.28}$$

For each $T \in 2^N\backslash\{N\}$ and $x_{-T} \in]0\,,1[^{N\backslash T}$, we pose $y = (\mathbf{1}_T,\, x_{-T})$ and $z = (\mathbf{0}_T,\, x_{-T})$. Then, $\mathbf{0}_y = \emptyset$ and $\mathbf{0}_z = T$. Now let compute $\Delta u_l\,(T,\, x_{-T})$ and $\Delta u_{l-1}\,(T,\, x_{-T})$. Two cases arise:

- If $T \neq S_l$, then $\mathbf{0}_z \neq S_l$ and one have:

$$\begin{aligned} \Delta u_l\,(T,\, x_{-T}) &= u_l(y) - u_l(z) \\ &= u_{l-1}(y) - u_{l-1}(z), \quad \text{by Equation (3.28)} \\ &= \Delta u_{l-1}\,(T,\, x_{-T}) \end{aligned}$$

- If $T = S_l$, then $\mathbf{0}_z = S_l$. Two subcases are considered.
  - If $x_{-S_l} \notin D_l$, then $y, z \notin [\alpha]_{e^{k+1}}$. Therefore, Equation (3.28) implies,

$$\begin{aligned} \Delta u_l(S_l,\, x_{-S_l}) &= u_l(y) - u_l(z) \\ &= u_{l-1}(y) - u_{l-1}(z) \\ &= \Delta u_{l-1}(S_l,\, x_{-S_l}) \end{aligned}$$

  - If $x_{-S_l} \in D_l$ then, $z \in [\alpha]_{e^{k+1}}$. Since $\mathbf{0}_z = S_l$. Thus by Equation (3.28),

$$\begin{aligned}
\Delta u_l(S_l, x_{-S_l}) &= u_l(y) - u_l(z) \\
&= u_{l-1}(y) - c(z) \\
&= u_{l-1}(y) - u_{l-1}(z) + u_{l-1}(z) - a_{e^{k+1}} \quad \text{since } z \in [\alpha]_{e^{k+1}} \\
&= \Delta u_{l-1}(S_l, x_{S_l^c}) + 1 - a_{e^{k+1}} \quad \text{due to Equations (3.25) and (3.28)} \\
&= \Delta u_{l-1}(S_l, x_{-S_l}) + \varepsilon_k
\end{aligned}$$

Therefore, $\Delta u_l(S_l, x_{-S_l}) = \Delta u_{l-1}(S_l, x_{-S_l}) + \varepsilon_k$ if $x_{-S_l} \in D_l$, and $\Delta u_l(S_l, x_{-S_l}) = \Delta u_{l-1}(S_l, x_{-S_l})$ if $x_{-S_l} \notin D_l$.

Finally, we conclude that, $u_{l-1} \xrightarrow{S_l,\, \varepsilon_k,\, D_l} u_l$.

On the one hand, note that $\Delta u_0 = \Delta c_k$. Thus u_1 is also an improvement of potentials in c_k. On the other hand, note that $\Delta u_q = \Delta c_{k+1}$. Thus c_{k+1} is also an improvement of potentials in u_{q-1}. One obtained the requested sequence by considering $m = q$, $f_0 = c_k$, $f_l = u_l$ for $1 \leq l < q$ and $f_q = c_{k+1}$.

Case 2: If $L(e^{k+1}) = \emptyset$ and $U\left(e^{k+1}\right) \neq \emptyset$. Pose $T_0 = \emptyset$ and consider a labelling $\{T_t,\, 1 \leq t \leq q'\}$ of $2^{U(e^{k+1})} \setminus \{N\}$ such that $|T_t| \leq |T_{t+1}|$ for all $1 \leq t < q'$. Let $(v_l)_{0 \leq l \leq q'}$ be the sequence defined by

$$v_l(x) = \begin{cases} c\,(x) & x \in \mathcal{B}^-(e^{k+1}) \text{ or } x \in [\alpha]_{e^{k+1}} \text{ and } \mathbf{1}_x = T_t \text{ for some } \quad t \leq l \\ c_k(x) & \text{otherwise} \end{cases} \tag{3.29}$$

Similarly, as in case 1, one can check that, $v_l \in \mathcal{CSG}_n$ and for all $l = 1, 2, \cdots, q'$, $v_l \xrightarrow{T_l,\, \varepsilon_k,\, E_l} v_{l-1}$; where $E_l = \bigtimes_{i \in N \setminus T_l} [\alpha]_{e_i^{k+1}}$. Moreover we can observe that, $v_{q'} = c_{k+1}$ and $\Delta v_0 = \Delta c_k$. Since $\Delta v_0 = \Delta c_k$, c_k is also a local improvement of potentials in v_1. One obtained a desired sequence by considering $m = q'$, $f_0 = c_k$, $f_l = v_l$ for $1 \leq l \leq q'$.

Case 3: If $L(e^{k+1}) \neq \emptyset$ and $U\left(e^{k+1}\right) \neq \emptyset$. This case combines the two previous one. In order to construct our desired sequence, consider the labelling of $2^{L(e^{k+1})} \setminus \{N\}$ and that one $2^{U(e^{k+1})} \setminus \{N\}$ given respectively in case 1 and case 2. Pose $r = q + q' + 1$ and $(w_l)_{0 \leq l \leq r}$ a sequence defined by :

$$w_l = \begin{cases} u_l & \text{if} \quad 0 \leq l \leq q \\ v_{l-q-1} & \text{if} \quad q+1 \leq l \leq r \end{cases} \tag{3.30}$$

where, u_l is defined by Equation (3.27) and v_l is defined by Equation (3.29) by replacing c_k with u_q. It follows from case 1 and case 2 that, for all $0 \leq l \leq m$, $w_l \in \mathcal{CSG}_n$ and satisfies:

$$\begin{cases} \Delta w_0 = \Delta c_k \quad \text{and} \quad w_{l-1} \xrightarrow{S_l,\, \varepsilon_k,\, D_l} w_l \quad \text{for all} \quad 1 \leq l \leq q \\ \Delta w_q = \Delta w_{q+1} \quad \text{and} \quad w_{l-1} \xrightarrow{T_{l-q-1},\, -\varepsilon_k,\, E_{l-q-1}} w_l \quad \text{for all} \quad q+2 \leq l \leq r \\ w_r = c_{k+1} \end{cases} \tag{3.31}$$

Therefore by Equations (3.30) and (3.31), we obtain a desired sequence by considering $m = q + q'$, $f_0 = c_k$, $f_l = w_l$ for $1 \leq l \leq q$ and $f_l = w_{l+1}$ for $q+1 \leq l \leq m$.

■

An illustration of the construction of a sequence of local improvements of potentials from c_k to c_{k+1} is provided in Appendix C for any 2-players discrete CSGs.

Theorem 3.2.3.

Consider $n \geq 3$ and let F be a power index on $\mathcal{CSG}_n$ satisfying (E), (NP), and (HIS). Then, $F(c) = \Psi(c)$ for every $c \in \mathcal{CSG}_n$ that is regular.

Proof. Consider $p \geq 2$, $\alpha \in \mathcal{D}_p$ and $c = (a_e)_{e \in \mathcal{A}_{p,n}} \in \Gamma_p^\alpha$. We prove that $\Psi(c) = F(c)$.

First suppose that $c = c_0$. By a direct computation, we have $\Psi(c_0) = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$. Consider $i \in N$ and the CSG w_i defined by $w_i(x) = 1$ if $x_{-i} \neq \mathbf{0}_{-i}$ and $w_i(x) = 0$ otherwise. One can easily check that, player i is a null player in w_i and $c_0 \xrightarrow{N\setminus\{i\},\, 1,\, [0,1]} w_i$. Thus, (HIS) and (NP) yield $F_i(c_0) = \gamma_F\left(N \setminus \{i\}\right)$. Since F meets (E), (HIS), (NP) and $n \geq 3$, then Lemma 3.1.4 implies,

$$F_i(c_0) = \gamma_F\left(N \setminus \{i\}\right) = \gamma_\Psi\left(N \setminus \{i\}\right) = \frac{1}{n} = \Psi_i(c_0)\,. \tag{3.32}$$

Hereafter, we assume that $c \neq c_0$. Then, the game c can be built up step by step by considering the sequence $(c_k)_{0 \leq k \leq p^*}$ defined by Equation (3.25). Given $0 \leq k \leq p^*$ consider the following assertion :

$$\mathcal{H}(k)\ : F(c_k) = \Psi(c_k).$$

We prove by induction that $\mathcal{H}(k)$ holds. According to Equation (3.32), $\mathcal{H}(0)$ holds. Now given $k \in \{0, 1, ..., p^* - 1\}$ such that for all $l \leq k$; $\mathcal{H}(l)$ holds. Let show that $\mathcal{H}(k+1)$ also holds, that is $F(c_{k+1}) = \Psi(c_{k+1})$.

If $a_{e^{k+1}} = 1$, then $\Delta c_k = \Delta c_{k+1}$. Since $c \neq c_0$, there exists some $l \in \{1\,, 2\,, \ldots\,, k\}$ such that $a_{e^l} \neq 1$ and $\Delta c_t = \Delta c_{t+1}$ for all $t \in \{l,\, l+1,\, \cdots,\, k\}$. Consequently, by Lemma 3.2.1, there exists a sequence $(f_s)_{0 \leq s \leq m}$ such that $f_0 = c_{l-1}$, $f_m = c_l$ and for all $s < m$, either f_{s+1} is an improvement of potentials in f_s or f_s is an improvement of potentials in f_{s+1}. Since $\Delta c_t = \Delta c_{t+1}$ for all $1 \leq t \leq k$, c_{k+1} is also a local improvement of potentials in f_{m-1}. Then, the sequence $(g_s)_{0 \leq s \leq m}$ with $g_s = f_s$ for all $0 \leq s < m$ and $g_m = c_{k+1}$: satisfies g_s is a local improvement of potentials in g_{s+1} or g_{s+1} is a local improvement of potentials in g_s. Although Ψ and F satisfy (E), (NP), (HIS) and $n \geq 3$, by Lemma 3.1.4 and Proposition 3.1.4 we obtain:

$$F\left(g_{s+1}\right)-F\left(g_s\right)=\Psi\left(g_{s+1}\right)-\Psi\left(g_s\right) \quad \text{for all } 0\leq s<m \tag{3.33}$$

Since, $g_0=c_{l-1}$ and $\mathcal{H}(l-1)$ holds by hypothesis of induction then $F(g_0)=\Psi(g_0)$. So, Equation (3.33) implies $F\left(c_{k+1}\right)=\Psi\left(c_{k+1}\right)$. That is $\mathcal{H}(k+1)$ holds.

Now suppose that $a_{e^{k+1}}\neq 1$ then, by Lemma 3.2.1 there exists a sequence $(f_s)_{0\leq s\leq m}$ of CSGs such that $f_0=c_k$, $f_m=c_{k+1}$ and for all $s<m$, either f_{s+1} is an improvement of potentials in f_l or f_s is an improvement of potentials in f_{s+1}. By Lemma 3.1.4 and Proposition 3.1.4,

$$F\left(f_{s+1}\right)-F\left(f_s\right)=\Psi\left(f_{s+1}\right)-\Psi\left(f_s\right) \quad \text{for all } 0\leq s<m \tag{3.34}$$

Since $\mathcal{H}(k)$ holds, then $F\left(c_k\right)=\Psi\left(c_k\right)$. Hence, Equation (3.34) implies $F\left(c_{k+1}\right)=\Psi\left(c_{k+1}\right)$. That is $\mathcal{H}(k+1)$ holds.

Finally, for all $k=0,1\cdots,p^*$; $F\left(c_k\right)=\Psi\left(c_k\right)$. In particular $F\left(c_{p^*}\right)=\Psi\left(c_{p^*}\right)$, i.e., $F(c)=\Psi(c)$. ■

Theorem 3.2.3 provides a characterization of Ψ by three axioms on the subset of all regular CSGs with at least three players. The additional axiom of discreteness permits to extends previous result to the whole set of all CSGs.

Theorem 3.2.4.

Consider $n\geq 3$ and let F be a power index on $\mathcal{CSG}_n$. *Then,* F satisfies (E), (NP), (HIS) and (D) if an only if $F=\Psi$.

Proof. Necessity: If $F=\Psi$, it follows from Lemmas 3.1.2 and 3.1.6 that F satisfies (HIS) and (D). The two others properties (E) and (NP) come directly from Lemma 3.1.1.

Sufficiency: Now assume that F satisfies (E), (NP), (HIS) and (D). Consider $c\in\mathcal{CSG}_n$, by Proposition 3.1.6, there exists a sequence of regular discretizations $(u_p)_{p\geq 2}$ of c compatible with potentials such that, $u_p\in\Gamma_p^{\alpha^p}$ and $\lim\limits_{p\longrightarrow+\infty}\omega\left(\alpha^p\right)=0$. Hence :

$$\begin{aligned} F(c)=&\lim_{p\longrightarrow+\infty}F(u_p) \quad \text{since } F \text{ is discretizable}\\ =&\lim_{p\longrightarrow+\infty}\Psi(u_p) \quad \text{by Theorem 3.2.3}\\ =&\Psi(c) \quad \text{since } \Psi \text{ is discretizable.}\end{aligned}$$

We conclude that $F=\Psi$. ■

3.2.3 Independence of the axioms of characterization

Theorems 3.2.2 and 3.2.4 characterize the Shapley Shubik index for CSGs. We now prove that none of those axioms can not be dropped and this highlights the independence and the non-redundancy of these axioms.

Efficiency can not be dropped

Proposition 3.2.1. *The power index $F^1 = 2 \cdot \Psi$ satisfies (NP), (HIS), and (D), but not (E).*

Proof. Directly from the definition, F^1 obviously satisfies (NP), (HIS) and(D); but not (E). Note that the (HIS) constants of F^1 is given by $\lambda_{F^1} = 2.\lambda_\Psi$ and $\gamma_{F^1} = 2.\gamma_\Psi$. ■

The null player property can not be dropped

Proposition 3.2.2. *Denote by ED the *equal division* power index defined for any $v \in \mathcal{CSG}_n$ by*

$$\mathrm{ED}(v) = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right).$$

Then, the power index $F^2 = \frac{1}{2}\Psi + \frac{1}{2}\mathrm{ED}$ satisfies (E), (HIS) and (D); but not (NP).

Proof. Consider $v \in \mathcal{CSG}_n$. Then, $\sum_{i\in N} F_i^2(v) = \frac{1}{2}\sum_{i\in N}\Psi_i(v) + \frac{1}{2} = 1$, thus F^2 is efficient. F^2 satisfies (HIS) with constants $\lambda_{F^2} = \frac{\lambda_\Psi}{2}$ and $\gamma_{F^2} = \frac{\gamma_\Psi}{2}$. Let (v_p) be a sequence of regular discretizations of v compatible with the potentials, then

$$\lim_{p\longrightarrow+\infty} F^2(v_p) = \frac{1}{2} \cdot \lim_{p\longrightarrow+\infty} \Psi(v_p) + \frac{1}{2n} = \frac{1}{2} \cdot \Psi(v) + \frac{1}{2n} = F^2(v).$$

We conclude that F^2 satisfies (D). By definition F^2 assigns to each null player in any CSG the power $\frac{1}{n}$, thus F^2 does not satisfy (NP). ■

Homogeneous increments sharing can not be dropped

In order to construct a power index that satisfies (E), (NP) and (D); but not (HIS), we define for any CSG v, the game v^2 by setting $v^2(x) = (v(x))^2$, for all $x \in I^n$.

Proposition 3.2.3. *The mapping F^3 that associates each game v to $\Psi(v^2)$ is a power index that satisfies (E), (NP) and (D), but not (HIS).*

Proof.

It is easy to check that F^3 satisfies (E) and (NP). Let v be a CSG on N and $(v_p)_{p\geq 2}$ a sequence of regular discretizations of v compatible with the potentials. We first show that, the series $(v_p^2)_{p\geq 2}$ is the sequence of discretizations of v^2 compatible with potentials. Consider $p \geq 2$ and $x \in [\alpha^p]_e$ for some $e \in \mathcal{A}_{p,n}$. Since v_p is a regular discretization of v, then

$$v_p(x) = v(c_e^p), \tag{3.35}$$

where c_e^p is the center of $[\alpha^p]_e$.

Equation (3.35) is equivalent to $v_p^2(x) = v^2(c_e^p)$, i.e, v_p^2 is a regular discretizaion of v^2. Moreover, for each $S \in 2^N \backslash \{N\}$, it follows by Equation (3.35) that

$$v_p^2(\mathbf{1}_S,\, x_{-S}) - v_p^2(\mathbf{0}_S,\, x_{-S}) = v^2(\mathbf{1}_S,\, (c_e^p)_{-S}) - v^2(\mathbf{0}_S,\, (c_e^p)_{-S})\,.$$

Therefore, v_p^2 is compatible with the potentials in v^2 and thus,

$$\lim_{p\longrightarrow+\infty} F^3(v_p) = \lim_{p\longrightarrow+\infty} \Psi(v_p^2) = \Psi(v^2) = F^3(v)\,.$$

Hence, F^3 is discretizable.

To prove that F^3 does not meet (HIS) consider a CSG v defined as follows $v(x) = x_1x_2^2$ for all $x \in I^n$. Then

$$F^3(v) = \left(\frac{13}{30}, \frac{17}{30}, 0, \cdots, 0\right) \neq \Psi(v) = \left(\frac{5}{12}, \frac{7}{12}, 0, \cdots, 0\right)\,.$$

Thus by Theorem 3.2.4, it appears that F^3 does not satisfy (HIS). ■

Discreteness can not be dropped

The construction of a power index that satisfies (E), (NP), and (HIS) but not (D) is a bit more technically involved. On $\mathcal{CSG}_n$ we can define an equivalence relation, where two CSGs are in the same class if one of them can be obtained from the other by a finite sequence of local improvements. It can be shown that there exists more than one equivalence class.

We define a power index F^4 as follows:

- for $v(x) = \prod_{i=1}^n x_i^i$, $F^4(v) = \frac{2}{n(n+1)} \cdot (1, 2, \ldots, n)$;
- for every CSG u within the same equivalence from v, $F^4(u)$ is defined by $F^4(v)$ via (HIS)
- for every continuous simple game u' that is contained in a different equivalence class than v, we set $F^4(u') = \Psi(u')$.

Proposition 3.2.4. *The mapping F^4 is a power index for CSGs that satisfies (E), (NP) and (HIS), but not (D).*

Proof. By construction of F^4, we cannot lose efficiency and no player can turn into a null player. So, F^4 satisfies (E), (NP) and (HIS). Since $F^4(v) \neq \Psi(v)$, then by Theorem 3.2.4, it follows that F^4 does not satisfy (D). ■

Propositions 3.2.1–3.2.4 prove that the four axioms in Theorem 3.2.4 are independent.

Independence of the axioms of characterization in Theorem 3.2.2

We remark that the power indices constructed in the Propositions 3.2.1–3.2.4 also satisfy the symmetry axiom. So, to prove the independence of axioms in Theorem 3.2.2, it is sufficient to construct a power index that satisfies (E), (NP), (HIS), (D) but not (S). For this, one can easily check that every power index F^a (with $a = (a_1, a_2) \in \mathbb{R}^2_{\geq 0}$ and $a_1 \neq a_2$) as defined in Lemma 3.1.5 satisfies (E), (NP), (HIS), (D) but not (S).

3.3 Alternative axiomatization

In this section published in Kurz et al. (2020), we transfer some notions introduced in Chapter 2 to CSGs. In particular, we introduce an operator that associates each CSG v with a TU-game $\widehat{v}$ called *average game*. This notion is used to formalize the continuous version of average convexity axiom which, combined with efficiency, null player property and symmetry leads to an alternative characterization of the Shapley-Shubik index for CSGs. We also show the independence of axioms.

3.3.1 Average game of a CSG

Similarly to the case of uniform (j, k) simple games, we introduce the average game operator for CSGs and study his properties.

Definition 3.3.1. *Let v be a CSG on N. The average game associated with v and denoted by $\widehat{v}$ is defined via*

$$\forall S \subseteq N,\, \widehat{v}(S) = \int_{I^n} [v(\mathbf{1}_S,\, x_{-S}) - v(\mathbf{0}_S,\, x_{-S})]dx = \mathcal{C}(v,\, S)\,. \tag{3.36}$$

Following Definition 3.1.4 of the potential function, $\widehat{v}(S)$ can be interpreted as the average potential of the coalition S in the game v. Using the definition of the average game of the CSG, Theorem 1.2.3, page 25 can be rewritten as,

Theorem 3.3.1.

For all CSGs v on N and for all $i \in N$,

$$\Psi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [\widehat{v}(S) - \widehat{v}\,(S \backslash \{i\})]. \tag{3.37}$$

In other words, for a given CSG v the power distribution $\Psi(v)$ is given by the Shapley value of its average game $\widehat{v}$. As with uniform (j,k) simple games, the average game operator for CSGs is not injective, i.e. two distinct CSGs may have the same average game as illustrated in the following example.

Example 3.3.1. *Consider the CSGs u and v defined on N respectively for all $x \in I^n$ by : $u(x) = 1$ if $x = \mathbf{1}$, and $u(x) = 0$ otherwise; $v(x) = 1$ if $x \neq \mathbf{0}$, and $v(x) = 0$ otherwise. It is clear that, $u \neq v$. But, Equation (3.36) gives $\widehat{u}(S) = \widehat{v}(S) = 1$ if $S = N$ and $\widehat{u}(S) = \widehat{v}(S) = 0$ otherwise. So, $\widehat{u} = \widehat{v}$.*

The average game operator preserves some properties of CSGs.

Proposition 3.3.1. *Given a CSG v on N,*

(a) *$\widehat{v}$ is a TU-game on N that is $[0,1]$-valued and monotone;*

(b) *any null player in v is a null player in $\widehat{v}$;*

(c) *any two symmetric players in v are symmetric players in $\widehat{v}$;*

(d) *if $v = \sum_{t=1}^{p} \alpha_t \cdot v_t$ is a convex combination for some $v_1, \ldots, v_p \in \mathcal{CSG}_n$ then $\widehat{v} = \sum_{t=1}^{p} \alpha_t \cdot \widehat{v}_t$.*

Proof.
Very similar to the one of Proposition 2.1.5, page 46. ■

For the remaining part of this thesis, we introduce some useful collection of CSGs. Given a coalition S, consider the CSG C^S defined for all $x \in I^n$ by :

$$C^S(x) = \begin{cases} 1 & \text{if } S \subseteq \mathbf{1}_x \\ 0 & \text{otherwise} \end{cases} \tag{3.38}$$

We can remark that, given a coalition S, any player who is not in S is null in C^S and the players whose belong to S are symmetric. Furthermore, we have the following proposition.

Proposition 3.3.2. *The collection of average games $\left(\widehat{C^S}\right)_{S \in 2^N}$ is a basis of Γ^N.*

Proof.
In order to prove this proposition, it is sufficient to show that, for each $S \in 2^N$, $\widehat{C^S} = \gamma_S$. For this, consider $S \in 2^N$ and $T \subseteq N$. For every $x \in [0,1[^n$, we pose $y^T(x) = (\mathbf{1}_T, x_{-T})$ and $z^T(x) = (\mathbf{0}_T, x_{-T})$, so $\mathbf{1}_{y^T(x)} = T$ and $\mathbf{1}_{z^T(x)} = \emptyset$. Therefore by definition, $C^S(z^T(x)) = 0$.

If $S \subseteq T = \mathbf{1}_{y^T(x)}$, for all $x \in [0,1[^n$ then Equation (3.38) yields $C^S(y^T(x)) = 1$. Thus,

$$\widehat{C^S}(T) = \int_{[0,1[^n} [C^S(y^T(x)) - C^S(z^T(x))]dx = 1 = \gamma_S(T).$$

Now assume that $S \not\subseteq T = \mathbf{1}_{y^T(x)}$, for all $x \in [0,1[^n$ then Equation (3.38) gives $C^S(y^T(x)) = 0$. So,

$$\widehat{C^S}(T) = \int_{[0,1[^n} [C^S(y^T(x)) - C^S(z^T(x))]dx = 0 = \gamma_S(T).$$

In both cases $\widehat{C^S}(T) = \gamma_S(T)$ for all $T \in 2^N$; that is $\widehat{C^S} = \gamma_S$. We conclude that $\left(\widehat{C^S}\right)_{S\in 2^N}$ is a basis of the vector space of all TU-games on N ■

PROPOSITION 3.3.3. *Let u be a CSG and i a given player. If i is a null player in u then, there exists a collection (x_S^u) of real numbers such that:*

$$\widehat{u} = \sum_{i\notin S\in 2^N} x_S^u \cdot \widehat{C^S}.$$

Proof. The proof is similar from that one of Proposition 2.1.4, page 44 since $\widehat{C^S} = \gamma_S$ for all $S \in 2^N$ and $\widehat{u} \in \Gamma^N$. ■

3.3.2 New result of axiomatization

We provide a new axiomatization of the Shapley-Shubik index (Ψ) for CSGs. Before this, notice that, the axioms introduced in Definition 2.1.5, page 51 and in Definition 2.2.1, page 54 can be easily extended to CSGs by substituting $\mathcal{U}_n$ with $\mathcal{CSG}_n$. The continuous version convexity (average convexity) axiom is denoted (C*) (resp. (AC*)).

PROPOSITION 3.3.4. *The Shapley-Shubik index Ψ satisfies (AC*).*

Proof. Use Theorem 3.3.1 and the linearity of the Shapley value. ■

LEMMA 3.3.1. *If F is a power index on $\mathcal{CSG}_n$ that satisfies (E), (S), and (NP) then, $F(C^S) = \Psi(C^S)$ for all $S \in 2^N$.*

Proof. It is clear that all players in S are symmetric in C^S, while all players outside S are null players in C^S. Since both F and Ψ satisfy (E), (S), and (NP), we conclude that $F_i(C^S) = \Psi_i(C^S) = \dfrac{1}{|S|}$ if $i \in S$ and $F_i(C^S) = \Psi_i(C^S) = 0$ otherwise. This proves that, $F(C^S) = \Psi(C^S)$. ■

Theorem 3.3.2.

A power index F for CSGs satisfies (E), (S), (NP) and (AC*) if and only if $F = \Psi$. ★

Proof.

Necessity: It was shown in Corollary 3.1.1 and in Proposition 3.3.4 that Ψ satisfies (E), (S), (NP), and (AC*).

