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**BIFURCATIONS OF A SECOND ORDER  
DIFFERENTIAL EQUATION WITH PERIODIC  
COEFFICIENTS**

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ABSTRACT. In this paper, we study the bifurcations for the second order differential equation  $\ddot{x} - a_0(x) - a_1(x)\dot{x} - a_2(x)\dot{x}^2 = 0$ , where  $a_k : \mathbb{R} \rightarrow \mathbb{R}$  are  $C^r$  one periodic functions.

**1. Introduction**

Consider the non linear differential equation

$$(1) \quad \ddot{x} - a_0(x) - a_1(x)\dot{x} - a_2(x)\dot{x}^2 = 0$$

where  $a_k : \mathbb{R} \rightarrow \mathbb{R}$  are  $C^r$  one periodic functions with  $r \geq 3$ . Equation (1) describes the dynamics of many biological, physical or chemical phenomena. The most known is the oscillations of a damped pendulum. One of the problems in connection with this equation is the study of the bifurcations of its phase portrait. This problem has been studied by some authors. A. Chenciner [2] also obtained equation (1), with  $a_1$  and  $a_2$  constants, to describe the elimination of the resonant form in a bifurcation problem of a discrete dynamical system. He gives a description of the phase portrait of the equation. In the case  $a_2 = 0$  and  $a_1$  constant, Xu Gang and al [9] used a two times scales ( $t$  and  $\tau = a_1^{-1}t$ ) method to obtain asymptotic solutions of the equation, when  $a_1 \rightarrow +\infty$ , in the form

$$x(t, \tau) = A(\tau) + B(\tau)\exp(a_1 t).$$

In [6], some structural stability properties of (1) are presented.

In this paper, we analyze the bifurcations of the solutions for equation (1) as functions  $a_k$  change. We begin (section 2) by studying two local bifurcations of equilibrium points of (1), namely the saddle-node

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and the Andronov-Hopf-Poincaré bifurcations. For the last one, we obtain the existence conditions for a homotopic periodic orbit of (1). In section 3, we study (1) by using techniques of Coll and al. [3] which permit to transform equation (1) to an Abel type equation. The problem to determine the periodic solutions is equivalent to determine the fixed points of a Poincaré operator associated with that Abel's equation. We prove the existence of periodic orbits of (1) and we give some properties on the number of these orbits. In connection with the structural stability problem, the behavior at infinity of the solutions is described by a compactification technique [6].

The second order scalar differential equation (1) is equivalent to the 2-dimensional system:

$$(2) \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= a_0(x) + a_1(x)y + a_2(x)y^2 \end{aligned}$$

where  $(x, y)$  belongs to the cylinder  $S^1 \times \mathbb{R}$ , with  $S^1 = \mathbb{R}/2\pi\mathbb{Z} = [0, 1] \text{ mod. } 1$ .

We denote by  $X$  the vector field defined on the cylinder  $S^1 \times \mathbb{R}$  by

$$X(x, y) = y \frac{\partial}{\partial x} + \sum_{k=0}^{k=2} a_k(x) y^k \frac{\partial}{\partial y},$$

with  $a_k \in C^r(\mathbb{R}/2\pi\mathbb{Z})$ ,  $r \geq 3$ , the space of  $C^r$  real valued functions defined on the circle  $S^1$ . So we can identify the vector field  $X$  with an element  $(a_0, a_1, a_2)$  of  $(C^r(\mathbb{R}/2\pi\mathbb{Z}))^3$ .

## 2. Local bifurcations

This section concerns the local bifurcations of (2), i.e. the bifurcations of equilibrium points. We study two types of those bifurcations: the saddle-node and the Andronov-Hopf-Poincaré bifurcations.