Sufficiency: Let F be a power index for CSGs that simultaneously satisfies (E), (S), (NP), and (AC*). Consider a CSG u, note that $\widehat{u}$ is a TU-game by Proposition 3.3.1. Thus by Proposition 3.3.2, there exists a collection of real numbers $(\alpha_S)_{S\in 2^N}$ such that

$$\widehat{u} = \sum_{S\in 2^N} \alpha_S \cdot \widehat{C^S} = \sum_{S\in E_1} \alpha_S \cdot \widehat{C^S} + \sum_{S\in E_2} \alpha_S \cdot \widehat{C^S}$$

where $E_1 = \{S \in 2^N : \alpha_S > 0\}$ and $E_2 = \{S \in 2^N : \alpha_S < 0\}$. Moreover, $E_1 \neq \emptyset$ since $\widehat{v}(N) = 1$. We set

$$\varpi = \sum_{S\in E_1} \alpha_S \cdot \widehat{C^S}(N) = \sum_{S\in E_1} \alpha_S > 0.$$

It follows that,

$$\frac{1}{\varpi}\widehat{u} + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi}\ \widehat{C^S} = \sum_{S\in E_1} \frac{\alpha_S}{\varpi}\ \widehat{C^S}. \tag{3.39}$$

Since (3.39) is an equality among two convex combinations, then by (AC*), we deduce that

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi}\ F(C^S) = \sum_{S\in E_1} \frac{\alpha_S}{\varpi}\ F(C^S). \tag{3.40}$$

Therefore, by Lemma 3.3.1 we have,

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi}\ \Psi(C^S) = \sum_{S\in E_1} \frac{\alpha_S}{\varpi}\ \Psi(C^S). \tag{3.41}$$

Since Ψ also satisfies (AC*), we get

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi}\ \Psi(C^S) = \frac{1}{\varpi}\Psi(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi}\ \Psi(C^S). \tag{3.42}$$

Hence $F(u) = \Psi(u)$, for all $u \in \mathcal{CSG}_n$. This means that $F = \Psi$. ∎

Remark 3.3.1. *In contrast with uniform (j, k) simple games, all convex combinations of CSGs are also CSGs. Thus, the axiom (AC*) in Theorem 3.3.2 can be split into two easier axioms: the standard axiom of convexity (C*) and the axiom of average equivalence (AE) stating that if F is a power index for CSGs, then any two CSGs that induce the same average game must have the same power distribution by F.*

3.3.3 Independence of axioms

We now prove that the characterization axioms in Theorem 3.3.2 are independent. To do this, let $i_0 \in N$ be a given player and P^{i_0} be a power index on $\mathcal{CSG}_n$ such that

$$P_p^{i_0}(C^N) = \begin{cases} \frac{2}{n+1} & \text{if } p = i_0 \\ \frac{1}{n+1} & \text{if } p \neq i_0 \end{cases}$$

Proposition 3.3.5.

1. *The power index $F^1 = 2 \cdot \Psi$ satisfies (NP), (S) and (AC*); but not (E).*

2. *The power index $F^2 = \frac{1}{2} \cdot \Psi + \frac{1}{2} \cdot \mathrm{ED}$ satisfies (E), (S) and (AC*); but not (NP).*

3. *The power index $F^3 = \boldsymbol{\Psi^1}$ (see, Definition 3.1.2, page 71) satisfies (E), (S), and (NP); but not(AC*).*

4. *Let u be any CSG on N and $(x_S^u)_{S\in 2^N}$ the coordinates of it average game in the basis $(\widehat{C^S})_{S\in 2^N}$. The power index F^4 defined on $\mathcal{CSG}_n$ by*

$$F^4(u) = \sum_{S\in 2^N} x_S^u \cdot F^4\big(C^S\big) \quad \textit{for all } u \in \mathcal{CSG}_n$$

where, for each $S \in 2^N \backslash \{N\}$, $F^4\big(C^S\big) = \Psi\big(C^S\big)$ and $F^4(C^N) = P^2(C^N)$ satisfies (E), (NP), (AC); but not (S).*

Proof. The proofs of items 1, 2 and 4 are respectively similar to the one of Propositions 2.2.5, 2.2.6 and 2.2.8 (see pages 59–61).

Now, we prove item 3. By Corollary 3.1.1 one concludes that F^3 satisfies (E), (NP) and (S). In order to show that F^3 does not satisfy (AC*), consider the CSG defined by $v(x_1, \cdots, x_n) = x_1 x_2^2$. It can be easily checked that, for all $T \subseteq N$,

$$\widehat{v}(T) = \begin{cases} 1 & \text{if} \quad 1 \in T \,\text{and}\, 2 \in T \\ \frac{2}{3} & \text{if} \quad 1 \in T \,\text{and}\, 2 \notin T \\ \frac{1}{2} & \text{if} \quad 1 \notin T \,\text{and}\, 2 \in T \\ 0 & \text{if} \quad 1 \notin T \,\text{and}\, 2 \notin T \end{cases}$$

Therefore, the decomposition of $\widehat{v}$ in the basis $(\widehat{C^S})_{S\in 2^N}$ is given by:

$$\widehat{v} = \frac{2}{3} \cdot \widehat{C^{\{1\}}} + \frac{1}{2} \cdot \widehat{C^{\{2\}}} - \frac{1}{6} \cdot \widehat{C^{\{1,2\}}} \Longleftrightarrow \frac{6}{7} \cdot \widehat{v} + \frac{1}{7} \cdot \widehat{C^{\{1,2\}}} = \frac{4}{7} \cdot \widehat{C^{\{1\}}} + \frac{3}{7} \cdot \widehat{C^{\{2\}}}. \tag{3.43}$$

Since $\boldsymbol{\Psi^1}$ satisfies (NP), (E) and (S), we easily compute $\boldsymbol{\Psi^1}\left(C^{\{1,2\}}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right)$, $\boldsymbol{\Psi^1}\left(C^{\{1\}}\right) = (1, 0, 0, \cdots, 0)$ and $\boldsymbol{\Psi^1}\left(C^{\{2\}}\right) = (0, 1, 0, \cdots, 0)$. We also have $\boldsymbol{\Psi^1}(v) = \left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right)$ (see the proof of Proposition 3.1.1). So,

$$\frac{6}{7} \cdot \boldsymbol{\Psi^1}(v) + \frac{1}{7} \cdot \boldsymbol{\Psi^1}\left(C^{\{1,2\}}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0\right), \tag{3.44}$$

and

$$\frac{4}{7} \cdot \Psi^{\mathbf{1}}\left(C^{\{1\}}\right) + \frac{3}{7} \cdot \Psi^{\mathbf{1}}\left(C^{\{2\}}\right) = \left(\frac{4}{7}, \frac{3}{7}, 0, \cdots, 0\right). \tag{3.45}$$

It follows from Equations 3.43–3.45 that $\Psi^{\mathbf{1}}$ does not satisfy (AC*). ■

We have presented two axiomatizations of the Shapley-Shubik index for CSGs. In the next chapter, we pay our attention to the qualitative approach of the power measurement in a CSG. More precisely, we extend to CSGs the influence relation introduced by Isbell (1958) on simple games.

⋆ ⋆ Chapter Four ⋆ ⋆

The influence relation for continuous simple games

Ranking players with respect to their ability to affect the collective decision is a major concern in analyzing the structure of voting power in a collective decision making. A well-known tool for this purpose, in the context of simple games, are the influence (desirability) relation introduced by Isbell (1958) and the weak desirability relation introduced by Carreras and Freixas (2008). On the class of voting games with abstention (i.e. (3, 2) simple games), the notion of influence relation was introduced and studied by Tchantcho et al. (2008). In the same innovative trend, Pongou et al. (2011) introduce several versions of this relation on (j, k) simple games and study their properties. A generalization of the influence relation to the context of CSGs was proposed by Kurz (2014). In this chapter we make an in-depth study of that relation.

The chapter is organised as follows. Section 4.1 is devoted to preliminary definitions and results. In Section 4.2, we study the properties of the influence relation of CSGs. We mainly characterize CSGs for which this relation is complete, and show that it is a preordering whenever it is complete. In Section 4.3, we compare the influence relation with the preordering induced by the Shapley-Shubik index for CSGs. Chiefly, we provide a sufficient condition for which these relations coincide.

4.1 Preliminaries

We recall the notion of binary relation and present the influence relation of simple games as well as that of CSGs.

4.1.1 Binary relations

Let E be any nonempty set.

Definition 4.1.1. *A binary relation $\mathcal{R}$ on E is a subset of the cartesian product $E \times E$, i.e. a set of ordered pairs (a, b) of elements of E.*

We will simple write $a\mathcal{R}b$ instead of $(a, b) \in \mathcal{R}$ and $\rceil(a\mathcal{R}b)$ instead of $(a, b) \notin \mathcal{R}$.

Here follows some properties of binary relations.

Definition 4.1.2. A binary relation $\mathcal{R}$ on the set E is said to be:

- *reflexive* if for all $a \in E$, $a\mathcal{R}a$;
- *symmetric* if for all $(a,b) \in E^2$, $b\mathcal{R}a$ whenever $a\mathcal{R}b$;
- *asymmetric* if for all $(a,b) \in E^2$, $a\mathcal{R}b$ implies $\rceil(b\mathcal{R}a)$;
- *transitive* if for all $a,b,c \in E$, $a\mathcal{R}b$ and $b\mathcal{R}c$ imply that $a\mathcal{R}c$;
- *complete* if for all $(a,b) \in E^2$, $a\mathcal{R}b$ or $b\mathcal{R}a$.

The following definition provides some particular binary relations.

Definition 4.1.3. A binary relation $\mathcal{R}$ on E is :

- a *preordering* if it is reflexive and transitive;
- an *equivalence* if it is reflexive, symmetric and transitive;
- a *strict preordering* if it is asymmetric and transitive.

Definition 4.1.4. Two binary relations $\mathcal{R}$ and $\mathcal{R}'$ on E are incompatible if for any $a,b \in E$, $a\mathcal{R}b$ if and only if $\rceil(a\mathcal{R}'b)$.

Note that, any preordering can be split into two binary relations that are incompatible. Those components are presented below.

Definition 4.1.5. Given a preordering $\mathcal{R}$ on E:

- the *symmetric* component of $\mathcal{R}$ denoted $\approx_{\mathcal{R}}$ is the binary relation defined as follows:
$$\text{for all } a,b \in E,\ a \approx_{\mathcal{R}} b \Longleftrightarrow ((a\mathcal{R}b) \quad \text{and} \quad (b\mathcal{R}a))\,;$$
- the *strict* component of $\mathcal{R}$ denoted $\succ_{\mathcal{R}}$ is the preordering defined as follows:
$$\text{for all } a,b \in E,\ a \succ_{\mathcal{R}} b \Longleftrightarrow ((a\mathcal{R}b) \quad \text{and} \quad \rceil(b\mathcal{R}a))\,.$$

Definition 4.1.6. Given two preorderings $\mathcal{R}_1$ and $\mathcal{R}_2$ on E,

- $\mathcal{R}_1$ is a *sub-preordering* of $\mathcal{R}_2$ and we denote $\mathcal{R}_1 \subseteq \mathcal{R}_2$ if
$$\text{for all } a,b \in E,\ (a \succ_{\mathcal{R}_1} b \Longrightarrow a \succ_{\mathcal{R}_2} b) \quad \text{and} \quad (a \approx_{\mathcal{R}_1} b \Longrightarrow a \approx_{\mathcal{R}_2} b)\,.$$
- $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide if for all $a,b \in E$, $a\mathcal{R}_1 b \Longleftrightarrow a\mathcal{R}_2 b$. In other words $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide is equivalent to $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $\mathcal{R}_2 \subseteq \mathcal{R}_1$.

Proposition 4.1.1. Given $\mathcal{R}_1$ and $\mathcal{R}_2$ two preorderings on E such that $\mathcal{R}_1 \subseteq \mathcal{R}_2$. If $\mathcal{R}_1$ is complete then $\mathcal{R}_2$ is complete and both preorderings coincide.

Proof.

Consider $\mathcal{R}_1$ and $\mathcal{R}_2$ two preorderings on E such that $\mathcal{R}_1$ is complete. Assume that $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

- **We first prove that $\mathcal{R}_2$ is complete.** Let $a, b \in E$, since $\mathcal{R}_1$ is complete, w.l.o.g assume that $a\mathcal{R}_1 b$, then $a \succ_{\mathcal{R}_1} b$ or $a \approx_{\mathcal{R}_1} b$. It follows that $a \succ_{\mathcal{R}_2} b$ or $a \approx_{\mathcal{R}_2} b$, since $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Thus $a\mathcal{R}_2 b$, i.e. $\mathcal{R}_2$ is complete.

- **We now prove that $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide.** It is sufficient to prove that $\mathcal{R}_2 \subseteq \mathcal{R}_1$ since $\mathcal{R}_1 \subseteq \mathcal{R}_2$ by hypothesis. Let $a, b \in E$, suppose that $a\mathcal{R}_2 b$ and $\rceil(a\mathcal{R}_1 b)$. Since $\mathcal{R}_1$ is complete, we necessary have $b \succ_{\mathcal{R}_1} a$; thus $b \succ_{\mathcal{R}_2} a$ since $\mathcal{R}_1 \subseteq \mathcal{R}_2$. Hence a contradiction arise since $a\mathcal{R}_2 b$. Therefore, $\mathcal{R}_2 \subseteq \mathcal{R}_1$.

■

4.1.2 The influence relation for simple games

To measure the power of players in a simple game, Isbell (1958) introduced the concept of influence relation which is a preordering that ranks players according to their capacity to impact the final outcome of the game. We recall below the definition of this relation.

Definition 4.1.7. *Let v be a simple game on N, i and j two players:*

- *i is said to be at least as influential (desirable) as j in v, denoted by $i \succsim_v j$ if for all coalition $S \subseteq N\backslash\{i, j\}$, $v(S \cup \{j\}) = 1 \Longrightarrow v(S \cup \{i\}) = 1$;*
- *i is said to be more influential (strictly influential) than j, denoted $i \succ_v j$ if $i \succsim_v j$ and $\rceil(j \succsim_v i)$;*
- *i is said to be as influential as j, denoted $i \sim_v j$, if $i \succsim_v j$ and $j \succsim_v i$.*

It can be easily checked that given a simple game v, the binary relation $\succsim_v$ is a preordering on N. $\succsim$ is the Isbell's *influence relation* on simple games.

Example 4.1.1. *Consider $N = \{1, 2, 3, 4\}$ and v a simple game on N with minimal winning coalitions $\{1, 4\}$ and $\{2, 3\}$. Then we can easily check that, $2 \sim_v 3$, $1 \sim_v 4$ and for any two distinct players i and j such that $(i, j) \notin \{(1, 4); (2, 3); (4, 1); (3, 2)\}$, i and j are not comparable according to the influence relation.*

As shown in Example 4.1.1, the influence relation for simple game is not in general complete; i.e. it does not always compare two players in certain cases. Taylor and Zwicker (1993) introduced the notion of swap robustness of simple games which provides a necessary and sufficient condition under which the influence relation is complete.

Definition 4.1.8. *Let v be a simple game on N. v is said to be swap-robust if for any $S, T \in \mathcal{W}(v)$ and for any $i, j \in N$ such that $i \in S\backslash T$ and $j \in T\backslash S$, we have $(S\backslash\{i\}) \cup \{j\} \in \mathcal{W}(v)$ or $(T\backslash\{j\}) \cup \{i\} \in \mathcal{W}(v)$.*

The following proposition from (Taylor and Pacelli, 2008, pp.278) characterizes the class of simple games with a complete influence relation.

Proposition 4.1.2. *The influence relation of a simple game is a complete preordering if and only if the simple game is swap-robust.*

4.1.3 The influence relation for continuous simple games

The Isbell's influence relation introduced in Definition 4.1.7 was extended to CSGs by Kurz (2014). We recall the definition of this generalized influence relation.

Definition 4.1.9. *Let $v : I^n \longrightarrow I$ be a CSG and $i, j \in N$ two players. Player i is said to be at least as influential as j in v, denoted $i \succcurlyeq_v j$ if*

$$\text{for any } x \in I^n,\ [x_i \leq x_j \Longrightarrow v(\theta_{ij}(x)) \geq v(x)]\,.$$

In words, player i is at least as influential as j if the collective decision increases whenever the two players exchange their level of approval in any profile where the level approval of player i is smaller than the one of the player j.

We write:

- $i \approx_v j$ if $i \succcurlyeq_v j$ and $j \succcurlyeq_v i$. The relation $\approx$ denoted the symmetric component of $\succcurlyeq$;
- $i \succ_v j$ if $i \succcurlyeq_v j$ and $\rceil(j \succcurlyeq_v i)$. The relation $\succ$ is the strict component of $\succcurlyeq$.

Example 4.1.2. *Let u and v be two CSGs on $N = \{1, 2, 3\}$ respectively defined by*

$$\text{for all } x \in I^3,\ u(x_1, x_2, x_3) = \frac{x_1 + x_2 + 2x_3^2}{3} \quad \text{and} \quad v(x_1, x_2, x_3) = x_1 x_2^2 x_3^3\,.$$

- *for any $x \in I^3$, $u(x_1, x_2, x_3) = u(x_2, x_1, x_3)$, so $1 \approx_u 2$. Now consider $i \in N\backslash\{3\}$, then*
$$\text{for any } x \in I^3, u(\theta_{i3}(x)) - u(x) = \frac{(x_3 - x_i)[1 - 2(x_3 + x_i)]}{3}\,. \tag{4.1}$$
For this reason, if we choose $x \in I^3$ in such a way that $x_i < x_3$ and $x_i + x_3 > \frac{1}{2}$ (resp. $x_3 < x_i$ and $x_i + x_3 < \frac{1}{2}$), then Equation (4.1) implies $u(\theta_{i3}(x)) < u(x)$ (resp. $u(\theta_{i3}(x)) > u(x)$). It follows that $\rceil(i \succcurlyeq_u 3)$ and $\rceil(3 \succcurlyeq_u i)$, i.e. player 3 is not comparable to any other player in the game u.
- *Similarly one establishes that $2 \succ_v 1$, $3 \succ_v 1$ and $3 \succ_v 2$. So one can rewrite $3 \succ_v 2 \succ_v 1$.*

The following proposition provides an alternative definition of the influence relation of CSGs.

Proposition 4.1.3. *Let v be a CSG on N and $i, j \in N$ two players,*

$$i \succcurlyeq_v j \Longleftrightarrow [\ \forall x \in I^n,\ x_i \geq x_j \Longrightarrow v(\theta_{ij}(x)) \leq v(x))]\,.$$

Proof. Let v be a CSG and $i, j \in N$ two players. Suppose that $i \succcurlyeq_v j$ and consider $x \in I^n$ such that $x_i \geq x_j$. Setting $y = \theta_{ij}(x)$, we have $y_i \leq y_j$. So, by Definition 4.1.9 we can conclude that $v(\theta_{ij}(y)) \geq v(y)$, that is $v(\theta_{ij}(x)) \leq v(x)$, since $x = \theta_{ij}(y)$.

Conversely, assume that, for all $x \in I^n$ such that $x_i \geq x_j$, $v(\theta_{ij}(x)) \leq v(x)$ and let show that $i \succcurlyeq_v j$. To prove this, consider $z \in I^n$ such that $z_i \leq z_j$ and pose $y = \theta_{ij}(z)$. One obtains $y_i \geq y_j$. So, by assumption, we can write $v(\theta_{ij}(y)) \leq v(y)$, i.e. $v(\theta_{ij}(z)) \geq v(z)$. We conclude that $i \succcurlyeq_v j$. ■

Proposition 4.1.4. *Let v be a CSG on N and $i, j \in N$ two players.*

$$i \approx_v j \Longleftrightarrow v(\theta_{ij}(x)) = v(x) \quad \text{for all } x \in I^n.$$

Proof. Let v be a CSG with a set of players N and $i, j \in N$ two players.

Assume that $i \approx_v j$ and consider $x \in I^n$. W.l.o.g. assume that $x_i \leq x_j$. Since $i \succcurlyeq_v j$ then $v(x) \leq v(\theta_{ij}x)$. Consider $y = \theta_{ij}(x)$ then $y_j = x_i \leq x_j = y_i$. Since $j \succcurlyeq_v i$ it follows that $v(y) \leq v(\theta_{ij}(y))$, i.e. $v(\theta_{ij}(x)) \leq v(x)$, so that $v(x) = v(\theta_{ij}x)$. The converse is obvious. ■

Definition 4.1.10. (see Carreras and Freixas (2008)) *Let F be a power index on $\mathcal{CSG}_n$. The separability relation of F is the preordering denoted $\succcurlyeq_F$ and defined as follows: for any $v \in \mathcal{CSG}_n$ and any $i, j \in N$,*

$$i \succcurlyeq_{F(v)} j \Longleftrightarrow F_i(v) \geq F_j(v)\,.$$

For instance, $\succcurlyeq_\Psi$ is the separability relation with respect to the Shapley-Shubik index. Note that $\succcurlyeq_F$ is always complete and we have $i \succ_{F(v)} j \Longleftrightarrow F_i(v) > F_j(v)$.

Example 4.1.3. *Consider the games of Example 4.1.2, then we have:*

$$\Psi(u) = \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) \quad \text{and} \quad \Psi(v) = \left(\frac{35}{144}, \frac{50}{144}, \frac{59}{144}\right).$$

Thus,

$$1 \approx_{\Psi(u)} 2 \succ_{\Psi(u)} 3 \quad \text{and} \quad 3 \succ_{\Psi(v)} 2 \succ_{\Psi(v)} 1\,.$$

4.2 Properties of the influence relation of CSGs

We study some properties of the influence relation ($\succcurlyeq$) of CSGs. Firstly, we show that this relation generalises the one defined on simple games by Isbell (1958). Secondly, we provide a necessary and sufficient condition for which $\succcurlyeq$ is complete. Furthermore, we show that the completeness of $\succcurlyeq$ is a sufficient condition to guarantee his transitivity.

4.2.1 Influence relation of CSGs as a generalization

We prove that the influence relation $\succcurlyeq$ of CSGs generalises the influence relation $\succsim$ of simple games. To this end, we use the collection $\{T^\tau,\ 0 < \tau \leq 1\}$ of embeddings mapping (see Proposition 1.3.1, page 28), which associate a simple game with a CSG. We need to reconsider a few preliminary notations. Consider $\tau \in]0,\ 1]$ and $x \in I^n$, pose $S_\tau(x) = \{i \in N,\ x_i \geq \tau\}$. For each simple game v, we associate the CSG v_τ defined by $v_\tau(x) = v\left(S_\tau(x)\right)$, for all $x \in I^n$.

The following result states that the influence relation of CSGs is a generalization of the same notion for simple games.

Proposition 4.2.1. *Let v be a simple game on N. For any $\tau \in]0,\ 1]$, for any $i, j \in N$,*

$$i \succcurlyeq_{v_\tau} j \Longleftrightarrow i \succsim_v j\,.$$

Proof. Let v be a simple game on N, $\tau \in]0,\ 1]$ and v_τ the CSG associated with v. Consider two players $i, j \in N$.

Assume that $i \succcurlyeq_{v_\tau} j$. To prove that $i \succsim_v j$, consider $T \subseteq N \backslash \{i, j\}$ and let us prove that $v(T \cup \{i\}) \geq v(T \cup \{j\})$. Pose $x = \left(\mathbf{1}_{T\cup\{j\}},\ \mathbf{0}_{-(T\cup\{j\})}\right) \in I^n$, then $\theta_{ij}(x) = \left(\mathbf{1}_{T\cup\{i\}},\ \mathbf{0}_{-(T\cup\{i\})}\right)$, $S_\tau(x) = T \cup \{j\}$ and $S_\tau(\theta_{ij}(x)) = T \cup \{i\}$. Since $0 = x_i \leq x_j = 1$ and $i \succcurlyeq_{v_\tau} j$ then $v_\tau(\theta_{ij}(x)) \geq v_\tau(x)$. That is $v\left(S_\tau(\theta_{ij}(x))\right) \geq v\left(S_\tau(x)\right)$, i.e. $v(T \cup \{i\}) \geq v(T \cup \{j\})$. It follows that $i \succsim_v j$.

Conversely, suppose that $i \succsim_v j$ and consider $x \in I^n$ such that $x_i < x_j$ (the case $x_i = x_j$ is obvious) let us show that $v_\tau(\theta_{ij}(x)) \geq v_\tau(x)$. Two cases arise:

Case 1 If $x_i < \tau \leq x_j$ then $i \notin S_\tau(x)$ and $j \in S_\tau(x)$. Pose $T = S_\tau(x) \backslash \{j\}$ then $T \subseteq N \backslash \{i,\ j\}$; $T \cup \{j\} = S_\tau(x)$ and $T \cup \{i\} = [S_\tau(x) \cup \{i\}] \backslash \{j\} = \theta_{ij}(S_\tau(x)) = S_\tau(\theta_{ij}(x))$. Since $i \succsim_v j$ and $T \subseteq N \backslash \{i,\ j\}$, $v(T \cup \{i\}) \geq v(T \cup \{j\})$. Hence, $v(S_\tau(\theta_{ij}(x))) \geq v(S_\tau(x))$, that is $v_\tau(\theta_{ij}(x)) \geq v_\tau(x)$.

Case 2 If $\tau \leq x_i < x_j$ (resp. $x_i < x_j < \tau$) then $i, j \in S_\tau(x)$ (resp. $i, j \notin S_\tau(x)$). So, $\theta_{ij}(S_\tau(x)) = S_\tau(x)$. It follows that, $v(S_\tau(\theta_{ij}(x))) = v(S_\tau(x))$, i.e. $v_\tau(\theta_{ij}(x)) = v_\tau(x)$.

From the two cases highlighted above we conclude that, for any $x \in I^n$ such that $x_i \leq x_j$, $v_\tau(\theta_{ij}(x)) \geq v_\tau(x)$, i.e. $i \succcurlyeq_{v_\tau} j$. ■

A direct consequence of Proposition 4.2.1 is that the influence relation $\succcurlyeq$ of CSGs is not complete in general, since $\succsim$ is not always complete[1].

[1] As illustrated in Example 4.1.1.

4.2.2 Transitivity and completeness of $\succcurlyeq$

We prove that the influence relation $\succcurlyeq$ of CSGs is neither transitive nor complete in general; but, it is transitive as soon as it is complete. Furthermore, we provide a necessary and sufficient condition that guarantees the completeness of $\succcurlyeq$. To do this, we extend to CSGs the notion of swap-robustness introduced by Taylor and Zwicker (1993) on simple games and generalized to voting games with abstention by Tchantcho et al. (2008).

We start the study of the properties of $\succcurlyeq$ by showing that the symmetric component $\approx$ is an equivalence relation on N.

Proposition 4.2.2. *For any CSG v on N, $\approx_v$ is an equivalence relation on N.*

Proof. Let v be a CSG on N. The relation $\approx_v$ is naturally reflexive and symmetric. To prove that $\approx_v$ is transitive, we assume that $|N| = n \geq 3$. Let i, j and k be three distinct players such that $i \approx_v j$ and $j \approx_v k$. Consider $x \in I^n$. By Proposition 4.1.4, we have:

$$\begin{aligned}
v(x) = \; & v(\theta_{ij}x) && \text{because } i \approx_v j \\
= \; & v(\theta_{jk}(\theta_{ij}x)) && \text{because } j \approx_v k \\
= \; & v[\theta_{ij}(\theta_{jk}(\theta_{ij}))x] && \text{because } i \approx_v j \\
= \; & v(\theta_{ik}x) && \text{because } \theta_{ij} \circ \theta_{jk} \circ \theta_{ij} = \theta_{ik}
\end{aligned}$$

Finally, for any $x \in I^n$, $v(x) = v(\theta_{ik}(x))$ i.e. $i \approx_v k$. ∎

Proposition 4.2.3. *The influence relation $\succcurlyeq$ is neither transitive nor complete in general.*

Proof. Consider the CSG c on $N = \{1, 2, 3\}$ defined as follows: for all $x \in I^3$,

$$c\,(x) = \frac{1}{3}x_1 + \frac{1}{3}x_1\sqrt{x_2} + \frac{1}{3}x_2\sqrt{x_3}$$

Here we use an equivalent definition of $\succcurlyeq$ given in Proposition 4.1.3.

We show that $1 \succcurlyeq_c 2$. Consider $x \in I^3$ such that $x_1 \geq x_2$.

$$\begin{aligned}
c\,(x_1, x_2, x_3) - c\,(x_2, x_1, x_3) =& \frac{1}{3}x_1 + \frac{1}{3}x_1\sqrt{x_2} + \frac{1}{3}x_2\sqrt{x_3} - \frac{1}{3}x_2 - \frac{1}{3}x_2\sqrt{x_1} - \frac{1}{3}x_1\sqrt{x_3} \\
=& \frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_1\sqrt{x_2} - \frac{1}{3}x_2\sqrt{x_1} + \frac{1}{3}x_2\sqrt{x_3} - \frac{1}{3}x_1\sqrt{x_3} \\
=& \frac{1}{3}\,(x_1 - x_2) + \frac{1}{3}\sqrt{x_1}\sqrt{x_2}\,(\sqrt{x_1} - \sqrt{x_2}) + \frac{1}{3}\sqrt{x_3}\,(x_2 - x_1) \\
=& \frac{1}{3}\,(x_1 - x_2)\,(1 - \sqrt{x_3}) + \frac{1}{3}\sqrt{x_1}\sqrt{x_2}\,(\sqrt{x_1} - \sqrt{x_2}) \geq 0
\end{aligned}$$

It follows that $1 \succcurlyeq_c 2$.

We show that $2 \succcurlyeq_c 3$. Consider $x \in I^3$ such that $x_2 \geq x_3$.

$$\begin{aligned}
c(x_1, x_2, x_3) - c(x_1, x_3, x_2) =& \frac{1}{3}x_1 + \frac{1}{3}x_1\sqrt{x_2} + \frac{1}{3}x_2\sqrt{x_3} - \frac{1}{3}x_1 - \frac{1}{3}x_1\sqrt{x_3} - \frac{1}{3}x_3\sqrt{x_2} \\
=& \frac{1}{3}x_1\sqrt{x_2} - \frac{1}{3}x_1\sqrt{x_3} + \frac{1}{3}x_2\sqrt{x_3} - \frac{1}{3}x_3\sqrt{x_2} \\
=& \frac{1}{3}x_1\left(\sqrt{x_2} - \sqrt{x_3}\right) + \frac{1}{3}\sqrt{x_2}\sqrt{x_3}\left(\sqrt{x_2} - \sqrt{x_3}\right) \\
=& \frac{1}{3}\left(\sqrt{x_2} - \sqrt{x_3}\right)\left(x_1 + \sqrt{x_2}\sqrt{x_3}\right) \geq 0
\end{aligned}$$

It is follow that $2 \succcurlyeq_c 3$.

We show that players 1 and 3 are not comparable. Consider $x = \left(\frac{1}{10}, 1, 0\right)$ and $x' = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$, then $x_1 \geq x_3$ and $x'_3 \geq x'_1$. Since,

$$c\left(\frac{1}{10}, 1, 0\right) = \frac{2}{30} < c\left(0, 1, \frac{1}{10}\right) = \frac{\sqrt{10}}{30} \quad \text{and} \quad c\left(\frac{1}{4}, 0, \frac{1}{2}\right) = \frac{1}{12} < c\left(\frac{1}{2}, 0, \frac{1}{4}\right) = \frac{1}{6},$$

it follows that $\rceil(1 \succcurlyeq_c 3)$ and $\rceil(3 \succcurlyeq_c 1)$.

In conclusion $1 \succcurlyeq_c 2$, $2 \succcurlyeq_c 3$ and players 1 and 3 are not comparable by $\succcurlyeq_c$. So, $\succcurlyeq_c$ is neither transitive nor complete.

■

It follows from the Proposition 4.2.2 that $\succcurlyeq$ is not a complete preordering in general. Nevertheless, we will show that this relation is a preordering whenever it is complete. To this end, we first characterize the set of CSGs for which $\succcurlyeq$ is complete.

The definition of swap robustness of a simple game given in Definition 4.1.8 can be rewritten using profiles as in the following remark:

Remark 4.2.1. *Let v be a simple game on N. v is swap-robust if for any $x, y \in \{0,1\}^n$ such that $v(x) = v(y) = 1$ and any $i, j \in N$ such that $x_i > x_j$ and $y_j > y_i$* [2]*, we have $v(\theta_{ij}(x)) \geq v(x)$ or $v(\theta_{ij}(y)) \geq v(y)$* [3].

The above remark leads to an extension of swap robustness concept to CSGs as follows:

Definition 4.2.1. *A CSG v on N is swap-robust if for all $\alpha, \beta \in]0,1]$ and all $x, y \in I^n$ such that $v(x) \geq \alpha$ and $v(y) \geq \beta$, it holds that for all $i, j \in N$ such that $(x_i > x_j$ and $y_j > y_i)$ implies $(v(\theta_{ij}(x)) \geq \alpha$ or $v(\theta_{ij}(y)) \geq \beta)$.*

Proposition 4.2.4. *A CSG v is swap-robust if and only if for any $x, y \in I^n$ and any $i, j \in N$*

$$x_i > x_j \text{ and } y_j > y_i \Longrightarrow v(\theta_{ij}(x)) \geq v(x) \text{ or } v(\theta_{ij}(y)) \geq v(y). \tag{4.2}$$

[2] i.e. $i \in \mathbf{1}_x \backslash \mathbf{1}_y$ and $j \in \mathbf{1}_y \backslash \mathbf{1}_x$.

[3] The definition of simple game using profiles is given in Definition 1.1.3, page 7.

Proof. Assume that v is swap-robust. Consider $x, y \in I^n$ and $i, j \in N$ such that $x_i > x_j$ and $y_i < y_j$. Suppose that $v(\theta_{ij}(x)) < v(x)$ and let us show that $v(\theta_{ij}(y)) \geq v(y)$. If $v(y) = 0$, there is noting to prove. Now, assume that $v(y) > 0$ and pose $\beta = v(y) > 0$. Since $v(\theta_{ij}(x)) < v(x)$ then $\alpha = v(x) > 0$. Likewise, v is swap-robust, $v(x) \geq \alpha$, $v(y) \geq \beta$ and $v(\theta_{ij}(x)) < \alpha = v(x)$. Then Definition 4.2.1 implies $v(\theta_{ij}(y)) \geq \beta = v(y)$.

Conversely, consider $\alpha, \beta \in]0,1]$; $x, y \in I^n$ with $v(x) \geq \alpha$, $v(y) \geq \beta$ and $i, j \in N$ such that $x_i > x_j$ and $y_j > y_i$. By Equation (4.2), one concludes that $v(\theta_{ij}(x)) \geq v(x) \geq \alpha$ or $v(\theta_{ij}(y)) \geq v(y) \geq \beta$, i.e. v is swap-robust. ■

The following example illustrates the definition of swap robustness of CSGs.

Example 4.2.1.

- The game c in the proof of Proposition 4.2.3 is not swap-robust. To see this, take $x = \left(\frac{1}{10}, 1, 0\right)$ and $y = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$. We have $x_1 > x_3$ and $y_3 > y_1$ but $c(x) < c(\theta_{13}(x))$ and $c(y) < c(\theta_{13}(y))$.

- The CSG v in Example 4.1.2 is swap-robust. Indeed, consider $x, y \in I^3$; $i, j \in N$ such that $x_i > x_j$ and $y_i < y_j$. We denote by k the unique player of $N \backslash \{i, j\}$. Therefore, $v(x)$ can be rewritten as $v(x) = x_i^i \, x_j^j \, x_k^k$. W.l.o.g. suppose that $i < j$ then $j = i + l$, for some $l = 1, 2$.

$$\begin{aligned} v(\theta_{ij}(x)) - v(x) &= x_j^i \, x_i^j \, x_k^k - x_i^i \, x_j^j \, x_k^k \\ &= x_j^i \, x_i^i \, x_k^k \, (x_i^l - x_j^l) \geq 0 \quad \text{since } x_i > x_j \end{aligned}$$

  It appears that for any $x \in I^3$ such that $x_i > x_j$, $v(\theta_{ij}(x)) \geq v(x)$. So, v is swap-robust.

Some important classes of CSGs that are robust-swap are given below.

Proposition 4.2.5. *Any CSG $v \in \mathbb{L}_n \cup \mathbb{W}_n \cup \mathbb{T}_n \cup \mathbb{E}_n$ is swap-robust.*

Proof. In this proof, for any non negative weight vector $w = (w_i)_{i \in N}$ we set,

$$w(z) := \sum_{p \in N} w_p \cdot z_p, \quad \text{for all } z \in I^n .$$

We show that any CSG linearly weighted game is swap-robust.

Consider $v \in \mathbb{L}_n$, there exists $w = (w_i)_{i \in N}$ such that $v(z) = w(z)$, for all $z \in I^n$. Consider $x, y \in I^n$ and $i, j \in N$ such that $x_i > x_j$ and $y_j > y_i$. W.l.o.g., assume that $w_i \leq w_j$ then,

$$v(\theta_{ij}(x)) - v(x) = (w_j - w_i)(x_i - x_j) \geq 0 \quad \text{since } w_i \leq w_j \text{ and } x_i > x_j.$$

One concludes that v is swap-robust.

We show that any CSG weighted game is swap-robust.

Consider $v \in \mathbb{W}_n$. Then, there exists $w = (w_i)_{i \in N}$ and a monotonic increasing function $f : [0,1] \longrightarrow [0,1]$ with $f(0) = 0$ and $f(1) = 1$ such that for all $z \in I^n$, $v(z) = f(w(z))$. Consider $x, y \in I^n$ and $i, j \in N$ such that $x_i > x_j$ and $y_j > y_i$. W.l.o.g., suppose that $w_i \leq w_j$. Following the previous case, $w(\theta_{ij}(x)) \geq w(x)$. Since f is an increasing monotonic function, then $v(\theta_{ij}(x)) = f(w(\theta_{ij}(x))) \geq v(x) = f(w(x))$. We conclude that v is swap-robust.

We prove that any CSG threshold game is swap-robust.

Consider $v \in \mathbb{T}_n$, then there exists $w = (w_i)_{i \in N}$ and a quota $q \in]0,1]$ such that for all $z \in I^n$, $v(z) = 1$ if $w(z) \geq q$ and $v(z) = 0$ otherwise. Consider $x, y \in I^n$ and $i, j \in N$ such that $x_i > x_j$ and $y_j > y_i$. W.l.o.g., suppose that $w_i \leq w_j$ and let us prove that $v(\theta_{ij}(x)) \geq v(x)$. If $v(x) = 0$, the result is obvious. Now assume that $v(x) = 1$, then by the definition of v, $w(x) \geq q$. But, we have $w(\theta_{ij}(x)) \geq w(x) \geq q$, therefore $v(\theta_{ij}(x)) = 1$. Hence, $v(\theta_{ij}(x)) \geq v(x)$ i.e. v is swap-robust.

We show that any CSG exponential product game is swap-robust.

Consider $v \in \mathbb{E}_n$. Then, there exists a vector $(\alpha_i)_{i \in N}$ of positive real numbers such that, for all $z \in I^n$, $v(z) = \prod_{p \in N} x_p^{\alpha_p}$. Consider $x, y \in I^n$ and $i, j \in N$ such that $x_i > x_j$ and $y_j > y_i$. W.l.o.g., suppose that $\alpha_i \leq \alpha_j$ then $\alpha_j = \alpha_i + \varepsilon$ for some $\varepsilon \geq 0$.

$$\begin{aligned} v(\theta_{ij}(x)) - v(x) &= \left(\prod_{p \in N \setminus \{i,j\}} x_p^{\alpha_p} \right) \left(x_j^{\alpha_i} x_i^{\alpha_j} - x_i^{\alpha_i} x_j^{\alpha_j} \right) \\ &= \left(\prod_{p \in N \setminus \{i,j\}} x_p^{\alpha_p} \right) \left(x_i^{\varepsilon} - x_j^{\varepsilon} \right) x_i^{\alpha_i} x_j^{\alpha_i} \geq 0 \quad \text{since } x_i > x_j \text{ and } \varepsilon = \alpha_j - \alpha_i \geq 0. \end{aligned}$$

We conclude that $v(\theta_{ij}(x)) \geq v(x)$, so v is swap-robust.

■

The following proposition states that, the embedding transformation T^τ, $\tau \in]0, 1]$, that associates a simple game v with the CSG v_τ preserves the swap-robustness property of simple games.

Proposition 4.2.6. *If v is a swap-robust simple game then for any $\tau \in]0, 1]$, v_τ is a swap-robust CSG.*

Proof.

Let v be a swap-robust simple game and $\tau \in]0, 1]$. To prove that v_τ is a swap-robust CSG, consider $x, y \in I^n$ and $i, j \in N$ such that $x_i > x_j$ and $y_j > y_i$, then two cases arise:

- If $\tau \leq x_j < x_i$ (resp. $x_j < x_i < \tau$) then $i, j \in S_\tau(x)$ (resp. $i, j \notin S_\tau(x)$). Hence $\theta_{ij}(S_\tau(x)) = S_\tau(\theta_{ij}(x)) = S_\tau(x)$. By the definition of v_τ, $v_\tau(\theta_{ij}(x)) = v_\tau(x)$.
- If $x_j < \tau \leq x_i$ then $i \in S_\tau(x)$ and $j \notin S_\tau(x)$. So, $\theta_{ij}(S_\tau(x)) = (S_\tau(x) \backslash \{i\}) \cup \{j\}$. Two cases arise:
  - If $\tau \leq y_i < y_j$ (resp. $y_i < y_j < \tau$) then $\theta_{ij}(S_\tau(y)) = S_\tau(y)$. Hence, by the definition of v_τ, $v_\tau(\theta_{ij}(y)) = v_\tau(y)$;
  - If $y_i < \tau \leq y_j$ then $j \in S_\tau(y)$ and $i \notin S_\tau(y)$ therefore, $\theta_{ij}(S_\tau(y)) = (S_\tau(y) \backslash \{j\}) \cup \{i\}$. Assume that $v(S_\tau(x)) = 0$ or $v(S_\tau(y)) = 0$. Then $v_\tau(\theta_{ij}(x)) \geq v_\tau(x) = 0$ or $v_\tau(\theta_{ij}(y)) \geq v_\tau(y) = 0$, by the definition of v_τ.

    Now, suppose that $v(S_\tau(x)) = 1$ and $v(S_\tau(y)) = 1$. Since v is swap-robust, then $v\left((S_\tau(y) \backslash \{i\}) \cup \{j\}\right) = 1$ or $v\left((S_\tau(y) \backslash \{j\}) \cup \{i\}\right) = 1$. It follows that, $v_\tau(\theta_{ij}(x)) \geq v_\tau(x)$ or $v_\tau(\theta_{ij}(y)) \geq v_\tau(y)$.

Finally, one concludes that $v_\tau(\theta_{ij}(x)) \geq v_\tau(x)$ or $v_\tau(\theta_{ij}(y)) \geq v_\tau(y)$. Consequently, v_τ is swap-robust. ■

Taylor(1995, pp.231) proves that the influence relation $\succsim_v$ for simple game v is complete if and only if the game v is swap-robust. This result was generalized to voting games with abstention by Tchantcho et al. (2008). The following theorem provides a similar characterization in the context of CSGs.

Theorem 4.2.1.

Let v be a CSG. The influence relation $\succcurlyeq_v$ is complete if and only if v is swap-robust. ★

Proof. Let v be a CSG with set of players N. Assume that $\succcurlyeq_v$ is complete and consider $x, y \in I^n$. Let $i, j \in N$ be two players such that $x_i > x_j$ and $y_j > y_i$. Since $\succcurlyeq_v$ is complete then $i \succcurlyeq_v j$ or $j \succcurlyeq_v i$:

- If $i \succcurlyeq_v j$ then $v(\theta_{ij}(y)) \geq v(y)$, since $y_i < y_j$;
- If $j \succcurlyeq_v i$ then $v(\theta_{ij}(x)) \geq v(x)$ since $x_j < x_i$.

It follows from Proposition 4.2.4 that v is swap-robust.

Conversely, suppose that v is swap-robust and consider two players $i, j \in N$ such that $\rceil(i \succcurlyeq_v j)$, let us prove that $j \succcurlyeq_v i$. Consider $x \in I^n$ such that $x_j < x_i$. Since $\rceil(i \succcurlyeq_v j)$

then there exists $y \in I^n$ such that $y_i < y_j$ and $v(\theta_{ij}(y)) < v(y)$. Hence, $x, y \in I^n$ such that $x_i > x_j$, $y_j > y_i$ and $v(\theta_{ij}(y)) < v(y)$. Since v is swap-robust, then Proposition 4.2.4 yields $v(\theta_{ij}(x)) \geq v(x)$. Therefore, $j \succcurlyeq_v i$ i.e. $\succcurlyeq_v$ is complete. ■

Corollary 4.2.1. *The influence relation $\succcurlyeq$ is complete on $\mathbb{L}_n \cup \mathbb{W}_n \cup \mathbb{T}_n \cup \mathbb{E}_n$. Furthermore, for any $i, j \in N$*

$$i \succcurlyeq_v j \Longleftrightarrow \begin{cases} w_i \geq w_j & \text{if } v \in \mathbb{L}_n \cup \mathbb{W}_n \cup \mathbb{T}_n \\ \alpha_i \geq \alpha_j & \text{if } v \in \mathbb{E}_n \end{cases} \tag{4.3}$$

where $w = (w_i)_{i \in N}$ is a non negative weight vector defining $v \in \mathbb{L}_n \cup \mathbb{W}_n \cup \mathbb{T}_n$ and $\alpha = (\alpha_i)_{i \in N}$ is a vector of positive real numbers defining an exponential product game.

Lemma 4.2.1. *Let v be a CSG. If $\succcurlyeq_v$ is complete then it is transitive.*

Proof.
Let v be a CSG on N (with $n \geq 3$). Suppose that $\succcurlyeq_v$ is complete and consider i, j and k three players such that $i \succcurlyeq_v j$ and $j \succcurlyeq_v k$. Let us show that $i \succcurlyeq_v k$.
Since $\succcurlyeq_v$ is complete then $i \succcurlyeq_v k$ or $k \succcurlyeq_v i$. If $i \succcurlyeq_v k$ then ends the prove. Assume that $k \succcurlyeq_v i$ and consider $x \in I^n$ such that $x_i \leq x_k$. Three cases arise:

Case 1 : $x_i \leq x_j \leq x_k$ **:**

$$\begin{aligned} v(x) \leq\ & v(\theta_{ij}(x)) && \text{because } x_i \leq x_j \text{ and } i \succcurlyeq_v j \\ \leq\ & v(\theta_{jk}(\theta_{ij}x)) && \text{because } (\theta_{ij}x)_j = x_i \leq x_k = (\theta_{ij}x)_k \text{ and } j \succcurlyeq_v k \\ \leq\ & v[\theta_{ij}(\theta_{ik}(\theta_{ij}x))] && \text{because } (\theta_{ik} \circ \theta_{jk}(x))_i = x_j \leq x_k = (\theta_{ik} \circ \theta_{ij}(x))_j \text{ and } i \succcurlyeq_v j \\ =\ & v(\theta_{ik}(x)) && \text{because } \theta_{ij} \circ \theta_{jk} \circ \theta_{ij} = \theta_{ik} \end{aligned}$$

It follows that $v(\theta_{ik}(x)) \geq v(x)$;

Case 2 : $x_i \leq x_k < x_j$ **:**

$$\begin{aligned} v(x) \leq\ & v(\theta_{ij}(x)) && \text{because } x_i \leq x_j \text{ and } i \succcurlyeq_v j \\ \leq\ & v(\theta_{ik}(\theta_{ij}x)) && \text{because } (\theta_{ij}x)_k = x_k \leq x_j = (\theta_{ij}x)_i \text{ and } k \succcurlyeq_v i \\ \leq\ & v[\theta_{jk}(\theta_{ik}(\theta_{ij}x))] && \text{because } [\theta_{ik} \circ \theta_{jk}]_j(x) = x_i \leq x_j = [\theta_{ik} \circ \theta_{ij}]_k(x) \text{ and } j \succcurlyeq_v k \\ =\ & v(\theta_{ik}(x)) && \text{because } \theta_{jk} \circ \theta_{ik} \circ \theta_{ij} = \theta_{ik} \end{aligned}$$

Hence, $v(\theta_{ik}(x)) \geq v(x)$;

Case 3 : $x_j \leq x_i \leq x_k$ **:**

$$\begin{aligned} v(x) \leq\ & v(\theta_{jk}(x =) && \text{because } x_j \leq x_k \text{ and } j \succcurlyeq_v k \\ \leq\ & v(\theta_{ik}(\theta_{jk}x)) && \text{because } (\theta_{jk}x)_k = x_j \leq x_i = (\theta_{jk}x)_i \text{ and } k \succcurlyeq_v i \\ \leq\ & v[\theta_{ij}(\theta_{ik}(\theta_{jk}x))] && \text{because } (\theta_{ik} \circ \theta_{jk}(x))_i = x_j \leq x_k = (\theta_{ik} \circ \theta_{jk}(x))_j \text{ and } i \succcurlyeq_v j \\ =\ & v(\theta_{ik}(x)) && \text{because } \theta_{ij} \circ \theta_{ik} \circ \theta_{jk} = \theta_{ik} \end{aligned}$$

So, $v(\theta_{ik}(x)) \geq v(x)$;

It comes from cases 1–3 that for all $x \in I^n$ such that $x_i \leq x_k$, $v(\theta_{ik}(x)) \geq v(x)$. Therefore, we conclude that $i \succcurlyeq_v k$ and then $\succcurlyeq_v$ is transitive. ■

Corollary 4.2.2. *Let v be a CSG. If v is swap-robust, then $\succcurlyeq_v$ is a complete preordering.*

4.3 Influence relation and Shapley-Shubik power index

The multiplicity of power theories raises the problem of their comparison. In reaction to that, researchers have attempted to identify classes of games for which two given power indices induce the same ordinal structure in the set of players. For example, in the context of simple games Diffo Lambo and Moulen (2002) generalize earlier result of Tomiyama (1987) by showing that the influence relations and the preorderings induced both by the Shapley-Shubik index and the Banzhaf-Coleman index (see, Banzhaf (1965)) coincide if and only if the game is swap-robust. On the class of voting games with abstention and that of (j, k) simple games, similar results were obtained respectively by Tchantcho et al. (2008) and by Pongou et al. (2014).

In this section, we conduct an ordinal comparison of the influence relation of CSGs with the preordering (separability relation) induced by the Shapley-Shubik index on the set of players. In other words, if player i is more influential than player j in a given CSG v, can we say that the Shapley-Shubik index $(\Psi_i(v))$ of player i in v is greater than the one of player j? The following example provides a response.

Example 4.3.1. *Consider a 2-players CSG v such that*

$$v(1,0) = \frac{1}{2}\,; v(0,1) = \frac{3}{4}\,;\ v(x_2, x_1) = v(x_1, x_2) \text{ for any } (x_1, x_2) \in I^2 \backslash \{(0,1),(1,0)\}$$

and for $x_1 \leq x_2$,

$$v(x_1, x_2) = \begin{cases} \frac{1}{2} & \text{if} \quad (x_1, x_2) \in]0,1[^2 \\ 1 & \text{if} \quad x_2 = 1 \quad \text{and} \quad x_1 \neq 0 \\ 0 & \text{if} \quad x_1 = 0 \quad \text{and} \quad x_2 \neq 1 \end{cases}$$

- *since $v(\theta_{12}(x)) \geq v(x)$ for any $x \in I^2$ such that $x_2 \leq x_1$ and $v(0,1) = \frac{3}{4} > v(1,0) = \frac{1}{2}$, then $2 \succ_v 1$;*
- *however $\widehat{v}(\{1\}) = \widehat{v}(\{2\}) = 1$[4], thus $\Psi_1(v) = \Psi_2(v) = \frac{1}{2}$.*

It follows that the preorderings induced by the Shapley-Shubik index does not reflect in general the influence relation of CSGs. In order to compare the influence relation and the separability relation of the Shapley-Shubik index, further notations are needed.

Given a CSG v on N, let us consider for each $i \in N$ and for each coalition S such that $i \in S$ the following notations:

$$\mathcal{N}_i = \{T \subseteq N,\, i \in T\} \quad \text{and} \quad \mathcal{C}_i(v, S) = \widehat{v}(S) - \widehat{v}(S \backslash \{i\})\,.$$

[4] $\widehat{v}$ is the average game introduced in Definition 3.3.1, page 98.

$\mathcal{C}_i(v,\, S)$ can be seen as the average marginal contribution of player i in coalition S. Using the previous notations, we obtain a useful expression of the Shapley-Shubik index $\Psi_i(v)$ of player i in the game v as follows:

$$\Psi_i(v) = \sum_{S\in\mathcal{N}_i} \frac{(s-1)!(n-s)!}{n!}\mathcal{C}_i(v,\, S)\,. \tag{4.4}$$

4.3.1 The Shapley-Shubik index weakly reflected the influence relation

We compare the influence relation $\succcurlyeq$ of CSGs with the separability relation $\succcurlyeq_\Psi$ of the Shapley-Shubik index. Before the main result, we prove below that two equally influential players have the same power distribution with respect to the Shapley-Shubik index

Proposition 4.3.1. *Let v be a CSG on N and $i, j \in N$ two players then*

$$i \approx_v j \Longrightarrow \Psi_i(v) = \Psi_j(v)\,.$$

Proof. Consider $v \in \mathcal{CSG}_n$ and $i, j \in N$ two players such $i \approx_v j$. Then for all $x \in I^n, v(\theta_{ij}(x)) = v(x)$ i.e. i and j are symmetric players in v. Since Ψ is symmetric, one concludes that $\Psi_i(v) = \Psi_j(v)$. ∎

This result implies that the influence relation coincides on complete anonymous CSGs with the separability relation of the Shapley-Shubik index.