The equilibrium points (singularities) of (2) have to be of the form  $(x_*, 0)$ , where  $x_*$  satisfies  $a_0(x_*) = 0$ , and are located on the circle  $S^1 \times \{0\}$ . If  $(x_*, 0)$  is an equilibrium point of (2), then the eigenvalues of  $dX(x_*, 0)$  are solutions of the characteristic equation

$$(3) \quad \lambda^2 - a_1(x_*)\lambda - a_0'(x_*) = 0$$

and are given by the relations

$$(4) \quad \lambda_1 = \frac{a_1(x_*) - \sqrt{a_1(x_*)^2 + 4a_0'(x_*)}}{2}, \quad \lambda_2 = \frac{a_1(x_*) + \sqrt{a_1(x_*)^2 + 4a_0'(x_*)}}{2}.$$

**2.1. Saddle-node bifurcation.** Let  $(x_*, 0)$  be a point of  $S^1 \times \mathbb{R}$ . On the neighborhood of  $x_*$ , we write

$$(5) \quad a_0(x) = \sum_{k=0}^{\infty} a_{0k} (x - x_*)^k$$

where  $a_{0k} = \frac{1}{k!} \frac{\partial^k a_0}{\partial x^k}(x_*)$ ,  $k \geq 0$ .

Let  $\mu \equiv a_{00}$ , then

$$(6) \quad a_0(x, \mu) = \mu + \sum_{k=1}^{\infty} a_{0k} (x - x_*)^k,$$

and the system (2) takes the form

$$(7) \quad \begin{cases} \dot{x} = y \\ \dot{y} = a_0(x, \mu) + a_1(x)y + a_2(x)y^2 \end{cases}$$

The point  $(x_*, 0)$  is an equilibrium of (2) when  $\mu = 0$ , and the eigenvalues of  $dX(x_*, 0)$  are

$$(8) \quad \lambda_1 = \frac{a_1(x_*) - \sqrt{a_1^2(x_*) + 4a_{01}}}{2}, \quad \lambda_2 = \frac{a_1(x_*) + \sqrt{a_1^2(x_*) + 4a_{01}}}{2}.$$

A saddle-node bifurcation occurs for  $\mu = 0$  if

$$\lambda_1 = 0, \quad \lambda_2 \neq 0 \quad \text{and} \quad a_{02} \neq 0.$$

The above conditions are equivalent to

$$(9) \quad a_{01} = 0, \quad a_1(x_*) \neq 0 \quad \text{and} \quad a_{02} \neq 0.$$

So we deduce the following proposition.

**Proposition 1.** *(Saddle-node bifurcation) The subset  $S$  of the function space  $(C^r(\mathbb{R}/2\pi\mathbb{Z}))^3$  defined by the conditions (9) is a saddle-node bifurcation set for the differential system (2).*

**2.2. Andronov-Hopf-Poincaré bifurcation.** Suppose that  $(0, 0)$  is an equilibrium point of (2). An Andronov-Hopf-Poincaré bifurcation can occur for the differential system (2) when (3) admits two pure imaginary complex eigenvalues. This is achieved if

$$(10) \quad a_1(0) = 0 \quad \text{and} \quad a'_0(0) < 0.$$

Then the eigenvalues of  $dX(0, 0)$  are given by

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}a_1(0) \pm i\omega, \\ \omega^2 &= -\frac{1}{4}(a_1(0))^2 - a'_0(0). \end{aligned}$$

By choosing

$$c = \frac{1}{2}a_1(0)$$

as the bifurcation parameter for (2), the pulsation  $\omega(c)$  and the eigenvalues  $\lambda_{1,2}$  take the form

$$(11) \quad \begin{aligned} \omega^2(c) &= -c^2 - a'_0(0) \\ \lambda_{1,2}(c) &= c \pm i\omega(c). \end{aligned}$$