Lemma 4.3.1. *Let v be a CSG on N and $i, j \in N$ two players. Then,*

$$i \succcurlyeq_v j \Longrightarrow \mathcal{C}_i(v, \theta_{ij}(S)) \geq \mathcal{C}_j(v, S), \quad \text{for all } S \in \mathcal{N}_j\,.$$

Proof. Consider $v \in \mathcal{CSG}_n$ and $i, j \in N$ two players such that $i \succcurlyeq_v j$. Let us show that for any $S \in \mathcal{N}_j$, $\mathcal{C}_i(v, \theta_{ij}(S)) \geq \mathcal{C}_j(v, S)$. We first note that for any $\mathbf{a}, x \in I^n$ and any $T \subseteq N$, $\theta_{ij}(\mathbf{a}_T, x_{-T}) = (\mathbf{a}_{\theta_{ij}(T)}, (\theta_{ij}x)_{-\theta_{ij}(T)})$. Consider $S \in \mathcal{N}_j$, then:

- If $i \in S$ then $\theta_{ij}(S) = S$ and we have:

$$\begin{array}{rcl}
\mathcal{C}_i(v,\theta_{ij}(S)) & = & \mathcal{C}_i(v,S) \\
& = & \displaystyle\int_{I^n} \left[v\left(\mathbf{1}_S, x_{-S}\right) - v\left(\mathbf{1}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})}\right) + v\left(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\ \setminus\{i\})}\right) - v\left(\mathbf{0}_S, x_{-S}\right)\right] dx \\
& \geq & \displaystyle\int_{I^n} \left[v\left(\mathbf{1}_S, x_{-S}\right) - v\left(\theta_{ij}(\mathbf{1}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})})\right)\right] dx \\
& & + \displaystyle\int_{I^n} \left[v\left(\theta_{ij}(\mathbf{0}_{S\setminus\{i\}}, x_{-(S\setminus\{i\})})\right) - v\left(\mathbf{0}_S, x_{-S}\right)\right] dx \quad \text{since } i \succcurlyeq_v j. \\
& = & \displaystyle\int_{I^n} \left[v\left(\mathbf{1}_S, x_{-S}\right) - \left(\mathbf{0}_S, x_{-S}\right)\right] dx \\
& & - \displaystyle\int_{I^n} \left[v\left(\mathbf{1}_{S\setminus\{j\}}, (\theta_{ij}x)_{-(S\setminus\{j\})}\right) - v\left(\mathbf{0}_{S\setminus\{j\}}, (\theta_{ij}x)_{-(S\setminus\{j\})}\right)\right] dx \\
& = & \widehat{v}(S) - \widehat{v}(S\setminus\{j\}) \quad \text{with } y = \theta_{ij}(x) \text{ in the second integral.} \\
& = & \mathcal{C}_j(v,S)
\end{array}$$

- If $i \notin S$ then $\theta_{ij}(S) = [S\setminus\{j\}] \cup \{i\}$. Pose $T = \theta_{ij}(S)$ then $T\setminus\{i\} = S\setminus\{j\}$. Thus,

$$\begin{aligned}
\mathcal{C}_i(v,\theta_{ij}(S)) &= \mathcal{C}_i(v,T)\\
&= \int_{I^n} \left[v\left(\mathbf{1}_T, x_{-T}\right) - v\left(\mathbf{1}_{T\setminus\{i\}}, x_{-(T\setminus\{i\})}\right) + v\left(\mathbf{0}_{T\setminus\{i\}}, x_{-(T\ \setminus\{i\})}\right) - v\left(\mathbf{0}_T, x_{-T}\right)\right] dx\\
&= \int_{I^n} \left[v\left(\mathbf{1}_T, x_{-T}\right) - v\left(\mathbf{1}_{S\setminus\{j\}}, x_{-(S\setminus\{j\})}\right) + v\left(\mathbf{0}_{S\setminus\{j\}}, x_{-(S\ \setminus\{j\})}\right) - v\left(\mathbf{0}_T, x_{-T}\right)\right] dx\\
&\geq \int_{I^n} \left[v\left(\theta_{ij}(\mathbf{1}_T, x_{-T})\right) - v\left(\mathbf{1}_{S\setminus\{j\}}, x_{-(S\setminus\{j\})}\right)\right] dx\\
&\quad + \int_{I^n} \left[v\left(\mathbf{0}_{S\setminus\{j\}}, x_{-(S\setminus\{j\})}\right) - v\left(\theta_{ij}(\mathbf{0}_T, x_{-T})\right)\right] dx \quad \text{since } i \succcurlyeq_v j.\\
&= \int_{I^n} \left[v\left(\mathbf{1}_S, (\theta_{ij}x)_{-S}\right) - \left(\mathbf{0}_S, (\theta_{ij}x)_{-S}\right)\right] dx\\
&\quad - \int_{I^n} \left[v\left(\mathbf{1}_{S\setminus\{j\}}, x_{-(S\setminus\{j\})}\right) - v\left(\mathbf{0}_{S\setminus\{j\}}, x_{-(S\setminus\{j\})}\right)\right] dx\\
&= \widehat{v}(S) - \widehat{v}(S\setminus\{j\}) \quad \text{with } y = \theta_{ij}(x) \text{ in the first integral.}\\
&= \mathcal{C}_j(v,S)
\end{aligned}$$

It follows that $\mathcal{C}_i(v,\theta_{ij}(S)) \geq \mathcal{C}_j(v,S)$ for all $S \in \mathcal{N}_j$. ■

A direct consequence of Lemma 4.3.1 is given below.

Proposition 4.3.2. *Let v be a CSG on N; i and j two distinct players. Then,*

$$i \succcurlyeq_v j \Longrightarrow i \succcurlyeq_{\Psi(v)} j\,.$$

Proof. Let v be a CSG on N, i and j two players such that $i \succcurlyeq_v j$. Then,

$$\begin{aligned}
\Psi_i(v) &= \sum_{S\in\mathcal{N}_i} \frac{(s-1)!(n-s)!}{n!}\mathcal{C}_i(v,S) \quad \text{by Equation (4.4)}\\
&= \sum_{S\in\mathcal{N}_j} \frac{(s-1)!(n-s)!}{n!}\mathcal{C}_i(v,\theta_{ij}(S)) \quad \text{since} \quad \theta_{ij} : \mathcal{N}_j \longrightarrow \mathcal{N}_i \quad \text{is one-to-one and onto}\\
&\geq \sum_{S\in\mathcal{N}_j} \frac{(s-1)!(n-s)!}{n!}\mathcal{C}_j(v,S) \quad \text{by Lemma 4.3.1}\\
&= \Psi_j(v)
\end{aligned}$$

We conclude that $\Psi_i(v) \geq \Psi_j(v)$, i.e. $i \succcurlyeq_{\Psi(v)} j$. ■

The results of Propositions 4.3.1 and 4.3.2 are quite encouraging for the comparison of the influence relations $\succcurlyeq$ and $\succcurlyeq_\Psi$. Nevertheless, in Example 4.3.1, we have shown that we can have two players, one been more influential than the other but the two have the same distribution of power according to Shapley-Shubik index, this can be qualified as unfair in power measurement theory. It is therefore necessary to identify necessary and/or sufficient conditions for which the influence relation coincide with the separability relation of the Shapley-Shubik index. We carry out this analysis in the next section.

4.3.2 A sufficient condition for the ordinal equivalence of $\succcurlyeq$ and $\succcurlyeq_\Psi$

We provide a sufficient condition for which the influence relations $\succcurlyeq$ and $\succcurlyeq_\Psi$ coincide. Before this, we prove that in the class of linearly weighted games and exponential product games, these two power measurement always coincide.

Proposition 4.3.3. *For every CSG $v \in \mathbb{L}_n \cup \mathbb{E}_n$, a influence relation $\succcurlyeq_v$ is the sub-preordering of $\succcurlyeq_\Psi$.*

Proof. Consider $v \in \mathbb{L}_n \cup \mathbb{E}_n$ and let us show that $\succcurlyeq_v$ is a sub-preordering of $\succcurlyeq_{\Psi(v)}$. Let $i, j \in N$ be two players. If $i \approx_v j$, then $i \approx_{\Psi(v)} j$, by Proposition 4.3.1.

Now suppose that $i \succ_v j$ and let us show that $i \succ_{\Psi(v)} j$.

If $v \in \mathbb{L}_n$, then there exists $w = (w_i)_{i\in N}$ a non negative weight vector such that for all $x \in I^n$, $v(x) = \sum_{p\in N} w_p x_p$. By Corollary 1.2.4, page 27 we have $\Psi_p(v) = w_p$, for any player p. Since $i \succ_v j$, Corollary 4.2.1 implies $w_i > w_j$, that is $i \succ_{\Psi(v)} j$.

Now suppose that $v \in \mathbb{E}_n$, then there exists $\alpha = (\alpha_i)_{i\in N}$ a positive real vector such that for all $x \in I^n$, $v(x) = \prod_{p\in N} x_p^{\alpha_p}$. So, Definition 3.3.1, page 98 give:

$$\widehat{v}(\{p\}) = \prod_{l\in N\setminus\{p\}} \frac{1}{(\alpha_l + 1)} \quad \text{for any player } p \tag{4.5}$$

Since $i \succ_v j$ then, $\alpha_i > \alpha_j$ by Corollary 4.2.1. It follows from Equation (4.5) that $\widehat{v}(\{i\}) > \widehat{v}(\{j\})$. That is

$$\mathcal{C}_i(v, \theta_{ij}(\{j\})) > \mathcal{C}_j(v, \{j\})\,. \tag{4.6}$$

Additionally by Lemma 4.3.1,

$$\mathcal{C}_i(v, \theta_{ij}(S)) \geq \mathcal{C}_j(v, S) \quad \text{for any } S \in \mathcal{N}_j. \tag{4.7}$$

Thus, Equations (4.4), (4.6) and (4.7) imply $\Psi_i(v) > \Psi_j(v)$, i.e. $i \succ_{\Psi(v)} j$. ∎

Proposition 4.3.4. *The preorderings $\succcurlyeq$ and $\succcurlyeq_\Psi$ coincide on $\mathbb{L}_n \cup \mathbb{E}_n$.*

Proof. Combine Corollary 4.2.1, Lemma 4.3.1 and Proposition 4.3.3. ∎

Definition 4.3.1. *Let v be a CSG on N.*

- *v is said to be topologically continuous (TC) on I^n if for all $x_0 \in I^n$ and all $\varepsilon > 0$, there exists $\alpha > 0$ such that:*

$$\text{for all } x \in I^n,\ \|x - x_0\|_1 < \alpha \Longrightarrow |v(x) - v(x_0)| < \varepsilon.$$

- v satisfies condition (Θ) if for all players $i, j \in N$ and all profiles $x \in I^n$ such that $x_i < x_j$,

$$v(\theta_{ij}(x)) > v(x) \Longrightarrow \exists y \in I^n,\ v(1, y_{-i}) - v(0, y_{-i}) > v(1, y_{-j}) - v(0, y_{-j}) \qquad (4.8)$$

- A given player i is more powerful than another player j in a profile x if $x_i < x_j$ and $v(\theta_{ij}(x)) > v(x)$.

In words, the topologically continuity of a CSG on I^n simply means that any small changes in the actions of players (possible minor errors) should not entail a big change in the final decision (output error); see (Grabisch et al., 2009, Definition 2.7). The set of topologically continuous games on I^n is denoted $\mathcal{TC}_n$.

Condition (Θ) inspired from (Pongou et al., 2014, Definition 4) identifies CSGs such that: whenever a player is more powerful in a profile than another player, there exists a voting situation in which this player still has a greater impact than the the other player as the level of approval of each of the two players goes from 0 to 1.

Hereafter for a given CSG v on N and $i, j \in N$, we define the mapping g^v_{ij} on I^n as follows:

$$\text{for all } x \in I^n, \quad g^v_{ij}(x) = \Delta v(\{i\}, x_{-i}) - \Delta v(\{j\}, x_{-j})^5\,. \qquad (4.9)$$

Proposition 4.3.5. *Consider a CSG $v \in \mathcal{TC}_n$ that satisfies condition (Θ) and $i, j \in N$ two players. If $i \succ_v j$ then:*

1. *for all $x \in I^n$, $g^v_{ij}(x) \geq 0$;*
2. *there exists $\mathcal{D} \subseteq I^n$, with $\mathrm{vol}(\mathcal{D}) > 0$ and $m > 0$ such that for all $x \in \mathcal{D}$, $g^v_{ij}(x) > m$.*

Proof. Consider $v \in \mathcal{TC}_n$ satisfying (Θ) and $i, j \in N$ two players such that $i \succ_v j$.

1. Consider $x \in I^n$, then $(1, x_{-i})_i = 1 \geq (1, x_{-i})_j = x_j$, $(0, x_{-i})_i = 0 \leq (1, x_{-i})_j = x_j$ and $\theta_{ij}(\alpha, x_{-i}) = (\alpha, x_{-j})$, $(\alpha = 0, 1)$. Since $i \succ_v j$ it follows that $v(1, x_{-i}) \geq v(1, x_{-j})$ and $v(0, x_{-i}) \leq v(0, x_{-j})$. Therefore, $\Delta v(\{i\}, x_{-i}) = v(1, x_{-i}) - v(0, x_{-i}) \geq v(1, x_{-j}) - v(0, x_{-j}) = \Delta v(\{j\}, x_{-j})$, i.e. $g^v_{ij}(x) \geq 0$.

2. Since $i \succ_v j$, $v(\theta_{ij}(x^0)) > v(x^0)$ for some $x^0 \in I^n$ such that $x^0_i < x^0_j$. Consequently, following condition (Θ) and g^v_{ij}, there exists $y \in I^n$ such that $g^v_{ij}(y) > 0$. Since v is (TC) on I^n then g^v_{ij} is a continuous mapping on I^n. So, for $\varepsilon = \frac{g^v_{ij}(y)}{2}$, there exists $\alpha > 0$ such that for all $x \in I^n$,

$$\|x - y\|_1 < \alpha \Longrightarrow |g^v_{ij}(x) - g^v_{ij}(y)| < \frac{g^v_{ij}(y)}{2} \Longleftrightarrow \frac{g^v_{ij}(y)}{2} < g^v_{ij}(x) < 3 \cdot \frac{g^v_{ij}(y)}{2}\,. \qquad (4.10)$$

[5] $\Delta v(\{k\}, x_{-k}) = v(1, x_{-k}) - v(0, x_{-k})$ is the potential of the player k in the game v given a profile $x \in I^n$, see Page 75.

Pose $\mathcal{D} = \{x \in I^n, \|x - y\|_1 < \alpha\}$ and $m = \varepsilon$, $\text{vol}(\mathcal{D}) > 0$ and by relation (4.10) one obtains $g^v_{ij}(x) > m$ for all $x \in \mathcal{D}$.

■

We show below that, the influence relation is strictly reflected by the Shapley-Shubik index on the subclass $\mathcal{TC}_n$ of CSGs satisfying condition (Θ).

Proposition 4.3.6. *Given $v \in \mathcal{TC}_n$ satisfying condition (Θ) and $i, j \in N$ two players. Then,*

$$i \succ_v j \Longrightarrow \mathcal{C}_i(v, \{i\}) > \mathcal{C}_j(v, \{j\})\,.$$

Proof. Consider $v \in \mathcal{TC}_n$ that satisfies condition (Θ) and $i, j \in N$ two players such that $i \succ_v j$. Then,

$$\begin{aligned}
\mathcal{C}_i(v, \{i\}) - \mathcal{C}_j(v, \{j\}) &= \int_{I^n} g^v_{ij}(x)\,dx \quad \text{by Definition 3.3.1, page 98.}\\
&= \int_{x\in\mathcal{D}} g^v_{ij}(x)\,dx + \int_{x\notin\mathcal{D}} g^v_{ij}(x)\,dx \quad \text{by Proposition 4.3.5 (item 2)}\\
&\geq \int_{\mathcal{D}} g^v_{ij}(x)\,dx \quad \text{by Proposition 4.3.5 (item 1)}\\
&\geq m \cdot \text{vol}(\mathcal{D}) > 0 \quad \text{by Proposition 4.3.5 (item 2)}
\end{aligned}$$

We conclude that $\mathcal{C}_i(v, \{i\}) - \mathcal{C}_j(v, \{j\}) > 0$, i.e. $\mathcal{C}_i(v, \{i\}) > \mathcal{C}_j(v, \{j\})$. ■

Theorem 4.3.1.

Let v be a swap-robust CSG. If $v \in \mathcal{TC}_n$ and satisfies condition (Θ) then $\succcurlyeq_v$ and $\succcurlyeq_\Psi$ coincide. ⋆

Proof. Let v be a swap-robust CSG that is (TC) that satisfies condition (Θ). Consider $i, j \in N$ two players.

- If $i \approx_v j$, Proposition 4.3.1 implies $\Psi_i(v) = \Psi_j(v)$, i.e. $i \approx_\Psi (v)j$.
- If $i \succ_v j$, then $\mathcal{C}_i(v, \{i\}) > \mathcal{C}_j(v, \{j\})$ by Proposition 4.3.6. Moreover, by Lemma 4.3.1, we have $\mathcal{C}_i(v, \theta_{ij}(S)) \geq \mathcal{C}_j(v, S)$ for any $S \in \mathcal{N}_j$.

  Therefore, it appears from Equation (4.4) that $\Psi_i(v) > \Psi_j(v)$, i.e. $i \succ_{\Psi(v)} j$.

We conclude that $\succcurlyeq_v$ is a sub-preordering of $\succcurlyeq_\Psi$. Furthermore, since v is swap-robust then $\succcurlyeq_v$ is complete (see Theorem 4.2.1). Thus, $\succcurlyeq_v$ and $\succcurlyeq_\Psi$ coincide by Proposition 4.1.1. ■

⋆ ⋆ Conclusion ⋆⋆

Decision-making that involves several participants poses a number of problems, including the measurement of decision-making power. In other words, how to formalize the capacity of an individual to affect the outcome of a collective decision in which he or she is an actor? This question has opened up a fairly wide field of research in Social Choice Theory. In this thesis, we have addressed some open issues about the power measures for (j, k) simple games as well as for CSGs. We have closely examined the possibility of axiomatizing the Shapley-Shubik index of these classes of games, (see Freixas (2005b) and Kurz (2014)), and also provided a detailed study of the influence relation of CSGs.

First of all, we have presented the models of simple games, (j, k) simple games and CSGs. It appears that the class of simple games and that of (j, k) simple games can be identified with subclasses of CSGs. This result provides a coherent story condensing the different variants for committee decisions in one common framework. Besides, we have shown that the Shapley-Shubik index for simple games, as well as for (j, k) simple games, is a discretization of that for CSGs, see Theorems 1.3.1–1.3.3. These results give some relevance to the Shapley-Shubik index generalized to CSGs by Kurz (2014).

In order to provide an axiomatization of the Shapley-Shubik index for (j, k) simple games, we introduced the notion of average game of a (j, k) simple game and the axiom of average convexity. The average game allows us to give the Shapley-Shubik index of a (j, k) simple game an explicit formula in terms of the characteristic function. More precisely, the Shapley-Shubik index of a (j, k) simple game as showing in Theorem 2.1.1, is simply the Shapley value of its average game. The average convexity axiom is the requirement that if two convex combinations of average games coincide, then the corresponding convex combinations of the power distributions of the underlying games also coincide. This property can be viewed as some form of linearity condition. The average convexity axiom combined with those of efficiency, symmetry and null player allowed us to obtain the first characterization of the Shapley-Shubik index for (j, k) simple games. Similarly, by extending the notion of average game to CSGs, we provided the first characterization of the Shapley-Shubik index on this class of games. In each case, we established the independence of the axioms.

We have also obtained a second characterization of the Shapley-Shubik index for CSGs by introducing two new axioms. For the newly introduced axiom (HIS), we gave some justification and remarked its similarity to the axiom (SymGL) for simple games. However, the implication of (HIS) are much more forereaching than the implications of (SymGL), which is a more direct axiom tailored for simple games. The idea behind the discreetness axiom is much more mathematical than intuitive. This property allows to extend a power index that is defined on the subclass of discrete CSGs to the whole set of all CSGs. We have shown that, the axiom of symmetry together with efficiency, null player property, homogeneous increments sharing and discreteness are sufficient to uniquely identify the Shapley-Shubik index on the set of 2-players CSGs. In the context of CSGs with at least three players, we proved that this result of characterization still holds even if the axiom of symmetry is dropped. The independence of those axioms of characterization has been established.

Concerning the influence relation of CSGs, the generalized version of the concept of swap-robust game due to Taylor and Zwicker (1993) allowed us to characterize the games for which this relation is complete. Furthermore, we have shown that it is transitive whenever it is complete. Our result therefore generalizes those of Taylor and Zwicker (1999) and Tchantcho et al. (2008) known respectively on simple games and on $(3, 2)$ simple games. To conduct an ordinal comparison of the influence relation and the Shapley-Shubik index, we introduced a sufficient condition (Θ) for which these two relations coincide. Nevertheless, we failed to show whether or not the condition is necessary. This remains an open issue.

The results obtained in this thesis suggest possible future research related to the topics of power measurements. The Shapley-Shubik index (Ψ) for CSGs as defined by Kurz (2014) is an n-dimensional integral. However, Theorem 1.2.3, page 25 tells us that only the values of the game on lower dimensional faces of $[0, 1]^n$ are essential in the definition of this index. Stated more directly, the values of a CSG v in the interior of its domain are more or less irrelevant for $\Psi(v)$. This property might be analyzed and criticized from a more general and non-technical point of view. Our rigorous technical analysis uncovers this fact for the first time, while it is also valid for the Shapley-Shubik index for (j, k) simple games. For example, in an uniform 3-players $(4, 4)$ simple game v, the value of $v(1, 2, 1)$ can be changed to 0, 1, 2, or 3 without any direct effect for the power distribution of the players. Of course monotonicity implies some possible indirect changes of other function values, which then can have an effect for the power distribution. For simple games there are no “internal” vote profiles. In any case this “boundary dependence” should be studied and interpreted in more detail. Only the Shapley-Shubik and Banzhaf-Coleman indices have been generalized to CSGs, the latter assigning a zero power to all players in some games (this finding is omitted in this thesis with focus on the Shapley-Subik index). The generalization of other indices as well as the correction of this flaw in the generalized Banzhaf-Coleman index might be a promising direction for further research. In the case of CSGs, it appears from Remark 3.3.1 that average convexity is equivalent to average equivalence and convexity. Whether this equivalence still holds for (j, k) simple games remains an open issue.

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⋆ ⋆ Appendices ⋆⋆

A Determination of $\mathcal{C}_v$ for Example 1.1.2, page 9

Recall that, given a uniform (j,k) simple game v its associated TU-game $\mathcal{C}_v$ is defined as follows: for any $S \in 2^N$,

$$\begin{aligned}\mathcal{C}_v(S) =& \frac{1}{j^n(k-1)} \sum_{x \in J^n} v((\mathbf{j}-\mathbf{1})_S\,, x_{-S}) - v(\mathbf{0}_S\,, x_{-S}) \\ =& \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S} \in J^{n-s}} v((\mathbf{j}-\mathbf{1})_S\,, x_{-S}) - v(\mathbf{0}_S\,, x_{-S})\,.\end{aligned}$$

For the game v of Example 1.1.2, page 9 we compute $\mathcal{C}_v(S)$, for any $S \subseteq \{1,2,3\}$. It this important to recall that, for each $x \in \{0,1,2\}^3$,

$$v(x) = \begin{cases} 3 & \text{if } 2 \in N_2(x),\ |N_2(x)| \geq 2 \text{ and } N_0(x) = \emptyset \\ 2 & \text{if } N_0(x) = \emptyset \text{ and } (N_2(x) = \{2\} \text{ or } N_1(x) = \{2\}) \\ 1 & \text{if } (|N_2(x)| = 1,\ |N_1(x)| = 2 \text{ and } 2 \in N_1(x)) \text{ or} \\ & (|N_2(x)| = 2,\ N_1(x) = \emptyset \text{ and } 1 \in N_2(x)) \\ 0 & \text{otherwise} \end{cases}$$

Computation of $\mathcal{C}_v(\{1\})$										
(x_2,x_3)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	$\mathcal{C}_v(\{1\})$
$v(2,x_2,x_3) - v(0,x_2,x_3)$	0	0	1	0	1	2	1	3	3	$11/27$
Computation of $\mathcal{C}_v(\{2\})$										
(x_1,x_3)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	$\mathcal{C}_v(\{2\})$
$v(x_1,2,x_3) - v(x_1,0,x_3)$	0	0	0	0	2	3	1	3	2	$11/27$
Computation of $\mathcal{C}_v(\{3\})$										
(x_1,x_2)	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$	$\mathcal{C}_v(\{3\})$
$v(x_1,x_2,2) - v(x_1,x_2,0)$	0	0	0	0	1	3	1	2	2	$9/27$

Computation of $\mathcal{C}_v(\{1,\,2\})$				
x_3	0	1	2	$\mathcal{C}_v(\{1,\,2\})$
$v(2,2,x_3)-v(0,0,x_3)$	1	3	3	7/9
Computation of $\mathcal{C}_v(\{1,\,3\})$				
x_2	0	1	2	$\mathcal{C}_v(\{1,\,3\})$
$v(2,x_2,2)-v(0,x_2,0)$	1	2	3	6/9
Computation of $\mathcal{C}_v(\{2,\,3\})$				
x_1	0	1	2	$\mathcal{C}_v(\{2,\,3\})$
$v(x_1,2,2)-v(x_1,0,0)$	0	3	3	6/9

B Freixas (2005b) error: counting of h-pivotal players

For Example 1.1.2 in page 9 we proceed to count the h-pivots for each step level $h = 1, 2, 3$, and for each player th := 1, pra := 2 and ex := 3, using Remark 1.1.1, page 13. The results according to the entry orders of players are presented in Tables 2–7. We summarize in the Table 1 the number of times, for each player to be a h-pivot at each approval level.

	1-pivot	2-pivot	3-pivot
Player 1	**84**	**51**	**42**
Player 2	**39**	**60**	**78**
Player 3	**39**	**51**	**42**
Total	**162**	**162**	**162**

Table 1: Number of times of each player to be a h-pivot, $h = 1, 2, 3$.

The Shapley-Shubik index of each player is computed as follows:

$$\Phi_1(v) = \frac{1}{3}\cdot\frac{84}{162} + \frac{1}{3}\cdot\frac{51}{162} + \frac{1}{3}\cdot\frac{42}{162} = \frac{59}{162}\,;$$

$$\Phi_2(v) = \frac{1}{3}\cdot\frac{39}{162} + \frac{1}{3}\cdot\frac{60}{162} + \frac{1}{3}\cdot\frac{78}{162} = \frac{59}{162}\,;$$

$$\Phi_3(v) = \frac{1}{3}\cdot\frac{39}{162} + \frac{1}{3}\cdot\frac{51}{162} + \frac{1}{3}\cdot\frac{42}{162} = \frac{44}{162}\,.$$

$x \in J^3$	player(i)	$v(x_{\pi_{\leq i}}, \mathbf{2}_{\pi_{>i}})$	$v(x_{\pi_{<i}}, \mathbf{2}_{\pi_{\geq i}})$	$v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\geq i}})$	$v(x_{\pi_{\leq i}}, \mathbf{0}_{\pi_{>i}})$	1-pivot	2-pivot	3-pivot
	1	**0**	**3**	0	0	✓	✓	✓
(0, 0, 0)	2	0	0	0	0			
	3							
	1	**0**	**3**	0	0	✓	✓	✓
(0, 0, 1)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 0, 2)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 1, 0)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 1, 1)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 1, 2)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 2, 0)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 2, 1)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 2, 2)	2	0	0	0	0			
	3	0	0	0	0			
	1	3	3	0	0			
(1, 0, 0)	2	**0**	**3**	0	0	✓	✓	✓
	3	0	0	0	0			
	1	3	3	0	0			
(1, 0, 1)	2	**0**	**3**	0	0	✓	✓	✓
	3	0	0	0	0			
	1	3	3	0	0			
(1, 0, 2)	2	**0**	**3**	0	0	✓	✓	✓
	3	0	0	0	0			
	1	3	3	0	0			
(1, 1, 0)	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
	1	3	3	0	0			
(1, 1, 1)	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
	1	3	3	0	0			
(1, 1, 2)	2	**1**	**3**	0	0		✓	✓
	3	1	1	**0**	**1**	✓		
	1	3	3	0	0			
(1, 2, 0)	2	3	3	0	0			
	3	**0**	**3**	0	0	✓	✓	✓
	1	3	3	0	0			
(1, 2, 1)	2	3	3	0	0			
	3	**2**	**3**	**0**	**2**	✓	✓	✓
	1	3	3	0	0			
(1, 2, 2)	2	3	3	0	0			
	3	3	3	**0**	**3**	✓	✓	✓
	1	3	3	0	0			
(2, 0, 0)	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
	1	3	3	0	0			
(2, 0, 1)	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
	1	3	3	0	0			
(2, 0, 2)	2	**1**	**3**	0	0		✓	✓
	3	1	1	**0**	**1**	✓		
	1	3	3	0	0			
(2, 1, 0)	2	**2**	**3**	0	0			✓
	3	**0**	**2**	0	0	✓	✓	
	1	3	3	0	0			
(2, 1, 1)	2	**2**	**3**	0	0			✓
	3	**1**	**2**	**0**	**1**	✓	✓	
	1	3	3	0	0			
(2, 1, 2)	2	**2**	**3**	0	0			✓
	3	2	2	**0**	**2**	✓	✓	
	1	3	3	0	0			
(2, 2, 0)	2	3	3	**0**	**1**	✓		
	3	**1**	**3**	1	1		✓	✓
	1	3	3	0	0			
(2, 2, 1)	2	3	3	**0**	**1**	✓		
	3	3	3	**1**	**3**		✓	✓
	1	3	3	0	0			
(2, 2, 2)	2	3	3	**0**	**1**	✓		
	3	3	3	**1**	**3**		✓	✓
	1					**9**	**9**	**9**
Total	2					**6**	**9**	**12**
	3					**12**	**9**	**6**

Table 2: Pivotal players in each level for $\pi = 123$

$x \in J^3$	player(i)	$v(x_{\pi_{\leq i}}, \mathbf{2}_{\pi_{>i}})$	$v(x_{\pi_{<i}}, \mathbf{2}_{\pi_{\geq i}})$	$v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\geq i}})$	$v(x_{\pi_{\leq i}}, \mathbf{0}_{\pi_{>i}})$	1-pivot	2-pivot	3-pivot
	1	**0**	**3**	0	0	✓	✓	✓
(0, 0, 0)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 0, 1)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 0, 2)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 1, 0)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 1, 1)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 1, 2)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 2, 0)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 2, 1)	2	0	0	0	0			
	3	0	0	0	0			
	1	**0**	**3**	0	0	✓	✓	✓
(0, 2, 2)	2	0	0	0	0			
	3	0	0	0	0			
	1	3	3	0	0			
(1, 0, 0)	2	0	0	0	0			
	3	**0**	**3**	0	0	✓	✓	✓
	1	3	3	0	0			
(1, 0, 1)	2	**0**	**2**	0	0	✓	✓	
	3	**2**	**3**	0	0			✓
	1	3	3	0	0			
(1, 0, 2)	2	**0**	**3**	0	0	✓	✓	✓
	3	3	3	0	0			
	1	3	3	0	0			
(1, 1, 0)	2	0	0	0	0			
	3	**0**	**3**	0	0	✓	✓	✓
	1	3	3	0	0			
(1, 1, 1)	2	**0**	**2**	0	0	✓	✓	
	3	**2**	**3**	0	0			✓
	1	3	3	0	0			
(1, 1, 2)	2	**1**	**3**	**0**	**1**	✓	✓	✓
	3	3	3	0	0			
	1	3	3	0	0			
(1, 2, 0)	2	0	0	0	0			
	3	**0**	**3**	0	0	✓	✓	✓
	1	3	3	0	0			
(1, 2, 1)	2	2	2	**0**	**2**	✓	✓	
	3	**2**	**3**	0	0			✓
	1	3	3	0	0			
(1, 2, 2)	2	3	3	**0**	**3**	✓	✓	✓
	3	3	3	0	0			
	1	3	3	0	0			
(2, 0, 0)	2	**0**	**1**	0	0	✓		
	3	**1**	**3**	0	0		✓	✓
	1	3	3	0	0			
(2, 0, 1)	2	**0**	**3**	0	0	✓	✓	✓
	3	3	3	0	0			
	1	3	3	0	0			
(2, 0, 2)	2	**1**	**3**	1	1		✓	✓
	3	3	3	**0**	**1**	✓		
	1	3	3	0	0			
(2, 1, 0)	2	**0**	**1**	0	0	✓		
	3	**1**	**3**	0	0		✓	✓
	1	3	3	0	0			
(2, 1, 1)	2	**1**	**3**	**0**	**1**	✓	✓	✓
	3	3	3	0	0			
	1	3	3	0	0			
(2, 1, 2)	2	**2**	**3**	**1**	**2**		✓	✓
	3	3	3	**0**	**1**	✓		
	1	3	3	0	0			
(2, 2, 0)	2	1	1	**0**	**1**	✓		
	3	**1**	**3**	0	0		✓	✓
	1	3	3	0	0			
(2, 2, 1)	2	3	3	**0**	**3**	✓	✓	✓
	3	3	3	0	0			
	1	3	3	0	0			
(2, 2, 2)	2	3	3	**1**	**3**		✓	✓
	3	3	3	**0**	**1**	✓		
	1					**9**	**9**	**9**
Total	2					**12**	**12**	**9**
	3					**6**	**6**	**9**

Table 3: Pivotal players in each level for $\pi = 132$

$x \in J^3$	player(i)	$v(x_{\pi_{\le i}}, \mathbf{2}_{\pi_{>i}})$	$v(x_{\pi_{<i}}, \mathbf{2}_{\pi_{\ge i}})$	$v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\ge i}})$	$v(x_{\pi_{\le i}}, \mathbf{0}_{\pi_{>i}})$	1-pivot	2-pivot	3-pivot
$(0,0,0)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	0	0	0	0			
$(0,0,1)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	0	0	0	0			
$(0,0,2)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	0	0	0	0			
$(0,1,0)$	1	**0**	**2**	0	0	✓	✓	
	2	**2**	**3**	0	0			✓
	3	0	0	0	0			
$(0,1,1)$	1	**0**	**2**	0	0	✓	✓	
	2	**2**	**3**	0	0			✓
	3	0	0	0	0			
$(0,1,2)$	1	**0**	**2**	0	0	✓	✓	
	2	**2**	**3**	0	0			✓
	3	0	0	0	0			
$(0,2,0)$	1	**0**	**3**	0	0	✓	✓	✓
	2	3	3	0	0			
	3	0	0	0	0			
$(0,2,1)$	1	**0**	**3**	0	0	✓	✓	✓
	2	3	3	0	0			
	3	0	0	0	0			
$(0,2,2)$	1	**0**	**3**	0	0	✓	✓	✓
	2	3	3	0	0			
	3	0	0	0	0			
$(1,0,0)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	0	0	0	0			
$(1,0,1)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	0	0	0	0			
$(1,0,2)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	0	0	0	0			
$(1,1,0)$	1	**1**	**2**	0	0		✓	
	2	**2**	**3**	0	0			✓
	3	**0**	**1**	0	0	✓		
$(1,1,1)$	1	**1**	**2**	0	0		✓	
	2	**2**	**3**	0	0			✓
	3	**0**	**1**	0	0	✓		
$(1,1,2)$	1	**1**	**2**	0	0		✓	
	2	**2**	**3**	0	0			✓
	3	1	1	**0**	**1**	✓		
$(1,2,0)$	1	3	3	0	0			
	2	3	3	0	0			
	3	**0**	**3**	0	0	✓	✓	✓
$(1,2,1)$	1	3	3	0	0			
	2	3	3	0	0			
	3	**2**	**3**	**0**	**2**	✓	✓	✓
$(1,2,2)$	1	3	3	0	0			
	2	3	3	0	0			
	3	3	3	**0**	**3**	✓	✓	✓
$(2,0,0)$	1	1	1	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(2,0,1)$	1	1	1	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(2,0,2)$	1	1	1	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	1	1	**0**	**1**	✓		
$(2,1,0)$	1	2	2	0	0			
	2	**2**	**3**	0	0			✓
	3	**0**	**2**	0	0	✓	✓	
$(2,1,1)$	1	2	2	0	0			
	2	**2**	**3**	0	0			✓
	3	**1**	**2**	**0**	**1**	✓	✓	
$(2,1,2)$	1	2	2	0	0			
	2	**2**	**3**	0	0			✓
	3	2	2	**0**	**2**	✓	✓	
$(2,2,0)$	1	3	3	**0**	**1**	✓		
	2	3	3	0	0			
	3	**1**	**3**	1	1		✓	✓
$(2,2,1)$	1	3	3	**0**	**1**	✓		
	2	3	3	0	0			
	3	3	3	**1**	**3**		✓	✓
$(2,2,2)$	1	3	3	**0**	**1**	✓		
	2	3	3	0	0			
	3	3	3	**1**	**3**		✓	✓
Total	1					**15**	**9**	**3**
	2					**0**	**9**	**18**
	3					**12**	**9**	**6**

Table 4: Pivotal players in each level for $\pi = 213$

$x \in J^3$	player(i)	$v(x_{\pi_{\le i}}, \mathbf{2}_{\pi_{>i}})$	$v(x_{\pi_{<i}}, \mathbf{2}_{\pi_{\ge i}})$	$v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\ge i}})$	$v(x_{\pi_{\le i}}, \mathbf{0}_{\pi_{>i}})$	1-pivot	2-pivot	3-pivot
$(0,0,0)$	1	0	0	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(0,0,1)$	1	0	0	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(0,0,2)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	1	1	0	0			
$(0,1,0)$	1	0	0	0	0			
	2	**2**	**3**	0	0			✓
	3	**0**	**2**	0	0	✓	✓	
$(0,1,1)$	1	**0**	**1**	0	0	✓		
	2	**2**	**3**	0	0			✓
	3	**1**	**2**	0	0		✓	
$(0,1,2)$	1	**0**	**2**	0	0	✓	✓	
	2	**2**	**3**	0	0			✓
	3	2	2	0	0			
$(0,2,0)$	1	**0**	**1**	0	0	✓		
	2	3	3	0	0			
	3	**1**	**3**	0	0		✓	✓
$(0,2,1)$	1	**0**	**3**	0	0	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
$(0,2,2)$	1	**0**	**3**	0	0	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
$(1,0,0)$	1	0	0	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(1,0,1)$	1	0	0	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(1,0,2)$	1	**0**	**1**	0	0	✓		
	2	**1**	**3**	0	0		✓	✓
	3	1	1	0	0			
$(1,1,0)$	1	0	0	0	0			
	2	**2**	**3**	0	0			✓
	3	**0**	**2**	0	0	✓	✓	
$(1,1,1)$	1	**0**	**1**	0	0	✓		
	2	**2**	**3**	0	0			✓
	3	**1**	**2**	0	0		✓	
$(1,1,2)$	1	**1**	**2**	**0**	**1**	✓	✓	
	2	**2**	**3**	0	0			✓
	3	2	2	0	0			
$(1,2,0)$	1	**0**	**1**	0	0	✓		
	2	3	3	0	0			
	3	**1**	**3**	0	0		✓	✓
$(1,2,1)$	1	**2**	**3**	**0**	**2**	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
$(1,2,2)$	1	3	3	**0**	**3**	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
$(2,0,0)$	1	0	0	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(2,0,1)$	1	0	0	0	0			
	2	**1**	**3**	0	0		✓	✓
	3	**0**	**1**	0	0	✓		
$(2,0,2)$	1	1	1	**0**	**1**	✓		
	2	**1**	**3**	0	0		✓	✓
	3	1	1	0	0			
$(2,1,0)$	1	0	0	0	0			
	2	**2**	**3**	0	0			✓
	3	**0**	**2**	0	0	✓	✓	
$(2,1,1)$	1	1	1	**0**	**1**	✓		
	2	**2**	**3**	0	0			✓
	3	**1**	**2**	0	0		✓	
$(2,1,2)$	1	2	2	**0**	**2**	✓	✓	
	2	**2**	**3**	0	0			✓
	3	2	2	0	0			
$(2,2,0)$	1	1	1	**0**	**1**	✓		
	2	3	3	0	0			
	3	**1**	**3**	0	0		✓	✓
$(2,2,1)$	1	3	3	**0**	**3**	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
$(2,2,2)$	1	3	3	**0**	**3**	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
Total	1					**18**	**9**	**6**
	2					**0**	**9**	**18**
	3					**9**	**9**	**3**

Table 5: Pivotal players in each level for $\pi = 231$

$x \in J^3$	player(i)	$v(x_{\pi_{\le i}}, \mathbf{2}_{\pi_{>i}})$	$v(x_{\pi_{<i}}, \mathbf{2}_{\pi_{\ge i}})$	$v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\ge i}})$	$v(x_{\pi_{\le i}}, \mathbf{0}_{\pi_{>i}})$	1-pivot	2-pivot	3-pivot
(0, 0, 0)	1	0	1	0	0	✓		
	2	0	0	0	0			
	3	1	3	0	0		✓	✓
(0, 0, 1)	1	0	3	0	0	✓	✓	✓
	2	0	0	0	0			
	3	3	3	0	0			
(0, 0, 2)	1	0	3	0	0	✓	✓	✓
	2	0	0	0	0			
	3	3	3	0	0			
(0, 1, 0)	1	0	1	0	0	✓		
	2	0	0	0	0			
	3	1	3	0	0		✓	✓
(0, 1, 1)	1	0	3	0	0	✓	✓	✓
	2	0	0	0	0			
	3	3	3	0	0			
(0, 1, 2)	1	0	3	0	0	✓	✓	✓
	2	0	0	0	0			
	3	3	3	0	0			
(0, 2, 0)	1	0	1	0	0	✓		
	2	0	0	0	0			
	3	1	3	0	0		✓	✓
(0, 2, 1)	1	0	3	0	0	✓	✓	✓
	2	0	0	0	0			
	3	3	3	0	0			
(0, 2, 2)	1	0	3	0	0	✓	✓	✓
	2	0	0	0	0			
	3	3	3	0	0			
(1, 0, 0)	1	0	1	0	0	✓		
	2	0	0	0	0			
	3	1	3	0	0		✓	✓
(1, 0, 1)	1	2	3	0	0			✓
	2	0	2	0	0	✓	✓	
	3	3	3	0	0			
(1, 0, 2)	1	3	3	0	0			
	2	0	3	0	0	✓	✓	✓
	3	3	3	0	0			
(1, 1, 0)	1	0	1	0	0	✓		
	2	0	0	0	0			
	3	1	3	0	0		✓	✓
(1, 1, 1)	1	2	3	0	0			✓
	2	0	2	0	0	✓	✓	
	3	3	3	0	0			
(1, 1, 2)	1	3	3	0	0			
	2	1	3	0	1	✓	✓	✓
	3	3	3	0	0			
(1, 2, 0)	1	0	1	0	0	✓		
	2	0	0	0	0			
	3	1	3	0	0		✓	✓
(1, 2, 1)	1	2	3	0	0			✓
	2	2	2	0	2	✓	✓	
	3	3	3	0	0			
(1, 2, 2)	1	3	3	0	0			
	2	3	3	0	3	✓	✓	✓
	3	3	3	0	0			
(2, 0, 0)	1	1	1	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(2, 0, 1)	1	3	3	0	0			
	2	0	3	0	0	✓	✓	✓
	3	3	3	0	0			
(2, 0, 2)	1	3	3	0	1	✓		
	2	1	3	1	1		✓	✓
	3	3	3	0	0			
(2, 1, 0)	1	1	1	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(2, 1, 1)	1	3	3	0	0			
	2	1	3	0	1	✓	✓	✓
	3	3	3	0	0			
(2, 1, 2)	1	3	3	0	1	✓		
	2	2	3	1	2		✓	✓
	3	3	3	0	0			
(2, 2, 0)	1	1	1	0	0			
	2	1	1	0	1	✓		
	3	1	3	0	0		✓	✓
(2, 2, 1)	1	3	3	0	0			
	2	3	3	0	3	✓	✓	✓
	3	3	3	0	0			
(2, 2, 2)	1	3	3	0	1	✓		
	2	3	3	1	3		✓	✓
	3	3	3	0	0			
Total	1					**15**	**6**	**9**
	2					**12**	**12**	**9**
	3					**0**	**9**	**9**

Table 6: Pivotal players in each level for $\pi = 312$

$x \in J^3$	player(i)	$v(x_{\pi_{\le i}}, \mathbf{2}_{\pi_{>i}})$	$v(x_{\pi_{<i}}, \mathbf{2}_{\pi_{\ge i}})$	$v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\ge i}})$	$v(x_{\pi_{\le i}}, \mathbf{0}_{\pi_{>i}})$	1-pivot	2-pivot	3-pivot
(0, 0, 0)	1	0	0	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(0, 0, 1)	1	0	0	0	0			
	2	0	3	0	0	✓	✓	✓
	3	3	3	0	0			
(0, 0, 2)	1	0	1	0	0	✓		
	2	1	3	0	0		✓	✓
	3	3	3	0	0			
(0, 1, 0)	1	0	0	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(0, 1, 1)	1	0	1	0	0	✓		
	2	1	3	0	0		✓	✓
	3	3	3	0	0			
(0, 1, 2)	1	0	2	0	0	✓	✓	
	2	2	3	0	0			✓
	3	3	3	0	0			
(0, 2, 0)	1	0	1	0	0	✓		
	2	1	1	0	0			
	3	1	3	0	0		✓	✓
(0, 2, 1)	1	0	3	0	0	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
(0, 2, 2)	1	0	3	0	0	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
(1, 0, 0)	1	0	0	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(1, 0, 1)	1	0	0	0	0			
	2	0	3	0	0	✓	✓	✓
	3	3	3	0	0			
(1, 0, 2)	1	0	1	0	0	✓		
	2	1	3	0	0		✓	✓
	3	3	3	0	0			
(1, 1, 0)	1	0	0	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(1, 1, 1)	1	0	1	0	0	✓		
	2	1	3	0	0		✓	✓
	3	3	3	0	0			
(1, 1, 2)	1	1	2	0	1	✓	✓	
	2	2	3	0	0			✓
	3	3	3	0	0			
(1, 2, 0)	1	0	1	0	0	✓		
	2	1	1	0	0			
	3	1	3	0	0		✓	✓
(1, 2, 1)	1	2	3	0	2	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
(1, 2, 2)	1	0	3	0	0	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
(2, 0, 0)	1	0	0	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(2, 0, 1)	1	0	0	0	0			
	2	0	3	0	0	✓	✓	✓
	3	3	3	0	0			
(2, 0, 2)	1	1	1	0	1	✓		
	2	1	3	0	0		✓	✓
	3	3	3	0	0			
(2, 1, 0)	1	0	0	0	0			
	2	0	1	0	0	✓		
	3	1	3	0	0		✓	✓
(2, 1, 1)	1	1	1	0	1	✓		
	2	1	3	0	0		✓	✓
	3	3	3	0	0			
(2, 1, 2)	1	1	1	0	2	✓	✓	
	2	2	3	0	0			✓
	3	3	3	0	0			
(2, 2, 0)	1	1	1	0	1	✓		
	2	1	1	0	0			
	3	1	3	0	0		✓	✓
(2, 2, 1)	1	3	3	0	3	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
(2, 2, 2)	1	3	3	0	3	✓	✓	✓
	2	3	3	0	0			
	3	3	3	0	0			
Total	1					**18**	**9**	**6**
	2					**9**	**9**	**12**
	3					**0**	**9**	**9**

Table 7: Pivotal players in each level for $\pi = 321$

C Moves from c_k to c_{k+1} by local improvement of potentials: case of 2-players CSG

Let $\alpha = (0, \alpha_1, \alpha_2, 1) \in \mathcal{D}_3$ and $c \in \Gamma_2^\alpha$ a 2-players CSG on $N = \{1, 2\}$ given by

$$c := \begin{array}{|c|c|c|}\hline a_{1,3} & a_{2,3} & a_{3,3} \\ \hline a_{1,2} & a_{2,2} & a_{3,2} \\ \hline a_{1,1} & a_{2,1} & a_{3,1} \\ \hline\end{array}$$

By Remark 3.1.3 and Equation (3.25) a sequence $(c_k)_{0 \leq k \leq 8}$ can be represented as follows:

$$c_0 := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array} \quad c_1 := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline a_{1,1} & 1 & 1 \\ \hline\end{array} \quad c_2 := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline a_{1,1} & a_{2,1} & 1 \\ \hline\end{array} \quad c_3 := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline a_{1,1} & a_{2,1} & a_{3,2} \\ \hline\end{array}$$

$$c_4 := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline a_{1,2} & 1 & 1 \\ \hline a_{1,1} & a_{2,1} & a_{3,2} \\ \hline\end{array} \quad c_5 := \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline a_{1,2} & a_{2,2} & a_{3,2} \\ \hline a_{1,1} & a_{2,1} & a_{3,2} \\ \hline\end{array} \quad c_6 := \begin{array}{|c|c|c|}\hline a_{1,3} & 1 & 1 \\ \hline a_{1,2} & a_{2,2} & a_{3,2} \\ \hline a_{1,1} & a_{2,1} & a_{3,2} \\ \hline\end{array}$$

$$c_7 := \begin{array}{|c|c|c|}\hline a_{1,3} & a_{2,3} & 1 \\ \hline a_{1,2} & a_{2,2} & a_{3,2} \\ \hline a_{1,1} & a_{2,1} & a_{3,2} \\ \hline\end{array} \quad c_8 := \begin{array}{|c|c|c|}\hline a_{1,3} & a_{2,3} & a_{3,3} \\ \hline a_{1,2} & a_{2,2} & a_{3,2} \\ \hline a_{1,1} & a_{2,1} & a_{3,2} \\ \hline\end{array}$$

To illustrate the difference cases highlighted in the proof of Lemma 3.2.1, we construct the sequence of local improvement from, c_0 to c_1; from c_4 to c_5 and from c_5 to c_6. We suppose that, $a_{i,j} \neq 1$ for all $(i, j) \in \mathcal{A}_{3,2}$. Note that a red color on a segment of a rectangular box allows to specify the value of the game on this segment.

- **Moves from c_0 to c_1 by local improvement of potentials**

  Here $e^1 = (1, 1)$, so $L(e^1) = \{1, 2\}$ and $U(e^1) = \emptyset$. We set $S_1 = \{1\}$, $S_2 = \{2, \}$ and $\varepsilon_0 = 1 - a_{1,1}$. Follows Equations (3.26) and (3.27) the games u_0, u_1 and u_2 are given below:

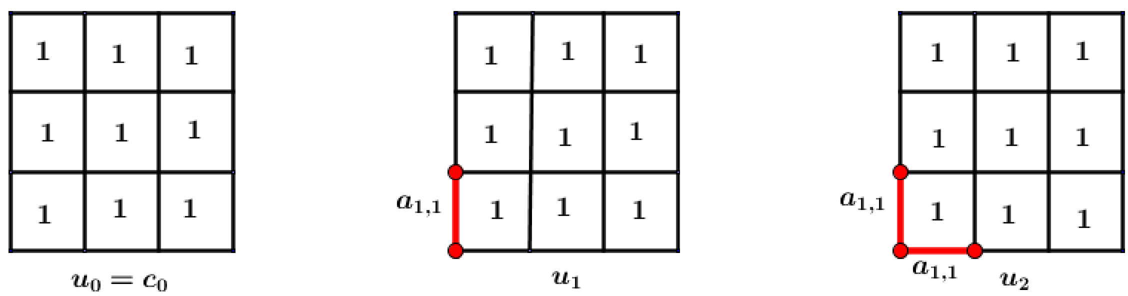

Figure 1: Moves from c_0 to c_1 by local improvement of potentials

From the Figure 1 and the definition of local improvement of potentials of two CSGs, we one get:

$$\left(c_0 = u_0 \xrightarrow{\{1\},\, \varepsilon_0,\, [0,\alpha_1]} u_1 \xrightarrow{\{2\},\, \varepsilon_0,\, [0,\alpha_1]} u_2 \quad \text{and} \quad \Delta u_2 = \Delta c_1\right) \Longrightarrow c_0 \xrightarrow{\{1\},\, \varepsilon_0,\, [0,\alpha_1]} u_1 \xrightarrow{\{2\},\, \varepsilon_0,\, [0,\alpha_1]} c_1\,.$$

- **Moves from c_4 to c_5 by local improvement of potentials**

  Here $k = 4$, $e^5 = (3,2)$, hence $L(e^5) = \emptyset$ and $U(e^4) = \{1\}$. We pose $\varepsilon_4 = 1 - a_{3,2}$. From Equation (3.29) the CSGs v_0 and v_1 are represented in Figure 2 below.

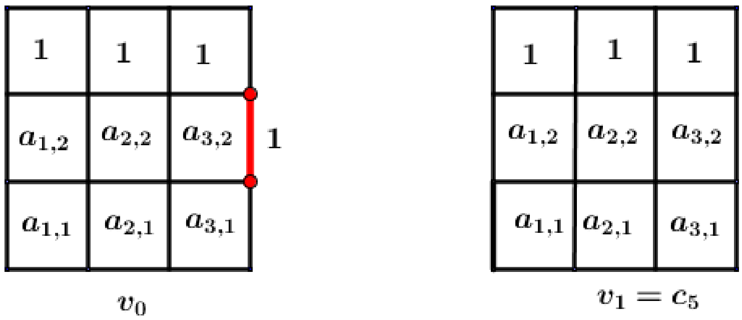

Figure 2: Moves from c_4 to c_5 by local improvement of potentials

By definition of potentials and local increment of two CSGs, we have

$$\left(\Delta c_4 = \Delta v_0 \quad \text{and} \quad v_0 \xrightarrow{\{1\},\, \varepsilon_4,\, [\alpha_1,\alpha_2]} v_1 = c_5\right) \Longrightarrow c_4 \xrightarrow{\{1\},\, \varepsilon_4,\, [\alpha_1,\alpha_2]} c_5$$

- **Moves from c_5 to c_6 by local improvement of potentials** In this case, $k = 5$, $e^6 = (1,3)$, $L(e^6) = \{1\}$ and $U(e^6) = \{2\}$. So, applying (3.30) the sequence $(w_l)_{0\leq l\leq 3}$ is presented as follows:

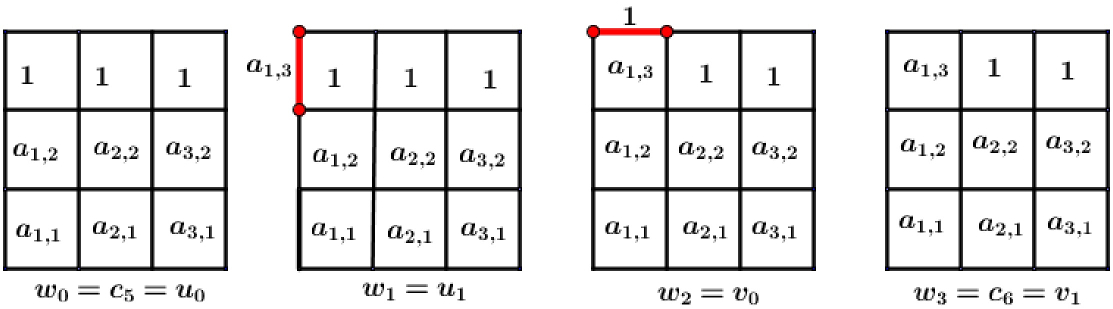

Figure 3: Moves from c_5 to c_6 by local improvement of potentials

Setting $\varepsilon_5 = 1 - a_{1,3} > 0$, we one get

$$c_5 = w_0 \xrightarrow{\{1\},\, \varepsilon_5,\, [\alpha_2,1]} w_1\,;\, \Delta w_1 = \Delta w_2 \quad \text{and} \quad w_2 \xrightarrow{\{2\},\, -\varepsilon_5,\, [0,\alpha_1]} w_3 = c_6$$

Therefore,

$$c_5 \xrightarrow{\{1\},\, \varepsilon_5,\, [\alpha_2,1]} w_2 \xrightarrow{\{2\},\, -\varepsilon_5,\, [0,\alpha_1]} c_6\,.$$

D Articles and project

Published article

- Kurz, S., Moyouwou, I. & Touyem, H. Axiomatizations for the Shapley-Shubik power index for games with several levels of approval in the input and output. Soc Choice Welf (2020). https://doi.org/10.1007/s00355-020-01296-6

Articles in progress

- Kurz Sascha, Issofa Moyouwou, and Hilaire Touyem. "An Axiomatization of the Shapley-Shubik Index for Interval Decisions." arXiv preprint arXiv: 1907.01323 (2019).
- Hilaire Touyem, Issofa Moyouwou and Bertrand Tchantcho. "On Riemann Integrability of Monotone Multivariate Real-Valued Functions"

Social Choice and Welfare
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ORIGINAL PAPER

Axiomatizations for the Shapley–Shubik power index for games with several levels of approval in the input and output

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Abstract
The Shapley–Shubik index is a specialization of the Shapley value and is widely applied to evaluate the power distribution in committees drawing binary decisions. It was generalized to decisions with more than two levels of approval both in the input and the output. The corresponding games are called (j, k) simple games. Here we present a new axiomatization for the Shapley–Shubik index for (j, k) simple games as well as for a continuous variant, which may be considered as the limit case.