Then the eigenvalues are of the form  $\lambda(c_*) = \pm i\omega_*$  when  $c = 0$ , and we have

$$\frac{d}{dc} \operatorname{Re}(\lambda_{1,2}(c)) = 1 \neq 0.$$

In view of the bifurcation analysis, we use the transformation given by the eigenvectors corresponding with the eigenvalues  $\lambda_{1,2}(c)$  of  $dX(0, 0, c)$ ,

$$x = u, \quad y = cu + \omega(c)v.$$

Then we obtain the canonical form of (2):

$$(12) \quad \begin{cases} \dot{u} = cu + \omega(c)v, \\ \dot{v} = -\omega(c)u + cv + \sum_{k+l=3}^{k+l=3} A_{kl}(c)u^k v^l + O(\|(u, v)\|^4). \end{cases}$$

where the coefficients  $A_{kl}(c)$  are given by the following relations

$$(13) \quad \begin{aligned} A_{20}(c) &= \frac{1}{\omega(c)} \left( \frac{1}{2}a''_0(0) + a'_1(0)c + a_2(0)c^2 \right), \\ A_{11}(c) &= a'_1(0) + 2\mu a_2(0), \\ A_{02}(c) &= a_2(0)\omega, \\ A_{30}(c) &= \frac{1}{\omega(c)} \left( \frac{1}{6}a'''_0(0) + \frac{1}{2}a''_1(0)c + a'_2(0)c^2 \right) \\ A_{21}(c) &= \frac{1}{2}a''_1(0) + \frac{1}{\omega(c)}a'_2(0)c^2, \\ A_{12}(c) &= a'_2(0)\omega(c), \\ A_{03}(c) &= 0. \end{aligned}$$

Using the polar coordinates  $(r, \theta)$ ,

$$(14) \quad u = r \cos \theta, \quad v = r \sin \theta,$$

(12) becomes

$$(15) \quad \begin{cases} \dot{r} = cr + \Omega_2(\theta, c)r^2 + \Omega_4(\theta, c)r^3 + \dots \\ \dot{\theta} = \omega(c) + \beta_1(\theta, c)r + \beta_2(\theta, c)r^2 + \dots \end{cases}$$

where

$$(16) \quad \begin{cases} \Omega_2(\theta, c) = \sum_{k+l=2} A_{kl}(c) \cos^k \theta \sin^{l+1} \theta, \\ \Omega_4(\theta, c) = \sum_{k+l=3} A_{kl}(c) \cos^k \theta \sin^{l+1} \theta, \\ \beta_1(\theta, c) = \sum_{k+l=2} A_{kl}(c) \cos^{k+1} \theta \sin^l \theta \\ \beta_2(\theta, c) = \sum_{k+l=3} A_{kl}(c) \cos^{k+1} \theta \sin^l \theta \end{cases}$$

From (15) and (16), we deduce that, in the region such that

$$(17) \quad \dot{\theta} = \omega(c) + \beta_1(\theta, c)r + \beta_2(\theta, c)r^2 + \dots \neq 0,$$

the differential system (15) is equivalent to the one dimensional equation

$$(18) \quad \frac{dr}{d\theta} = \frac{cr + \frac{1}{16} (a_1''(0) + 4a_2'(0)c) r^3 + R(r, \theta)}{\omega(c) + \frac{3}{8\omega(c)} \left( \frac{1}{16} a_0'''(0) + \frac{1}{2} a_1''(0)c + a_2'(0)c^2 \right) r^2 + \Theta(r, \theta)},$$

where  $R$  and  $\Theta$  are  $2\pi$ -periodic functions in  $\theta$  with zero averages.