1 Introduction

Shapley (1953) introduced a function that could be interpreted as the expected utility of a game from each of its positions via the axiomatic approach—the so-called Shapley value. A bit later, see Shapley and Shubik (1954), it was restricted to games with binary decisions, i.e., simple games. An axiomatization of this so-called Shapley–Shubik index was given quite a few years later by Dubey (1975). Nowadays, the Shapley–Shubik index is one of the most established power indices for committees drawing binary decisions. However, not all decisions are binary. Abstaining from a vote might be seen as a third option for the committee members. In general, there might also be any number $j \geq 2$ of alternatives that can be chosen from.

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To this end, simple games were generalized to (j, k) simple games, see Freixas and Zwicker (2003), where j is the number of ordered alternatives in the input, i.e., the voting possibilities, and k the number of ordered alternatives for the group decision.[1] A Shapley–Shubik index for these (j, k) simple games was introduced in Freixas (2005) generalizing earlier attempts for special cases, see e.g. (Felsenthal and Machover (1998), pp. 291–293). However, also other variants have been introduced in the literature, see e.g. Amer et al. (1998); Friedman and Parker (2018); Hsiao and Raghavan (1993). Here, we will only consider the variant from Freixas (2005). A corresponding axiomatization is given in Freixas (2019) for $(j, 2)$ simple games and for j-cooperative games (when outputs are real numbers).

If we normalize the input and output levels to numbers between zero and one, we can consider the limit if j and k tend to infinity for (j, k) simple games. More precisely we can consider the input levels $i/(j-1)$ for $0 \le i \le j-1$ and the output levels $i/(k-1)$ for $0 \le i \le k-1$. Then those games are discrete approximations for games with input and output levels freely chosen from the real interval $[0, 1]$. The latter games were called simple aggregation functions in Kurz (2018), linking to the literature on aggregation functions in Grabisch et al. (2009), and interval simple games in Kurz et al. (2019). A Shapley–Shubik like index for those games was motivated and introduced in Kurz (2014); an axiomatization is given in Kurz et al. (2019). So, results for interval simple games might be obtained from results for (j, k) simple games via a suitable limit argument. We do not use this approach in our paper.

The success story of the Shapley–Shubik index for simple games, initiated by Shapley (1953) and Shapley and Shubik (1954), triggered a huge amount of modifications and generalizations to different types of games, see e.g. Algaba et al. (2019) for some current research directions. We think that the variants from Freixas (2005), for (j, k) simple games, and from Kurz (2014), for interval simple games, form one consistent way to generalize the Shapley–Shubik index for simple games. Here we mainly focus on an axiomatic justification. This is the object of our main result in Theorem 2. More precisely, we replace the axiom of additivity and the recent axiom of level change effect on unanimity games introduced in Freixas (2019) by the new axiom of average convexity. The underlying idea is to associate a TU game $\tilde{v}$, called the average game, to every (j, k) simple game v, see Definition 7. The axiom then says that if the convex combinations of average games coincide then also the power distributions for the same convex combinations of the underlying (j, k) simple games should coincide. The average game itself seems to be a very natural object on its own and has some nice properties. Indeed, the Shapley value of the TU game $\tilde{v}$ coincides with the Shapley–Shubik index of the (j, k) simple game v. Moreover, we present another formula for the Shapley–Shubik index for (j, k) simple games which is better suited for computation issues, see Lemma 1 and Theorem 1. For a generalization of the Banzhaf index a similar result was obtained in Pongou et al. (2012). As

[1] In our definition of a (j, k) simple game we will deviate from Freixas and Zwicker (2003) by numbering the input and output levels starting from 0 instead of 1 and assuming that lower numbers correspond to a lower level of approval.

the title of the preface of Algaba et al. (2019) names it, the idea of the Shapley value is the root of a still ongoing research agenda.

The remaining part of this paper is organized as follows. In Sect. 2 we introduce the necessary preliminaries and present the first few basic results. A Shapley–Shubik index $\boldsymbol{\Phi}$ for general (j, k) simple games is introduced in Sect. 3. Moreover, we study the first basic properties of $\boldsymbol{\Phi}$. In Sect. 4 we introduce the average game, which is a TU game associated to each (j, k) simple game. This notion is then used to formulate the new axiom of average convexity, which culminates in an axiomatic characterization of $\boldsymbol{\Phi}$ in Sect. 5. In Sect. 6 we transfer all notions and the axiomatic characterization to interval simple games. We close with a brief conclusion in Sect. 7.

2 Preliminaries

Let $N = \{1, 2, \dots, n\}$ be a finite set of voters. Any subset S of N is called a coalition and the set of all coalitions of N is denoted by the power set 2^N. For given integers $j, k \geq 2$ we denote by $J = \{0, \dots, j-1\}$ the possible input levels and by $K = \{0, \dots, k-1\}$ the possible output levels, respectively. We write $x \leq y$ for $x, y \in \mathbb{R}^n$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For each $\emptyset \subseteq S \subseteq N$ we write x_S for the restriction of $x \in \mathbb{R}^n$ to $\left(x_i\right)_{i \in S}$. As an abbreviation, we write $x_{-S} = x_{N \backslash S}$. Instead of $x_{\{i\}}$ and $x_{-\{i\}}$ we write x_i and x_{-i}, respectively. Slightly abusing notation we write $\mathbf{a} \in \mathbb{R}^n$, for the vector that entirely consists of a's, e.g., $\mathbf{0}$ for the zero vector.

A *TU game* with player set N is a mapping v from the set of all subsets N to $\mathbb{R}$ such that $v(\emptyset) = 0$. In particular, a *simple game* with player set N is a $\{0, 1\}$-valued TU game v such that $v(N) = 1$, and $v(S) \leq v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$. A coalition $S \subseteq N$ is said to be a *winning coalition* if $v(S) = 1$ and a *losing coalition* if $v(S) = 0$. The interpretation in the voting context is as follows. Those elements $i \in N$, called *voters* or *players*, that are contained in a coalition S are those that are in favor of a certain proposal. The other voters, i.e., those in $N \backslash S$, are against the proposal. If $v(S) = 1$ then the proposal is implemented and otherwise the status quo persists. A simple game v is *weighted* if there exists a *quota* $q \in \mathbb{R}_{>0}$ and *weights* $w_i \in \mathbb{R}_{\geq 0}$ for all $i \in N$ such that $v(S) = 1$ iff $w(S) := \sum_{i \in S} w_i \geq q$. As notation we use $\left[q; w_1, \dots w_n\right]$ for a weighted (simple) game. An example is given by $v = [4; 3, 2, 1, 1]$, also called ''My Aunt and I'', see e.g. (Osborne and Rubinstein (1994), page 297), with winning coalitions $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 2, 3, 4\}$, and $\{2, 3, 4\}$. As an example, the coalition $\{1, 3\}$ is winning since player 1 has a weight of $w_1 = 3$ and player 3 has a weight of $w_3 = 1$, so that $w(\{1, 3\}) = w_1 + w_3 = 4 \geq q$. A simple game v is a *unanimity game* if there exists a coalition $\emptyset \neq T \subseteq N$ such that $v(S) = 1$ iff $T \subseteq S$. As an abbreviation we use the notation γ_T for a unanimity game with defining coalition T. It is well known that each simple game admits a representation as disjunctions of a finite list of unanimity games. Calling a winning coalition *minimal* if all proper subsets are losing; such a list is given by the minimal winning coalitions, i.e., by $\{1, 2\}, \{1, 3\}, \{1, 4\}$, and $\{2, 3, 4\}$ in the above example.

If being part of a coalition is modeled as voting ''yes'' and ''no'' otherwise, represented as 1 and 0, respectively, then one can easily reformulate and generalize the definition of a simple game[2]:

Definition 1 A (j, k) *simple game for n players*, where $j, k \geq 2$ and $n \geq 1$ are integers, is a mapping $v : J^n \to K$ with $v(\mathbf{0}) = 0$, $v(\mathbf{j-1}) = k - 1$, and $v(x) \leq v(y)$ for all $x, y \in J^n$ with $x \leq y$. The set of all (j, k) simple games on N is denoted by $\mathcal{U}_n^{j,k}$ or by $\mathcal{U}_n$, whenever j and k are clear from the context.

So, (2, 2) simple games are in one-to-one correspondence to simple games. We use the usual ordering of J (and K) as a set of integers, i.e., $0 < 1 < \cdots < j - 1$. In words, in the input set, 0 is the lowest level of approval, followed by 1 and so on. In general, we call a function $f : \mathbb{R}^n \supseteq U \to \mathbb{R}$ *monotone* if we have $f(x) \leq f(y)$ for all $x, y \in U$ with $x \leq y$. We remark that Freixas (2005) considers a more general definition of a (j, k) simple game than we have presented here. Additionally the j input levels and the k output levels are given by a so-called numeric evaluation. Our case is called uniform numeric evaluation there, which motivated the notation $\mathcal{U}_n$ for (j, k) simple games for n players. We also call a vector $x \in J^n$ a *profile*. In words, we assume that the input and output levels are cardinal quantities and not ordinal quantities that are qualitatively ordered. To be more precise, we assume that the distance between two input levels is equally sized. This is in contrast to the more general interpretation of (j, k) simple games as introduced in Freixas and Zwicker (2003).

Definition 2 Given a (j, k) simple game v with player set N, we call a player $i \in N$ a *null* player iff $v(x) = v(x_{-i}, y_i)$ for all $x \in J^n$ and all $y_i \in J$.[3] Two players $i, h \in N$ are called *equivalent* if $v(x) = v(x')$ for all $x, x' \in J^n$ with $x_l = x'_l$ for all $l \in N \backslash \{i, h\}$, $x_i = x'_h$, and $x_h = x'_i$.

In words, a player i is a null player if its input y_i does not alter the output $v(x)$. If interchanging the input x_i and x_h of two players does never alter the output $v(x)$, then players i and h are equivalent. By π_{ih} we denote the transposition on N interchanging i and h, so that the previous condition reads $v(x) = v(\pi_{ih}x)$ for all $x \in J^n$. By $\mathcal{S}_n$ we denote the set of permutations of length n, i.e., the bijections on N.

Now let us introduce a subclass of (j, k) simple games with the property that for each profile x, the collective decision $v(x)$ is either 0 *(the lowest level of approval)* or it is $k - 1$ *(the highest level of approval)* depending on whether some given voters report some minimum approval levels. For example, when any full support of the proposal necessitates a full support of each voter in a given coalition S, players in S are each empowered with a veto. One may require from each player in S only a

[2] Note that we slightly deviate from the original definition of a (j, k) simple game in Freixas and Zwicker (2003), see Footnote 1. With this, we ensure that (2, 2) simple games are in one-to-one correspondence to simple games encoding ''no'' as 0 and ''yes'' as 1.

[3] Using the notation introduced at the beginning of Sect. 2, we have $v(x_{-i}, y_i) = v(x_{N\backslash\{i\}}, y_i) = v(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

certain level of approval for a full support of the proposal. All such games will be called (j, k) simple games with point-veto.

Definition 3 A (j, k) simple game with a *point-veto* is a (j, k) simple game v such that there exists some $a \in J^n \backslash \{\mathbf{0}\}$ satisfying $v(x) = k - 1$ if $a \le x$ and $v(x) = 0$ otherwise for all $x \in J^n$. In this case, a is the veto and the game v is denoted by u^a. For each coalition $S \in 2^N$ we use the abbreviation $w^S = u^a$ for a game, where $a_i = j - 1$ for all $i \in S$ and $a_i = 0$ otherwise.

We remark that (2, 2) simple games with a point veto are in one-to-one correspondence to the subclass of unanimity games within simple games.

The set of all players who report a non-null approval level is denoted by N^a, i.e., $N^a = \left\{ i \in N : 0 < a_i \le j - 1 \right\}$. Every player in N^a will be called a *vetoer* of the game u^a. Note that for the vector a defined via $w^S = u^a$ we have $N^a = S$.

Null players as well as equivalent players can be identified easily in a given (j, k) simple game with point-veto:

Proposition 1 *Let $a \in J^n \backslash \{\mathbf{0}\}$. A player $i \in N$ is a null player of u^a iff $i \in N \backslash N^a$. Two players $i, h \in N$ are equivalent in u^a iff $a_i = a_h$.*

Proof For every $a \in J^n \backslash \{\mathbf{0}\}$ and every $i \in N \backslash N^a$ we have $a_i = 0$ by the definition of N^a. Now let $i \in N \backslash N^a$. For every $x \in J^n$ and every $y_i \in J$ we have $a \le x$ iff $a \le \left(x_{-i}, y_i\right)$. Thus, $u^a(x) = u^a\left(x_{-i}, y_i\right)$ and i is a null player in u^a. Now let $i \in N^a$, i.e., $a_i > 0$. Since $v(a) = k - 1 \ne 0 = v\left(a_{-i}, \mathbf{0}_i\right)$, player i is not a null player in u^a.

Assume that $a_i = a_h$ and consider an arbitrary $x \in J^n$. Then we have $a \le x$ if and only if $a \le \pi_{ih} x$. The definition of u^a directly gives $u^a(x) = u^a(\pi_{ih} x)$, so that the players i and h are equivalent in u^a. Now suppose that the players i and h are equivalent in u^a. Since $a \le a$, we obtain $u^a(a) = u^a(\pi_{ih} a) = k - 1$. This implies that $a \le \pi_{ih} a$. Therefore $a_i \le a_h$ and $a_h \le a_i$, that is $a_i = a_h$. □

Note that (j, k) simple games can be combined using the disjunction ($\vee$) or the conjunction ($\wedge$) operations to obtain new games. This generalizes the union and intersection operations for simple games.

Definition 4 Let v' and v'' be two (j, k) simple games with player set N. By $v' \vee v''$ we denote the (j, k) simple game v defined by $v(x) = \max\{v'(x), v''(x)\}$ for all $x \in J^n$. Similarly, by $v' \wedge v''$ we denote the (j, k) simple game v defined by $v(x) = \min\{v'(x), v''(x)\}$ for all $x \in J^n$.

We remark that the defining properties of a (j, k) simple game, i.e., monotonicity, domain, and codomain, can be easily checked. This can be specialized to the subclass of (j, k) simple games with point veto, i.e., (j, k) simple games with point-veto can be combined using the disjunction ($\vee$) or the conjunction ($\wedge$) operations to obtain new games. To see this, consider a non-empty subset E of $J^n \backslash \{\mathbf{0}\}$ and define the (j, k) simple game denoted by u^E by $u^E(x) = k - 1$ if $a \le x$ for some $a \in E$ and

$u^E(x) = 0$ otherwise, where $x \in J^n$ is arbitrary. Note that the notational simplification $u^{\{a\}} = u^a$, where $a \in J^n \backslash \{\mathbf{0}\}$, goes in line with Definition 3.

Proposition 2 *Let E and E' be two non-empty subsets of $J^n \backslash \{\mathbf{0}\}$. Then, we have $u^E \vee u^{E'} = u^{E \cup E'}$ and $u^E \wedge u^{E'} = u^{E''}$, where $E'' = \{c \in J^n \,:\, c_i = max(a_i, b_i)$ for all $i \in N$and some $a \in E, b \in E'\}$.*

Proof In order to prove $u^E \vee u^{E'} = u^{E \cup E'}$ we consider an arbitrary $x \in J^n$. If $u^{E \cup E'}(x) = k-1$, then there exists $a \in E \cup E'$ such that, $a \le x$. Therefore $u^E(x) = k-1$ or $u^{E'}(x) = k-1$ and $(u^E \vee u^{E'})(x) = k-1$. Now suppose that $u^{E \cup E'}(x) = 0$. Then, for all $a \in E \cup E'$ we have $a \not\le x$. Since $E \subseteq E \cup E'$ and $E' \subseteq E \cup E'$ we have $b \not\le x$ and $c \not\le x$ for all $b \in E$ and all $c \in E'$. This implies that $u^E(x) = u^{E'}(x) = 0$ and $(u^E \vee u^{E'})(x) = 0$. Thus, $u^E \vee u^{E'} = u^{E \cup E'}$.

Similarly, in order to prove $u^E \wedge u^{E'} = u^{E''}$ we consider an arbitrary $x \in J^n$. If $u^{E''}(x) = k-1$, then there exists $c \in E''$ such that $c \le x$. But, by definition of E'', $c = \max(a, b)$ for some $a \in E$ and $b \in E'$, that is $a \le c \le x$ and $b \le c \le x$. Hence, $u^E(x) = u^{E'}(x) = k-1$ and $(u^E \wedge u^{E'})(x) = k-1$. Now assume that $u^{E''}(x) = 0$ and $(u^E \wedge u^{E'})(x) \neq 0$. By definition of u^E and $u^{E'}$, we have $(u^E \wedge u^{E'})(x) = k-1$. Thus, there exists $a \in E$ and $b \in E'$ such that $a \le x$ and $b \le x$. It follows that $c = \max(a, b) \le x$, which is a contradiction to $u^{E''}(x) = 0$. This proves $u^E \wedge u^{E'} = u^{E''}$. ☐

For $(j, k) = (5, 3)$ and $n = 3$ an example is given by $E = \{(1,2,3), (2,1,2)\}$, $E' = \{(4,1,1), (1,1,3)\}$. With this, $E'' = \{(4,2,3), (1,2,3), (2,1,3), (4,1,2)\}$. Note that we may remove (4, 2, 3) from that list since $(4,2,3) \ge (1,2,3)$ (or $(4,2,3) \ge (4,1,2)$).

Especially, Proposition 2 yields that every $(j,\, k)$ simple game of the form u^E is a disjunction of some $(j,\, k)$ simple games with point-veto. So, each $(j,\, k)$ simple game of the form u^E will be called a $(j,\, k)$ simple game with *veto*. In the game u^E, E can be viewed as some minimum requirements (or thresholds) on the approval levels of voters' inputs for the full support of the proposal. It is worth noticing that u^E is $\{0, k-1\}$-valued; the final decision at all profiles is either a no-support or a full-support. The set of all veto $(j,\, k)$ simple games on N is denoted $\mathcal{V}_n$. Note that Proposition 2 shows that $\mathcal{V}_n$ is a lattice.

The sum of two $(j,\, k)$ simple games cannot be a $(j,\, k)$ simple game itself. However, we will show that each $(j,\, k)$ simple game is a convex combination of $(j,\, k)$ simple games with veto.

Definition 5 A convex combination of the games $v_1, v_2, \dots, v_p \in \mathcal{U}_n$ is given by $v = \sum_{t=1}^{p} \alpha_t v_t$ for some non-negative numbers α_t, where $t = 1, 2, \dots, p$, that sum to 1.

Note that not all convex combinations of $(j,\, k)$ simple games are $(j,\, k)$ simple games.

Proposition 3 *For each (j, k) simple game v there exists a collection of positive numbers α_t, where $t = 1, 2, \ldots, p$, that sum to 1 and a collection $F_t(v)$, where $t = 1, 2, \ldots, p$, of non-empty subsets of J^n such that $v = \sum_{t=1}^{p} \alpha_t u^{F_t(v)}$.*

Proof Let $v \in \mathcal{U}_n$ and $\mathcal{F}(v) = \{x \in J^n, v(x) > 0\}$. Since J^n is finite and v is monotone, the elements of $\mathcal{F}(v)$ can be labeled in such a way that $\mathcal{F}(v) = \{x^1, x^2, \ldots, x^p\}$, where $x^p = \mathbf{j-1}$, $v(x^t) \le v(x^{t+1})$ for all $1 \le t < p$, and $t \le s$ whenever $x^t \le x^s$. Now, set $x^0 = \mathbf{0}$ and $F_t(v) = \{x^s, t \le s \le p\}$, $\alpha_t = \frac{v(x^t)-v(x^{t-1})}{k-1}$ for all $1 \le t \le p$. By our assumption on x^t we have $\alpha_t \ge 0$ for all $1 \le t \le p$. Moreover, it can be easily checked that $\sum_{t=1}^{p} \alpha_t = \frac{v(x^p)-v(x^0)}{k-1} = 1$. Set $u = \sum_{t=1}^{p} \alpha_t u^{F_t(v)}$.

In order to prove that $v = u$, we consider an arbitrary $x \in J^n$. First suppose that $x \notin \mathcal{F}(v)$. Since v is monotone, there is no $a \in \mathcal{F}(v)$ such that $a \le x$. By definition, it follows that $v^{F_t(v)}(x) = 0$ for all $t = 1, 2, \ldots, p$. Therefore $v(x) = u(x) = 0$. Now suppose that $x \in \mathcal{F}(v)$. Then $x = x^s$ for some $s = 1, 2, \ldots, p$. It follows that for all $t = 1, 2, \ldots, p$ we have $v^{F_t(v)}(x) = k-1$ if $1 \le t \le s$ and $v^{F_t(v)}(x) = 0$ otherwise. Therefore

$$u(x) = \sum_{t=1}^{s} \alpha_t = \sum_{t=1}^{s} \left(\frac{v(x^t) - v(x^{t-1})}{k-1} \cdot (k-1) \right) = v(x^s) = v(x).$$

Clearly, the game v is a convex combination of the games $u^{F_t(v)}$, where $t = 1, 2, \ldots, p$. ☐

Proposition 3 underlines the importance of (j, k) simple games with veto, i.e., every (j, k) simple game can be obtained from (j, k) simple games with veto as a convex combination.

Now let us consider a continuous version of (j, k) simple games normalized to the real interval $I := [0, 1]$ for the input as well as the output levels. Following Kurz (2018) and using the name from Kurz et al. (2019), we call a mapping $v : [0, 1]^n \to [0, 1]$ an *interval simple game* if $v(\mathbf{0}) = 0$, $v(\mathbf{1}) = 1$, and $v(x) \le v(y)$ for all $x, y \in [0, 1]^n$ with $x \le y$. The set of all interval simple games on N is denoted by $\mathcal{ISG}_n$. Replacing both J and K by [0, 1] in Definition 2 we can transfer the concept of a null player and that of equivalent players to interval simple games.

3 The Shapley–Shubik index for simple and (*j*, *k*) simple games

Since in a typical simple game v not all players are equivalent, the question of influence of a single player i on the final group decision $v(S)$ arises. Even if v can be represented as a weighted game, i.e., $v = [q;w]$, the relative individual influence is not always reasonably reflected by the weights w_i. This fact is well-known and triggered the invention of power indices, i.e., mappings from a simple game on n players to $\mathbb{R}^n$ reflecting the influence of a player on the final group decision. One of the most established power indices is the *Shapley–Shubik index*, see Shapley and Shubik (1954). It can be defined via

$$\mathrm{SSI}_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot \left[v(S) - v(S \backslash \{i\})\right] \tag{1}$$

for all players $i \in N$, where $s = |S|$. If $v(S) - v(S\backslash\{i\}) = 1$, then we have $v(S) = 1$ and $v(S\backslash\{i\}) = 0$ in a simple game and voter i is called a *swing voter*.

We remark that the Shapley–Shubik index is a restriction of the Shapley value to simple games. Both, the Shapley value and the Shapley–Shubik index have compelling axiomatizations. Besides that, Shapley and Shubik (1954) motivate the Shapley–Shubik index by the following interpretation. Assume that the n voters line up in a row in a given ordering of the players and sequentially declare to be part in the coalition of ''yes''-voters. The player that first guarantees that a proposal can be put through is then called *pivotal*. Considering all $n!$ orderings $\pi \in \mathcal{S}_n$ of the players with equal probability then gives a probability for being pivotal for a given player $i \in N$ that equals its Shapley–Shubik index. So we can rewrite Eq. (1) as $\mathrm{SSI}_i(v) =$

$$\frac{1}{n!} \cdot \sum_{\pi \in \mathcal{S}_n} \Big(v(\{j \in N \ : \ \pi(j) \le \pi(i)\}) - v(\{j \in N \ : \ \pi(j) < \pi(i)\}) \Big). \tag{2}$$

Setting $S_\pi^i := \{j \in N \ : \ \pi(j) \le \pi(i)\}$ we have $S_\pi^i = S$ for exactly $(s-1)!(n-s)!$ permutations $\pi \in \mathcal{S}_n$ and an arbitrary set $\{i\} \subseteq S \subseteq N$, so that Eq. (1) is just a simplification of Eq. (2).

Instead of assuming that all players vote ''yes'' one can also assume that all players vote ''no''. Mann and Shapley (1964) mention that the model also yields the same result if we assume that all players independently vote ''yes'' with a fixed probability $p \in [0, 1]$. This was further generalized to probability measures p on $\{0, 1\}^n$ where vote vectors with the same number of ''yes'' votes have the same probability, see Hu (2006). In other words, individual votes may be interdependent but must be exchangeable. That no further probability measures lead to the Shapley–Shubik index was finally shown in Kurz and Napel (2018). For the most symmetric case $p = \frac{1}{2}$ we have

$$\mathrm{SSI}_i(v) = \frac{1}{n! \cdot 2^n} \cdot \sum_{(\pi,x) \in \mathcal{S}_n \times \{0,1\}^n} M(v, (\pi, x), i), \tag{3}$$

where $M(v, (\pi, x), i)$ is one if player i is pivotal for ordering π and vote vector x in v, see Kurz and Napel (2018), and zero otherwise. To be more precise, consider a given ordering π of the players and a vector $x \in \{0, 1\}^n$ encoding a sequence of ''yes'' and ''no'' votes. Then, we call a player i pivotal, with respect to π and x, if player i is the first player such that the votes of player i and her predecessors uniquely determine the outcome $v(x)$ of the group decision. As an example we again consider the weighted game $v = [4;3, 2, 1, 1]$. For ordering $\pi = (2, 1, 3, 4)$ and vote vector $x = (1, 1, 0, 1)$ player 1 is the pivotal player that determines the outcome $v(x) = 1$. If $\bar{x} \in \{0, 1\}^4$ coincides with x for player 1 and her predecessors, then we have $v(\bar{x}) \ge v(1, 1, 0, 0) = 1$, i.e., the outcome is determined to be 1 in any case. Player 1 is the first such player since for player 2 we may consider the continued vote vector $\bar{x} = (0, 1, 0, 0)$ with $v(\bar{x}) = 0$. For ordering $\pi = (2, 1, 4, 3)$ and vote vector $x = (0, 1, 1, 0)$ player 4 is the pivotal player that determines the outcome $v(x) = 0$.

This line of reasoning can be used to motivate a definition of a Shapley–Shubik index for (j, k) simple games as defined by Freixas (2005), c.f. Kurz (2014). Suppose that voters successively and independently each choose a level of approval in J with equal probability. Such a vote scenario is modeled by a roll-call (π, x) that consists in a permutation π of the voters and a profile $x \in J^n$ such for all $i \in N$; the integer $\pi(i) \in \{1, 2, \dots, n\}$ is the entry position of voter i and x_i is his approval level. Given an index $h \in \{1, \dots, k-1\}$, a voter i is an h-pivotal voter if the vote of player i, according to the ordering π and the approval levels of his predecessors, pushes the outcome to at least h or pushes the outcome to at most $h-1$.

Example 1 Let v be the (3, 3) simple game v for 2 players defined by $v(0,0) = v(1,0) = 0$, $v(1,1) = v(0,1) = 1$, and $v(2,0) = v(0,2) = v(2,1) = v(1,2) = v(2,2) = 2$. As an example, consider the ordering $\pi = (2, 1)$, i.e., player 2 is first, and the vote vector $x = (2, 1)$. Before player 2 announces his vote $x_2 = 1$ all outcomes in $K = \{0, 1, 2\}$ are possible. After the announcement the outcome 0 is impossible, since $v(2,1) \geq v(1,1) \geq v(0,1) = 1$, while the outcomes 1 and 2 are still possible. Thus, player 2 is the 1-pivotal voter. Finally, after the announcement of $x_1 = 2$, the outcome is determined to be $v(2,1) = 2$, so that player 1 is the 2-pivotal voter.

Going in line with the above motivation and the definition from Freixas (2005), the Shapley–Shubik index for (j, k) simple games is defined for all $v \in \mathcal{U}_n$ and for all $i \in N$ by:

$$
\begin{aligned}
\Phi_i(v) =& \frac{1}{n! \cdot j^n \cdot (k-1)} \\
& \times \sum_{h=1}^{k-1} \left| \left\{ (\pi, x) \in \mathcal{S}_n \times J^n \ : \ \text{i is an h-pivot for} \pi \text{ and } x \text{ in } v \right\} \right|.
\end{aligned}
\tag{4}
$$

Since several different definitions of a Shapley–Shubik index for (j, k) simple games have been introduced in the literature, we prefer to use the notation $\Phi_i(v)$ instead of the more suggestive notation $\mathrm{SSI}_i(v)$. For the (j, k) simple game v from Example 1 we have

$$
\Phi(v) = \left(\Phi_1(v), \Phi_2(v)\right) = \left(\frac{5}{12}, \frac{7}{12}\right).
$$

Hereafter, some properties of Φ are explored. To achieve this, we introduce further definitions and axioms for power indices on (j, k) simple games. First of all, we simplify Eq. (4) to a more handy formula that will allow us to associate a TU game $\tilde{v}$, called the average game, to each (j, k) simple game v in Sect. 4 such that the Shapley value of $\tilde{v}$ coincides with $\Phi(v)$.

Lemma 1 *For each (j, k) simple game $v \in \mathcal{U}_n$ and each player $i \in N$ we have*

$$\Phi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot \left[C(v,S) - C(v,S\backslash\{i\}) \right], \tag{5}$$

where $s = |S|$ and

$$C(v,T) = \frac{1}{j^n(k-1)} \cdot \sum_{x \in J^n} \left(v((\mathbf{j}-\mathbf{1})_T, x_{-T}) - v(\mathbf{0}_T, x_{-T}) \right) \tag{6}$$

for all $T \subseteq N$.

Proof For a given permutation $\pi \in \mathcal{S}_n$ and a player $i \in N$, we set $\pi_{<i} = \{h \in N \,:\, \pi(h) < \pi(i)\}$, $\pi_{\le i} = \{h \in N \,:\, \pi(h) \le \pi(i)\}$, $\pi_{>i} = \{h \in N \,:\, \pi(h) > \pi(i)\}$, and $\pi_{\ge i} = \{h \in N \,:\, \pi(h) \ge \pi(i)\}$. With this, we can rewrite $n! \cdot j^n \cdot (k-1)$ times the right hand side of Eq. (4) to

$$\begin{aligned} \sum_{(\pi,x) \in \mathcal{S}_n \times J^n} \Big(& \left[v(x_{\pi_{<i}}, (\mathbf{j}-\mathbf{1})_{\pi_{\ge i}}) - v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\ge i}}) \right] \\ & - \left[v(x_{\pi_{\le i}}, (\mathbf{j}-\mathbf{1})_{\pi_{>i}}) - v(x_{\pi_{\le i}}, \mathbf{0}_{\pi_{>i}}) \right] \Big). \end{aligned} \tag{7}$$

The interpretation is as follows. Since v is monotone, before the vote of player i exactly the values in $\left\{ v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\ge i}}), \dots, v(x_{\pi_{<i}}, (\mathbf{j}-\mathbf{1})_{\pi_{\ge i}}) \right\}$ are still possible as final group decision. After the vote of player i this interval eventually shrinks to $\left\{ v(x_{\pi_{\le i}}, \mathbf{0}_{\pi_{>i}}), \dots, v(x_{\pi_{\le i}}, (\mathbf{j}-\mathbf{1})_{\pi_{>i}}) \right\}$. The difference in (7) just computes the difference between the lengths of both intervals, i.e., the number of previously possible outputs that can be excluded for sure after the vote of player i.

As in the situation where we simplified the Shapley–Shubik index of a simple game given by Eq. (2) to Eq. (1), we observe that it is sufficient to know the sets $\pi_{\ge i}$ and $\pi_{>i}$ for every permutation $\pi \in \mathcal{S}_n$. So we can condense all permutations that lead to the same set and can simplify the expression in (7) and obtain Eq. (5). $\square$

For the (3, 3) simple game v from Example 1 we obtain $C(v,\emptyset) = 0$, $C(v,\{1\}) = \frac{1}{2}$, $C(v,\{2\}) = \frac{2}{3}$, and $C(v,\{1,2\}) = 1$, so that $\Phi_1(v) = \frac{1}{2} \cdot \left(\frac{1}{2} - 0\right) + \frac{1}{2} \cdot \left(1 - \frac{2}{3}\right) = \frac{5}{12}$ and $\Phi_2(v) = \frac{1}{2} \cdot \left(\frac{2}{3} - 0\right) + \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right) = \frac{7}{12}$.

While we think that the roll-call motivation stated above for Eq. (4) is a valid justification on its own, we also want to pursue the more rigorous path to characterize power indices, i.e., we want to give an axiomatization. A set of properties that are satisfied by the Shapley–Shubik index for simple games and uniquely characterize the index is given, e.g., in Dubey (1975). In order to obtain a similar result for (j, k) simple games, we consider a *power index* F as a map form v to $\mathbb{R}^n$ for all (j, k) simple games $v \in \mathcal{U}_n$.

Definition 6 A power index F for (j, k) simple games satisfies

– Positivity (P) if $F(v) \neq \mathbf{0}$ and $F_i(v) \ge 0$ for all $i \in N$ and all $v \in \mathcal{U}_n$;

- Anonymity (A) if $F_{\pi(i)}(\pi v) = F_i(v)$ for all permutations π of N, $i \in N$, and $v \in \mathcal{U}_n$, where $\pi v(x) = v(\pi(x))$ and $\pi(x) = \left(x_{\pi(i)}\right)_{i\in N}$;
- Symmetry (S) if $F_i(v) = F_j(v)$ for all $v \in \mathcal{U}_n$ and all voters $i, j \in N$ that are equivalent in v;
- Efficiency (E) if $\sum_{i\in N} F_i(v) = 1$ for all $v \in \mathcal{U}_n$;
- the Null player property (NP) if $F_i(v) = 0$ for every null voter i of an arbitrary game $v \in \mathcal{U}_n$;
- the *transfer property (T)* if for all $u, v \in \mathcal{U}_n$ and all $i \in N$ we have $F_i(u) + F_i(v) = F_i(u \vee v) + F_i(u \wedge v)$, where $(u \vee v)(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = \min\{u(x), v(x)\}$ for all $x \in J^n$, see Definition 4 and Proposition 2;
- Convexity (C) if $F(w) = \alpha F(u) + \beta F(v)$ for all $u, v \in \mathcal{U}_n$ and all $\alpha, \beta \in \mathbb{R}_{\geq 0}$ with $\alpha + \beta = 1$, where $w = \alpha u + \beta v \in \mathcal{U}_n$;
- *Linearity* (L) if $F(w) = \alpha F(u) + \beta F(v)$ for all $u, v \in \mathcal{U}_n$ and all $\alpha, \beta \in \mathbb{R}$, where $w = \alpha u + \beta v \in \mathcal{U}_n$.

Note that $\alpha \cdot u + \beta \cdot v$ does not need to be a (j, k) simple game for $u, v \in \mathcal{U}_n$, where $\alpha \cdot u$ is defined via $(\alpha \cdot u)(x) = \alpha \cdot u(x)$ for all $x \in J^n$ and all $\alpha \in \mathbb{R}$. This is already true for $(j, k) = (2, 2)$, i.e. simple games, so that (T) was introduced in Dubey (1975). We remark that, obviously, (L) implies (C) and (L) implies (T). Also (S) is implied by (A) since it is a restriction. Some of the properties of Definition 6 have been proven to be valid for $\boldsymbol{\Phi}$ in Freixas (2005). However, for the convenience of the reader we give an extended result and a full proof next:

Proposition 4 *The power index $\boldsymbol{\Phi}$, defined in Eq. (4), satisfies the axioms (P), (A), (S), (E), (NP), (T), (C), and (L).*

Proof We use the notation from the proof of Lemma 1 and let v be an arbitrary (j, k) simple game with n players.

For each $x \in J^n$, $\pi \in \mathcal{S}_n$, and $i \in N$, we have $v(x_{\pi_{<i}}, (\mathbf{j-1})_{\pi_{\geq i}}) \geq v(x_{\pi_{\leq i}}, (\mathbf{j-1})_{\pi_{>i}})$ and $v(x_{\pi_{\leq i}}, \mathbf{0}_{\pi_{>i}}) \geq v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\geq i}})$, so that $\boldsymbol{\Phi}_i(v) \geq 0$ due to Eq. (7). Since we will show that $\boldsymbol{\Phi}$ is efficient, we especially have $\boldsymbol{\Phi}(v) \neq \mathbf{0}$, so that $\boldsymbol{\Phi}$ is positive.

For any permutation $\pi \in \mathcal{S}_n$ and any $0 \leq h \leq n$ let $\pi|h := \{\pi(i) \,:\, 1 \leq i \leq h\}$, i.e., the first h players in ordering π. Then, for any profile $x \in J^n$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\left(v(x_{\pi_{<i}}, (\mathbf{j-1})_{\pi_{\geq i}}) - v(x_{\pi_{\leq i}}, (\mathbf{j-1})_{\pi_{>i}}) + v(x_{\pi_{\leq i}}, \mathbf{0}_{\pi_{>i}}) - v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\geq i}})\right)\\
&\quad= \sum_{h=1}^{n}\left(v(x_{\pi|h-1}, (\mathbf{j-1})_{-\pi|h-1}) - v(x_{\pi|h}, (\mathbf{j-1})_{-\pi|h})\right)\\
&\qquad+ \sum_{h=1}^{n}\left(v(x_{\pi|h}, \mathbf{0}_{-\pi|h}) - v(x_{\pi|h-1}, \mathbf{0}_{-\pi|h-1})\right)\\
&\quad= v(x_{\pi|0}, (\mathbf{j-1})_{-\pi|0}) - v(x_{\pi|n}, (\mathbf{j-1})_{-\pi|n}) + v(x_{\pi|n}, \mathbf{0}_{-\pi|n}) - v(x_{\pi|0}, \mathbf{0}_{-\pi|0})\\
&\quad= v((\mathbf{j-1})) - v(x) + v(x) - v(\mathbf{0}) = k - 1 - 0 = k - 1,
\end{aligned}
$$

so that Eq. (7) gives $\sum_{i=1}^{n} \boldsymbol{\Phi}_i(v) = 1$, i.e., $\boldsymbol{\Phi}$ is efficient.

The definition of $\boldsymbol{\Phi}$ is obviously anonymous, so that it is also symmetric. If player $i \in N$ is a null player and $\pi \in \mathcal{S}_n$ arbitrary, then $v(x_{\pi_{<i}}, \mathbf{0}_{\pi_{\geq i}}) = v(x_{\pi_{\leq i}}, \mathbf{0}_{\pi_{>i}})$ and $v(x_{\pi_{<i}}, (\mathbf{j-1})_{\pi_{\geq i}}) = v(x_{\pi_{\leq i}}, (\mathbf{j-1})_{\pi_{>i}})$, so that $\boldsymbol{\Phi}_i(v) = 0$, i.e., $\boldsymbol{\Phi}$ satisfies the null player property. Since Eq. (7) is linear in the involved (j, k) simple game, $\boldsymbol{\Phi}$ satisfies (L) as well as (C), which is only a relaxation. Since $x + y = \max\{x, y\} + \min\{x, y\}$ for all $x, y \in \mathbb{R}$, $\boldsymbol{\Phi}$ also satisfies the transfer axiom (T). □

Actually the proof of Proposition 4 is valid for a larger class of power indices for (j, k) simple games. To this end we associate each vector $a \in J^n$ with the function v_a defined by

$$v_a(S) = \frac{1}{k-1} \cdot [v((\mathbf{j-1})_S, a_{-S}) - v(\mathbf{0}_S, a_{-S})]$$

for all $S \subseteq N$. With this, we define the mapping $\boldsymbol{\Phi}^a$ on $\mathcal{U}_n$ by

$$\boldsymbol{\Phi}_i^a(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v_a(S) - v_a(S \backslash \{i\})] \tag{8}$$

for all $i \in N$. We remark that it can be easily checked that v_a is a TU game, c.f. Sect. 2.

Similar as in the proof of Lemma 4, we conclude:

Proposition 5 *For every $a \in J^n$ such that $a_i = a_h$ for all $i, h \in N$, the mapping $\boldsymbol{\Phi}^a$ satisfies the axioms* (P), (A), (S), (E), (NP), (T), (C), *and* (L).

While the Shapley–Shubik index for simple games is the unique power index that is symmetric, efficient, satisfies both the null player property and the transfer property, see Dubey (1975), this result does not transfer to general (j, k) simple games.

Proposition 6 *When $j \geq 3$, there exists some $a \in J^n$ such that $\boldsymbol{\Phi}^a \neq \boldsymbol{\Phi}$.*

Proof For $b = (1, j-1, 0, \ldots, 0) \in J^n$ let u^b be the (j, k) simple game with point-veto b. From Equation (8) we conclude $\boldsymbol{\Phi}^a(u^b) = (0,\ 1,\ 0,\ \ldots,\ 0)$, where we set $a = (j-2, j-2, \ldots, j-2) = \mathbf{j-2} \in J^n$. Using Eq. (5) we easily compute $\boldsymbol{\Phi}(u^b) = \left(\frac{1}{j}, \frac{j-1}{j}, 0,\ \ldots,\ 0\right) \neq \boldsymbol{\Phi}^a(u^b)$. □

We remark that the condition $j \geq 3$ is necessary in Proposition 6, since for (2, 2) simple games the roll-call interpretation of Mann and Shapley, see Mann and Shapley (1964), for the Shapley–Shubik index for simple games yields $\boldsymbol{\Phi}^{\mathbf{0}} = \boldsymbol{\Phi}^{\mathbf{1}} = \boldsymbol{\Phi}$.

4 The average game of a (*j*, *k*) simple game

Equation (5) in Lemma 1 has the important consequence that $\Phi(v)$ equals the Shapley value of the TU game $C(v, \cdot)$, where a TU game is a mapping $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. To this end, we introduce an operator that associates each (j, k) simple game v with a TU game $\widetilde{v}$ as follows.

Definition 7 Let $v \in \mathcal{U}_n$ be an arbitrary (j, k) simple game. The *average game*, denoted by $\widetilde{v}$, associated to v is defined by

$$\widetilde{v}(S) = \frac{1}{j^n(k-1)} \sum_{x \in J^n} \left[v((\mathbf{j}-\mathbf{1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S}) \right] \tag{9}$$

for all $S \subseteq N$.

For the (j, k) simple game v from Example 1 the average simple game is given by

$$\tilde{v}(\emptyset) = 0, \tilde{v}(\{1\}) = \frac{1}{2}, \tilde{v}(\{2\}) = \frac{2}{3}, \text{ and } \tilde{v}(N) = 1.$$

We remark that Definition 7 depends on our assumption that the gaps between the output levels of a (j, k) simple game are equally sized; see our remark on the uniform numeric evaluation in the introduction.

With the notation of Definition 7, i.e., $C(v, T) = \tilde{v}(T)$ our above remark on Eq. (5) in Lemma 1 reads:

Theorem 1 *For every (j, k) simple game v the vector $\Phi(v)$ equals the Shapley value of $\widetilde{v}$.*

Before giving some properties of the average game operator we note that two distinct (j, k) simple games may have the same average game, as illustrated in the following example.

Example 2 Consider the (j, k) simple games $u, v \in \mathcal{U}_n$ defined by

- $u(x) = k - 1$ if $x = \mathbf{j} - \mathbf{1}$ and $u(x) = 0$ otherwise;
- $v(x) = k - 1$ if $x \neq \mathbf{0}$ and $v(\mathbf{0}) = 0$

for all $x \in J^n$. Obviously, $u \neq v$. A simple calculation, using Eq. (9), gives $\tilde{u}(S) = \tilde{v}(S) = \frac{1}{j^{n-s}}$ for all $S \in 2^N$.

In other words, the mapping from a (j, k) simple game v to its average game $\tilde{v}$ is not injective and it would be interesting to characterize inclusion maximal sets of (j, k) simple games with the same average game. Observe that the mapping is not surjective since, e.g., $\tilde{v}$ attains only rational values by construction.

The average game operator has some nice properties among which are the following:

Proposition 7 *Given a* (j, k) *simple game* $v \in \mathcal{U}_n$,

(a) $\widetilde{v}$ *is a TU game on* N *that is* $[0, 1]$*-valued and monotone*;
(b) *any null player in* v *is a null player in* $\widetilde{v}$;
(c) *any two equivalent players in* v *are equivalent in* $\widetilde{v}$;
(d) *if* $v = \sum_{t=1}^{p} \alpha_t v_t$ *is a convex combination for some* $v_1, \dots, v_p \in \mathcal{U}_n$, *then* $\widetilde{v} = \sum_{t=1}^{p} \alpha_t \widetilde{v}_t$ *is a TU game.*

Proof Let $v \in \mathcal{U}_n$. All mentioned properties of $\widetilde{v}$ are more or less transferred from the corresponding properties of v via Eq. (9). More precisely:

(a) Note that $\widetilde{v}(\emptyset) = \frac{1}{j^n(k-1)} \sum_{x \in J^n} [v(x) - v(x)] = 0$ and

$$\widetilde{v}(N) = \frac{1}{j^n(k-1)} \sum_{x \in J^n} [v(\mathbf{j-1}) - v(\mathbf{0})] = 1.$$

Since v is monotone and $\mathbf{0} \le x \le \mathbf{j-1}$ for all $x \in J^n$, we have

$$v((\mathbf{j-1})_S, x_{-S}) \le v((\mathbf{j-1})_T, x_{-T}) \text{ and } v(\mathbf{0}_S, x_{-S}) \ge v(\mathbf{0}_T, x_{-T})$$

for all $\emptyset \subseteq S \subseteq T \subseteq N$. Thus, we can conclude $0 \le \widetilde{v}(S) \le \widetilde{v}(T) \le 1$ from Eq. (9).

(b) Let $i \in N$ be a null player in v and $S \subseteq N \backslash \{i\}$ be a coalition, so that $v((\mathbf{j-1})_{S\cup\{i\}}, x_{-(S\cup\{i\})}) = v((\mathbf{j-1})_S, x_{-S})$ and $v(\mathbf{0}_{S\cup\{i\}}, x_{-(S\cup\{i\})}) = v(\mathbf{0}_S, x_{-S})$. Thus, we have that $\widetilde{v}(S \cup \{i\}) = \widetilde{v}(S)$, i.e., player i is a null player in $\widetilde{v}$.

(c) Let $i, h \in N$ be two equivalent players in v, $S \subseteq N \backslash \{i, h\}$, and $\pi_{ih} \in \mathcal{S}_n$ the transposition that interchanges i and h. Since $v((\mathbf{j-1})_{S\cup\{i\}}, x_{-(S\cup\{i\})}) = v((\mathbf{j-1})_{S\cup\{h\}}, \left(\pi_{ih} x\right)_{-S\cup\{h\}})$ and

$$v(\mathbf{0}_{S\cup\{i\}}, x_{-(S\cup\{i\})}) = v(\mathbf{0}_{S\cup\{h\}}, \left(\pi_{ih} x\right)_{-S\cup\{h\}}),$$

we have $\widetilde{v}(S \cup \{i\}) = \widetilde{v}(S \cup \{h\})$, i.e., players i and h are equivalent in $\widetilde{v}$.

(d) Now suppose that $v = \sum_{t=1}^{p} \alpha_t v_t$ is a convex combination for some games $v_1, v_2, \dots, v_p \in \mathcal{U}_n$. Since $v((\mathbf{j-1})_S, x_{-S}) = \sum_{t=1}^{p} \alpha_t v_t((\mathbf{j-1})_S, x_{-S})$ and $v(\mathbf{0}_S, x_{-S}) = \sum_{t=1}^{p} \alpha_t v_t(\mathbf{0}_S, x_{-S})$, applying Eq. (9) gives $\widetilde{v}(S) = \sum_{t=1}^{p} \alpha_t \widetilde{v}_t(S)$ for all $\emptyset \subseteq S \subseteq N$.

□

The operator that associates each (j, k) simple game v with its average game $\widetilde{v}$ can be seen as a coalitional representation of (j, k) simple games. Moreover, Proposition 7 suggests that this representation preserves some properties of the initial game. The average game of a (j, k) simple game with a point-veto is provided by:

Proposition 8 *Given* $a \in J^n \backslash \{\mathbf{0}\}$, *the average game* $\widetilde{u^a}$ *satisfies for every coalition* $S \neq N$

$$\widetilde{u^a}(S) = \begin{cases} \prod_{i \in N\setminus S} \left(\frac{j - a_i}{j}\right) & \text{if } S \cap N^a \neq \emptyset \\ 0 & \text{if } S \cap N^a = \emptyset \end{cases}$$

Proof Let $a \in J^n \setminus \{\mathbf{0}\}$ and $\emptyset \subsetneq S \subsetneq N$.

First suppose that $S \cap N^a = \emptyset$. Then, for all $x \in J^n$ we have $a \leq ((\mathbf{j-1})_S, x_{-S})$ iff $a \leq (\mathbf{0}_S, x_{-S})$. Thus, $u^a(((\mathbf{j-1})_S, x_{-S})) = u^a((\mathbf{0}_S, x_{-S}))$. It then follows from (9) that $\widetilde{u^a}(S) = 0$.

Now suppose that $S \cap N^a \neq \emptyset$. Then, for all $x \in J^n$ we have $a \nleq (\mathbf{0}_S, x_{-S})$. Thus, $u^a((\mathbf{0}_S, x_{-S})) = 0$. Note that $a \leq ((\mathbf{j-1})_S, x_{-S})$ iff $a_{-S} \leq x_{-S}$. Hence,

$$\begin{aligned}
\widetilde{u^a}(S) =& \frac{1}{j^n(k-1)} \sum_{x \in J^n} u^a((\mathbf{j-1})_S, x_{-S}) \\
=& \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S} \in J^{-S}} u^a((\mathbf{j-1})_S, x_{-S}) \\
=& \frac{1}{j^{n-s}(k-1)} \sum_{x_{-S} \in J^{-S} \wedge a_{-S} \leq x_{-S}} u^a((\mathbf{j-1})_S, x_{-S}) \\
=& \frac{1}{k-1} \cdot (k-1) \frac{|\{x_{-S} \in J^{-S},\, a_{-S} \leq x_{-S}\}|}{j^{n-s}} = \prod_{i \in N\setminus S} \left(\frac{j-a_i}{j}\right).
\end{aligned}$$

□

In Proposition 10 we will show that the average game of each (j, k) simple game can be written as the linear combination of the average games of (j, k) simple games with a point-veto of the form $a \in \{0, j-1\}^n$. Before we prove this, recall that the average game associated with each (j, k) simple game is a TU game on N. The set of all TU games on N is vector space and a commonly used basis consists in all unanimity games $(\gamma_S)_{S \in 2^N}$, where $\gamma_S(T) = 1$ if $S \subseteq T$ and $\gamma_S(T) = 0$ otherwise.[4]

In Definition 3 we have introduced the notation $w^S = u^a$ for a coalition $S \in 2^N$, where $a \in J^n$ is specified by $a_i = j - 1$ if $i \in S$ and $a_i = 0$ otherwise.