Since the right hand side of (18) is  $2\pi$ -periodic in the variable  $\theta$ , we can apply the averaging method [5], and we obtain the averaged equation of (18)

$$(19) \quad \frac{d\tilde{r}}{d\theta} = \frac{1}{\omega_0} \left( c + R_4(c)\tilde{r}^2 + \dots \right) \tilde{r}$$

where

$$(20) \quad \begin{aligned} \omega_0 &= \omega(0) = \sqrt{-a_0'(0)}, \\ R_4(c) &= \frac{1}{16} \left[ a_1''(0) + 4a_2'(0)c + O(c^2) \right]. \end{aligned}$$

The averaged equation (19) admits  $r = 0$  as a trivial equilibrium state. Under the condition

$$(21) \quad a_1''(0) \neq 0,$$

equation (19) has a non zero equilibrium point at order  $c$  given by

$$(22) \quad \tilde{r}_* = \sqrt{\frac{1}{|a_1''(0)|}} c^{1/2} + O(c), \quad \text{if } c > 0,$$

or

$$(23) \quad \tilde{r}_* = \sqrt{\frac{1}{|a_1''(0)|}} (-c)^{1/2} + O(|c|), \quad \text{if } c < 0.$$

Then the equation (1) admits a limit cycle of amplitude  $r \simeq \tilde{r}_*$ . The limit cycle is stable for  $c > 0$  and unstable for  $c < 0$ .

The above analysis leads to the following proposition.

**Proposition 2.** (*Andronov-Hopf-Poincaré bifurcation*) *The subset  $P_H$  of the function space  $(C^1(\mathbb{R}/2\pi\mathbb{Z}))^3$  defined by*

$$P_H = \left\{ (a_0, a_1, a_2) : a_0(0) = 0, a_0'(0) < 0, a_1(0) = 0 \text{ and } a_1''(0) \neq 0 \right\}$$

*is a Andronov-Hopf-Poincaré bifurcation set for the differential system (2).*

*Proof.* Let  $(x_*, 0)$  be an equilibrium point of (2). Without loss of generality, we suppose  $x_* = 0$ . Then the eigenvalues of  $dX(0, 0)$  are purely imaginary if and only if

$$a_1(0) = 0 \text{ and } a_0'(0) < 0.$$

So the bifurcation conditions are given by

$$(24) \quad a_0(0) = 0, a_1(0) = 0, a_0'(0) < 0 \text{ and } a_0''(0) < 0,$$

and the theorem follows from [5]. ■

### 3. Existence of non-homotopic periodic orbits

In this section, we study the existence problem for non-homotopic to zero periodic orbits of (2). We recall that a periodic orbit of  $X$  is non-homotopic to zero if it circles the cylinder without intercepting the  $x$ -axis. For such orbits, we define a Poincaré map (first return map), and we reduce the problem of determining these orbits to the problem of finding fixed points of the Poincaré map. The stability property of the solutions is also deduced from that of the fixed points of the Poincaré map.

**3.1. Poincaré map.** A non homotopic periodic orbit of (2) is entirely located either on the half cylinder  $y > 0$ , or on the half cylinder  $y < 0$ . In these regions, the differential system (2) is equivalent to the first order differential equation with periodic coefficients

$$(25) \quad \frac{dy}{dx} = \frac{1}{y}(a_0(x) + a_1(x)y + a_2(x)y^2).$$

Since (25) is not defined for  $y = \infty$ , we use a compactification  $M_c$  of  $M^* = S^1 \times \mathbb{R}^*$  for the study of the orbits of the dynamical system

(2) at the infinity of the cylinder  $S^1 \times \mathbb{R}$ . We defined the coordinate transformation  $g : M^* \rightarrow M^*$ , where

$$(26) \quad \begin{aligned} g(x, y) &= \left(x, \frac{1}{y}\right) \equiv (x, u), \\ g(x, \infty) &= (x, 0). \end{aligned}$$

Under the map  $g$ , the points of the infinity of  $M$ , i.e.  $y = \infty$ , become the circle  $C_0 = S^1 \times \{0\}$  and the differential equation (25) takes the form

$$(27) \quad \frac{du}{dx} = -(a_2(x)u + a_1(x)u^2 + a_0(x)u^3) \equiv F(x, u).$$

Equation (27) is a particular case of the Abel differential equation. We shall use it to describe some properties of the bifurcated periodic orbits of (2).

For any  $\eta \neq 0$ , we denote by  $u(x, \eta)$  the solution of (27) such that  $u(0, \eta) = \eta$ . Let

$$\Sigma^+ = \{(\xi, \eta) : \xi = 0 \pmod{1}, \eta > 0\}$$

and

$$\Sigma^- = \{(\xi, \eta) : \xi = 0 \pmod{1}, \eta < 0\}$$

Then  $\Sigma^\pm$  are transversal sections to the vector field  $X$ . The elements of  $\Sigma^\pm$  will be denoted simply by  $\eta$ . We define the Poincaré map  $P$  associated with  $\Sigma^\pm$  by

$$(28) \quad P(\eta) = \begin{cases} u(1, \eta), & \text{if } \eta > 0; \\ u(-1, \eta), & \text{if } \eta < 0. \end{cases}$$

A non homotopic periodic orbit of (2) in the half-cylinder  $y > 0$  (respectively  $y < 0$ ) corresponds to a solution  $u(x, \eta)$  of the initial value problem

$$(29) \quad \frac{du}{dx} = -(a_2(x)u + a_1(x)u^2 + a_0(x)u^3),$$

$$(30) \quad u(0) = u(1) \quad (\text{respectively } u(0) = u(-1)).$$

This implies that  $u(x, \eta)$  defines a non homotopic periodic orbit if and only if  $\eta$  is a fixed point of the Poincaré map  $P$ , i.e.  $P(\eta) = \eta$ . The following lemma gives the expressions for the first three derivatives of  $P$ .

**Lemma 3** (See [3]). *For  $\eta > 0$ , the first three derivatives of the Poincaré map satisfy the following formulae:*

$$\begin{aligned} P'(\eta) &= \exp\left(\int_0^1 \frac{\partial F}{\partial u}(x, u(x, \eta)) dx\right), \\ P''(\eta) &= P'(\eta) \int_0^1 \left[ \frac{\partial^2 F}{\partial u^2}(x, u(x, \eta)) \exp\left(\int_0^x \frac{\partial F}{\partial u}(\zeta, u(\zeta, \eta)) d\zeta\right) \right] dx, \\ P'''(\eta) &= P'(\eta) \left[ \frac{3}{2} \left(\frac{P'(\eta)}{P'(\eta)}\right)^2 - \int_0^1 [6a_0(x) \exp\left(2 \int_0^x \frac{\partial F}{\partial u}(\zeta, u(\zeta, \eta)) d\zeta\right)] dx \right]. \end{aligned}$$

*Proof.* By differentiating the relation

$$\frac{du}{dx}(x, \eta) = F(x, u(x, \eta))$$

with respect to the initial condition  $\eta$ , we obtain the variation equation:

$$\frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial \eta}(x, \eta) \right) = \frac{\partial F}{\partial u}(x, u(x, \eta)) \left( \frac{\partial u}{\partial \eta}(x, \eta) \right)$$

with initial condition

$$\frac{\partial u}{\partial \eta}(0, \eta) = 1.$$

Then

$$\frac{\partial u}{\partial \eta}(x, \eta) = \exp\left(\int_0^x \frac{\partial F}{\partial u}(\xi, u(\xi, \eta)) d\xi\right),$$

and the expression for  $P'$  follows from the relation

$$P'(\eta) = \frac{\partial u}{\partial \eta}(1, \eta).$$

The formulae for  $P''(\eta)$  and  $P'''(\eta)$  are obtained by successive derivations of  $P'(\eta)$ . ■

**Corollary 4.** *Let  $u(x, \eta)$  be a periodic solution of (2 $\gamma$ ), and  $\gamma$  be the non-homotopic orbit defined by  $u(x, \eta)$ . Then*

$$(31) \quad P'(\eta) = \exp(2N(\gamma)\tilde{a}_2 + T(\eta)\tilde{a}_1)$$

where  $N(\gamma) = +1$  (respectively  $N(\gamma) = -1$ ) if  $x$  varies from 0 to 1 (respectively  $x$  varies from 1 to 0),  $T(\eta)$  is the period of the solution  $\left(x(t, 0), y(t, \frac{1}{\eta})\right)$ , and  $\tilde{a}_i$ ,  $i = 1, 2$ , means the average of the periodic function  $a_i$ .