Proposition 9 *For every coalition $C \in 2^N$, there exists a collection of real numbers $(y_S)_{S \in 2^C}$ such that*

$$\widetilde{w^C} = \sum_{S \in 2^C} y_S \gamma_S,$$

where

[4] The definition of unanimity games has already been given in the second paragraph of Sect. 2.

$$y_S = \frac{(j-1)^s - (-1)^s}{j^c}. \tag{10}$$

Proof Note that $\widetilde{w^C}$ is a TU game on N. Therefore, we have

$$\widetilde{w^C} = \sum_{S \in 2^N} y_S \gamma_S. \tag{11}$$

where the coefficients

$$y_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} \widetilde{w^C}(T) \tag{12}$$

are the well-known Harsanyi dividends, see Harsanyi (1963). This proves the result for $C = N$. Now, suppose that $C \neq N$. Consider $S \subseteq N$ such that $S \nsubseteq C$. Thus S contains some voter i such that $i \notin C$. By Proposition 1 and Proposition 7, voter i is a null player in $\widetilde{w^C}$. Thus, by rewriting y_S from (12), one gets

$$y_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} \widetilde{w^C}(T) = \sum_{i \in T \subseteq S} (-1)^{|S|-|T|} (\widetilde{w^C}(T) - \widetilde{w^C}(T \backslash \{i\})) = 0.$$

Finally we can rewrite Eq. (11) as

$$\widetilde{w^C} = \sum_{S \in 2^N} y_S \gamma_S = \sum_{S \in 2^C} y_S \gamma_S + \sum_{S \nsubseteq C} y_S \gamma_S = \sum_{S \in 2^C} y_S \gamma_S.$$

Moreover for all $S \subseteq C$, y_S can be clearly determined using Proposition 8 as follows:

$$y_S = \sum_{t=1}^{s} (-1)^{s-t} \binom{s}{t} \left(\frac{1}{j}\right)^{c-t} = \frac{(j-1)^s - (-1)^s}{j^c}.$$

This completes the proof. ☐

Proposition 10 *For every* (j,k) *simple game* $u \in \mathcal{U}_n$, *there exists a collection of real numbers* $(x_S)_{S \in 2^N}$ *such that*

$$\tilde{u} = \sum_{S \in 2^N} x_S \widetilde{w^S}. \tag{13}$$

Proof The result is straightforward when $j = 2$ since J reduces to $J = \{0, 1\}$. In the rest of the proof, we assume that $j \geq 3$. Note that all TU games on N can be written as a linear combination of unanimity games $(\gamma_S)_{S \in 2^N}$. It is then sufficient to only prove that each TU game γ_C for $C \in 2^N$ is a linear combination of the TU games $(\widetilde{w^S})_{S \in 2^C}$. The proof is by induction on $1 \leq k = |C| \leq n$. More precisely, we prove the assertion $\mathcal{A}(k)$ that for all $C \in 2^N$ such that $|C| \leq k$, there exists a collection $(z_S)_{S \in 2^C}$ such that

$$\gamma_C = \sum_{S \in 2^C} z_S \widetilde{w^S}. \tag{14}$$

First assume that $k = 1$. Using Proposition 8, it can be easily checked that we have $\gamma_{\{i\}} = \widetilde{w^{\{i\}}}$ for all $i \in N$. Therefore $\mathcal{A}(1)$ holds. Now, consider a coalition C such that $|C| = k \in \{2, \dots, n\}$ and assume that $\mathcal{A}(l)$ holds for all l such that $1 \leq l < k$. By Proposition 9, there exists some real numbers $(\alpha_S)_{S\in 2^C}$ and $(\beta_S)_{S\in 2^C\setminus\{C\}}$ such that

$$\widetilde{w^C} = \sum_{S\in 2^C} \alpha_S\gamma_S = \alpha_C\gamma_C + \sum_{S\in 2^C\setminus\{C\}} \alpha_S\gamma_S = \alpha_C\gamma_C + \sum_{S\in 2^C\setminus\{C\}} \beta_S\widetilde{w^S}.$$

where the last equality holds by the induction hypothesis. Moreover, since $j - 1 \geq 2$, from Eq. (10), we have,

$$\alpha_C = \frac{(j-1)^c - (-1)^c}{j^c} \neq 0.$$

Therefore we get

$$\gamma_C = \sum_{S\in 2^C} z_S\widetilde{w^S}$$

where for all $S \in 2^C$, $z_S = -\frac{1}{\alpha_C}$ if $S = C$ and $z_S = -\frac{\beta_S}{\alpha_C}$ otherwise. This gives $\mathcal{A}(k)$. In summary, for each coalition $S \in 2^N$ the game γ_S is a linear combination of the games $\widetilde{w^C}$, where $C \in 2^N$. Thus, the proof is completed since $\tilde{u}$ is a linear combination of the games γ_S, where $S \in 2^N$. $\square$

Before we continue, note that by Eq. (14), for $C \in 2^N$ each TU game γ_C is a linear combination of the TU games $\left(\widetilde{w^S}\right)_{S\in 2^N}$. Since $\left(\gamma_S\right)_{S\in 2^N}$ is a basis of the vector space of all TU games on N, it follows that $\left(\widetilde{w^S}\right)_{S\in 2^N}$ is also a basis of the vector space of all TU games on N.

5 A characterization of the Shapley–Shubik index for (*j*, *k*) simple games

As shown in Proposition 5 the axioms of Definition 6 are not sufficient to uniquely characterize the power index $\boldsymbol{\Phi}$ for the class of (j, k) simple games. Therefore we introduce an additional axiom.

Definition 8 A power index F for (j, k) simple games is *averagely convex* (AC) if we always have

$$\sum_{t=1}^{p} \alpha_t F(u_t) = \sum_{t=1}^{q} \beta_t F(v_t) \tag{15}$$

whenever

$$\sum_{t=1}^{p} \alpha_t \widetilde{u}_t = \sum_{t=1}^{q} \beta_t \widetilde{v}_t, \tag{16}$$

where $u_1, u_2, \dots, u_p, v_1, v_2, \dots, v_q \in \mathcal{U}_n$ and $(\alpha_t)_{1 \le t \le p}$, $(\beta_t)_{1 \le t \le q}$ are non-negative numbers that sum to 1 each.

One may motivate the axiom (AC) as follows. In a game, the *a priori strength* of a coalition, given the profile of the other individuals, is the difference between the outputs observed when all of her members respectively give each her *maximum support* and her *minimum support*. The *average strength* game associates each coalition with her expected strength when the profile of other individuals uniformly varies. Average convexity for power indices is the requirement that whenever two convex combinations of average games coincide, the corresponding convex combinations of the power distributions also coincide.

We remark that the axiom of average convexity is much stronger than the axiom of convexity. A minor technical point is that $\sum_{t=1}^{p} \alpha_t u_t$ as well as $\sum_{t=1}^{q} \beta_t v_t$ do not need to be (j, k) simple games. However, the more important issue is that

$$\widetilde{\sum_{t=1}^{p} \alpha_t u_t} \overset{\text{Proposition (7).(d)}}{=} \sum_{t=1}^{p} \alpha_t \widetilde{u}_t = \sum_{t=1}^{q} \beta_t \widetilde{v}_t \overset{\text{Proposition (7).(d)}}{=} \widetilde{\sum_{t=1}^{q} \beta_t v_t},$$

i.e., Eq. (16), is far less restrictive than

$$\sum_{t=1}^{p} \alpha_t u_t = \sum_{t=1}^{q} \beta_t v_t$$

since two different (j, k) simple games may have the same average game, see Example 2. Further evidence is given by the fact that the parametric power indices $\boldsymbol{\Phi}^a$, defined in Eq. (8), do not all satisfy (AC).

Proposition 11 *When $j \ge 3$, there exists some $a \in J^n$ such that $\boldsymbol{\Phi}^a$ does not satisfy (AC).*

Proof As in the proof of Proposition 6, consider the (j, k) simple game with point-veto $b = (1, j-1, 0, \dots, 0) \in J^n$ and let $a = (j-2, j-2, \dots, j-2)$. It can be easily checked that, for all subsets $T \subseteq N$ we have

$$\widetilde{u^b}(T) = \begin{cases} 1 & \text{if } 1, 2 \in T \\ (j-1)/j & \text{if } 2 \in T \subseteq N \backslash \{1\} \\ 1/j & \text{if } 1 \in T \subseteq N \backslash \{2\} \\ 0 & \text{if } T \subseteq N \backslash \{1, 2\} \end{cases}$$

and that

$$\widetilde{u^b} = \frac{1}{j} \cdot \widetilde{w^{\{1\}}} + \frac{j-1}{j} \cdot \widetilde{w^{\{2\}}} \tag{17}$$

holds. Since $\boldsymbol{\Phi}^a$ satisfies (NP), (E), (S) we can easily compute $\boldsymbol{\Phi}^a\left(w^{\{1\}}\right) = (1, 0, \dots, 0)$ and $\boldsymbol{\Phi}^a\left(w^{\{2\}}\right) = (0, 1, 0, \dots, 0)$. Therefore,

$$\frac{1}{j} \cdot \boldsymbol{\Phi}^a\left(w^{\{1\}}\right) + \frac{j-1}{j} \cdot \boldsymbol{\Phi}^a\left(w^{\{2\}}\right) = \left(\frac{1}{j}, \frac{j-1}{j}, 0, \dots, 0\right). \tag{18}$$

Using (8), one gets $\boldsymbol{\Phi}^a(u^b) = (0, 1, 0, \dots, 0)$. It then follows from Eqs. (17) and (18) that $\boldsymbol{\Phi}^a$ does not satisfy (AC). $\square$

We remark that proving that a power index does not satisfy (AC) can always be done by suitable examples. For the other direction we, unfortunately, do not know an algorithmic method aside from verifying Eq. (15) directly.

As a preliminary step to our characterization result in Theorem 2 we state:

Lemma 2 *If a power index F for the class $\mathcal{U}_n$ of (j, k) simple games satisfies (E), (S), and (NP), then we have $F(w^C) = \boldsymbol{\Phi}(w^C)$ for all $C \in 2^N$.*

Proof Let F be a power index on $\mathcal{U}_n$ that satisfies (E), (S), (NP) and let $C \in 2^N$ be arbitrary.

According to Proposition 1, all players $i, h \in C$ are equivalent in w^C and those outside of C are null players in the game w^C. Since both F and $\boldsymbol{\Phi}$ satisfy (E), (S), and (NP), we have $F_i(w^C) = \boldsymbol{\Phi}_i(w^C) = \frac{1}{|C|}$ if $i \in C$ and $F_i(w^C) = \boldsymbol{\Phi}_i(w^C) = 0$ otherwise. $\square$

Theorem 2 *A power index F for the class $\mathcal{U}_n$ of (j, k) simple games satisfies (E), (S), (NP), and (AC) if and only if $F = \boldsymbol{\Phi}$.*

Proof Necessity: As shown in Proposition 4, $\boldsymbol{\Phi}$ satisfies (E), (S), and (NP). For (AC) the proof follows from Theorem 1 since the average game operator is linear by Proposition 7.

Sufficiency: Consider a power index F for (j, k) simple games that satisfies (E), (S), (NP), and (AC). Next, consider an arbitrary (j, k) simple game $u \in \mathcal{U}_n$. By Proposition 10, there exists a collection of real numbers $(x_S)_{S \in 2^N}$ such that

$$\widetilde{u} = \sum_{S \in 2^N} x_S \widetilde{w^S} = \sum_{S \in E_1} x_S \widetilde{w^S} + \sum_{S \in E_2} x_S \widetilde{w^S}, \tag{19}$$

where $E_1 = \{S \in 2^N : x_S > 0\}$ and $E_2 = \{S \in 2^N : x_S < 0\}$. Note that $E_1 \neq \emptyset$ since $\widetilde{u}(N) = 1$. As an abbreviation we set

$$\varpi = \sum_{S \in E_1} x_S \widetilde{w^S}(N) = \sum_{S \in E_1} x_S > 0. \tag{20}$$

It follows that

$$\frac{1}{\varpi}\widetilde{u} + \sum_{S\in E_2} \frac{-x_S}{\varpi}\,\widetilde{w^S} = \sum_{S\in E_1} \frac{x_S}{\varpi}\,\widetilde{w^S}. \tag{21}$$

Since (21) is an equality among two convex combinations, axiom (AC) yields

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\,F(w^S) = \sum_{S\in E_1} \frac{x_S}{\varpi}\,F(w^S).$$

Therefore by Lemma 2,

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\,\boldsymbol{\Phi}(w^S) = \sum_{S\in E_1} \frac{x_S}{\varpi}\,\boldsymbol{\Phi}(w^S). \tag{22}$$

Since $\boldsymbol{\Phi}$ also satisfies (AC), we obtain

$$\frac{1}{\varpi}F(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\,\boldsymbol{\Phi}(w^S) = \frac{1}{\varpi}\boldsymbol{\Phi}(u) + \sum_{S\in E_2} \frac{-x_S}{\varpi}\,\boldsymbol{\Phi}(w^S), \tag{23}$$

so that $F(u) = \boldsymbol{\Phi}(u)$. □

Proposition 12 *For $j \geq 3$, the four axioms in Theorem 2 are independent.*

Proof For each of the four axioms in Theorem 2, we provide a power index on $\mathcal{U}_n$ that meets the three other axioms but not the chosen one.

- The power index $2 \cdot \boldsymbol{\Phi}$ satisfies (NP), (S), and (AC) but not (E).
- Denote by ED the *equal division* power index which assigns $\frac{1}{n}$ to each player for every (j, k) simple game v. Then, the power index $\frac{1}{2} \cdot \boldsymbol{\Phi} + \frac{1}{2} \cdot \mathrm{ED}$ satisfies (E), (S) and (AC), but not (NP).
- In Proposition 5 we have constructed a parametric series of power indices that satisfiy (E), (S), and (NP). For $j \geq 3$, at least one example does not satisfy (AC), see Proposition 11.
- Recall that $\left(\widetilde{w^S}\right)_{S\in 2^N}$ is a basis of the vector space of all TU games on N. Thus given a (j, k) simple game u, there exists a unique collection of real numbers $\left(x^u_S\right)_{S\in 2^N}$ such that

$$\widetilde{u} = \sum_{S\in 2^N} x^u_S \widetilde{w^S}. \tag{24}$$

Consider some $i_0 \in N$ and set

$$F(u) = \sum_{S\in 2^N} x^u_S \cdot F\left(w^S\right). \tag{25}$$

For each $S \in 2^N \backslash \{N\}$ we set $F_i\left(w^S\right) = \Phi\left(w^S\right)$. For $S = N$ we set $F_i\left(w^N\right) = \dfrac{2}{n+1}$ if $i = i_0$ and $F_i\left(w^N\right) = \dfrac{1}{n+1}$ otherwise. We can easily check that F satisfies (E), (NP), (AC), but not (S).

This proves that the four axioms in Theorem 2 are independent. □

If we compare the axiomatization given in Theorem 2 with the one from Freixas (2019) the only difference is our axiom (AC) and the axiom of level change effect on unanimity games introduced in Freixas (2019). In Freixas (2019, Lemma 1.3) it is made very transparent when the axioms (E), (A), and (NP) are sufficient to determine the power index on a unanimity game. The same statement is true for the axioms (E), (S), and (NP). Using axiom (AC) we can start from an arbitrary (j, k) simple game v, write its average game $\tilde{v}$ as a linear combination of unanimity TU games, and find for each unanimity TU game γ_S a (j, k) simple game u_S which satisfies the conditions of Freixas (2019, Lemma 1.3) and has γ_S as its average game.

6 Axiomatization of the Shapley–Shubik index for interval simple games

As with (j, k) simple games, a Shapley–Shubik like index for interval simple games can be constructed from the idea of the roll-call model; see Definition 9. As introduced in Sect. 2, an *interval simple game* is a mapping $v : [0, 1]^n \to [0, 1]$ with $v(\mathbf{0}) = 0$, $v(\mathbf{1}) = 1$, and $v(x) \le v(y)$ for all $\mathbf{0} \le x \le y \le \mathbf{1}$. In Theorem 4 we will show that this index is uniquely characterized by the axioms (E), (S), (NP) and (AC), see Definition 10 for the definition of the average game. The technical details are rather similar to our considerations for (j, k) simple games, so that we will mainly skip the proofs.

Definition 9 (cf. Kurz 2014, Definition 6.2)

Let v be an interval simple game with player set N and $i \in N$ an arbitrary player. We set

$$\begin{aligned}\Psi_i(v) =& \frac{1}{n!}\sum_{\pi\in\mathcal{S}_n}\int_0^1\cdots\int_0^1\left[v(x_{\pi_{<i}},\mathbf{1}_{\pi_{\ge i}}) - v(x_{\pi_{<i}},\mathbf{0}_{\pi_{\ge i}})\right]\\ &- \left[v(x_{\pi_{\le i}},\mathbf{1}_{\pi_{>i}}) - v(x_{\pi_{\le i}},\mathbf{0}_{\pi_{>i}})\right]\mathrm{d}x_1\cdots\mathrm{d}x_n.\end{aligned} \tag{26}$$

In this section, we give a similar axiomatization for Ψ (for interval simple games) as we did for (j, k) simple games and Φ. By a power index for interval simple games we understand a mapping from the set of interval simple games for n players to $\mathbb{R}^n$. Replacing both J and K by $I = [0, 1]$ in Definition 6, allows us to directly transfer the properties of power indices for (j, k) simple games to the present situation. Also Proposition 4 is valid for interval simple games and Ψ. More precisely, Ψ satisfies (P), (A), (S), (E), (NP), and (T), see (Kurz (2018), Lemma 6.1). The proof for (C) and (L) goes along the same lines as the proof of Proposition 4. Also the generalization of the power index to a parametric class can be done just as the one for (j, k) simple games in Eq. (8).

Proposition 13 *For every $\alpha \in [0,1]$ the mapping Ψ^a, where $a = (\alpha, \dots, \alpha) \in [0,1]^n$, defined by $\Psi_i^a(v) =$*

$$\frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \left(\left[v(a_{\pi_{<i}}, \mathbf{1}_{\pi_{\geq i}}) - v(a_{\pi_{<i}}, \mathbf{0}_{\pi_{\geq i}}) \right] - \left[v(a_{\pi_{\leq i}}, \mathbf{1}_{\pi_{>i}}) - v(a_{\pi_{\leq i}}, \mathbf{0}_{\pi_{>i}}) \right] \right)$$

for all $i \in N$, satisfies (P), (A), (S), (E), (NP), (T), (C), and (L).

Again, there exist vectors $a \in [0,1]^n$ and interval simple games v with $\Psi^a(v) \neq \Psi(v)$. Also the simplified formula for Φ for (j, k) simple games in Lemma 1 can be mimicked for interval simple games and Ψ, see Kurz et al. (2019).

Proposition 14 *For every interval simple game v with player set N and every player $i \in N$ we have*

$$\Psi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \cdot \left[C(v, S) - C(v, S \backslash \{i\}) \right], \tag{27}$$

where $C(v, T) = \int_{[0,1]^n} v(\mathbf{1}_T, x_{-T}) - v(\mathbf{0}_T, x_{-T})\, \mathrm{d}x$ for all $T \subseteq N$.

This triggers:

Definition 10 Let v be an interval simple game on N. The *average game* associated with v and denoted by $\widehat{v}$ is defined via

$$\forall S \subseteq N,\ \widehat{v}(S) = \int_{I^n} [v(\mathbf{1}_S, x_{-S}) - v(\mathbf{0}_S, x_{-S})] dx. \tag{28}$$

Theorem 3 *For all every interval simple game v on N and for all $i \in N$,*

$$\Psi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [\widehat{v}(S) - \widehat{v}(S \backslash \{i\})] \tag{29}$$

In other words, for a given interval simple game v the power distribution $\Psi(v)$ is given by the Shapley value of its average game $\hat{v}$.

As with (j, k) simple games, two distinct interval simple games may have the same average game as illustrated in the following example.

Example 3 Consider the interval simple games u and v defined on N respectively for all $x \in [0,1]^n$ by : $u(x) = 1$ if $x = \mathbf{1}$, and $u(x) = 0$ otherwise; $v(x) = 1$ if $x \neq \mathbf{0}$, and $v(\mathbf{0}) = 0$. It is clear that, $u \neq v$. But, Eq. (28) and a simple calculation give $\widehat{u}(S) = \widehat{v}(S) = 1$ if $S = N$ and $\widehat{u}(S) = \widehat{v}(S) = 0$ otherwise.

We can also transfer Proposition 7, i.e., the average game operator preserves the following nice properties of interval simple games.

Proposition 15 *Given an interval simple game $v \in \mathcal{ISG}_n$,*

(a) *$\widehat{v}$ is a TU game on N that is $[0,1]$-valued and monotone;*
(b) *any null voter in v is null player in $\widehat{v}$;*
(c) *any two symmetric voters in v are symmetric players in $\widehat{v}$;*
(d) *if $v = \sum_{t=1}^{p} \alpha_t v_t$ is a convex combination for some $v_1, \dots, v_p \in \mathcal{ISG}_n$ then $\widehat{v} = \sum_{t=1}^{p} \alpha_t \widehat{v}_t$ is a TU game.*

Proof Very similar to the one of Proposition 7. □

From Theorem 3 we can directly conclude that Ψ also satisfies *average convexity* (AC), which is defined as in Definition 8.

For the remaining part of this section we introduce some further notation. For all $x \in I^n$, let $\mathbf{1}_x = \{i \in N,\ x_i = 1\}$; and given a coalition S, let C^S be the interval simple game defined for all $x \in I^n$ by $C^S(x) = 1$ if $S \subseteq \mathbf{1}_x$ and $C^S(x) = 0$ otherwise.

Proposition 16 *For all $T \in 2^N$, the average game $\widehat{C^S}$ (see Definition 10) equals γ_S.*

Proof Consider $S, T \subseteq N$. If $S \subseteq T$ then for all $x \in [0,1]^n$, $S \subseteq T \subseteq \{i \in N, (\mathbf{1}_T, x_{-T})_i = 1\}$ and $S \cap \{i \in N, (\mathbf{0}_T, x_{-T})_i = 1\} = \emptyset$. Then by definition of C^S, $C^S(\mathbf{1}_T, x_{-T}) = 1$ and $C^S(\mathbf{0}_T, x_{-T}) = 0$. Therefore,

$$\widehat{C^S}(T) = \int_{[0,1]^n} [C^S(\mathbf{1}_T, , x_{-T}) - C^S(\mathbf{0}_T, x_{-T})] dx = 1 = \gamma_S(T).$$

Now assume that $S \nsubseteq T$. Let $x \in [0,1)^n$. Note that $\{i \in N, (\mathbf{1}_T, x_{-T})_i = 1\} = T$ and $\{i \in N, (\mathbf{0}_T, x_{-T})_i = 1\} = \emptyset$. Therefore, $S \nsubseteq \{i \in N, (\mathbf{1}_T, x_{-T})_i = 1\}$ and $S \nsubseteq \{i \in N, (\mathbf{0}_T, x_{-T})_i = 1\}$. By the definition of C^S, it follows that $C^S(\mathbf{1}_T, x_{-T}) = C^S(\mathbf{0}_T, x_{-T}) = 0$. Hence

$$\widehat{C^S}(T) = \int_{[0,1]^n} [C^S(\mathbf{1}_T, x_{-T}) - C^S(\mathbf{0}_T, x_{-T})] dx = 0 = \gamma_S(T).$$

In both cases $\widehat{C^S}(T) = \gamma_S(T)$ for all $T \in 2^N$; that is $\widehat{C^S} = \gamma_S$. □

Theorem 4 *A power index Ψ' for interval simple games satisfies (E), (S), (NP) and (AC) if and only if $\Psi' = \Psi$.*

Proof *Necessity*: We have already remarked that Ψ satisfies (E), (S), (AC), and (NP).

Sufficiency: Let Ψ' be a power index for interval simple games on N that simultaneously satisfies (E), (S), (AC), and (NP). Consider an interval simple game u. Note that $\widehat{u}$ is a TU game by Proposition 15. Thus by Proposition 16, there exists a collection of real numbers $(\alpha_S)_{S\in 2^N}$ such that

$$\widehat{u} = \sum_{S\in 2^N} \alpha_S \cdot \widehat{C^S} = \sum_{S\in E_1} \alpha_S \cdot \widehat{C^S} + \sum_{S\in E_2} \alpha_S \cdot \widehat{C^S} \tag{30}$$

where $E_1 = \{S \in 2^N : \alpha_S > 0\}$ and $E_2 = \{S \in 2^N : \alpha_S < 0\}$. Moreover, $E_1 \neq \emptyset$ since $\widehat{v}(N) = 1$. We set

$$\varpi = \sum_{S\in E_1} \alpha_S \cdot \widehat{C^S}(N) = \sum_{S\in E_1} \alpha_S > 0. \tag{31}$$

It follows that

$$\frac{1}{\varpi}\widehat{u} + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi} \widehat{C^S} = \sum_{S\in E_1} \frac{\alpha_S}{\varpi} \widehat{C^S}. \tag{32}$$

Since (32) is an equality among two convex combinations, then by (AC), we deduce that

$$\frac{1}{\varpi}\Psi'(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi} \Psi'(C^S) = \sum_{S\in E_1} \frac{\alpha_S}{\varpi} \Psi'(C^S). \tag{33}$$

Note that given $S \in 2^N$, all voters in S are equivalent in C^S while all voters outside S are null players in C^S. Since Ψ' and Ψ satisfy (E), (S), and (NP), it follows that $\Psi'(C^S) = \Psi(C^S)$. Thus,

$$\frac{1}{\varpi}\Psi'(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi} \Psi(C^S) = \sum_{S\in E_1} \frac{\alpha_S}{\varpi} \Psi(C^S). \tag{34}$$

Since Ψ also satisfies (AC), we get

$$\frac{1}{\varpi}\Psi'(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi} \Psi(C^S) = \frac{1}{\varpi}\Psi(u) + \sum_{S\in E_2} \frac{-\alpha_S}{\varpi} \Psi(C^S). \tag{35}$$

Hence $\Psi'(u) = \Psi(u)$, which proves that $\Psi' = \Psi$. □

Proposition 17 *The four axioms in Theorem 4 are independent.*

Proof

- The power index $2 \cdot \Psi$ satisfies (NP), (S), (AC), but not (E).
- Denote by ED the *equal division* power index which assigns $\frac{1}{n}$ to each player for every interval simple game. Then the power index $\frac{1}{2} \cdot \Psi + \frac{1}{2} \cdot \mathrm{ED}$ satisfies (E), (S) and (AC), but not (NP).
- In Proposition 13 (c.f. (Kurz et al. (2019), Proposition 4)) we have stated a parametric classes of power indices for interval simple games that satisfy (E), (S), and (NP). In Kurz et al. (2019) it was also proved that there is at least one param-

eter $\mathbf{a} \in I^n$ for which the parameterized index $\Psi^{\mathbf{a}} \neq \Psi$. Thus, by Theorem 4 we can conclude that $\Psi^{\mathbf{a}}$ does not satisfies (AC). (Also Proposition 11 for (j, k) simple games can be adjusted easily.)

– Note that by Proposition 16 the set $\left(\widehat{C^S}\right)_{S\in 2^N}$ is a basis of the vector space of all TU games on N. Thus, given an interval simple game u, there exists a unique collection of real numbers $\left(y_S^u\right)_{S\in 2^N}$ such that

$$\hat{u} = \sum_{S\in 2^N} y_S^u \widehat{C^S}. \tag{36}$$

Consider some $i_0 \in N$ and set

$$F(u) = \sum_{S\in 2^N} y_S^u \cdot F\left(C^S\right). \tag{37}$$

For each $S \in 2^N \setminus \{N\}$ we set $F_i\left(C^S\right) = \Phi\left(C^S\right)$. For $S = N$ we set $F_i\left(C^N\right) = \frac{2}{n+1}$ if $i = i_0$ and $F_i\left(C^N\right) = \frac{1}{n+1}$ otherwise. We can easily check that F satisfies (E), (NP), (AC), but not (S).

This proves that the four axioms in Theorem 4 are independent. □

In contrast with (j, k) simple games, all convex combinations of interval simple games are also interval simple games. Thus, the axiom (AC) in Theorem 4 can be split into two easier axioms: the standard axiom of convexity (C) and the axiom of average equivalence (AE) stating that if F is a power index for interval simple games, then any two interval simple games that induce the same average game must have the same power distribution by F.

7 Conclusion

Freixas (2005) extends the Shapley–Shubik index from simple games to some wider classes of interesting games with several levels of inputs and outputs. Related axiomatization results comprise Freixas (2019) for $(j, 2)$ simple games and for j-cooperative games (outputs are real numbers); and Kurz et al. (2019) for interval simple games. In this paper, we provide new axiomatizations of the Shapley–Shubik index for (j, k) simple games as well as for interval simple games.

We introduce the notion of average game of a (j, k) simple game and the axiom of average convexity. The average game allows us to give the Shapley–Shubik index of a (j, k) simple game an explicit formula in terms of the characteristic function. More precisely, the Shapley–Shubik index of a (j, k) simple game is simply the Shapley value of its average game. Theorem 2 differs from previous works essentially on the new axiom of average convexity. The axiom is the requirement that if two convex combinations of average games coincide, then do the corresponding convex combinations of the power distributions of the underlying games. Average convexity condition can be viewed as some form of linearity condition. It is an interesting open question whether this axiom can be decomposed into some weaker axioms. In the case of interval simple games, it appears that average convexity is equivalent to average equivalence and convexity. Whether this equivalence still holds for (j, k) simple games remains an open issue since a convex combination of (j, k) simple games may not be a (j, k) simple game.

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