*Proof.* Let  $u(x, \eta)$  be a periodic solution of (27), with  $\eta > 0$ , then  $x$  varies from 0 to 1. By using Lemma 3, we obtain

$$\begin{aligned} P'(\eta) &= \exp\left(\int_0^1 \frac{\partial F}{\partial u}(x, u(x, \eta))dx\right) \\ &= \exp\int_0^1 - (a_2(x) + 2a_1(x)u(x, \eta) + 3a_0(x)u^2(x, \eta)) dx. \end{aligned}$$

From the relation

$$a_0(x)u^2 = - \left[ \frac{du}{udx} + a_2(x) + a_1(x)u \right],$$

we deduce that

$$\begin{aligned} P'(\eta) &= \exp\int_0^1 \left( 2a_2(x) + a_1(x)u(x, \eta) + 3a_0(x)\frac{du}{udx}(x, \eta) \right) dx \\ &= \exp(2\tilde{a}_2 + T(\eta)a_1). \end{aligned}$$

The case where  $\eta < 0$  follows by similar considerations. ■

**3.2. Bifurcations of non-homotopic orbits.** The following theorem gives the behavior of the orbits of (27) at the infinity of  $M_c$ , and describes some bifurcation properties of non-homotopic to zero periodic orbits of (2).

**Theorem 5.** *Consider the differential system (2) where the functions  $a_k$  are one periodic. Then the following properties hold.*

- (i)  $C_0$  is a limit cycle of (27) with characteristic multiplier equal to  $exp(-\tilde{a}_2)$ .
- (ii) If  $a_1 \equiv 0$ , then the Poincaré map  $P$  undergoes a saddle node bifurcation at  $\tilde{a}_0 = \tilde{a}_{0c}$ , leading to the existence of two non-homotopic periodic orbits of (2) located on  $S^1 \times \mathbb{R}^+$  and  $S^1 \times \mathbb{R}^-$  respectively, symmetric with respect to the circle  $C_0$ , one stable and the other unstable.
- (iii) If the function  $a_1$  is negative on  $[0, 1]$ , then equation (2) admits at most three non-homotopic to zero periodic solutions. When  $\tilde{a}_1$  varies on  $\mathbb{R}^+$  or  $\mathbb{R}^-$ , a stable periodic orbit of (2) never disappears.

*Proof.* (i) The property follows from the relations

$$\begin{aligned} F(x, 0) &= 0, \\ P'(0) &= \exp\left(\int_0^1 \frac{\partial F}{\partial u}(x, 0)dx\right) = \exp\left(-\int_0^1 a_2(x)dx\right) \\ &= exp(-\tilde{a}_2) \end{aligned}$$

(ii) Suppose that  $a_1 \equiv 0$ . Let  $\rho = u^2$ , then (27) becomes

$$(32) \quad \frac{d\rho}{dx} = -2(a_2(x)\rho + a_0(x)).$$

The solution  $\rho(x; x_0, \rho_0)$  of (32) such that  $\rho(x; x_0, \rho_0) = \rho_0$  is given by the relation :

$$(33) \quad \rho(x; x_0, \rho_0) = \exp(-2 \int_{x_0}^x a_2(\tau) d\tau) \\ \times \left[ \rho_0 - \int_{x_0}^x 2a_0(\tau) \exp\left(2 \int_{x_0}^{\tau} a_2(\xi) d\xi\right) d\tau \right]$$

So, the equation of the orbit of (2) issued from the point  $(x_0, y_0)$ ,  $y_0 \neq 0$ , is given by the relation

$$(34) \quad y^2(x, x_0, y_0) = y_0^2 \exp(2 \int_{x_0}^x a_2(\tau) d\tau) \\ \times \left[ 1 - 2y_0^2 \int_{x_0}^x a_0(\tau) \exp(2 \int_{x_0}^{\tau} a_2(\xi) d\xi) d\tau \right]^{-1}$$

$$y(x, x_0, y_0) = y_0 \exp\left(\int_{x_0}^x a_2(\tau) d\tau\right) \\ \times \left[ 1 - 2y_0^2 \int_{x_0}^x a_0(\tau) \exp(2 \int_{x_0}^{\tau} a_2(\xi) d\xi) d\tau \right]^{-1/2}.$$

From (34) we deduce that the phase portrait of (2) is symmetric with respect to the x-axis. Then the non homotopic orbits appear by couples on  $S^1 \times \mathbb{R}^+$  (respectively  $S^1 \times \mathbb{R}^-$ ). A solution  $y(x, 0, y_0)$  defines a non homotopic to zero orbit if and only if  $y(1, 0, y_0) = y_0$ . Then  $y_0$  must be a solution of the equation

$$(35) \quad y_0 \left\{ \exp(\tilde{a}_2) - \left[ 1 - 2y_0^2 \times \int_0^1 a_0(\zeta) \exp\left(\int_0^{\zeta} (-2a_2(\xi) d\xi)\right) d\zeta \right]^{1/2} \right\} = 0$$

i.e

$$(36) \quad \left( \exp(2\tilde{a}_2) - 1 \right) - 2y_0^2 \int_0^1 a_0(\zeta) \exp\left(\int_0^{\zeta} (-2a_2(\xi) d\xi)\right) d\zeta = 0.$$

The equation (36) is a canonical normal form of a saddle node bifurcation for the Poincaré map and admits two real solutions if and only if

$$(37) \quad \left( \exp(\tilde{a}_2) - 1 \right) \left( \int_0^1 a_0(\tau) \exp(-2 \int_0^{\tau} a_2(\xi) d\xi) d\tau \right) > 0.$$

When the condition (37) is fulfilled, the equation (2) admits two non-homotopic to zero periodic orbits which are symmetric with respect to the x-axis, i.e. to  $C_0$ . The stability property follows from (31) .

- (iii) Let  $d$  denote the displacement function defined on the transversal section  $\Sigma^+$  by

$$d(\eta) = P(\eta) - \eta.$$

Zeros of  $d$  correspond to initial conditions which define periodic orbits of (2). If  $a_0 < 0$ , then using lemma 3, we have

$$d''' = P''' > 0.$$

The first part of the property (iii) follows from the fact that, if the third derivative of the displacement function  $d$  has the same sign, then this function admits at most three zeros. For the second part of the property, the relation (31) implies that the characteristic multiplier  $P'$  decreases when  $\tilde{a}_1$  moves from 0 to  $-\infty$ . Then if  $P'(\eta)$  remains inferior to one as  $\tilde{a}_1$  moves from 0 to  $-\infty$ , a stable orbit cannot disappear.

■

**Remark 6.** *The proof of statement (ii) of the above theorem also follows using averaging theory [5].*

#### 4. CONCLUSION

In this paper, we considered a nonlinear second order differential system on the cylinder  $S^1 \times \mathbb{R}$ . Through section 2, we analyzed the conditions in order to obtain an Andronov-Hopf-Poincaré bifurcation leading to the existence of a homotopic to zero orbit. Next, in section 3, we used a Poincaré operator associated with an Abel's equation equivalent to the system for obtain some existence and stability theorems for the non homotopic orbits. More considerations on global homotopic to zero orbits obtained by using a Poincaré operator as for the non homotopic case are left for further analysis.

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